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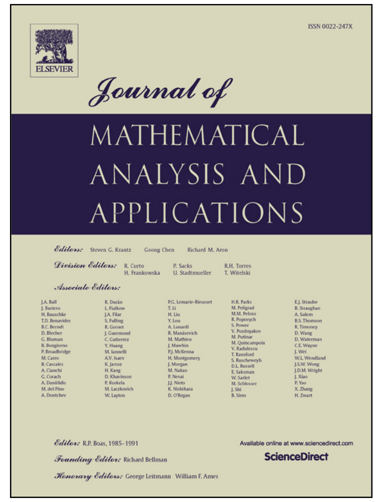

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Two-Time-Scales Hyperbolic-Parabolic Equations Driven by Poisson Random Measures: Existence, Uniqueness and Averaging Principles

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Abstract

In this article, we are concerned with averaging principle for stochastic hyperbolic-parabolic equations driven by Poisson random measures with slow and fast time-scales. We first establish the existence and uniqueness of weak solutions of the stochastic hyperbolic-parabolic equations. Then, under suitable conditions, we prove that there is a limit process in which the fast varying process is averaged out and the limit process which takes the form of the stochastic wave equation is an average with respect to the stationary measure of the fast varying process. Finally, we derive the rate of strong convergence for the slow component towards the solution of the averaged equation.

Keywords. Averaging principles, stochastic hyperbolic-parabolic equations, Poisson random measures, two-time-scales.

Mathematics subject classification. 60G51, 60H15, 70K70.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition. Fix $L > 0$ arbitrarily, we denote $D := (0, L)$, i.e., D is a fixed, open, bounded interval of the real line $\mathbb{R}$. Let $\mathbb{H}$ denote the Hilbert space $\mathbb{L}^2(D)$ equipped with the inner product $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ and the corresponding norm $\|\cdot\|$. Let $T > 0$ be fixed arbitrarily. In this paper, we are concerned with the following stochastic hyperbolic-parabolic (i.e., wave-heat) equations driven by both Brownian motions and Poisson random measures,

$$\begin{cases} \frac{\partial^2 X_t^\varepsilon(\xi)}{\partial t^2} = \Delta X_t^\varepsilon(\xi) + f(X_t^\varepsilon(\xi), Y_t^\varepsilon(\xi)) + g(X_t^\varepsilon(\xi))\dot{W}_t^1 \\ \qquad\qquad + \int_{\mathbb{Z}} h(X_{t-}^\varepsilon(\xi), z)\dot{\tilde{N}}_1(t, dz), \\ \frac{\partial Y_t^\varepsilon(\xi)}{\partial t} = \frac{1}{\varepsilon}\Delta Y_t^\varepsilon(\xi) + \frac{1}{\varepsilon}F(X_t^\varepsilon(\xi), Y_t^\varepsilon(\xi)) + \frac{1}{\sqrt{\varepsilon}}G(X_t^\varepsilon(\xi), Y_t^\varepsilon(\xi))\dot{W}_t^2 \\ \qquad\qquad + \int_{\mathbb{Z}} H(X_{t-}^\varepsilon(\xi), Y_{t-}^\varepsilon(\xi), z)\dot{\tilde{N}}_2^\varepsilon(t, dz), \\ X_t^\varepsilon(\xi) = Y_t^\varepsilon(\xi) = 0, \quad (\xi, t) \in \partial D \times (0, T], \\ X_0^\varepsilon(\xi) = X_0(\xi), \quad Y_0^\varepsilon(\xi) = Y_0(\xi), \quad \frac{\partial X_t^\varepsilon(\xi)}{\partial t}|_{t=0} = \dot{X}_0(\xi), \xi \in D, \end{cases} \tag{1.1}$$

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for $\varepsilon > 0$ and for $(\xi, t) \in D \times [0, T]$, where the drift coefficients $f(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $F(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and the diffusion coefficients $g(\cdot) : \mathbb{R} \to \mathbb{R}, G(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, h(\cdot,\cdot) : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}, H(\cdot,\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$ are real-valued measurable functions. The detailed conditions on them will be specified in the next section. Here, $\{W_t^1\}_{t\geq 0}$ and $\{W_t^2\}_{t\geq 0}$ are given independent real-valued $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motions, and $\tilde{N}_1(dt, dz)$ and $\tilde{N}_2^\varepsilon(dt, dz)$ are compensated martingale measures associated with given mutually independent Poisson random measures $N_1(dt, dz)$ and $N_2^\varepsilon(dt, dz)$, respectively. We assume that $N_1(dt, dz)$ and $N_2^\varepsilon(dt, dz)$ are also mutually independent of $\{W_t^1\}_{t\geq 0}$ and $\{W_t^2\}_{t\geq 0}$. Before proceeding, let us explicate the Poisson random measures $\tilde{N}_1(dt, dz)$ and $\tilde{N}_2^\varepsilon(dt, dz)$. Let $(\mathbb{Z}, \mathcal{B}(\mathbb{Z}))$ be a given measurable space and $v(dz)$ be a σ-finite measure on it. Let $D_{p_t^i}, i = 1, 2$, be two countable subsets of $\mathbb{R}_+$. Furthermore, let $p_t^1, t \in D_{p_t^1}$, be a stationary $\mathcal{F}_t$-adapted Poisson point process on $\mathbb{Z}$ with characteristic v and let $p_t^2, t \in D_{p_t^2}$ be a stationary $\mathcal{F}_t$-adapted Poisson point process on $\mathbb{Z}$ with characteristic $\frac{v}{\varepsilon}$. Denote by $N^i(dt, ds)$ the Poisson random (counting) measures associated with $p_t^i, i = 1, 2$, respectively, i.e., for $i = 1, 2$

$$N^i(t, A) := \sum_{s \in D_{p_t^i}, s \leq t} I_A(p_t^i), \quad t \geq 0,\ A \in \mathcal{B}(\mathbb{Z}).$$

The corresponding compensated Poisson martingale measures are respectively defined by the following

$$\tilde{N}_1(dt, dz) := N^1(dt, dz) - v(dz)dt$$

and

$$\tilde{N}_2^\varepsilon(dt, dz) := N^2(dt, dz) - \frac{1}{\varepsilon} v(dz)dt.$$

The reader is referred to [11, 25] for more detailed descriptions of the stochastic integrals with respect to (cylindrical) Wiener processes and Poisson martingale measures. It is well known nowadays that much evidence has been gathered that Poisson jumps are ubiquitous in modelling uncertainty in many diverse fields of science [1, 25, 13]. By now it is well established that stochastic dynamical systems driven by Poisson jump noises are much more suitable for capturing sudden bursty fluctuations, large scale moves and unpredictable events than classical diffusion modelling systems, see, e.g., [25, 23, 2, 15].

Note that the system (1.1) is an abstract model for random vibration of an elastic string with external force on a large time scale. More generally, the slow-fast nonlinear coupled wave-heat equations could model thermoelastic wave propagations in a random medium [9], describe wave phenomena which are heat generating or are temperature related [20], as well as model biological problems with uncertainty [10, 5, 27]. Taking advantages of the fast and slow motions, in this paper, we focus on the limit behavior of the slow-fast nonlinear coupled wave-heat equations driven by both Brownian motions and Poisson random measures, in which the original complex system is replaced by a much simpler averaged system.

There is an extensive literature on averaging principles for stochastic differential equations, see for example, Freidlin and Wentzell [14], Khasminskii [21], Duan [12], Thompson [26], Xu and his co-workers [28, 29, 30, 31, 32]. To obtain the effective approximation for the two-time-scales stochastic partial differential equations (SPDEs), the averaging approach for SPDEs begun to receive more attention recently. In [6], Cerrai and Freidlin showed an averaged result for stochastic parabolic equations with additive noise. In [7], Cerrai succeeded with the case of multiplicative noise. The concerned convergence in the latter two works, however, is in sense of convergence in probability (which implies weak convergence), and the rate of convergence has not been given. On the other hand, Bréhier [4] derived explicit convergence rates in both strong and weak convergences for averaging of stochastic parabolic equations. Xu, Miao and Liu [33] established averaging principles for two time-scale SPDEs driven by Poisson random measures in the sense of mean-square. Very recently, Fu et al [17] established an averaging principle for stochastic hyperbolic-parabolic equations driven by additive noise (Wiener process) with two-time-scales and obtained the rate of strong convergence for the slow component towards the solution of the averaging equation as a byproduct.

To the best of our knowledge, the averaging principle for stochastic hyperbolic-parabolic equations driven by jump-diffusion processes are not yet fully addressed. In this article, our main objective is to establish an effective approximation for slow process of the original system (1.1). To be more precise, the slow component X_t^ε of original system (1.1) can be approximated by the solution process $\bar{X}_t$, which governed by the following stochastic wave equation

$$\begin{cases} \frac{\partial^2 \bar{X}_t(\xi)}{\partial t^2} &= \Delta \bar{X}_t(\xi) + \bar{f}(\bar{X}_t(\xi)) + g(\bar{X}_t(\xi))\dot{W}_t^1 + \int_{\mathbb{Z}} h(\bar{X}_{t-}(\xi), z)\dot{\tilde{N}}_1(t, dz), \\ \bar{X}_t(\xi) &= 0, (\xi, t) \in \partial D \times (0, T], \\ \bar{X}_0(\xi) &= X_0(\xi), \frac{\partial \bar{X}_t(\xi)}{\partial t}|_{t=0} = \dot{X}_0(\xi), \xi \in D, \end{cases} \tag{1.2}$$

where

$$\bar{f}(x) = \int_{\mathbb{H}} f(x, y)\mu^x(dy), x \in \mathbb{H},$$

and μ^x denotes the unique invariant measure which will be introduced in Appendix B. The main novelty of this article is the model itself and how to treat the term with Poisson random measures is the key of the paper. Moreover, we will work in the framework of Green functions, which is a little different from the previous works [6, 7, 16, 18] investigating the coupled hyperbolic-parabolic equations.

The paper is organised as follows. In Section 2, we will present our main results and state some well-known facts for the later use. In Section 3, the existence, uniqueness and energy identity for an abstract hyperbolic-parabolic equation driven by Poisson random measures will be proved. In Section 4, some priori estimates will be derived. Section 5 is devoted to establishing the stochastic averaging principle in sense of strong convergence, with a determination of the explicit error bounds on the difference between the solution of the slow component and the solution of the approximating reduction equation (1.2). The paper ends with two appendices where an important lemma (Lemma 5.1) is proved in Appendix A and the ergodicity of the fast motion is discussed in Appendix B.

Throughout this paper, $C > 0$ will denote a generic constant whose value may vary in different occasions.

2. Preliminary

Recall that $\mathbb{H} = \mathbb{L}^2(D)$. Let us denote abstractly $\mathbb{A} = \partial_{\xi\xi}$ with zero Dirichlet boundary condition on $\partial D = \{0, L\}$. Let $\{e_k(\xi)\}_{k \in \mathbb{N}}$ be a complete orthornormal system of eigenvectors in $\mathbb{H}$ such that, for $k = 1, 2\cdots$,

$$\mathbb{A}e_k = -\alpha_k e_k, e_k|_{\partial D=0},$$

with $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \leq \cdots$. For $s \in \mathbb{R}$, we introduced the space $H_0^s := D((-\mathbb{A})^{s/2})$, which equipped with norm

$$\|\phi\|_s = \left\{\sum_{k=1}^{\infty} \alpha_k^s \langle \phi, e_k\rangle_{\mathbb{H}}^2\right\}^{\frac{1}{2}}.$$

Let $\mathbb{V}$ denote the Sobolev space $\mathbb{H}_0^1$ of order 1 with zero Dirichlet boundary condition, which is densely and continuously embedded in the Hilbert space $\mathbb{H}$. It is clear that for $\Lambda \in \mathbb{V} \subset \mathbb{H}$, $\|\Lambda\| \leq \alpha_1^{-\frac{1}{2}}\|\Lambda\|_{\mathbb{V}}$. Obviously, $\|\cdot\|_{\mathbb{V}} = \|\cdot\|_1$. Identifying $\mathbb{H}$ with its dual space, we obtain the following Gelfand triple

$$\mathbb{V} \subset \mathbb{H} \cong \mathbb{H}^* \subset \mathbb{V}^*.$$

Then, Poincaré inequality yields that $\langle \mathbb{A}u, u\rangle = -\|\nabla u\|^2 \leq -\alpha_1\|u\|_{\mathbb{V}}^2$, where $\langle\cdot,\cdot\rangle$ denotes the dual pair of $\mathbb{V}$ and $\mathbb{V}^*$.

Note that the Green function $U(\xi, \zeta, t)$ for the deterministic equation $(\partial/\partial t - \mathbb{A})X(t, \xi) = 0$ can be expressed as

$$U(\xi, \zeta, t) = \sum_{k=1}^{\infty} e^{-\alpha_k t} e_k(\xi)e_k(\zeta).$$

Thus, the associated Green's operator is given by the following

$$U_t\Lambda(\xi) = \int_D U(\xi, \zeta, t)\Lambda(\zeta)d\zeta = \sum_{k=1}^{\infty} e^{-\alpha_k t} e_k(\xi)\langle e_k, \Lambda\rangle_{\mathbb{H}},\ \Lambda(\xi) \in \mathbb{H}.$$

It is straightforward that $\{U_t\}_{t\geq 0}$ forms a contractive semigroup on $\mathbb{H}$ and one has that $\|U_t\Lambda(\xi)\| \leq \|\Lambda(\xi)\|$.

