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The Brouwer Fixed Point Theorem revisited*

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Abstract. We revisit the investigation of the computational content of the Brouwer Fixed Point Theorem in [7], and answer the two open questions from that work. First, we show that the computational hardness is independent of the dimension, as long as it is greater than 1 (in [7] this was only established for dimension greater than 2). Second, we show that restricting the Brouwer Fixed Point Theorem to L-Lipschitz functions for any $L > 1$ also does not change the computational strength, which together with prior results establishes a trichotomy for $L > 1$, $L = 1$ and $L < 1$.

1 Introduction

In this paper we continue with the programme to classify the computational content of mathematical theorems in the Weihrauch lattice (see [8, 4, 3, 18, 17, 5, 11, 13]). This lattice is induced by Weihrauch reducibility, which is a reducibility for partial multi-valued functions $f :\subseteq X \rightrightarrows Y$ on represented spaces X, Y. Intuitively, $f \leq_{\mathrm{W}} g$ reflects the fact that the function f can be realized with a single application of the function g as an oracle. Hence, if two functions are equivalent in the sense that they are mutually reducible to each other, then they are equivalent as computational resources, as far as computability is concerned.

Many theorems in mathematics are actually of the logical form

$$(\forall x \in X)(\exists y \in Y)\ P(x, y)$$

and such theorems can straightforwardly be represented by a multi-valued function $f : X \rightrightarrows Y$ with $f(x) := \{y \in Y : P(x, y)\}$ (sometimes partial f are needed,

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where the domain captures additional requirements that this input x has to satisfy). In some sense the multi-valued function f directly reflects the computational task of the theorem to find some suitable y for any x. Hence, in a very natural way the classification of a theorem can be achieved via a classification of the corresponding multi-valued function that represents the theorem. In this paper we attempt to classify the Brouwer Fixed Point Theorem.

Theorem 1 (Brouwer Fixed Point Theorem 1911). *Every continuous function $f : [0,1]^n \to [0,1]^n$ has a fixed point $x \in [0,1]^n$.*

The fact that Brouwer's Fixed Point Theorem cannot be proved constructively has been confirmed in many different ways; most relevant for us is the counterexample in Russian constructive analysis by Orevkov [16], which was transferred into computable analysis by Baigger [1].

Constructions similar to those used for the above counterexamples have been utilized in order to prove that the Brouwer Fixed Point Theorem is equivalent to Weak Kőnig's Lemma in reverse mathematics [22, 21] and to analyze computability properties of fixable sets [14], but a careful analysis of these reductions reveals that none of them can be straightforwardly transferred into a *uniform* reduction in the sense that we are seeking here. The results cited above essentially characterize the complexity of fixed points themselves, whereas we want to characterize the complexity of finding the fixed point, given the function. This requires full uniformity.

In the Weihrauch lattice the Brouwer Fixed Point Theorem of dimension n is represented by the multi-valued function $\mathsf{BFT}_n : \mathcal{C}([0,1]^n, [0,1]^n) \rightrightarrows [0,1]^n$ that maps any continuous function $f : [0,1]^n \to [0,1]^n$ to the set of its fixed points $\mathsf{BFT}_n(f) \subseteq [0,1]^n$. The question now is where BFT_n is located in the Weihrauch lattice?

In order to approach this question, we introduce a choice principle CC_n that we call *connected choice* and which is just the closed choice operation restricted to connected subsets. That is, in the sense discussed above, CC_n is the multi-valued function that represents the following mathematical statement: every non-empty connected closed set $A \subseteq [0,1]^n$ has a point $x \in A$. Since closed sets are represented by negative information (i.e. by an enumeration of open balls that exhaust the complement), the computational task of CC_n consists in finding a point in a closed set $A \subseteq [0,1]^n$ that is promised to be non-empty and connected and that is given by negative information.

