Paper:
Ji, Y. & Yuan, C. (2017). Tamed EM scheme of neutral stochastic differential delay equations. *Journal of Computational and Applied Mathematics, 326*, 337-357.
http://dx.doi.org/10.1016/j.cam.2017.06.002

Accepted Manuscript

Tamed EM scheme of neutral stochastic differential delay equations

Yanting Ji, Chenggui Yuan

PII: S0377-0427(17)30302-3
DOI: http://dx.doi.org/10.1016/j.cam.2017.06.002
Reference: CAM 11180

To appear in: *Journal of Computational and Applied Mathematics*

Received date : 22 March 2016
Revised date : 2 June 2017

Tamed EM Scheme of Neutral Stochastic Differential Delay Equations

Yanting Ji[a] and **Chenggui Yuan**[b*]
[a]Bryant Zhuhai
Beijing Institute of Technology, Zhuhai, P.R.C.
[b]Department of Mathematics
Swansea University, Swansea, SA2 8PP, U.K.

Abstract

In this paper, we investigate the convergence of the tamed Euler-Maruyama (EM) scheme for a class of neutral stochastic differential delay equations. The strong convergence results of the tamed EM scheme are presented under global and local non-Lipschitz conditions, respectively. Moreover, under the global Lipschitz condition, we provide the convergence rate of tamed EM scheme, which could be the same as the convergence rate of classical EM scheme one half.

MSC 2010: 65C30 (65L20, 60H10)

Key Words and Phrases: neutral stochastic differential delay equations, non-Lipschitz, monotonicity, tamed EM scheme, rate of convergence, pure jumps, Poission processes.

1 Introduction

The Euler-Maruyama (EM) scheme is of vital importance in numerical approximation for stochastic differential equations (SDEs). In [21], Kloeden and Platen illustrated that, if the coefficients of an SDE are globally Lipschitz continuous, then the EM approximation converges to the exact solution of the SDE in both strong and weak sense, the convergence rates for both cases are provided as well. In the same book, they also mentioned that the Milstein scheme converges to exact solution of SDE in both strong and weak sense with different orders under certain conditions including the global Lipschitz condition. It is the first time that Higham, Mao and Stuart [12] established strong convergence results under the super-linear condition and the moment boundedness condition, however, it remained an open question whether the moment of the EM approximation is bounded within finite time if the coefficients of an SDE are not globally Lipschitz continuous. Recently, Hutzenthaler, Jentzen and Kloeden [10] have found that once the global Lipschitz condition was replaced by the super-linear condition, the moment of the EM scheme could be infinity within finite time. To tackle this problem, in the paper [11], Hutzenthaler,

*Contact e-mail address: C.Yuan@swansea.ac.uk

Jentzen and Kloeden introduced a new approximation scheme, which is the so-call tamed EM scheme. By employing the tamed scheme, the drift coefficient is tamed so that it is uniformly bounded. With such an approach, it has been proved that the tamed EM scheme converges to the exact solution of the SDE under the super-linear condition of the drift coefficients. The tamed EM scheme is later extended to SDEs with locally Lipschitz condition of the diffusion by Dareiotis et al. [7] and Sabanis [31].

On the other hand, stochastic differential delay equations (SDDEs) and neutral stochastic differential delay equations (NSDDEs) describe a wide variety of natural and man-made systems. For the theories and applications of SDDEs and NSDDEs, we here only mention [1, 2, 4, 5, 6, 8, 9, 15, 18, 22, 27, 28], to name a few. Since most SDDEs and NSDDEs can not be solved explicitly, numerical methods have become essential. Recently, an extensive literature has emerged in investigating the strong convergence, weak convergence and sample path convergence of numerical schemes for SDDEs and NSDDEs, for example, [13, 18, 20, 26]. We should point out that the strong convergence of EM schemes for SDDEs is, in general, discussed under a linear growth condition or bounded moments of analytic and numerical solutions, e.g., [18, 20, 26]. However, similar to the SDEs case, it remained an open question whether the EM scheme converges to the exact solution if the coefficients of the SDDEs and NSDDEs are under the super-linear condition. The main aim of this paper is to answer this question by extending the tamed EM method to NSDDEs. Because of the neutral term, the technical details increased significantly.

The remainder of this paper will be organised as follows. In Section 2, some notation and preliminaries are introduced. In Section 3, p-th moment boundedness, convergence of EM scheme under global and local monotonicity conditions are provided with detailed proofs respectively. Moreover, the rate of convergence is provided under the global monotonicity condition. In Section 4, we present similar results as in Section 3 while the Brownian motion is replaced by the pure jump processes.

2 Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition (i.e. it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $\tau > 0$ be a constant and denote $C([-\tau, 0]; \mathbb{R}^n)$ the space of all continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\phi\| = \sup_{-\tau\leq\theta\leq 0} |\phi(\theta)|$. Let $B(t)$ be a standard m-dimensional Brownian motion.

Consider an n-dimensional neutral stochastic differential delay equation

$$d[X(t) - D(X(t-\tau))] = b(X(t), X(t-\tau))dt + \sigma(X(t), X(t-\tau))dB(t), \tag{2.1}$$

on $t \geq 0$, where

$$D : \mathbb{R}^n \to \mathbb{R}^n,\ b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,\ \sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m},$$

we assume that D, b and σ are Borel-measurable, and the initial data satisfies the following condition: for any $p \geq 2$

$$\{X(\theta) : -\tau \leq \theta \leq 0\} = \xi \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n), \tag{2.2}$$

that is ξ is an $\mathcal{F}_0$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-valued random variable and $E\|\xi\|^p < \infty$. Now, fix $T > \tau > 0$, without loss of generality, we assume that T and τ are rational numbers, and the step size $h \in (0,1)$ be fraction of τ and T, so that there exist two positive integers $M, \bar{M}$ such that $h = T/M = \tau/\bar{M}$. Throughout the paper, unless otherwise stated, let $C > 0$ denote a generic constant, independent of h, whose value may change from line to line.

For the future use, we assume that:

(A1) There exists a positive constant $\tilde{K}$ such that for $\forall\, x, y \in \mathbb{R}^n$,

$$\langle x - D(y), b(x,y)\rangle \vee ||\sigma(x,y)||^2 \le \tilde{K}(1 + |x|^2 + |y|^2), \tag{2.3}$$

and $b(x,y)$ is continuous in both x and y.

(A2) $D(0) = 0$ and there exists a constant $\kappa \in (0,1)$ such that

$$|D(x) - D(\bar{x})| \le \kappa|x - \bar{x}| \text{ for all } x, y \in \mathbb{R}^n. \tag{2.4}$$

(A3) For any $R > 0$, there exist two positive constants $\tilde{K}_R$ and K_R such that

$$\begin{aligned} &\langle x - D(y) - \bar{x} + D(\bar{y}), b(x,y) - b(\bar{x},\bar{y})\rangle \vee ||\sigma(x,y) - \sigma(\bar{x},\bar{y})||^2 \\ &\le \tilde{K}_R(|x - \bar{x}|^2 + |y - \bar{y}|^2), \end{aligned} \tag{2.5}$$

for all $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \le R$.

(A4) There exist two positive constants l and L such that,

$$\langle x - D(y) - \bar{x} + D(\bar{y}), b(x,y) - b(\bar{x},\bar{y})\rangle + ||\sigma(x,y) - \sigma(\bar{x},\bar{y})||^2 \le L(|x-\bar{x}|^2 + |y - \bar{y}|^2),$$

and

$$|b(x,y) - b(\bar{x},\bar{y})| \le L(1 + |x|^l + |y|^l + |\bar{x}|^l + |\bar{y}|^l)(|x - \bar{x}| + |y - \bar{y}|),$$

for all x, y, $\bar{x}$ and $\bar{y} \in \mathbb{R}^n$.

(A5) For any $s, t \in [-\tau, 0]$ and $q > 0$, let $\bar{L}$ be a positive number, then

$$\mathbb{E}\|\xi(t) - \xi(s)\|^q \le \bar{L}|t - s|^q.$$

Remark 2.1 Assume that **(A1)-(A3)** hold, then NSDDE (2.1) with initial (2.2) admits a unique strong global solution $X(t)$, $t \in [0, T]$. The proof details of such existence and uniqueness result can be found in [19]. If assumption **(A3)** is replaced by **(A4)**, the theorem of existence and uniqueness still holds.

Now, we define

$$b_h(x,y) := \frac{b(x,y)}{1 + h^\alpha |b(x,y)|}, \tag{2.6}$$

for all $x, y \in \mathbb{R}^n$ and $\alpha \in (0, \frac{1}{2}]$. The reason why such value of α is chosen will be revealed later in Section 3. By observation, one has

$$|b_h(x,y)| \le \min(h^{-\alpha}, |b(x,y)|).$$

Remark 2.2 It is easy to verify that $b_h(x,y) = \frac{b(x,y)}{1+h^\alpha|b(x,y)|}$ satisfies all assumptions **(A1)**, **(A3)** and **(A4)**. For the assumption **(A1)**, we have

$$\begin{aligned} 2\langle x - D(y), b_h(x,y)\rangle &= \frac{2}{1+h^\alpha|b(x,y)|}\langle x - D(y), b(x,y)\rangle \\ &\le \frac{\tilde{K}}{1+h^\alpha|b(x,y)|}(1+|x|^2+|y|^2) \\ &\le \tilde{K}(1+|x|^2+|y|^2). \end{aligned}$$

For the assumption **(A3)**, which is a local assumption, we can derive that

$$\begin{aligned} &2\langle x - D(y) - \bar{x} + D(\bar{y}), b_h(x,y) - b_h(\bar{x},\bar{y})\rangle \\ &\le \bar{K}_R(|x-\bar{x}|^2 + |y-\bar{y}|^2), \end{aligned}$$

where $\bar{K}_R = \frac{\tilde{K}_R}{1+h^\alpha(|b(\bar{x},\bar{y})|\wedge|b(x,y)|)}$. The verification detail for assumption **(A4)** is omitted, since it is similar to **(A3)**.

Now, we can define the discrete-time tamed EM scheme:
For every integer $n = -\bar{M}, \cdots, 0$, $Y_h^{(n)} = \xi(nh)$. For every integer $n = 0, \cdots, M-1$,

$$Y_h^{(n+1)} - D(Y_h^{(n+1-\bar{M})}) = Y_h^{(n)} - D(Y_h^{(n-\bar{M})}) + b_h(Y_h^{(n)}, Y_h^{(n-\bar{M})})h + \sigma(Y_h^{(n)}, Y_h^{(n-\bar{M})})\Delta B_h^{(n)}, \tag{2.7}$$

where $\Delta B_h^{(n)} = B((n+1)h) - B(nh)$. For every integer $n = 0, \cdots, M-1$, the discrete-time tamed EM scheme (2.7) can be rewritten as

$$\begin{aligned} Y_h^{(n+1)} =& D(Y_h^{(n+1-\bar{M})}) + \xi(0) - D(\xi(-\tau)) + \sum_{i=0}^{n} b_h(Y_h^{(i)}, Y_h^{(i-\bar{M})})h \\ &+ \sum_{i=0}^{n} \sigma(Y_h^{(i)}, Y_h^{(i-\bar{M})})\Delta B_h^{(i)}. \end{aligned} \tag{2.8}$$

For $t \in [nh, (n+1)h)$, we set $\bar{Y}(t) := Y_h^{(n)}$. Since $\tau = \bar{M}h$, $\bar{Y}(t-\tau) = Y_h^{(n-\bar{M})}$. For the sake of simplicity, we define the corresponding continuous-time tamed EM approximate solution $Y(t)$ as follows. For any $\theta \in [-\tau, 0]$, $Y(\theta) = \xi(\theta)$. For any $t \in [0,T]$,

$$\begin{aligned} Y(t) =& D(\bar{Y}(t-\tau)) + \xi(0) - D(\xi(-\tau)) + \int_0^t b_h(\bar{Y}(s), \bar{Y}(s-\tau))ds \\ &+ \int_0^t \sigma(\bar{Y}(s), \bar{Y}(s-\tau))dB(s). \end{aligned} \tag{2.9}$$

Noting that for any $t \in [0,T]$, there exists a positive integer n, $0 \le n \le M-1$, such that for $t \in [nh, (n+1)h)$, we have

$$\begin{aligned} Y(t) &= D(\bar{Y}(t-\tau)) + \xi(0) - D(\xi(-\tau)) + \int_0^{nh} b_h(\bar{Y}(s), \bar{Y}(s-\tau))ds \\ &\quad + \int_0^{nh} \sigma(\bar{Y}(s), \bar{Y}(s-\tau))dB(s) + \int_{nh}^t b_h(\bar{Y}(s), \bar{Y}(s-\tau))ds \\ &\quad + \int_{nh}^t \sigma(\bar{Y}(s), \bar{Y}(s-\tau))dB(s) \\ &= Y(nh) + \int_{nh}^t b_h(\bar{Y}(s), \bar{Y}(s-\tau))ds + \int_{nh}^t \sigma(\bar{Y}(s), \bar{Y}(s-\tau))dB(s). \end{aligned} \tag{2.10}$$

This means the continuous-time tamed EM approximate solution $Y(t)$ coincides with the discrete-time tamed approximation solution $\bar{Y}(t)$ at grid points $t = nh$, $n = 0, 1, \cdots, M-1$.

