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Strong convergence of a tamed theta scheme for NSDDEs with one-sided Lipschitz drift

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Abstract

This paper is concerned with strong convergence of a tamed theta scheme for neutral stochastic differential delay equations with one-sided Lipschitz drift. Strong convergence rate is revealed under a global one-sided Lipschitz condition, while for a local one-sided Lipschitz condition, the tamed theta scheme is modified to ensure the well-posedness of implicit numerical schemes, then we show the convergence of the numerical solutions.

MSC 2010: 65C30, 65L20
Key Words: tamed theta scheme; neutral stochastic differential delay equations; one-sided Lipschitz; strong convergence

1 Introduction

Numerical analysis plays an important role in studying stochastic differential equations (SDEs) because most equations can not be solved explicitly. The most commonly used method for approximating SDEs is the explicit Euler-Maruyama (EM) method. There are a lot of literature concerning with the explicit EM scheme for all kinds of SDEs, e.g., Hairer et

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al. [1], Maruyama [9], Milstein [10], and Kloeden and Platen [6]. Most of the early works on explicit EM scheme were about the SDEs with the globally Lipschitz continuous coefficients, since the explicit EM scheme solutions may not converge in the strong sense to the exact solutions with one-sided Lipschitz continuous and superlinearly growing drift coefficients. Moreover, Hutzenthaler et al. [3] pointed out that the absolute moments of the EM scheme at a finite time could diverge to infinity. In order to cope with these difficulties, Higham et.al [2] studied a split-step backward Euler method for nonlinear SDEs, they showed that the implicit EM scheme converged if the drift coefficient satisfied a one-sided Lipschitz condition and the diffusion coefficient was globally Lipschitz. Hutzenthaler et al. [4] proposed a tamed EM scheme in which the drift term is modified to guarantee the boundness of moments. Later, Sabanis [11, 12] studied the strong convergence of the tamed EM scheme and extend the tamed EM scheme to SDEs with superlinearly growing drift and diffusion coefficients, respectively. Zong et al. [16] proposed a semi-tamed Euler scheme to solve the SDEs with the drift coefficient equipped with the Lipschitz continuous part and non-Lipschitz continuous part. Although additional computational effort is needed for implicit analysis, the implicit EM schemes have been showed better than the explicit EM scheme which converges strongly to the exact solution of SDEs under non-globally Lipschitz conditions. The implicit EM methods including the backward EM scheme, the split-step backward EM scheme and the theta scheme have been extensively studied, for example, Mao and Szpruch [8] studied strong convergence and almost sure stability of the backward EM scheme and the theta scheme to SDEs with non-linear and non-Lipschitzian coefficients, to name a few.

Recently, numerical analysis for neutral stochastic differential delay equations (NSDDEs) has also received a great deal of attention, see e.g., Lan and Yuan [7], Wu and Mao [13], Zhou [15], Zong et al. [17], Zong and Wu [18], and the references therein. However, the existing literature are difficult to deal with one-sided Lipschitz and superlinearly drift. To fill the gap, in this paper, we are going to introduce a tamed theta scheme and discuss the strong convergence of this scheme for NSDDEs in which the drift coefficients are one-sided Lipschitz and superlinearly.

The content of our paper is organized as follows. In section 2, we consider NSDDEs with global one-sided Lipschitz drift, the tamed theta scheme is introduced and strong convergence is investigated. We reveal that the tamed theta solution converges to the exact solution with order $\alpha$ (see (B1) below)under the global one-sided Lipschitz and the superlinearly growth condition. In section 3, the global one-sided Lipschitz drift is replaced by the local one-sided Lipschitz drift, under which we show the convergence of the numerical solutions. In order to guarantee the well-posedness of the implicit tamed scheme, we impose a modified tamed theta scheme with a truncated skill.

## 2 Global One-sided Lipschitz Drift

For a fixed positive integer $n$, let $(\mathbb{R}^n, \langle \cdot, \cdot \rangle, |\cdot|)$ be an $n$-dimensional Euclidean space. Denote $\mathbb{R}^n \otimes \mathbb{R}^d$ by the set of all $n \times d$ matrices endowed with Hilbert-Schmidt norm $||A|| := \sqrt{\text{trace}(A^*A)}$ for every $A \in \mathbb{R}^n \otimes \mathbb{R}^d$, in which $A^*$ is the transpose of $A$. For a fixed $\tau \in (0, \infty)$,
which will be referred to as the delay or memory, let \( C = C([-\tau, 0]; \mathbb{R}^n) \) be all continuous functions from \([-\tau, 0]\) to \(\mathbb{R}^n\), equipped with the uniform norm \(\|\xi\|_\infty := \sup_{-\tau \leq \theta \leq 0} |\xi(\theta)|\) for every \(\xi \in C\). By a filtered probability space, we mean a quadruple \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\), where \(\mathcal{F}\) is a \(\sigma\)-algebra on the outcome space \(\Omega\), \(\mathbb{P}\) is a probability measure on the measurable space \((\Omega, \mathcal{F})\), and \(\{\mathcal{F}_t\}_{t \geq 0}\) is a filtration of sub-\(\sigma\)-algebra of \(\mathcal{F}\), where the usual conditions are satisfied, i.e., \((\Omega, \mathcal{F}, \mathbb{P})\) is a complete probability space, and \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-null sets of \(\mathcal{F}\) and \(\mathcal{F}_{t+} := \bigcap_{s \geq t} \mathcal{F}_s = \mathcal{F}_t\). Let \(\{W(t)\}_{t \geq 0}\) be a \(d\)-dimensional Brownian motion defined on the filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\).

In this paper, we consider the following NSDDE

\[
(2.1) \quad \frac{d\langle X(t) - D(X(t - \tau))\rangle}{dt} = b(X(t), X(t - \tau))dt + \sigma(X(t), X(t - \tau))dW(t), \quad t \geq 0
\]

with initial data

\[
X_0 = \xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\} \in \mathcal{L}_{\mathcal{F}_0}^p([-\tau, 0]; \mathbb{R}^n), \quad p \geq 2,
\]

that is, \(\xi\) is an \(\mathcal{F}_0\)-measurable \(C\)-valued random variable such that \(E\|\xi\|_\infty^p < \infty\) for \(p \geq 2\).

Here, \(D : \mathbb{R}^n \to \mathbb{R}^n\), and \(b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\), \(\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^d\) are continuous in \(x\) and \(y\). Fix \(T > \tau > 0\), assume that \(T\) and \(\tau\) are rational numbers, and the step size \(\Delta \in (0, 1)\) be fraction of \(T\) and \(\tau\), so that there exist two positive integers \(M, m\) such that \(\Delta = T/M = \tau/m\). Throughout the paper, we shall denote \(C\) by a generic positive constant, whose value may change from line to line. Further, for any \(x, y, \bar{x}, \bar{y} \in \mathbb{R}^n\), we shall assume that:

**A1** For any \(s, t \in [-\tau, 0]\) and \(q > 0\), there exists a positive constant \(K_1\) such that

\[
E\|\xi(s) - \xi(t)\|_\infty^q \leq K_1|s - t|^q.
\]

**A2** \(D(0) = 0\), and there exists a positive constant \(\kappa \in (0, 1/2)\) such that

\[
|D(x) - D(\bar{x})| \leq \kappa|x - \bar{x}|.
\]

**A3** There exists a positive constant \(K_2\) such that

\[
\langle x - D(y), b(x, y) \rangle \vee \|\sigma(x, y)\|^2 \leq K_2(1 + |x|^2 + |y|^2).
\]

**A4** There exist positive constants \(l, K_3\) and \(K_4\) such that for \(p \geq 2\)

\[
2\langle x - D(y) - \bar{x} + D(\bar{y}), b(x, y) - b(\bar{x}, \bar{y}) \rangle + (p - 1)\|\sigma(x, y) - \sigma(\bar{x}, \bar{y})\|^2 \\
\leq K_3(|x - \bar{x}|^2 + |y - \bar{y}|^2),
\]

and

\[
|b(x, y) - b(\bar{x}, \bar{y})| \leq K_4(1 + |x|^l + |\bar{x}|^l + |y|^l + |\bar{y}|^l)(|x - \bar{x}| + |y - \bar{y}|).
\]

**Remark 2.1.** Due to the existence of implicitness and the neutral term, scopes of \(\Delta\) and \(\kappa\) in assumption (A2) are given in order to guarantee rationality.
Remark 2.2. If $b(x, y)$ satisfies (A4), then, for any $x, y \in \mathbb{R}^n$, we have
\[
|b(x, y)| \leq |b(x, y) - b(0, 0)| + |b(0, 0)| \leq K_4(1 + |x|^l + |y|^l)(|x| + |y|) + |b(0, 0)| \\
\leq C(1 + |x| + |x|^{l+1} + |y| + |y|^{l+1}),
\]
where $C = K_4 \vee |b(0, 0)|$. If the coefficients satisfy (A2) and (A4), then one has
\[
(p - 1)\left\|\sigma(x, y) - \sigma(\overline{x}, \overline{y})\right\|^2 \\
\leq K_3(|x - \overline{x}|^2 + |y - \overline{y}|^2) + 2|x - D(y) - \overline{x} + D(\overline{y})||b(x, y) - b(\overline{x}, \overline{y})| \\
\leq C(1 + |x|^l + |x|^l + |y|^l + |\overline{y}|^l)(|x - \overline{x}|^2 + |y - \overline{y}|^2).
\]

Remark 2.3. There are many examples such that the assumptions can be verified. For example, let
\[
D(y) = -ay, \quad b(x, y) = x - x^3 + ay - a^3y^3, \quad \sigma(x, y) = x + ay,
\]
for $x, y \in \mathbb{R}$, where $a$ is a constant such that $|a| < 1/2$. We can check that assumptions (A2)-(A4) are satisfied. Since (A2) and (A3) are obvious, we only check (A4) here. By computation, we have
\[
2(x - D(y) - \overline{x} + D(\overline{y}), b(x, y) - b(\overline{x}, \overline{y})) + (p - 1)\left\|\sigma(x, y) - \sigma(\overline{x}, \overline{y})\right\|^2 \\
\leq 2(x + ay - \overline{x} - a\overline{y})(x - x^3 + ay - a^3y^3 - \overline{x} + \overline{x}^3 - a\overline{y} + a^3\overline{y}^3) + (p - 1)(x + ay - \overline{x} - a\overline{y})^2 \\
\leq 2[x - \overline{x} + a(y - \overline{y})][(x - \overline{x})(1 - x^2 - \overline{x}^2 - x\overline{x}) \\
+ a(y - \overline{y})(1 - a^2y^2 - a^2\overline{y}^2 - 2a^2y\overline{y})] + (p - 1)[x - \overline{x} + a(y - \overline{y})]^2 \\
\leq 4|x - \overline{x}|^2 + 4a^2|y - \overline{y}|^2 + 2(p - 1)(x - \overline{x})^2 + a^2(y - \overline{y})^2 \\
\leq 2(p + 1)(|x - \overline{x}| + |y - \overline{y}|^2),
\]
and
\[
|b(x, y) - b(\overline{x}, \overline{y})| = |x - x^3 + ay - a^3y^3 - \overline{x} + \overline{x}^3 - a\overline{y} + a^3\overline{y}^3| \\
= |(x - \overline{x})(1 - x^2 - \overline{x}^2 - x\overline{x}) + a(y - \overline{y})(1 - a^2y^2 - a^2\overline{y}^2 - 2a^2y\overline{y})| \\
\leq (1 + |x| + |\overline{x}| + |y| + |\overline{y}|)(|x - \overline{x}| + |y - \overline{y}|).
\]

Lemma 2.1. Let (A1)-(A4) hold, the NSDDE (2.1) admits a unique strong global solution $X(t), t \in [0, T]$, and
\[
\mathbb{E}\left(\sup_{0 \leq t \leq T} |X(t)|^p\right) \leq C
\]
for $p \geq 2$. One can consult [5] for more details.