One of our main results, presented in Section 3, is that the Brouwer Fixed Point Theorem is equivalent to connected choice for each fixed dimension n, i.e. $\mathsf{BFT}_n \equiv_{\mathrm{W}} \mathsf{CC}_n$. This result allows us to study the Brouwer Fixed Point Theorem in terms of the function CC_n that is easier to handle since it involves neither function spaces nor fixed points. This is also another instance of the observation that several important theorems are equivalent to certain choice principles (see [3]) and many important classes of computable functions can be calibrated in terms of choice (see [2]). For instance, closed choice on Cantor space $\mathsf{C}_{\{0,1\}^{\mathbb{N}}}$ and on the unit cube $\mathsf{C}_{[0,1]^n}$ are both easily seen to be equivalent to Weak

Kőnig's Lemma WKL, i.e. $\mathsf{WKL} \equiv_{\mathrm{W}} \mathsf{C}_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{C}_{[0,1]^n}$ for any $n \geq 1$. Studying the Brouwer Fixed Point Theorem in the form of CC_n now amounts to comparing $\mathsf{C}_{[0,1]^n}$ with its restriction CC_n.

Our second main result, given in Section 5, is that from dimension two onwards connected choice is equivalent to Weak Kőnig's Lemma, i.e. $\mathsf{CC}_n \equiv_{\mathrm{W}} \mathsf{C}_{[0,1]}$ for $n \geq 2$.

This refutes an earlier conjecture [7] by some of the authors that connected choice in dimension two be computationally simpler than connected choice in dimension three. We then also consider the restriction of Brouwer's Fixed Point theorem to Lipschitz functions in Section 4. In the following Section 2 we start with a short summary of relevant definitions and results regarding the Weihrauch lattice.

This extended abstract does not contain any proofs. Sections 1, 2 and 3 are taken mostly from [7]. An extended version including the omitted proofs can be found as [6].

2 The Weihrauch Lattice

In this section we briefly recall some basic results and definitions regarding the Weihrauch lattice. The original definition of Weihrauch reducibility is due to Weihrauch and has been studied for many years (see [23–25, 9]). Only recently it has been noticed that a certain variant of this reducibility yields a lattice that is very suitable for the classification of mathematical theorems (see [8, 4, 3, 18, 2, 17, 5, 10]). The basic reference for all notions from computable analysis is [26], alternatively see [19].

The Weihrauch lattice is a lattice of multi-valued functions on represented spaces. A *representation* δ of a set X is just a surjective partial map $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$. In this situation we call (X, δ) a *represented space*. In general we use the symbol "$\subseteq$" in order to indicate that a function is potentially partial. Using represented spaces we can define the concept of a realizer. We denote the composition of two (multi-valued) functions f and g either by $f \circ g$ or by fg.

Definition 1 (Realizer). *Let $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ be a multi-valued function on represented spaces. A function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is called a* realizer *of f, in symbols $F \vdash f$, if $\delta_Y F(p) \in f\delta_X(p)$ for all $p \in \mathrm{dom}(f\delta_X)$.*

Realizers allow us to transfer the notions of computability and continuity and other notions available for Baire space to any represented space; a function between represented spaces will be called *computable* if it has a computable realizer, etc. Now we can define Weihrauch reducibility.

Definition 2 (Weihrauch reducibility). *Let f, g be multi-valued functions on represented spaces. Then f is said to be* Weihrauch reducible *to g, in symbols $f \leq_{\mathrm{W}} g$, if there are computable functions $K, H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $K\langle \mathrm{id}, GH\rangle \vdash f$ for all $G \vdash g$. Moreover, f is said to be* strongly Weihrauch reducible *to g, in symbols $f \leq_{\mathrm{sW}} g$, if there are computable functions K, H such that $KGH \vdash f$ for all $G \vdash g$.*

Here $\langle , \rangle$ denotes some standard pairing on Baire space. We note that the relations $\leq_{\mathrm{W}}$, $\leq_{\mathrm{sW}}$ and $\vdash$ implicitly refer to the underlying representations, which we mention explicitly only when necessary. It is known that these relations only depend on the underlying equivalence classes of representations, but not on the specific representatives (see Lemma 2.11 in [4]). We use $\equiv_{\mathrm{W}}$ and $\equiv_{\mathrm{sW}}$ to denote the respective equivalences regarding $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sW}}$, and by $<_{\mathrm{W}}$ and $<_{\mathrm{sW}}$ we denote strict reducibility.