3 Tamed EM Method of NSDDEs driven by Brownian Motion

In this section, we show that the tamed EM scheme converges to the exact solution under certain conditions, i.e. we have the following main results:

Theorem 3.1 *Suppose that* **(A1)**, **(A2)**, **(A4)** *and* **(A5)** *hold, then the tamed EM scheme* (2.9) *converges to the exact solution of* (2.1) *such that for any* $p \geq 2$,

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}|X(t)-Y(t)|^p\Big] \leq Ch^{\alpha p}. \tag{3.1}$$

The next theorem states that the convergence result still holds if the global monotonicity condition **(A4)** is replaced by its local counterpart **(A3)**. However, we are unable to provide the convergence rate under this weaker condition.

Theorem 3.2 *Suppose that* **(A1)**-**(A3)** *and* **(A5)** *hold, then the tamed EM scheme* (2.9) *converges to the exact solution of* (2.1) *such that for any* $p \geq 2$,

$$\lim_{h\to 0}\mathbb{E}\Big[\sup_{0\leq t\leq T}|X(t)-Y(t)|^p\Big] = 0. \tag{3.2}$$

3.1 Moment Bounds

Before the proof of our main results, we investigate the boundedness of moments of both exact solution and EM approximation in this subsection.

Lemma 3.1 *Consider the continuous-time tamed EM scheme given by equation* (2.10). *If for some* $p \geq 2$,

$$\sup_{0\leq t\leq T}\mathbb{E}(|Y(t)|^p) \leq C, \tag{3.3}$$

and **(A1)** *hold, then it holds that*

$$\mathbb{E}\Big[\sup_{0\leq n\leq M-1}\ \sup_{nh\leq t\leq (n+1)h}|Y(t)-Y(nh)|^p\Big] \leq Ch^{p/2}, \tag{3.4}$$

and

$$\mathbb{E}\Big[\sup_{0\leq n\leq M-1}\ \sup_{nh\leq t\leq (n+1)h}|Y(t)-Y(nh)|^p|b_h(\bar{Y}(t),\bar{Y}(t-\tau))|^p\Big] \leq C. \tag{3.5}$$

Proof: By the definition of the tamed EM scheme, we have for $nh \leq t \leq (n+1)h$,

$$\begin{aligned}\mathbb{E}\Big[\sup_{nh\leq t\leq (n+1)h}|Y(t)-Y(nh)|^p\Big] = \mathbb{E}\Big[\sup_{nh\leq t\leq (n+1)h}\Big|\int_{nh}^t b_h(\bar{Y}(s),\bar{Y}(s-\tau))ds \\ + \int_{nh}^t \sigma(\bar{Y}(s),\bar{Y}(s-\tau))dB(s)\Big|^p\Big].\end{aligned}$$

Therefore, due to Hölder's inequality,

$$\mathbb{E}\Big[\sup_{nh\le t\le(n+1)h}|Y(t)-Y(nh)|^p\Big]\le 2^{p-1}h^{p-1}\mathbb{E}\Big[\int_{nh}^{(n+1)h}\Big|b_h(\bar{Y}(s),\bar{Y}(s-\tau))\Big|^p ds\Big]$$
$$+2^{p-1}\mathbb{E}\Big[\sup_{nh\le t\le(n+1)h}\Big|\int_{nh}^{t}\sigma(\bar{Y}(s),\bar{Y}(s-\tau))dB(s)\Big|^p\Big]. \tag{3.6}$$

Using the Burkholder-Davis-Gundy(BDG) inequality [23, Theorem 1.7.3 page 40] and (3.3), for some $p\ge 2$, we derive that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{nh\le t\le(n+1)h}\big|\int_{nh}^{t}\sigma(\bar{Y}(s),\bar{Y}(s-\tau))dB(s)\big|^p\Big]\\ &\le C\mathbb{E}\left[\int_{nh}^{(n+1)h}\|\sigma(\bar{Y}(s),\bar{Y}(s-\tau))\|^2ds\right]^{p/2}\\ &\le C\mathbb{E}\left[\int_{nh}^{(n+1)h}(1+|\bar{Y}(s)|^2+|\bar{Y}(s-\tau)|^2)ds\right]^{p/2}\\ &\le Ch^{p/2}.\end{aligned}$$

This, together with $|b_h(\bar{Y}(s),\bar{Y}(s-\tau))|\le h^{-\alpha}$, yields

$$\mathbb{E}\Big[\sup_{nh\le t\le(n+1)h}|Y(t)-Y(nh)|^p\Big]\le 2^{p-1}h^{(1-\alpha)p}+Ch^{p/2}\le Ch^{p/2}. \tag{3.7}$$

Therefore (3.4) holds. Moreover,

$$\begin{aligned}&\mathbb{E}\Big[\sup_{nh\le t\le(n+1)h}|Y(t)-Y(nh)|^p|b_h(\bar{Y}(s),\bar{Y}(s-\tau))|^p\Big]\\ &\le \mathbb{E}\Big[\sup_{nh\le t\le(n+1)h}|Y(t)-Y(nh)|^p\Big]h^{-\alpha p}\\ &\le Ch^{(1/2-\alpha)p}\le C.\end{aligned} \tag{3.8}$$

In (3.7) and (3.8), we have used the fact that $\alpha\in(0,1/2]$. The proof is therefore complete. $\Box$

Lemma 3.2 *Assume that* **(A1)**, **(A2)** *and* **(A5)** *hold. Then there exists a positive constant C independent of h such that for any $p\ge 2$,*

$$\mathbb{E}[\sup_{0\le t\le T}|X(t)|^p]\vee\mathbb{E}[\sup_{0\le t\le T}|Y(t)|^p]\le C. \tag{3.9}$$

Proof. For every integer $k\ge 1$, define the stopping time

$$\tau_k=T\wedge\inf\{t\in[0,T]:|X(t)|\ge k\}.$$

Clearly, $\tau_k\to T$ as $k\to\infty$ almost surely. Now, for any $t\in[0,T]$, by [23, Lemma 4.4, p212] and assumption **(A2)**, we know that for any $p\ge 2$,

$$\sup_{0\le s\le t\wedge\tau_k}|X(s)|^p\le\frac{\kappa}{1-\kappa}||\xi||^p+\frac{1}{(1-\kappa)^p}\sup_{0\le s\le t\wedge\tau_k}|X(s)-D(X(s-\tau))|^p. \tag{3.10}$$

An application of Itô's formula yields,

$$
\begin{aligned}
&|X(t)-D(X(t-\tau))|^p \le |\xi(0)-D(\xi(-\tau))|^p \\
&+p\int_0^t |X(s)-D(X(s-\tau))|^{p-2}\langle X(s)-D(X(s-\tau)), b(X(s),X(s-\tau))\rangle ds \\
&+\frac{p(p-1)}{2}\int_0^t |X(s)-D(X(s-\tau))|^{p-2}||\sigma(X(s),X(s-\tau))||^2 ds \\
&+p\int_0^t |X(s)-D(X(s-\tau))|^{p-2}(X(s)-D(X(s-\tau)))^T(\sigma(X(s),X(s-\tau))dB(s)) \\
&=: |\xi(0)-D(\xi(-\tau))|^p + H_1(t)+H_2(t)+H_3(t).
\end{aligned}
\tag{3.11}
$$

Under **(A1)** and **(A2)**, one has

$$
\begin{aligned}
&\mathbb{E}(\sup_{0\le s\le t\wedge\tau_k} H_1(s)) + \mathbb{E}(\sup_{0\le s\le t\wedge\tau_k} H_2(s)) \\
&\le C\mathbb{E}\int_0^{t\wedge\tau_k} |X(s)-D(X(s-\tau))|^{p-2}(1+|X(s)|^2+|X(s-\tau)|^2)ds \\
&\le C\mathbb{E}\int_0^{t\wedge\tau_k} (|X(s)|^{p-2}+|D(X(s-\tau))|^{p-2})(1+|X(s)|^2+|X(s-\tau)|^2)ds \\
&\le C\mathbb{E}\int_0^{t\wedge\tau_k} (1+|X(s)|^p+|X(s-\tau)|^p)ds \le C + C\int_0^t \mathbb{E}(\sup_{0\le u\le s\wedge\tau_k}|X(u)|^p)ds.
\end{aligned}
\tag{3.12}
$$

By the Burkholder-Davis-Gundy(BDG) inequality and the Young inequality, we derive that

$$
\begin{aligned}
&\mathbb{E}(\sup_{0\le s\le t\wedge\tau_k} H_3(s)) \le C\mathbb{E}\Big(\int_0^{t\wedge\tau_k}\big(|X(s)-D(X(s-\tau))|^{2p-2}\|\sigma(X(s),X(s-\tau))\|^2\big)ds\Big)^{1/2} \\
&\le C\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_k}|X(s)-D(X(s-\tau))|^{p-1}\big(\int_0^{t\wedge\tau_k}\|\sigma(X(s),X(s-\tau))\|^2 ds\big)^{1/2}\Big) \\
&\le \frac{1}{4}\mathbb{E}(\sup_{0\le s\le t\wedge\tau_k}|X(s)-D(X(s-\tau))|^p) + C(\int_0^{t\wedge\tau_k}\|\sigma(X(s),X(s-\tau))\|^2 ds)^{p/2} \\
&\le \frac{1}{4}\mathbb{E}(\sup_{0\le s\le t\wedge\tau_k}|X(s)-D(X(s-\tau))|^p) + C(\int_0^{t\wedge\tau_k}(1+|X(s)|^2+|X(s-\tau)|^2 ds)^{p/2} \\
&\le \frac{1}{4}\mathbb{E}(\sup_{0\le s\le t\wedge\tau_k}|X(s)-D(X(s-\tau))|^p) + C\int_0^{t\wedge\tau_k}\mathbb{E}(1+|X(s)|^p+|X(s-\tau)|^p)ds \\
&\le \frac{1}{4}\mathbb{E}(\sup_{0\le s\le t\wedge\tau_k}|X(s)-D(X(s-\tau))|^p) + C + C\int_0^t \mathbb{E}(\sup_{0\le u\le s\wedge\tau_k}|X(u)|^p)ds.
\end{aligned}
\tag{3.13}
$$

Now, substituting (3.12) and (3.13) into (3.11), we then have

$$
\mathbb{E}(\sup_{0\le s\le t\wedge\tau_k}|X(s)-D(X(s-\tau))|^p) \le C + C\int_0^t \mathbb{E}(\sup_{0\le u\le s\wedge\tau_k}|X(u)|^p)ds. \tag{3.14}
$$

This, together with (3.10), implies that

$$
\mathbb{E}(\sup_{0\le s\le t\wedge\tau_k}|X(s)|^p) \le C + C\int_0^t \mathbb{E}(\sup_{0\le u\le s\wedge\tau_k}|X(u)|^p)ds.
$$

The desired assertion for the exact solution follows an application of Gronwall's inequality and letting $k \to \infty$.

In order to estimate the p-th moment of the tamed EM scheme (2.9), an inductive argument is used below. Firstly, we claim that there exists a constant C such that:

$$\sup_{0\le t\le T} \mathbb{E}|Y(t)|^2 \le C. \tag{3.15}$$

Similarly, we define another stopping time: for every integer $k \ge 1$, define a stopping time

$$\bar{\tau}_k = T \wedge \inf\{t \in [0,T] : |Y(t)| \ge k\}.$$

Clearly, $\bar{\tau}_k \to T$ as $k \to \infty$ almost surely. Now, for any $t \in [0,T]$, an application of the Itô formula yields

$$\begin{aligned}
&|Y(t) - D(\bar{Y}(t-\tau))|^2 \\
&= |\xi(0) - D(\xi(-\tau))|^2 + 2\int_0^t \langle Y(s) - D(\bar{Y}(s-\tau)), b_h(\bar{Y}(s), \bar{Y}(s-\tau))\rangle ds \\
&+ \int_0^t ||\sigma(\bar{Y}(s), \bar{Y}(s-\tau))||^2 ds + 2\int_0^t \langle Y(s) - D(\bar{Y}(s-\tau)), \sigma(\bar{Y}(s), \bar{Y}(s-\tau))dB(s)\rangle \\
&= |\xi(0) - D(\xi(-\tau))|^2 + 2\int_0^t \langle \bar{Y}(s) - D(\bar{Y}(s-\tau)), b_h(\bar{Y}(s), \bar{Y}(s-\tau))\rangle ds \\
&+ 2\int_0^t \langle Y(s) - \bar{Y}(s), b_h(\bar{Y}(s), \bar{Y}(s-\tau))\rangle ds + \int_0^t ||\sigma(\bar{Y}(s), \bar{Y}(s-\tau))||^2 ds \\
&+ 2\int_0^t \langle Y(s) - D(\bar{Y}(s-\tau)), \sigma(\bar{Y}(s), \bar{Y}(s-\tau))dB(s)\rangle \\
&=: |\xi(0) - D(\xi(-\tau))|^2 + \hat{H}_1(t) + \hat{H}_2(t) + \hat{H}_3(t) + \hat{H}_4(t).
\end{aligned} \tag{3.16}$$

By **(A1)** and **(A2)**, we compute

$$\begin{aligned}
&\sup_{0\le s\le t} \mathbb{E}(\hat{H}_1(s\wedge\bar{\tau}_k)) + \mathbb{E}(\hat{H}_3(s\wedge\bar{\tau}_k)) \le C\mathbb{E}\int_0^{t\wedge\bar{\tau}_k} 1 + |\bar{Y}(s)|^2 + |\bar{Y}(s-\tau)|^2 ds \\
&\le C + C\int_0^t \sup_{0\le u\le s} \mathbb{E}(|Y(u\wedge\bar{\tau}_k)|^2) ds.
\end{aligned} \tag{3.17}$$

