Meixner class of orthogonal polynomials of a non-commutative monotone Lévy noise

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Abstract

Let $(X_t)_{t\ge0}$ denote a non-commutative monotone Lévy process. Let $\omega=(\omega(t))_{t\ge0}$ denote the corresponding monotone Lévy noise, i.e., formally $\omega(t)=\frac{d}{dt}X_t$. A continuous polynomial of ω is an element of the corresponding non-commutative L^2-space $L^2(\tau)$ that has the form $\sum_{i=0}^n\langle\omega^{\otimes i},f^{(i)}\rangle$, where $f^{(i)}\in C_0(\mathbb R_+^i)$. We denote by **CP** the space of all continuous polynomials of ω. For $f^{(n)}\in C_0(\mathbb R_+^n)$, the orthogonal polynomial $\langle P^{(n)}(\omega),f^{(n)}\rangle$ is defined as the orthogonal projection of the monomial $\langle\omega^{\otimes n},f^{(n)}\rangle$ onto the subspace of $L^2(\tau)$ that is orthogonal to all continuous polynomials of ω of order $\le n-1$. We denote by **OCP** the linear span of the orthogonal polynomials. Each orthogonal polynomial $\langle P^{(n)}(\omega),f^{(n)}\rangle$ depends only on the restriction of the function $f^{(n)}$ to the set $\{(t_1,\dots,t_n)\in\mathbb R_+^n\mid t_1\ge t_2\ge\dots\ge t_n\}$. The orthogonal polynomials allow us to construct a unitary operator $J:L^2(\tau)\to\mathbb F$, where $\mathbb F$ is an extended monotone Fock space. Thus, we may think of the monotone noise ω as a distribution of linear operators acting in $\mathbb F$. We say that the orthogonal polynomials belong to the Meixner class if **CP** = **OCP**. We prove that each system of orthogonal polynomials from the Meixner class is characterized by two parameters: $\lambda\in\mathbb R$ and $\eta\ge0$. In this case, the monotone Lévy noise has the representation $\omega(t)=\partial_t^\dagger+\lambda\partial_t^\dagger\partial_t+\partial_t+\eta\partial_t^\dagger\partial_t\partial_t$. Here, $\partial_t^\dagger$ and ∂_t are the (formal) creation and annihilation operators at $t\in\mathbb R_+$ acting in $\mathbb F$.

Keywords: Monotone independence, monotone Lévy noise, monotone Lévy process, Meixner class of orthogonal polynomials.

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1 Introduction

The Meixner class of orthogonal polynomials on $\mathbb R$ was originally derived by Meixner [28] as the class of all Sheffer sequences of monic polynomials that are orthogonal with respect to a probability measure on $\mathbb R$ with infinite support. They include the Hermite polynomials, the Charlier polynomials, the Laguerre polynomials, the Meixner polynomials of the first kind, and the Meixner polynomials of the second kind (also called the Meixner–Polaczek polynomials).

Let $(p_n)_{n=0}^{\infty}$ be a polynomial sequence from the Meixner class. Let us assume that the measure of orthogonality of these polynomials is centered. Let $\partial^\dagger$ and ∂ be linear operators acting on polynomials on $\mathbb{R}$ that satisfy $\partial^\dagger p_n = p_{n+1}$ and $\partial p_n = np_{n-1}$ for all n. Then there exist three parameters, $\lambda \in \mathbb{R}$, $\eta \geq 0$, $k > 0$, such that

$$x = \partial^\dagger + \lambda\partial^\dagger\partial + k\partial + \eta\partial^\dagger\partial\partial. \tag{1}$$

In this formula, x denotes the operator of multiplication by the variable x, considered as a linear operator acting on polynomials.

For each Meixner sequence of polynomials, its measure of orthogonality is infinitely divisible. Thus, the Meixner class is related to Lévy processes.

It appears that the notion of the Meixner class of orthogonal polynomials admits several generalizations. The first generalization is related to changing the definition of the operators $\partial^\dagger$ and ∂. Recall that, for $q \in [-1, 1]$, one defines the q-numbers $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$. Then, the q-Meixner class of orthogonal polynomials on $\mathbb{R}$ is defined as the class of the polynomial sequences $(p_n)_{n=0}^{\infty}$ satisfying formula (1) in which $\partial^\dagger p_n = p_{n+1}$ and $\partial p_n = [n]_q\, p_{n-1}$, see [16, 17] and the references therein.

Furthermore, it is also possible to generalize the notion of the Meixner class of orthogonal polynomials to the (classical) infinite dimensional setting, as well as to some non-commutative settings.

Let us briefly describe the extension to the classical infinite dimensional setting, see [10, 25, 26] for details and [1, 23, 24, 27, 33, 34] for related topics. Consider the Gel'fand triple

$$\mathcal{D} \subset L^2(\mathbb{R}_+, dt) \subset \mathcal{D}'.$$

Here $\mathcal{D}$ is the nuclear space of smooth compactly supported functions on $\mathbb{R}_+ := [0, \infty)$ and $\mathcal{D}'$ is the dual space of $\mathcal{D}$, where the dual pairing between $\mathcal{D}'$ and $\mathcal{D}$ is obtained by continuously extending the inner product in $L^2(\mathbb{R}_+, dt)$. A continuous polynomial P of $\omega \in \mathcal{D}'$ is a function $P : \mathcal{D}' \to \mathbb{R}$ of the form

$$P(\omega) = \sum_{i=0}^{n} \langle \omega^{\otimes i}, f^{(i)} \rangle, \quad \omega \in \mathcal{D}',$$

where $f^{(i)} \in \mathcal{D}^{\odot i}$ (i.e., $f^{(i)}$ belongs to the ith symmetric tensor power of $\mathcal{D}$) and $\langle \omega^{\otimes i}, f^{(i)} \rangle$ denotes the dual pairing of $\omega^{\otimes i} \in \mathcal{D}'^{\odot i}$ and $f^{(i)} \in \mathcal{D}^{\odot i}$. We denote by **CP** the space of all continuous polynomials of ω.

Let μ be a Lévy white noise measure on $\mathcal{D}'$. Thus, μ is a probability measure on $\mathcal{D}'$ whose Fourier transform has the Kolmogorov representation

$$\int_{\mathcal{D}'} e^{i\langle \omega, h \rangle}\, d\mu(\omega) = \exp\left(\int_{\mathbb{R}_+} \int_{\mathbb{R}} (e^{ish(t)} - 1 - ish(t)) \frac{1}{s^2}\, d\nu(s)\, dt \right), \quad h \in \mathcal{D}. \tag{2}$$

Here ν is a finite measure on $\mathbb{R}$. We assume that ν has all moments finite. We also assume, for simplicity, that ν is a probability measure.

For $f^{(n)} \in \mathcal{D}^{\odot n}$, we define the orthogonal polynomial $\langle P^{(n)}(\omega), f^{(n)}\rangle$ as the orthogonal projection of the monomial $\langle \omega^{\otimes n}, f^{(n)}\rangle$ onto the subspace of $L^2(\mathcal{D}', \mu)$ that is orthogonal to all continuous polynomials of ω of order $\leq n-1$. We denote by **OCP** the linear span of all orthogonal polynomials.

By using the orthogonal polynomials, one constructs a unitary operator

$$J : L^2(\mathcal{D}', \mu) \to \mathbb{F},$$

where

$$\mathbb{F} = \mathbb{R} \oplus \bigoplus_{n=1}^{\infty} L^2_{\mathrm{sym}}(\mathbb{R}_+^n, m_n)$$

is an extended symmetric Fock space. Here m_n is a Radon measure on $\mathbb{R}_+^n$ (which depends on the Kolmogorov measure ν in formula (2)) and $L^2_{\mathrm{sym}}(\mathbb{R}_+^n, m_n)$ denotes the subspace of all symmetric functions from $L^2(\mathbb{R}_+^n, m_n)$. The unitary operator J is given by

$$J\langle P^{(n)}(\cdot), f^{(n)}\rangle = (0, \dots, 0, \underbrace{f^{(n)}}_{n\text{th place}}, 0, 0, \dots) \in \mathbb{F}.$$

For $h \in \mathcal{D}$, let us preserve the notation $\langle \omega, h\rangle$ for the operator of multiplication by $\langle \omega, h\rangle$ in $L^2(\mathcal{D}', \mu)$. In terms of the unitary operator J, we may also think of $\langle \omega, h\rangle$ as a linear operator in $\mathbb{F}$.

We say that the orthogonal polynomials $\langle P^{(n)}(\omega), f^{(n)}\rangle$ belong to the Meixner class if **CP** $=$ **OCP**. Each system of orthogonal polynomials from the Meixner class is characterized by two parameters: $\lambda \in \mathbb{R}$ and $\eta \geq 0$. For each choice of such parameters, we have the following equality for the action of the operator $\langle \omega, h\rangle$ in $\mathbb{F}$: for each $f \in \mathcal{D}$:

$$\begin{aligned} \langle \omega, h\rangle f^{\otimes n} &= h \odot f^{\otimes n} + \lambda n (hf) \odot f^{\otimes(n-1)} \\ &\quad + n \int_{\mathbb{R}_+} h(u) f(u)\, du\, f^{\otimes(n-1)} + \eta n(n-1)(hf^2) \odot f^{\otimes(n-2)}. \end{aligned} \tag{3}$$

The choice $\lambda = \eta = 0$ gives the infinite dimensional Hermite polynomials with μ being Gaussian white noise measure; the choice $\lambda \neq 0$, $\eta = 0$ gives the infinite dimensional Charlier polynomials with μ being a centered Poisson random measure; the choice $|\lambda| = 2\sqrt{\eta}$, $\eta > 0$ gives the infinite dimensional Laguerre polynomials with μ being the centered gamma random measure; the choice $|\lambda| > 2\sqrt{\eta}$, $\eta > 0$ gives the infinite dimensional Meixner polynomials of the first kind with μ being a centered negative binomial random measure; and the choice $|\lambda| < 2\sqrt{\eta}$, $\eta > 0$ gives the infinite dimensional Meixner polynomials of the second kind with μ being the Meixner measure on $\mathcal{D}'$.