For the deterministic wave equation $(\partial^2/\partial t^2 - \mathbb{A})Y(t, \xi) = 0$, its Green's function is given by (cf. e.g.,[8])

$$S(\xi, \zeta, t) = \sum_{k=1}^{\infty} \frac{\sin\{\sqrt{\alpha_k}t\}}{\sqrt{\alpha_k}} e_k(\xi)e_k(\zeta).$$

It is easy to shown that the above series converge in $\mathbb{L}^2(D \times D)$ and the associated Green's operator is defined by

$$S_t\Lambda(\xi) = \int_D S(\xi, \zeta, t)\Lambda(\zeta)d\zeta = \sum_{k=1}^{\infty} \frac{\sin\{\sqrt{\alpha_k}t\}}{\sqrt{\alpha_k}} e_k(\xi)\langle e_k, \Lambda\rangle_{\mathbb{H}},\ \Lambda(\xi) \in \mathbb{H}.$$

In order to present our results in a clear manner, it is convenient to formulate our equations in an abstract setting, where system (1.1) can be rewritten as following

$$\begin{cases} \frac{d^2X_t^\varepsilon}{dt^2} & = \mathbb{A}X_t^\varepsilon + f(X_t^\varepsilon, Y_t^\varepsilon) + g(X_t^\varepsilon)\dot{W}_t^1 + \int_{\mathbb{Z}} h(X_{t-}^\varepsilon, z)\dot{\tilde{N}}_1(t, dz), \\ \frac{dY_t^\varepsilon}{dt} & = \frac{1}{\varepsilon}\mathbb{A}Y_t^\varepsilon + \frac{1}{\varepsilon}F(X_t^\varepsilon, Y_t^\varepsilon) + \frac{1}{\sqrt{\varepsilon}}G(X_t^\varepsilon, Y_t^\varepsilon)\dot{W}_t^2 \\ & \quad + \int_{\mathbb{Z}} H(X_{t-}^\varepsilon, Y_{t-}^\varepsilon, z)\dot{\tilde{N}}_2^\varepsilon(t, dz), \\ & X_0 \in \mathbb{V}, Y_0 \in \mathbb{H}, \frac{dX_t^\varepsilon}{dt}|_{t=0} = \dot{X}_0 \in \mathbb{H}. \end{cases} \tag{2.1}$$

The system (2.1) is understood in terms of the following two integral equations

$$\begin{cases} X_t^\varepsilon &= S_t' X_0 + S_t \dot{X}_0 + \int_0^t S_{t-s} f(X_s^\varepsilon, Y_s^\varepsilon) ds + \int_0^t S_{t-s} g(X_s^\varepsilon) dW_s^1 \\ & \quad + \int_0^t \int_{\mathbb{Z}} S_{t-s} h(X_{s-}^\varepsilon, z) \tilde{N}_1(ds, dz), \\ Y_t^\varepsilon &= U_{t/\varepsilon} Y_0 + \frac{1}{\varepsilon} \int_0^t U_{(t-s)/\varepsilon} F(X_s^\varepsilon, Y_s^\varepsilon) ds + \frac{1}{\sqrt{\varepsilon}} \int_0^t U_{(t-s)/\varepsilon} G(X_s^\varepsilon, Y_s^\varepsilon) dW_s^2 \\ & \quad + \int_0^t \int_{\mathbb{Z}} U_{(t-s)/\varepsilon} H(X_{s-}^\varepsilon, Y_{s-}^\varepsilon, z) \tilde{N}_2^\varepsilon(ds, dz), \end{cases} \tag{2.2}$$

where $S_t' = \frac{d}{dt} S_t$ is the derived Green's operator with integral kernel

$$K'(\xi, \zeta, t) = \sum_{k=1}^{\infty} \cos\{\sqrt{\alpha_k} t\} e_k(\xi) e_k(\zeta).$$

We now give the definition of mild solutions of (2.1)

Definition 2.1. The pair $(X_t^\varepsilon, Y_t^\varepsilon)$ of two adapted processes over $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is called a mild solution of (2.1), *if for any $t > 0$, the integral equations (2.2) hold true $\mathbb{P}$-a.s..*

Next, let us introduce the globally Lipschitz condition for (2.1). We assume the following

(A1) The coefficients of (2.1) are globally Lipschitz continuous in x, y, i.e., $\forall x_1, x_2, y_1, y_2 \in \mathbb{R}$, there exist six positive constants $C_f, C_g, C_h, C_F, C_G, C_H$, we have

$$\begin{aligned} &|f(x_1, y_1) - f(x_2, y_2)|^2 \le C_f(|x_1 - x_2|^2 + |y_1 - y_2|^2), \\ &|g(x_1) - g(x_2)|^2 \le C_g |x_1 - x_2|^2, \\ &\int_{\mathbb{Z}} |h(x_1, z) - h(x_2, z)|^2 v(dz) \le C_h |x_1 - x_2|^2, \end{aligned}$$

and

$$\begin{aligned} &|F(x_1, y_1) - F(x_2, y_2)|^2 \le C_F(|x_1 - x_2|^2 + |y_1 - y_2|^2), \\ &|G(x_1, y_1) - G(x_2, y_2)|^2 \le C_G(|x_1 - x_2|^2 + |y_1 - y_2|^2), \\ &\int_{\mathbb{Z}} |H(x_1, y_1, z) - H(x_2, y_2, z)|^2 v(dz) \le C_H(|x_1 - x_2|^2 + |y_1 - y_2|^2). \end{aligned}$$

(A2) $\eta' := \alpha_1 - C_F' - C_G - C_H > 0$, $C_F' = \max\{C_F, 1\}$. This condition is a strong dissipative condition, it is very important to prove the ergodicity for the fast motion. The detailed proofs will be given in Appendix B.

Remark 2.2. With assumption (A1), it immediately follows,

$$\begin{aligned} &|f(x_1, y_1)|^2 + |g(x_1)|^2 + |F(x_1, y_1)|^2 + |G(x_1, y_1)|^2 \\ &\quad + \int_{\mathbb{Z}} |h(x_1, z)|^2 v(dz) + \int_{\mathbb{Z}} |H(x_1, y_1, z)|^2 v(dz) \\ &\le 2(C_f + C_g + C_h + C_F + C_G + C_H)|x_1|^2 \\ &\quad + 2(C_f + C_F + C_G + C_H)|y_1|^2 \\ &\quad + 2(|f(0,0)|^2 + |g(0)|^2 + \int_{\mathbb{Z}} |h(0, z)|^2 v(dz)), \\ &\quad + 2(|F(0,0)|^2 + |G(0,0)|^2 + \int_{\mathbb{Z}} |H(0, 0, z)|^2 v(dz)), \end{aligned}$$

for all $x_1, y_1 \in \mathbb{R}$.

Remark 2.3. Since for each $t \geq 0, S_t$ is a Green's operator, the cosine family of operators $\{S'_t : t \in [0,T]\}$ and the corresponding sine family of operators $\{S_t : t \in [0,T]\}$ satisfy that $\|S'_t\|^2 \leq M$ and $\|S_t\|^2 \leq M$ for a positive constant M.

Theorem 2.4. *Let* (A1)-(A2) *hold, then for any* $X_0 \in \mathbb{V}$, $\dot{X}_0, Y_0 \in \mathbb{H}$ *and* $T > 0$, *we have*

$$\mathbb{E}(\sup_{0\leq t\leq T} \|\dot{X}^\varepsilon_t - \dot{\bar{X}}_t\|^2 + \sup_{0\leq t\leq T} \|X^\varepsilon_t - \bar{X}_t\|^2_{\mathbb{V}}) \leq C\sqrt{\varepsilon}, \tag{2.3}$$

where $\bar{X}_t$ *is the solution of the effective dynamical system (1.2).*

3. Existence, Uniqueness and Energy Equality

For the separable Hilbert space $\mathbb{H}$, we use $\mathbb{M}^2([0,T];\mathbb{H})$ to denote the Hilbert space of progressively measurable, square integrable, $\mathbb{H}$-valued processes equipped with the inner product

$$\langle u, u'\rangle_{\mathbb{M}} := \mathbb{E}\int_0^T \langle u(t), u'(t)\rangle_{\mathbb{H}} dt.$$

We also define $\mathbb{M}^{v,2}([0,T]\times\mathbb{Z},\mathbb{H})$ to be the totality of all predictable mappings $\Phi(s,z,\omega)$: $[0,T]\times\mathbb{Z}\times\Omega \to \mathbb{H}$ such that

$$\mathbb{E}\int_0^T\int_{\mathbb{Z}} \|\Phi(s,z,\omega)\|^2 v(dz)ds < \infty.$$

In addition, we denote by $\mathbb{D}([0,T];\mathbb{H})$ the space of all càdlàg paths from $[0,T]$ into $\mathbb{H}$.

3.1. Weak Solution of the Linear Hyperbolic-Parabolic Equations

Consider the following linear equations

$$\frac{d^2X_t}{dt^2} = \mathbb{A}X_t + f_t + g_t\dot{W}^1_t + \int_{\mathbb{Z}} h(t,z)\dot{\tilde{N}}_1(t,dz), X_0 = x_0, \frac{dX_t}{dt}|_{t=0} = \dot{x}_0, \tag{3.1}$$

and

$$\frac{dY_t}{dt} = \mathbb{A}Y_t + F_t + G_t\dot{W}^2_t + \int_{\mathbb{Z}} H(t,z)\dot{\tilde{N}}_2(t,dz), Y_0 = y_0, \tag{3.2}$$

where $f,g,F,G \in \mathbb{M}^2([0,T],\mathbb{H}), h, H \in \mathbb{M}^{v,2}([0,T]\times\mathbb{Z},\mathbb{H})$.

By Itô's formula, one can get the following Lemmas 3.1 and 3.2 (cf. [3, 22])

Lemma 3.1. *Assume that* $f, g \in \mathbb{M}^2([0,T],\mathbb{H}), h \in \mathbb{M}^{v,2}([0,T]\times\mathbb{Z},\mathbb{H})$. *Then there is a unique weak solution* $(X_t,\dot{X}_t) \in \mathbb{L}^2(\Omega;C([0,T];\mathbb{V}))\times(\mathbb{D}([0,T];\mathbb{H})\cap\mathbb{L}^2(\Omega\times[0,T];\mathbb{H}))$ *of (3.1) such that the following holds a.s.*

$$\begin{aligned}\|\dot{X}_t\|^2 &= \|\dot{X}_0\|^2 + \langle\mathbb{A}X_t, X_t\rangle - \langle\mathbb{A}X_0, X_0\rangle + 2\int_0^t\langle f_s, \dot{X}_s\rangle_{\mathbb{H}}ds + 2\int_0^t\langle g_s,\dot{X}_s\rangle_{\mathbb{H}}dW^1_s \\ &\quad + \int_0^t\|g_s\|^2ds + \int_0^t\int_{\mathbb{Z}}[\|h(s,z)\|^2 + 2\langle h(s,z),\dot{X}_{s-}\rangle_{\mathbb{H}}]\tilde{N}_1(ds,dz) \\ &\quad + \int_0^t\int_{\mathbb{Z}}\|h(s,z)\|^2v(dz)ds. \end{aligned} \tag{3.3}$$

Lemma 3.2. *Assume that $F, G \in \mathbb{M}^2([0,T],\mathbb{H}), H \in \mathbb{M}^{v,2}([0,T]\times\mathbb{Z},\mathbb{H})$. Then there is a unique weak solution $Y_t \in \mathbb{D}([0,T];\mathbb{H}) \cap \mathbb{M}^2(\Omega\times[0,T];\mathbb{V})$ of (3.2) such that the following holds a.s.*

$$\begin{aligned}\|Y_t\|^2 &= \|Y_0\|^2 + \int_0^t \langle \mathbb{A}Y_s, Y_s\rangle ds + 2\int_0^t \langle F_s, Y_s\rangle_{\mathbb{H}} ds + 2\int_0^t \langle G_s, Y_s\rangle_{\mathbb{H}} dW_s^2 \\ &\quad + \int_0^t \|G_s\|^2 ds + \int_0^t \int_{\mathbb{Z}} [\|H(s,z)\|^2 + 2\langle H(s,z), Y_{s-}\rangle_{\mathbb{H}}]\tilde{N}_2(ds,dz) \\ &\quad + \int_0^t \int_{\mathbb{Z}} \|H(s,z)\|^2 v(dz)ds. \end{aligned} \tag{3.4}$$

3.2. Weak Solution of the Stochastic Nonlinear Hyperbolic-Parabolic Equations

For fixed $x_0 \in \mathbb{V}, \dot{x}_0, y_0 \in \mathbb{H}$, we now discuss the existence and uniqueness results for the nonlinear hyperbolic-parabolic equations

$$\begin{cases} \frac{d^2X_t}{dt^2} &= \mathbb{A}X_t + f(X_t,Y_t) + g(X_t)\dot{W}_t^1 + \int_{\mathbb{Z}} h(X_{t-},z)\dot{\tilde{N}}_1(t,dz), \\ \frac{dY_t}{dt} &= \mathbb{A}Y_t + F(X_t,Y_t) + G(X_t,Y_t)\dot{W}_t^2 + \int_{\mathbb{Z}} H(X_{t-},Y_{t-},z)\dot{\tilde{N}}_2(t,dz), \end{cases} \tag{3.5}$$

where $X_0 = x_0, \frac{dX_t}{dt}|_{t=0} = \dot{x}_0, Y_0 = y_0$.