Using the definition of $Y(s)$ and $\bar{Y}(s)$, together with the property of conditional expectation, we have

$$\begin{aligned}
&\sup_{0\le s\le t}\mathbb{E}(\hat{H}_2(s\wedge\bar{\tau}_k)) = 2\sup_{0\le s\le t}\Big\{\mathbb{E}\int_0^{s\wedge\bar{\tau}_k}\langle\int_{[\frac{u}{h}]h}^{u} b_h(\bar{Y}(r),\bar{Y}(r-\tau))dr, b_h(\bar{Y}(u),\bar{Y}(u-\tau))\rangle du\\
&+\mathbb{E}\int_0^{s}\langle\int_{[\frac{u\wedge\bar{\tau}_k}{h}]h}^{u\wedge\bar{\tau}_k}\sigma(\bar{Y}(r),\bar{Y}(r-\tau))dB(r), b_h(\bar{Y}(u\wedge\bar{\tau}_k),\bar{Y}(u\wedge\bar{\tau}_k-\tau))\rangle du\Big\}\\
&= 2\sup_{0\le s\le t}\Big\{\mathbb{E}\int_0^{s\wedge\bar{\tau}_k}\langle\int_{[\frac{u}{h}]h}^{u} b_h(\bar{Y}(r),\bar{Y}(r-\tau))dr, b_h(\bar{Y}(u),\bar{Y}(u-\tau))\rangle du\\
&+\mathbb{E}\int_0^{s}\mathbb{E}\langle\int_{[\frac{u\wedge\bar{\tau}_k}{h}]h}^{u\wedge\bar{\tau}_k}\sigma(\bar{Y}(r),\bar{Y}(r-\tau))dB(r), b_h(\bar{Y}(u\wedge\bar{\tau}_k),\bar{Y}(u\wedge\bar{\tau}_k-\tau))\rangle|\mathcal{F}_{[\frac{u\wedge\bar{\tau}_k}{h}]h}du\Big\}\\
&= 2\sup_{0\le s\le t}\mathbb{E}\int_0^{s\wedge\bar{\tau}_k}\langle\int_{[\frac{u}{h}]h}^{u} b_h(\bar{Y}(r),\bar{Y}(r-\tau))dr, b_h(\bar{Y}(u),\bar{Y}(u-\tau))\rangle du\\
&\le Cth^{1-2\alpha}\le C,
\end{aligned}\tag{3.18}$$

where we have used $\alpha\in(0,1/2]$. Using the Young inequality, we have

$$\begin{aligned}
&\sup_{0\le s\le t}\mathbb{E}(|Y(s\wedge\bar{\tau}_k)|^2) = \sup_{0\le s\le t}\mathbb{E}(|Y(s\wedge\bar{\tau}_k)-D(\bar{Y}((s-\tau)\wedge\bar{\tau}_k)) + D(\bar{Y}((s-\tau)\wedge\bar{\tau}_k))|^2)\\
&\le C+\frac{1}{2}\sup_{0\le s\le t}\mathbb{E}(|Y(s\wedge\bar{\tau}_k)|^2) + C\sup_{0\le s\le t}\mathbb{E}(|Y(s\wedge\bar{\tau}_k)-D(\bar{Y}((s-\tau)\wedge\bar{\tau}_k))|^2),
\end{aligned}\tag{3.19}$$

Now, substituting (3.17) and (3.18) into (3.19), we can derive that

$$\begin{aligned}
&\sup_{0\le s\le t}\mathbb{E}(|Y(s\wedge\bar{\tau}_k)|^2)\le C+C\big(\sup_{0\le s\le t}\mathbb{E}(|Y(s\wedge\bar{\tau}_k)-D(\bar{Y}((s-\tau)\wedge\bar{\tau}_k))|^2)\big)\\
&\le C+C\int_0^t\sup_{0\le u\le s}\mathbb{E}(|Y(u\wedge\bar{\tau}_k)|^2)ds.
\end{aligned}$$

The assertion (3.15) follows an application of the Gronwall inequality and letting $k\to\infty$.

In the sequel, we are now going to show that there exists a constant $C>0$ such that for any $p\ge 2$,

$$\mathbb{E}[\sup_{0\le t\le T}|Y(t)|^p]\le C.\tag{3.20}$$

Letting $p=4$ and using the Itô formula, we have for any $t\in[0,T]$,

$$\begin{aligned}
&|Y(t)-D(\bar{Y}(t-\tau))|^p = |\xi(0)-D(\xi(-\tau))|^p + p\int_0^t|Y(s)-D(\bar{Y}(s-\tau))|^{p-2}\\
&\qquad\times\langle\bar{Y}(s)-D(\bar{Y}(s-\tau)), b_h(\bar{Y}(s),\bar{Y}(s-\tau))\rangle ds\\
&+p\int_0^t|Y(s)-D(\bar{Y}(s-\tau))|^{p-2}\langle Y(s)-\bar{Y}(s), b_h(\bar{Y}(s),\bar{Y}(s-\tau))\rangle ds\\
&+\frac{p(p-1)}{2}\int_0^t|Y(s)-D(\bar{Y}(s-\tau))|^{p-2}\|\sigma(\bar{Y}(s),\bar{Y}(s-\tau))\|^2ds\\
&+p\int_0^t|Y(s)-D(\bar{Y}(s-\tau))|^{p-2}\langle Y(s)-D(\bar{Y}(s-\tau)),\sigma(\bar{Y}(s),\bar{Y}(s-\tau))dB(s)\rangle\\
&=: |\xi(0)-D(\xi(-\tau))|^p+\bar{H}_1(t)+\bar{H}_2(t)+\bar{H}_3(t)+\bar{H}_4(t).
\end{aligned}\tag{3.21}$$

Under assumptions **(A1)** and **(A2)**, there exists a constant $C > 0$ such that

$$\begin{aligned}
&\mathbb{E}(\sup_{0\le s\le t\wedge\bar{\tau}_k}\bar{H}_1(s)) + \mathbb{E}(\sup_{0\le s\le t\wedge\bar{\tau}_k}\bar{H}_3(s))\\
&\le C\mathbb{E}\int_0^{t\wedge\bar{\tau}_k}|Y(s)-D(\bar{Y}(s-\tau))|^{p-2}(1+|\bar{Y}(s)|^2+|\bar{Y}(s-\tau)|^2)ds\\
&\le C\mathbb{E}\int_0^{t\wedge\bar{\tau}_k}(|Y(s)|^{p-2}+|D(\bar{Y}(s-\tau))|^{p-2})(1+|\bar{Y}(s)|^2+|\bar{Y}(s-\tau)|^2)ds\\
&\le C\mathbb{E}\int_0^{t\wedge\bar{\tau}_k}(1+|Y(s)|^p+|\bar{Y}(s)|^p+|\bar{Y}(s-\tau)|^p)ds \le C + C\int_0^t \mathbb{E}(\sup_{0\le u\le s\wedge\bar{\tau}_k}|Y(u)|^p)ds.
\end{aligned} \tag{3.22}$$

Using Young's inequality, we derive that

$$\begin{aligned}
&\mathbb{E}(\sup_{0\le s\le t\wedge\bar{\tau}_k}\bar{H}_2(s))\\
&\le C\mathbb{E}\int_0^{t\wedge\bar{\tau}_k}|Y(s)-D(\bar{Y}(s-\tau))|^{p-2}|\langle Y(s)-\bar{Y}(s), b_h(\bar{Y}(s),\bar{Y}(s-\tau))\rangle|ds\\
&\le C\mathbb{E}\big(\sup_{0\le s\le t\wedge\bar{\tau}_k}|Y(s)-D(\bar{Y}(s-\tau))|^{p-2}\times\big[|\int_0^{t\wedge\bar{\tau}_k}\langle Y(s)-\bar{Y}(s), b_h(\bar{Y}(s),\bar{Y}(s-\tau))\rangle ds|^{1/2}\big]^2\big)\\
&\le \frac{1}{4}\mathbb{E}\sup_{0\le s\le t\wedge\bar{\tau}_k}|Y(s)-D(\bar{Y}(s-\tau))|^p + C\mathbb{E}\int_0^{t\wedge\bar{\tau}_k}|Y(s)-\bar{Y}(s)|^{p/2}|b_h(\bar{Y}(s),\bar{Y}(s-\tau))|^{p/2}ds\\
&\le \frac{1}{4}\mathbb{E}\sup_{0\le s\le t\wedge\bar{\tau}_k}|Y(s)-D(\bar{Y}(s-\tau))|^p + C,
\end{aligned} \tag{3.23}$$

where we have applied result from obtained Lemma 3.1 to the second term above.

By the BDG inequality again, we have

$$\begin{aligned}
&\mathbb{E}(\sup_{0\le s\le t\wedge\bar{\tau}_k}\bar{H}_4(s))\\
&\le \mathbb{E}(\sup_{0\le s\le t\wedge\bar{\tau}_k} p\int_0^s|Y(u)-D(\bar{Y}(u-\tau))|^{p-2}|\langle Y(u)-D(\bar{Y}(u-\tau)), \sigma(\bar{Y}(u),\bar{Y}(u-\tau))dB(u)\rangle|)\\
&\le C\mathbb{E}\Big(\int_0^{t\wedge\bar{\tau}_k}\big(|Y(s)-D(\bar{Y}(s-\tau))|^{2p-2}\|\sigma(\bar{Y}(s),\bar{Y}(s-\tau))\|^2\big)ds\Big)^{1/2}\\
&\le C\mathbb{E}\Big(\sup_{0\le s\le t\wedge\bar{\tau}_k}|Y(s)-D(\bar{Y}(s-\tau))|^{p-1}\big(\int_0^{t\wedge\bar{\tau}_k}\|\sigma(\bar{Y}(s),\bar{Y}(s-\tau))\|^2ds\big)^{1/2}\Big)\\
&\le \frac{1}{4}\mathbb{E}(\sup_{0\le s\le t\wedge\bar{\tau}_k}|Y(s)-D(\bar{Y}(s-\tau))|^p) + C + C\int_0^t\mathbb{E}(\sup_{0\le u\le s\wedge\bar{\tau}_k}|Y(u)|^p)ds.
\end{aligned} \tag{3.24}$$

Substituting (3.23) and (3.24) into (3.21) and using (3.10), for $p = 4$, we can derive that

$$\begin{aligned}
\mathbb{E}(\sup_{0\le s\le t\wedge\bar{\tau}_k}|Y(s)|^p) &\le C + C\mathbb{E}(\sup_{0\le s\le t\wedge\bar{\tau}_k}|Y(s)-D(\bar{Y}(s-\tau))|^p)\\
&\le C + C\int_0^t\mathbb{E}(\sup_{0\le u\le s\wedge\bar{\tau}_k}|Y(u)|^p)ds.
\end{aligned}$$

The required assertion follows from an application of the Grownwall inequality and letting $k \to \infty$. By repeating the same procedure, the desired result (3.9) can be obtained by induction. □

3.2 Proof of Main Results

In this subsection, we give proofs for the main results.

Proof of Theorem 3.1: For any $0 \le t \le T$, by [23, Lemma 6.4.1] as well as the assumption **(A2)**, we have, for $p \ge 2$ and $\varepsilon > 0$,

$$\begin{aligned} |X(t) - Y(t)|^p &= |X(t) - Y(t) - D(X(t-\tau)) + D(\bar{Y}(t-\tau)) \\ &\quad + D(X(t-\tau)) - D(\bar{Y}(t-\tau))|^p \\ &\le \left[1 + \varepsilon^{\frac{1}{p-1}}\right]^{p-1} \Big(\frac{|D(X(t-\tau)) - D(\bar{Y}(t-\tau))|^p}{\varepsilon} \\ &\quad + |X(t) - D(X(t-\tau)) - Y(t) + D(\bar{Y}(t-\tau))|^p \Big) \\ &\le \left[1 + \varepsilon^{\frac{1}{p-1}}\right]^{p-1} \Big(\frac{\kappa^p |X(t-\tau) - \bar{Y}(t-\tau)|^p}{\varepsilon} \\ &\quad + |X(t) - D(X(t-\tau)) - Y(t) + D(\bar{Y}(t-\tau))|^p \Big). \end{aligned} \tag{3.25}$$