Let δ_t denote the delta function at t. For each $t \in \mathbb{R}_+$, we formally define operators $\partial_t^\dagger$, ∂_t on $\mathbb{F}$ by

$$\partial_t^\dagger f^{\otimes n} := \delta_t \odot f^{\otimes n}, \quad \partial_t f^{\otimes n} := n f(t) f^{\otimes(n-1)}.$$

(To be more precise, $\partial_t^\dagger$ is a formal operator, while the operator ∂_t is rigorously defined on a subspace of $\mathbb{F}$.) Then formula (3) can be formally written in the form

$$\omega(t) = \partial_t^\dagger + \lambda \partial_t^\dagger \partial_t + \partial_t + \eta \partial_t^\dagger \partial_t \partial_t, \tag{4}$$

compare with (1). (Note that $k = 1$ in our case since we chose ν to be a probability measure.)

In non-commutative probability, monomials $\langle \omega, h \rangle$ are replaced with non-commutative operators $\langle \omega, h \rangle$ (so that $\omega(t)$ can be thought of as a non-commutative noise), while the probability measure μ is replaced by a state on the algebra of polynomials generated by the monomials $\langle \omega, h \rangle$. There are several non-commutative generalizations of independence. The most studied one is free independence, see e.g. [32, 36]. In the framework of free probability, the Meixner class of orthogonal polynomials of a free Lévy noise was studied in [11,12]. This study led to a formula similar to (4). For other studies of the free Meixner-type Lévy processes and the free Meixner polynomials on $\mathbb{R}$ we refer to [2–8, 15, 18, 19, 35].

Furthermore, in [14], the notion of a non-commutative Lévy noise was introduced for the anyon statistics, and in [13], the corresponding Meixner class of non-commutative orthogonal polynomials was studied. Quite unexpectedly, this class was again fully described by a formula similar to (4), albeit its meaning was quite different. Note that the Kolmogorov measures ν of the corresponding anyon noises are the same as in the case of the classical Meixner noises.

In this paper, we will deal with another important example of non-commutative independence: the monotone independence. This notion was introduced and studied by Muraki [29–31], see also [20,21]. The Lévy processes of the monotone independence were studied in [21,22]. Note that there is also the related notion of the anti-monotone independence. We will not discuss it in this paper but only mention that, when trivially modified, all the results of the present paper hold in the anti-monotone case.

The main result of the paper is a characterization of the Meixner class of orthogonal polynomials of a monotone Lévy noise. More precisely, let $(X_t)_{t \geq 0}$ be a (non-commutative) monotone Lévy process. Let $\omega = (\omega(t))_{t \geq 0}$ denote the corresponding monotone Lévy noise, i.e., formally $\omega(t) = \frac{d}{dt} X_t$. A continuous polynomial of ω is an element of the corresponding non-commutative L^2-space $L^2(\tau)$ that has the form $\sum_{i=0}^{n} \langle \omega^{\otimes i}, f^{(i)} \rangle$, where $f^{(i)} \in C_0(\mathbb{R}_+^i)$. We denote by **CP** the space of all continuous polynomials of ω. For $f^{(n)} \in C_0(\mathbb{R}_+^n)$, the orthogonal polynomial $\langle P^{(n)}(\omega), f^{(n)} \rangle$ is defined as the orthogonal projection of the monomial $\langle \omega^{\otimes n}, f^{(n)} \rangle$ onto the subspace of $L^2(\tau)$ that is orthogonal to all continuous polynomials of ω of order $\leq n-1$. We denote by **OCP** the linear span of all orthogonal polynomials. Each orthogonal polynomial $\langle P^{(n)}(\omega), f^{(n)} \rangle$ depends only on the restriction of the function $f^{(n)}$ to the set

$$T_n := \{(t_1, \dots, t_n) \in \mathbb{R}_+^n \mid t_1 \geq t_2 \geq \dots \geq t_n\}. \tag{5}$$

The orthogonal polynomials allow us to construct a unitary operator $J : L^2(\tau) \to \mathbb{F}$, where $\mathbb{F}$ now denotes an extended monotone Fock space:

$$\mathbb{F} = \mathbb{R} \oplus \bigoplus_{n=1}^{\infty} L^2(T_n, m_n). \tag{6}$$

Here m_n is a Radon measure on T_n, determined by the Kolmogorov measure ν of the monotone noise ω. By using this operator J, we may think of the monotone noise ω as a distribution of linear operators acting in $\mathbb{F}$.

We say that the orthogonal polynomials of ω belong to the Meixner class if $\mathbf{CP} = \mathbf{OCP}$. We prove that each system of orthogonal polynomials from the Meixner class is again characterized by two parameters: $\lambda \in \mathbb{R}$ and $\eta \geq 0$. In this case, the monotone Lévy noise has again representation (4). Here, for a point $t \in \mathbb{R}_+$, $\partial_t^\dagger$ and ∂_t are the (formal) creation and annihilation operators at t acting in $\mathbb{F}$. It is worth noting that the corresponding Kolmogorov measures ν appear to be the same as in the case of free independence.

Let us mention a drastic difference between the monotone case and all the other cases mentioned above, see Remark 15 below for details. In the monotone case, it is possible that a continuous monomial $\langle \omega^{\otimes n}, f^{(n)} \rangle$ is a non-zero element of $L^2(\tau)$ but belongs to the space of all continuous polynomials of order $\leq n-1$. A necessary condition for this is that the restriction of the function $f^{(n)}$ to T_n is equal to zero.

The paper is organized as follows. In Section 2, we briefly recall the notion of monotone independence, define a monotone Lévy noise and the corresponding non-commutative L^2-space. In Section 3, we formulate the main results. Finally, in Section 4, we prove the results.

Among numerous open problems related to the monotone Meixner orthogonal polynomials, let us mention only one: the explicit form of their generating function; compare with the form of the generating function of the Meixner orthogonal polynomials in the free setting [12], see also [2, 4, 5, 35].

2 Monotone Lévy noise

Let us first recall the notion of monotone independence, cf. [30]. Let $\mathcal{F}$ be a real Hilbert space and let $\mathcal{L}(\mathcal{F})$ denote the space of all continuous linear operators in $\mathcal{F}$. Let $\Omega \in \mathcal{F}$ be a unit vector, and define a state $\tau : \mathcal{L}(\mathcal{F}) \to \mathbb{R}$ by $\tau(A) := (A\Omega, \Omega)_{\mathcal{F}}$. Subalgebras (not necessarily unital) $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_r$ of $\mathcal{L}(\mathcal{F})$ are called *monotonically independent with respect to* τ if, for any $i < j > k$ and $A \in \mathcal{A}_i$, $B \in \mathcal{A}_j$, $C \in \mathcal{A}_k$,

$$ABC = \tau(B)AC,$$

and any $i_1 > \cdots > i_m > j < k_1 < \cdots < k_n$, $A_1 \in \mathcal{A}_{i_1}, \dots, A_m \in \mathcal{A}_{i_m}$, $B \in \mathcal{A}_j$, $C_1 \in \mathcal{A}_{k_1}, \dots, C_n \in \mathcal{A}_{k_n}$,

$$\tau(A_1 \cdots A_m B C_1 \cdots C_n) = \tau(A_1) \cdots \tau(A_m) \tau(B) \tau(C_1) \cdots \tau(C_n).$$

Operators $A_1, \dots, A_r \in \mathcal{L}(\mathcal{F})$ are called *monotonically independent with respect to* τ if the subalgebras $\mathcal{A}_i = \mathrm{l.\,s.}(A_i^k \mid k \in \mathbb{N})$, $i = 1, 2, \dots, r$, are monotonically independent. Here l. s. denotes the linear span.

Let $B_0(\mathbb{R}_+)$ denote the linear space of all measurable bounded functions on $\mathbb{R}_+$ with compact support. We endow $B_0(\mathbb{R}_+)$ with a topology such that a sequence $(h_n)_{n=1}^\infty$ converges to a function h in $B_0(\mathbb{R}_+)$ if all functions h_n vanish outside a compact set in $\mathbb{R}_+$ and $\sup_{t \in \mathbb{R}_+} |h_n(t) - h(t)| \to 0$ as $n \to \infty$.

Let $(\langle \omega, h \rangle)_{h \in B_0(\mathbb{R}_+)}$ be a family of operators from $\mathcal{L}(\mathcal{F})$. We assume that the operators $\langle \omega, h \rangle$ depend on h linearly, and if $h_n \to h$ in $B_0(\mathbb{R}_+)$, then $\langle \omega, h_n \rangle \to \langle \omega, h \rangle$ strongly in $\mathcal{L}(\mathcal{F})$. We may formally think of ω as an operator-valued distribution.

We will say that ω is a *monotone Lévy noise* if the following conditions are satisfied.

(i) Let $0 \le t_0 < t_1 < t_2 < \cdots < t_r$ and let functions $h_1, h_2, \dots, h_r \in B_0(\mathbb{R}_+)$ be such that, for each $i = 1, 2, \dots, r$, $h_i = h_i \, \chi_{[t_{i-1}, t_i]}$. (Here χ_Δ denotes the indicator function of a set Δ.) Then the operators $\langle \omega, h_1 \rangle, \dots, \langle \omega, h_r \rangle$ are monotonically independent.