Lemma 3.3. *Assume that the conditions* (A1)-(A2) *are satisfied. Given $X_0 \in \mathbb{V}, \dot{X}_0, Y_0 \in \mathbb{H}$, then there is a unique weak solution (also mild solution) $(X_t, Y_t) \in \mathbb{L}^2(\Omega; C([0,T];\mathbb{V})) \times (\mathbb{D}([0,T];\mathbb{H}) \cap \mathbb{M}^2(\Omega\times[0,T];\mathbb{V}))$ of (3.5) such that for $t\in[0,T]$, the following two energy identities hold a.s.*

$$\begin{aligned}\|\dot{X}_t\|^2 &= \|\dot{X}_0\|^2 + \langle \mathbb{A}X_t, X_t\rangle - \langle \mathbb{A}X_0, X_0\rangle + 2\int_0^t \langle f(X_s,Y_s), \dot{X}_s\rangle_{\mathbb{H}} ds \\ &\quad +2\int_0^t \langle g(X_s), \dot{X}_s\rangle_{\mathbb{H}} dW_s^1 + \int_0^t \|h(X_s)\|^2 ds \\ &\quad + \int_0^t \int_{\mathbb{Z}} [\|h(X_{s-},z)\|^2 + 2\langle h(X_{s-},z), \dot{X}_{s-}\rangle_{\mathbb{H}}]\tilde{N}_1(ds,dz) \\ &\quad + \int_0^t \int_{\mathbb{Z}} \|h(X_s,z)\|^2 v(dz)ds, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\|Y_t\|^2 &= \|Y_0\|^2 + 2\int_0^t \langle \mathbb{A}Y_s, Y_s\rangle ds + 2\int_0^t \langle F(X_s,Y_s), Y_s\rangle_{\mathbb{H}} ds \\ &\quad +2\int_0^t \langle G(X_s,Y_s), Y_s\rangle_{\mathbb{H}} dW_s^2 + \int_0^t \|G(X_s,Y_s)\|^2 ds \\ &\quad + \int_0^t \int_{\mathbb{Z}} [\|H(X_{s-},Y_{s-},z)\|^2 + 2\langle H(X_{s-},Y_{s-},z), Y_{s-}\rangle_{\mathbb{H}}]\tilde{N}_2(ds,dz) \\ &\quad + \int_0^t \int_{\mathbb{Z}} \|H(X_s,Y_s,z)\|^2 v(dz)ds. \end{aligned} \tag{3.7}$$

Proof: We will verify the existence by utilising successive approximations. Let $u_t = X_t, v_t = \dot{X}_t, w_t = Y_t$ and

$$\begin{cases} u_t^0 &= x_0 + \int_0^t \dot{x}_0 ds, \\ v_t^0 &= \dot{x}_0, w_t^0 = y_0. \end{cases}$$

For $n \geq 1$, let (u_t^n, v_t^n, w_t^n) be the unique weak solution to the follow

$$\begin{cases} u_t^{n+1} &= x_0 + \int_0^t v_s^{n+1} ds, \\ v_t^{n+1} &= \dot{x}_0 + \int_0^t \mathbb{A} u_s^{n+1} ds + \int_0^t f(u_s^n, w_s^n) ds + \int_0^t g(u_s^n) dW_s^1 \\ & \quad + \int_0^t \int_{\mathbb{Z}} h(u_{s-}^n, z) \tilde{N}_1(ds, dz), \\ w_t^{n+1} &= y_0 + \int_0^t \mathbb{A} w_s^{n+1} ds + \int_0^t F(u_s^n, w_s^n) ds + \int_0^t G(u_s^n, w_s^n) dW_s^2 \\ & \quad + \int_0^t \int_{\mathbb{Z}} H(u_{s-}^n, w_{s-}^n, z) \tilde{N}_2(ds, dz). \end{cases} \tag{3.8}$$

Using the energy equality (3.3), it follows that

$$\begin{aligned} \|v_t^{n+1} - v_t^n\|^2 &= \langle \mathbb{A}(u_t^{n+1} - u_t^n), u_t^{n+1} - u_t^n \rangle \\ &\quad +2 \int_0^t \langle f(u_s^n, w_s^n) - f(u_s^{n-1}, w_s^{n-1}), v_s^{n+1} - v_s^n \rangle_{\mathbb{H}} ds \\ &\quad +2 \int_0^t \langle g(u_s^n) - g(u_s^{n-1}), v_s^{n+1} - v_s^n \rangle_{\mathbb{H}} dW_s^1 + \int_0^t \|g(u_s^n) - g(u_s^{n-1})\|^2 ds \\ &\quad + \int_0^t \int_{\mathbb{Z}} \|h(u_{s-}^n, z) - h(u_{s-}^{n-1}, z)\|^2 \tilde{N}_1(ds, dz) \\ &\quad +2 \int_0^t \int_{\mathbb{Z}} \langle h(u_{s-}^n, z) - h(u_{s-}^{n-1}, z), v_{s-}^{n+1} - v_{s-}^n \rangle_{\mathbb{H}} \tilde{N}_1(ds, dz) \\ &\quad + \int_0^t \int_{\mathbb{Z}} \|h(u_s^n, z) - h(u_s^{n-1}, z)\|^2 v(dz) ds \\ &\leq -\alpha_1 \|u_t^{n+1} - u_t^n\|_{\mathbb{V}}^2 + \sum_{i=1}^6 J_{i,t}. \end{aligned} \tag{3.9}$$

By Condition (A1) and the inequality $|ab| \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2, \varepsilon > 0$, it turns out

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} |J_{1,s}| &= 2\mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s \langle f(u_r^n, w_r^n) - f(u_r^{n-1}, w_r^{n-1}), v_r^{n+1} - v_r^n \rangle_{\mathbb{H}} dr \right| \\ &\leq C\mathbb{E} \int_0^t (\|v_r^{n+1} - v_r^n\|^2 + \|u_r^n - u_r^{n-1}\|_{\mathbb{V}}^2 + \|w_r^n - w_r^{n-1}\|^2) dr, \end{aligned} \tag{3.10}$$

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} |J_{2,s}| &= 2\mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s \langle g(u_r^n) - g(u_r^{n-1}), v_r^{n+1} - v_r^n \rangle_{\mathbb{H}} dW_r^1 \right| \\ &\leq \varepsilon C \mathbb{E} \sup_{0 \leq s \leq t} \|v_s^{n+1} - v_s^n\|^2 + CC_\varepsilon \mathbb{E} \int_0^t \|u_r^n - u_r^{n-1}\|_{\mathbb{V}}^2 dr, \end{aligned} \tag{3.11}$$

$$\mathbb{E} \sup_{0 \leq s \leq t} |J_{3,s}| = \mathbb{E} \sup_{0 \leq s \leq t} \int_0^s \|g(u_r^n) - g(u_r^{n-1})\|^2 dr$$

$$\leq \quad C\mathbb{E}\int_0^t \|u_r^n - u_r^{n-1}\|^2 dr, \tag{3.12}$$

$$\begin{aligned}
\mathbb{E}\sup_{0\leq s\leq t}|J_{6,s}| &= \mathbb{E}\sup_{0\leq s\leq t}\int_0^s\int_{\mathbb{Z}}\|h(u_r^n,z)-h(u_r^{n-1},z)\|^2 v(dz)dr\\
&\leq C\mathbb{E}\int_0^t\|u_r^n-u_r^{n-1}\|^2dr.
\end{aligned}\tag{3.13}$$

For $J_{4,s}$, $J_{5,s}$, by Burkholer's inequality, we have

$$\begin{aligned}
\mathbb{E}\sup_{0\leq s\leq t}|J_{4,s}| &\leq C[J_4,J_4]_t^{\frac{1}{2}}\\
&\leq C\mathbb{E}\left\{\sum_{s\in D_{p_t^1},s\leq t}(\|h(u_s^n,p_s^1)-h(u_s^{n-1},p_s^1)\|^2)^2\right\}^{\frac{1}{2}}\\
&\leq C\mathbb{E}\sum_{s\in D_{p_t^1},s\leq t}\|h(u_s^n,p_s^1)-h(u_s^{n-1},p_s^1)\|^2\\
&= C\mathbb{E}\int_0^t\int_{\mathbb{Z}}\|h(u_s^n,z)-h(u_s^{n-1},z)\|^2v(dz)ds\\
&\leq C\mathbb{E}\int_0^t\|u_s^n-u_s^{n-1}\|_{\mathbb{V}}^2ds.
\end{aligned}\tag{3.14}$$

$$\begin{aligned}
\mathbb{E}\sup_{0\leq s\leq t}|J_{5,s}| &\leq C[J_5,J_5]_t^{\frac{1}{2}}\\
&\leq C\mathbb{E}\left\{\sum_{s\in D_{p_t^1},s\leq t}\langle h(u_s^n,p_s^1)-h(u_s^{n-1},p_s^1),v_s^{n+1}-v_s^n\rangle_{\mathbb{H}}^2\right\}^{\frac{1}{2}}\\
&\leq C\mathbb{E}\left\{\sum_{s\in D_{p_t^1},s\leq t}\|h(u_s^n,p_s^1)-h(u_s^{n-1},p_s^1)\|^2\|v_s^{n+1}-v_s^n\|^2\right\}^{\frac{1}{2}}\\
&\leq C\varepsilon\mathbb{E}\sup_{0\leq s\leq t}\|v_s^{n+1}-v_s^n\|^2+CC_\varepsilon\mathbb{E}\sum_{s\in D_{p_t^1},s\leq t}\|h(u_s^n,p_s^1)-h(u_s^{n-1},p_s^1)\|^2\\
&\leq C\varepsilon\mathbb{E}\sup_{0\leq s\leq t}\|v_s^{n+1}-v_s^n\|^2+CC_\varepsilon\mathbb{E}\int_0^t\int_{\mathbb{Z}}\|h(u_s^n,z)-h(u_s^{n-1},z)\|^2v(dz)ds\\
&\leq C\varepsilon\mathbb{E}\sup_{0\leq s\leq t}\|v_s^{n+1}-v_s^n\|^2+CC_\varepsilon\mathbb{E}\int_0^t\|u_s^n-u_s^{n-1}\|_{\mathbb{V}}^2ds.
\end{aligned}\tag{3.15}$$

Therefore, gathering (3.10)-(3.15) and choosing $\varepsilon > 0$ sufficiently small, it yields that

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq s\leq t}(\|v_s^{n+1}-v_s^n\|^2+\|u_s^{n+1}-u_s^n\|_{\mathbb{V}}^2)\\
&\quad\leq C\mathbb{E}\int_0^t(\|v_r^{n+1}-v_r^n\|^2+\|u_r^n-u_r^{n-1}\|_{\mathbb{V}}^2+\|w_r^n-w_r^{n-1}\|^2)dr.
\end{aligned}\tag{3.16}$$

To proceed, using the energy equality (3.4), and the fact that

$$\langle \mathbb{A}(w_t^{n+1} - w_t^n), w_t^{n+1} - w_t^n \rangle \leq 0,$$

we obtain

$$\begin{aligned}
\|w_t^{n+1} - w_t^n\|^2 \leq\ & 2\int_0^t \langle F(u_s^n, w_s^n) - F(u_s^{n-1}, w_s^{n-1}), w_s^{n+1} - w_s^n \rangle_{\mathbb{H}} ds \\
&+2\int_0^t \langle G(u_s^n, w_s^n) - G(u_s^{n-1}, w_s^{n-1}), w_s^{n+1} - w_s^n \rangle_{\mathbb{H}} dW_s^2 \\
&+ \int_0^t \|G(u_s^n, w_s^n) - G(u_s^{n-1}, w_s^{n-1})\|^2 ds \\
&+ \int_0^t \int_{\mathbb{Z}} \|H(u_{s-}^n, w_{s-}^n, z) - H(u_{s-}^{n-1}, w_{s-}^{n-1}, z)\|^2 \tilde{N}_2(ds, dz) \\
&+2\int_0^t \int_{\mathbb{Z}} \langle H(u_{s-}^n, w_{s-}^n, z) - H(u_{s-}^{n-1}, w_{s-}^{n-1}, z), w_{s-}^{n+1} - w_{s-}^n \rangle_{\mathbb{H}} \tilde{N}_2(ds, dz) \\
&+ \int_0^t \int_{\mathbb{Z}} \|H(u_s^n, w_s^n, z) - H(u_s^{n-1}, w_s^{n-1}, z)\|^2 v(dz) ds.
\end{aligned}$$

By a similar calculation as in (3.16), it follows that

$$\mathbb{E} \sup_{0\leq s\leq t} \|w_s^{n+1} - w_s^n\|^2 \leq C\mathbb{E} \int_0^t (\|w_r^{n+1} - w_r^n\|^2 + \|u_r^n - u_r^{n-1}\|_{\mathbb{V}}^2 + \|w_r^n - w_r^{n-1}\|^2) dr. \quad (3.17)$$

Putting (3.16) and (3.17) together, we then have

$$\begin{aligned}
&\mathbb{E} \sup_{0\leq s\leq t} (\|v_s^{n+1} - v_s^n\|^2 + \|u_s^{n+1} - u_s^n\|_{\mathbb{V}}^2 + \|w_s^{n+1} - w_s^n\|^2) \\
&\leq C\mathbb{E} \int_0^t (\|v_r^n - v_r^{n-1}\|^2 + \|u_r^n - u_r^{n-1}\|_{\mathbb{V}}^2 + \|w_r^n - w_r^{n-1}\|^2) dr \\
&\quad +C\mathbb{E} \int_0^t (\|w_r^{n+1} - w_r^n\|^2 + \|v_r^{n+1} - v_r^n\|^2 + \|u_r^{n+1} - u_r^n\|_{\mathbb{V}}^2) dr.
\end{aligned}$$

Let $\Gamma_s^{n+1} = \|v_s^{n+1} - v_s^n\|^2 + \|u_s^{n+1} - u_s^n\|_{\mathbb{V}}^2 + \|w_s^{n+1} - w_s^n\|^2$, we have

$$\mathbb{E} \sup_{0\leq s\leq t} \Gamma_s^{n+1} \leq C\mathbb{E} \int_0^t \Gamma_s^n ds + C\mathbb{E} \int_0^t \Gamma_s^{n+1} ds, \quad (3.18)$$

where $\Gamma_s^0 = \|v_s^1 - v_s^0\|^2 + \|u_s^1 - u_s^0\|_{\mathbb{V}}^2 + \|w_s^1 - w_s^0\|^2$.

