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Functional SPDE with Multiplicative Noise and Dini Drift

Functional SPDE with Multiplicative Noise and Dini Drift

Xing Huang [(1)] and Feng-Yu Wang [(1)] [(2)]

Abstract. — Existence, uniqueness and non-explosion of the mild solution are proved for a class of semi-linear functional SPDEs with multiplicative noise and Dini continuous drifts. In the finite-dimensional and bounded time delay setting, the log-Harnack inequality and L^2-gradient estimate are derived. As the Markov semigroup is associated to the functional solution of the equation, one needs to make analysis on the path space of the solution in the time interval of delay.

Résumé. — Dans cet article, nous établissons l'existence, l'unicité et la non-explosion de la solution douce pour une classe d'équations aux dérivées partielles stochastiques semi-linéaires dont le bruit est multiplicatif et le drift satisfait la condition de Dini. Dans le cas de dimension finie et du temps de retard borné, nous montrons l'inégalité de Harnack logarithmique et une estimée de gradient dans L^2 pour la solution douce. Comme le semi-groupe Markovien est associé à la solution fonctionnelle de l'équation, nous devons étudier l'analyse sur l'espace des chemins des solutions définies sur l'intervalle du temps de retard.

1. Introduction

It is well known by Dominique Bakry and his collaborators that the curvature lower bound condition of a diffusion process is equivalent to a number of gradient inequalities for the associated Markov semigroup, see e.g. the recent monographs [4, 15]. Among many other equivalent inequalities, the L^2-gradient estimate of type

$$|\nabla P_t f|^2 \leqslant C(t) P_t f^2, \quad t > 0$$

has been extended to more general situations without curvature conditions, see e.g. [9, 10, 11, 12, 17] and references within for the study of SDEs/SPDEs with non-Lipschitz coefficients. The L^2-gradient estimate links to the log-Harnack inequality which has further applications in analysis of Markov operators, see [14] and references within. Recently, by constructing a Zvonkin type transformation in Hilbert spaces, the L^2-gradient estimate and log-Harnack inequality have been derived in [16] for semi-linear SPDEs with Dini drifts. In the present paper we aim to extend these results to SPDEs with delay.

We will consider semi-linear SPDEs with delay in a separable Hilbert space $(\mathbb{H}, \langle \cdot , \cdot \rangle, | \cdot |)$. To describe the time delay, let ν be a non-trivial measure on $(-\infty, 0)$ such that

$$\nu \text{ is locally finite and } \nu(\cdot - t) \leqslant \kappa(t)\nu(\cdot), \quad t > 0 \tag{1.1}$$

for some increasing function $\kappa : (0, \infty) \to (0, \infty)$. This condition is crucial to prove the pathwise (see the proof of Proposition 2.2 below), and to determine the state space of the segment solutions (see Remark 1.1 below). Obviously, (1.1) holds for $\nu(\mathrm{d}\theta) := 1_{(-\infty,0)}(\theta)\rho(\theta)\,\mathrm{d}\theta$ with density $\rho \geqslant 0$ satisfying $\rho(\theta - t) \leqslant \kappa(t)\rho(\theta)$, $t > 0$ for $\theta < 0$, which is the case if, for instance, $\rho(\theta) = \mathrm{e}^{\lambda\theta}1_{[-r_0,0)}(\theta)$ for some constants $\lambda \in \mathbb{R}$ and $r_0 \in (0, \infty)$. Then the state space of the segment process under study is given by

$$\mathscr{C}_\nu := \left\{ \xi : (-\infty, 0] \to \mathbb{H} \text{ is measurable with } \nu(|\xi|^2) < \infty \right\},$$

where $\nu(f) := \int_{-\infty}^0 f(\theta)\nu(\mathrm{d}\theta)$ for $f \in L^1(\nu)$. Let

$$\|\xi\|_{\mathscr{C}_\nu} = \sqrt{\nu(|\xi|^2) + |\xi(0)|^2}, \quad \xi \in \mathscr{C}_\nu.$$

Throughout the paper, we identify ξ and η in $\mathscr{C}_\nu$ if $\xi = \eta$ ν-a.e. and $\xi(0) = \eta(0)$, so that $\mathscr{C}_\nu$ is a separable Hilbert space with inner product

$$\langle \xi, \eta \rangle_{\mathscr{C}_\nu} := \nu(\langle \xi, \eta \rangle) + \xi(0)\eta(0), \quad \xi, \eta \in \mathscr{C}_\nu.$$

For a map $X : \mathbb{R} \to \mathbb{H}$ and $t \geqslant 0$, let $X_t : (-\infty, 0] \to \mathbb{H}$ be defined by

$$X_t(\theta) = X(t + \theta), \quad \theta \in (-\infty, 0],$$

which describes the path of X from $-\infty$ to time t. We call X_t the segment of X at time t.

Consider the following semi-linear SPDE on $\mathbb{H}$:

$$\mathrm{d}X(t) = \big\{ AX(t) + b(t, X(t)) + B(t, X_t) \big\}\,\mathrm{d}t + Q(t, X(t))\,\mathrm{d}W(t), \tag{1.2}$$

where

- $(A, \mathbb{D}(A))$ is a negative definite self-adjoint operator on $\mathbb{H}$;

- $B : [0, \infty) \times \mathscr{C}_\nu \to \mathbb{H}$ and $b : [0, \infty) \times \mathbb{H} \to \mathbb{H}$ are measurable and locally bounded (i.e. bounded on bounded sets);
- $W = (W(t))_{t \geqslant 0}$ is a cylindrical Brownian motion on a separable Hilbert space $\bar{\mathbb{H}}$, with respect to a complete filtration probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geqslant 0}, \mathbb{P})$. More precisely, $W(t) := \sum_{n=1}^\infty B^n(t)\bar{e}_n$ for a sequence of independent one-dimensional Brownian motions $\{B^n(t)\}_{n \geqslant 1}$ with respect to $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geqslant 0}, \mathbb{P})$, and an orthonormal basis $\{\bar{e}_n\}_{n \geqslant 1}$ on $\bar{\mathbb{H}}$;
- $Q : [0, \infty) \times \mathbb{H} \to \mathscr{L}(\bar{\mathbb{H}}; \mathbb{H})$ is measurable, where $\mathscr{L}(\bar{\mathbb{H}}; \mathbb{H})$ is the space of bounded linear operators from $\bar{\mathbb{H}}$ to $\mathbb{H}$.

DEFINITION 1.1. — *For any $\xi \in \mathscr{C}_\nu$, an adapted continuous process $(X(t))_{t \in [0, \zeta)}$ on $\mathbb{H}$ is called a mild solution to (1.2) with initial value $X_0 = \xi$ and life time ζ, if ζ is a stopping time such that $\limsup_{t \uparrow \zeta} |X(t)| = \infty$ holds on $\{\zeta < \infty\}$, the Lebesgue integral $\int_0^t \mathrm{e}^{(t-s)A}\{b(s, X(s)) + B(s, X_s)\} \, \mathrm{d}s$ and the Itô integral $\int_0^t \mathrm{e}^{(t-s)A} Q(s, X(s)) \, \mathrm{d}W(s)$ are well defined on $\mathbb{H}$ for $t \in [0, \zeta)$, and*

$$X(t) = \mathrm{e}^{At}\xi(0) + \int_0^t \mathrm{e}^{(t-s)A}\{b(s, X(s)) + B(s, X_s)\} \, \mathrm{d}s$$
$$+ \int_0^t \mathrm{e}^{(t-s)A} Q(s, X(s)) \, \mathrm{d}W(s), \quad t \in [0, \zeta) \quad (1.3)$$

holds. Here, due to $X_0 = \xi$, X is extended to $(-\infty, 0)$ with $X(\theta) = \xi(\theta)$ for $\theta \leqslant 0$. If the solution exists uniquely, we denote it by $(X^\xi(t))_{t \in [0, \zeta)}$. The solution is called non-explosive if $\zeta = \infty$ a.s.