A particularly useful multi-valued function in the Weihrauch lattice is closed choice (see [8, 4, 3, 2]) and it is known that many notions of computability can be calibrated using the right version of choice. We will focus on closed choice for computable metric spaces, which are separable metric spaces such that the distance function is computable on the given dense subset. We assume that computable metric spaces are represented via their Cauchy representation (see [26] for details).

By $\mathcal{A}_-(X)$ we denote the set of closed subsets of a metric space X, where the index "$-$" indicates that we work with negative information. This information is given by a representation $\psi_- : \mathbb{N}^{\mathbb{N}} \to \mathcal{A}_-(X)$, defined by $\psi_-(p) := X \setminus \bigcup_{i=0}^{\infty} B_{p(i)}$, where B_n is some standard enumeration of the open balls of X with center in the dense subset and rational radius. The computable points in $\mathcal{A}_-(X)$ are called *co-c.e. closed sets*. We now define closed choice for the case of computable metric spaces.

Definition 3 (Closed Choice). *Let* $\mathbf{X}$ *be a computable metric space. Then the* closed choice *operation* $\mathsf{C}_{\mathbf{X}} :\subseteq \mathcal{A}_-(X) \rightrightarrows X$ *of this space is defined by* $\mathrm{dom}(\mathsf{C}_X) := \{A \in \mathcal{A}_-(X) : A \neq \emptyset\}$ *and* $x \in \mathsf{C}_{\mathbf{X}}(A)$ *iff* $x \in A$.

Intuitively, C_X takes as input a non-empty closed set in negative representation (i.e. given by ψ_-) and it produces an arbitrary point of this set as output. For short we use the notation $\mathcal{A}_n := \{A \in \mathcal{A}_-([0,1]^n) : A \neq \emptyset\}$ for the space of non-empty closed subsets with representation ψ_- in the following.

3 Brouwer's Fixed Point Theorem and Connected Choice

In this section we want to show that the Brouwer Fixed Point Theorem is computably equivalent to connected choice for any fixed dimension. We first define these two operations. By $\mathcal{C}(X, Y)$ we denote the *set of continuous functions* $f : X \to Y$ and for short we write $\mathcal{C}_n := \mathcal{C}([0,1]^n, [0,1]^n)$.

Definition 4 (Brouwer Fixed Point Theorem). *By* $\mathsf{BFT}_n : \mathcal{C}_n \rightrightarrows [0,1]^n$ *we denote the operation defined by* $\mathsf{BFT}_n(f) := \{x \in [0,1]^n : f(x) = x\}$ *for* $n \in \mathbb{N}$.

We note that BFT_n is well-defined, i.e. $\mathsf{BFT}_n(f)$ is non-empty for all f, since by the Brouwer Fixed Point Theorem every $f \in \mathcal{C}_n$ admits a fixed point x, i.e. with $f(x) = x$. We now define connected choice.

Definition 5 (Connected choice). *By* $\mathsf{CC}_n :\subseteq \mathcal{A}_n \rightrightarrows [0,1]^n$ *we denote the operation defined by* $\mathsf{CC}_n(A) := A$ *for all non-empty connected closed* $A \subseteq [0,1]^n$ *and* $n \in \mathbb{N}$. *We call* CC_n connected choice *(of dimension* n*).*

Hence, connected choice is just the restriction of closed choice $\mathsf{C}_{[0,1]^n}$ to connected sets. We also use the following notation for the set of fixed points of a function $f \in \mathcal{C}_n$.

Definition 6 (Set of fixed points). *By* $\mathsf{Fix}_n : \mathcal{C}_n \to \mathcal{A}_n$ *we denote the function with* $\mathsf{Fix}_n(f) := \{x \in [0,1]^n : f(x) = x\}$.

It is easy to see that Fix_n is computable, since $\mathsf{Fix}_n(f) := (f - \mathrm{id})^{-1}\{0\}$ and it is well-known that closed sets in $\mathcal{A}_n$ can also be represented as zero sets of continuous functions (see [26]).