2.1 The Tamed Theta Scheme

Now we introduce a tamed theta scheme for (2.1). For $k = -m, \cdots, 0$, set $y_{tk} = \xi(k\Delta)$; For $k = 0, 1, \cdots, M - 1$, we form
\[
y_{tk+1} - D(y_{tk+1-m}) = y_{tk} - D(y_{tk-m}) + \theta b\Delta(y_{tk+1}, y_{tk+1-m}) \Delta \\
+ (1 - \theta)b\Delta(y_{tk}, y_{tk-m}) \Delta + \sigma(y_{tk}, y_{tk-m}) \Delta W_t,
\]

(2.2)
where \( t_k = k\Delta \), and \( \Delta W_{t_k} = W(t_{k+1}) - W(t_k) \). Here \( b_\Delta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuous function and satisfy some conditions given below. Besides, \( \theta \in [0, 1] \) is an additional parameter that allows us to control the implicitness of the numerical scheme. Since it is convenient to work with a continuous extension of a numerical method, we now define the equivalent continuous form for (2.2). Let \( Y_\Delta(t) = \xi(t), t \in [-\tau, 0] \). For \( t \in [0, T] \), we define the corresponding continuous-time tamed theta scheme by

\[
Y_\Delta(t) = D(\overline{Y}_\Delta(t - \tau)) + \xi(0) - D(\xi(-\tau)) + \theta \int_{0}^{t} b_\Delta(\overline{Y}_{\Delta+}(s), \overline{Y}_{\Delta+}(s - \tau))ds
\]

\[
+ (1 - \theta) \int_{0}^{t} b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s - \tau))ds + \int_{0}^{t} \sigma(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s - \tau))dW(s),
\]

here \( \overline{Y}_\Delta(t) \) is defined by

\[
(2.3) \quad \overline{Y}_\Delta(t) = y_{t_k} \text{ and } \overline{Y}_{\Delta+}(t) = y_{t_{k+1}} \quad \text{for} \quad t \in [t_k, t_{k+1}),
\]

thus \( \overline{Y}_\Delta(t - \tau) = y_{t_{k-m}} \), and \( \overline{Y}_{\Delta+}(t - \tau) = y_{t_{k+1-m}} \). However, this \( Y_\Delta(t) \) is not \( \mathcal{F}_t \)-adapted, it does not meet the fundamental requirement in the Itô stochastic analysis. To avoid Malliavin calculus, we use the discrete split-step theta scheme introduced by Zong et al. [17] as follows: For \( k = -m, \cdots, -1 \), set \( z_{t_k} = \xi(k\Delta) \). For \( k = 0, 1, \cdots, M - 1 \), we reformulate the scheme (2.2) as follows

\[
(2.4) \quad \begin{cases}
y_{t_k} = D(y_{t_{k-m}}) + z_{t_k} - D(z_{t_{k-m}}) + \theta b_\Delta(y_{t_k}, y_{t_{k-m}})\Delta, \\
z_{t_{k+1}} = D(z_{t_{k+1-m}}) + z_{t_k} - D(z_{t_{k-m}}) + b_\Delta(y_{t_k}, y_{t_{k-m}})\Delta + \sigma(y_{t_k}, y_{t_{k-m}})\Delta W_{t_k}.
\end{cases}
\]

This scheme can also be rewritten as

\[
z_{t_{k+1}} - D(z_{t_{k+1-m}}) = z_{t_0} - D(z_{t_{-m}}) + \sum_{i=0}^{k} b_\Delta(y_{t_i}, y_{t_{i-m}})\Delta + \sum_{i=0}^{k} \sigma(y_{t_i}, y_{t_{i-m}})\Delta W_{t_i}
\]

\[
= \xi(0) - D(\xi(-\tau)) - \theta b_\Delta(\xi(0), \xi(-\tau))\Delta + \sum_{i=0}^{k} b_\Delta(y_{t_i}, y_{t_{i-m}})\Delta + \sum_{i=0}^{k} \sigma(y_{t_i}, y_{t_{i-m}})\Delta W_{t_i}.
\]

In order to simplify the computation, we define the corresponding continuous-time split-step tamed theta solution \( Z_\Delta(t) \) as follows: For any \( t \in [-\tau, 0] \), \( Z_\Delta(t) = \xi(t) \), \( Z_\Delta(0) = \xi(0) - \theta b_\Delta(\xi(0), \xi(-\tau))\Delta \). For any \( t \in [0, T] \),

\[
(2.5) \quad d[Z_\Delta(t) - D(Z_\Delta(t - \tau))] = b_\Delta(\overline{Y}_\Delta(t), \overline{Y}_\Delta(t - \tau))dt + \sigma(\overline{Y}_\Delta(t), \overline{Y}_\Delta(t - \tau))dW(t),
\]

where \( \overline{Y}_\Delta(t) \) is defined by (2.3). With the split-step tamed theta scheme (2.4), the continuous form of the split-step tamed theta solution \( Z_\Delta(t) \) and the tamed theta solution \( Y_\Delta(t) \) have the following relation:

\[
Y_\Delta(t) - D(Y_\Delta(t - \tau)) - \theta b_\Delta(Y_\Delta(t), Y_\Delta(t - \tau))\Delta = Z_\Delta(t) - D(Z_\Delta(t - \tau)).
\]
Denote \( \tilde{Y}_\Delta(t) = Y_\Delta(t) - D(Y_\Delta(t) - \theta b_\Delta(Y_\Delta(t), Y_\Delta(t - \tau)) \Delta, \) we can rewrite (2.5) as

\[
(2.6) \quad \tilde{Y}_\Delta(t) = \tilde{Y}_\Delta(0) + \int_0^t b_\Delta(\tilde{Y}_\Delta(s), \tilde{Y}_\Delta(s - \tau)) ds + \int_0^t \sigma(\tilde{Y}_\Delta(s), \tilde{Y}_\Delta(s - \tau)) dW(s), \quad t \in [0, T],
\]

where \( \tilde{Y}_\Delta(0) = \xi(0) - D(\xi(-\tau)) - \theta b_\Delta(\xi(0), \xi(-\tau)) \Delta. \) It is easy to see \( \tilde{Y}_\Delta(t) \) coincides with \( Y_\Delta(t) - D(Y_\Delta(t - \tau)) - \theta b_\Delta(Y_\Delta(t), Y_\Delta(t - \tau)) \Delta \) at grid points \( t = k\Delta, k = 0, 1, \ldots, M - 1, \) this also means that the continuous-time tamed theta solution \( Y_\Delta(t) \) coincides with the discrete-time tamed theta solution \( \tilde{Y}_\Delta(t) \) at grid points \( t = k\Delta, k = 0, 1, \ldots, M - 1. \)

We need some assumptions on \( b_\Delta(x, y) \). We assume that there exists an \( \alpha \in (0, 1/2] \) such that for any \( x, y, \bar{x}, \bar{y} \in \mathbb{R}^n \), the following conditions hold:

**B1** There exist a positive constant \( K_5 \geq 1 \) such that

\[
|b_\Delta(x, y)| \leq \min(K_5 \Delta^{-\alpha}(1 + |x| + |y|), |b(x, y)|).
\]

**B2** There exists a positive constant \( K_3 \) such that

\[
\langle x - D(y) - \bar{x} + D(\bar{y}), b_\Delta(x, y) - b_\Delta(\bar{x}, \bar{y}) \rangle \leq K_3(|x - \bar{x}|^2 + |y - \bar{y}|^2).
\]

**B3** There exist positive constants \( l, K_6 \) such that

\[
|b(x, y) - b_\Delta(x, y)|^p \leq K_6 \Delta^{\alpha p}[1 + |x|^{(2l+1)p} + |y|^{(2l+1)p}].
\]

**Remark 2.4.** With assumptions (B1)-(B3), (2.2) is well defined under some constraints on time step \( \Delta. \) Furthermore, by (A3) and (B2), we have

\[
(2.7) \quad \langle x - D(y), b_\Delta(x, y) \rangle \lor \|\sigma(x, y)\|^2 \leq \tilde{K}_2(1 + |x|^2 + |y|^2),
\]

where \( \tilde{K}_2 \) is a positive constant.

In order to ensure the implicitness of scheme (2.2) is well defined, an additional restriction is required on time step, i.e. \( \theta \Delta \tilde{K}_3 < 1, \) where \( \tilde{K}_3 \) is defined in (B2) (see [14] for more details). For \( \theta \in (0, 1], \) denote \( \Delta_1 = \frac{1}{\theta \tilde{K}_3}. \) Further, in order to guarantee the boundedness of the \( p \)-th moment of numerical solutions, the step size is also required to satisfy \( \theta^p \Delta < 6^{1-p}(2^{p-1} - \kappa^p)/(\kappa K_5^p) \) for \( p \geq 2 \) where \( \kappa \) and \( K_5 \) are defined in (A2) and (B1). Denote \( \Delta_2 = 6^{1-p}(2^{p-1} - \kappa^p)/(\theta^p K_5^p) \) for \( \theta \in (0, 1]. \) Thus in this section, we set \( \Delta^* \in (0, \Delta_1 \land \Delta_2), \) and let \( 0 < \Delta \leq \Delta^* \) for \( \theta \in (0, 1], \) while for \( \theta = 0, \) we may set \( \Delta \in (0, 1). \)

**Remark 2.5.** Under conditions (A2)-(A4), the set of sequences of functions which satisfy (B1)-(B3) are non-empty. For example, let \( b(x, y), \sigma(x, y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \) define

\[
b_\Delta(x, y) = \frac{b(x, y)}{1 + \Delta^\alpha |b(x, y)|}.
\]
It is easy to see $|b_\Delta(x, y)| \leq |b(x, y)|$, and on the other hand, we have

$$\frac{|b(x, y)|}{\Delta^{-\alpha} + |b(x, y)|} \leq \Delta^{-\alpha} \leq K_5 \Delta^{-\alpha}(1 + |x| + |y|).$$