(ii) For any $h_1, \dots, h_n \in B_0(\mathbb{R}_+)$ and any $u > 0$,

$$\tau\big(\langle \omega, h_1 \rangle \cdots \langle \omega, h_n \rangle\big) = \tau\big(\langle \omega, \mathcal{S}_u h_1 \rangle \cdots \langle \omega, \mathcal{S}_u h_n \rangle\big).$$

Here, for $h \in B_0(\mathbb{R}_+)$ and $u > 0$, we define $\mathcal{S}_u h \in B_0(\mathbb{R}_+)$ by

$$(\mathcal{S}_u h)(t) := \begin{cases} 0, & \text{if } 0 \le t < u, \\ h(t-u), & \text{if } t \ge u. \end{cases}$$

If ω is a monotone Lévy noise, we define, for $t \ge 0$, $X_t := \langle \omega, \chi_{[0,t]} \rangle$. Then $(X_t)_{t \ge 0}$ is a monotone Lévy process, cf. [22, Section 4].

By analogy with [21,22], we will now present an explicit construction of a monotone Lévy noise. Let ν be a probability measure on $\mathbb{R}$ with compact support. For each $n \in \mathbb{N}$, let T_n be defined by (5) and we denote

$$\begin{aligned} S_n &:= \big\{(t_1, s_1, \dots, t_n, s_n) \in (\mathbb{R}_+ \times \mathbb{R})^n \mid (t_1, \dots, t_n) \in T_n\big\}, \\ \mathcal{F}^{(n)} &:= L^2(S_n, dt_1 \, d\nu(s_1) \cdots dt_n \, d\nu(s_n)). \end{aligned}$$

Let also $\mathcal{F}^{(0)} := \mathbb{R}$. We define the *monotone Fock space* by $\mathcal{F} := \bigoplus_{n=0}^\infty \mathcal{F}^{(n)}$. The vector $\Omega = (1, 0, 0, \dots) \in \mathcal{F}$ will be called the *vacuum*. As usual, we will identify

$f^{(n)} \in \mathcal{F}^{(n)}$ with the corresponding element $(0,\dots,0,\underbrace{f^{(n)}}_{n\text{th place}},0,\dots)$ of $\mathcal{F}$. We define the *vacuum state on* $\mathcal{L}(\mathcal{F})$ by $\tau(A) := (A\Omega,\Omega)_{\mathcal{F}}$ for $A \in \mathcal{L}(\mathcal{F})$.

Let $h \in B_0(\mathbb{R}_+)$. We define a *creation operator* $a^+(h)$, a *neutral operator* $a^0(h)$, and an *annihilation operator* $a^-(h)$ as bounded linear operators in $\mathcal{F}$ that satisfy the following conditions. For the creation operator, we have $a^+(h)\Omega = h$ (here $h(t,s) := h(t)$), and for $f^{(n)} \in \mathcal{F}^{(n)}$, $n \in \mathbb{N}$, we have $a^+(h)f^{(n)} \in \mathcal{F}^{(n+1)}$,

$$\big(a^+(h)f^{(n)}\big)(t_1,s_1,\dots,t_{n+1},s_{n+1}) = h(t_1)f^{(n)}(t_2,s_2,\dots,t_{n+1},s_{n+1}).$$

For the neutral operator, $a^0(h)\Omega = 0$ and for each $f^{(n)} \in \mathcal{F}^{(n)}$, $n \in \mathbb{N}$, we have $a^0(h)f^{(n)} \in \mathcal{F}^{(n)}$,

$$\big(a^0(h)f^{(n)}\big)(t_1,s_1,\dots,t_n,s_n) = h(t_1)s_1f^{(n)}(t_1,s_1,\dots,t_n,s_n).$$

For the annihilation operator, $a^-(h)\Omega = 0$ and for $f^{(n)} \in \mathcal{F}^{(n)}$, $a^-(h)f^{(n)} \in \mathcal{F}^{(n-1)}$ and

$$\big(a^-(h)f^{(n)}\big)(t_1,s_1,\dots,t_{n-1},s_{n-1}) = \int_{t_1}^{\infty}\int_{\mathbb{R}} h(u)f^{(n)}(u,v,t_1,s_1,\dots,t_{n-1},s_{n-1})\,d\rho(v)\,du.$$

As easily seen, the annihilation operator $a^-(h)$ is the adjoint of the creation operator $a^+(h)$, while the neutral operator $a^0(h)$ is self-adjoint. Thus, for each $h \in B_0(\mathbb{R}_+)$, we define a self-adjoint operator

$$\langle\omega,h\rangle := a^+(h) + a^0(h) + a^-(h).$$

Note that $\tau(\langle\omega,h\rangle) = 0$. It can be easily checked that ω is a monotone Lévy noise. In fact, it follows from [22] that we have just described essentially all centered monotone Lévy noises up to equivalence. (To construct all centered monotone Lévy processes, one needs to assume that the measure ν is finite rather than probability)

Let $\mathbf{A}$ denote the real algebra generated by the operators $(\langle\omega,h\rangle)_{h\in B_0(\mathbb{R}_+)}$ and the identity operator in $\mathcal{F}$. We define an inner product on $\mathbf{A}$ by

$$(P_1,P_2)_{L^2(\tau)} := \tau(P_2^*P_1) = (P_1\Omega,P_2\Omega)_{\mathcal{F}}, \quad P_1,P_2 \in \mathbf{A}.$$

Let $\widetilde{\mathbf{A}} := \{P \in \mathbf{A} \mid (P,P)_{L^2(\tau)} = 0\}$. We define the non-commutative L^2-space $L^2(\tau)$ as the completion of the quotient space $\mathbf{A}/\widetilde{\mathbf{A}}$ with respect to the norm generated by the scalar product $(\cdot,\cdot)_{L^2(\tau)}$. Elements $P \in \mathbf{A}$ are considered as representatives of the equivalence classes from $\mathbf{A}/\widetilde{\mathbf{A}}$, and so $\mathbf{A}$ becomes a dense subspace of $L^2(\tau)$.

Note that, for each $h \in B_0(\mathbb{R}_+)$,

$$(\langle\omega,h\rangle P_1,P_2)_{L^2(\tau)} = (P_1,\langle\omega,h\rangle P_2)_{L^2(\tau)}, \quad P_1,P_2 \in \mathbf{A}.$$

Hence, $\mathbf{A} \ni P \mapsto \langle\omega,h\rangle P \in L^2(\tau)$ is a well defined linear operator in $L^2(\tau)$, i.e., $\langle\omega,h\rangle P = 0$ for each $P \in \widetilde{\mathbf{A}}$, see e.g. [9, Ch. 5, Sect. 5, subsec. 2]. Furthermore, we may extend this operator by continuity to a bounded self-adjoint linear operator in $L^2(\tau)$. With an abuse of notation, we will denote this operator of left multiplication by $\langle\omega,h\rangle$ in $L^2(\tau)$ by $\langle\omega,h\rangle$.

3 The main results

We will now present the main results of the paper.

Theorem 1. *The vacuum vector* Ω *is cyclic for the operator family* $\big(\langle\omega,h\rangle\big)_{h\in B_0(\mathbb{R}_+)}$, *i.e., the set* $\{P\Omega \mid P\in\mathbf{A}\}$ *is dense in* $\mathcal{F}$. *Hence, the mapping* $\mathbf{A}\ni P\mapsto IP:=P\Omega\in\mathcal{F}$ *extends by continuity to a unitary operator* $I: L^2(\tau)\to\mathcal{F}$.

Note that the operator of left multiplication by $\langle\omega,h\rangle$ in $L^2(\tau)$, which we denoted by $\langle\omega,h\rangle$, is equal to $I^{-1}\langle\omega,h\rangle I$, in the latter expression the operator $\langle\omega,h\rangle$ acting in $\mathcal{F}$.

For any $(h_1,\dots,h_n)\in B_0(\mathbb{R}_+)^n$, the function

$$(h_1\otimes\dots\otimes h_n)(t_1,\dots,t_n)=h(t_1)\cdots h(t_n)$$

belongs to $B_0(\mathbb{R}_+^n)$. As easily seen, we can extend the mapping

$$B_0(\mathbb{R}_+)^n\ni(h_1,\dots,h_n)\mapsto\langle\omega,h_1\rangle\cdots\langle\omega,h_n\rangle=:\langle\omega^{\otimes n},h_1\otimes\dots\otimes h_n\rangle\in\mathcal{L}(\mathcal{F})$$

by linearity and strong continuity to a mapping

$$B_0(\mathbb{R}_+^n)\ni f^{(n)}\mapsto\langle\omega^{\otimes n},f^{(n)}\rangle\in\mathcal{L}(\mathcal{F}).$$

Furthermore, we may think of $\langle\omega^{\otimes n},f^{(n)}\rangle$ as the element of $L^2(\tau)$ defined by $I^{-1}\langle\omega^{\otimes n},f^{(n)}\rangle\Omega$.

We will call $\langle\omega^{\otimes n},f^{(n)}\rangle$ a *monomial of* ω *of order* n. Sums of such operators and (real) constants form the space $\mathbf{P}$ of *polynomials of* ω. Since $\mathbf{A}\subset\mathbf{P}$, $\mathbf{P}$ is dense in $L^2(\tau)$.

It is now standard to introduce orthogonal polynomials. Indeed, we denote by $\mathbf{P}^{(n)}$ the linear space of all polynomials of ω of order $\le n$. Let $\overline{\mathbf{P}^{(n)}}$ denote the closure of $\mathbf{P}^{(n)}$ in $L^2(\tau)$, and let $\mathbf{S}^{(n)}:=\overline{\mathbf{P}^{(n)}}\ominus\overline{\mathbf{P}^{(n-1)}}$. We thus get $L^2(\tau)=\bigoplus_{n=0}^\infty\mathbf{S}^{(n)}$.

Let $f^{(n)}\in B_0(\mathbb{R}_+^n)$. We denote the orthogonal projection of the monomial $\langle\omega^{\otimes n},f^{(n)}\rangle$ onto $\mathbf{S}^{(n)}$ by $\langle P^{(n)}(\omega),f^{(n)}\rangle$. We define the subspace $\mathbf{OP}$ of $L^2(\tau)$ as the linear span of the identity operator and the orthogonal polynomials $\langle P^{(n)}(\omega),f^{(n)}\rangle$ with $f^{(n)}\in B_0(\mathbb{R}_+^n)$ (the space of *orthogonal polynomials of* ω). One can easily check that $\mathbf{OP}$ is dense in $L^2(\tau)$.