Iterating (3.18), we obtain

$$\mathbb{E} \sup_{0\leq s\leq t} \Gamma_s^{n+1} \leq C\frac{(C_T T)^n}{n!}.$$

This implies that there exists $(u_t, w_t) \in \mathbb{L}^2(\Omega; C([0,T]; \mathbb{V})) \times (\mathbb{D}([0,T]; \mathbb{H}) \cap \mathbb{M}^2(\Omega \times [0,T]; \mathbb{V}))$ such that

$$\lim_{n\to\infty} \mathbb{E} \sup_{0\leq s\leq t} (\|u_s^n - u_s\|_{\mathbb{V}}^2 + \|v_s^n - v_s\|^2 + \|w_s^n - w_s\|^2) \to 0.$$

Letting $n \to \infty$ in (3.8), we claim that (u_t, w_t) is a weak solution of (3.5). The uniqueness is a directive consequence of the energy equalities and Gronwall's inequality. To verify the energy equalities, one has the following convergence in mean square as $n \to \infty$ for all $0 \le t \le T$,

$$\begin{aligned}
&v_t^n \to v_t,\\
&u_t^n \to u_t,\\
&\int_0^t \langle f(u_s^n, w_s^n), v_s^n\rangle_{\mathbb{H}} ds \to \int_0^t \langle f(u_s, w_s), v_s\rangle_{\mathbb{H}} ds,
\end{aligned} \tag{3.19}$$

and hence

$$\begin{aligned}
&\int_0^t \langle g(u_s^n), v_s^n\rangle_{\mathbb{H}} dW_s^1 \to \int_0^t \langle g(u_s), v_s\rangle_{\mathbb{H}} dW_s^1,\\
&\int_0^t \|g(u_s^n)\|^2 ds \to \int_0^t \|g(u_s)\|^2 ds,\\
&\int_0^t \int_{\mathbb{Z}} \|h(u_{s-}^n, z)\|^2 \tilde{N}_1(ds, dz) \to \int_0^t \int_{\mathbb{Z}} \|h(u_{s-}, z)\|^2 \tilde{N}_1(ds, dz),\\
&\int_0^t \int_{\mathbb{Z}} \langle h(u_{s-}^n, z), v_{s-}^n\rangle_{\mathbb{H}} \tilde{N}_1(ds, dz) \to \int_0^t \int_{\mathbb{Z}} \langle h(u_{s-}, z), v_{s-}\rangle_{\mathbb{H}} \tilde{N}_1(ds, dz),\\
&\int_0^t \int_{\mathbb{Z}} \|h(u_s^n, z)\|^2 v(dz) ds \to \int_0^t \int_{\mathbb{Z}} \|h(u_s, z)\|^2 v(dz) ds.
\end{aligned} \tag{3.20}$$

in mean as $n \to \infty$ for all $0 \le t \le T$. Then by taking a subsequence converging $\mathbb{P}$-a.s. for (3.20), one can obtain the energy equality given by (3.6). By a similar calculation we can get the energy equality (3.7). □

4. A priori bounds for the solution

The following three lemmas provide mean square estimates for the process X_t^ε and Y_t^ε with bounds independent of ε.

Lemma 4.1. *Assume that the conditions* (A1)-(A2) *are satisfied. Given* $X_0 \in \mathbb{V}, \dot{X}_0, Y_0 \in \mathbb{H}$*, then there exists a constant* $C > 0$ *such that*

$$\sup_{0\le t\le T} \mathbb{E}(\|\dot{X}_t^\varepsilon\|^2 + \|X_t^\varepsilon\|_{\mathbb{V}}^2) \le C. \tag{4.1}$$

and

$$\sup_{0\le t\le T} \mathbb{E}\|Y_t^\varepsilon\|^2 \le C. \tag{4.2}$$

Proof: By the energy equality (3.6), we have

$$\begin{aligned}
\mathbb{E}(\|\dot{X}_t^\varepsilon\|^2 + \alpha_1\|X_t^\varepsilon\|_{\mathbb{V}}^2) \quad \le \quad & \mathbb{E}(\|\dot{X}_0\|^2 + \alpha_1\|X_0\|_{\mathbb{V}}^2) + C\mathbb{E}\int_0^t \langle f(X_s^\varepsilon, Y_s^\varepsilon), \dot{X}_s^\varepsilon\rangle_{\mathbb{H}} ds\\
& + C\mathbb{E}\int_0^t \|g(X_s^\varepsilon)\|^2 ds + C\mathbb{E}\int_0^t \int_{\mathbb{Z}} \|h(X_s^\varepsilon, z)\|^2 v(dz) ds
\end{aligned}$$

$$\begin{aligned}&\leq \mathbb{E}(\|\dot{X}_0\|^2+\alpha_1\|X_0\|^2_{\mathbb{V}})+C\mathbb{E}\int_0^t(\|\dot{X}^\varepsilon_s\|^2+\alpha_1\|X^\varepsilon_s\|^2_{\mathbb{V}})ds\\&\quad+C\mathbb{E}\int_0^t(1+\|Y^\varepsilon_s\|^2)ds.\end{aligned}$$

Thus

$$\mathbb{E}(\|\dot{X}^\varepsilon_t\|^2+\alpha_1\|X^\varepsilon_t\|^2_{\mathbb{V}})\leq e^{Ct}\mathbb{E}(\|\dot{X}_0\|^2+\alpha_1\|X_0\|^2_{\mathbb{V}})+C\int_0^t e^{C(t-s)}(1+\mathbb{E}\|Y^\varepsilon_s\|^2)ds. \tag{4.3}$$

Hence, we have for $\|Y^\varepsilon_t\|^2$

$$\begin{aligned}\frac{d}{dt}\mathbb{E}\|Y^\varepsilon_t\|^2 &\leq \frac{2}{\varepsilon}\mathbb{E}\langle\mathbb{A}Y^\varepsilon_t,Y^\varepsilon_t\rangle+\frac{2}{\varepsilon}\mathbb{E}\langle F(X^\varepsilon_t,Y^\varepsilon_t)-F(X^\varepsilon_t,0),Y^\varepsilon_t\rangle_{\mathbb{H}}+\frac{2}{\varepsilon}\mathbb{E}\langle F(X^\varepsilon_t,0),Y^\varepsilon_t\rangle_{\mathbb{H}}\\&\quad+\frac{2}{\varepsilon}\mathbb{E}\|G(X^\varepsilon_t,Y^\varepsilon_t)-G(X^\varepsilon_t,0)\|^2+\frac{2}{\varepsilon}\mathbb{E}\|G(X^\varepsilon_t,0)\|^2\\&\quad+\frac{2}{\varepsilon}\mathbb{E}\int_{\mathbb{Z}}\|H(X^\varepsilon_t,Y^\varepsilon_t,z)-H(X^\varepsilon_t,0,z)\|^2v(dz)+\frac{2}{\varepsilon}\mathbb{E}\int_{\mathbb{Z}}\|H(X^\varepsilon_t,0,z)\|^2v(dz)\\&\leq -\frac{2\alpha_1}{\varepsilon}\mathbb{E}\|Y^\varepsilon_t\|^2+\frac{C_F+1}{\varepsilon}\mathbb{E}\|Y^\varepsilon_t\|^2+\frac{C}{\varepsilon}\mathbb{E}\|F(X^\varepsilon_t,0)\|^2\\&\quad+\frac{\alpha_1-1-C_G-C_H}{\varepsilon}\mathbb{E}\|Y^\varepsilon_t\|^2+\frac{2C_G+2C_H}{\varepsilon}\mathbb{E}\|Y^\varepsilon_t\|^2\\&\quad+\frac{2}{\varepsilon}\mathbb{E}\|G(X^\varepsilon_t,0)\|^2+\frac{2}{\varepsilon}\mathbb{E}\int_{\mathbb{Z}}\|H(X^\varepsilon_t,0,z)\|^2v(dz)\\&\leq -\frac{\eta}{\varepsilon}\mathbb{E}\|Y^\varepsilon_t\|^2+\frac{C}{\varepsilon}(1+\mathbb{E}\|X^\varepsilon_t\|^2_{\mathbb{V}}),\end{aligned}$$

where $\eta=\alpha_1-C_F-C_G-C_H>0$. Next, according to (4.3), we have

$$\begin{aligned}\mathbb{E}\|Y^\varepsilon_t\|^2 &\leq e^{-\frac{\eta}{\varepsilon}t}\mathbb{E}\|Y_0\|^2+\frac{C}{\varepsilon}\mathbb{E}\int_0^t e^{-\frac{\eta}{\varepsilon}(t-s)}(1+\|X^\varepsilon_s\|^2_{\mathbb{V}})ds\\&\leq C\mathbb{E}(1+\|Y_0\|^2+\|\dot{X}_0\|^2+\|X_0\|^2_{\mathbb{V}})+\frac{C}{\varepsilon}\mathbb{E}\int_0^t e^{-\frac{\eta}{\varepsilon}(t-s)}\int_0^s\|Y^\varepsilon_r\|^2drds.\end{aligned}$$

By change of variables, we get

$$\begin{aligned}\mathbb{E}\|Y^\varepsilon_t\|^2 &\leq C\mathbb{E}(1+\|Y_0\|^2+\|\dot{X}_0\|^2+\|X_0\|^2_{\mathbb{V}})+C\int_0^t\mathbb{E}\|Y^\varepsilon_r\|^2\left[\int_0^{\frac{t-r}{\varepsilon}}e^{-\eta v}dv\right]dr\\&\leq C\mathbb{E}(1+\|Y_0\|^2+\|\dot{X}_0\|^2+\|X_0\|^2_{\mathbb{V}})+C\int_0^t\mathbb{E}\|Y^\varepsilon_r\|^2dr.\end{aligned}$$

Finally, by Gronwall's inequality, we get

$$\mathbb{E}\|Y^\varepsilon_t\|^2\leq C\mathbb{E}(1+\|Y_0\|^2+\|\dot{X}_0\|^2+\|X_0\|^2_{\mathbb{V}}).$$

which give the estimate (4.2).

By replacing the estimate above in (4.3) and using the Gronwall's inequality once more, we derive the first estimate (4.1). This completes the proof. □

Lemma 4.2. *Assume that the conditions* (A1)-(A2) *are satisfied. Given* $X_0 \in \mathbb{V}, \dot{X}_0, Y_0 \in \mathbb{H}$, *then there exists a constant* $C > 0$ *such that*

$$\mathbb{E}\|X^{\varepsilon}_{t+h} - X^{\varepsilon}_t\|^2 \leq Ch^2.$$

Proof: Clearly, by (4.1), we have

$$\mathbb{E}\|X^{\varepsilon}_{t+h} - X^{\varepsilon}_t\|^2 = \mathbb{E}\left\|\int_t^{t+h} \dot{X}^{\varepsilon}_s ds\right\|^2 \leq h\mathbb{E}\int_t^{t+h} \|\dot{X}^{\varepsilon}_s\|^2 ds \leq Ch^2.$$