That is, (B1) is verified. We are now going to check (B2), we divide it into two cases. For $b(x, y) \cdot b(x, y) < 0$,

$$\langle x - D(y) - \bar{x} + D(\bar{y}), b_\Delta(x, y) - b_\Delta(\bar{x}, \bar{y}) \rangle = \langle x - D(y) - \bar{x} + D(\bar{y}), \frac{b(x, y)}{1 + \Delta^\alpha|b(x, y)|} - \frac{b(x, y)}{1 + \Delta^\alpha|b(x, y)|} \rangle \leq \frac{1}{2} K_3 (|x - \bar{x}|^2 + |y - \bar{y}|^2).$$

For $b(x, y) \cdot b(x, y) > 0$,

$$\langle x - D(y) - \bar{x} + D(\bar{y}), b_\Delta(x, y) - b_\Delta(\bar{x}, \bar{y}) \rangle = \left\langle x - D(y) - \bar{x} + D(\bar{y}), \frac{b(x, y) - b(x, y)}{(1 + \Delta^\alpha|b(x, y)|)(1 + \Delta^\alpha|b(x, y)|)} \right\rangle + \left\langle x - D(y) - \bar{x} + D(\bar{y}), \frac{\Delta^\alpha|b(x, y)|b(x, y) - |b(x, y)|b(x, y)}{(1 + \Delta^\alpha|b(x, y)|)(1 + \Delta^\alpha|b(x, y)|)} \right\rangle \leq \frac{1}{2} K_3 (|x - \bar{x}|^2 + |y - \bar{y}|^2).$$

Due to Remark 2.2, we see that

$$|b(x, y) - b_\Delta(x, y)|^p \leq \Delta^{\alpha p}|b(x, y)|^{2p} \leq K_6 \Delta^{\alpha p} \left[1 + |x|^{2(p + 1)} + |y|^{2(p + 1)}\right].$$

That is to say, (B3) is also satisfied.

### 2.2 Moment Bounds

In order to prove the main results, we now give some estimates for the numerical solution $Y_\Delta(t)$.

**Lemma 2.2.** Let (A1)-(A3) and (B1)-(B2) hold. Then it holds that for $p \geq 2$,

$$\sup_{0 \leq t \leq T} \mathbb{E}|Y_\Delta(t)|^p \leq C,$$

where the positive constant $C$ is independent of $\Delta$. 

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Proof. For $a > 0$, let $\lfloor a \rfloor$ be the integer part of $a$. Applying the Itô formula to $[1 + |\tilde{Y}_\Delta(t)|^2]^2$, we obtain
\[
\mathbb{E}[1 + |\tilde{Y}_\Delta(t)|^2]^2 \leq \mathbb{E}[1 + |\tilde{Y}_\Delta(0)|^2]^2 + \frac{p}{2} \mathbb{E} \int_0^t \left[ 1 + |\tilde{Y}_\Delta(s)|^2 \right]^{p-2} 2\langle \tilde{Y}_\Delta(s), b\Delta(\tilde{Y}_\Delta(s), \tilde{Y}_\Delta(s-\tau)) \rangle ds
+ \frac{1}{2} p(p-1) \mathbb{E} \int_0^t [1 + |\tilde{Y}_\Delta(s)|^2]^{p-2} \|\sigma(\tilde{Y}_\Delta(s), \tilde{Y}_\Delta(s-\tau))\|^2 ds
\]
\[
\leq \mathbb{E}[1 + |\tilde{Y}_\Delta(0)|^2]^2 + \frac{p}{2} \mathbb{E} \int_0^t [1 + |\tilde{Y}_\Delta(s)|^2]^{p-2} \langle \tilde{Y}_\Delta(s), -D(\tilde{Y}_\Delta(s-\tau)), b\Delta(\tilde{Y}_\Delta(s), \tilde{Y}_\Delta(s-\tau)) \rangle ds
\]
\[
+ p\mathbb{E} \int_0^t \left[ |\tilde{Y}_\Delta(s)|^2 \right]^{p-2} \langle \tilde{Y}_\Delta(s) - \tilde{Y}_\Delta(s), b\Delta(\tilde{Y}_\Delta(s), \tilde{Y}_\Delta(s-\tau)) \rangle ds
= \mathbb{E}[1 + |\tilde{Y}_\Delta(0)|^2]^2 + E_1(t) + E_2(t) + E_3(t),
\]
where $\tilde{Y}_\Delta(t) = \tilde{Y}_\Delta(t) - D(\tilde{Y}_\Delta(t-\tau)) - \theta b\Delta(\tilde{Y}_\Delta(t), \tilde{Y}_\Delta(t-\tau)) \Delta$. With (A2), (B1) and (2.7), we have
\[
E_1(t) + E_2(t) \leq C \mathbb{E} \int_0^t \left[ |\tilde{Y}_\Delta(s)|^2 \right]^{p-2} (1 + |\tilde{Y}_\Delta(s)|^2 + |\tilde{Y}_\Delta(s-\tau)|^2) ds
\]
\[
\leq C \mathbb{E} \int_0^t \left[ |\tilde{Y}_\Delta(s)|^2 \right]^{p-2} + |\tilde{Y}_\Delta(s)|^p + |\tilde{Y}_\Delta(s-\tau)|^p ds
\]
\[
\leq C \mathbb{E} \int_0^t \left[ |\tilde{Y}_\Delta(s)|^p + |\tilde{Y}_\Delta(s-\tau)|^p + |\theta b\Delta(\tilde{Y}_\Delta(s), \tilde{Y}_\Delta(s-\tau)) \Delta|^p
+ |\tilde{Y}_\Delta(s)|^p + |\tilde{Y}_\Delta(s-\tau)|^p \right] ds
\]
\[
\leq C \mathbb{E} \int_0^t \left[ |\tilde{Y}_\Delta(s)|^p + |\tilde{Y}_\Delta(s-\tau)|^p + |\tilde{Y}_\Delta(s)|^p + |\tilde{Y}_\Delta(s-\tau)|^p \right] ds
+ C \Delta^{(1-a)p} \mathbb{E} \int_0^t (1 + |\tilde{Y}_\Delta(s)|^p + |\tilde{Y}_\Delta(s-\tau)|^p) ds
\]
\[
\leq C + C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}[|\tilde{Y}_\Delta(u)|^p] ds.
\]
Furthermore, it is easy to observe that,
\[
E_3(t) = p \mathbb{E} \int_0^t \left[ |\tilde{Y}_\Delta(s)|^2 \right]^{p-2} \langle \tilde{Y}_\Delta(s) - \tilde{Y}_\Delta(s), b\Delta(\tilde{Y}_\Delta(s), \tilde{Y}_\Delta(s-\tau)) \rangle ds
\]
\[
+ p\mathbb{E} \int_0^t \left\{ |\tilde{Y}_\Delta(s)|^2 \right\}^{p-2} - \left[ 1 + |\tilde{Y}_\Delta(s)|^2 \right]^{p-2} \langle \tilde{Y}_\Delta(s) - \tilde{Y}_\Delta(s), b\Delta(\tilde{Y}_\Delta(s), \tilde{Y}_\Delta(s-\tau)) \rangle ds
\]
\[
=: pE_{31}(t) + pE_{32}(t),
\]
where
\[
\tilde{Y}_\Delta(s) - \tilde{Y}_\Delta(s) = \int_{[s] \Delta} b\Delta(\tilde{Y}_\Delta(u), \tilde{Y}_\Delta(u-\tau)) du + \int_{[s] \Delta} \sigma(\tilde{Y}_\Delta(u), \tilde{Y}_\Delta(u-\tau)) dW(u).
\]
Due to (B1) and the Young inequality,

\[ E_{31}(t) = \mathbb{E} \int_0^t \left[ 1 + |\bar{Y}_\Delta(s)|^2 \right]^{p-2} \left\langle \int_{[\frac{s}{\Delta}]} b_\Delta(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau)) du, b_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) \right\rangle ds \]

\[ + \mathbb{E} \int_0^t \left[ 1 + |\bar{Y}_\Delta(s)|^2 \right]^{p-2} \left\langle \mathbb{E} \int_{[\frac{s}{\Delta}]} \sigma(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau)) dW(u) \bigg| b_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) \right\rangle ds \]

\[ \leq \mathbb{E} \int_0^t \left[ 1 + |\bar{Y}_\Delta(s)|^2 \right]^{p-2} \left| b_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) \right| ds \]

\[ \leq C_{\Delta}^{1-2\alpha} \mathbb{E} \int_0^t \left( 1 + |\bar{Y}_\Delta(s)|^p + |\bar{Y}_\Delta(s-\tau)|^p \right) ds \]

\[ + C_{\Delta}^{1-2\alpha} (1-\alpha)p \mathbb{E} \int_0^t \left( 1 + |\bar{Y}_\Delta(s)|^p + |\bar{Y}_\Delta(s-\tau)|^p \right) ds \]

\[ \leq C + C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|Y_\Delta(u)|^p ds. \]

Applying the Itô formula again, we obtain

\[ \left[ 1 + |\bar{Y}_\Delta(s)|^2 \right]^{\frac{p-2}{2}} \]

\[ \leq \left[ 1 + |\bar{Y}_\Delta(0)|^2 \right]^{\frac{p-2}{2}} + \left( p - 2 \right) \int_0^s \left[ 1 + |\bar{Y}_\Delta(u)|^2 \right]^{\frac{p-4}{2}} \left\langle \bar{Y}_\Delta(u), b_\Delta(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau)) \right\rangle du \]

\[ + \frac{1}{2} (p - 2) (p - 3) \int_0^s \left[ 1 + |\bar{Y}_\Delta(u)|^2 \right]^{\frac{p-4}{2}} \| \sigma(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau)) \|^2 du \]

\[ + (p - 2) \int_0^s \left[ 1 + |\bar{Y}_\Delta(u)|^2 \right]^{\frac{p-4}{2}} \left\langle \bar{Y}_\Delta(u), \sigma(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau)) \right\rangle dW(u). \]

Thus we have

\[ \left[ 1 + |\bar{Y}_\Delta(s)|^2 \right]^{\frac{p-2}{2}} \]

\[ \leq \left[ 1 + |\bar{Y}_\Delta(0)|^2 \right]^{\frac{p-2}{2}} + \left( p - 2 \right) \int_{[\frac{s}{\Delta}]} \left[ 1 + |\bar{Y}_\Delta(u)|^2 \right]^{\frac{p-4}{2}} \left\langle \bar{Y}_\Delta(u), b_\Delta(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau)) \right\rangle du \]

\[ + \frac{1}{2} (p - 2) (p - 3) \int_{[\frac{s}{\Delta}]} \left[ 1 + |\bar{Y}_\Delta(u)|^2 \right]^{\frac{p-4}{2}} \| \sigma(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau)) \|^2 du \]

\[ + (p - 2) \int_{[\frac{s}{\Delta}]} \left[ 1 + |\bar{Y}_\Delta(u)|^2 \right]^{\frac{p-4}{2}} \left\langle \bar{Y}_\Delta(u), \sigma(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau)) \right\rangle dW(u). \]
Hence,

\[ E_{32}(t) \leq (p-2) \mathbb{E} \int_0^t \int_s^t [1 + |\tilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} (\tilde{Y}_\Delta(u), b_\Delta(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u - \tau))) du \]
\[ \times (\tilde{Y}_\Delta(s) - \tilde{Y}_\Delta(s), b_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s - \tau))) ds \]
\[ + \frac{1}{2} (p-2)(p-3) \mathbb{E} \int_0^t \int_s^t \left[1 + |\tilde{Y}_\Delta(u)|^2\right]^{\frac{p-4}{2}} \|\sigma(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u - \tau))\|^2 du \]
\[ \times (\tilde{Y}_\Delta(s) - \tilde{Y}_\Delta(s), b_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s - \tau))) ds \]
\[ + (p-2) \mathbb{E} \int_0^t \int_s^t [1 + |\tilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} (\tilde{Y}_\Delta(u), \sigma(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u - \tau))) dW(u) \]
\[ \times (\tilde{Y}_\Delta(s) - \tilde{Y}_\Delta(s), b_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s - \tau))) ds \]
\[ =: (p-2) E_{321} + \frac{1}{2} (p-2)(p-3) E_{322} + (p-2) E_{323}. \]