Definition 7 (Connectedness components). *By* $\mathsf{Con}_n : \mathcal{A}_n \rightrightarrows \mathcal{A}_n$ *we denote the map with* $\mathsf{Con}_n(A) := \{C : C$ *is a connectedness component of* $A\}$ *for every* $n \geq 1$.

Theorem 2 (Connectedness components). $\mathsf{Con}_n \equiv_{\mathrm{sW}} \mathsf{WKL}$ *for* $n \geq 1$.

We note that the Brouwer Fixed Point Theorem can be decomposed to $\mathsf{BFT}_n = \mathsf{CC}_n \circ \mathsf{Con}_n \circ \mathsf{Fix}_n$.

The main result of this section will be that the Brouwer Fixed Point Theorem and connected choice are (strongly) equivalent for any fixed dimension n (see Theorem 3 below).

The direction $\mathsf{CC}_n \leq_{\mathrm{sW}} \mathsf{BFT}_n$ can be seen as a uniformization of an earlier construction of Baigger [1] that is in turn built on results of Orevkov [16]. This part of the construction was explained in some detail by Potgieter in [20].

For the other direction $\mathsf{BFT}_n \leq_{\mathrm{sW}} \mathsf{CC}_n$ of the reduction we uniformize ideas from the third author's PhD thesis [14]. A central technique is topological degree theory. For the uniform aspects of both directions, a representation of closed sets via trees of rational complexes is employed.

The first observation is that the map $\mathsf{Con}_n \circ \mathsf{Fix}_n$ is computable (which might be surprising in light of Theorem 2).

Proposition 1. $\mathsf{Con}_n \circ \mathsf{Fix}_n : \mathcal{C}_n \rightrightarrows \mathcal{A}_n$ *is computable for all* $n \in \mathbb{N}$.

Since $\mathsf{BFT}_n \supseteq \mathsf{CC}_n \circ \mathsf{Con}_n \circ \mathsf{Fix}_n$ we can directly conclude $\mathsf{BFT}_n \leq_{\mathrm{sW}} \mathsf{CC}_n$ for all n. Together with $\mathsf{CC}_n \leq_{\mathrm{sW}} \mathsf{BFT}_n$ we obtain the following theorem.

Theorem 3 (Brouwer Fixed Point Theorem). $\mathsf{BFT}_n \equiv_{\mathrm{sW}} \mathsf{CC}_n$ *for all* n.

It is easy to see that in general the Brouwer Fixed Point Theorem and connected choice are not independent of the dimension. In case of $n = 0$ the space $[0,1]^n$ is the one-point space $\{0\}$ and hence $\mathsf{BFT}_0 \equiv_{\mathrm{sW}} \mathsf{CC}_0$ are both computable. In case of $n = 1$ connected choice was already studied in [3] and it was proved that it is equivalent to the Intermediate Value Theorem IVT (see Definition 6.1 and Theorem 6.2 in [3]).

Corollary 1 (Intermediate Value Theorem). $\mathsf{IVT} \equiv_{\mathrm{sW}} \mathsf{BFT}_1 \equiv_{\mathrm{sW}} \mathsf{CC}_1$.

It is also easy to see that the Brouwer Fixed Point Theorem BFT_2 in dimension two is more complicated than in dimension one. For instance, it is known that the Intermediate Value Theorem IVT always offers a computable function value for a computable input, whereas this is not the case for the Brouwer Fixed Point Theorem BFT_2 by Baigger's counterexample [1]. We continue to discuss this topic in Section 5.

Here we point out that Proposition 1 implies that the fixed point set $\mathsf{Fix}_n(f)$ of every computable function $f : [0,1]^n \to [0,1]^n$ has a co-c.e. closed connectedness component. The converse direction is true, too, and in a uniform way: We denote by $(f,g) :\subseteq X \rightrightarrows Y \times Z$ the *juxtaposition* of two functions $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq X \rightrightarrows Z$, defined by $(f,g)(x) = (f(x), g(x))$.