This completes the proof of Lemma 4.2. □

Our goal is to estimate the difference between X^{ε}_t, the slow component of (1.1), and $\bar{X}_t$, the solution of the effective dynamics. To this end, we introduce an auxiliary process $(\hat{X}^{\varepsilon}_t, \hat{Y}^{\varepsilon}_t) \in \mathbb{V} \times \mathbb{H}$. Considering a partition of $[0, T]$ consisting of intervals of the same length δ (δ is sufficiently small and is fixed), that is, $[0, T] = [0, \delta] \cup \{\cup_{k=1}^{\lfloor \frac{T}{\delta} \rfloor - 1}(k\delta, \min\{(k+1)\delta, T\}]\}$, where $\lfloor x \rfloor$ stands for the integer part of real number $x \in \mathbb{R}$. We then construct auxiliary processes $\hat{Y}^{\varepsilon}_t$ and $\hat{X}^{\varepsilon}_t$ as

$$\begin{aligned} \hat{Y}^{\varepsilon}_t &= Y^{\varepsilon}_{k\delta} + \frac{1}{\varepsilon}\int_{k\delta}^t [\mathbb{A}\hat{Y}^{\varepsilon}_s + F(X^{\varepsilon}_{k\delta}, \hat{Y}^{\varepsilon}_s)]ds + \frac{1}{\sqrt{\varepsilon}}\int_{k\delta}^t G(X^{\varepsilon}_{k\delta}, \hat{Y}^{\varepsilon}_s)dW^2_s \\ &\quad + \int_{k\delta}^t \int_{\mathbb{Z}} H(X^{\varepsilon}_{k\delta}, \hat{Y}^{\varepsilon}_{s-}, z)\tilde{N}^{\varepsilon}_2(ds, dz) \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \hat{X}^{\varepsilon}_t &= S'_t X_0 + S_t \dot{X}_0 + \int_0^t S_{t-s} f(X^{\varepsilon}_{\lfloor s/\delta \rfloor \delta}, \hat{Y}^{\varepsilon}_s)ds + \int_0^t S_{t-s} g(X^{\varepsilon}_s)dW^1_s \\ &\quad + \int_0^t \int_{\mathbb{Z}} S_{t-s} h(X^{\varepsilon}_{s-}, z)\tilde{N}_1(ds, dz). \end{aligned} \tag{4.5}$$

Lemma 4.3. *For* $t \in [k\delta, (k+1)\delta]$, *we have*

$$\mathbb{E}\|Y^{\varepsilon}_t - \hat{Y}^{\varepsilon}_t\|^2 \leq C\delta^2 \tag{4.6}$$

and

$$\mathbb{E}\sup_{0 \leq t \leq T}(\|\dot{X}^{\varepsilon}_t - \dot{\hat{X}}^{\varepsilon}_t\|^2 + \|X^{\varepsilon}_t - \hat{X}^{\varepsilon}_t\|^2_{\mathbb{V}}) \leq C\delta^2. \tag{4.7}$$

Proof: By the energy equality (3.4), we have

$$\begin{aligned} \frac{d}{dt}\mathbb{E}\|Y^{\varepsilon}_t - \hat{Y}^{\varepsilon}_t\|^2 &= \frac{2}{\varepsilon}\mathbb{E}\langle \mathbb{A}Y^{\varepsilon}_t - \mathbb{A}\hat{Y}^{\varepsilon}_t, Y^{\varepsilon}_t - \hat{Y}^{\varepsilon}_t\rangle + \frac{2}{\varepsilon}\mathbb{E}\langle F(X^{\varepsilon}_t, Y^{\varepsilon}_t) - F(X^{\varepsilon}_{k\delta}, \hat{Y}^{\varepsilon}_t), Y^{\varepsilon}_t - \hat{Y}^{\varepsilon}_t\rangle_{\mathbb{H}} \\ &\quad + \frac{1}{\varepsilon}\mathbb{E}\|G(X^{\varepsilon}_t, \hat{Y}^{\varepsilon}_t) - G(X^{\varepsilon}_{k\delta}, Y^{\varepsilon}_t)\|^2 \\ &\quad + \frac{1}{\varepsilon}\mathbb{E}\int_{\mathbb{Z}} \|H(X^{\varepsilon}_t, Y^{\varepsilon}_t, z) - H(X^{\varepsilon}_{k\delta}, \hat{Y}^{\varepsilon}_t, z)\|^2 v(dz) \end{aligned}$$

$$
\begin{aligned}
&\leq -\frac{2\alpha_1}{\varepsilon}\mathbb{E}\|Y_t^\varepsilon-\hat{Y}_t^\varepsilon\|^2+\frac{C_F+1}{\varepsilon}\mathbb{E}\|Y_t^\varepsilon-\hat{Y}_t^\varepsilon\|^2+\frac{C_G+C_H}{\varepsilon}\mathbb{E}\|Y_t^\varepsilon-\hat{Y}_t^\varepsilon\|^2\\
&\quad+\frac{C}{\varepsilon}\mathbb{E}\|X_t^\varepsilon-X_{k\delta}^\varepsilon\|^2\\
&\leq -\frac{2\alpha_1-C_F-C_G-C_H-1}{\varepsilon}\mathbb{E}\|Y_t^\varepsilon-\hat{Y}_t^\varepsilon\|^2+\frac{C}{\varepsilon}\delta^2,
\end{aligned}
$$

where $2\alpha_1-C_F-C_G-C_H-1>0$, and further by Gronwall's inequality, we get

$$\mathbb{E}\|Y_t^\varepsilon-\hat{Y}_t^\varepsilon\|^2\leq C\delta^2.$$

Next, by (4.6), (A1)-(A2), Lemma 4.2 and [8, Lemma 3.2], we have the following

$$
\begin{aligned}
&\mathbb{E}\sup_{0\leq s\leq t}(\|\dot{X}_s^\varepsilon-\dot{\hat{X}}_s^\varepsilon\|^2+\|X_s^\varepsilon-\hat{X}_s^\varepsilon\|_{\mathbb{V}}^2)\\
&\quad\leq C\mathbb{E}\sup_{0\leq s\leq t}\left\|\int_0^s S_{s-r}f(X_r^\varepsilon,Y_r^\varepsilon)-f(X_{\lfloor r/\delta\rfloor\delta}^\varepsilon,\hat{Y}_r^\varepsilon)dr\right\|_{\mathbb{V}}^2\\
&\quad\quad+C\mathbb{E}\sup_{0\leq s\leq t}\left\|\int_0^s S'_{s-r}f(X_r^\varepsilon,Y_r^\varepsilon)-f(X_{\lfloor r/\delta\rfloor\delta}^\varepsilon,\hat{Y}_r^\varepsilon)dr\right\|^2\\
&\quad\leq C\mathbb{E}\int_0^t\|f(X_s^\varepsilon,Y_s^\varepsilon)-f(X_{\lfloor s/\delta\rfloor\delta}^\varepsilon,\hat{Y}_s^\varepsilon)\|^2ds\\
&\quad\leq C\mathbb{E}\int_0^t(\|X_s^\varepsilon-X_{\lfloor s/\delta\rfloor\delta}^\varepsilon\|^2+\|Y_s^\varepsilon-\hat{Y}_s^\varepsilon\|^2)ds\\
&\quad\leq C\delta^2.
\end{aligned}
$$

This completes the proof of Lemma 4.3. □

5. Averaging Principles

The mild solution $\bar{X}_t$ of (1.2) is formulated in the following manner

$$
\begin{aligned}
\bar{X}_t &= S'_tX_0+S_t\dot{X}_0+\int_0^t S_{t-s}\bar{f}(\bar{X}_s)ds+\int_0^t S_{t-s}g(\bar{X}_s)dW_s^1\\
&\quad+\int_0^t\int_{\mathbb{Z}} S_{t-s}h(\bar{X}_{s-},z)\tilde{N}_1(ds,dz) \qquad (5.1)
\end{aligned}
$$

where $\bar{f}$ (introduced in Section 1) satisfies the global Lipschitz condition (due to the global Lipschitz condition (A1) for f given in Section 2). By similar arguments as before, the above integral equation (5.1) admits a unique mild solution $\bar{X}_t$. The time derivative of $\bar{X}_t$ is denoted by $\dot{\bar{X}}_t$ then, $(\bar{X}_t,\dot{\bar{X}}_t)\in\mathbb{L}^2(\Omega;C([0,T];\mathbb{V}))\times(\mathbb{D}([0,T];\mathbb{H})\cap\mathbb{L}^2(\Omega\times[0,T];\mathbb{H}))$.

Lemma 5.1. *Suppose that* (A1)-(A2) *hold. Then there exists a constant $C>0$ such that*

$$
\begin{aligned}
\mathcal{Q}_k^\varepsilon&=\sum_{i=1}^\infty\mathbb{E}\left[\int_0^\delta\sin\{\sqrt{\alpha}_i(s+k\delta)\}\langle f(X_{k\delta}^\varepsilon,Y_{\frac{s}{\varepsilon}}^{X_{k\delta}^\varepsilon,Y_{k\delta}^\varepsilon})-\bar{f}(X_{k\delta}^\varepsilon),e_i\rangle_{\mathbb{H}}ds\right]^2\leq C\delta\varepsilon,\\
\tilde{\mathcal{Q}}_k^\varepsilon&=\sum_{i=1}^\infty\mathbb{E}\left[\int_0^\delta\cos\{\sqrt{\alpha}_i(s+k\delta)\}\langle f(X_{k\delta}^\varepsilon,Y_{\frac{s}{\varepsilon}}^{X_{k\delta}^\varepsilon,Y_{k\delta}^\varepsilon})-\bar{f}(X_{k\delta}^\varepsilon),e_i\rangle_{\mathbb{H}}ds\right]^2\leq C\delta\varepsilon,
\end{aligned}
$$

for $k=0,1,\cdots,\lfloor T/\delta\rfloor-1$.

Proof: Please see Appendix A.

Lemma 5.2. *Suppose that* (A1)-(A2) *hold. Then, for* $T > 0$*, we have*

$$\mathbb{E}\sup_{0\le t\le T}(\|\Xi_t^\varepsilon\|_{\mathbb{V}}^2 + \|\dot{\Xi}_t^\varepsilon\|^2) \le C(\delta + \frac{\varepsilon}{\delta})$$

where

$$\begin{aligned}\Xi_t^\varepsilon &= \int_0^t S_{t-s}(f(X_{\lfloor s/\delta\rfloor\delta}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{f}(X_s^\varepsilon))ds,\\ \dot{\Xi}_t^\varepsilon &= \int_0^t S'_{t-s}(f(X_{\lfloor s/\delta\rfloor\delta}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{f}(X_s^\varepsilon))ds.\end{aligned}$$

Proof: For any $t \in [0, T]$, there exists an $n_t = \lfloor t/\delta\rfloor$ such that $t \in [n_t\delta, (n_t+1)\delta \wedge T]$. Hence, we have the following representation

$$\Xi_t^\varepsilon = I_1(t,\varepsilon) + I_2(t,\varepsilon) + I_3(t,\varepsilon)$$

where

$$\begin{aligned}I_1(t,\varepsilon) &= \sum_{k=0}^{\lfloor t/\delta\rfloor-1}\int_{k\delta}^{(k+1)\delta} S_{t-s}(f(X_{k\delta}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{f}(X_{k\delta}^\varepsilon))ds,\\ I_2(t,\varepsilon) &= \sum_{k=0}^{\lfloor t/\delta\rfloor-1}\int_{k\delta}^{(k+1)\delta} S_{t-s}(\bar{f}(X_{k\delta}^\varepsilon) - \bar{f}(X_s^\varepsilon))ds\\ &= \int_0^{\lfloor t/\delta\rfloor\delta} S_{t-s}(\bar{f}(X_{\lfloor s/\delta\rfloor\delta}^\varepsilon) - \bar{f}(X_s^\varepsilon))ds,\end{aligned}$$

and

$$I_3(t,\varepsilon) = \int_{\lfloor t/\delta\rfloor\delta}^t S_{t-s}(f(X_{\lfloor t/\delta\rfloor\delta}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{f}(X_s^\varepsilon))ds.$$

Let us first deal with $I_2(t,\varepsilon)$. Due to the Lipschitz continuity of $\bar{f}$, (A2) and [8, Lemma 3.2], we have the following

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\|I_2(t,\varepsilon)\|_{\mathbb{V}}^2 &= \mathbb{E}\sup_{0\le t\le T}\left\|\int_0^{\lfloor t/\delta\rfloor\delta} S_{t-s}(\bar{f}(X_{\lfloor s/\delta\rfloor\delta}^\varepsilon) - \bar{f}(X_s^\varepsilon))ds\right\|_{\mathbb{V}}^2\\ &\le T\mathbb{E}\int_0^T \|\bar{f}(X_{\lfloor s/\delta\rfloor\delta}^\varepsilon) - \bar{f}(X_s^\varepsilon)\|^2 ds\\ &\le CT\mathbb{E}\int_0^T \|X_{\lfloor s/\delta\rfloor\delta}^\varepsilon - X_s^\varepsilon\|^2 ds\\ &\le C\delta^2.\end{aligned}$$

Next, for $I_3(t,\varepsilon)$, due to Hölder inequality, Remark 2.2, (4.1)-(4.2) and [8, Lemma 3.2], we obtain

$$\mathbb{E}\sup_{0\le t\le T}\|I_3(t,\varepsilon)\|_{\mathbb{V}}^2 = \mathbb{E}\sup_{0\le t\le T}\left\|\int_{\lfloor t/\delta\rfloor\delta}^t S_{t-s}(f(X_{\lfloor t/\delta\rfloor\delta}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{f}(X_s^\varepsilon))ds\right\|_{\mathbb{V}}^2$$

$$
\begin{aligned}
&\le C\delta \mathbb{E}\sup_{0\le t\le T}\int_{\lfloor t/\delta\rfloor\delta}^{t} \|(f(X^\varepsilon_{\lfloor t/\delta\rfloor\delta},\hat{Y}^\varepsilon_s)-\bar{f}(X^\varepsilon_s))\|^2 ds\\
&\le C\delta + C\delta \mathbb{E}\sup_{0\le t\le T}\int_{\lfloor t/\delta\rfloor\delta}^{t} (\|X^\varepsilon_{\lfloor t/\delta\rfloor\delta}\|^2_{\mathbb{V}}+\|\hat{Y}^\varepsilon_s\|^2+\|X^\varepsilon_s\|^2_{\mathbb{V}})ds\\
&\le C\delta + C\delta \mathbb{E}\sup_{0\le t\le T}\int_{\lfloor t/\delta\rfloor\delta}^{t} \|X^\varepsilon_{\lfloor t/\delta\rfloor\delta}\|^2_{\mathbb{V}} ds.
\end{aligned}
$$