Using (A3) and (B1), the Young inequality, the Hölder inequality and the Burkholder-Davis-
Using the same techniques in the way to estimate $E_{321}(t)$, we get

$$E_{322}(t) \leq C + C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|Y_{\Delta}(u)|^p ds.$$
Furthermore, by (A3) and (B1), we have
\[ E_{323}(t) = \mathbb{E} \int_0^t \int_{[\frac{s}{\Delta}, \frac{t}{\Delta}]} [1 + |\tilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} \langle \tilde{Y}_\Delta(u), \sigma(\tilde{Y}_\Delta(u), \tilde{Y}_\Delta(u - \tau)) \rangle dW(u) \]
\[ \times \left\langle \int_{[\frac{s}{\Delta}, \frac{t}{\Delta}]} b_\Delta(\tilde{Y}_\Delta(u), \tilde{Y}_\Delta(u - \tau)) du, b_\Delta(\tilde{Y}_\Delta(s), \tilde{Y}_\Delta(s - \tau)) \right\rangle ds \]
\[ + \mathbb{E} \int_0^t \int_{[\frac{s}{\Delta}, \frac{t}{\Delta}]} [1 + |\tilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} \langle \tilde{Y}_\Delta(u), \sigma(\tilde{Y}_\Delta(u), \tilde{Y}_\Delta(u - \tau)) \rangle dW(u) \]
\[ \times \left\langle \int_{[\frac{s}{\Delta}, \frac{t}{\Delta}]} \sigma(\tilde{Y}_\Delta(u), \tilde{Y}_\Delta(u - \tau)) du, b_\Delta(\tilde{Y}_\Delta(s), \tilde{Y}_\Delta(s - \tau)) \right\rangle ds \]
\[ = \mathbb{E} \int_0^t \int_{[\frac{s}{\Delta}, \frac{t}{\Delta}]} [1 + |\tilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} \langle \tilde{Y}_\Delta(u), \sigma(\tilde{Y}_\Delta(u), \tilde{Y}_\Delta(u - \tau)) \rangle dW(u) \]
\[ \times \left\langle \int_{[\frac{s}{\Delta}, \frac{t}{\Delta}]} \sigma(\tilde{Y}_\Delta(u), \tilde{Y}_\Delta(u - \tau)) du, b_\Delta(\tilde{Y}_\Delta(s), \tilde{Y}_\Delta(s - \tau)) \right\rangle ds \]
\[ \leq \mathbb{E} \int_0^t \int_{[\frac{s}{\Delta}, \frac{t}{\Delta}]} [1 + |\tilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} \| \sigma(\tilde{Y}_\Delta(u), \tilde{Y}_\Delta(u - \tau)) \| ds b_\Delta(\tilde{Y}_\Delta(s), \tilde{Y}_\Delta(s - \tau)) ds \]
\[ \leq C \Delta^{1-\alpha} \mathbb{E} \int_0^t (|Y_\Delta(s)|^p + |Y_\Delta(s - \tau)|^p + |\tilde{Y}_\Delta(s)|^p + |\tilde{Y}_\Delta(s - \tau)|^p) ds \]
\[ + C \Delta^{1-\alpha} (1-\alpha)^p \mathbb{E} \int_0^t (|Y_\Delta(s)|^p + |Y_\Delta(s - \tau)|^p) ds \]
\[ \leq C + C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|Y_\Delta(u)|^p ds. \]

By sorting these equations, we conclude that
\[ E_3(t) \leq C + C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|Y_\Delta(u)|^p ds. \]

Thus, the estimate of \( E_1(t) - E_3(t) \) results in
\[ \sup_{0 \leq u \leq t} \mathbb{E}|\tilde{Y}_\Delta(u)|^p \leq \sup_{0 \leq u \leq t} \mathbb{E}[1 + |\tilde{Y}_\Delta(u)|^2]^{\frac{p}{2}} \leq C + C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|Y_\Delta(u)|^p ds. \]

Since \(|x - y|^p \geq 2^{1-p}|x|^p - |y|^p\), we have
\[ |\tilde{Y}_\Delta(t)|^p \geq 2^{1-p}|Y_\Delta(t) - D(Y_\Delta(t - \tau))|^p - |\theta b_\Delta(Y_\Delta(t), Y_\Delta(t - \tau))\Delta|^p \]
\[ \geq 2^{1-p}[2^{1-p}|Y_\Delta(t)|^p - |D(Y_\Delta(t - \tau))|^p] - |\theta b_\Delta(Y_\Delta(t), Y_\Delta(t - \tau))\Delta|^p. \]

This, combining with (A2) and (B1), yields that
\[ |\tilde{Y}_\Delta(t)|^p \geq (2^{2-2p} - \tilde{C}_\Delta)|Y_\Delta(t) - Y_\Delta(t - \tau)|^p - (2^{2-p}K^p + \tilde{C}_\Delta)|Y_\Delta(t - \tau)|^p - \tilde{C}_\Delta, \]
where \( C_\Delta = \theta p K_5 \frac{p-1}{p} \Delta \). Consequently,

\[
\sup_{0 \leq u \leq t} \mathbb{E}|Y_\Delta(u)|^p \leq (2^{2-2p} - 2^{2-p} \kappa p - 2 \tilde{C}_\Delta)^{-1} \left[ \sup_{0 \leq u \leq t} \mathbb{E}|\tilde{Y}_\Delta(u)|^p + \tilde{C}_\Delta + (2^{2-p} \kappa p + \tilde{C}_\Delta) \mathbb{E} \|\xi\|_\infty^p \right].
\]

This, together with (2.8), implies

\[
\sup_{0 \leq u \leq t} \mathbb{E}|Y_\Delta(u)|^p \leq C + C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|Y_\Delta(u)|^p ds.
\]

Finally, the desired result is obtained by the Gronwall inequality. \(\square\)

**Lemma 2.3.** Let (A1)-(A3), (B1)-(B2) hold. Then, we have, for \( p \geq 2 \)

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \sup_{t \leq s < t+1} |Y_\Delta(t) - Y_\Delta(t_k)|^p \right] \leq C \Delta^\frac{p}{2},
\]

where \( C \) is a positive constant independent of \( \Delta \).

**Proof.** From the definition of numerical scheme (2.6), one sees that for \( t \in [t_k, t_{k+1}] \),

\[
\tilde{Y}_\Delta(t) - \tilde{Y}_\Delta(t_k) = \int_{t_k}^t b\left( \tilde{Y}_\Delta(s), \tilde{Y}_\Delta(s - \tau) \right) ds + \int_{t_k}^t \sigma\left( \tilde{Y}_\Delta(s), \tilde{Y}_\Delta(s - \tau) \right) dW(s).
\]

By the elementary inequality \(|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p), p \geq 1, \) we compute

\[
\mathbb{E} \left[ \sup_{t_k \leq t < t_{k+1}} |\tilde{Y}_\Delta(t) - \tilde{Y}_\Delta(t_k)|^p \right] \leq 2^{p-1} \mathbb{E} \left[ \sup_{t_k \leq t < t_{k+1}} \left( \int_{t_k}^t b\left( \tilde{Y}_\Delta(s), \tilde{Y}_\Delta(s - \tau) \right) ds \right)^p \right] + 2^{p-1} \mathbb{E} \left[ \sup_{t_k \leq t < t_{k+1}} \left( \int_{t_k}^t \sigma\left( \tilde{Y}_\Delta(s), \tilde{Y}_\Delta(s - \tau) \right) dW(s) \right)^p \right].
\]

With (A3), (B1), Lemma 2.2, the Hölder inequality and the BDG inequality, we derive

\[
\mathbb{E} \left[ \sup_{t_k \leq t < t_{k+1}} |\tilde{Y}_\Delta(t) - \tilde{Y}_\Delta(t_k)|^p \right] \leq 2^{p-1} \Delta^{p-1} \mathbb{E} \int_{t_k}^{t_{k+1}} |b\left( \tilde{Y}_\Delta(s), \tilde{Y}_\Delta(s - \tau) \right)|^p ds + C \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \| \sigma\left( \tilde{Y}_\Delta(s), \tilde{Y}_\Delta(s - \tau) \right) \|^2 ds \right]^{\frac{p}{2}} \leq C \Delta^{(1-\alpha)p} + C \Delta^\frac{p}{2} \leq C \Delta^\frac{p}{2}.
\]

Denoting by \( \bar{D}(t, t_k) := D(Y_\Delta(t - \tau)) - D(Y_\Delta(t_k - \tau)) \) and \( \bar{b}_\Delta(t, t_k) := b\left( Y_\Delta(t), Y_\Delta(t - \tau) \right) - b\left( Y_\Delta(t_k), Y_\Delta(t_k - \tau) \right) \), with (A2), we arrive at

\[
|\tilde{Y}_\Delta(t) - \tilde{Y}_\Delta(t_k)|^p \geq 2^{1-p}|Y_\Delta(t) - Y_\Delta(t_k) - \bar{D}(t, t_k)|^p - \theta p \Delta^p |\bar{b}_\Delta(t, t_k)|^p \geq 2^{2-2p}|Y_\Delta(t) - Y_\Delta(t_k)|^p - 2^{1-p} \kappa p |Y_\Delta(t - \tau) - Y_\Delta(t_k - \tau)|^p - \theta p \Delta^p |\bar{b}_\Delta(t, t_k)|^p.
\]
Obviously, for $0 \leq t < t_1 = \Delta$, we have $t - \tau < t_1 - \tau < 0$, then we see from (2.9) and Lemma 2.2 that
\[
\mathbb{E} \left[ \sup_{0 \leq t < t_1} |Y_{\Delta}(t) - Y_{\Delta}(t_0)|^p \right] \leq C \mathbb{E} \left[ \sup_{0 \leq t < t_1} |\tilde{Y}_{\Delta}(t) - \tilde{Y}_{\Delta}(t_0)|^p \right] + C \Delta^{(1-\alpha)p} \leq C \Delta^{\frac{p}{2}}.
\]

For $t_1 \leq t < t_2$, (2.9) and Lemma 2.2 lead to
\[
\mathbb{E} \left[ \sup_{t_1 \leq t < t_2} |Y_{\Delta}(t) - Y_{\Delta}(t_1)|^p \right] \leq C \mathbb{E} \left[ \sup_{t_1 \leq t < t_2} |\tilde{Y}_{\Delta}(t) - \tilde{Y}_{\Delta}(t_1)|^p \right] + C \Delta^{(1-\alpha)p}
\]
\[
\leq C \Delta^{\frac{p}{2}}.
\]

Consequently, the induction method yields,
\[
\mathbb{E} \left[ \sup_{t_k \leq t < t_{k+1}} |Y_{\Delta}(t) - Y_{\Delta}(t_k)|^p \right] \leq C \mathbb{E} \left[ \sup_{t_k \leq t < t_{k+1}} |\tilde{Y}_{\Delta}(t) - \tilde{Y}_{\Delta}(t_k)|^p \right] + C \Delta^{(1-\alpha)p}
\]
\[
\leq C \Delta^{\frac{p}{2}}.
\]

The proof is therefore complete. \(\square\)

### 2.3 Strong Convergence Rate

The following theorem reveals that the continuous form $Y_{\Delta}(t)$ of the tamed theta scheme (2.2) converges strongly to the exact solution $X(t)$.