Remark 1.2. — We note that condition (1.1) ensures that the segment solution $(X_t^\xi)_{t \in [0, \zeta)}$ is an adapted process on $\mathscr{C}_\nu$. If, for any initial value the equation (1.2) has a unique non-explosive mild solution, then let

$$P_t f(\xi) = \mathbb{E} f(X_t^\xi), \quad t \geqslant 0, f \in \mathbf{B}_b(\mathscr{C}_\nu),$$

where $\mathbf{B}_b(\mathscr{C}_\nu)$ is the set of all bounded measurable functions on $\mathscr{C}_\nu$. In the time-homogenous case (i.e. $b(s, \cdot), B(s, \cdot)$ and $Q(s, \cdot)$ do not depend on s), P_t is a Markov semigroup on $\mathbf{B}_b(\mathscr{C}_\nu)$. In general, for any $s \geqslant 0$, let $X_s^\xi(t)$ be the mild solution of the equation (1.2) for $t \geqslant s$ with $X_s = \xi$, then the associated Markov semigroup $\{P_{s,t}\}_{t \geqslant s \geqslant 0}$ is defined by

$$P_{s,t} f(\xi) = \mathbb{E} f(X_{s,t}^\xi), \quad t \geqslant s, f \in \mathbf{B}_b(\mathscr{C}_\nu),$$

where $X_{s,t}^\xi$ is the segment of $X_s^\xi(\cdot)$ at time t.

The remainder of the paper is organized as follows. In Section 2 we study the existence, uniqueness and non-explosion of the mild solution. In Section 3 we investigate the log-Harnack inequality and L^2-gradient estimate for $P_{s,t}$

when $\mathbb{H}$ is finite-dimensional and $\mathrm{supp}\,\nu \subset [-r_0, 0]$ for some constant $r_0 \in (0, \infty)$. Explanations on making these restrictions are given in the beginning of Section 3.

2. Existence, uniqueness and non-explosion

Let $\|\cdot\|$ and $\|\cdot\|_{HS}$ denote, respectively, the operator norm and the Hilbert–Schmidt norm for linear operators, and let $\mathscr{L}_{HS}(\bar{\mathbb{H}}; \mathbb{H})$ be the space of Hilbert–Schmidt linear operators from $\bar{\mathbb{H}}$ to $\mathbb{H}$. Moreover, let

$$\mathbb{D} = \left\{ \phi : [0, \infty) \to [0, \infty) \text{ is increasing, } \phi^2 \text{ is concave, } \int_0^1 \frac{\phi(s)}{s}\,\mathrm{d}s < \infty \right\}.$$

As in [16], we will use this class of functions to characterize the Dini continuity of the drift b. Note that the condition $\int_0^1 \frac{\phi(s)}{s}\,\mathrm{d}s < \infty$ is known as Dini condition, due to the notion of Dini continuity. Moreover, for simplicity, for a map f defined on $[0, \infty) \times \mathbb{H}$, $\nabla f(t, x) := \nabla f(t, \cdot)(x)$, $(t, x) \in [0, \infty) \times \mathbb{H}$.

To ensure the existence and uniqueness of solutions, we make the following assumptions:

(A1) There exists $\varepsilon \in (0, 1)$ such that $(-A)^{\varepsilon-1}$ is of trace class; i.e. $\sum_{n=1}^\infty \lambda_n^{\varepsilon-1} < \infty$ for $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots\cdots$ being all eigenvalues of $-A$ counting multiplicities. Let $\{e_n\}_{n\geqslant 1}$ be the eigenbasis of $(-A)$ associated with $\{\lambda_n\}_{n\geqslant 1}$.

(A2) $Q(t, \cdot) \in C^2(\mathbb{H}; \mathscr{L}(\bar{\mathbb{H}}; \mathbb{H}))$ for $t \in [0, \infty)$, $(QQ^*)(t, x)$ is invertible for $(t, x) \in [0, \infty) \times \mathbb{H}$, and

$$\|\nabla Q(t, x)\| + \|\nabla^2 Q(t, x)\| + \|Q(t, x)\| + \|(QQ^*)^{-1}(t, x)\|$$

is locally bounded in $(t, x) \in [0, \infty) \times \mathbb{H}$. Moreover, for any $x \in \mathbb{H}$ and $t \geqslant 0$,

$$\lim_{n\to\infty} \|Q(t, x) - Q(t, \pi_n x)\|_{HS}^2$$
$$:= \lim_{n\to\infty} \sum_{k\geqslant 1} |(Q(t, x) - Q(t, \pi_n x))\bar{e}_k|^2 = 0, \quad (2.1)$$

where $\pi_n : \mathbb{H} \to \mathbb{H}_n := \mathrm{span}\{e_i : 1 \leqslant i \leqslant n\}$ are orthogonal projections for $n \geqslant 1$.

(A3) For any $n \geqslant 1$, there exits $\phi_n \in \mathbb{D}$ such that

$$|b(t, x) - b(t, y)| \leqslant \phi_n(|x - y|),$$
$$t \in [0, n], x, y \in \mathbb{H} \text{ with } |x| \vee |y| \leqslant n. \quad (2.2)$$

(A4) For any $n \geqslant 1$ there exists a constant $C_n \in (0, \infty)$ such that

$$|B(t, \xi) - B(t, \eta)|^2 \leqslant C_n \|\xi - \eta\|_{\mathscr{C}_\nu}^2,$$
$$t \in [0, n], \xi, \eta \in \mathscr{C}_\nu \text{ with } \|\xi\|_{\mathscr{C}_\nu} \vee \|\eta\|_{\mathscr{C}_\nu} \leqslant n.$$

When the delay term B vanishes, the existence and uniqueness of mild solutions have been proved in [16] under assumptions (A1)–(A3). The additional assumption (A4) means that the delay term is locally Lipschitzian in $\mathscr{C}_\nu$. Note that this condition allows unbounded time delay, i.e. supp ν might be unbounded.

Fo $T > 0$, let $\| \cdot \|_{T,\infty}$ denote the uniform norm on $[0, T] \times \mathbb{H}$. The main result of this section is the following.

THEOREM 2.1. — *Assume that* (1.1) *and* (A1)–(A4) *hold. Then:*

(1) *For any initial value* $X_0 \in \mathscr{C}_\nu$, *the equation* (1.2) *has unique mild solution* $(X(t))_{t \in [0, \zeta)}$ *with life time* ζ.
(2) *Let* $\|Q\|_{T,\infty} < \infty$ *for* $T \in (0, \infty)$. *If there exist two positive increasing functions* $\Phi, h : [0, \infty) \times [0, \infty) \to (0, \infty)$ *such that* $\int_1^\infty \frac{\mathrm{d}s}{\Phi_t(s)} = \infty$ *for* $t > 0$ *and*

$$\langle B(t, \xi + \eta) + b(t, (\xi + \eta)(0)), \xi(0) \rangle \leqslant \Phi_t(\|\xi\|_{\mathscr{C}_\nu}^2) + h_t(\|\eta\|_{\mathscr{C}_\nu}),$$
$$\xi, \eta \in \mathscr{C}_\nu, t \geqslant 0, \quad (2.3)$$

then the mild solution is non-explosive.

In Subsection 2.1 we investigate the pathwsie uniqueness, which, together with the weak existence, implies the existence and uniqueness of mild solutions according to the Yamada–Watanabe principle. Complete proof of Theorem 2.1 is addressed in Subsection 2.2.