For the second term on the right hand side above, we further derive that

$$
\begin{aligned}
C\delta \mathbb{E}\sup_{0\le t\le T}\int_{\lfloor t/\delta\rfloor\delta}^{t} \|X^\varepsilon_{\lfloor t/\delta\rfloor\delta}\|^2_{\mathbb{V}} ds &\le C\delta\left[\sum_{k=0}^{\lfloor T/\delta\rfloor}\mathbb{E}\int_{k\delta}^{(k+1)\delta}\|X^\varepsilon_{k\delta}\|^2_{\mathbb{V}}ds+\mathbb{E}\int_{\lfloor T/\delta\rfloor\delta}^{T}\|X^\varepsilon_{\lfloor T/\delta\rfloor\delta}\|^2_{\mathbb{V}}ds\right]\\
&\le C\delta,
\end{aligned}
$$

where we have used the inequality (4.1) in the derivation. Hence, we have the following estimation

$$
\mathbb{E}\sup_{0\le t\le T}\|I_3(t,\varepsilon)\|^2_{\mathbb{V}}\le C\delta.
$$

For estimation of $I_1(t,\varepsilon)$, we start with the following series representation of Green's function

$$
I_1(t,\varepsilon)=\sum_{k=0}^{\lfloor t/\delta\rfloor-1}\mathcal{I}(t,k,\varepsilon)
$$

where

$$
\begin{aligned}
\mathcal{I}(t,k,\varepsilon) &= \int_{k\delta}^{(k+1)\delta}\sum_{i=1}^{\infty}\frac{\sin\{\sqrt{\alpha_i}(t-s)\}}{\sqrt{\alpha_i}}\langle f(X^\varepsilon_{k\delta},\hat{Y}^\varepsilon_s)-\bar{f}(X^\varepsilon_{k\delta}),e_i\rangle_{\mathbb{H}}e_i ds\\
&= \sum_{i=1}^{\infty}\frac{\sin\{\sqrt{\alpha_i}t\}}{\sqrt{\alpha_i}}e_i\int_{k\delta}^{(k+1)\delta}\cos\{\sqrt{\alpha_i}s\}\langle f(X^\varepsilon_{k\delta},\hat{Y}^\varepsilon_s)-\bar{f}(X^\varepsilon_{k\delta}),e_i\rangle_{\mathbb{H}}ds\\
&\quad -\sum_{i=1}^{\infty}\frac{\cos\{\sqrt{\alpha_i}t\}}{\sqrt{\alpha_i}}e_i\int_{k\delta}^{(k+1)\delta}\sin\{\sqrt{\alpha_i}s\}\langle f(X^\varepsilon_{k\delta},\hat{Y}^\varepsilon_s)-\bar{f}(X^\varepsilon_{k\delta}),e_i\rangle_{\mathbb{H}}ds,
\end{aligned}
$$

for $k=0,1,\cdots,\lfloor t/\delta\rfloor-1$. We have

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\|I_1(t,\varepsilon)\|^2_{\mathbb{V}}\le C\frac{T}{\delta}\sum_{k=0}^{\lfloor T/\delta\rfloor-1}\mathbb{E}\|\mathcal{I}(t,k,\varepsilon)\|^2_{\mathbb{V}}\\
&\le \frac{C}{\delta}\sum_{k=0}^{\lfloor T/\delta\rfloor-1}\mathbb{E}\sum_{i=1}^{\infty}\left[\int_{k\delta}^{(k+1)\delta}\cos\{\sqrt{\alpha_i}s\}\langle f(X^\varepsilon_{k\delta},\hat{Y}^\varepsilon_s)-\bar{f}(X^\varepsilon_{k\delta}),e_i\rangle_{\mathbb{H}}ds\right]^2\\
&\quad+\frac{C}{\delta}\sum_{k=0}^{\lfloor T/\delta\rfloor-1}\mathbb{E}\sum_{i=1}^{\infty}\left[\int_{k\delta}^{(k+1)\delta}\sin\{\sqrt{\alpha_i}s\}\langle f(X^\varepsilon_{k\delta},\hat{Y}^\varepsilon_s)-\bar{f}(X^\varepsilon_{k\delta}),e_i\rangle_{\mathbb{H}}ds\right]^2\\
&\le \frac{C}{\delta}\sum_{k=0}^{\lfloor T/\delta\rfloor-1}\mathbb{E}\sum_{i=1}^{\infty}\left[\int_{0}^{\delta}\cos\{\sqrt{\alpha_i}(s+k\delta)\}\langle f(X^\varepsilon_{k\delta},\hat{Y}^\varepsilon_{s+k\delta})-\bar{f}(X^\varepsilon_{k\delta}),e_i\rangle_{\mathbb{H}}ds\right]^2
\end{aligned}
$$

$$+\frac{C}{\delta}\sum_{k=0}^{\lfloor T/\delta\rfloor-1}\mathbb{E}\sum_{i=1}^{\infty}\left[\int_0^{\delta}\sin\{\sqrt{\alpha_i}(s+k\delta)\}\langle f(X_{k\delta}^{\varepsilon},\hat{Y}_{s+k\delta}^{\varepsilon})-\bar{f}(X_{k\delta}^{\varepsilon}),e_i\rangle_{\mathbb{H}}ds\right]^2.$$

By a time shift transformation, it follows from the definition of $\hat{Y}_s^{\varepsilon}$ that, for $s\in[0,\delta]$, the process $\hat{Y}_{k\delta+s}^{\varepsilon}$ coincides in distribution with the process, $Y_{\frac{s}{\varepsilon}}^{X_{k\delta}^{\varepsilon},Y_{k\delta}^{\varepsilon}}$ which is defined by (B.1) in Appendix B. We have

$$\begin{aligned}
\hat{Y}_{s+k\delta}^{\varepsilon} &= Y_{k\delta}^{\varepsilon}+\frac{1}{\varepsilon}\int_{k\delta}^{k\delta+s}\mathbb{A}\hat{Y}_u^{\varepsilon}du+\frac{1}{\varepsilon}\int_{k\delta}^{k\delta+s}F(X_{k\delta}^{\varepsilon},\hat{Y}_u^{\varepsilon})du\\
&\quad+\frac{1}{\sqrt{\varepsilon}}\int_{k\delta}^{k\delta+s}G(X_{k\delta}^{\varepsilon},\hat{Y}_u^{\varepsilon})dW_u^2+\int_{k\delta}^{k\delta+s}\int_{\mathbb{Z}}H(X_{k\delta}^{\varepsilon},\hat{Y}_{u-}^{\varepsilon},z)\tilde{N}_2^{\varepsilon}(du,dz)\\
&= Y_{k\delta}^{\varepsilon}+\frac{1}{\varepsilon}\int_0^{s}\mathbb{A}\hat{Y}_{u+k\delta}^{\varepsilon}du+\frac{1}{\varepsilon}\int_0^{s}F(X_{k\delta}^{\varepsilon},\hat{Y}_{u+k\delta}^{\varepsilon})du\\
&\quad+\frac{1}{\sqrt{\varepsilon}}\int_0^{s}G(X_{k\delta}^{\varepsilon},\hat{Y}_{u+k\delta}^{\varepsilon})dW_u^{2*}+\int_0^{s}\int_{\mathbb{Z}}H(X_{k\delta}^{\varepsilon},\hat{Y}_{(u+k\delta)-}^{\varepsilon},z)\tilde{N}_2^{\varepsilon,*}(du,dz),
\end{aligned}\tag{5.2}$$

where $W_u^{2*}=W_{u+k\delta}^{2*}-W_{k\delta}^2$ and $p_u^{2*}=p_{u+k\delta}^2-p_{k\delta}^2$ are the shift of W_u^2 and p_u^2, respectively. Let $\bar{W}_u$ be a Wiener process which is independent of W_u^1 and W_u^2, and let $\bar{p}_u$ be a simple Poisson Process which is independent of p_u^1 and p_u^2. We construct process $Y^{X_{k\delta}^{\varepsilon},Y_{k\delta}^{\varepsilon}}$ as follows

$$\begin{aligned}
Y_{\frac{s}{\varepsilon}}^{X_{k\delta}^{\varepsilon},Y_{k\delta}^{\varepsilon}} &= Y_{k\delta}^{\varepsilon}+\int_0^{\frac{s}{\varepsilon}}\mathbb{A}Y_u^{X_{k\delta}^{\varepsilon},Y_{k\delta}^{\varepsilon}}du+\int_0^{\frac{s}{\varepsilon}}F(X_{k\delta}^{\varepsilon},Y_u^{X_{k\delta}^{\varepsilon},Y_{k\delta}^{\varepsilon}})du\\
&\quad+\int_0^{\frac{s}{\varepsilon}}G(X_{k\delta}^{\varepsilon},Y_u^{X_{k\delta}^{\varepsilon},Y_{k\delta}^{\varepsilon}})d\bar{W}_u+\int_0^{\frac{s}{\varepsilon}}\int_{\mathbb{Z}}H(X_{k\delta}^{\varepsilon},Y_{u-}^{X_{k\delta}^{\varepsilon},Y_{k\delta}^{\varepsilon}},z)\bar{N}(du,dz)\\
&= Y_{k\delta}^{\varepsilon}+\frac{1}{\varepsilon}\int_0^{s}\mathbb{A}Y_{\frac{u}{\varepsilon}}^{X_{k\delta}^{\varepsilon},Y_{k\delta}^{\varepsilon}}du+\frac{1}{\varepsilon}\int_0^{s}F(X_{k\delta}^{\varepsilon},Y_{\frac{u}{\varepsilon}}^{X_{k\delta}^{\varepsilon},Y_{k\delta}^{\varepsilon}})du\\
&\quad+\frac{1}{\sqrt{\varepsilon}}\int_0^{s}G(X_{k\delta}^{\varepsilon},Y_{\frac{u}{\varepsilon}}^{X_{k\delta}^{\varepsilon},Y_{k\delta}^{\varepsilon}})d\bar{\bar{W}}_u+\int_0^{s}\int_{\mathbb{Z}}H(X_{k\delta}^{\varepsilon},Y_{(\frac{u}{\varepsilon})-}^{X_{k\delta}^{\varepsilon},Y_{k\delta}^{\varepsilon}},z)\bar{\bar{N}}(du,dz)
\end{aligned}\tag{5.3}$$

where $\bar{\bar{W}}_u=\sqrt{\varepsilon}\bar{W}_{u/\varepsilon}$ and $\bar{\bar{p}}_u=\bar{p}_{u/\varepsilon}$ are the scaled version of $\bar{W}_u$ and $\bar{p}_u$, respectively.

Comparing (5.2) and (5.3) yields that

$$(X_{k\delta}^{\varepsilon},\hat{Y}_{u+k\delta}^{\varepsilon})\sim(X_{k\delta}^{\varepsilon},Y_{\frac{u}{\varepsilon}}^{X_{k\delta}^{\varepsilon},Y_{k\delta}^{\varepsilon}}),u\in[0,\delta)\tag{5.4}$$

where $\sim$ means the coincidence in the distribution sense.