**Theorem 2.4.** Let (A1)-(A4) and (B1)-(B3) hold, then it holds that for $p \geq 2$,
\[
\mathbb{E} \sup_{0 \leq t \leq T} |X(t) - Y_{\Delta}(t)|^p \leq C \Delta^{\alpha p},
\]
where $\alpha$ is defined in (B1) and $C$ is a positive constant independent of $\Delta$. That is, the strong convergence rate of the tamed theta scheme (2.2) is $\alpha$.

**Proof.** Denote $I(t) = Y_{\Delta}(t) - D(Y_{\Delta}(t - \tau)) - \theta b_{\Delta}(Y_{\Delta}(t), Y_{\Delta}(t - \tau)) \Delta - X(t) + D(X(t - \tau))$, then
\[
I(t) = I(0) + \int_0^t \left[ b_{\Delta}(\overline{Y}_{\Delta}(s), \overline{Y}_{\Delta}(s - \tau)) - b(X(s), X(s - \tau)) \right] ds
\]
\[
+ \int_0^t \left[ \sigma(\overline{Y}_{\Delta}(s), \overline{Y}_{\Delta}(s - \tau)) - \sigma(X(s), X(s - \tau)) \right] dW(s),
\]
\[
\mathbb{E} \sup_{0 \leq t \leq T} |X(t) - Y_{\Delta}(t)|^p \leq C \Delta^{\alpha p},
\]
where $\alpha$ is defined in (B1) and $C$ is a positive constant independent of $\Delta$. That is, the strong convergence rate of the tamed theta scheme (2.2) is $\alpha$. 

**Proof.** Denote $I(t) = Y_{\Delta}(t) - D(Y_{\Delta}(t - \tau)) - \theta b_{\Delta}(Y_{\Delta}(t), Y_{\Delta}(t - \tau)) \Delta - X(t) + D(X(t - \tau))$, then
\[
I(t) = I(0) + \int_0^t \left[ b_{\Delta}(\overline{Y}_{\Delta}(s), \overline{Y}_{\Delta}(s - \tau)) - b(X(s), X(s - \tau)) \right] ds
\]
\[
+ \int_0^t \left[ \sigma(\overline{Y}_{\Delta}(s), \overline{Y}_{\Delta}(s - \tau)) - \sigma(X(s), X(s - \tau)) \right] dW(s),
\]
\[
\leq C \Delta^{\frac{p}{2}}.
\]
where \( I(0) = -\theta b\Delta(\xi(0), \xi(-\tau))\Delta \). An application of the Itô formula yields,

\[
|I(t)|^p \leq |I(0)|^p + p \int_0^t |I(s)|^{p-2} \langle I(s), b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) - b(X(s), X(s-\tau)) \rangle ds \\
+ \frac{1}{2} p(p-1) \int_0^t |I(s)|^{p-2} \|\sigma(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) - \sigma(X(s), X(s-\tau))\|^2 ds \\
+ p \int_0^t |I(s)|^{p-2} \langle I(s), \sigma(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) - \sigma(X(s), X(s-\tau)) \rangle dW(s)
\]

\[
\leq |I(0)|^p + \sum_{i=1}^6 H_i(t),
\]

where

\[
H_1(t) := p \int_0^t |I(s)|^{p-2} \langle I(s), b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) - b(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) \rangle ds,
\]

\[
H_2(t) := p \int_0^t |I(s)|^{p-2} \langle I(s), b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) - b(Y_\Delta(s), Y_\Delta(s-\tau)) \rangle ds,
\]

\[
H_3(t) := p \int_0^t |I(s)|^{p-2} \langle I(s), b(Y_\Delta(s), Y_\Delta(s-\tau)) - b(X(s), X(s-\tau)) \rangle ds,
\]

\[
H_4(t) := p(p-1) \int_0^t |I(s)|^{p-2} \|\sigma(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) - \sigma(Y_\Delta(s), Y_\Delta(s-\tau))\|^2 ds,
\]

\[
H_5(t) := p(p-1) \int_0^t |I(s)|^{p-2} \|\sigma(Y_\Delta(s), Y_\Delta(s-\tau)) - \sigma(X(s), X(s-\tau))\|^2 ds,
\]

\[
H_6(t) := p \int_0^t |I(s)|^{p-2} \langle I(s), \sigma(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) - \sigma(X(s), X(s-\tau)) \rangle dW(s).
\]

By (A2), (B1), (B3), Lemma 2.2, and the Hölder inequality,

\[
\sup_{0 \leq u \leq t} E[H_1(u)] \leq C\mathbb{E} \int_0^t |I(s)|^p ds + C\mathbb{E} \int_0^t |b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) - b(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))|^p ds
\]

\[
\leq C\mathbb{E} \int_0^t [\|X(s-\tau) - X(s)\|^p + \|X_\Delta(s-\tau) - X(s-\tau)\|^p + \theta^p \Delta^p |b_\Delta(Y_\Delta(s), Y_\Delta(s-\tau))|^p] ds
\]

\[
+ C\Delta^{\alpha p} \mathbb{E} \int_0^t (1 + |\overline{Y}_\Delta(s)|^{2(l+1)p} + |\overline{Y}_\Delta(s-\tau)|^{2(l+1)p}) ds
\]

\[
\leq C \int_0^t \sup_{0 \leq u \leq s} E[Y_\Delta(u) - X(u)]^p ds + C\Delta^{(1-\alpha)p} + C\Delta^{\alpha p}.
\]
By (A2), (A4), (B1), Lemmas 2.2-2.3, and the Hölder inequality,
\[
\sup_{0 \leq u \leq t} \mathbb{E}H_2(u) \leq C \mathbb{E} \int_0^t |I(s)|^p ds + C \mathbb{E} \int_0^t |b(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s - \tau)) - b(Y_\Delta(s), Y_\Delta(s - \tau))|^p ds
\]
\[
\leq C \mathbb{E} \int_0^t [\overline{Y}_\Delta(s) - X(s)]^p + |Y_\Delta(s - \tau) - X(s - \tau)|^p + \theta^p |b_\Delta(Y_\Delta(s), Y_\Delta(s - \tau))|^p ds
\]
\[
+ C \mathbb{E} \int_0^t (1 + |\overline{Y}_\Delta(s)|^p + |Y_\Delta(s - \tau)|^p + |Y_\Delta(s)|^p + |Y_\Delta(s - \tau)|^p)^p
\]
\[
\times (|\overline{Y}_\Delta(s) - Y_\Delta(s)| + |\overline{Y}_\Delta(s - \tau) - Y_\Delta(s - \tau)|)^p ds
\]
\[
\leq C \mathbb{E} \int_0^t (|Y_\Delta(s) - X(s)|^p + |Y_\Delta(s - \tau) - X(s - \tau)|^p) ds + C \Delta^{1-\alpha} p
\]
\[
+ C \mathbb{E} \int_0^t [\mathbb{E}(1 + |\overline{Y}_\Delta(s)|^p + |Y_\Delta(s - \tau)|^p + |Y_\Delta(s)|^p + |Y_\Delta(s - \tau)|^p)^{\frac{p}{2}}
\]
\[
\times [\mathbb{E}(|\overline{Y}_\Delta(s) - Y_\Delta(s)| + |\overline{Y}_\Delta(s - \tau) - Y_\Delta(s - \tau)|)^{2p}]^{\frac{1}{2}} ds
\]
\[
\leq C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|Y_\Delta(u) - X(u)|^p ds + C \Delta^{1-\alpha} p + C \Delta^{\frac{\alpha}{2}}.
\]

Due to (A2), (B1), (2.7), Lemma 2.2, and the Hölder inequality,
\[
\sup_{0 \leq u \leq t} \mathbb{E}H_3(u) + \sup_{0 \leq u \leq t} \mathbb{E}H_5(u)
\]
\[
\leq C \mathbb{E} \int_0^t |I(s)|^{p-2}[|Y_\Delta(s) - X(s)|^2 + |Y_\Delta(s - \tau) - X(s - \tau)|^2] ds
\]
\[
+ C \mathbb{E} \int_0^t [I(s)]^{p-2}|b_\Delta(Y_\Delta(s), Y_\Delta(s - \tau))\Delta|b(Y_\Delta(s), Y_\Delta(s - \tau)) - b(X(s), X(s - \tau))| ds
\]
\[
\leq C \mathbb{E} \int_0^t [|Y_\Delta(s) - X(s)|^p + \theta^p |b_\Delta(Y_\Delta(s), Y_\Delta(s - \tau))|^p] ds
\]
\[
+ C \Delta^{1-\alpha} \mathbb{E} \int_0^t [I(s)]^{p-2}(1 + |Y_\Delta(s)| + |Y_\Delta(s - \tau)|) \times
\]
\[
(1 + |Y_\Delta(s)|^p + |Y_\Delta(s - \tau)|^p + |X(s)|^p + |X(s - \tau)|^p) \times
\]
\[
(|Y_\Delta(s) - X(s)| + |Y_\Delta(s - \tau) - X(s - \tau)|) ds
\]
\[
\leq C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|Y_\Delta(u) - X(u)|^p ds + C \Delta^{1-\alpha} p.
\]

In the same way as the estimate of $H_1(t)$ and $H_2(t)$, we arrive at
\[
\sup_{0 \leq u \leq t} \mathbb{E}H_4(u) \leq C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|Y_\Delta(u) - X(u)|^p ds + C \Delta^{1-\alpha} p + C \Delta^{\frac{\alpha}{2}}.
\]