Our next aim is to calculate the $L^2(\tau)$-norm of $\langle P^{(n)}(\omega),f^{(n)}\rangle$ for $f^{(n)}\in B_0(\mathbb{R}_+^n)$. Let $(p_k)_{k=0}^\infty$ denote the system of monic orthogonal polynomials in $L^2(\mathbb{R},\nu)$. (If the support of ν is finite and consists of N points, we set $p_k:=0$ for $k\ge N$.) Hence, $(p_k)_{k=0}^\infty$ satisfy the recursion formula

$$sp_k(s)=p_{k+1}(s)+b_kp_k(s)+a_kp_{k-1}(s),\quad k\in\mathbb{N}_0,\tag{7}$$

with $p_{-1}(s) := 0$, $a_k > 0$, and $b_k \in \mathbb{R}$. (If the support of ν has N points, $a_k = 0$ for $k \geq N$.) We define

$$c_k := \int_{\mathbb{R}} p_{k-1}(s)^2 \, \nu(ds) = a_0 a_1 \cdots a_{k-1}, \quad k \in \mathbb{N},$$

where $a_0 := 1$. Note that $c_1 = 1$ and $c_k = 0$ for $k \geq 2$ if and only if the measure ν is concentrated at one point.

We denote by M the set of all multi-indices of the form $(l_1, \ldots, l_i) \in \mathbb{N}_0^i$, $i \in \mathbb{N}$. Fix any $n \in \mathbb{N}$. For each $(l_1, \ldots, l_i) \in M$, $l_1 + \cdots + l_i + i = n$, we denote

$$\begin{aligned} &T^{(l_1,\ldots,l_i)} := \big\{(t_1, \ldots, t_n) \in \mathbb{R}_+^n \mid t_1 = t_2 = \cdots = t_{l_1+1} > \\ &> t_{l_1+2} = t_{l_1+3} = \cdots = t_{l_1+l_2+2} > \cdots > t_{l_1+l_2+\cdots+l_{i-1}+i} = t_{l_1+l_2+\cdots+l_{i-1}+i+1} = \cdots = t_n\big\}. \end{aligned}$$

The sets $T^{(l_1,\ldots,l_i)}$ with $(l_1, \ldots, l_i) \in M$, $l_1 + \cdots + l_i + i = n$, form a set partition of T_n. Consider the bijection

$$T^{(l_1,\ldots,l_i)} \ni (t_1, \ldots, t_n) \mapsto (t_{l_1+1}, t_{l_1+l_2+2}, t_{l_1+l_2+l_3+3}, \ldots, t_n) \in \widetilde{T}_i, \tag{8}$$

where $\widetilde{T}_i := \{(t_1, \ldots, t_i) \in \mathbb{R}_+^i \mid t_1 > t_2 > \cdots > t_i\}$. Note that the set $T_i \setminus \widetilde{T}_i$ is of null Lebesgue measure. We denote by $m^{(l_1,\ldots,l_i)}$ the pre-image of the measure $c_{l_1} \cdots c_{l_i} dt_1 \cdots dt_i$ on T_i under the mapping (8). We then extend $m^{(l_1,\ldots,l_i)}$ by zero to the whole space T_n. We define a measure m_n on T_n by

$$m_n := \sum_{(l_1,\ldots,l_i) \in M,\ l_1+\cdots+l_i+i=n} m^{(l_1,\ldots,l_i)}.$$

Theorem 2. *For any $f^{(n)}, g^{(n)} \in B_0(\mathbb{R}_+^n)$, $n \in \mathbb{N}$, we have*

$$\big(\langle P^{(n)}(\omega), f^{(n)}\rangle, \langle P^{(n)}(\omega), g^{(n)}\rangle\big)_{L^2(\tau)} = (f^{(n)}, g^{(n)})_{L^2(T_n, m_n)}.$$

We define a Hilbert space $\mathbb{F}$ by (6). We will preserve the notation Ω for the vacuum vector $(1, 0, 0, \ldots) \in \mathbb{F}$.

Note that the space $L^2(T_n, dt_1 \cdots dt_n)$ can be identified with the subspace $L^2(T^{(0,\ldots,0)}, m_n)$ of $L^2(T_n, m_n)$. Hence, the space $\mathbb{F}$ contains the subspace

$$\mathbb{R} \oplus \bigoplus_{n=1}^{\infty} L^2(T_n, dt_1 \cdots dt_n),$$

which is a monotone Fock space. So it is natural to call $\mathbb{F}$ an *extended monotone Fock space.*

Note that Theorem 2 implies, in particular, that $\langle P^{(n)}(\omega), f^{(n)}\rangle \in L^2(\tau)$ is completely determined by the restriction of the function $f^{(n)} \in B_0(\mathbb{R}_+^n)$ to T_n, i.e., by

$f^{(n)} \in B_0(T_n)$. Note, however, that such a statement does not hold, in general, for the monomials $\langle \omega^{\otimes n}, f^{(n)} \rangle$ in $L^2(\tau)$.

Clearly, the set $B_0(T_n)$ is dense in $L^2(T_n, m_n)$. Hence, by Theorem 2, the mapping

$$J^{(n)} \langle P^{(n)}(\omega), f^{(n)} \rangle := f^{(n)} \in L^2(T_n, m_n) \tag{9}$$

can be extended by continuity to a unitary operator $J^{(n)} : \mathbf{S}^{(n)} \to L^2(T_n, m_n)$. We can now define a unitary operator $J : L^2(\tau) \to \mathbb{F}$ whose restriction to each $\mathbf{S}^{(n)}$ is equal to $J^{(n)}$. (Here $J^{(0)}$ is the identity operator in $\mathbb{R}$.) Note that we get the following diagram of the unitary operators:

$$\mathcal{F} \xleftarrow{I} L^2(\tau) \xrightarrow{J} \mathbb{F}. \tag{10}$$

Again with an abuse of notations, we will denote by $\langle \omega, h \rangle$ the operator $J \langle \omega, h \rangle J^{-1}$ in $\mathbb{F}$.

Theorem 3. *For each $h \in B_0(\mathbb{R}_+)$, consider $\langle \omega, h \rangle$ as a continuous linear operator acting in $\mathbb{F}$. Then*

$$\langle \omega, h \rangle = A^+(h) + B^0(h) + B^-(h).$$

Here, $A^+(h)$ is a creation operator: $A^+(h)\Omega = h$ and for $f^{(n)} \in L^2(T_n, m_n)$, $A^+(h) f^{(n)} \in L^2(T_{n+1} m_{n+1})$,

$$\big(A^+(h) f^{(n)}\big)(t_1, \dots, t_{n+1}) = h(t_1) f^{(n)}(t_2, \dots, t_{n+1}),$$

$B^0(h)$ is a neutral operator: $B^0(h)\Omega = 0$, for each $f^{(n)} \in L^2(T_n, m_n)$ we have $B^0(h) f^{(n)} \in L^2(T_n, m_n)$ and for each $(l_1, \dots, l_i) \in M$, $l_1 + \dots + l_i + i = n$ and $(t_1, \dots, t_n) \in T^{(l_1, \dots, l_i)}$,

$$\big(B^0(h) f^{(n)}\big)(t_1, \dots, t_n) = b_{l_1} h(t_1) f^{(n)}(t_1, \dots, t_n),$$

and $B^-(h)$ is an annihilation operator: $B^-(h)\Omega = 0$, for $f^{(1)} \in L^2(T_1, m_1) = L^2(\mathbb{R}_+, dt)$

$$B^-(h) f^{(1)} = \int_{\mathbb{R}_+} h(u) f^{(1)}(u)\, du\, \Omega,$$

and for $n \geq 2$ and $f^{(n)} \in L^2(T^n, m_n)$, we have $B^-(h) f^{(n)} \in L^2(T_{n-1}, m_{n-1})$, and for each $(l_1, \dots, l_i) \in M$, $l_1 + \dots + l_i + i = n - 1$ and $(t_1, \dots, t_{n-1}) \in T^{(l_1, \dots, l_i)}$,

$$\begin{aligned} &\big(B^-(h) f^{(n)}\big)(t_1, \dots, t_{n-1}) \\ &\qquad = \int_{t_1}^{\infty} h(u) f^{(n)}(u, t_1, \dots, t_{n-1})\, du + a_{l_1+1} h(t_1) f^{(n)}(t_1, t_1, t_2, \dots, t_{n-1}). \end{aligned}$$

Let $C_0(\mathbb{R}_+^n)$ denote the space of all continuous functions on $\mathbb{R}_+^n$ with compact support. Obviously, $C_0(\mathbb{R}_+^n) \subset B_0(\mathbb{R}_+^n)$. If a polynomial $P = f^{(0)} + \sum_{i=1}^{n} \langle \omega^{\otimes i}, f^{(i)} \rangle \in \mathbf{P}$ is such that $f^{(i)} \in C_0(\mathbb{R}_+^n)$ for all $i = 1, \dots, n$, we call P a *a continuous polynomial of*

ω. We denote by $\mathbf{CP}$ the space of all continuous polynomials of ω, and by $\mathbf{CP}^{(n)}$ the space of all continuous polynomials of ω of order $\le n$. It follows immediately from the proof of Theorem 1 that $\mathbf{CP}^{(n)}$ is dense in $\overline{\mathbf{P}^{(n)}}$. Hence, $\mathbf{CP}$ is dense in $L^2(\tau)$.

Further we define the subspace $\mathbf{OCP}$ of $L^2(\tau)$ as the linear span of the identity operator and the orthogonal polynomials $\langle P^{(n)}(\omega), f^{(n)}\rangle$ with $f^{(n)} \in C_0(T_n)$ (the space of *orthogonal polynomials with continuous coefficients*). Clearly, $\mathbf{OCP}$ is dense in $L^2(\tau)$.