Furthermore, in view of (5.4) and Lemma 5.1, we have

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\|I_1(t,\varepsilon)\|_{\mathbb{V}}^2\\
&\le\frac{C}{\delta}\sum_{k=0}^{\lfloor T/\delta\rfloor-1}\mathbb{E}\sum_{i=1}^{\infty}\left[\int_0^{\delta}\cos\{\sqrt{\alpha_i}(s+k\delta)\}\langle f(X_{k\delta}^{\varepsilon},Y_{\frac{s}{\varepsilon}}^{X_{k\delta}^{\varepsilon},Y_{k\delta}^{\varepsilon}})-\bar{f}(X_{k\delta}^{\varepsilon}),e_i\rangle_{\mathbb{H}}ds\right]^2\\
&\quad+\frac{C}{\delta}\sum_{k=0}^{\lfloor T/\delta\rfloor-1}\mathbb{E}\sum_{i=1}^{\infty}\left[\int_0^{\delta}\sin\{\sqrt{\alpha_i}(s+k\delta)\}\langle f(X_{k\delta}^{\varepsilon},Y_{\frac{s}{\varepsilon}}^{X_{k\delta}^{\varepsilon},Y_{k\delta}^{\varepsilon}})-\bar{f}(X_{k\delta}^{\varepsilon}),e_i\rangle_{\mathbb{H}}ds\right]^2\\
&\le\frac{C}{\delta}\sum_{k=0}^{\lfloor T/\delta\rfloor-1}\mathbb{E}\mathcal{Q}_k^{\varepsilon}+\frac{C}{\delta}\sum_{k=0}^{\lfloor T/\delta\rfloor-1}\mathbb{E}\tilde{\mathcal{Q}}_k^{\varepsilon}
\end{aligned}$$

$$\leq C\frac{\varepsilon}{\delta}.$$

The estimation of $\|\dot{\Xi}_t^\varepsilon\|^2$ can be done analogously to $\|\Xi_t^\varepsilon\|_{\mathbb{V}}^2$ as following

$$\dot{\Xi}_t^\varepsilon = \dot{I}_1(t,\varepsilon) + \dot{I}_2(t,\varepsilon) + \dot{I}_3(t,\varepsilon)$$

where

$$\begin{aligned}
\dot{I}_1(t,\varepsilon) &= \sum_{k=0}^{\lfloor t/\delta\rfloor-1}\int_{k\delta}^{(k+1)\delta} S'_{t-s}(f(X_{k\delta}^\varepsilon,\hat{Y}_s^\varepsilon)-\bar{f}(X_{k\delta}^\varepsilon))ds,\\
\dot{I}_2(t,\varepsilon) &= \sum_{k=0}^{\lfloor t/\delta\rfloor-1}\int_{k\delta}^{(k+1)\delta} S'_{t-s}(\bar{f}(X_{k\delta}^\varepsilon)-\bar{f}(X_s^\varepsilon))ds\\
&= \int_0^{\lfloor t/\delta\rfloor\delta} S'_{t-s}(\bar{f}(X_{\lfloor s/\delta\rfloor\delta}^\varepsilon)-\bar{f}(X_s^\varepsilon))ds,\\
\dot{I}_3(t,\varepsilon) &= \int_{\lfloor t/\delta\rfloor\delta}^{t} S'_{t-s}(f(X_{\lfloor t/\delta\rfloor\delta}^\varepsilon,\hat{Y}_s^\varepsilon)-\bar{f}(X_s^\varepsilon))ds.
\end{aligned}$$

For $\dot{I}_1(t,\varepsilon)$, we have

$$\dot{I}_1(t,\varepsilon) = \sum_{k=0}^{\lfloor t/\delta\rfloor-1}\dot{\mathcal{I}}(t,k,\varepsilon)$$

where

$$\begin{aligned}
\dot{\mathcal{I}}(t,k,\varepsilon) &= \int_{k\delta}^{(k+1)\delta}\sum_{i=1}^{\infty}\cos\{\sqrt{\alpha_i}(t-s)\}\langle f(X_{k\delta}^\varepsilon,\hat{Y}_s^\varepsilon)-\bar{f}(X_{k\delta}^\varepsilon),e_i\rangle_{\mathbb{H}}e_i ds\\
&= \sum_{i=1}^{\infty}\cos\{\sqrt{\alpha_i}t\}e_i\int_{k\delta}^{(k+1)\delta}\cos\{\sqrt{\alpha_i}s\}\langle f(X_{k\delta}^\varepsilon,\hat{Y}_s^\varepsilon)-\bar{f}(X_{k\delta}^\varepsilon),e_i\rangle_{\mathbb{H}}ds\\
&\quad -\sum_{i=1}^{\infty}\sin\{\sqrt{\alpha_i}t\}e_i\int_{k\delta}^{(k+1)\delta}\sin\{\sqrt{\alpha_i}s\}\langle f(X_{k\delta}^\varepsilon,\hat{Y}_s^\varepsilon)-\bar{f}(X_{k\delta}^\varepsilon),e_i\rangle_{\mathbb{H}}ds,
\end{aligned}$$

for $k=0,1,\cdots,\lfloor t/\delta\rfloor-1$.

Similarly to the proof of $I_1(t,\varepsilon), I_2(t,\varepsilon), I_3(t,\varepsilon)$, we have the following

$$\begin{aligned}
\mathbb{E}\sup_{0\leq t\leq T}\|\dot{I}_1(t,\varepsilon)\|^2 &\leq C\frac{\varepsilon}{\delta},\\
\mathbb{E}\sup_{0\leq t\leq T}\|\dot{I}_2(t,\varepsilon)\|^2 &\leq C\delta^2,\\
\mathbb{E}\sup_{0\leq t\leq T}\|\dot{I}_3(t,\varepsilon)\|^2 &\leq C\delta.
\end{aligned}$$

This completes the proof of Lemma 5.2. □

Lemma 5.3. *Let* (A1)-(A2) *hold, then, we have*

$$\mathbb{E}(\sup_{0\leq t\leq T}\|\dot{\hat{X}}_t^\varepsilon-\dot{\bar{X}}_t\|^2+\sup_{0\leq t\leq T}\|\hat{X}_t^\varepsilon-\bar{X}_t\|_{\mathbb{V}}^2)\leq C(\delta+\frac{\varepsilon}{\delta}). \tag{5.5}$$

Proof: By Lemma 4.3, B-D-G inequality, Hölder inequality, (A1) and [8, Lemma 3.2, Lemma 3.3, Lemma 3.6], we have

$$\begin{aligned}
\mathbb{E}\sup_{0\le s\le t}\|\hat{X}^\varepsilon_s-\bar{X}_s\|^2_{\mathbb{V}} &\le C\mathbb{E}\sup_{0\le s\le t}\|\Xi^\varepsilon_s\|^2_{\mathbb{V}}+C\mathbb{E}\sup_{0\le s\le t}\left\|\int_0^s S_{s-u}(\bar{f}(X^\varepsilon_u)-\bar{f}(\hat{X}^\varepsilon_u))du\right\|^2_{\mathbb{V}}\\
&\quad+C\mathbb{E}\sup_{0\le s\le t}\left\|\int_0^s S_{s-u}(\bar{f}(\hat{X}^\varepsilon_u)-\bar{f}(\bar{X}_u))du\right\|^2_{\mathbb{V}}\\
&\quad+C\mathbb{E}\sup_{0\le s\le t}\left\|\int_0^s S_{s-u}(g(X^\varepsilon_u)-g(\hat{X}^\varepsilon_u))dW^1_u\right\|^2_{\mathbb{V}}\\
&\quad+C\mathbb{E}\sup_{0\le s\le t}\left\|\int_0^s S_{s-u}(g(\hat{X}^\varepsilon_u)-g(\bar{X}_u))dW^1_u\right\|^2_{\mathbb{V}}\\
&\quad+C\mathbb{E}\sup_{0\le s\le t}\left\|\int_0^s\int_{\mathbb{Z}} S_{s-u}(h(X^\varepsilon_{u-},z)-h(\hat{X}^\varepsilon_{u-},z))\tilde{N}_1(du,dz)\right\|^2_{\mathbb{V}}\\
&\quad+C\mathbb{E}\sup_{0\le s\le t}\left\|\int_0^s\int_{\mathbb{Z}} S_{s-u}(h(\hat{X}^\varepsilon_{u-},z)-h(\bar{X}_{u-},z))\tilde{N}_1(du,dz)\right\|^2_{\mathbb{V}}\\
&\le C(\delta+\frac{\varepsilon}{\delta})+C\int_0^t\mathbb{E}\|\hat{X}^\varepsilon_s-X^\varepsilon_s\|^2_{\mathbb{V}}ds+C\int_0^t\mathbb{E}\|\hat{X}^\varepsilon_s-\bar{X}_s\|^2_{\mathbb{V}}ds.
\end{aligned}$$

Further by Grownwall's inequality, we obtain

$$\mathbb{E}\sup_{0\le s\le t}\|\hat{X}^\varepsilon_s-\bar{X}_s\|^2_{\mathbb{V}}\le C(\delta+\frac{\varepsilon}{\delta})e^{CT}. \tag{5.6}$$

Next, by Lemma 4.3, B-D-G inequality, Hölder inequality, (A1) and Remark 2.3, as well as by Grownwall's inequality, we have

$$\begin{aligned}
\mathbb{E}\sup_{0\le s\le t}\|\dot{\hat{X}}^\varepsilon_s-\dot{\bar{X}}_s\|^2 &\le C\mathbb{E}\sup_{0\le s\le t}\|\dot{\Xi}^\varepsilon_s\|^2+C\mathbb{E}\sup_{0\le s\le t}\left\|\int_0^s S'_{s-u}(\bar{f}(X^\varepsilon_u)-\bar{f}(\hat{X}^\varepsilon_u))du\right\|^2\\
&\quad+C\mathbb{E}\sup_{0\le s\le t}\left\|\int_0^s S'_{s-u}(\bar{f}(\hat{X}^\varepsilon_u)-\bar{f}(\bar{X}_u))du\right\|^2\\
&\quad+C\mathbb{E}\sup_{0\le s\le t}\left\|\int_0^s S'_{s-u}(g(X^\varepsilon_u)-g(\hat{X}^\varepsilon_u))dW^1_u\right\|^2\\
&\quad+C\mathbb{E}\sup_{0\le s\le t}\left\|\int_0^s S'_{s-u}(g(\hat{X}^\varepsilon_u)-g(\bar{X}_u))dW^1_u\right\|^2\\
&\quad+C\mathbb{E}\sup_{0\le s\le t}\left\|\int_0^s\int_{\mathbb{Z}} S'_{s-u}(h(X^\varepsilon_{u-},z)-h(\hat{X}^\varepsilon_{u-},z))\tilde{N}_1(du,dz)\right\|^2\\
&\quad+C\mathbb{E}\sup_{0\le s\le t}\left\|\int_0^s\int_{\mathbb{Z}} S'_{s-u}(h(\hat{X}^\varepsilon_{u-},z)-h(\bar{X}_{u-},z))\tilde{N}_1(du,dz)\right\|^2\\
&\le C(\delta+\frac{\varepsilon}{\delta})e^{CT}.
\end{aligned}$$

This completes the proof of Lemma 5.3. □

The proof of Theorem 2.3: As a consequence of Lemma 4.3 and Lemma 5.3, we clearly get

$$\mathbb{E}(\sup_{0\le t\le T}\|\dot{X}^\varepsilon_t-\dot{\bar{X}}_t\|^2+\sup_{0\le t\le T}\|X^\varepsilon_t-\bar{X}_t\|^2_{\mathbb{V}})\le C(\delta+\frac{\varepsilon}{\delta}).$$

Next, by choosing $\delta = \sqrt{\varepsilon}$, the above then yields

$$\mathbb{E}(\sup_{0\le t\le T} \|\dot{X}_t^\varepsilon - \dot{\bar{X}}_t\|^2 + \sup_{0\le t\le T} \|X_t^\varepsilon - \bar{X}_t\|_{\mathbb{V}}^2) \le C\sqrt{\varepsilon}.$$

This completes the proof of Theorem 2.4. We are done. □

Remark 5.4. We would like to emphasize that we have confined ourselves to the case that the diffusion coefficients of the slow dynamics do not depend on the fast component, that is, $g(x,y) = g(x), h(x,y,z) = h(x,z)$. In fact, a simple example (see [19]) indicates that strong convergence does not hold where the noise coefficients of the slow equation depend on fast variable.

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Appendix A: Proof for Lemma 5.1

Recall that we have defined $X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon, Y_s^{X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon}$. Fix $x \in \mathbb{H}$ and given $y \in \mathbb{H}$, let $\mathbb{Q}^y$ denote the probability law of the diffusion process $\{Y_t^{x,y}\}_{t\ge0}$ which is determined by the following stochastic differential equation

$$dY_t^x = [\mathbb{A}Y_t^x + F(x, Y_t^x)]dt + G(x, Y_t^x)d\bar{W}_t + \int_{\mathbb{Z}} H(x, Y_{t-}^x, z)\bar{N}(dt, dz), Y_0^x = y, \tag{A.1}$$

where $\bar{W}_t$ is a Wiener process, and $\bar{N}(dt, dz)$ is a Poisson random measure with the compensator $v(dz)dt$ which is independent of $\bar{W}_t$. Both $\bar{W}_t$ and $\bar{N}(dt, dz)$ are defined on the given filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\ge0})$. The expectation with respect to $\mathbb{Q}^y$ is denoted by $\mathbb{E}^y$. Hence, we have

$$\mathbb{E}^y(\Psi(Y_t^x)) = \mathbb{E}(\Psi(Y_t^{x,y})),$$

for all bounded function Ψ. The reader is referred to [24] for more details about $\mathbb{Q}^y$. Recall that $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of $\mathbb{H}$ defined in Section 2, by the Fourier expansion and the Fubini theorem, we have

$$\begin{aligned}
\mathcal{Q}_k^\varepsilon &= \sum_{i=1}^\infty \mathbb{E}\left[\int_0^\delta \sin\{\sqrt{\alpha_i}(s+k\delta)\}\langle f(X_{k\delta}^\varepsilon, Y_{\frac{s}{\varepsilon}}^{X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon}) - \bar{f}(X_{k\delta}^\varepsilon), e_i\rangle_{\mathbb{H}} ds\right]^2 \\
&= 2\int_0^\delta \int_\tau^\delta \sum_{i=1}^\infty \mathbb{E} \sin\{\sqrt{\alpha_i}(s+k\delta)\}\langle f(X_{k\delta}^\varepsilon, Y_{\frac{s}{\varepsilon}}^{X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon}) - \bar{f}(X_{k\delta}^\varepsilon), e_i\rangle_{\mathbb{H}} \\
&\quad \times \sin\{\sqrt{\alpha_i}(\tau+k\delta)\}\langle f(X_{k\delta}^\varepsilon, Y_{\tau/\varepsilon}^{X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon}) - \bar{f}(X_{k\delta}^\varepsilon), e_i\rangle_{\mathbb{H}} ds d\tau
\end{aligned}$$

$$\leq 2\int_0^\delta \int_\tau^\delta \sum_{i=1}^\infty \mathbb{E}\left|\langle f(X_{k\delta}^\varepsilon, Y_{\frac{s}{\varepsilon}}^{X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon}) - \bar{f}(X_{k\delta}^\varepsilon), e_i\rangle_{\mathbb{H}}\right| \times \left|\langle f(X_{k\delta}^\varepsilon, Y_{\tau/\varepsilon}^{X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon}) - \bar{f}(X_{k\delta}^\varepsilon), e_i\rangle_{\mathbb{H}}\right| ds d\tau$$

for $k = 0, 1, \cdots, \lfloor T/\delta \rfloor - 1$. For $i = 1, 2, \cdots$, we set

$$\mathcal{Q}_i(s, \tau, x, y) = \mathbb{E}\left[\left|\langle f(x, Y_s^{x,y}) - \bar{f}(x), e_i\rangle_{\mathbb{H}}\right| \times \left|\langle f(x, Y_\tau^{x,y}) - \bar{f}(x), e_i\rangle_{\mathbb{H}}\right|\right].$$