2.1. Pathwise uniqueness

In this section, we prove the pathwise uniqueness of the mild solution to (1.2) under (A1) and the following stronger versions of (A2)–(A4):

(A2′) In addition to (A2) there holds

$$\|\nabla Q\|_{T,\infty} + \|\nabla^2 Q\|_{T,\infty} + \|Q\|_{T,\infty} + \|(QQ^*)^{-1}\|_{T,\infty} < \infty, \quad T > 0.$$

(A3′) For any $T > 0$, $\|b\|_{T,\infty} < \infty$ and there exits $\phi \in \mathbb{D}$ such that

$$|b(t, x) - b(t, y)| \leqslant \phi(|x - y|), \quad t \in [0, T], x, y \in \mathbb{H}.$$

(A4′) For any $T > 0$ there exists a constant $C \in (0, \infty)$ such that

$$|B(t, \xi) - B(t, \eta)|^2 \leqslant C\|\xi - \eta\|_{\mathscr{C}_\nu}^2, \quad t \in [0, T], \xi, \eta \in \mathscr{C}_\nu.$$

PROPOSITION 2.2. — *Assume* (1.1), (A1) *and* (A2′)–(A4′). *Let* $X(t)_{t \geqslant 0}$ *and* $Y(t)_{t \geqslant 0}$ *be two adapted continuous processes on* $\mathbb{H}$ *with* $X_0 = Y_0 = \xi \in \mathscr{C}_\nu$. *For any* $n \geqslant 1$, *let*

$$\tau_n^X = n \wedge \inf\{t \geqslant 0 : |X(t)| \geqslant n\}, \quad \tau_n^Y = n \wedge \inf\{t \geqslant 0 : |Y(t)| \geqslant n\}.$$

If for all $t \in [0, \tau_n^X \wedge \tau_n^Y]$

$$
\begin{aligned}
X(t) &= \mathrm{e}^{At}\xi(0) + \int_0^t \mathrm{e}^{A(t-s)}(b(s, X(s)) + B(s, X_s))\,\mathrm{d}s \\
&\qquad + \int_0^t \mathrm{e}^{A(t-s)}Q(s, X(s))\,\mathrm{d}W(s), \\
Y(t) &= \mathrm{e}^{At}\xi(0) + \int_0^t \mathrm{e}^{A(t-s)}(b(s, Y(s)) + B(s, Y_s))\,\mathrm{d}s \\
&\qquad + \int_0^t \mathrm{e}^{A(t-s)}Q(s, Y(s))\,\mathrm{d}W(s),
\end{aligned}
\tag{2.4}
$$

then $X(t) = Y(t)$ *for all* $t \in [0, \tau_n^X \wedge \tau_n^Y]$. *In particular,* $\tau_n^X = \tau_n^Y$.

We will prove this result by using the Zvonkin type transform constructed in [16]. Let $\{P_{s,t}^0\}_{t \geqslant s \geqslant 0}$ be the Markov semigroup associated to the O-U type SPDE on $\mathbb{H}$:

$$\mathrm{d}Z_s(t) = AZ_s(t)\,\mathrm{d}t + Q(t, Z_s(t))\,\mathrm{d}W(t), \quad t \geqslant s.$$

Given $\lambda, T > 0$, consider the following equation on $\mathbb{H}$:

$$u(s, \cdot) = \int_s^T \mathrm{e}^{-\lambda(t-s)} P_{s,t}^0 \{\nabla_{b(t,\cdot)} u(t, \cdot) + b(t, \cdot)\}\,\mathrm{d}t, \quad s \in [0, T]. \tag{2.5}$$

The next result is essentially due to [16], where the first assertion follows from [16, Lemma 2.3] and the second can be proved as in the proof of [16, Proposition 2.5] by taking into account the additional drift B. So, to save space we skip the proof.

LEMMA 2.3 ([16]). — *Assume* (A1) *and* (A2′)–(A4′). *For any* $T > 0$, *there exists a constant* $\lambda(T) > 0$ *such that the following assertions hold for* $\lambda \geqslant \lambda(T)$:

(1) (2.5) *has a unique solution* $u \in C([0, T]; C_b^1(\mathbb{H}; \mathbb{H}))$ *and*

$$\lim_{\lambda \to \infty} \{\|u\|_{T,\infty} + \|\nabla u\|_{T,\infty} + \|\nabla^2 u\|_{T,\infty}\} = 0.$$

(2) *Let τ be a stopping time. If an adapted continuous process on $\mathbb{H}$ $(X(t))_{t\in[0,T\wedge\tau]}$ satisfies*

$$X(t) = e^{At}X(0) + \int_0^t e^{A(t-s)}(b(s,X(s)) + B(s,X_s))\,\mathrm{d}s$$

$$+ \int_0^t e^{A(t-s)}Q(s,X(s))\,\mathrm{d}W(s)$$

for all $t\in[0,\tau\wedge T]$, then

$$X(t) = e^{At}\{X(0) + u(0,X(0))\} - u(t,X(t))$$

$$+ \int_0^t e^{A(t-s)}\{I + \nabla u(s,X(s))\}Q(s,X(s))\,\mathrm{d}W(s)$$

$$+ \int_0^t \Big\{(\lambda - A)e^{A(t-s)}u(s,X(s))$$

$$+ e^{A(t-s)}\{I + \nabla u(s,X(s))\}B(s,X_s)\Big\}\,\mathrm{d}s$$

holds for all $t\in[0,\tau\wedge T]$, where I is the identity operator.

Proof of Proposition 2.2. — For any $m\geqslant 1$, let $\tau_m = \tau_m^X \wedge \tau_m^Y$. It suffices to prove that for any $T > 0$ and $m\geqslant 1$,

$$\int_0^T \mathbb{E}\{1_{\{s<\tau_m\}}|X(s) - Y(s)|^2\}\,\mathrm{d}s = 0. \tag{2.6}$$

Let $\lambda > 0$ be such that assertions in Lemma 2.3 hold. Due to (2.4), Lemma 2.3(2) with $\tau = \tau_m$ implies

$$X(t) - Y(t) = \Lambda(t) + \Xi(t), \quad t\in[0,\tau_m\wedge T], \tag{2.7}$$

where

$$\Lambda(t) := \int_0^t e^{A(t-s)}\Big\{\big(I + \nabla u(s,X(s))\big)\big(B(s,X_s) - B(s,Y_s)\big)$$

$$+ \big(\nabla u(s,X(s)) - \nabla u(s,Y(s))\big)B(s,Y_s)\Big\}\,\mathrm{d}s,$$

$$\Xi(t) := u(t,Y(t)) - u(t,X(t))$$

$$+ \int_0^t (\lambda - A)e^{A(t-s)}\{u(s,X(s)) - u(s,Y(s))\}\,\mathrm{d}s$$

$$+ \int_0^t e^{A(t-s)}\{\nabla u(s,X(s)) - \nabla u(s,Y(s))\}Q(s,X(s))\,\mathrm{d}W(s)$$

$$+ \int_0^t e^{A(t-s)}\big(\nabla u(s,Y(s)) + I\big)\big(Q(s,X(s)) - Q(s,Y(s))\big)\,\mathrm{d}W(s),$$

$$t\in[0,\tau_m\wedge T].$$

According to the proof of [16, Proposition 3.1] (see the inequality before (3.8) therein), when $\lambda \geqslant \lambda(T)$ is large enough there exists a constant $C_0 \in (0, \infty)$ such that

$$\int_0^r e^{-2\lambda t} \mathbb{E}\big[1_{\{t < \tau_m\}} |\Xi(t)|^2\big] \, dt \leqslant \frac{3}{4}\Gamma(r) + C_0 \int_0^r \Gamma(t) \, dt, \quad r \in [0, T] \quad (2.8)$$

holds for

$$\Gamma(t) := \int_0^t e^{-2\lambda s} \mathbb{E}\big[1_{\{s < \tau_m\}} |X(s) - Y(s)|^2\big] \, ds, \quad t \in [0, T], \quad\quad (2.9)$$

which is denoted by η_t in [16]. So, to prove (2.6), it remains to estimate the corresponding term for $\Lambda(t)$ in place of $\Xi(t)$. Noting that $X_0 = Y_0$ in $\mathscr{C}_\nu$ implies $X = Y$ ν-a.e. on $(-\infty, 0)$, by (1.1) we have

$$\int_{-\infty}^{-s} |X(s+q) - Y(s+q)|^2 \nu(dq) = \int_{-\infty}^0 |X(\theta) - Y(\theta)|^2 \nu(d\theta - s) = 0, \quad s \geqslant 0.$$