Thus, we have constructed the subspaces $\mathbf{CP}$ and $\mathbf{OCP}$ of $L^2(\tau)$. We will say that the orthogonal polynomials of monotone Lévy noise ω belong to the *Meixner class* if $\mathbf{CP} = \mathbf{OCP}$.

Theorem 4. *The orthogonal polynomials of monotone Lévy noise ω belong to the Meixner class if and only if there exist $\lambda \in \mathbb{R}$ and $\eta \ge 0$ such that, in formula (7), $b_k = \lambda$ for all $k \in \mathbb{N}_0$ and $a_k = \eta$ for all $k \in \mathbb{N}$. In the latter case, for each $h \in B_0(\mathbb{R}_+)$, the operator $\langle \omega, h\rangle$ in $\mathbb{F}$ has the following representation:*

$$\langle \omega, h\rangle = A^+(h) + \lambda A^0(h) + A_1^-(h) + \eta A_2^-(h). \tag{11}$$

Here, $A^+(h)$ is the creation operator defined in Theorem 3, $A^0(h)$ is a neutral operator: $A^0(h)\Omega = 0$, and for $f^{(n)} \in L^2(T_n, m_n)$, $A^0(h) f^{(n)} \in L^2(T_n, m_n)$,

$$\big(A^0(h) f^{(n)}\big)(t_1, \dots, t_n) = h(t_1) f^{(n)}(t_1, \dots, t_n),$$

$A_1^-(h)$ is an annihilation operator of the first kind: $A_1^-(h)\Omega = 0$ and for $f^{(n)} \in L^2(T_n, m_n)$, $A_1^-(h) f^{(n)} \in L^2(T_{n-1}, m_{n-1})$ (for $n = 1$ the latter space being $\mathbb{R}$),

$$\big(A_1^-(h) f^{(n)}\big)(t_1, \dots, t_{n-1}) = \int_{t_1}^{\infty} h(u) f^{(n)}(u, t_1, \dots, t_{n-1})\, du,$$

and $A_2^-(h)$ is an annihilation operator of the second kind: $A_2^-(h) = 0$ on $\mathbb{R} \oplus L^2(\mathbb{R}_+, dt)$, and for each $f^{(n)} \in L^2(T_n, m_n)$, $n \ge 2$, $A_2^-(h) \in L^2(T_{n-1}, m_{n-1})$,

$$\big(A_2^-(h) f^{(n)}\big)(t_1, \dots, t_{n-1}) = h(t_1) f^{(n)}(t_1, t_1, t_2, t_3, \dots, t_{n-1}).$$

Following [29], for functions $f^{(m)} : T_m \to \mathbb{R}$ and $g^{(n)} : T_n \to \mathbb{R}$, we define their *monotone tensor product* $f^{(m)} \rhd g^{(n)}$ as the function from T_{m+n} to $\mathbb{R}$ given by

$$\big(f^{(m)} \rhd g^{(n)}\big)(t_1, \dots, t_{m+n}) := f^{(m)}(t_1, \dots, t_m) g^{(n)}(t_{m+1}, \dots, t_{m+n}).$$

This operation is obviously associative.

Corollary 5. *Consider the system of orthogonal polynomials from the Mexiner class that corresponds to parameters $\lambda \in \mathbb{R}$ and $\eta \ge 0$. Then, for each $h, h_1, h_2 \in C_0(\mathbb{R}_+)$,*

$$\langle P^{(1)}(\omega), h\rangle = \langle \omega, h\rangle, \quad \langle P^{(2)}(\omega), h_1 \rhd h_2\rangle = \langle \omega^{\otimes 2}, h_1 \otimes h_2\rangle - \lambda \langle \omega, h_1 h_2\rangle - \int_{\mathbb{R}_+} h_1(u) h_2(u)\, du,$$

and for $n \geq 2$ and $h_1, \dots, h_n \in C_0(\mathbb{R}_+)$, the following recursion formula holds:

$$\begin{aligned}
&\langle P^{(n)}(\omega), h_1 \triangleright \cdots \triangleright h_n \rangle = \langle \omega, h_1 \rangle \langle P^{(n-1)}(\omega), h_2 \triangleright \cdots \triangleright h_n \rangle \\
&\quad - \lambda \langle P^{(n-1)}(\omega), (h_1 h_2) \triangleright h_3 \triangleright \cdots \triangleright h_n \rangle - \langle P^{(n-2)}(\omega), \mathcal{I}(h_1, h_2, h_3) \triangleright h_4 \triangleright \cdots \triangleright h_n \rangle \\
&\quad - \eta \langle P^{(n-2)}(\omega), (h_1 h_2 h_3) \triangleright h_4 \triangleright \cdots \triangleright h_n \rangle,
\end{aligned}$$

where the mapping $\mathcal{I} : C_0(\mathbb{R}_+)^3 \to C_0(\mathbb{R}_+)$ is given by

$$\big(\mathcal{I}(h_1, h_2, h_3)\big)(t) := \int_t^\infty h_1(u) h_2(u)\, du\, h_3(t). \tag{12}$$

Let us now show that formula (11) admits a formal interpretation as in formula (4). For each $t \in \mathbb{R}_+$, we set formally

$$\partial_t^\dagger := A^+(\delta_t), \quad \partial_t := A_1^-(\delta_t),$$

so that, for each $h \in B_0(\mathbb{R}_+)$,

$$A^+(h) = \int_{\mathbb{R}_+} h(t) \partial_t^\dagger \, dt, \quad A_1^-(h) = \int_{\mathbb{R}_+} h(t) \partial_t \, dt.$$

Thus, for $f^{(n)} \in L^2(T_n, m_n)$,

$$\begin{aligned}
\big(\partial_t^\dagger f^{(n)}\big)(t_1, \dots, t_{n+1}) &= \big(\delta_t \triangleright f^{(n)}\big)(t_1, \dots, t_{n+1}) \\
&= \chi_{[t_2,\infty)}(t) \delta_t(t_1) f^{(n)}(t_2, \dots, t_{n+1})
\end{aligned}$$

and

$$\big(\partial_t f^{(n)}\big)(t_1, \dots, t_{n-1}) = \chi_{[t_1,\infty)}(t) f^{(n)}(t, t_1, t_2, \dots, t_{n-1}).$$

We then calculate, for $h \in B_0(\mathbb{R}_+)$ and $(t_1, \dots, t_n) \in T_n$,

$$\begin{aligned}
&\left(\int_{\mathbb{R}_+} dt\, h(t) \partial_t^\dagger \partial_t f^{(n)} \right)(t_1, \dots, t_n) = \int_{\mathbb{R}_+} dt\, h(t) \big(\partial_t^\dagger \partial_t f^{(n)}\big)(t_1, \dots, t_n) \\
&\quad = \int_{\mathbb{R}_+} dt\, h(t) \chi_{[t_2,\infty)}(t) \delta_t(t_1) \big(\partial_t f^{(n)}\big)(t_2, \dots, t_n) \\
&\quad = \int_{\mathbb{R}_+} dt\, h(t) \chi_{[t_2,\infty)}^2(t) \delta_t(t_1) f^{(n)}(t, t_2, \dots, t_n) \\
&\quad = \int_{t_2}^\infty dt\, h(t) \delta_t(t_1) f^{(n)}(t, t_2, \dots, t_n) \\
&\quad = h(t_1) f^{(n)}(t_1, t_2, \dots, t_n),
\end{aligned}$$

since $t_1 \geq t_2$. Thus,

$$\int_{\mathbb{R}_+} dt\, h(t) \partial_t^\dagger \partial_t = A^0(h).$$

Analogously, it can be shown that

$$\int_{\mathbb{R}_+} dt\, h(t)\partial_t^\dagger\partial_t\partial_t = A_2^-(h).$$

Thus, we can formally write formula (11) in the form

$$\langle\omega, h\rangle = \int_{\mathbb{R}_+} h(t)\big(\partial_t^\dagger + \lambda\partial_t^\dagger\partial_t + \partial_t + \eta\partial_t^\dagger\partial_t\partial_t\big)\, dt. \tag{13}$$

If we also formally write

$$\langle\omega, h\rangle = \int_{\mathbb{R}_+} \omega(t)h(t)\, dt, \tag{14}$$

then formulas (13) and (14) imply (4).

4 Proofs

Some parts of the proofs below take their ideas from the case of the free Meixner orthogonal polynomials [11]. For the reader's convenience, we will still present self-contained proofs of the results from Section 3.

Below, for open intervals $\Delta_1, \Delta_2 \subset \mathbb{R}_+$, we write $\Delta_1 > \Delta_2$ if for any $t_1 \in \Delta_1$ and $t_2 \in \Delta_2$, we have $t_1 > t_2$. This particularly implies that $\Delta_1 \cap \Delta_2 = \varnothing$.

In the lemma below, c. l. s. stands for the closed linear span.

Lemma 6. *For $n \in \mathbb{N}$, we define closed subspaces $\mathcal{X}^{(n)}$, $\mathcal{Y}^{(n)}$, and $\mathcal{Z}^{(n)}$ of $\mathcal{F}$ by*

$$\begin{aligned}
&\mathcal{X}^{(n)} := \text{c. l. s.}\big\{\Omega,\ \langle\omega, h_1\rangle\cdots\langle\omega, h_i\rangle\Omega \mid h_1,\dots,h_i \in B_0(\mathbb{R}_+),\ i \in \{1,\dots,n\}\big\},\\
&\mathcal{Y}^{(n)} := \text{c. l. s.}\big\{\Omega,\ (\chi_{\Delta_1}\otimes s^{l_1}) \rhd \cdots \rhd (\chi_{\Delta_i}\otimes s^{l_i}) \mid (l_1,\dots,l_i) \in M,\\
&\quad l_1+\cdots+l_i+i \le n,\ \Delta_1,\dots,\Delta_i \subset \mathbb{R}_+ \textit{ are open intervals, } \Delta_1 > \Delta_2 > \cdots > \Delta_i\big\},\\
&\mathcal{Z}^{(n)} := \text{l. s.}\big\{\Omega,\ f^{(i)}(t_1,\dots,t_i)q_{l_1}(s_1)\cdots q_{l_i}(s_i) \mid (l_1,\dots,l_i) \in M,\ l_1+\cdots+l_i+i \le n,\\
&\quad f^{(i)} \in L^2(T_i, dt_1\cdots dt_i),\ \textit{each } q_{l_j} \textit{ is a polynomial on } \mathbb{R} \textit{ of order } l_j\big\}.
\end{aligned}$$

Then $\mathcal{X}^{(n)} = \mathcal{Y}^{(n)} = \mathcal{Z}^{(n)}$.