Letting $\varepsilon = [\frac{\kappa}{1-\kappa}]^{p-1}$, by (3.25) we obtain

$$\begin{aligned} &\sup_{0 \le s \le t} |X(s) - Y(s)|^p \\ &\le \kappa \sup_{0 \le s \le t} |X(s-\tau) - \bar{Y}(s-\tau)|^p \\ &+ \frac{1}{(1-\kappa)^{p-1}} \sup_{0 \le s \le t} |X(s) - D(X(s-\tau)) - Y(s) + D(\bar{Y}(s-\tau))|^p \\ &\le \kappa \sup_{-\tau \le s \le 0} |X(s) - \bar{Y}(s)|^p + \kappa \sup_{0 \le s \le t} |X(s) - Y(s) + Y(s) - \bar{Y}(s)|^p \\ &+ \frac{1}{(1-\kappa)^{p-1}} \sup_{0 \le s \le t} |X(s) - D(X(s-\tau)) - Y(s) + D(\bar{Y}(s-\tau))|^p \\ &\le \kappa \sup_{-\tau \le s \le 0} |X(s) - \bar{Y}(s)|^p + \kappa_c \sup_{0 \le s \le t} |X(s) - Y(s)|^p + C \sup_{0 \le s \le t} |Y(s) - \bar{Y}(s)|^p \\ &+ \frac{1}{(1-\kappa)^{p-1}} \sup_{0 \le s \le t} |X(s) - D(X(s-\tau)) - Y(s) + D(\bar{Y}(s-\tau))|^p, \end{aligned} \tag{3.26}$$

where $\kappa_c \in (0, 1)$ is a constant. This, together with (**A5**), implies

$$\begin{aligned} \mathbb{E}\big(\sup_{0 \le s \le t} |X(s) - Y(s)|^p \big) \le \frac{1}{(1-\kappa)^{p-1}(1-\kappa_c)} &\mathbb{E}\big(\sup_{0 \le s \le t} |X(s) - D(X(s-\tau)) \\ &- Y(s) + D(\bar{Y}(s-\tau))|^p \big) + C h^{p/2}. \end{aligned} \tag{3.27}$$

An application of Itô's formula yields

$$
\begin{aligned}
&|X(t)-D(X(t-\tau))-Y(t)+D(\bar{Y}(t-\tau))|^p \\
&\le p\int_0^t |X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^{p-2} \\
&\times \langle X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau)), b(X(s),X(s-\tau))-b_h(\bar{Y}(s),\bar{Y}(s-\tau))\rangle ds \\
&+\frac{p(p-1)}{2}\int_0^t |X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^{p-2} \\
&\times \|\sigma(X(s),X(s-\tau))-\sigma(\bar{Y}(s),\bar{Y}(s-\tau))\|^2 ds \\
&+p\int_0^t |X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^{p-2} \\
&\times \langle X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau)), \\
&\qquad (\sigma(X(s),X(s-\tau))-\sigma(\bar{Y}(s),\bar{Y}(s-\tau)))dB(s)\rangle \\
&= p\int_0^t |X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^{p-2} \\
&\times \langle X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau)), b(X(s),X(s-\tau))-b(Y(s),\bar{Y}(s-\tau))\rangle ds \\
&+p\int_0^t |X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^{p-2} \\
&\times \langle X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau)), b(Y(s),\bar{Y}(s-\tau))-b(\bar{Y}(s),\bar{Y}(s-\tau))\rangle ds \\
&+p\int_0^t |X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^{p-2} \\
&\times \langle X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau)), b(\bar{Y}(s),\bar{Y}(s-\tau))-b_h(\bar{Y}(s),\bar{Y}(s-\tau))\rangle ds \\
&+\frac{p(p-1)}{2}\int_0^t |X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^{p-2} \\
&\times ||\sigma(X(s),X(s-\tau))-\sigma(\bar{Y}(s),\bar{Y}(s-\tau))||^2 ds \\
&+p\int_0^t |X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^{p-2} \\
&\times \langle X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau)), \\
&\qquad (\sigma(X(s),X(s-\tau))-\sigma(\bar{Y}(s),\bar{Y}(s-\tau)))dB(s)\rangle \\
&=: I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t).
\end{aligned}
\tag{3.28}
$$

By assumptions **(A2)**, **(A4)** and **(A5)**, we have

$$\begin{aligned}
&\mathbb{E}(\sup_{0\le s\le t}(I_1(s))) \le C\mathbb{E}\int_0^t |X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^{p-2} \\
&\times(|X(s)-Y(s)|^2+|X(s-\tau)-\bar{Y}(s-\tau)|^2)ds \\
&\le C\mathbb{E}\int_0^t \big(|X(s)-Y(s)|^{p-2}+|D(X(s-\tau))-D(\bar{Y}(s-\tau))|^{p-2}\big) \\
&\times(|X(s)-Y(s)|^2+|X(s-\tau)-\bar{Y}(s-\tau)|^2)ds \\
&\le C\int_0^t \mathbb{E}(\sup_{0\le u\le s}|X(u)-Y(u)|^p)ds + C\int_0^t \mathbb{E}(\sup_{0\le u\le s}|X(u)-Y(u)+Y(u)-\bar{Y}(u)|^p)ds \\
&+C\int_{-\tau}^0 \mathbb{E}(\sup_{-\tau\le\theta\le 0}|X(\theta)-\bar{Y}(\theta)|^p)ds \le C\int_0^t \mathbb{E}(\sup_{0\le u\le s}|X(u)-Y(u)|^p)ds + Ch^p + Ch^{p/2}.
\end{aligned} \tag{3.29}$$

In the same way as in (3.29), we can estimate $I_4(t)$ such that

$$\mathbb{E}(\sup_{0\le s\le t}(I_4(s))) \le C\int_0^t \mathbb{E}(\sup_{0\le u\le s}|X(u)-Y(u)|^p)ds + Ch^p + Ch^{p/2}. \tag{3.30}$$

Using assumption **(A4)**, Hölder's inequality and (3.4), we arrive at

$$\begin{aligned}
\mathbb{E}(\sup_{0\le s\le t} I_2(s)) \le\ & \mathbb{E}\bigg(\int_0^t |X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^{p-1} \\
& \times L(1+|\bar{Y}(s)|^l+|Y(s)|^l+2|\bar{Y}(s-\tau)|^l)(|Y(s)-\bar{Y}(s)|)ds\bigg) \\
\le\ & \mathbb{E}\bigg(\sup_{0\le s\le t}|X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^{p-1} \\
& \times\int_0^t L(1+|\bar{Y}(s)|^l+|Y(s)|^l+2|\bar{Y}(s-\tau)|^l)(|Y(s)-\bar{Y}(s)|)ds\bigg) \\
\le\ & \frac{1}{4}\mathbb{E}(\sup_{0\le s\le t}|X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^p) \\
& +C\mathbb{E}\bigg[\int_0^t (1+|\bar{Y}(s)|^l+|Y(s)|^l+|\bar{Y}(s-\tau)|^l)(|Y(s)-\bar{Y}(s)|)ds\bigg]^p \\
\le\ & \frac{1}{4}\mathbb{E}(\sup_{0\le s\le t}|X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^p) \\
& +C\bigg[\int_0^t \mathbb{E}(1+|\bar{Y}(s)|^l+|Y(s)|^l+|\bar{Y}(s-\tau)|^l)^p(|Y(s)-\bar{Y}(s)|)^p ds\bigg] \\
\le\ & \frac{1}{4}\mathbb{E}(\sup_{0\le s\le t}|X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^p) \\
& +C[\int_0^t (\mathbb{E}(1+|\bar{Y}(s)|^l+|Y(s)|^l+|\bar{Y}(s-\tau)|^l)^{2p})^{1/2}(\mathbb{E}|Y(s)-\bar{Y}(s)|^{2p})^{1/2}ds] \\
\le\ & \frac{1}{4}\mathbb{E}(\sup_{0\le s\le t}|X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^p) + Ch^{p/2}.
\end{aligned} \tag{3.31}$$

We now estimate I_3,

$$\begin{aligned}
\mathbb{E}(\sup_{0\le s\le t} I_3(s)) &\le p\mathbb{E}\int_0^t |X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^{p-1} \\
&\quad \times |b(\bar{Y}(s),\bar{Y}(s-\tau))-b_h(\bar{Y}(s),\bar{Y}(s-\tau))|ds \\
&\le \frac{1}{4}\mathbb{E}(\sup_{0\le s\le t}|X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^p) \\
&\quad + C\mathbb{E}\int_0^t |b(\bar{Y}(s),\bar{Y}(s-\tau))-b_h(\bar{Y}(s),\bar{Y}(s-\tau))|^p ds \\
&\le \frac{1}{4}\mathbb{E}(\sup_{0\le s\le t}|X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^p)+Ch^{\alpha p},
\end{aligned} \tag{3.32}$$

where we have used the following fact

$$\begin{aligned}
&\mathbb{E}\int_0^t |b(\bar{Y}(s),\bar{Y}(s-\tau))-b_h(\bar{Y}(s),\bar{Y}(s-\tau))|^p ds \\
&\le h^{\alpha p}\mathbb{E}\Big[\int_0^t \frac{|b(\bar{Y}(s),\bar{Y}(s-\tau))|^{2p}}{(1+h^\alpha|b(\bar{Y}(s),\bar{Y}(s-\tau))|)^p}ds\Big] \\
&\le h^{\alpha p}\int_0^t \mathbb{E}\Big[|L(1+|\bar{Y}(s)|^l+|\bar{Y}(s-\tau)|^l)(|\bar{Y}(s)|+|\bar{Y}(s-\tau)|)+C|^{2p}\Big]ds \\
&\le Ch^{\alpha p}.
\end{aligned} \tag{3.33}$$

Moreover, the BGD inequality yields that

$$\begin{aligned}
&\mathbb{E}(\sup_{0\le s\le t} I_5(s)) \le C\mathbb{E}\Big(\int_0^t |X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^{2p-2} \\
&\times \|\sigma(X(s),X(s-\tau))-\sigma(\bar{Y}(s),\bar{Y}(s-\tau))\|^2 ds\Big)^{1/2} \\
&\le \frac{1}{4}\mathbb{E}(\sup_{0\le s\le t}|X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^p) \\
&+ C\int_0^t \mathbb{E}(|X(s)-\bar{Y}(s)|^p+|X(s-\tau)-\bar{Y}(s-\tau)|^p)ds \\
&\le \frac{1}{4}\mathbb{E}(\sup_{0\le s\le t}|X(s)-D(X(s-\tau))-Y(s)+D(\bar{Y}(s-\tau))|^p) \\
&+ C\int_0^t \mathbb{E}(\sup_{0\le u\le s}|X(u)-Y(u)|^p)ds+Ch^p.
\end{aligned} \tag{3.34}$$

Substituting (3.29), (3.30), (3.31), (3.32) and (3.34) into (3.27), then for any $t\in[0,T]$, we have

$$\begin{aligned}
\mathbb{E}\big(\sup_{0\le s\le t}|X(s)-Y(s)|^p\big) &\le Ch^{\alpha p}+Ch^{p/2}+Ch^p+C\int_0^t \mathbb{E}(\sup_{0\le u\le s}|X(u)-Y(u)|^p)ds \\
&\le Ch^{\alpha p}+C\int_0^t E(\sup_{0\le u\le s}|X(u)-Y(u)|^p)ds.
\end{aligned} \tag{3.35}$$

The desired result follows by the Grownwall inequality and the fact $\alpha\in(0,1/2]$. □

In the sequel, we want to prove Theorem 3.2, in which the monotonicity condition **(A4)** is replaced by its local counterpart **(A3)**. The techniques of this proof have been developed in Higham, Mao and Stuart [16] where they showed the strong convergence of the EM method for the SDE under a local Lipschitz condition. Therefore only the sketched proof is provided here.

Proof of Theorem 3.2: For every $R > 0$, we define the stopping times

$$\tau_R := \inf\{t \geq 0 : |X(t)| \geq R\}, \quad \rho_R := \inf\{t \geq 0 : |Y(t)| \geq R\}, \tag{3.36}$$

and denote that $\theta_R = \tau_R \wedge \rho_R$, and $e(t) = X(t) - Y(t)$. By the virtue of Young's inequality, for $q > p$ and $\eta > 0$ we have for any $t \in [0, T]$

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq s\leq t}|e(s)|^p\right] \leq \mathbb{E}\left[\sup_{0\leq s\leq t}|e(s)|^p I_{\{\tau_R \leq t \text{ or } \rho_R \leq t\}}\right] + \mathbb{E}\left[\sup_{0\leq s\leq t}|e(s\wedge\theta_R)|^p\right] \\
&= \mathbb{E}\left[\left(\eta^{p/q}\sup_{0\leq s\leq t}|e(s)|^p\right)\left(\eta^{-p/q} I_{\{\tau_R \leq t \text{ or } \rho_R \leq t\}}\right)\right] + \mathbb{E}\left[\sup_{0\leq s\leq t}|e(s\wedge\theta_R)|^p\right] \\
&\leq \frac{p\eta}{q}\mathbb{E}\left[\sup_{0\leq s\leq t}|X(s)-Y(s)|^q\right] + \frac{q-p}{q\eta^{p/(q-p)}} P(\tau_R \leq t \text{ or } \rho_R \leq t) \\
&+ \mathbb{E}\left[\sup_{0\leq s\leq t}|e(s\wedge\theta_R)|^p\right] \\
&\leq \frac{p\eta}{q}2^q\mathbb{E}\left\{\left[\sup_{0\leq s\leq t}|X(s)|^q\right] + \left[\sup_{0\leq s\leq t}|Y(s)|^q\right]\right\} + \frac{q-p}{q\eta^{p/(q-p)}}\left\{\mathbb{E}\left[\frac{|X(\tau_R)|}{R^p}\right] + \mathbb{E}\left[\frac{|Y(\rho_R)|}{R^p}\right]\right\} \\
&+ \mathbb{E}\left[\sup_{0\leq s\leq t}|e(s\wedge\theta_R)|^p\right] \\
&\leq \frac{p\eta}{q}2^q C + \frac{q-p}{q\eta^{p/(q-p)}R^p} C + \mathbb{E}\left[\sup_{0\leq s\leq t}|e(s\wedge\theta_R)|^p\right].
\end{aligned}$$

In the similar way as Theorem 3.1 was proved, we can show that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|e(t\wedge\theta_R)|^p\right] \leq C_R h^{\alpha p}. \tag{3.37}$$

Finally, given an $\epsilon > 0$, there exist some η small enough that

$$\frac{p\eta}{q}2^q C < \frac{\epsilon}{3},$$

choose R large enough that

$$\frac{q-p}{q\eta^{p/(q-p)}R^p}2C < \frac{\epsilon}{3},$$

and h small enough that

$$\mathbb{E}[\sup_{0\leq t\leq T}|e(t\wedge\theta_R)|^p] < \frac{\epsilon}{3},$$

Hence we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|X(t)-Y(t)|^p\right] < \epsilon.$$

□

4 Tamed EM Method of NSDDEs driven by Pure Jump Processes

In this section, we investigate NSDDEs driven by pure jump processes. Similar to NSDDEs driven by Brownian motion, most NSDDEs driven by pure jumps have no explicit solutions. Therefore, it is important to investigate the numerical approximation of NSDDEs driven by pure jumps.