So, by $\|e^{A(t-s)}\| \leqslant 1$ for $t \geqslant s$, Lemma 2.3(1) and (A4′), we may find constants $C_1, C_2 \in (0, \infty)$ such that

$$|\Lambda(t)|^2$$
$$\leqslant C_1 \int_0^t \big\{|B(s, X_s) - B(s, Y_s)|^2 + |X(s) - Y(s)|^2\big\} \, ds$$
$$\leqslant C_2 \int_0^t |X(s) - Y(s)|^2 \, ds + C_2 \int_0^t ds \int_{-\infty}^0 |X(s+q) - Y(s+q)|^2 \nu(dq)$$
$$= C_2 \int_0^t |X(s) - Y(s)|^2 \, ds + C_2 \int_0^t ds \int_{-s}^0 |X(s+q) - Y(s+q)|^2 \nu(dq)$$
$$= C_2 \int_0^t |X(s) - Y(s)|^2 \, ds + C_2 \int_{-t}^0 \nu(dq) \int_{-q}^t |X(s+q) - Y(s+q)|^2 \, ds$$
$$\leqslant K(T) \int_0^t |X(s) - Y(s)|^2 \, ds, \qquad t \in [0, T],$$

where $K(T) := C_2 + C_2 \nu([-T, 0)) < \infty$ since ν is locally finite by (1.1). Thus,

$$\int_0^r e^{-2\lambda t} \mathbb{E}\big[1_{\{t < \tau_m\}} |\Lambda(t)|^2\big] \, dt$$
$$\leqslant K(T) \mathbb{E} \int_0^r e^{-2\lambda t} 1_{\{t < \tau_m\}} \, dt \int_0^t |X(s) - Y(s)|^2 \, ds$$
$$\leqslant K(T) \int_0^r \Gamma(t) \, dt, \qquad r \in [0, T].$$

Combining this with (2.7)–(2.9), we arrive at

$$\Gamma(r) \leqslant \int_0^r e^{-2\lambda t} \mathbb{E}\Big\{ 1_{\{t<\tau_m\}} \Big(8|\Lambda(t)|^2 + \frac{8}{7}|\Xi(t)|^2 \Big) \Big\}\,\mathrm{d}t$$

$$\leqslant \frac{6}{7}\Gamma(r) + \frac{8}{7}C_0 \int_0^r \Gamma(t)\,\mathrm{d}t + 8K(T)\int_0^r \Gamma(t)\,\mathrm{d}t$$

$$\leqslant \frac{6}{7}\Gamma(r) + 8(C_0 + K(T))\int_0^r \Gamma(t)\,\mathrm{d}t, \quad r \in [0,T].$$

Since by the definitions of Γ and τ_m we have $\Gamma(t) < \infty$ for $t \in [0,T]$, it follows from Gronwall's inequality that $\Gamma(T) = 0$. Therefore, (2.6) holds and the proof is finished. $\qquad\square$

Remark 2.4. — Due to the unbounded term $\lambda - A$ in the definition of $\Xi(t)$, even when the time delay is bounded we are not able to prove Proposition 2.2 with the following weaker condition in place of (A4$'$):

$$|B(t,\xi) - B(t,\eta)| \leqslant C\|\xi - \eta\|_\infty, \quad t \in [0,T], \xi, \eta \in C([-r_0, 0]; \mathbb{H}). \quad (2.10)$$

However, when $\mathbb{H}$ is finite-dimesional, A becomes bounded so that the proof of Proposition 2.2 can be modified by using (2.10) in place of (A4$'$). This will be discussed in a forthcoming paper.

2.2. Proof of Theorem 2.1

Let $X_0 = \xi \in \mathscr{C}_\nu$ be fixed.

(a). — We first assume that (1.1), (A1) and (A2$'$)–(A4$'$) hold. Consider the following O-U type SPDE on $\mathbb{H}$:

$$\mathrm{d}Z^\xi(t) = AZ^\xi(t)\,\mathrm{d}t + Q(t, Z^\xi(t))\,\mathrm{d}W(t), \quad Z^\xi(0) = \xi(0).$$

It is classical by [5] that our assumptions imply the existence, uniqueness and non-explosion of the mild solution:

$$Z^\xi(t) = e^{At}\xi(0) + \int_0^t e^{A(t-s)}Q(s, Z^\xi(s))\,\mathrm{d}W(s), \quad t \geqslant 0.$$

Letting $Z_0^\xi = \xi$ (i.e. $Z^\xi(\theta) = \xi(\theta)$ for $\theta \leqslant 0$), and taking

$$W^\xi(t) = W(t) - \int_0^t \psi(s)\,\mathrm{d}s,$$

$$\psi(s) = \{Q^*(QQ^*)^{-1}\}(s, Z^\xi(s))\{b(s, Z^\xi(s)) + B(s, Z_s^\xi)\}, \quad s, t \in [0,T],$$

we have

$$Z^\xi(t) = \mathrm{e}^{At}\xi(0) + \int_0^t \mathrm{e}^{A(t-s)}B(s, Z_s^\xi)\,\mathrm{d}s + \int_0^t \mathrm{e}^{A(t-s)}b(s, Z^\xi(s))\,\mathrm{d}s$$

$$+ \int_0^t \mathrm{e}^{A(t-s)}Q(s, Z^\xi(s))\,\mathrm{d}W^\xi(s), \quad t \in [0, T].$$

By the Girsanov theorem, $W^\xi(t)_{t\in[0,T]}$ is a cylindrical Brownian motion on $\bar{\mathbb{H}}$ under probability $\mathrm{d}\mathbb{Q}^\xi = R^\xi\,\mathrm{d}\mathbb{P}$, where

$$R^\xi = \exp\left[\int_0^T \langle \psi(s), \mathrm{d}W(s)\rangle_{\bar{\mathbb{H}}} - \frac{1}{2}\int_0^T |\psi(s)|_{\bar{\mathbb{H}}}^2\,\mathrm{d}s\right].$$

Then, under the probability $\mathbb{Q}^\xi$, $(Z^\xi(t), W^\xi(t))_{t\in[0,T]}$ is a weak mild solution to (1.2). On the other hand, by Proposition 2.2, the pathwise uniqueness holds for the mild solution to (1.2). So, by the Yamada–Watanabe principle [18] (see [7, Theorem 2] or [8] for the result in infinite dimensions), the equation (1.2) has a unique mild solution. Moreover, in this case the solution is non-explosive.