To proceed, by invoking the Markov property of $Y_t^{x,y}$, for $i = 1, 2, \cdots$, we have

$$\begin{aligned}\mathcal{Q}_i(s, \tau, x, y) &= \mathbb{E}^y\left\{\mathbb{E}^y\left[\left|\langle f(x, Y_s^x) - \bar{f}(x), e_i\rangle_{\mathbb{H}}\right| \times \left|\langle f(x, Y_\tau^x) - \bar{f}(x), e_i\rangle_{\mathbb{H}}\right| |\mathcal{M}_\tau^x\right]\right\} \\ &= \mathbb{E}^y\left\{\left|\langle f(x, Y_\tau^x) - \bar{f}(x), e_i\rangle_{\mathbb{H}}\right| \times \mathbb{E}^{Y_\tau^{x,y}}\left|\langle f(x, Y_{s-\tau}^x) - \bar{f}(x), e_i\rangle_{\mathbb{H}}\right|\right\}\end{aligned}$$

where $\mathcal{M}_t^x$ stands for the σ-field generated by $\{Y_r^x, r \leq t\}$ and $\mathbb{E}^{Y_\tau^{x,y}}\langle f(x, Y_{s-\tau}^x) - \bar{f}(x), e_i\rangle_{\mathbb{H}}$ means the function $\mathbb{E}^{\bar{y}}[\langle f(x, Y_{s-\tau}^x) - \bar{f}(x), e_i\rangle_{\mathbb{H}}]$ evaluated at $\bar{y} = Y_\tau^{x,y}$.

Using Hölder inequality and the Lipschitz continuity of $\bar{f}$, B.3, B.5 in Appendix B, we then obtain the following

$$\begin{aligned}\sum_{i=1}^\infty \mathcal{Q}_i(s, \tau, x, y) &\leq \left\{\mathbb{E}^y\|f(x, Y_\tau^x) - \bar{f}(x))\|^2\right\}^{\frac{1}{2}} \times \left\{\mathbb{E}^y\left(\|\mathbb{E}^{\bar{y}} f(x, Y_{s-\tau}^x) - \bar{f}(x))\|^2_{\bar{y}=Y_\tau^{x,y}}\right)\right\}^{\frac{1}{2}} \\ &\leq C\left\{\mathbb{E}^y\|f(x, Y_\tau^x) - \bar{f}(x))\|^2\right\}^{\frac{1}{2}} \times \left\{\mathbb{E}^y(1 + \|x\|^2 + \|Y_{s-\tau}^x\|^2)\right\}^{\frac{1}{2}} e^{-\frac{1}{2}(s-\tau)\eta} \\ &\leq C(1 + \|x\|^2 + \|y\|^2)e^{-\frac{1}{2}(s-\tau)\eta}.\end{aligned}$$

Let $\mathcal{M}_{k\delta}^\varepsilon$ be the σ-field generated by $X_{k\delta}^\varepsilon$ and $Y_{k\delta}^\varepsilon$, which is independent of $\{Y_r^{x,y} : r \geq 0\}$. By adopting the approach in [24] (cf. Theorem 7.1.2 therein). We have

$$\begin{aligned}\mathcal{Q}_k^\varepsilon &= 2\int_0^\delta \int_\tau^\delta \sum_{i=1}^\infty \mathbb{E}(\mathbb{E}[|\langle f(X_{k\delta}^\varepsilon, Y_{\frac{s}{\varepsilon}}^{X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon}) - \bar{f}(X_{k\delta}^\varepsilon), e_i\rangle_{\mathbb{H}}| \\ &\quad \times|\langle f(X_{k\delta}^\varepsilon, Y_{\tau/\varepsilon}^{X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon}) - \bar{f}(X_{k\delta}^\varepsilon), e_i\rangle_{\mathbb{H}}||\mathcal{M}_{k\delta}^x])dsd\tau \\ &= 2\int_0^\delta \int_\tau^\delta \mathbb{E}[(\sum_{i=1}^\infty \mathcal{Q}_i(s/\varepsilon, \tau/\varepsilon, x, y))|_{(x,y)=(X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon)}]dsd\tau \\ &\leq C\int_0^\delta \int_\tau^\delta e^{-\frac{1}{2}(s-\tau)\eta} dsd\tau \\ &\leq C\delta\varepsilon\end{aligned}$$

for $k = 0, 1, \cdots, \lfloor \frac{T}{\delta} \rfloor - 1$.

Analogously, we derive that $\dot{\mathcal{Q}}_k^\varepsilon \leq C\delta\varepsilon$, for $k = 0, 1, \cdots, \lfloor \frac{T}{\delta} \rfloor - 1$. This then completes the proof. □

Appendix B: The Ergodicity of The Fast Motion

For fixed $x \in \mathbb{H}$, we consider the problem associates to the fast motion with frozen component

$$dY_t = [\mathbb{A}Y_t + F(x, Y_t)]dt + G(x, Y_t)d\bar{W}_t + \int_{\mathbb{Z}} H(x, Y_{t-}, z)\bar{N}(dt, dz), Y_0 = y, \quad \text{(B.1)}$$

where $\bar{W}_t$ and $\bar{N}(dt,dz)$ are given as before. Then, for any fixed $x\in\mathbb{H}$ and any $y\in\mathbb{H}$, there exists a unique mild solution of (B.1), which is denoted by $Y_t^{x,y}$. By the energy equality (3.7), we get

$$\begin{aligned}\|Y_t^{x,y}\|^2 &= \|y\|^2+2\int_0^t\langle\mathbb{A}Y_s^{x,y},Y_s^{x,y}\rangle ds+2\int_0^t\langle F(x,Y_s^{x,y}),Y_s^{x,y}\rangle_{\mathbb{H}}ds\\ &\quad+2\int_0^t\langle G(x,Y_s^{x,y}),Y_s^{x,y}\rangle_{\mathbb{H}}d\bar{W}_s+\int_0^t\|G(x,Y_s^{x,y}\|^2ds\\ &\quad+\int_0^t\int_{\mathbb{Z}}[\|H(x,Y_{s-}^{x,y},z)\|^2+2\langle H(x,Y_{s-}^{x,y},z),Y_{s-}^{x,y}\rangle_{\mathbb{H}}]\bar{N}(ds,dz)\\ &\quad+\int_0^t\int_{\mathbb{Z}}\|H(x,Y_s^{x,y},z)\|^2v(dz)ds. \end{aligned}\tag{B.2}$$

Then by (A1)-(A2) and Remark 2.2, we have

$$\begin{aligned}\frac{d}{dt}\mathbb{E}\|Y_t^{x,y}\|^2 &\le 2\mathbb{E}\langle\mathbb{A}Y_t^{x,y},Y_t^{x,y}\rangle+2\mathbb{E}\langle F(x,Y_t^{x,y})-F(x,0),Y_t^{x,y}\rangle_{\mathbb{H}}+2\mathbb{E}\langle F(x,0),Y_t^{x,y}\rangle_{\mathbb{H}}\\ &\quad+2\mathbb{E}\|G(x,Y_t^{x,y})-G(x,0)\|^2+2\mathbb{E}\|G(x,0)\|^2\\ &\quad+2\mathbb{E}\int_{\mathbb{Z}}\|H(x,Y_t^{x,y},z)-H(x,0,z)\|^2v(dz)+2\mathbb{E}\int_{\mathbb{Z}}\|H(x,0,z)\|^2v(dz)\\ &\le -2\alpha_1\mathbb{E}\|Y_t^{x,y}\|^2+(C_F+1)\mathbb{E}\|Y_t^{x,y}\|^2+C\mathbb{E}\|F(x,0)\|^2\\ &\quad+(\alpha_1-1-C_G-C_H)\mathbb{E}\|Y_t^{x,y}\|^2+(2C_G+2C_H)\mathbb{E}\|Y_t^{x,y}\|^2\\ &\quad+2\mathbb{E}\|G(x,0)\|^2+2\mathbb{E}\int_{\mathbb{Z}}\|H(x,0,z)\|^2v(dz)\\ &\le -\eta\mathbb{E}\|Y_t^{x,y}\|^2+C(1+\|x\|^2)\end{aligned}$$

where $\eta=\alpha_1-C_F-C_G-C_H>0$.

Moreover, by Gronwall's inequality, we have

$$\mathbb{E}\|Y_t^{x,y}\|^2\le\|y\|^2e^{-\eta t}+C(1+\|x\|^2).\tag{B.3}$$

Next, let $Y_t^{x,y'}$ be a solution of (B.1) with the initial value $Y_0=y'$, with (A1) and (A2), we have the following derivation

$$\begin{aligned}\|Y_t^{x,y}-Y_t^{x,y'}\|^2 &= \|y-y'\|^2+2\int_0^t\langle\mathbb{A}(Y_s^{x,y}-Y_s^{x,y'}),Y_s^{x,y}-Y_s^{x,y'}\rangle ds\\ &\quad+2\int_0^t\langle F(x,Y_s^{x,y})-F(x,Y_s^{x,y'}),Y_s^{x,y}-Y_s^{x,y'}\rangle_{\mathbb{H}}ds\\ &\quad+2\int_0^t\langle G(x,Y_s^{x,y})-G(x,Y_s^{x,y'}),Y_s^{x,y}-Y_s^{x,y'}\rangle_{\mathbb{H}}d\bar{W}_s\\ &\quad+\int_0^t\|G(x,Y_s^{x,y}-G(x,Y_s^{x,y'})\|^2ds\\ &\quad+\int_0^t\int_{\mathbb{Z}}\|H(x,Y_{s-}^{x,y},z)-H(x,Y_{s-}^{x,y'},z)\|^2\bar{N}(ds,dz)\\ &\quad+2\int_0^t\int_{\mathbb{Z}}\langle H(x,Y_{s-}^{x,y},z)-H(x,Y_{s-}^{x,y'},z),Y_{s-}^{x,y}-Y_{s-}^{x,y'}\rangle_{\mathbb{H}}\bar{N}(ds,dz)\end{aligned}$$

$$+\int_0^t\int_{\mathbb{Z}}\|H(x,Y_s^{x,y},z)-H(x,Y_s^{x,y'},z)\|^2v(dz)ds.$$

Furthermore, with the aid of the energy equality (3.7) and the conditions (A1) and (A2), we obtain the following

$$\mathbb{E}\|Y_t^{x,y}-Y_t^{x,y'}\|^2\le\|y-y'\|^2e^{-\eta t} \tag{B.4}$$

where $\eta=\alpha_1-C_F-C_G-C_H$.

Next, for any $x\in\mathbb{H}$, we use P_t^x to denote the Markov semigroup associated to (B.1) which is defined by the following

$$P_t^x\Psi(y')=\mathbb{E}\Psi(Y_t^{x,y'}),t\ge 0,y'\in\mathbb{H},$$

for any $\Psi\in\mathcal{B}_b(\mathbb{H})$, where $\mathcal{B}_b(\mathbb{H})$ is the space of bounded measurable functions on $\mathbb{H}$. We recall that a probability measure μ^x on $\mathbb{H}$ is an *invariant measure* for $(P_t^x)_{t\ge0}$, if

$$\int_{\mathbb{H}}P_t^x\Psi d\mu^x=\int_{\mathbb{H}}\Psi d\mu^x,\quad t\ge 0$$

for any $\Psi\in\mathcal{B}_b(\mathbb{H})$. As in [6, 7], it is possible to show the existence of a unique invariant measure μ^x for the semigroup $(P_t^x)_{t\ge0}$ satisfying the following

$$\int_{\mathbb{H}}\|y'\|^2\mu^x(dy')\le(1+\|x\|^2),\quad x\in\mathbb{H}.$$

Finally, according to the global Lipschitz assumption on f and the condition (B.4), we end up with the following

$$\begin{aligned}\left\|\mathbb{E}f(x,Y_t^{x,y})-\int_{\mathbb{H}}f(x,y')\mu^x(dy')\right\| &= \left\|\int_{\mathbb{H}}[\mathbb{E}f(x,Y_t^{x,y})-\mathbb{E}f(x,Y_t^{x,y'})]\mu^x(dy')\right\|\\ &\le C\int_{\mathbb{H}}\mathbb{E}\left\|Y_t^{x,y}-Y_t^{x,y'}\right\|\mu^x(dy')\\ &\le Ce^{-\frac{1}{2}\eta t}\int_{\mathbb{H}}\|y-y'\|\,\mu^x(dy')\\ &\le Ce^{-\frac{1}{2}\eta t}(1+\|x\|+\|y\|).\end{aligned}\tag{B.5}$$

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