We need to introduce more notation. Let $(\mathbb{Y}, \mathcal{B}(\mathbb{Y}))$ be a measurable space, and $p : D_p \mapsto \mathbb{Y}$ an adapted Poisson point process, where D_p is a countable subset of $\mathbb{R}_+$. Then, as in Ikeda-Watanabe [17, p.59], the Poisson random measure $N(\cdot,\cdot) : \mathcal{B}(\mathbb{R}_+ \times \mathbb{Y}) \times \Omega \mapsto \mathbb{N} \cup \{0\}$, defined on the complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, can be represented by

$$N((0,t] \times \Gamma) = \sum_{s\in D_p, s\leq t} \mathbf{1}_\Gamma(p(s)), \quad \Gamma \in \mathcal{B}(\mathbb{Y}).$$

In this case, we say that N is the Poisson random measure generated by p. Let $\lambda(\cdot) = \mathbb{E}N((0,1] \times \cdot)$. Then, the compensated Poisson random measure

$$\tilde{N}(dt, dz) := N(dt, dz) - dt\lambda(dz) \quad \text{is a martingale.}$$

In what follows, we further assume that $\int_{\mathbb{Y}} |u|^p \lambda(du) < \infty$ for any $p \geq 1$. In this section, we consider the following NSDDE driven by pure jump processes on $\mathbb{R}^n$:

$$d[x(t) - G(x(t-\tau))] = f(x(t), x(t-\tau))dt + \int_{\mathbb{Y}} g(x(t-), x((t-\tau)-), u)\tilde{N}(dt, du), \ t \geq 0, \tag{4.1}$$

with initial data $\{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n), p \geq 2$, $x(t-) := \lim_{s\uparrow t} x(t)$, where $G : \mathbb{R}^n \mapsto \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$, and $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{Y} \mapsto \mathbb{R}^n$ are Borel measurable.

Again, we assume that the step size $h \in (0,1)$ be fraction of two positive rational numbers τ and T, so that there exist two positive integers $M, \bar{M}$ such that $h = T/M = \tau/\bar{M}$.

For the future use, we assume:

(B1) There exists a positive constant K_1 such that

$$2\langle x - G(y), f(x,y)\rangle \leq K_1(1+|x|^2+|y|^2) \text{ and } \int_{\mathbb{Y}} |g(x,y,u)|^p \lambda(du) \leq K_1(1+|x|^p+|y|^p) \tag{4.2}$$

for $\forall\, x, y \in \mathbb{R}^n, p \geq 2$, and $f(x,y)$ is continuous in both x and y.

(B2) $G(0) = 0$ and there exists a constant $\kappa \in (0,1)$ such that

$$|G(x) - G(\bar{x})| \leq \kappa|x - \bar{x}| \text{ for all } x, y \in \mathbb{R}^n. \tag{4.3}$$

(B3) For any $R > 0$, there exists a positive constant $\tilde{K}_R$ such that for all $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq R$, $p \geq 2$

$$\begin{aligned} &2\langle x - G(y) - \bar{x} + G(\bar{y}), f(x,y) - f(\bar{x},\bar{y})\rangle \leq \tilde{K}_R(|x-\bar{x}|^2 + |y-\bar{y}|^2), \\ &\int_{\mathbb{Y}} |g(x,y,u) - g(\bar{x},\bar{y},u)|^p \lambda(du) \leq \tilde{K}_R(|x-\bar{x}|^p + |y-\bar{y}|^p). \end{aligned} \tag{4.4}$$

(B4) There exist two positive constants l and L such that for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n, p \geq 2$

$$2\langle x - G(y) - \bar{x} + G(\bar{y}), f(x,y) - f(\bar{x},\bar{y})\rangle \leq L(|x-\bar{x}|^2 + |y-\bar{y}|^2)$$
$$\int_{\mathbb{Y}} |g(x,y,u) - g(\bar{x},\bar{y},u)|^p \lambda(du) \leq L(|x-\bar{x}|^p + |y-\bar{y}|^p)$$

and

$$|f(x,y) - f(\bar{x},\bar{y})| \leq L(1 + |x|^l + |y|^l + |\bar{x}|^l + |\bar{y}|^l)(|x-\bar{x}| + |y-\bar{y}|).$$

(B5) For every $p > 0$, there exists a positive integer K, such that

$$\mathbb{E}\|\xi(t) - \xi(s)\|^p \leq K|t-s|^p, \text{ for any } s, t \in [-\tau, 0].$$

Remark 4.1 Under assumptions **(B1)**, **(B2)** and **(B4)**, in the same way as that of [24, Theorem A.1] we can prove that the NSDDE (4.1) with the initial data $x(0) = \xi$ satisfying $\mathbb{E}\|\xi\|^2 < \infty$ has a pathwise unique strong solution. If the condition **(B4)** is replaced by the condition **(B3)**, the existence and uniqueness theorem still holds.

Set

$$f_h(x,y) = \frac{f(x,y)}{1 + h^\alpha |f(x,y)|}, \ \alpha \in (0, 1/2].$$

Similarly to the Brownian motion case, the discrete-time tamed EM scheme associated with (4.1) can be defined as following: For every integer $n = -\bar{M}, \cdots, 0$, $z_h^{(n)} = \xi(nh)$. For every integer $n = 0, \cdots, M-1$,

$$z_h^{(n+1)} - G(z_h^{(n+1-\bar{M})}) = z_h^{(n)} - G(z_h^{(n-\bar{M})}) + f_h(z_h^{(n)}, z_h^{(n-\bar{M})})h + \int_{\mathbb{Y}} g(z_h^{(n)}, z_h^{(n-\bar{M})}, u)\Delta\tilde{N}_h^n(du), \tag{4.5}$$

where $\Delta\tilde{N}_h^n(du) := \tilde{N}((n+1)h, du) - \tilde{N}(nh, du)$, the increment of compensated Poisson process. We may rewrite the discrete tamed EM scheme as follows:

$$\begin{aligned} z_h^{(n+1)} =& G(z_h^{(n+1-\bar{M})}) + \xi(0) - G(\xi(-\tau)) + \sum_{i=0}^{n} f_h(z_h^{(i)}, z_h^{(i-\bar{M})})h \\ &+ \sum_{i=0}^{n} \int_{\mathbb{Y}} g(z_h^{(i)}, z_h^{(i-\bar{M})}, u)\Delta\tilde{N}_h^i(du). \end{aligned} \tag{4.6}$$

For $t \in [nh, (n+1)h)$, denote that $\bar{z}(t) := z_h^{(n)}$, and then $\bar{z}(t-\tau) = z_h^{(n-\bar{M})}$. It is more convenient to define the continuous-time tamed EM approximate solution $z(t)$ associated with (4.1) as below:

For any $\theta \in [-\tau, 0]$, $z(\theta) = \xi(\theta)$. For any $t \in [0, T]$,

$$\begin{aligned} z(t) =& G(\bar{z}(t-\tau)) + \xi(0) - G(\xi(-\tau)) + \int_0^t f_h(\bar{z}(s), \bar{z}(s-\tau))ds \\ &+ \int_0^t \int_{\mathbb{Y}} g(\bar{z}(s-), \bar{z}((s-\tau)-), u)\tilde{N}(ds, du). \end{aligned} \tag{4.7}$$

Since for any $t > 0$ there exists a positive integer n, $0 \leq n \leq M-1$, such that $t \in [nh, (n+1)h)$, we have

$$z(t) = z(nh) + \int_{nh}^t f_h(\bar{z}(s), \bar{z}(s-\tau))ds + \int_{nh}^t \int_{\mathbb{Y}} g(\bar{z}(s-), \bar{z}((s-\tau)-), u)\tilde{N}(ds, du). \tag{4.8}$$

Clearly, the continuous-time tamed EM approximate solution $z(t)$ coincides with the discrete-time tamed approximation solution $\bar{z}(t)$ at the grid points $t = nh$, i.e. $\bar{z}(t) = z_h^n = z(nh)$.

We now state our main results of this section.

Theorem 4.1 *Assume that* **(B1)**, **(B2)**, **(B4)** *and* **(B5)** *hold. Assume also that* $p \geq 2$ *and* $\alpha \in (0, 1/p)$, *then the tamed EM scheme* (4.7) *converges to the exact solution of* (4.1) *such that*

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |z(t) - x(t)|^p\Big] \leq Ch^\gamma, \tag{4.9}$$

where $\gamma = 1/2 \wedge \alpha p$.

Theorem 4.2 *Assume that* **(B1)-(B3)** *and* **(B5)** *hold. Assume also that* $p \geq 2$ *and* $\alpha \in (0, 1/p)$, *then the tamed EM scheme* (4.7) *converges to the exact solution of* (4.1) *such that*

$$\lim_{h\to 0} \mathbb{E}\Big[\sup_{0\leq t\leq T} |z(t) - x(t)|^p\Big] = 0. \tag{4.10}$$

4.1 Boundedness of Moments

In order to show our main results, we need the following inequality [25, Theorem 1].

Lemma 4.1 *Let* $\varphi : \mathbb{R}_+ \times \mathbb{Y} \times \Omega \to \mathbb{R}^n$ *be a progressively measurable process and assume that*

$$\int_0^t \int_{\mathbb{Y}} |\varphi(s, u)|^2 \lambda(du)ds < \infty, \quad t \geq 0 \quad a.s..$$

Then there exist a constant $C_p > 0$ *such that for any* $p \geq 1$

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\leq s\leq t} \Big|\int_0^s \int_{\mathbb{Y}} \varphi(r, u)\tilde{N}(dr, du)\Big|^p\Big) \leq C_p \Big[&\mathbb{E}\Big(\int_0^t \int_{\mathbb{Y}} |\varphi(s, u)|^2 \lambda(du)ds\Big)^{p/2} \\ &+ \mathbb{E}\int_0^t \int_{\mathbb{Y}} |\varphi(s, u)|^p \lambda(du)ds\Big].\end{aligned}$$

It is known that if $1 \leq p \leq 2$, then the second term on the right hand side can be eliminated.

Lemma 4.2 *Consider the continuous-time tamed EM scheme given by equation* (4.8). *Assume that* $p \geq 2, \alpha \in (0, 1/p)$, *and*

$$\sup_{0\leq t\leq T} \mathbb{E}(|z(t)|^p) \leq C. \tag{4.11}$$

Assume also that **(B1)** *holds, then the following two inequalities hold*

$$\mathbb{E}\Big[\sup_{0\leq n\leq M-1}\ \sup_{nh\leq t\leq (n+1)h} |z(t) - z(nh)|^p\Big] \leq Ch, \tag{4.12}$$

and

$$\mathbb{E}\Big[\sup_{0\leq n\leq M-1}\ \sup_{nh\leq t\leq (n+1)h} |z(t) - z(nh)|^p |f_h(\bar{z}(t), \bar{z}(t-\tau))|^p\Big] \leq C. \tag{4.13}$$

Proof: By the definition of $z(t)$, we can write that

$$\mathbb{E}\Big[\sup_{nh\le t\le (n+1)h}|z(t)-z(nh)|^p\Big]=\mathbb{E}\Big[\sup_{nh\le t\le (n+1)h}\Big|\int_{nh}^t f_h(\bar z(s),\bar z(s-\tau))ds \\ +\int_{nh}^t\int_{\mathbb{Y}} g(\bar z(s-),\bar z((s-\tau)-),u)\tilde N(ds,du)\Big|^p\Big].$$

Therefore, due to Hölder's inequality,

$$\begin{aligned}&\mathbb{E}\Big[\sup_{nh\le t\le (n+1)h}|z(t)-z(nh)|^p\Big]\\ &\le 2^{p-1}h^{p-1}\mathbb{E}\Big[\int_{nh}^{(n+1)h}|f_h(\bar z(s),\bar z(s-\tau))|^p ds\Big]\\ &+2^{p-1}\mathbb{E}\Big[\sup_{nh\le t\le (n+1)h}\Big|\int_{nh}^t\int_{\mathbb{Y}} g(\bar z(s-),\bar z((s-\tau)-),u)\tilde N(ds,du)\Big|^p\Big].\end{aligned}\tag{4.14}$$