(b). — In general, take $\psi \in C_b^\infty([0,\infty))$ such that $0 \leqslant \psi \leqslant 1$, $\psi(r) = 1$ for $r \in [0,1]$ and $\psi(r) = 0$ for $r \in [2,\infty)$. For any $m \geqslant 1, t \geqslant 0, z \in \mathbb{H}, \xi \in \mathscr{C}_\nu$, let

$$b^{[m]}(t,z) = b(t,z)\psi(m^{-1}|z|),$$
$$Q^{[m]}(t,z) = Q(t, \psi(m^{-1}|z|)z),$$
$$B^{[m]}(t,\xi) = B(t,\xi)\psi(m^{-1}\|\xi\|_{\mathscr{C}_\nu}).$$

Then (A2)–(A4) imply that $B^{[m]}, Q^{[m]}, b^{[m]}$ satisfy (A2′)–(A4′). Here, we only verify (A4′) for $B^{[m]}$ since the other two conditions are obvious for $Q^{[m]}$ and $b^{[m]}$ in place of Q and b. For any $\xi, \eta \in \mathscr{C}_\nu$, let for instance $\|\xi\|_{\mathscr{C}_\nu} \geqslant \|\eta\|_{\mathscr{C}_\nu}$. By the choice of ψ, (A4) implies

$$|B^{[m]}(t,\xi) - B^{[m]}(t,\eta)|$$
$$\leqslant \psi(m^{-1}\|\xi\|_{\mathscr{C}_\nu})|B(t,\eta) - B(t,\xi)|$$
$$+ |B(t,\eta)| \cdot \left|\psi(m^{-1}\|\xi\|_{\mathscr{C}_\nu}) - \psi(m^{-1}\|\eta\|_{\mathscr{C}_\nu})\right|$$
$$= 1_{\{\|\xi\|_{\mathscr{C}_\nu} \leqslant 2m\}}\psi(m^{-1}\|\xi\|_{\mathscr{C}_\nu})|B(t,\eta) - B(t,\xi)|$$
$$+ 1_{\{\|\eta\|_{\mathscr{C}_\nu} \leqslant 2m\}}|B(t,\eta)| \cdot \left|\psi(m^{-1}\|\xi\|_{\mathscr{C}_\nu}) - \psi(m^{-1}\|\eta\|_{\mathscr{C}_\nu})\right|$$
$$\leqslant C(m)\|\xi - \eta\|_{\mathscr{C}_\nu}$$

for some constant $C(m) > 0$. Thus, (A4′) holds for $B^{[m]}$ in place of B.

Now, by (a), equation (1.2) for $B^{[m]}, Q^{[m]}, b^{[m]}$ in place of B, Q, b has a unique non-explosive mild solution $X^{[m]}(t)$ starting at $X_0 = \xi$. Let

$$\tau_0 := 0, \quad \tau_m = m \wedge \inf\{t \geqslant 0 : \|X_t^{[m]}\|_{\mathscr{C}_\nu} \geqslant m\}, \ m \geqslant 1.$$

Since $B^{[m]}(s, \xi) = B(s, \xi)$, $Q^{[m]}(s, \xi(0)) = Q(s, \xi(0))$ and $b^{[m]}(s, \xi(0)) = b(s, \xi(0))$ hold for $s \leqslant m$ and $\|\xi\|_{\mathscr{C}_\nu} \leqslant m$, Proposition 3.3 implies $X^{[m]}(t) = X^{[n]}(t)$ for any $n, m \geqslant 1$ and $t \in [0, \tau_m \wedge \tau_n]$. In particular, τ_m is increasing in m. Let $\zeta = \lim_{m \to \infty} \tau_m$ and

$$X(t) = \sum_{m=1}^{\infty} 1_{[\tau_{m-1}, \tau_m)} X^{[m]}(t), \quad t \in [0, \zeta).$$

It is easy to see that $X(t)_{t \in [0, \zeta)}$ is a mild solution to (1.2) with lifetime ζ, since condition (1.1) and the definition of ζ imply $\lim_{t \uparrow \zeta} |X(t)| = \infty$ on $\{\zeta < \infty\}$. Finally, by Proposition 2.2, the mild solution is unique. Then the proof of Theorem 2.1(1) is finished.

(c). — Under the conditions of Theorem 2.1(2), for a mild solution $X(t)_{t \in [0, \zeta)}$ to (1.2) with lifetime ζ, we intend to prove $\zeta = \infty$ a.s. Obviously,

$$\bar{X}(t) := \int_0^t e^{A(t-s)} Q(s, X(s)) \, dW(s), t \in [0, \zeta), \quad t \geqslant 0$$

is an adapted continuous process on $\mathbb{H}$ up to the lifetime ζ. Let $\bar{X}(t) = 0$ for $t \in (-\infty, 0]$. We see that $Y(t) := X(t) - \bar{X}(t)$ is a mild solution to the equation

$$dY(t) = (AY(t) + b(t, Y(t) + \bar{X}(t)) + B(t, Y_t + \bar{X}_t)) \, dt, \quad Y_0 = X_0, \ t \in [0, \zeta).$$

Due to (2.3), the increasing property of h and Φ, and noting that $A \leqslant 0$, this implies that for any $T > 0$,

$$d|Y(t)|^2 \leqslant 2\langle Y(t), b(t, Y(t) + \bar{X}(t)) + B(t, Y_t + \bar{X}_t) \rangle \, dt$$
$$\leqslant 2\{\Phi_{\zeta \wedge T}(\|Y_t\|_{\mathscr{C}_\nu}^2) + h_T(\|\bar{X}_t\|_{\mathscr{C}_\nu})\} \, dt, \quad Y_0 = X_0, \ t \in [0, \zeta \wedge T).$$

Then

$$|Y(t)|^2 \leqslant |X(0)|^2 + 2 \int_0^t h_T(\|\bar{X}_s\|_{\mathscr{C}_\nu}) \, ds + 2 \int_0^t \Phi_T(\|Y_s\|_{\mathscr{C}_\nu}^2) \, ds,$$
$$t \in [0, T \wedge \zeta). \quad (2.11)$$

Since $Y_0 = X_0$, (1.1) implies

$$\|Y_s\|_{\mathscr{C}_\nu}^2 = |Y(s)|^2 + \int_{-\infty}^{-s} |Y(s+\theta)|^2 \nu(\mathrm{d}\theta) + \int_{-s}^{0} |Y(s+\theta)|^2 \nu(\mathrm{d}\theta)$$

$$\leqslant \left\{1 + \nu([-s,0))\right\} \sup_{r\in[0,s]} |Y(r)|^2 + \kappa(s) \int_{-\infty}^{0} |X_0(\theta)|^2 \nu(\mathrm{d}\theta)$$

$$\leqslant \kappa(T)\|X_0\|_{\mathscr{C}_\nu}^2 + \left\{1 + \nu([-T,0))\right\} \sup_{r\in[0,s]} |Y(r)|^2$$

$$=: K_1(T) + K_2(T) \sup_{r\in[0,s]} |Y(r)|^2, \quad s \in [0,T].$$

So, by letting

$$H(t) = \sup_{r\in[0,t]} |Y(r)|^2, \quad \alpha(T) = |X(0)|^2 + 2\int_0^T h_T(\|\bar{X}_s\|_{\mathscr{C}_\nu})\,\mathrm{d}s,$$

we obtain from (2.11) that

$$H(t) \leqslant \alpha(T) + 2\int_0^t \Phi_T\big(K_1(T) + K_2(T)H(s)\big)\,\mathrm{d}s, \quad t \in [0,T\wedge\zeta).$$

Taking

$$\Psi_T(s) = \int_1^s \frac{\mathrm{d}r}{2\Phi_T(K_1(T)+K_2(T)r)}, \quad s \geqslant 0,$$

we have $\lim_{s\to\infty}\Psi_T(s) = \infty$ due to the assumption on Φ, so that by Biharis' inequality,

$$H(t) \leqslant \Psi_T^{-1}(\alpha(T)+T) < \infty, \quad t \in [0,\zeta\wedge T).$$

Since $\sup_{t\in[0,T\wedge\zeta)} |\bar{X}(t)|^2 < \infty$ a.s., on the set $\{\zeta \leqslant T\}$ we have

$$\infty = \lim_{t\uparrow\zeta} H(t) \leqslant \Psi_T^{-1}(\alpha(T)+T) < \infty, \quad \text{a.s.}$$

This means $\mathbb{P}(\zeta \leqslant T) = 0$ for all $T > 0$ and hence, $\zeta = \infty$ a.s. $\qquad\square$

3. Log-Harnack inequality and gradient estimate

Throughout of this section, we assume that $\mathbb{H}$ is finite-dimensional and the length of time delay is finite. Since the log-Harnack inequality implies the strong Feller property (see [14, Theorem 1.4.1]), and it is easy to see that P_T is strong Feller only if $\mathrm{supp}\,\nu \subset [-T,0]$, we see that the restriction on bounded time delay is essential for the study. On the other hand, although the restriction on finite-dimensions might be technical rather than necessary, we are not able to drop it in the moment. The reason is that we

adopt the argument of [13] using coupling by change of measures, for which the Hilbert–Schmidt norm of the diffusion coefficient is used. This reduces the framework to finite-dimensions as the diffusion coefficient in (3.6) below is merely Lipschitz in the operator norm. We remark that for SPDEs with Dini drifts but without delay, the log-Harnack inequality is presented in [16] by using finite-dimensional approximation and Itô's formula as in [10, 17]. However, in the case with delay the Markov semigroup is associated to the segment solution, for which the corresponding Itô formula is not yet available.