Since $\mathbb{E}H_6(t) = 0$, by sorting $H_1(t) - H_6(t)$ together, we derive
\[
\sup_{0 \leq u \leq t} \mathbb{E}|I(u)|^p \leq C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|Y_\Delta(u) - X(u)|^p ds + C \Delta^{\alpha p}.
\]
By the definition of \( I(t) \), we have

\[
|I(t)|^p \geq 2^{1-p}|Y_\Delta(t) - X(t) - D(Y_\Delta(t - \tau)) + D(X(t - \tau))|^p - |\theta b_\Delta(Y_\Delta(t), Y_\Delta(t - \tau))\Delta|^p \\
\geq 2^{1-p}[2^{1-p}|Y_\Delta(t) - X(t)|^p - |D(Y_\Delta(t - \tau)) - D(X(t - \tau))|^p] - |\theta b_\Delta(Y_\Delta(t), Y_\Delta(t - \tau))\Delta|^p,
\]

this, together with (A2), leads to

\[
|I(t)|^p \geq 2^{2-2p}|Y_\Delta(t) - X(t)|^p - 2^{1-p}\kappa^p|Y_\Delta(t - \tau) - X(t - \tau)|^p - |\theta b_\Delta(Y_\Delta(t), Y_\Delta(t - \tau))\Delta|^p.
\]

Taking (B1) and Lemma 2.2 into consideration yields

\[
\sup_{0 \leq u \leq \theta} \mathbb{E}|Y(u) - X(u)|^p \leq C \sup_{0 \leq u \leq \theta} \mathbb{E}|I(u)|^p + C\Delta^{(1-\alpha)p} \\
\leq C\Delta^{\alpha p} + C \int_0^\theta \sup_{0 \leq u \leq s} \mathbb{E}|Y(u) - X(u)|^p ds.
\]

The desired result follows from the Gronwall inequality. \( \square \)

**Remark 2.6.** If we replace (A3) by the following (A3’):

**A3’** There exists a positive constant \( K_2 \) such that

\[
2\langle x - D(y), b(x, y) \rangle + (p - 1)\| \sigma(x, y) \|^2 \leq K_2(1 + \|x\|^2 + \|y\|^2),
\]

and add another assumption (B4):

**B4** There exists a positive constant \( \tilde{K}_2 \) such that

\[
2\langle x - D(y), b_\Delta(x, y) \rangle + (p - 1)\| \sigma(x, y) \|^2 \leq \tilde{K}_2(1 + \|x\|^2 + \|y\|^2),
\]

we can also show that under assumptions (A1)-(A2), (A3’), (A4), (B1)-(B4), the tamed theta scheme \( Y_\Delta(t) \) converges strongly to the exact solution \( X(t) \) with order \( \alpha \).

## 3 Local One-sided Lipschitz Drift

In this section, instead of the global one-sided Lipschitz condition (A4), we impose the following local one-sided Lipschitz condition:

**A5** For every \( R > 0 \), there exists a positive constant \( L_R \) such that

\[
\langle x - D(y) - \bar{x} + D(\bar{y}), b(x, y) - b(\bar{x}, \bar{y}) \rangle \vee \| \sigma(x, y) - \sigma(\bar{x}, \bar{y}) \|^2 \\
\leq L_R(\|x - \bar{x}\|^2 + \|y - \bar{y}\|^2)
\]

for all \( |x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq R. \)
Remark 3.1. The local one-sided Lipschitz condition is a weaker than the classical local Lipschitz condition, since if \( b \) satisfies the local Lipschitz condition such that,

\[
|b(x, y) - b(x, \overline{y})|^2 \leq L_R(|x - \overline{x}|^2 + |y - \overline{y}|^2), |x| \lor |\overline{x}| \lor |y| \lor |\overline{y}| \leq R,
\]

we then have

\[
\langle x - \overline{x}, b(x, y) - b(x, \overline{y}) \rangle \leq \frac{1}{2}|x - \overline{x}|^2 + \frac{1}{2}|b(x, y) - b(x, \overline{y})|^2
\]

\[
\leq \frac{1}{2}|x - \overline{x}|^2 + \frac{1}{2}L_R(|x - \overline{x}|^2 + |y - \overline{y}|^2) \leq \tilde{L}_R(|x - \overline{x}|^2 + |y - \overline{y}|^2),
\]

where \( \tilde{L}_R = \max(\frac{1}{2}, \frac{1}{2}L_R) \). This implies that the local one-sided Lipschitz condition holds.

On the other hand, if \( b \) satisfies the local one-sided Lipschitz condition, it need not satisfy the classical local Lipschitz condition. For example, let \( b(x, y) = x^3 - x^{\frac{3}{2}} + y \), because of \( x^{\frac{1}{2}} \), \( b \) does not satisfy the classical local Lipschitz condition. However, noting that

\[
(x - \overline{x})(x^{\frac{1}{2}} - \overline{x}^{\frac{1}{2}}) \geq 0 \text{ for all } x, \overline{x},
\]

we see that

\[
\langle x - \overline{x}, x^3 - \sqrt{x} + y - \overline{x}^3 + \sqrt{\overline{x}} - \overline{y} \rangle
\]

\[
\leq |x - \overline{x}|^2(x^2 + x\overline{x} + \overline{x}^2) + \frac{1}{2}(|x - \overline{x}|^2 + |y - \overline{y}|^2) - (x - \overline{x})(x^{\frac{1}{2}} - \overline{x}^{\frac{1}{2}})
\]

\[
\leq |x - \overline{x}|^2(x^2 + x\overline{x} + \overline{x}^2) + \frac{1}{2}(|x - \overline{x}|^2 + |y - \overline{y}|^2)
\]

\[
\leq \tilde{L}_R(|x - \overline{x}|^2 + |y - \overline{y}|^2), |x| \lor |\overline{x}| \lor |y| \lor |\overline{y}| \leq R,
\]

where \( \tilde{L}_R = \max(\frac{1}{2}, 3R^2) \). This means that \( b \) satisfies the local one-sided Lipschitz condition.

Remark 3.2. Due to the continuity of \( b(x, y) \), for every \( R > 0 \), there exists a positive constant \( \mathcal{L}_R \) such that

\[
\sup_{|x| \lor |y| \leq R} |b(x, y)| \leq \mathcal{L}_R.
\]

Remark 3.3. There are many examples such that the assumptions can be verified. For example, if we set

\[
D(y) = \frac{1}{4} \cos y, \quad b(x, y) = x - x^3 + \cos y, \quad \sigma(x, y) = y \sin x + x \sin y,
\]

then assumptions (A2)-(A3) and (A5) hold.

Consider the following tamed theta scheme imposed in Section 2:

\[
y_{t_{k+1}} - D(y_{t_{k+1-m}}) = y_k - D(y_{t_{k-m}}) + \theta b\Delta(y_{t_{k+1}}, y_{t_{k+1-m}})\Delta + (1 - \theta)b\Delta(y_{t_k}, y_{t_{k-m}})\Delta W_{t_k}.
\]
Generally speaking, for a given \( y_{tk} \), to guarantee a unique solution \( y_{tk+1} \) is to assume that there exists a positive constant \( L \) such that

\[
\langle x - D(y) - \bar{x} + D(\bar{y}), b_\Delta(x, y) - b_\Delta(\bar{x}, \bar{y}) \rangle \leq L(|x - \bar{x}|^2 + |y - \bar{y}|^2)
\]

as in Section 2. Moreover, as shown in Mao and Szpruch [8], this condition is somehow hard to relax. While in our assumption \((A5)\), the drift coefficient \( b \) is local one-sided Lipschitz, thus in this case, the tamed drift \( b_\Delta \) is hardly to be global one-sided Lipschitz. That is, we do not know if the tamed theta scheme \((2.2)\) is well defined under assumptions \((A2)-(A3)\) and \((A5)\). In the following, we will provide an improved tamed theta scheme to ensure the well-posedness of implicit equations.

3.1 The Improved Tamed Theta Scheme

For any \( R > 0 \), define a smooth, non-negative function such that

\[
\zeta_R(x, y) = \begin{cases} 
1, & \text{for } |x|, |y| \leq R, \\
0, & \text{for } |x| \text{ or } |y| > R + 1,
\end{cases}
\]

and \( \zeta_R(x, y) \leq 1 \) for all \( x, y \in \mathbb{R}^n \). It is obvious that \( \zeta_R(x, y) \) is Lipschitz with some constant \( C_\zeta \). Now we introduce the improved tamed theta scheme for \((2.1)\). For \( k = -m, \ldots, 0 \), set \( y_{tk} = \xi(k\Delta) \); For \( k = 0, 1, \ldots, M - 1 \), we form

\[
y_{tk+1} - D(y_{tk+1-m}) = y_{tk} - D(y_{tk-m}) + \theta \tilde{b}_\Delta(y_{tk+1}, y_{tk+1-m}) \zeta_R(y_{tk+1}, y_{tk+1-m}) \Delta \\
+ (1 - \theta) \tilde{b}_\Delta(y_{tk}, y_{tk-m}) \zeta_R(y_{tk}, y_{tk-m}) \Delta + \sigma(y_{tk}, y_{tk-m}) \Delta W_{tk},
\]

where \( t_k = k\Delta \), and \( \Delta W_{tk} = W(t_{k+1}) - W(t_k) \). Here \( \tilde{b}_\Delta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuous function. Besides, \( \theta \in [0, 1] \) is an additional parameter that allows us to control the implicitness of the numerical scheme. Denote

\[
\tilde{b}_\Delta(x, y) = \bar{b}_\Delta(x, y) \zeta_R(x, y),
\]

then, \((3.1)\) can be rewritten as

\[
y_{tk+1} - D(y_{tk+1-m}) = y_{tk} - D(y_{tk-m}) + \theta \tilde{b}_\Delta(y_{tk+1}, y_{tk+1-m}) \Delta \\
+ (1 - \theta) \tilde{b}_\Delta(y_{tk}, y_{tk-m}) \Delta + \sigma(y_{tk}, y_{tk-m}) \Delta W_{tk},
\]

which is exactly the form of \((2.2)\). According to \((3.2)\) we define \( \bar{Y}_\Delta(t), Y_\Delta(t), \tilde{Y}_\Delta(t), Z_\Delta(t) \) by using the same notation as in Section 2. Instead of constraints on \( \tilde{b}_\Delta(x, y) \), we impose some assumptions on \( \tilde{b}_\Delta(x, y) \). Assume that there exists an \( \alpha \in (0, 1/2] \) such that for any \( x, y, \bar{x}, \bar{y} \in \mathbb{R}^n \), the following conditions hold:

(C1) There exists a positive constant \( K_5 \geq 1 \) such that

\[
|\tilde{b}_\Delta(x, y)| \leq \min(K_5 \Delta^{-\alpha}(1 + |x| + |y|), |b(x, y)|).
\]
(C2) There exists a positive constant \( \tilde{K}_2 \) such that
\[
\langle x - D(y), \tilde{b}_\Delta(x, y) \rangle \leq \tilde{K}_2(1 + |x|^2 + |y|^2).
\]

(C3) For any \( R > 0 \), there exists a positive constant \( M_R \) such that
\[
\langle x - D(y) - \bar{x} + D(\bar{y}), \tilde{b}_\Delta(x, y) - \tilde{b}_\Delta(\bar{x}, \bar{y}) \rangle \leq M_R(|x - \bar{x}|^2 + |y - \bar{y}|^2).
\]
for all \( |x| \lor |y| \lor |\bar{x}| \lor |\bar{y}| \leq R \).