Proof. By definition, $\mathcal{Y}^{(n)} \subset \mathcal{Z}^{(n)}$. Since the Lebesgue measure is non-atomic, it can be easily shown that

$$\begin{aligned}
\mathcal{Y}^{(n)} = \text{l. s.}\big\{&\Omega,\ f^{(i)}(t_1,\dots,t_i)s_1^{l_1}\cdots s_i^{l_i} \mid (l_1,\dots,l_i) \in M,\ l_1+\cdots+l_i+i \le n,\\
&f^{(i)} \in L^2(T_i, dt_1\cdots dt_i)\big\}.
\end{aligned}$$

Hence, $\mathcal{Y}^{(n)} = \mathcal{Z}^{(n)}$. Using the definition of the operator $\langle\omega, h\rangle$ in $\mathcal{F}$, one can easily show by induction on n that $\mathcal{X}^{(n)} \subset \mathcal{Z}^{(n)}$. Thus, to prove the lemma, it suffices to prove the inclusion $\mathcal{Y}^{(n)} \subset \mathcal{X}^{(n)}$. We prove this by induction on n. The statement is obviously true for $n = 1$. Assume that it is true for up to n, and let us prove it for $n+1$. Let $(l_1, \dots, l_i) \in M$, $l_1 + \dots + l_i + i = n+1$. If $l_1 = 0$, we get (using the obvious notations):

$$\begin{aligned}
&\chi_{\Delta_1} \rhd (\chi_{\Delta_2} \otimes s^{l_2}) \rhd \cdots \rhd (\chi_{\Delta_i} \otimes s^{l_i}) = a^+(\chi_{\Delta_1})(\chi_{\Delta_2} \otimes s^{l_2}) \rhd \cdots \rhd (\chi_{\Delta_i} \otimes s^{l_i}) \\
&\quad = \langle \omega, \chi_{\Delta_1} \rangle (\chi_{\Delta_2} \otimes s^{l_2}) \rhd \cdots \rhd (\chi_{\Delta_i} \otimes s^{l_i}) \in \mathcal{X}^{(n+1)}.
\end{aligned} \tag{15}$$

If $l_1 \geq 1$,

$$\begin{aligned}
&(\chi_{\Delta_1} \otimes s^{l_1}) \rhd \cdots \rhd (\chi_{\Delta_i} \otimes s^{l_i}) \\
&\quad = a^0(\chi_{\Delta_1})(\chi_{\Delta_1} \otimes s^{l_1 - 1}) \rhd (\chi_{\Delta_2} \otimes s^{l_2}) \rhd \cdots \rhd (\chi_{\Delta_i} \otimes s^{l_i}) \\
&\quad = \langle \omega, \chi_{\Delta_1} \rangle (\chi_{\Delta_1} \otimes s^{l_1 - 1}) \rhd (\chi_{\Delta_2} \otimes s^{l_2}) \rhd \cdots \rhd (\chi_{\Delta_i} \otimes s^{l_i}) \\
&\qquad - \big(a^+(\chi_{\Delta_1}) + a^-(\chi_{\Delta_1})\big)(\chi_{\Delta_1} \otimes s^{l_1 - 1}) \rhd (\chi_{\Delta_2} \otimes s^{l_2}) \rhd \cdots \rhd (\chi_{\Delta_i} \otimes s^{l_i}).
\end{aligned} \tag{16}$$

We get

$$\begin{aligned}
&a^+(\chi_{\Delta_1})(\chi_{\Delta_1} \otimes s^{l_1 - 1}) \rhd (\chi_{\Delta_2} \otimes s^{l_2}) \rhd \cdots \rhd (\chi_{\Delta_i} \otimes s^{l_i}) \\
&\quad = \chi_{\Delta_1} \rhd (\chi_{\Delta_1} \otimes s^{l_1 - 1}) \rhd (\chi_{\Delta_2} \otimes s^{l_2}) \rhd \cdots \rhd (\chi_{\Delta_i} \otimes s^{l_i}).
\end{aligned} \tag{17}$$

It follows from (15) by approximation that the vector on the right hand side of (17) belongs to $\mathcal{X}^{(n+1)}$. Furthermore,

$$\begin{aligned}
&a^-(\chi_{\Delta_1})(\chi_{\Delta_1} \otimes s^{l_1 - 1}) \rhd (\chi_{\Delta_2} \otimes s^{l_2}) \rhd \cdots \rhd (\chi_{\Delta_i} \otimes s^{l_i}) \\
&\quad = \int_{\mathbb{R}} v^{l_1 - 1}\, d\nu(v) (g \otimes s^{l_2}) \rhd (\chi_{\Delta_3} \otimes s^{l_3}) \rhd \cdots \rhd (\chi_{\Delta_i} \otimes s^{l_i}),
\end{aligned} \tag{18}$$

where

$$g(t) := \chi_{\Delta_2}(t) \int_{\Delta_1 \cap (t, \infty)} du = \chi_{\Delta_2}(t) \int_{\Delta_1} du,$$

since $\Delta_2 > \Delta_1$. Hence, the vector on the right hand side of (18) belongs to $\mathcal{X}^{(n-1)}$. Therefore, the vector on the right nand side of (16) belongs to $\mathcal{X}^{(n+1)}$. $\square$

Proof of Theorem 1. Since the probability measure ν on $\mathbb{R}$ has compact support, the set of polynomials on $\mathbb{R}$ is dense in $L^2(\mathbb{R}, \nu)$. Therefore, the set $\bigcup_{n=1}^{\infty} \mathcal{Z}^{(n)}$ is dense in $\mathcal{F}$. Hence, by Lemma 6, the set $\bigcup_{n=1}^{\infty} \mathcal{X}^{(n)}$ is dense in $\mathcal{F}$, which implies the theorem. $\square$

Lemma 7. *For $(l_1, \dots, l_i) \in M$, let $\mathcal{H}_{l_1, \dots, l_i}$ denote the following subspace of $\mathcal{F}$:*

$$\mathcal{H}_{l_1, \dots, l_i} = \big\{ f^{(i)}(t_1, \dots, t_i) p_{l_1}(s_1) \cdots p_{l_i}(s_i) \mid f^{(i)} \in L^2(T_i, dt_1 \cdots dt_i) \big\}.$$

Then

$$\mathcal{F} = \mathcal{F}^{(0)} \oplus \bigoplus_{(l_1, \dots, l_i) \in M} \mathcal{H}_{l_1, \dots, l_i}. \tag{19}$$

Proof. The statement that the subspaces $\mathcal{H}_{l_1,\dots,l_i}$ are orthogonal to each other follows from the definition of the scalar product in $\mathcal{F}$ and the fact that the polynomials $(p_k)_{k=0}^{\infty}$ are orthogonal in $L^2(\mathbb{R},\nu)$. Since the polynomials are dense in $L^2(\mathbb{R},\nu)$, the statement follows. $\square$

We denote $\mathcal{H}^{(0)} := \mathcal{F}^{(0)}$, and for $n\in\mathbb{N}$ we denote

$$\mathcal{H}^{(n)} := \bigoplus_{\substack{(l_1,\dots,l_i)\in M\\ l_1+\dots+l_i+i=n}} \mathcal{H}_{l_1,\dots,l_i}. \tag{20}$$

Using Lemma 7, we get

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}. \tag{21}$$

Lemma 8. *For each $n\in\mathbb{N}_0$, we have $I\big(\mathbf{OP}^{(n)}\big) = \mathcal{H}^{(n)}$.*

Proof. In view of (21), the statement of the lemma is equivalent to the statement

$$I\left(\overline{\mathbf{P}^{(n)}}\right) = \bigoplus_{i=0}^{n} \mathcal{H}^{(i)}. \tag{22}$$

By (20) and Lemma 6,

$$\bigoplus_{i=0}^{n} \mathcal{H}^{(i)} = \mathcal{F}^{(0)} \oplus \bigoplus_{\substack{(l_1,\dots,l_i)\in M\\ l_1+\dots+l_i+i\le n}} \mathcal{H}_{l_1,\dots,l_i} = \mathcal{Z}^{(n)} = \mathcal{X}^{(n)},$$

which implies (22). $\square$

For each $h\in B_0(\mathbb{R}_+)$, we will now represent the neutral operator $a^0(h)$ as a sum of three operators. To this end, we define bounded linear operators $a^{0+}(h)$, $a^{00}(h)$, and $a^{0-}(h)$ in $\mathcal{F}$ by

$$a^{0+}(h)\Omega = a^{00}(h)\Omega = a^{0-}(h)\Omega = 0$$

and for any $(l_1,\dots,l_i)\in M$ and $f^{(i)}\in L^2(T_i, dt_1\cdots dt_i)$,

$$a^{0+}(h)f^{(i)}(t_1,\dots,t_i)p_{l_1}(s_1)\cdots p_{l_i}(s_i) = h(t_1)f^{(i)}(t_1,\dots,t_i)p_{l_1+1}(s_1)p_{l_2}(s_2)\cdots p_{l_i}(s_i),$$
$$a^{00}(h)f^{(i)}(t_1,\dots,t_i)p_{l_1}(s_1)\cdots p_{l_i}(s_i) = h(t_1)f^{(i)}(t_1,\dots,t_i)b_{l_1}p_{l_1}(s_1)\cdots p_{l_i}(s_i),$$
$$a^{0-}(h)f^{(i)}(t_1,\dots,t_i)p_{l_1}(s_1)\cdots p_{l_i}(s_i) = h(t_1)f^{(i)}(t_1,\dots,t_i)a_{l_1}p_{l_1-1}(s_1)p_{l_2}(s_2)\cdots p_{l_i}(s_i).$$