Let $r_0 \in (0, \infty)$ such that $\operatorname{supp} \nu \subset [-r_0, 0]$. In this case $\mathscr{C}_\nu$ is reformulated as

$$\mathscr{C}_\nu = \left\{ \xi : [-r_0, 0] \to \mathbb{H} \text{ is measurable with } \nu(|\xi|^2) := \int_{-r_0}^0 |\xi(\theta)|^2 \nu(\mathrm{d}\theta) < \infty \right\}.$$

For $f \in \mathbf{B}_b(\mathscr{C}_\nu)$, the length of the gradient of f at point $\xi \in \mathscr{C}_\nu$ is defined by

$$|\nabla f|_{\mathscr{C}_\nu}(\xi) = \limsup_{\eta \to \xi} \frac{|f(\eta) - f(\xi)|}{\|\xi - \eta\|_{\mathscr{C}_\nu}}.$$

THEOREM 3.1. — *Let $\mathbb{H}$ be finite-dimensional and $\operatorname{supp} \nu \subset [-r_0, 0]$ for some $r_0 \in (0, \infty)$. Assume (1.1), (A1) and (A2′)–(A4′). Then for any $s \geqslant 0$ there exists a constant $C > 0$ such that the log-Harnack inequality*

$$P_{s, T+s+r_0} \log f(\eta) \leqslant \log P_{s, T+s+r_0} f(\xi) + \frac{C}{T \wedge 1} \|\xi - \eta\|_{\mathscr{C}_\nu}^2, \quad \xi, \eta \in \mathscr{C}_\nu, T > 0 \tag{3.1}$$

holds for strictly positive functions $f \in \mathbf{B}_b(\mathscr{C}_\nu)$. Consequently,

$$|\nabla P_{s, s+r_0+T} f|_{\mathscr{C}_\nu}^2 \leqslant \frac{C}{T \wedge 1} \left\{ P_{s, s+r_0+T} f^2 - (P_{s, s+r_0+T} f)^2 \right\},$$
$$T > 0, s \geqslant 0, f \in \mathbf{B}_b(\mathscr{C}_\nu). \tag{3.2}$$

Proof. — According to [3, Proposition 2.3], the L^2-gradient estimate (3.2) follows from the log-Harnack inequality (3.1). Moreover, according to the semigroup property and the Jensen inequality, it suffices to prove the log-Harnack inequality for $T \in (0, 1]$. Finally, without loss of generality, we may and do assume that $s = 0$.

Now, we use Lemma 2.3 to transform (1.2) into a SDE with regular coefficients. Let $\{u(t, \cdot)\}_{t \in [0, T]}$ be in Lemma 2.3 for fixed $T > 0$ and let $u(\theta, \cdot) = u(0, \cdot)$ for $\theta \in [-r_0, 0]$. Define

$$\Theta(t, x) = x + u(t, x), \quad t \in [-r_0, T], x \in \mathbb{H},$$
$$(\Theta_t(\xi))(\theta) = \Theta(t + \theta, \xi(\theta)), \quad \theta \in [-r_0, 0], t \in [0, T], \xi \in \mathscr{C}_\nu.$$

Then there exists $\lambda(T) > 0$ such that for any $\lambda \geqslant \lambda(T)$, $\Theta(t, \cdot)$ is a diffeomorphism on $\mathbb{H}$ for $t \in (-\infty, T]$ such that

$$\|\nabla u\|_{T,\infty} \leqslant \frac{1}{2}, \quad \|\nabla\Theta\|_{T,\infty} + \|\nabla\Theta^{-1}\|_{T,\infty} \leqslant 2, \tag{3.3}$$

where and in what follows, denote $\Theta^{-1}(t,x) = \{\Theta(t,\cdot)\}^{-1}(x)$. Obviously, $\Theta_t : \mathscr{C}_\nu \to \mathscr{C}_\nu$ is invertible with

$$\{\Theta_t^{-1}(\xi)\}(\theta) = \Theta^{-1}(t+\theta, \xi(\theta)), \quad \theta \in [-r_0, 0], t \in [0, T].$$

Moreover, for a mild solution $X^\xi(t)$ to (1.2) with $X_0 = \xi$, Lemma 2.3(2) implies that $Y^\xi(t) := \Theta(t, X^\xi(t))$ solves the equation

$$
\begin{aligned}
Y^\xi(t) = \ & \mathrm{e}^{At}Y^\xi(0) \\
& + \int_0^t \mathrm{e}^{A(t-s)}\Big\{\nabla\Theta(s, \Theta^{-1}(s, Y^\xi(s)))\Big\}Q(s, \Theta^{-1}(s, Y^\xi(s)))\,\mathrm{d}W(s) \\
& + \int_0^t (\lambda - A)\mathrm{e}^{A(t-s)}u\big(s, \Theta^{-1}(s, Y^\xi(s))\big)\,\mathrm{d}s \\
& + \int_0^t \mathrm{e}^{A(t-s)}\big\{\nabla\Theta\big(s, \Theta^{-1}(s, Y^\xi(s))\big)\big\}B(s, \Theta_s^{-1}(Y_s^\xi))\,\mathrm{d}s, \quad t \in [0, T].
\end{aligned}
\tag{3.4}
$$

So, letting

$$
\begin{aligned}
\mathtt{Q}(t,x) = \ & \big(\nabla\Theta(t, \Theta^{-1}(t,x))\big)Q(t, \Theta^{-1}(t,x)), \quad t \in [0, T], x \in \mathbb{H}; \\
\mathtt{B}(t,\xi) = \ & A\xi(0) + (\lambda - A)u\big(t, \Theta^{-1}(t, \xi(0))\big) \\
& + \big\{\nabla\Theta\big(t, \Theta^{-1}(t, \xi(0))\big)\big\}B\big(t, \Theta_t^{-1}(\xi)\big), \quad t \in [0, T], \xi \in \mathscr{C}_\nu,
\end{aligned}
\tag{3.5}
$$

we have

$$\mathrm{d}Y^\xi(t) = \mathtt{B}(t, Y_t^\xi)\,\mathrm{d}t + \mathtt{Q}(t, Y^\xi(t))\,\mathrm{d}W(t), \quad t \in [0, T], Y_0^\xi = \Theta_0(\xi).$$

Thus, $\mathtt{X}^\xi(t) := Y^{\Theta_0^{-1}(\xi)}(t)$ solves the following SDE with delay:

$$\mathrm{d}\mathtt{X}^\xi(t) = \mathtt{B}(t, \mathtt{X}_t^\xi)\,\mathrm{d}t + \mathtt{Q}(t, \mathtt{X}^\xi(t))\,\mathrm{d}W(t), \quad t \in [0, T], \mathtt{X}_0^\xi = \xi. \tag{3.6}$$