Using Lemma 4.1, Hölder's inequality and (4.11), for some $p\ge 2$ we have

$$\begin{aligned}&\mathbb{E}\Big[\sup_{nh\le t\le (n+1)h}\Big|\int_{nh}^t\int_{\mathbb{Y}} g(\bar z(s-),\bar z((s-\tau)-),u)\tilde N(ds,du)\Big|^p\Big]\\ &\le C\big[\mathbb{E}\big(\int_{nh}^{(n+1)h}\int_{\mathbb{Y}}|g(\bar z(s),\bar z(s-\tau),u)|^2\lambda(du)ds\big)^{p/2}\\ &+\mathbb{E}\int_{nh}^{(n+1)h}\int_{\mathbb{Y}}|g(\bar z(s),\bar z(s-\tau),u)|^p\lambda(du)ds\big]\\ &\le C\mathbb{E}\int_{nh}^{(n+1)h}(1+|\bar z(s)|^p+|\bar z(s-\tau)|^p)ds\\ &\le Ch.\end{aligned}$$

This, together with $|f_h(\bar z(s),\bar z(s-\tau),s)|\le h^{-\alpha}$, yields

$$\mathbb{E}\Big[\sup_{nh\le t\le (n+1)h}|z(t)-z(nh)|^p\Big]\le 2^{p-1}h^{(1-\alpha)p}+Ch\le Ch.\tag{4.15}$$

Hence (4.12) holds. Moreover,

$$\begin{aligned}&\mathbb{E}\Big[\sup_{nh\le t\le (n+1)h}|z(t)-z(nh)|^p|f_h(\bar z(s),\bar z(s-\tau))|^p\Big]\\ &\le\mathbb{E}\Big[\sup_{nh\le t\le (n+1)h}|z(t)-z(nh)|^p\Big]h^{-\alpha p}\le Ch^{1-\alpha p}\le C,\end{aligned}\tag{4.16}$$

as required. The proof is therefore complete. □

The result of boundedness is given below:

Lemma 4.3 *Assume that* **(B1)** *and* **(B2)** *hold. Assume also $p\ge 2,\alpha\in(0,1/p)$, then there exists a constant C such that*

$$\mathbb{E}[\sup_{0\le t\le T}|x(t)|^p]\vee\mathbb{E}[\sup_{0\le t\le T}|z(t)|^p]\le C.\tag{4.17}$$

Proof: Similarly, we can give a proof for $p=2$, and then for $p>2$ as the case of NSDDEs driven by Brownian motion, here we only give a proof for $p \geq 4$. For every integer $k \geq 1$, we define a stopping time as follows

$$\hat{\tau}_k = T \wedge \inf\{t \in [0,T] : |x(t)| > k\}.$$

Clearly, $\hat{\tau}_k \to T$ as $k \to \infty$ almost surely. Now, for any $t \in [0,T]$, we have

$$\begin{aligned}
&|x(t) - G(x(t-\tau))|^p = (|x(t) - G(x(t-\tau))|^2)^{p/2} \\
&\leq C\Bigg(|\xi(0) - G(\xi(-\tau))|^p + \bigg| \int_0^t \langle x(s) - G(x(s-\tau)), f(x(s), x(s-\tau))\rangle ds \bigg|^{p/2} \\
&+ \bigg| \int_0^t \int_{\mathbb{Y}} |x(s) - G(x(s-\tau)) + g(x(s), x(s-\tau), u)|^2 - |x(s) - G(x((s-\tau))|^2 \\
&- 2\langle x(s) - G(x(s-\tau)), g(x(s), x(s-\tau), u)\rangle \lambda(du) ds \bigg|^{p/2} \\
&+ \bigg| \int_0^t \int_{\mathbb{Y}} \big(|x(s-) - G(x((s-\tau))-) + g(x(s-), x((s-\tau)-), u)|^2 \\
&\qquad\qquad - |x(s-) - G(x((s-\tau)-))|^2 \big) \tilde{N}(ds, du) \bigg|^{p/2} \Bigg) \\
&=: J_1(t) + J_2(t) + J_3(t) + J_4(t).
\end{aligned}$$

Due to assumptions **(B1)**, we derive that

$$\begin{aligned}
&\mathbb{E}(\sup_{0 \leq s \leq t \wedge \hat{\tau}_k} J_2(s)) \\
&\leq C\mathbb{E} \bigg| \int_0^{t \wedge \hat{\tau}_k} \langle x(s) - G(x(s-\tau)), f(x(s), x(s-\tau))\rangle ds \bigg|^{p/2} \\
&\leq C\mathbb{E} \bigg| \int_0^{t \wedge \hat{\tau}_k} (1 + |x(s)|^2 + |x(s-\tau)|^2) ds \bigg|^{p/2} \\
&\leq C + C\mathbb{E} \int_0^{t \wedge \hat{\tau}_k} |x(s)|^p ds = C + C\mathbb{E} \int_0^{t \wedge \hat{\tau}_k} |x(s-)|^p ds
\end{aligned} \tag{4.18}$$

where the Hölder inequality is employed in the last second step. By the Taylor expansion, one gets

$$|x+y|^p - |x|^p - p\langle x, y\rangle |x|^{p-2} \leq C(|x|^{p-2}|y|^2 + |y|^p). \tag{4.19}$$

This, together with **(B1)**, **(B2)** and the Hölder inequality, implies

$$\begin{aligned}
&\mathbb{E}(\sup_{0\le s\le t\wedge\hat{\tau}_k} J_3(s)) \\
&\le C\mathbb{E}\Big|\int_0^{t\wedge\hat{\tau}_k}\int_{\mathbb{Y}} |x(s)-G(x(s-\tau))+g(x(s),x(s-\tau),u)|^2-|x(s)-G(x((s-\tau))|^2 \\
&\qquad -2\langle x(s)-G(x(s-\tau)),g(x(s),x(s-\tau),u)\rangle\lambda(du)ds\Big|^{p/2} \\
&\le C\mathbb{E}\Big|\int_0^{t\wedge\hat{\tau}_k}\int_{\mathbb{Y}}\big[|x(s)-G(x(s-\tau))|^2+|g(x(s),x(s-\tau),u)|^2\big]\lambda(du)ds\Big|^{p/2} \\
&\le C\mathbb{E}\Big|\int_0^{t\wedge\hat{\tau}_k}(1+|x(s)|^2+|x(x-\tau)|^2)ds\Big|^{p/2} \\
&\le C+C\mathbb{E}\int_0^{t\wedge\hat{\tau}_k}|x(s)|^p ds = C+C\mathbb{E}\int_0^{t\wedge\hat{\tau}_k}|x(s-)|^p ds.
\end{aligned}\tag{4.20}$$

Using Lemma 4.1, Taylor's expansion and noting $p\ge 4$, one has

$$\begin{aligned}
&\mathbb{E}(\sup_{0\le s\le t\wedge\hat{\tau}_k} J_4(s)) \\
&\le C\mathbb{E}\int_0^{t\wedge\hat{\tau}_k}\int_{\mathbb{Y}}\Big||x(s-)-G(x((s-\tau)-))+g(x(s-),x((s-\tau)-),u)|^2 \\
&\qquad\qquad -|x(s-)-G(x((s-\tau)-))|^2\Big|^{p/2}\lambda(du)ds \\
&+C\mathbb{E}\Big(\int_0^{t\wedge\hat{\tau}_k}\int_{\mathbb{Y}}\Big||x(s-)-G(x((s-\tau)-))+g(x(s-),x((s-\tau)-),u)|^2 \\
&\qquad\qquad -|x(s-)-G(x((s-\tau)-))|^2\Big|^2\lambda(du)ds\Big)^{p/4} \\
&\le C\mathbb{E}\int_0^{t\wedge\hat{\tau}_k}\int_{\mathbb{Y}}\Big||x(s-)-G(x((s-\tau)-))+g(x(s-),x((s-\tau)-),u)|^2 \\
&\qquad\qquad -|x(s-)-G(x((s-\tau)-))|^2\Big|^{p/2}\lambda(du)ds \\
&\le C\mathbb{E}\int_0^{t\wedge\hat{\tau}_k}\int_{\mathbb{Y}}\big(|x(s-)-G(x((s-\tau)-))|^p+|g(x(s-),x((s-\tau)-),u)|^p\big)\lambda(du)ds \\
&\le C+C\mathbb{E}\int_0^{t\wedge\hat{\tau}_k}|x(s-)|^p ds.
\end{aligned}\tag{4.21}$$

Applying [23, Lemma 6.4.4] (also see (3.10)), we derive that

$$\mathbb{E}\big(\sup_{0\le s\le t\wedge\hat{\tau}_k}|x(s)|^p\big)\le C+C\mathbb{E}\int_0^{t\wedge\hat{\tau}_k}|x(s-)|^p ds<\infty.$$

This implies

$$\mathbb{E}\big(\sup_{0\le s\le t\wedge\hat{\tau}_k}|x(s)|^p\big)\le C+C\int_0^t\mathbb{E}\big(\sup_{0\le v\le s\wedge\hat{\tau}_k}|x(v)|^p\big)ds.$$

By the Gronwall inequality and letting $k \to \infty$, we have

$$\mathbb{E}\big(\sup_{0\le t\le T}|x(t)|^p\big) \le C.$$

The proof of boundedness of EM approximation is analogous to its Brownian motion counterpart, we first claim that there exists a constant $C > 0$ such that:

$$\sup_{0\le t\le T}\mathbb{E}(|z(t)|^2) \le C. \tag{4.22}$$

For every integer $k \ge 1$, define the stopping time

$$\tilde{\tau}_k = T \wedge \inf\{t \in [0,T], |z(t)| > k\}.$$

Clearly, $\tilde{\tau}_k \to T$ as $k \to \infty$ almost surely. Now, for any $t \in [0,T]$, an application of the Itô formula yields

$$\begin{aligned}
&|z(t) - G(\bar{z}(t-\tau))|^2 = |\xi(0) - G(\xi(-\tau))|^2 \\
&+ 2\int_0^t \langle \bar{z}(s) - G(\bar{z}(s-\tau)), f_h(\bar{z}(s), \bar{z}(s-\tau))\rangle ds \\
&+ 2\int_0^t \langle z(s) - \bar{z}(s), f_h(\bar{z}(s), \bar{z}(s-\tau))\rangle ds \\
&+ \bigg(\int_0^t \int_{\mathbb{Y}} |z(s) - G(\bar{z}(s-\tau)) + g(\bar{z}(s), \bar{z}(s-\tau), u)|^2 - |z(s) - G(\bar{z}(s-\tau))|^2 \\
&- 2\langle z(s) - G(\bar{z}(s-\tau)), g(\bar{z}(s), \bar{z}(s-\tau), u)\rangle\bigg)\lambda(du)ds \\
&+ \int_0^t \int_{\mathbb{Y}} \big(|z(s-) - G(\bar{z}((s-\tau)-)) + g(\bar{z}(s-), \bar{z}((s-\tau)-), u)|^2 \\
&\qquad\qquad - |z(s-) - G(\bar{z}((s-\tau)-))|^2\big)\tilde{N}(ds,du) \\
&=: |\xi(0) - G(\xi(-\tau))|^2 + \bar{J}_1(t) + \bar{J}_2(t) + \bar{J}_3(t) + \bar{J}_4(t).
\end{aligned} \tag{4.23}$$

By **(B1)**, we compute

$$\begin{aligned}
&\sup_{0\le s\le t}\mathbb{E}(\bar{J}_1(s\wedge\hat{\tau}_k)) \le \mathbb{E}\int_0^{t\wedge\hat{\tau}_k} C(1 + |\bar{z}(s)|^2 + \bar{z}(s-\tau)|^2)ds \\
&\le C + C\mathbb{E}\int_0^{t\wedge\hat{\tau}_k} |z(s)|^2 ds = C + C\mathbb{E}\int_0^{t\wedge\hat{\tau}_k} |z(s-)|^2 ds.
\end{aligned} \tag{4.24}$$

Using the definition of $z(s)$ and $\bar{z}(s)$, together with the property of conditional expectation, we have

$$\begin{aligned}
&\sup_{0\le s\le t}\mathbb{E}(\bar{J}_2(s\wedge\hat{\tau}_k)) = 2\Big\{\sup_{0\le s\le t}\mathbb{E}\int_0^{s\wedge\hat{\tau}_k}\langle f_h(\bar{z}(v),\bar{z}(v-\tau)),\int_{[\frac{v}{h}]h}^{v} f_h(\bar{z}(r),\bar{z}(r-\tau))dr\rangle dv\\
&+\mathbb{E}\int_0^{s\wedge\hat{\tau}_k}\langle\int_{[\frac{v}{h}]h}^{v} g_h(\bar{z}(r-),\bar{z}((r-\tau)-),u)\tilde{N}(du,dr), f_h(\bar{z}(v),\bar{z}(v-\tau))\rangle dv\Big\}\\
&=2\Big\{\sup_{0\le s\le t}\mathbb{E}\int_0^{s\wedge\hat{\tau}_k}\langle f_h(\bar{z}(v),\bar{z}(v-\tau)),\int_{[\frac{v}{h}]h}^{v} f_h(\bar{z}(r),\bar{z}(r-\tau))dr\rangle du\\
&+\mathbb{E}\int_0^{s}\mathbb{E}\langle\int_{[\frac{v\wedge\hat{\tau}_k}{h}]h}^{v\wedge\hat{\tau}_k} g_h(\bar{z}(r-),\bar{z}((r-\tau)-),u)\tilde{N}(du,dr), f_h(\bar{z}(v),\bar{z}(v-\tau))\rangle\big|\mathcal{F}_{[\frac{v\wedge\hat{\tau}_k}{h}]h}dv\Big\}\\
&=2\mathbb{E}\int_0^{t}\langle\int_{[\frac{v}{h}]h}^{v} f_h(\bar{z}(r),\bar{z}(r-\tau))dr, f_h(\bar{z}(v),\bar{z}(v-\tau)\rangle dv\\
&\le Cth^{1-2\alpha}\le C.
\end{aligned}\tag{4.25}$$