(C4) For any \( R > 0 \), there exists a positive constant \( N_R \) such that
\[
\sup_{|x| \lor |y| \leq R} |b(x, y) - \tilde{b}_\Delta(x, y)|^p \leq N_R \Delta^{\alpha p} \to 0 \quad \text{as} \quad \Delta \to 0.
\]

Lemma 3.1. Let (A2), (C1)-(C4) hold, then \( \tilde{b}_\Delta \) satisfies (C1), (C2), (C4) and the following (C3'):

(C3') There exists an \( M_{R_0} \) such that for all \( x, y, \bar{x}, \bar{y} \in \mathbb{R}^n \)
\[
\langle x - D(y) - \bar{x} + D(\bar{y}), \tilde{b}_\Delta(x, y) - \tilde{b}_\Delta(\bar{x}, \bar{y}) \rangle \leq M_{R_0}(|x - \bar{x}|^2 + |y - \bar{y}|^2),
\]
where \( M_{R_0} = M_{R_0} + 2C\zeta \bar{L}_{R_0} \).

Proof. By the relationship between \( \tilde{b}_\Delta \) and \( b_\Delta \), (C1) and (C2) can be verified easily. Noting that for \( |x| \lor |y| \leq R, \zeta_R(x, y) = 1 \), thus we get
\[
\sup_{|x| \lor |y| \leq R} \left[ |b(x, y) - \tilde{b}_\Delta(x, y)|^p \right] = \sup_{|x| \lor |y| \leq R} \left[ |b(x, y) - \tilde{b}_\Delta(x, y)\zeta(x, y)|^p \right]
\]
\[
= \sup_{|x| \lor |y| \leq R} \left[ |b(x, y) - \tilde{b}_\Delta(x, y)|^p \right] \leq N_R \Delta^{\alpha p} \to 0, \quad \text{as} \quad \Delta \to 0,
\]
then (C4) holds for \( \tilde{b}_\Delta(x, y) \). Now we are going to check (C3'). Divide it into four cases.

Case a: None of \( |x|, |y|, |\bar{x}|, |\bar{y}| \) bigger than \( R+1 \). In this case, we see \( 0 \leq \zeta_R(x, y), \zeta_R(\bar{x}, \bar{y}) \leq 1 \). Rewrite \( \tilde{b}_\Delta \) with \( \tilde{b}_\Delta \), we have
\[
\langle x - D(y) - \bar{x} + D(\bar{y}), \tilde{b}_\Delta(x, y) - \tilde{b}_\Delta(\bar{x}, \bar{y}) \rangle
\]
\[
= \langle x - D(y) - \bar{x} + D(\bar{y}), \tilde{b}_\Delta(x, y) - \tilde{b}_\Delta(\bar{x}, \bar{y}) \rangle \zeta_R(x, y)
\]
\[
+ \langle x - D(y) - \bar{x} + D(\bar{y}), \zeta_R(x, y) - \zeta_R(\bar{x}, \bar{y}) \rangle \tilde{b}_\Delta(\bar{x}, \bar{y})
\]
\[
=: q_1 + q_2.
\]
Since \( 0 \leq \zeta_R(x, y) \leq 1 \), thus by (C3),
\[
q_1 \leq M_{R+1}(|x - \bar{x}|^2 + |y - \bar{y}|^2).
\]
Further, noting that $\zeta_R$ is Lipschitz with constant $C_\zeta$ and for $|\bar{x}| \vee |\bar{y}| \leq R + 1$, we see from (C1) and Remark 3.2 that $|\tilde{b}_\Delta(x, y)| \leq |b(\bar{x}, \bar{y})| \leq \bar{L}_{R+1}$, then (A2) leads to

$$q_2 \leq 2C_\zeta \bar{L}_{R+1}(|x - \bar{x}|^2 + |y - \bar{y}|^2).$$

Combining the estimation of $q_1$ and $q_2$, we get

$$\langle x - D(y) - \bar{x} + D(\bar{y}), \tilde{b}_\Delta(x, y) - \tilde{b}_\Delta(\bar{x}, \bar{y}) \rangle \leq (M_{R+1} + 2C_\zeta \bar{L}_{R+1})(|x - \bar{x}|^2 + |y - \bar{y}|^2).$$

**Case b:** One of $|x|, |y|, |ar{x}|, |ar{y}|$ bigger than $R + 1$. Assume $|x| > R + 1$ and $|y|, |ar{x}|, |ar{y}| \leq R + 1$. In this case, we have $\zeta_R(x, y) = 0$ and $0 \leq \zeta_R(\bar{x}, \bar{y}) \leq 1$. Similar to Case a, we have

$$\langle x - D(y) - \bar{x} + D(\bar{y}), \tilde{b}_\Delta(x, y) - \tilde{b}_\Delta(\bar{x}, \bar{y}) \rangle \leq 2C_\zeta \bar{L}_{R+1}(|x - \bar{x}|^2 + |y - \bar{y}|^2).$$

**Case c:** Two of $|x|, |y|, |ar{x}|, |ar{y}|$ bigger than $R + 1$. We divide it into two cases.

i): Both $|x|, |y|$ bigger than $R + 1$ or both $|ar{x}|, |ar{y}|$ bigger than $R + 1$. Consider one of the case $|x|, |y| > R + 1$ while $|ar{x}|, |ar{y}| \leq R + 1$. It is obvious that $\zeta_R(x, y) = 0$ and $0 \leq \zeta_R(\bar{x}, \bar{y}) \leq 1$. By taking similar steps as Case a, we can get

$$\langle x - D(y) - \bar{x} + D(\bar{y}), \tilde{b}_\Delta(x, y) - \tilde{b}_\Delta(\bar{x}, \bar{y}) \rangle \leq 2C_\zeta \bar{L}_{R+1}(|x - \bar{x}|^2 + |y - \bar{y}|^2).$$

ii): One of $|x|, |y|$ bigger than $R + 1$ and one of $|ar{x}|, |ar{y}|$ bigger than $R + 1$. Consider the case of $|x| > R + 1, |y| \leq R + 1, |ar{x}| > R + 1, |ar{y}| \leq R + 1$. Then $\zeta_R(x, y) = \zeta_R(\bar{x}, \bar{y}) = 0$ and

$$\langle x - D(y) - \bar{x} + D(\bar{y}), \tilde{b}_\Delta(x, y) - \tilde{b}_\Delta(\bar{x}, \bar{y}) \rangle = 0.$$

**Case d:** Three or four of $|x|, |y|, |ar{x}|, |ar{y}|$ bigger than $R + 1$. Since we have $\zeta_R(x, y) = \zeta_R(\bar{x}, \bar{y}) = 0$, the result is obvious.

Taking Cases a-d into consideration, there exists an $\bar{M}_{R_0}$ such that (C3') satisfies for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$.

**Remark 3.4.** Lemma 3.1 shows that with assumptions (C1)-(C3), (3.2) is well defined under some constraints on time step $\Delta$. It is worth mentioning that (C3) and (C3') are merely used to guarantee the uniqueness of numerical solutions.

**Remark 3.5.** Under assumptions (A2)-(A3), (A5), we now give an example such that the set of sequences of functions satisfy (C1)-(C4). Let $b(x, y), \sigma(x, y)$ be one-dimensional and define

$$\tilde{b}_\Delta(x, y) = \frac{1}{1 + \Delta^\alpha |b(x, y)| + \Delta^{\alpha/2} \|\sigma(x, y)\|} b(x, y)$$

for any $x, y \in \mathbb{R}$. It is easy to see $|\tilde{b}_\Delta(x, y)| \leq |b(x, y)|$, and on the other hand, we have

$$|\tilde{b}_\Delta(x, y)| = \frac{\Delta^{-\alpha} |b(x, y)|}{\Delta^{-\alpha} + |b(x, y)| + \Delta^{-\alpha/2} \|\sigma(x, y)\|} \leq K_5 \Delta^{-\alpha} \leq K_5 \Delta^{-\alpha} (1 + |x| + |y|).$$

Furthermore, due to (A3),

$$\langle x - D(y), \tilde{b}_\Delta(x, y) \rangle = \frac{\langle x - D(y), b(x, y) \rangle}{1 + \Delta^\alpha |b(x, y)| + \Delta^{\alpha/2} \|\sigma(x, y)\|} \leq K_2 (1 + |x|^2 + |y|^2).$$
That is to say, (C2) is satisfied. In order to show (C3), we have to divide it into several cases. Denote by \( \Gamma(x, y) = 1 + \Delta^\alpha |b(x, y)| + \Delta^{\alpha/2} \|\sigma(x, y)\| \).

**Case a:** \( b(x, y) \cdot b(x, y) < 0 \). We divide this into four classes.

i): For \( b(x, y) > 0 \), \( b(x, y) \leq 0 \), and \( x - D(y) - \bar{x} + D(y) \geq 0 \),

\[
\langle x - D(y) - \bar{x} + D(y), \bar{b}_\Delta(x, y) - \bar{b}_\Delta(x, y) \rangle = \langle x - D(y) - \bar{x} + D(y), \frac{b(x, y) - b(x, y)}{\Gamma(x, y) - \Gamma(x, y)} \rangle
\]

\[
\leq \langle x - D(y) - \bar{x} + D(y), b(x, y) - b(x, y) \rangle \leq L_R(|x - \bar{x}|^2 + |y - \bar{y}|^2).
\]

ii): For \( b(x, y) > 0 \), \( b(x, y) < 0 \), and \( x - D(y) - \bar{x} + D(y) < 0 \), the result is obvious.

iii): For \( b(x, y) < 0 \), \( b(x, y) > 0 \), and \( x - D(y) - \bar{x} + D(y) < 0 \),

\[
\langle x - D(y) - \bar{x} + D(y), \bar{b}_\Delta(x, y) - \bar{b}_\Delta(x, y) \rangle
\]

\[
\leq \langle x - D(y) - \bar{x} + D(y), b(x, y) - b(x, y) \rangle \leq L_R(|x - \bar{x}|^2 + |y - \bar{y}|^2).
\]

iv): For \( b(x, y) < 0 \), \( b(x, y) > 0 \), and \( x - D(y) - \bar{x} + D(y) \geq 0 \), the result is also obvious.