Since (A2′), (A4′) and (3.3) imply

$$
\begin{aligned}
\|\mathtt{Q}\|_{T,\infty} + \|(\mathtt{Q}\mathtt{Q}^*)^{-1}\|_{T,\infty} \leqslant \ & K, \\
\|\mathtt{Q}(t,x) - \mathtt{Q}(t,y)\| \leqslant \ & K|x - y|, \quad x, y \in \mathbb{H}, t \in [0, T], \\
|\mathtt{B}(t,\xi) - \mathtt{B}(t,\eta)| \leqslant \ & K\|\xi - \eta\|_{\mathscr{C}_\nu}, \quad \xi, \eta \in \mathscr{C}_\nu, t \in [0, T]
\end{aligned}
\tag{3.7}
$$

for some constant $K > 0$, this equation has a unique non-explosive mild solution for any initial value $\xi \in \mathscr{C}_\nu$. Let $\mathtt{P}_t f(\xi) = \mathbb{E}f(\mathtt{X}_t^\xi)$. Since $X^\xi(t) =$

$\Theta^{-1}(t, Y^\xi(t)) = \Theta^{-1}(t, \mathtt{X}^{\Theta_0(\xi)}(t))$, we have

$$
\begin{aligned}
P_t f(\xi) := \mathbb{E}f(X_t^\xi) &= \mathbb{E}(f \circ \Theta_t^{-1})(Y_t^\xi) = \mathbb{E}(f \circ \Theta_t^{-1})(\mathtt{X}_t^{\Theta_0(\xi)}) \\
&= \mathtt{P}_t(f \circ \Theta_t^{-1})(\Theta_0(\xi)), \quad \xi \in \mathscr{C}_\nu, t \in (0, T], f \in \mathbf{B}_b(\mathscr{C}_\nu).
\end{aligned}
$$

Therefore, by (3.3), the desired log-Harnack inequality for P_{r_0+T} follows from the corresponding inequality for $\mathtt{P}_{r_0+T}$, which is ensured by the following Lemma 3.2. $\square$

The following result is parallel to [14, Theorem 4.3.1] where the uniform norm on the segment space is used instead of $\| \cdot \|_{\mathscr{C}_\nu}$.

LEMMA 3.2. — *Let* $\mathtt{P}_t$ *be associated to* (3.6) *with coefficients satisfying* (3.7). *Then there exists a constant* $C > 0$ *such that the log-Harnack inequality*

$$
\mathtt{P}_{T+r_0} \log f(\eta) \leqslant \log \mathtt{P}_{T+r_0} f(\xi) + C\left(\frac{1}{T}|\xi(0) - \eta(0)|^2 + \|\xi - \eta\|_{\mathscr{C}_\nu}^2\right)
$$

holds for all $\xi, \eta \in \mathscr{C}_\nu, T \in (0, 1]$, *and strictly positive functions* $f \in \mathbf{B}_b(\mathscr{C}_\nu)$.

Proof. — The result can be proved in a similar way as in the proof of [14, Theorem 4.3.1] using coupling by change of measures, the only difference is to use $\| \cdot \|_{\mathscr{C}_\nu}$ in place of $\| \cdot \|_\infty$. We remark that the couplings by change of measures are developed in [1] and [12] to prove the dimension-free Harnack inequality and the log-Harnack inequality respectively, see [14] for more results and discussions. For completeness, we include below a brief proof.

(a). — For any $\xi, \eta \in \mathscr{C}_\nu$, let $X(t) = \mathtt{X}^\xi(t)$ solve (3.6), and let $Y(t)$ solve the following SDE with delay:

$$
\begin{aligned}
\mathrm{d}Y(t) = &\mathtt{B}(t, X_t)\,\mathrm{d}t + \mathtt{Q}(t, Y(t))\,\mathrm{d}W(t) \\
&+ \frac{\mathbf{1}_{[0,T)}(t)}{\gamma(t)}\mathtt{Q}(t, Y(t))\big\{\mathtt{Q}^*(\mathtt{Q}\mathtt{Q}^*)^{-1}\big\}(t, X(t))\big\{X(t) - Y(t)\big\}\,\mathrm{d}t, \\
&\hspace{6cm} t \geqslant 0, Y_0 = \eta, \quad (3.8)
\end{aligned}
$$

where

$$
\gamma(t) := \frac{1}{K^2}\big(1 - \mathrm{e}^{(t-T)K^2}\big), \quad t \in [0, T].
$$

Here, following the line of [6], we take the delay term $\mathtt{B}(t, X_t)$ instead of $\mathtt{B}(t, Y_t)$ such that the SDE for $X(t) - Y(t)$ does not have time delay. Hence, for any solution $\{Y(t)\}_{t\geqslant 0}$ to (3.8) with coupling time:

$$
\tau := \inf\{t \in [0, T]: \ X(t) = Y(t)\}, \quad \inf \emptyset := \infty,
$$

the modified process

$$
\mathtt{Y}(t) := Y(t)\mathbf{1}_{\{t<\tau\}} + X(t)\mathbf{1}_{\{t\geqslant\tau\}}
$$

solves (3.8) as well. Using $\mathtt{Y}$ to replace Y we may and do assume $Y(t) = X(t)$ for $t \geqslant \tau$, so that $X_{T+r_0} = Y_{T+r_0}$ provided $\tau \leqslant T$. This is crucial to derive

the log-Harnack inequality. Moreover, we take the additional unbounded drift term in (3.8) to ensure $\tau \leqslant T$, and the idea comes from [13].

(b). — Since the coefficients in (3.8) are Lipschtiz continuous in the space variable locally uniformly in $t \in [0, T)$, it has a unique solution up to time T. To construct a solution for all $t \geqslant 0$, we reformulate the equation as (3.6) using Girsanov transform. Let

$$\phi(t) = \big\{ \mathbb{Q}^*(\mathbb{Q}\mathbb{Q}^*)^{-1} \big\}(t, Y(t)) \big\{ \mathbb{B}(t, Y_t) - \mathbb{B}(t, X_t) \big\}$$
$$- \frac{1}{\gamma(t)} \big\{ \mathbb{Q}^*(\mathbb{Q}\mathbb{Q}^*)^{-1} \big\}(t, X(t)) \big\{ X(t) - Y(t) \big\}, \quad t \in [0, T). \quad (3.9)$$

By (3.7) we have

$$|\phi(t)|_{\bar{\mathbb{H}}} \leqslant K_0 \| X_t - Y_t \|_{\mathscr{C}_\nu} + \frac{K_0 |X(t) - Y(t)|}{\gamma(t)}, \quad t \in [0, T) \quad (3.10)$$

for some constant $K_0 > 0$. Since (3.7) implies assumption (A4.4) in [14] for $K_4 = K^2$, it follows from [14, (i) on p. 92] that

$$R(t) := \exp \left[\int_0^t \langle \phi(s), \mathrm{d}W(s) \rangle_{\bar{\mathbb{H}}} - \frac{1}{2} \int_0^t |\phi(s)|_{\bar{\mathbb{H}}}^2 \, \mathrm{d}s \right], \quad t \in [0, T) \quad (3.11)$$

is a uniformly integrable martingale such that

$$R := \lim_{t \uparrow T} R(t) \quad (3.12)$$

exists, and $\mathrm{d}\mathbb{Q} := R \, \mathrm{d}\mathbb{P}$ is a probability measure on Ω. Moreover, by the Girsanov theorem,

$$\mathbb{W}(t) := W(t) - \int_0^{t \wedge T} \phi(s) \, \mathrm{d}s, \quad t \geqslant 0 \quad (3.13)$$

is a cylindrical Brownian motion on $\bar{\mathbb{H}}$ under probability $\mathbb{Q}$, and according to [14, (ii) on p. 92] we have $\tau \leqslant T, \mathbb{Q}$-a.s. So, as explained in the end of (a), we have

$$X_{T+r_0} = Y_{T+r_0}, \quad \mathbb{Q}\text{-a.s.} \quad (3.14)$$

Now, by (3.9) and (3.13) we reformulate (3.8) as

$$\mathrm{d}Y(t) = \mathbb{B}(t, Y_t) \, \mathrm{d}t + \mathbb{Q}(t, Y(t)) \, \mathrm{d}\mathbb{W}(t), \quad Y_0 = \eta. \quad (3.15)$$

By the weak uniqueness of (3.6), we have $\mathbb{P}_{T+r_0}(\log f)(\eta) = \mathbb{E}_{\mathbb{Q}}[\log f(Y_{T+r_0})]$.