By using (4.19), we have

$$\begin{aligned}
&\sup_{0\le s\le t}\mathbb{E}(\bar{J}_3(s\wedge\hat{\tau}_k))\le C\mathbb{E}\int_0^t(1+|\bar{z}(s\wedge\hat{\tau}_k)|^2+|\bar{z}((s-\tau)\wedge\hat{\tau}_k)|^2)ds\\
&\le C+C\mathbb{E}\int_0^{t\wedge\hat{\tau}_k}|z(s)|^2ds = C+C\mathbb{E}\int_0^{t\wedge\hat{\tau}_k}|z(s-)|^2ds.
\end{aligned}\tag{4.26}$$

By taking the expectation of $\bar{J}_4(t)$, we know that it is a local martingale with $\mathbb{E}(\bar{J}_4(t))=0$. Therefore, we have

$$\begin{aligned}
&\sup_{0\le s\le t\wedge\tilde{\tau}_k}\mathbb{E}(|z(s)|^2)\le C+C\Big(\sup_{0\le s\le t}\mathbb{E}(|z(s\wedge\tilde{\tau}_k)-D(\bar{z}((s-\tau)\wedge\hat{\tau}_k))|^2)\Big)\\
&\le C+C\mathbb{E}\int_0^{t\wedge\hat{\tau}_k}|z(s-)|^2ds<\infty.
\end{aligned}$$

This means

$$\sup_{0\le s\le t\wedge\tilde{\tau}_k}\mathbb{E}(|z(s)|^2)\le C+C\int_0^t\sup_{0\le v\le s\wedge\hat{\tau}_k}\mathbb{E}(|z(v)|^2)ds.$$

Letting $k\to\infty$, the required result (4.22) follows an application of the Gronwall inequality.

Now, letting $p=4$ and using the Itô formula, we have

$$
\begin{aligned}
&|z(t)-G(\bar z(t-\tau))|^p=\big(|z(t)-G(\bar z(t-\tau))|^2\big)^{p/2}\\
&\le C\bigg(|\xi(0)-G(\xi(-\tau))|^p+\Big|\int_0^t\langle z(s)-G(\bar z(s-\tau)),f_h(\bar z(s),\bar z(s-\tau))\rangle ds\Big|^{p/2}\\
&+\Big|\int_0^t\int_{\mathbb{Y}}\big(|z(s)-G(\bar z(s-\tau))+g(\bar z(s),\bar z(s-\tau),u)|^2-|z(s)-G(\bar z(s-\tau))|^2\\
&-2\langle z(s)-G(\bar z(s-\tau)),g(\bar z(s),\bar z(s-\tau),u)\rangle\big)\lambda(du)ds\Big|^{p/2}\\
&+\Big|\int_0^t\int_{\mathbb{Y}}\big(|z(s-)-G(\bar z((s-\tau)-))+g(\bar z(s-),\bar z((s-\tau)-),u)|^2\\
&\qquad-|z(s-)-G(\bar z((s-\tau)-))|^2\big)\tilde N(ds,du)\Big|^{p/2}\bigg)\\
&=:F_1(t)+F_2(t)+F_3(t)+F_4(t).
\end{aligned}
\tag{4.27}
$$

By using assumption **(B1)**, (4.13) and (4.22), we arrive at

$$
\begin{aligned}
\mathbb{E}(\sup_{0\le s\le t\wedge\tilde\tau_k}F_2(s))&\le C\mathbb{E}\Big|\int_0^{t\wedge\tilde\tau_k}(1+|\bar z(s)|^2+|\bar z(s-\tau)|^2)ds\Big|^{p/2}\\
&+\mathbb{E}\Big|\int_0^{t\wedge\tilde\tau_k}|z(s)-\bar z(s)||f_h(\bar z(s),\bar z(s-\tau))|ds\Big|^{p/2}\\
&\le C\mathbb{E}\int_0^{t\wedge\tilde\tau_k}(1+|\bar z(s)|^p+|\bar z(s-\tau)|^p)ds\\
&+C\mathbb{E}\int_0^{t\wedge\tilde\tau_k}(|z(s)-\bar z(s)|^{p/2}|f_h(\bar z(s),\bar z(s-\tau))|^{p/2})ds\\
&\le C+C\mathbb{E}\int_0^{t\wedge\tilde\tau_k}|z(s)|^pds=C+C\mathbb{E}\int_0^{t\wedge\tilde\tau_k}|z(s-)|^pds,
\end{aligned}
\tag{4.28}
$$

where the Hölder inequality is also applied. By **(B1)**, **(B2)** and (4.19), we obtain

$$
\begin{aligned}
&\mathbb{E}(\sup_{0\le s\le t\wedge\tilde\tau_k}F_3(s))\\
&\le C\mathbb{E}\Big|\int_0^{t\wedge\tilde\tau_k}\int_{\mathbb{Y}}|z(s)-G(\bar z(s-\tau))+g(\bar z(s),\bar z(s-\tau),u)|^2-|z(s)-G(\bar z(s-\tau))|^2\\
&\qquad-2\langle z(s)-G(\bar z(s-\tau)),g(\bar z(s),\bar z(s-\tau),u)\rangle\lambda(du)ds\Big|^{p/2}\\
&\le C\mathbb{E}\Big|\int_0^{t\wedge\tilde\tau_k}\int_{\mathbb{Y}}\big[|z(s)-G(\bar z(s-\tau))|^2+|g(\bar z(s),\bar z(s-\tau),u)|^2\big]\lambda(du)ds\Big|^{p/2}\\
&\le C+C\mathbb{E}\int_0^{t\wedge\tilde\tau_k}|z(s)|^pds=C+C\mathbb{E}\int_0^{t\wedge\tilde\tau_k}|z(s-)|^pds.
\end{aligned}
\tag{4.29}
$$

Using Lemma 4.1 and Young's inequality, we derive that

$$\begin{aligned}
&\mathbb{E}(\sup_{0\le s\le t\wedge\tilde{\tau}_k} F_4(s)) \\
&\le C\mathbb{E}\int_0^{t\wedge\tilde{\tau}_k}\int_{\mathbb{Y}}\Big||z(s-)-G(\bar{z}((s-\tau)-))+g(\bar{z}(s-),\bar{z}((s-\tau)-),u)|^2 \\
&\qquad\qquad -|z(s-)-G(\bar{z}((s-\tau)-))|^2\Big|^{p/2}\lambda(du)ds \\
&+C\mathbb{E}\Big(\int_0^{t\wedge\tilde{\tau}_k}\int_{\mathbb{Y}}\Big||z(s-)-G(\bar{z}((s-\tau)-))+g(\bar{z}(s-),\bar{z}((s-\tau)-),u)|^2 \\
&\qquad\qquad -|z(s-)-G(\bar{z}((s-\tau)-))|^2\Big|^{2}\lambda(du)ds\Big)^{p/4} \\
&\le C\mathbb{E}\int_0^{t\wedge\tilde{\tau}_k}\int_{\mathbb{Y}}\Big||z(s-)-G(\bar{z}((s-\tau)-))+g(\bar{z}(s-),\bar{z}((s-\tau)-),u)|^2 \\
&\qquad\qquad -|z(s-)-G(\bar{z}((s-\tau)-))|^2\Big|^{p/2}\lambda(du)ds \\
&\le C\mathbb{E}\int_0^{t\wedge\tilde{\tau}_k}\int_{\mathbb{Y}}\big(|z(s-)-G(\bar{z}((s-\tau)-))|^p+|g(\bar{z}(s-),\bar{z}((s-\tau)-),u)|^p\big)\lambda(du)ds \\
&\le C+C\mathbb{E}\int_0^{t\wedge\tilde{\tau}_k}|z(s-)|^p ds.
\end{aligned} \tag{4.30}$$

Now substituting (4.28), (4.29) and (4.30) into (4.27), we obtain that

$$\mathbb{E}(\sup_{0\le s\le t\wedge\tilde{\tau}_k}|z(s)-G(\bar{z}(s-\tau))|^p)\le C+C\mathbb{E}\int_0^{t\wedge\tilde{\tau}_k}|z(s-)|^p ds.$$

By applying [23, Lemma 6.4.4] (also see (3.10)), we derive that

$$\mathbb{E}\big(\sup_{0\le s\le t\wedge\tilde{\tau}_k}|z(s)|^p\big)\le C+C\mathbb{E}\int_0^{t\wedge\tilde{\tau}_k}|z(s-)|^p ds<\infty.$$

This implies

$$\mathbb{E}\big(\sup_{0\le s\le t\wedge\tilde{\tau}_k}|z(s)|^p\big)\le C+C\int_0^t\mathbb{E}\big(\sup_{0\le v\le s\wedge\tilde{\tau}_k}|z(v)|^p\big)ds.$$

By the Gronwall inequality, we have

$$\mathbb{E}\big(\sup_{0\le t\le T\wedge\tilde{\tau}_k}|z(t)|^p\big)\le C.$$

The required assertion follows for $p=4$ by letting $k\to\infty$. Repeating the same procedure above, we can obtain the result (4.17). $\square$

4.2 Proof of the Main Results

In this subsection, we shall prove our main results.

Proof of Theorem 4.1: By using the similar approach as (3.26), we have

$$
\begin{aligned}
\sup_{0\le t\le T}|x(t)-z(t)|^p \le\ & \kappa \sup_{0\le t\le T}|x(t-\tau)-\bar z(t-\tau)|^p \\
&+\frac{1}{(1-\kappa)^{p-1}}\sup_{0\le t\le T}|x(t)-G(x(t-\tau))-z(t)+G(\bar z(t-\tau))|^p \\
&\le \kappa \sup_{-\tau\le t\le 0}|x(t)-\bar z(t)|^p+\kappa\sup_{0\le t\le T}|x(t)-z(t)+z(t)-\bar z(t)|^p \\
&+\frac{1}{(1-\kappa)^{p-1}}\sup_{0\le t\le T}|x(t)-G(x(t-\tau))-z(t)+G(\bar z(t-\tau))|^p \\
&\le \kappa \sup_{-\tau\le t\le 0}|x(t)-\bar z(t)|^p+\bar\kappa_c\sup_{0\le t\le T}|x(t)-z(t)|^p+C\sup_{0\le t\le T}|z(t)-\bar z(t)|^p \\
&+\frac{1}{(1-\kappa)^{p-1}}\sup_{0\le t\le T}|x(t)-G(x(t-\tau))-z(t)+G(\bar z(t-\tau))|^p.
\end{aligned}
\tag{4.31}
$$

where $\bar\kappa_c\in(0,1)$ is a constant. This, together with **(B5)**, implies that

$$
\begin{aligned}
\mathbb{E}\big(\sup_{0\le s\le t}|x(s)-z(s)|^p\big)\le\ & \frac{1}{(1-\kappa)^{p-1}(1-\bar\kappa_c)}\mathbb{E}\big(\sup_{0\le s\le t}|x(s)-G(x(s-\tau)) \\
&-z(s)+G(\bar z(s-\tau))|^p\big)+Ch^p+Ch.
\end{aligned}
\tag{4.32}
$$

An application of the Itô formula yields that for any $p\ge 2$,

$$
\begin{aligned}
&|x(t)-G(x(s-\tau))-z(t)+G(\bar z(t-\tau))|^p \\
&=\big(|x(t)-G(x(s-\tau))-z(t)+G(\bar z(t-\tau))|^2\big)^{p/2} \\
&\le C\Bigg(\Big|\int_0^t\langle x(s)-G(x(s-\tau))-z(s)+G(\bar z(s-\tau)), \\
&\qquad f(x(s),x(s-\tau))-f_h(\bar z(s),\bar z(s-\tau))\rangle ds\Big|^{p/2} \\
&+\Big|\int_0^t\int_{\mathbb{Y}}|x(s)-G(x(s-\tau))-z(s)+G(\bar z(s-\tau)) \\
&\quad+(g(x(s),x(s-\tau),u)-g(\bar z(s),\bar z(s-\tau),u))|^2 \\
&\quad-|x(s)-G(x(s-\tau))-z(s)+G(\bar z(s-\tau))|^2 \\
&\quad-2\langle x(s)-G(x(s-\tau))-z(s)+G(\bar z(s-\tau)), \\
&\qquad g(x(s),x(s-\tau),u)-g(\bar z(s),\bar z(s-\tau),u)\rangle\lambda(du)ds\Big|^{p/2} \\
&+\Big|\int_0^t\int_{\mathbb{Y}}|x(s)-G(x(s-\tau))-z(s)+G(\bar z(s-\tau)) \\
&\quad+(g(x(s-),x((s-\tau)-),u)-g(\bar z(s-),\bar z((s-\tau)-),u))|^2 \\
&\quad-|x(s)-G(x(s-\tau))-z(s)+G(\bar z(s-\tau))|^2\tilde N(ds,du)\Big|^{p/2}\Bigg) \\
&=:\bar F_1(t)+\bar F_2(t)+\bar F_3(t)
\end{aligned}
\tag{4.33}
$$