Theorem 4 (Fixability). $(\mathsf{Fix}_n, \mathsf{Con}_n \circ \mathsf{Fix}_n)$ *is computable and has a multi-valued computable right inverse for all* $n \in \mathbb{N}$.

Roughly speaking a closed set $A \in \mathcal{A}_n$ together with one of its connectedness components is as good as a continuous function $f \in \mathcal{C}_n$ with A as set of fixed points. As a non-uniform corollary we obtain immediately Miller's original result.

Corollary 2 (Fixable sets, Miller 2002). *A set* $A \subseteq [0,1]^n$ *is the set of fixed points of a computable function* $f : [0,1]^n \to [0,1]^n$ *if and only if it is non-empty and co-c.e. closed and contains a co-c.e. closed connectedness component.*

4 The Lipschitz Trichotomy

It seems to be a natural question[5] to what extent finding fixed points becomes easier if the class of functions to be considered is further restricted. In particular we will denote by L-LBFT_n the restriction of BFT_n to L-Lipschitz functions.

Proposition 2. *For* $L_1, L_2 > 1$ *we find that* L_1-$\mathsf{LBFT}_n \equiv_{\mathrm{W}} L_2$-$\mathsf{LBFT}_n$.

Proof. If f *is* L_1*-Lipschitz and* $L_2 > 1$*, then* $\mathrm{id} + \frac{L_2-1}{L_1+1}(f - \mathrm{id})$ *is* L_2*-Lipschitz and has the same fixed points as* f.

With some additional constructions and a careful analysis, the proof of Theorem 3 can be adapted to yield:

Theorem 5. 2-$\mathsf{LBFT}_n \equiv_{\mathrm{W}} \mathsf{BFT}_n \equiv_{\mathrm{W}} \mathsf{CC}_n$.

Being L-Lipschitz for $L < 1$ implies the uniqueness of the fixed point, which in turn implies the computability of L-LBFT_n for $L < 1$. The remaining $L = 1$ case is also a special (since finite-dimensional) case of the Browder-Goehde-Kirk Fixed Point theorem. Its Weihrauch degree was studied by Neumann in [15], and shown to be equivalent to XC_n – closed choice for convex sets in $[0,1]^n$.

Theorem 6 (Le Roux & Pauly [12]).

$$\mathsf{CC}_1 \equiv_{\mathrm{W}} \mathsf{XC}_1 <_{\mathrm{W}} \mathsf{XC}_2 <_{\mathrm{W}} \mathsf{XC}_3 <_{\mathrm{W}} \ldots <_{\mathrm{W}} \mathsf{C}_{[0,1]}$$

[5] Which was put to the authors by Kohlenbach.

Corollary 3 (Lipschitz dichotomy in dimension 1).

- L-$\mathsf{LBFT}_1 \equiv_{\mathrm{W}} \mathrm{id}$, *iff* $L < 1$
- L-$\mathsf{LBFT}_1 \equiv_{\mathrm{W}} \mathsf{CC}_1$, *iff* $L \geq 1$

Corollary 4 (Lipschitz trichotomy). *Let* $n > 1$.

- L-$\mathsf{LBFT}_n \equiv_{\mathrm{W}} \mathrm{id}$, *iff* $L < 1$
- L-$\mathsf{LBFT}_n \equiv_{\mathrm{W}} \mathsf{XC}_n$, *iff* $L = 1$
- L-$\mathsf{LBFT}_n \equiv_{\mathrm{W}} \mathsf{C}_{[0,1]}$, *iff* $L > 1$

5 Classifying Connected Choice

In this section we want to discuss the degree of connected choice, in particular in relation to the dimension of the ambient space. We will consider three geometric constructions: The one employed in the original proof by Orevkov/Baigger – this construction is insufficient for the uniform aspects. Then a simple construction showing that connected choice is computably complete from dimension three onwards in the sense that it is strongly equivalent to Weak Kőnig's Lemma. Finally, a significantly more involved construction shows even connected choice in two dimensions to be computably complete, too.