In view of (7), we therefore get

$$a^0(h) = a^{0+}(h) + a^{00}(h) + a^{0-}(h). \tag{23}$$

Lemma 9. *For any $h_1, \dots, h_n \in B_0(\mathbb{R}_+)$, we have*

$$I\langle P^{(n)}(\omega), h_1 \otimes \cdots \otimes h_n\rangle = (a^+(h_1) + a^{+0}(h_1)) \cdots (a^+(h_{n-1}) + a^{+0}(h_{n-1}))a^+(h_n)\Omega.$$

Proof. Recall that $\langle P^{(n)}(\omega), h_1 \otimes \cdots \otimes h_n\rangle$ is the orthogonal projection in $L^2(\tau)$ of the monomial $\langle \omega^{\otimes n}, h_1 \otimes \cdots \otimes h_n\rangle = \langle \omega, h_1\rangle \cdots \langle \omega, h_n\rangle$ onto $\mathbf{OP}^{(n)}$. Hence, by Lemma 8, $I\langle P^{(n)}(\omega), h_1 \otimes \cdots \otimes h_n\rangle$, is the orthogonal projection in $\mathcal{F}$ of the vector $\langle \omega, h_1\rangle \cdots \langle \omega, h_n\rangle\Omega$ onto $\mathcal{H}^{(n)}$. By (23), for each $h \in B_0(\mathbb{R}_+)$,

$$\langle \omega, h\rangle = a^+(h) + a^{0+}(h) + a^{00}(h) + a^{0-}(h) + a^-(h). \tag{24}$$

From here the statement easily follows. □

Lemma 10. *For any $h_1, \dots, h_n \in B_0(\mathbb{R}_+)$, we have*

$$I\langle P^{(n)}(\omega), h_1 \otimes \cdots \otimes h_n\rangle = \sum_{\substack{(l_1,\dots,l_i)\in M\\ l_1+\cdots+l_i+i=n}} \big((h_1 \cdots h_{l_1+1}) \otimes p_{l_1}\big) \rhd \big((h_{l_1+2} \cdots h_{l_1+l_2+2}) \otimes p_{l_2}\big)$$
$$\rhd \cdots \rhd \big((h_{l_1+l_2+\cdots+l_{i-1}+i} \cdots h_n) \otimes p_{l_i}\big).$$

Proof. By Lemma 9,

$$I\langle P^{(n)}(\omega), h_1 \otimes \cdots \otimes h_n\rangle = \sum_{\substack{(l_1,\dots,l_i)\in M\\ l_1+\cdots+l_i+i=n}} a^{0+}(h_1) \cdots a^{0+}(h_{l_1})a^+(h_{l_1+1})$$
$$\times\, a^{0+}(h_{l_1+2}) \cdots a^{0+}(h_{l_1+l_2+1})a^+(h_{l_1+l_2+2}) \cdots a^{0+}(h_{l_1+l_2+\cdots+l_{i-1}+i}) \cdots a^{0+}(h_{n-1})a^+(h_n)\Omega.$$

From here the statement follows. □

Proof of Theorem 2. It suffices to show that, for any $h_1, \dots, h_n \in B_0(\mathbb{R}_+)$,

$$\|\langle P^{(n)}(\omega), h_1 \otimes \cdots \otimes h_n\rangle\|^2_{L^2(\tau)} = \|h_1 \rhd \cdots \rhd h_n\|^2_{L^2(T_n, m_n)},$$

or equivalently

$$\|I\langle P^{(n)}(\omega), h_1 \otimes \cdots \otimes h_n\rangle\|^2_{\mathcal{F}} = \|h_1 \rhd \cdots \rhd h_n\|^2_{L^2(T_n, m_n)}.$$

But the latter formula follows immediately from Lemma 10 and the construction of the measure m_n. □

Recall the diagram (10). We define a unitary operator $U : \mathcal{F} \to \mathbb{F}$ by $U := JI^{-1}$. We will now present en explicit form of the action of U. To this end, we recall the orthogonal decomposition (19) of $\mathcal{F}$.

Corollary 11. *Let $(l_1,\dots,l_i)\in M$ and let $f^{(i)}\in L^2(T_i,dt_1\cdots dt_i)$. Denote*

$$F(t_1,s_1,\dots,t_i,s_i)=f^{(i)}(t_1,\dots,t_i)p_{l_1}(s_1)\cdots p_{l_i}(s_i)\in\mathcal{H}_{l_1,\dots,l_i}.$$

Let $n=l_1+\dots+l_i+i$. Define a function $f^{(n)}:T_n\to\mathbb{R}$ by

$$f^{(n)}(\underbrace{t_1,\dots,t_1}_{l_1+1\ \text{times}},\underbrace{t_2,\dots,t_2}_{l_2+1\ \text{times}},\dots,\underbrace{t_i,\dots,t_i}_{l_i+1\ \text{times}}):=f^{(i)}(t_1,\dots,t_i)\quad \text{if } t_1>t_2>\dots>t_i\ge 0,$$

and $f^{(n)}(t_1,\dots,t_n)=0$ otherwise. Then $UF=f^{(n)}$. Furthermore, $U\Omega=\Omega$.

Proof. The statement $U\Omega=\Omega$ is trivial. To prove that $UF=f^{(n)}$, it is sufficient to consider the case where $f^{(i)}\in B_0(T_i)$. Then, the function $f^{(n)}$ defined in Corollary 11 belongs to $B_0(T_n)$. It follows from Lemma 10 by approximation that

$$I\langle P^{(n)}(\omega),f^{(n)}\rangle=f^{(i)}(t_1,\dots,t_i)p_{l_1}(s_1)p_{l_2}(s_2)\cdots p_{l_i}(s_i)=F.$$

Thus, $I^{-1}F=\langle P^{(n)}(\omega),f^{(n)}\rangle$. Hence, by (9), $UF=JI^{-1}F=f^{(n)}$. □

Proof of Theorem 3. Recall formula (24). Denote, for $h\in B_0(\mathbb{R}_+)$,

$$\alpha^+(h):=a^+(h)+a^{0+}(h),\quad \alpha^0(h):=a^{00}(h),\quad \alpha^-(h):=a^{0-}(h)+a^-(h),$$

so that

$$\langle\omega,h\rangle=\alpha^+(h)+\alpha^0(h)+\alpha^-(h).$$

Recall formula (20). It is easy to see that $\alpha^+(h)$ maps $\mathcal{H}^{(n)}$ into $\mathcal{H}^{(n+1)}$, $\alpha^0(h)$ maps $\mathcal{H}^{(n)}$ into itself, and $\alpha^-(h)$ maps $\mathcal{H}^{(n)}$ into $\mathcal{H}^{(n-1)}$. Furthermore, by using Corollary 11, one easily shows that

$$U\alpha^+(h)U^{-1}=A^+(h),\quad U\alpha^0(h)U^{-1}=B^0(h),\quad U\alpha^-(h)U^{-1}=B^-(h).$$

□

Lemma 12. *Assume that, in formula (7), $b_k=\lambda$ for all $k\in\mathbb{N}_0$ and some $\lambda\in\mathbb{R}$ and $a_k=\eta$ for all $k\in\mathbb{N}$ for some $\eta\ge 0$. Then formula (11) holds.*

Proof. Immediate from Theorem 3. □

For $n\in\mathbb{N}$ and $i=0,1,\dots,n$, consider a continuous linear operator $R_{i,n}:B_0(\mathbb{R}_+^n)\to B_0(\mathbb{R}_+^i)$. (For $i=0$, we set $B_0(\mathbb{R}_+^i):=\mathbb{R}$.) Then we define a mapping $\mathbf{1}\otimes R_{i,n}:B_0(\mathbb{R}_+^{n+1})\to B_0(\mathbb{R}_+^{i+1})$ by

$$\big(\mathbf{1}\otimes R_{i,n}f^{(n+1)}\big)(t_1,\dots,t_{n+1})=\big(R_{i,n}f^{(n+1)}(t_1,\cdot)\big)(t_2,\dots,t_{n+1}).$$

As easily seen, the mapping $\mathbf{1}\otimes R_{i,n}$ is continuous and linear.

We define mappings $D_{n-1,n} : B_0(\mathbb{R}_+^n) \to B_0(\mathbb{R}_+^{n-1})$, $D_{n-2,n} : B_0(\mathbb{R}_+^n) \to B_0(\mathbb{R}_+^{n-2})$, and $\mathcal{I}_{n-2,n} : B_0(\mathbb{R}_+^n) \to B_0(\mathbb{R}_+^{n-2})$ by

$$\begin{aligned}
\big(D_{n-1,n}f^{(n)}\big(t_1,\dots,t_{n-1}) &:= f^{(n)}(t_1,t_1,t_2,\dots,t_{n-1}),\\
\big(D_{n-2,n}f^{(n)}\big(t_1,\dots,t_{n-2}) &:= f^{(n)}(t_1,t_1,t_1,t_2,\dots,t_{n-2}),\\
\big(\mathcal{I}_{n-2,n}f^{(n)}\big)(t_1,\dots,t_{n-2}) &:= \int_{t_1}^{\infty} f^{(n)}(u,u,t_1,t_2,\dots,t_{n-2})\,du.
\end{aligned}$$

(For $n=2$, $\mathcal{I}_{0,2}f^{(2)} := \int_{\mathbb{R}_+} f^{(2)}(u,u)\,du$.)