Using the similar approach in(3.28), we derive

$$\begin{aligned}\bar{F}_1(t) = \Big| \int_0^t &\langle x(s) - G(x(s-\tau)) - z(s) + G(\bar{z}(s-\tau)), \\ &\quad f(x(s), x(s-\tau)) - f(z(s), \bar{z}(s-\tau))\rangle ds \\ + \int_0^t &\langle x(s) - G(x(s-\tau)) - z(s) + G(\bar{z}(s-\tau)), \\ &\quad f(z(s), \bar{z}(s-\tau)) - f(\bar{z}(s), \bar{z}(s-\tau))\rangle ds \\ + \int_0^t &\langle x(s) - G(x(s-\tau)) - z(s) + G(\bar{z}(s-\tau)), \\ &\quad f(\bar{z}(s), \bar{z}(s-\tau)) - f_h(\bar{z}(s), \bar{z}(s-\tau))\rangle ds \Big|^{p/2}.\end{aligned}$$

By the definition of f_h and Lemma 4.2, we have

$$\begin{aligned}&C\mathbb{E}\int_0^t |f(\bar{z}(s), \bar{z}(s-\tau)) - f_h(\bar{z}(s), \bar{z}(s-\tau))|^p ds \\ &\leq h^{\alpha p}\mathbb{E}\Big[\int_0^t \frac{|f(\bar{z}(s), \bar{z}(s-\tau))|^{2p}}{(1 + h^{\alpha}|f(\bar{z}(s), \bar{z}(s-\tau))|)^p} ds\Big] \\ &\leq Ch^{\alpha p}\mathbb{E}\Big[(1 + |\bar{z}(s)|^l + |\bar{z}(s-\tau)|^l)(|\bar{z}(s)| + |\bar{z}(s-\tau)| + 1)\Big] \\ &\leq h^{\alpha p}.\end{aligned}$$

This, together with **(B1)**, **(B2)** and **(B4)**, yields

$$\begin{aligned}&\mathbb{E}(\sup_{0\leq s\leq t} \bar{F}_1(t)) \leq C\mathbb{E}\int_0^t \big(\sup_{0\leq v\leq s} |x(v) - z(v)|^p\big) ds + C\mathbb{E}\int_{-\tau}^0 \big(\sup_{-\tau\leq\theta\leq 0} |x(\theta) - \bar{z}(\theta)|^p\big) ds \\ &+ C\mathbb{E}\int_0^t |f(\bar{z}(s), \bar{z}(s-\tau)) - f_h(\bar{z}(s), \bar{z}(s-\tau))|^p ds \\ &+ C\mathbb{E}\int_0^t (1 + |\bar{z}(s)|^l + |z(s)|^l + 2|\bar{z}(s-\tau)|^l)^p (|z(s) - \bar{z}(s|)^p) ds \\ &\leq C + C\int_0^t \mathbb{E}[\sup_{0\leq v\leq s} |x(v) - z(v)|^p] ds \\ &+ Ch^{1/2} + C\mathbb{E}\int_0^t |f(\bar{z}(s), \bar{z}(s-\tau)) - f_h(\bar{z}(s), \bar{z}(s-\tau))|^p ds \\ &\leq C\int_0^t \mathbb{E}[\sup_{0\leq v\leq s} |x(v) - z(v)|^p] ds + Ch^{\gamma},\end{aligned} \tag{4.34}$$

where $\gamma = 1/2 \wedge \alpha p$. By **(B2)**, **(B4)** and (4.19), we arrive at

$$\begin{aligned}&\mathbb{E}(\sup_{0\leq s\leq t} \bar{F}_2(s)) \\ &\leq C\mathbb{E}\int_0^t |x(s) - G(x(s-\tau)) - z(s) + G(\bar{z}(s-\tau))|^p \\ &|g(x(s), x(s-\tau), u) - g(\bar{z}(s), \bar{z}(s-\tau), u)|^p ds \\ &\leq Ch^p + C\int_0^t \mathbb{E}[\sup_{0\leq v\leq s} |x(v) - z(v)|^p] ds.\end{aligned} \tag{4.35}$$

and

$$
\begin{aligned}
&\mathbb{E}(\sup_{0\le s\le t} \bar{F}_3(s)) \\
&\le \mathbb{E}\int_0^t \int_{\mathbb{Y}} \Big| |x(s) - G(x(s-\tau)) - z(s) + G(\bar{z}(s-\tau)) \\
&\qquad + (g(x(s), x(s-\tau), u) - g(\bar{z}(s), \bar{z}(s-\tau), u))|^2 \\
&\qquad\quad - |x(s) - G(x(s-\tau)) - z(s) + G(\bar{z}(s-\tau))|^2 \Big|^{p/2} \lambda(du)ds \\
&+ \mathbb{E}\Big(\int_0^t \int_{\mathbb{Y}} \Big| |x(s) - G(x(s-\tau)) - z(s) + G(\bar{z}(s-\tau)) \\
&\qquad + (g(x(s), x(s-\tau), u) - g(\bar{z}(s), \bar{z}(s-\tau), u))|^2 \\
&\qquad\quad - |x(s) - G(x(s-\tau)) - z(s) + G(\bar{z}(s-\tau))|^2 \Big|^2 \lambda(du)ds \Big)^{p/4} \\
&\le \mathbb{E}\int_0^t \int_{\mathbb{Y}} \Big| |x(s) - G(x(s-\tau)) - z(s) + G(\bar{z}(s-\tau)) \\
&\qquad + (g(x(s), x(s-\tau), u) - g(\bar{z}(s), \bar{z}(s-\tau), u))|^2 \\
&\qquad\quad - |x(s) - G(x(s-\tau)) - z(s) + G(\bar{z}(s-\tau))|^2 \Big|^{p/2} \lambda(du)ds \\
&\le \mathbb{E}\int_0^t \int_{\mathbb{Y}} \big(|x(s) - G(x(s-\tau)) - z(s) + G(\bar{z}(s-\tau))|^p \\
&\qquad + |(g(x(s), x(s-\tau), u) - g(\bar{z}(s), \bar{z}(s-\tau), u))|^p \big) \lambda(du)ds \\
&\le Ch^p + C\int_0^t \mathbb{E}[\sup_{0\le v\le s} |x(v) - z(v)|^p]ds.
\end{aligned} \tag{4.36}
$$

Now, substituting (4.34), (4.35) and (4.36) into (4.33), and using (4.32), we have

$$
\begin{aligned}
\mathbb{E}\big(\sup_{0\le s\le t} |x(s) - z(s)|^p\big) &\le Ch^\gamma + Ch^p + C\mathbb{E}\int_0^t \sup_{0\le v\le s} |x(v) - z(v)|^p ds \\
&\le Ch^\gamma + C\int_0^t \mathbb{E}[\sup_{0\le v\le s} |x(v) - z(v)|^p]ds.
\end{aligned} \tag{4.37}
$$

An application of the Grownwall inequality yields the desired result. □

Proof of Theorem 4.2: Noting the fact that all sample paths associated with (4.1) are discontinuous, we define the following stopping times as follows, for every $R > 0$,

$$
\tilde{\tau}_R := \inf\{t \ge 0 : |x(t)| > R\}, \quad \tilde{\rho}_R := \inf\{t \ge 0 : |z(t)| > R\}. \tag{4.38}
$$

The remainder of the proof is similar to the that of Theorem 3.2, we omit details here. □

Acknowledgements

The authors would like to thank anonymous referees and editors for helpful comments and suggestions, which greatly improved the quality of this paper.

References

[1] Arriojas, M., Hu, Y., Mohammed, S-E. A. and Pap, G., A delayed Black and Scholes formula, *Stoch. Anal. and Appl.*, **25** (2007), 471-492.

[2] Bao, J. and Yuan, C., Large deviations for neutral functional SDEs with jumps, *Stochastics*, **87**(2015), 48-70.

[3] Bao, J., Bötcher, B., Mao, X. and Yuan, C., Convergence rate of numerical solutions to SFDEs with jumps, *J. Comput. Appl. Math.*, **236** (2011), 119-131.

[4] Bell, D. R. and Mohammed, S-E. A., Smooth densities for degenerate stochastic delay equations with hereditary drift, *Ann. Prob.*, **23**(1995) 1875-1894.

[5] Buckwar, E., Pikovsky, A. and Scheutzow, M., Stochastic dynamics with delay and memory, *Stochastics and Dynamics*, **5**(2005) Special Issue, June.

[6] Boufoussi, B. and Hajji, S., Successive approximation of neutral functional stochastic differential equations with jumps, *Statist. Probab. Lett.*, **80** (2010), 324-332.

[7] Dareiotis, K., Kumar, C. and Sabanis, S., On tamed Euler approximations of SDEs driven by Lévy noise with applications to delay equations, *SIAM J. Numer. Anal.*, **54** (2016), 1840-1872.

[8] Elsanosi, I., Oksendal, B. and Sulem, A., Some solvable stochastic control problems with delay, *Stoch. and Stoch. Rep.*, **71** (2000), 69-89.

[9] Es-Sarhir, A., Scheutzow, M. and van Gaans, O., Invariant measures for stochastic functional differential equations with superlinear drift term, *Differential Integral Equations,* **23** (2010), 189-200.

[10] Hutzenthaler, M., Jentzen, A. and Kloeden, P. E., Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients, *Proc. R. Soc. A*, **467**(2010) 1563-1576.

[11] Hutzenthaler, M., Jentzen, A. and Kloeden, P. E., Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients, *Ann. Appl. Probab.* **22** (2012) 1611-1641.

[12] Higham, D. J., Mao, X. and Stuart, A.M., Strong convergence of Euler type methods for nonlinear stochastic differential equations, *SIAM J. Numer. Anal.*, **40**(2002), 1041-1063.

[13] Hu, Y., Semi-implicit Euler-Maruyama scheme for stiff stochastic equations, *The Silvri Workshop, Progr. Probab. 38, H. Koerezlioglu, ed.,* Birkhauser, Boston **43** (1996), 183-202.

[14] Hu, Y., Salah-Eldin A. M. and Feng, Y., Discrete-time approximation of stochastic delay equations: the Milstein scheme, *Ann.Probab.,* **32** (2004), 265-314.

[15] Hairer, M., Mattingly, J., C. and Scheutzow, M., Asymptotic coupling and a general form of Harris' theorem with applications to stochastic delay equations, *Probab. Theory Related Fields*, **149** (2011), 223-259.

[16] Higham, D. J., Mao, X. and Stuart, A.M., Strong convergence of numerical methods for nonlinear stochastic differential equations, *SIAM Journal on Numerical Analysis*, **56** (2002), 113–123.

[17] Ikeda, N. and Watanabe, S., *Stochastic Differential Equations and Diffusion Processes*, North-Holland, 1981.

[18] Jacob, N., Wang, Y. and Yuan, C., Numerical solutions of stochastic differential delay equations with jumps, *Stoch. Anal. Appl.*, **27** (2009), 825–853.

[19] Ji,Y., Song, Q. and Yuan, C., Neutral Stochastic Differential Delay Equations with Locally Monotone Coefficients, *Dyn. Contin. Discrete Impuls. Syst. Ser. A: Math. Anal.*, **24** (2017), 195-217.

[20] Küchler, U. and Platen, E., Strong discrete time approximation of stochastic differential equations with time delay, *Math. Comput. Simulation*, **54** (2000), 189–205.

[21] Kloeden, P. E. and Platen, E., *Numerical Solution of Stochastic Differential Equations*, Springer, Berlin, 1992.

[22] Liao, X. and Mao,X., Almost sure exponential stability of neutral differential equations with demand stochastic perturbations, *J. Math. Anal. Appl.,* **212** (1997), 554-570.

[23] Mao, X., *Stochastic Differential Equations and Applications*, Second Edition. Horwood Publishing Limited, 2008. Chichester.

[24] Mao, X., Shen, Y. and Yuan, C., Almost surely asymptotic stability of neutral stochastic differential equations with Markovian switching, *Stochastic Process. Appl.,* **118** (2008) 1385-1406.

[25] Marinelli, C. and Röckner, M., On Maximal inequalities for purely discountinuous martingales in infinite dimensional, Sèminnaire de Probabilitès XLVI, Lecture Notes in Mathematics 2123 (2014), 293-316.

[26] Mao, X. and Sabanis, S., Numerical solutions of stochastic differential delay equations under local Lipschitz condition, *J. Comput. Appl. Math.*, **151** (2003), 215–227.

[27] Mao, X., Yuan, C. and Zou, J., Stochastic differential delay equations of population dynamics, *J. Math. Anal. Appl.* **304** (2005), 296-320.

[28] McWilliams,N. and Sabanis, S., Arithmetic Asian options under stochastic delay models, *Appl. Math. Fin.* **18** (2011), 423-446.

[29] Mohammed, S-E. A., *Stochastic Functional Differential Equations*, Longman Scientific and Technical, 1984.

[30] Øksendal, B. and Sulem, A., *Applied Stochastic Control of Jump Diffusions*, Springer Berlin Heidelberg, 2007.

[31] Sabanis, S., A note on tamed Euler approximations, *Electron. Commin. Probab.*, **18**(2013), 1-10.

[32] Situ, R., *Theory of Stochastic Differential Equations with Jumps and Applications*, Mathematical and Analytical Techniques with Applications to Engineering. Springer, New York, 2005.