A superficial reading of the results of Orevkov [16] and Baigger [1] can lead to the wrong conclusion that they actually provide a reduction of Weak Kőnig's Lemma to the Brouwer Fixed Point Theorem BFT_n of any dimension $n \geq 2$. However, this is only correct in a non-uniform way and the corresponding uniform result does not follow from the known constructions. The Orevkov-Baigger result is built on the following fact.

Proposition 3 (Mixed cube). *The function* $M :\subseteq \mathcal{A}_-[0,1] \to \mathcal{A}_2$ *with* $M(A) = (A \times [0,1]) \cup ([0,1] \times A)$ *is computable and maps non-empty closed sets* $A \subseteq [0,1]$ *to non-empty connected closed sets* $M(A) \subseteq [0,1]^2$.

It follows straightforwardly from the definition that the pairs $(x, y) \in M(A)$ are such that one out of two components x, y is actually in A. In order to express the uniform content of this fact, we introduce the concept of a fraction.

Definition 8 (Fractions). *Let* $f :\subseteq X \rightrightarrows Y$ *be a multi-valued function and* $0 < n \leq m \in \mathbb{N}$. *We define the* fraction $\frac{n}{m}f :\subseteq X \rightrightarrows Y^m$ *such that* $\frac{n}{m}f(x)$ *is the set of all* $(y_1, ..., y_m) \in \mathrm{range}(f)^m$ *with* $|\{i : y_i \in f(x)\}| \geq n$ *for all* $x \in \mathrm{dom}(\frac{n}{m}f) := \mathrm{dom}(f)$.

The idea of a fraction $\frac{n}{m}f$ is that it provides m potential answers for f, at least $n \leq m$ of which have to be correct. The uniform content of the Orevkov-Baigger construction is then summarized in the following result.

Proposition 4 (Dimension two). $\frac{1}{2}\mathsf{C}_{[0,1]} \leq_{\mathrm{sW}} \mathsf{CC}_2$.

However, the following results shows that the uniform content of the preceding proposition is very weak, as it cannot even solve closed choice on the two-point space **2** (which is equivalent to LLPO):

Proposition 5. $\mathsf{C}_{\mathbf{2}} \not\leq_{\mathrm{W}} \frac{1}{2}\mathsf{C}_{[0,1]}$

That is, given a closed set $A \subseteq [0,1]$ we can utilize connected choice CC_2 of dimension 2 in order to find a pair of points (x,y) one of which is in A. This result directly implies the counterexample of Baigger [1] because the fact that there are non-empty co-c.e. closed sets $A \subseteq [0,1]$ without computable points immediately implies that $\frac{1}{2}\mathsf{C}_{[0,1]}$ is not non-uniformly computable (i.e. there are computable inputs without computable outputs) and hence CC_2 is also not non-uniformly computable.

Corollary 5 (Orevkov 1963, Baigger 1985). *There exists a computable function $f : [0,1]^2 \to [0,1]^2$ that has no computable fixed point $x \in [0,1]^2$. There exists a non-empty connected co-c.e. closed subset $A \subseteq [0,1]^2$ without computable point.*

Instead, we shall use a different construction to classify connected choice from three dimensions upwards:

Proposition 6 (Twisted cube). *The function $T :\subseteq \mathcal{A}_-[0,1] \to \mathcal{A}_3$ with $T(A) = (A \times [0,1] \times \{0\}) \cup (A \times A \times [0,1]) \cup ([0,1] \times A \times \{1\})$ is computable and maps non-empty closed sets $A \subseteq [0,1]$ to non-empty connected closed sets $T(A) \subseteq [0,1]^3$.*

Here tuples $(x_1, x_2, x_3) \in T(A)$ have the property that at least one of the first two components provide a solution $x_i \in A$, but the third component provides the additional information which one surely does. If x_3 is close to 1, then surely $x_2 \in A$ and if x_3 is close to 0, then surely $x_1 \in A$. If x_3 is neither close to 0 nor 1, then both $x_1, x_2 \in A$. Hence, there is a computable function H such that $\mathsf{C}_{[0,1]} = H \circ \mathsf{CC}_3 \circ T$, which proves $\mathsf{C}_{[0,1]} \leq_{\mathrm{sW}} \mathsf{CC}_3$. Together with Theorem 3 we obtain the following conclusion.