According to Young inequality (see [2, Lemma 2.4]), this together with (3.14) implies

$$P_{T+r_0}(\log f)(\eta) = \mathbb{E}[R \log f(Y_{T+r_0})] = \mathbb{E}[R \log f(X_{T+r_0})]$$
$$\leqslant \log \mathbb{E}f(X_{T+r_0}) + \mathbb{E}[R \log R] = \log P_{T+r_0}f(\xi) + \mathbb{E}_{\mathbb{Q}} \log R. \tag{3.16}$$

(c). — To estimate $\mathbb{E}_{\mathbb{Q}} \log R$, by (3.9) and (3.13) we reformulate the equation (3.6) for $X(t)$ as

$$dX(t) = B(t, X_t)\, dt + Q(t, X(t))\, dW(t)$$
$$- Q(t, X(t))\{Q^*(QQ^*)^{-1}\}(t, Y(t))\{B(t, X_t) - B(t, Y_t)\}\, dt$$
$$- \frac{X(t) - Y(t)}{\gamma(t)}\, dt, \quad t \in [0, T).$$

Combining this with (3.15) we obtain

$$d(X(t) - Y(t))$$
$$= \{Q(t, X(t)) - Q(t, Y(t))\}\, dW(t)$$
$$+ \left[I - Q(t, X(t))\{Q^*(QQ^*)^{-1}\}(t, Y(t))\right]\{B(t, X_t) - B(t, Y_t)\}\, dt$$
$$- \frac{X(t) - Y(t)}{\gamma(t)}\, dt, \quad t \in [0, T).$$

By Itô's formula and (3.7), there exists a constant $C_0 > 0$ such that

$$d|X(t) - Y(t)|^2$$
$$\leqslant \left\{ C_0 \|X_t - Y_t\|_{\mathscr{C}_\nu} |X(t) - Y(t)| + K^2 |X(t) - Y(t)|^2 \right.$$
$$\left. - \frac{2|X(t) - Y(t)|^2}{\gamma(t)} \right\} dt + dM(t),$$
$$t \in [0, T), \quad (3.17)$$

where $dM(t) := 2\langle (Q(t, X(t)) - Q(t, Y(t)))\, dW(t), X(t) - Y(t)\rangle$ is a $\mathbb{Q}$-martingale. Since for some constant $C_1 > 0$ we have

$$C_0 \|X_t - Y_t\|_{\mathscr{C}_\nu} |X(t) - Y(t)| + K^2 |X(t) - Y(t)|^2 - \frac{2|X(t) - Y(t)|^2}{\gamma(t)}$$
$$\leqslant C_0 \|X_t - Y_t\|_{\mathscr{C}_\nu} |X(t) - Y(t)| - K^2 |X(t) - Y(t)|^2$$
$$\leqslant C_1 \|X_t - Y_t\|_{\mathscr{C}_\nu}^2, \quad t \in [0, T),$$

(3.17) and (1.1) imply

$$\mathbb{E}_{\mathbb{Q}}\|X_t - Y_t\|_{\mathscr{C}_\nu}^2$$

$$= \mathbb{E}_{\mathbb{Q}}|X(t) - Y(t)|^2 + \int_{-r_0}^{0} \mathbb{E}_{\mathbb{Q}}|X(t+\theta) - Y(t+\theta)|^2 \nu(\mathrm{d}\theta)$$

$$\leqslant |\xi(0) - \eta(0)|^2 + C_1 \int_0^t \mathbb{E}_{\mathbb{Q}}\|X_s - Y_s\|_{\mathscr{C}_\nu}^2 \, \mathrm{d}s$$

$$+ \kappa(T) \int_{-r_0}^{0} |\xi(\theta) - \eta(\theta)|^2 \nu(\mathrm{d}\theta) + \nu([-r_0, 0)) \int_0^t \mathbb{E}_{\mathbb{Q}}|X(s) - Y(s)|^2 \, \mathrm{d}s$$

$$\leqslant C_2 \|\xi - \eta\|_{\mathscr{C}_\nu}^2 + C_2 \int_0^t \mathbb{E}_{\mathbb{Q}}\|X_s - Y_s\|_{\mathscr{C}_\nu}^2 \, \mathrm{d}s, \quad t \in [0, T)$$

for some constant $C_2 > 0$. Therefore, by Gronwall's lemma,

$$\mathbb{E}_{\mathbb{Q}}\|X_t - Y_t\|_{\mathscr{C}_\nu}^2 \leqslant C_2 \mathrm{e}^{C_2 T} \|\xi - \eta\|_{\mathscr{C}_\nu}^2, \quad t \in [0, T). \tag{3.18}$$

On the other hand, by (3.17) and noting that $2 + \gamma' - K^2\gamma = 1$, we have

$$\mathrm{d}\left\{ \frac{|X(t) - Y(t)|^2}{\gamma(t)} \right\}$$

$$\leqslant \left\{ \frac{C_0 \|X_t - Y_t\|_{\mathscr{C}_\nu} |X(t) - Y(t)|}{\gamma(t)} - \frac{|X(t) - Y(t)|^2}{\gamma(t)^2} \left(2 + \gamma'(t) - K^2\gamma(t)\right) \right\} \mathrm{d}t$$

$$+ \frac{1}{\gamma(t)} \, \mathrm{d}M(t)$$

$$\leqslant \left\{ \frac{C_0^2}{2} \|X_t - Y_t\|_{\mathscr{C}_\nu}^2 - \frac{|X(t) - Y(t)|^2}{2\gamma(t)^2} \right\} \mathrm{d}t + \frac{1}{\gamma(t)} \, \mathrm{d}M(t), \quad t \in [0, T).$$

So, it follows from (3.18) that

$$\mathbb{E}_{\mathbb{Q}} \int_0^T \frac{|X(t) - Y(t)|^2}{\gamma(t)^2} \, \mathrm{d}t \leqslant \frac{2|\xi(0) - \eta(0)|^2}{\gamma(0)} + C_0^2 \int_0^T \mathbb{E}_{\mathbb{Q}}\|X_t - Y_t\|_{\mathscr{C}_\nu}^2 \, \mathrm{d}t.$$

Combining this with (3.10), (3.11), (3.12) and (3.18), we arrive at

$$\mathbb{E}_{\mathbb{Q}} \log R = \lim_{t \uparrow T} \mathbb{E}_{\mathbb{Q}} \left\{ \int_0^t \langle \phi(s), \, \mathrm{d}W(s) \rangle_{\mathbb{H}} + \frac{1}{2} \int_0^t |\phi(s)|_{\mathbb{H}}^2 \, \mathrm{d}s \right\}$$

$$= \frac{1}{2} \int_0^T \mathbb{E}_{\mathbb{Q}}|\phi(s)|_{\mathbb{H}}^2 \, \mathrm{d}s \leqslant C \left(\frac{1}{T}|\xi(0) - \eta(0)|^2 + \|\xi - \eta\|_{\mathscr{C}_\nu}^2 \right)$$

for some constant $C > 0$. Then the proof is finished by (3.16). $\qquad\square$

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