Let $\lambda \in \mathbb{R}$ and $\eta \geq 0$ be fixed. For $n \in \mathbb{N}$ and $i \in \mathbb{N}_0$, $i \leq n$, we define continuous linear operators $R_{i,n} : B_0(\mathbb{R}_+^n) \to B_0(\mathbb{R}_+^i)$ by the following recursion formulas:

$$\begin{aligned}
&R_{1,1} = \mathbf{1}, \quad R_{0,1} = \mathbf{0},\\
&R_{2,2} = \mathbf{1}, \quad R_{1,2} = -\lambda D_{1,2}, \quad R_{0,2} = -\mathcal{I}_{0,2},\\
&R_{i,n} = \mathbf{1} \otimes R_{i-1,n-1} - \lambda R_{i,n-1}D_{n-1,n} - R_{i,n-2}\mathcal{I}_{n-2,n} - \eta R_{i,n-2}D_{n-2,n}, \quad n \geq 2. \qquad (25)
\end{aligned}$$

(In the above formula, we assume that $R_{i,n} = 0$ if $i > n$.) Note hat $R_{n,n} = \mathbf{1}$ for all $n \in \mathbb{N}$.

For each $f^{(n)} \in B_0(\mathbb{R}_+^n)$, we define

$$\langle R^{(n)}(\omega), f^{(n)} \rangle := \sum_{i=0}^{n} \langle \omega^{\otimes i}, R_{i,n}f^{(n)} \rangle. \qquad (26)$$

Lemma 13. *Assume that the condition of Lemma 12 is satisfied. Then, for each $f^{(n)} \in B_0(\mathbb{R}_+^n)$,*

$$\langle R^{(n)}(\omega), f^{(n)} \rangle = \langle P^{(n)}(\omega), f^{(n)} \rangle, \qquad (27)$$

the equality in $L^2(\tau)$.

Remark 14. Since $\langle P^{(n)}(\omega), f^{(n)} \rangle$ depends only on the restriction of the function $f^{(n)}$ to T_n, formula (27) means that the element of $L^2(\tau)$ given by formula (26) also depends only on the restriction of the function $f^{(n)}$ to T_n. However, this statement is not true about each individual term of the sum on the right hand side of (26). Take, for example, the term corresponding to $i = n$, i.e., $\langle \omega^{\otimes n}, f^{(n)} \rangle$. As easily seen, for $n \geq 4$, this monomial does depend on the values of the function $f^{(n)}$ outside T_n.

Remark 15. It follows from Lemma 13 and formula (26) that, for each $f^{(n)} \in B_0(\mathbb{R}_+^n)$

$$\langle \omega^{\otimes n}, f^{(n)} \rangle = \langle P^{(n)}(\omega), f^{(n)} \rangle - \sum_{i=0}^{n-1} \langle \omega^{\otimes i}, R_{i,n}f^{(i)} \rangle.$$

Assume that $f^{(n)} = 0$ m_n-a.e. on T_n. Then $\langle P^{(n)}(\omega), f^{(n)} \rangle = 0$, so that

$$\langle \omega^{\otimes n}, f^{(n)} \rangle = -\sum_{i=0}^{n-1} \langle \omega^{\otimes i}, R_{i,n}f^{(i)} \rangle \in \mathbf{P}^{(n-1)}.$$

Note that the above monomial $\langle\omega^{\otimes n}, f^{(n)}\rangle$ is not necessarily equal to 0 as an element of $L^2(\tau)$.

Proof of Lemma 13. As easily seen, it suffices to prove formula (27) in the case where $f^{(n)} = h_1\otimes\cdots\otimes h_n$ for some $h_1, \dots, h_n \in B_0(\mathbb{R}_+)$. In this case, formula (27) is obviously true for $n = 1, 2$. Furthermore, it follows from the definition of $\langle R^{(n)}(\omega), f^{(n)}\rangle$ that the following recursion relation holds, for $n \geq 3$,

$$\begin{aligned}
&\langle R^{(n)}(\omega), h_1 \otimes \cdots \otimes h_n\rangle = \langle \omega, h_1\rangle\langle P^{(n-1)}(\omega), h_2 \otimes \cdots \otimes h_n\rangle \\
&\quad - \lambda\langle R^{(n-1)}(\omega), (h_1h_2) \otimes h_3 \otimes \cdots \otimes h_n\rangle - \langle R^{(n-2)}(\omega), \mathcal{I}_{3,1}(h_1, h_2, h_3) \otimes h_4 \otimes \cdots \otimes h_n\rangle \\
&\quad - \eta\langle R^{(n-2)}(\omega), (h_1h_2h_3) \otimes h_4 \otimes \cdots \otimes h_n\rangle, \qquad (28)
\end{aligned}$$

where the mapping $\mathcal{I} : B_0(\mathbb{R}_+)^3 \to B_0(\mathbb{R}_+)$ is given by formula (12). From here and Lemma 12 the statement follows by induction on n. □

Lemma 16. *Assume that the condition of Lemma 12 is satisfied. Then* **CP** = **OCP***, i.e., the corresponding orthogonal polynomials belong to the Meixner class.*

Proof. For each $n \in \mathbb{N}$, we consider the topology on $C_0(\mathbb{R}_+^n)$ that is induced by the topology on $B_0(\mathbb{R}_+)$. Thus, a sequence $(h_n)_{n=1}^\infty$ converges to a function h in $C_0(\mathbb{R}_+)$ if all functions h_n vanish outside a compact set in $\mathbb{R}_+$ and $\sup_{t\in\mathbb{R}_+} |h_n(t) - h(t)| \to 0$ as $n \to \infty$. For $n = 0$, we will also set $C_0(\mathbb{R}_+^n) := \mathbb{R}$.

Let $n \in \mathbb{N}$ and $i = 0, 1, \dots, n$. If $R_{i,n} : C_0(\mathbb{R}_+^n) \to C_0(\mathbb{R}_+^i)$ is a continuous linear operator, then so is the mapping $\mathbf{1} \otimes R_{i,n} : C_0(\mathbb{R}_+^{n+1}) \to C_0(\mathbb{R}_+^{i+1})$. Hence, we easily conclude that the (restrictions of the) operators $R_{i,n}$ defined by formula (25) are continuous linear operators acting from $C_0(\mathbb{R}_+^n)$ to $C_0(\mathbb{R}_+^i)$, respectively. Therefore, by (26), for each $f^{(n)} \in C_0(\mathbb{R}_+^n)$, $\langle R^{(n)}(\omega), f^{(n)}\rangle \in$ **CP**. Now, Lemma 13 implies that $\langle P^{(n)}(\omega), f^{(n)}\rangle \in$ **CP**.

Let $g^{(n)} \in C_0(T_n)$. Choose any $f^{(n)} \in C_0(\mathbb{R}_+^n)$ such that the restriction of $f^{(n)}$ to T_n is equal to $g^{(n)}$. Then, by the proved above

$$\langle P^{(n)}(\omega), g^{(n)}\rangle = \langle P^{(n)}(\omega), f^{(n)}\rangle \in \mathbf{CP},$$

hence **OCP** $\subset$ **CP**.

Let us now prove the inverse inclusion. Since $R_{n,n} = \mathbf{1}$ for each n, it easily follows from (26) and Lemma 13 by induction on n that, for each $f^{(n)} \in C_0(\mathbb{R}_+^n)$,

$$\langle\omega^{\otimes n}, f^{(n)}\rangle = \sum_{i=0}^{n} \langle P^{(i)}(\omega), L_{i,n}f^{(n)}\rangle,$$

where $L_{i,n} : C_0(\mathbb{R}_+^n) \to C_0(\mathbb{R}_+^i)$ are continuous linear operators. Let $g^{(i)} \in C_0(T_i)$ denote the restriction of the function $L_{i,n}f^{(n)}$ to T_i. Then

$$\langle\omega^{\otimes n}, f^{(n)}\rangle = \sum_{i=0}^{n} \langle P^{(i)}(\omega), g^{(i)}\rangle.$$

Hence, **CP** $\subset$ **OCP**. $\square$

Proof of Theorem 4. By Lemmas 12 and 16, it remains to prove that, if **CP** = **OCP**, then the condition of Lemma 12 is satisfied.

So we assume **CP** = **OCP**. Let $f^{(n)} \in C_0(T_n)$. Then $\langle P^{(n)}(\omega), f^{(n)}\rangle \in$ **CP**. Since **CP** is an algebra under multiplication of two continuous polynomials, we conclude that, for each $h \in C_0(\mathbb{R}_+)$, $\langle \omega, h\rangle\langle P^{(n)}(\omega), f^{(n)}\rangle \in$ **OCP**. Hence, by Theorem 3, there exist continuous functions $g^{(n)} \in C_0(T_n)$ and $g^{(n-1)} \in C_0(T_{n-1})$ such that

$$B^-(h)f^{(n)} = g^{(n-1)}\ m_{n-1}\text{-a.e.} \tag{29}$$

$$B^0(h)f^{(n)} = g^{(n)}\ m_n\text{-a.e.} \tag{30}$$

If $a_k = 0$ for all $k \in \mathbb{N}$, we set $\eta = 0$ and $\lambda = b_0$, and the condition of Lemma 12 is satisfied. So, we only have to consider the case where $a_1 > 0$. Set $\eta = a_1$. Using the construction of the measure m_n, the definition of the operator $B^-(h)$ and formula (29), we get by induction on k that $a_k = \eta$ for all $k \in \mathbb{N}$. But this also implies that $c_k > 0$ for all $k \in \mathbb{N}$. Now set $\lambda = b_0$. Using the definition of $B^0(h)$ and (30), we deduce from (30) by induction on k that $b_k = \lambda$ for all $k \in \mathbb{N}_0$. $\square$

Proof of Corollary 5. Immediate from Lemma 13 and formula (28). $\square$

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