Theorem 7 (Completeness of three dimensions). *For $n \geq 3$ we obtain $\mathsf{CC}_n \equiv_{\mathrm{sW}} \mathsf{BFT}_n \equiv_{\mathrm{sW}} \mathsf{WKL} \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]}$.*

We note that the reduction $\mathsf{CC}_n \leq_{\mathrm{sW}} \mathsf{C}_{[0,1]^n}$ holds for all $n \in \mathbb{N}$, since connected choice is just a restriction of closed choice and $\mathsf{C}_{[0,1]^n} \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]} \equiv_{\mathrm{sW}} \mathsf{WKL}$ is known for all $n \geq 1$ (see [2]).

Originally, three of the authors had conjectured in [7] that $\mathsf{CC}_2 <_{\mathrm{W}} \mathsf{C}_{[0,1]}$. However, a more involved construction actually establishes that:

Theorem 8 (Completeness of two dimensions). $\mathsf{CC}_2 \equiv_{\mathrm{W}} \mathsf{C}_{[0,1]}$

The proof of Theorem 8 exhibits a reduction $\widehat{\mathsf{C}_{\mathbf{2}}} \leq_{\mathrm{W}} \mathsf{CC}_2$ instead, using the equivalence $\widehat{\mathsf{C}_{\mathbf{2}}} \equiv_{\mathrm{W}} \mathsf{C}_{\{0,1\}^{\mathbb{N}}}$ from [4]. The geometric pattern constructed produces an infinitely long line which is then subdivided based on both the information obtained about the input to $\widehat{\mathsf{C}_{\mathbf{2}}}$, as well as the order in which this information is found. A glimpse of the construction might be gained from Figure 1.

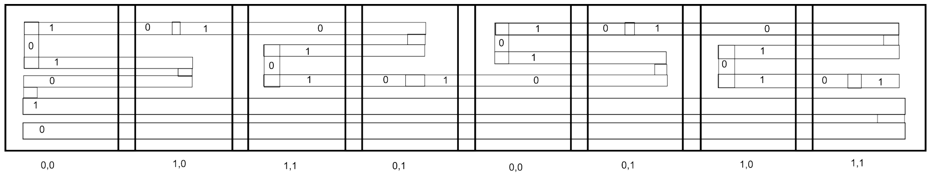

Fig. 1. The geometric pattern after the third round

6 Two Versus Three Dimensions

A noticeable difference between the construction from the proof of Theorem 8 and Proposition 6 is that the latter yields even a path-connected set, whereas the former does not. Thus, *path-connected choice* is computably-complete from dimension three onwards, but might be simpler in dimension two.

While the status of path-connected choice in dimension two remains open, we can exhibit a related choice principle distinguishing two from three dimensions.

Definition 9. *We say that* $A \in \mathcal{A}_2$ *has a straight cross, if there are* $x, y \in [0,1]$, $\delta > 0$ *s.t* $\forall \varepsilon \in (-\delta, \delta) \quad (x+\varepsilon, y) \in A \wedge (x, y+\varepsilon) \in A$. *Let* $\dagger\mathsf{C}_{[0,1]^2}$ *be choice for sets having a straight cross.*

Proposition 7. $\dagger\mathsf{C}_{[0,1]^2} \leq_{\mathrm{W}} \frac{1}{2}\mathsf{C}_{[0,1]} \star \mathsf{C}_{\mathbb{N}}$.

Corollary 6. $\dagger\mathsf{C}_{[0,1]^2} <_{\mathrm{W}} \mathsf{CC}_2$

Proof. Combine Proposition 7 with the Fractal Absorption Theorem from [12].

An analogous argument would not succeed in dimension 3, as $\frac{2}{3}\mathsf{C}_{[0,1]} \equiv_{\mathrm{W}} \mathsf{C}_{[0,1]}$ by a majority-voting argument.

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