**Case b:** \( b(x, y) \cdot b(x, y) > 0 \). We compute

\[
\langle x - D(y) - \bar{x} + D(y), \bar{b}_\Delta(x, y) - \bar{b}_\Delta(x, y) \rangle = \langle x - D(y) - \bar{x} + D(y), \frac{b(x, y) - b(x, y)}{\Gamma(x, y) - \Gamma(x, y)} \rangle
\]

\[
+ \langle x - D(y) - \bar{x} + D(y), \frac{\Delta^\alpha |b(x, y)| b(x, y) - b(x, y) b(x, y)|}{\Gamma(x, y) - \Gamma(x, y)} \rangle
\]

\[
+ \langle x - D(y) - \bar{x} + D(y), \frac{\Delta^{\alpha/2} \|\sigma(x, y)\| |b(x, y) - b(x, y)|}{\Gamma(x, y) - \Gamma(x, y)} \rangle
\]

\[
+ \langle x - D(y) - \bar{x} + D(y), \frac{\Delta^{\alpha/2} b(x, y) |\|\sigma(x, y)\| - \|\sigma(x, y)\||}{\Gamma(x, y) - \Gamma(x, y)} \rangle
\]

\[= \overline{\eta}_1 + \overline{\eta}_2 + \overline{\eta}_3 + \overline{\eta}_4.\]

Obviously, \( \overline{\eta}_2 = 0 \). Noticing that \( \Gamma(x, y) \geq 1, \Gamma(x, y) \geq 1 \) and \( 0 < \frac{\Delta^{\alpha/2} \|\sigma(x, y)\|}{\Gamma(x, y)} \leq 1 \), we then derive from (A2), (A5) and Remark 3.2 that

\[
\overline{\eta}_1 + \overline{\eta}_3 \leq 2L_R(|x - \bar{x}|^2 + |y - \bar{y}|^2),
\]

and

\[
\overline{\eta}_4 \leq \frac{1}{2} |x - D(y) - \bar{x} + D(y)|^2 \frac{\Delta^\alpha |b(x, y)|^2 \|\sigma(x, y)\| - \|\sigma(x, y)\|^2}{2\Gamma^2(x, y) \Gamma^2(x, y)}
\]

\[
\leq |x - \bar{x}|^2 + |y - \bar{y}|^2 + |b(x, y)||\sigma(x, y) - \sigma(x, y)||^2
\]

\[
\leq (1 + L_R \overline{\Gamma}_R)(|x - \bar{x}|^2 + |y - \bar{y}|^2).\]

This shows that (C3) is satisfied. Thanks to (A3) and Remark 3.2, we see that

\[
\sup_{|x|, |y| \leq R} |b(x, y) - \bar{b}_\Delta(x, y)|^p \leq \Delta^{\alpha p} \sup_{|x|, |y| \leq R} \frac{(|b(x, y)| + \|\sigma(x, y)\|^2)^p |b(x, y)|^p}{(1 + \Delta^\alpha |b(x, y)| + \Delta^\alpha \|\sigma(x, y)\|^2)^p} \leq C \Delta^{\alpha p} \rightarrow 0,
\]

Thus, (C4) is shown.
Condition (C3') shows that $\tilde{b}_{\Delta}$ is global one-sided Lipschitz. According to the monotone operator, the implicit scheme (3.2) is well defined with $\theta M R_0 \Delta < 1$. Define $\Delta_3 = \frac{1}{\theta M R_0} \Delta$ for $\theta \in (0, 1]$. Thus in the following section, we set $\Delta^* \in (0, \Delta_3 \wedge \Delta_2)$, and let $0 < \Delta \leq \Delta^*$ for $\theta \in (0, 1]$ while for $\theta = 0$, we set $\Delta \in (0, 1)$.

### 3.2 Convergence of the Numerical Solutions

We need the following lemma.

**Lemma 3.2.** Let (A1)-(A3) and (A5) hold, then it holds that

$$\sup_{0 \leq t \leq T} \mathbb{E}|X(t)|^p \vee \sup_{0 \leq t \leq T} \mathbb{E}|Y_{\Delta}(t)|^p \leq C,$$

for $p \geq 2$.

**Remark 3.6.** Through the derivation process of Lemmas 2.2 and 2.3, we see that what we really used are assumptions (A1)-(A3), (B1) and (2.7) in Lemmas 2.2 and 2.3. In this section, $\tilde{b}_{\Delta}$ and $\sigma$ satisfy (C1)-(C2), which implies that the corresponding $\tilde{b}_{\Delta}$ and $\sigma$ also satisfy (B1) and (2.7). Thus, Lemmas 2.2 and 2.3 proposed in Section 2 still hold in this section.

We now state the main result in this Section.

**Theorem 3.3.** Let (A1)-(A3), (A5) and (C1)-(C4) hold, then the continuous form $Y_{\Delta}(t)$ of the tamed theta scheme (3.2) converges strongly to the exact solution $X(t)$ of (2.1), that is,

$$\lim_{\Delta \to 0} \sup_{0 \leq t \leq T} \mathbb{E}|X(t) - Y_{\Delta}(t)|^2 = 0.$$

**Proof.** Denote $e(t) = Y_{\Delta}(t) - X(t)$, and for any $R > 0$ define the following stopping time

$$\tau_R = \inf\{t \geq 0 : |Y_{\Delta}(t)| \geq R\}, \rho_R = \inf\{t \geq 0 : |X(t)| \geq R\}, \nu_R = \tau_R \wedge \rho_R.$$

For any $\eta > 0$, by the Young inequality,

$$\sup_{0 \leq u \leq T} \mathbb{E}|e(u)|^2 = \sup_{0 \leq u \leq T} \mathbb{E}(|e(u)|^2 \mathbb{I}_{\{\tau_R > T, \rho_R > T\}}) + \sup_{0 \leq u \leq T} \mathbb{E}(|e(u)|^2 \mathbb{I}_{\{\tau_R \leq T \text{ or } \rho_R \leq T\}}) \leq \sup_{0 \leq u \leq T} \mathbb{E}(|e(u) \wedge \nu_R|^2 \mathbb{I}_{\{\nu_R > T\}}) + \frac{2\eta}{p} \sup_{0 \leq u \leq T} \mathbb{E}|e(u)|^p$$

$$+ \frac{p - 2}{p\eta^{p-2}} \mathbb{P}(\tau_R \leq T \text{ or } \rho_R \leq T). \quad (3.3)$$

Due to Lemma 2.2,

$$\mathbb{P}(\tau_R \leq T) = \mathbb{E}\left(\mathbb{I}_{\{\tau_R \leq T\}} |Y_{\Delta}(\tau_R)|^p \right) \leq \frac{1}{R^p} \sup_{0 \leq u \leq T} \mathbb{E}|Y_{\Delta}(u)|^p \leq \frac{C}{R^p}.$$
where here and in the following, we emphasize that $C$ is a positive constant independent of $\Delta, R$ and $\varepsilon$, while $C_R$ will be a positive constant depending on $R$. Similarly, we derive from Lemma 3.2 that

\begin{equation}
\mathbb{P}(\tau_R \leq T \text{ or } \rho_R \leq T) \leq \mathbb{P}(\tau_R \leq T) + \mathbb{P}(\rho_R \leq T) \leq \frac{2C}{R^p}.
\end{equation}

On the other hand, Lemma 2.2 and Lemma 3.2 yield

\begin{equation}
\sup_{0 \leq u \leq T} \mathbb{E}|e(u)|^p \leq 2^{p-1} \sup_{0 \leq u \leq T} \mathbb{E}(|Y_\Delta(u)|^p + |X(u)|^p) \leq C.
\end{equation}

Denote by $I(t) = Y_\Delta(t) - D(Y_\Delta(t) - \theta \tilde{b}_\Delta(Y_\Delta(t), Y_\Delta(t - \tau))) - X(t) + D(X(t - \tau))$. Applying the Itô formula,

\begin{align*}
\mathbb{E}[I(T \wedge \nu R)]^2 &= |I(0)|^2 + 2 \mathbb{E} \int_0^{T \wedge \nu R} \langle I(s), \tilde{b}_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s - \tau)) - b(X(s), X(s - \tau)) \rangle \, ds \\
&\quad + \mathbb{E} \int_0^{T \wedge \nu R} \| \sigma(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s - \tau)) - \sigma(X(s), X(s - \tau)) \|^2 \, ds \\
&\leq |I(0)|^2 + 2 \mathbb{E} \int_0^{T \wedge \nu R} \langle \overline{Y}_\Delta(s) - D(\overline{Y}_\Delta(s - \tau)) - X(s) + D(X(s - \tau)), \rangle \, ds \\
&\quad + 2 \mathbb{E} \int_0^{T \wedge \nu R} \langle \tilde{b}_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s - \tau)) - b(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s - \tau)) \rangle \, ds \\
&\quad + 2 \mathbb{E} \int_0^{T \wedge \nu R} \langle Y_\Delta(s) - D(Y_\Delta(s - \tau)) - \overline{Y}_\Delta(s) + D(\overline{Y}_\Delta(s - \tau)), \rangle \, ds \\
&\quad + 2 \mathbb{E} \int_0^{T \wedge \nu R} \langle \tilde{b}_\Delta(Y_\Delta(s), Y_\Delta(s - \tau)), \tilde{b}_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s - \tau)) - b(X(s), X(s - \tau)) \rangle \, ds \\
&\quad - 2\theta \Delta \mathbb{E} \int_0^{T \wedge \nu R} \langle \tilde{b}_\Delta(Y_\Delta(s), Y_\Delta(s - \tau)), \tilde{b}_\Delta(Y_\Delta(s), Y_\Delta(s - \tau)) - b(X(s), X(s - \tau)) \rangle \, ds \\
&\quad + 2 \mathbb{E} \int_0^{T \wedge \nu R} \| \sigma(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s - \tau)) - \sigma(X(s), X(s - \tau)) \|^2 \, ds \\
&\leq |I(0)|^2 + \sum_{i=1}^5 I_i(T),
\end{align*}

where $I(0) = -\theta \tilde{b}_\Delta(\xi(0), \xi(-\tau)) \Delta$. By assumption (A5) and Lemma 2.3,

\begin{equation}
\sup_{0 \leq u \leq T} (|I_1(u) + I_5(u)|) \leq C_R \int_0^{T \wedge \nu R} [\| \overline{Y}_\Delta(s) - X(s) \|^2 + \| \overline{Y}_\Delta(s - \tau) - X(s - \tau) \|^2] \, ds
\end{equation}

\begin{align*}
&\leq C_R \Delta + \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|e(u \wedge \nu R)|^2 \, ds.
\end{align*}
Obviously, due to (C4), Lemma 2.1 and Lemma 2.2,
\begin{equation}
\sup_{0 \leq u \leq T} |I_2(u)| \leq C_R \Delta^\alpha + C_R \Delta^{2\alpha}.
\end{equation}
By (A2) and Lemma 2.3, we obtain,
\begin{equation}
\sup_{0 \leq u \leq T} |I_3(u)| \leq C \int_0^{T \wedge \nu_R} \left( \mathbb{E}[|Y_\Delta(s) - \bar{Y}_\Delta(s)|^2 + |Y_\Delta(s - \tau) - \bar{Y}_\Delta(s - \tau)|^2] \right)^{\frac{1}{2}}
\left( \mathbb{E}[^\hat{b}_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s - \tau)) - b(X(s), X(s - \tau))|^2] \right)^{\frac{1}{2}} ds \leq C_R \Delta^{\frac{1}{2}}.
\end{equation}
Furthermore, by Remark 3.2 and (C1)
\begin{equation}
\sup_{0 \leq u \leq T} |I_4(u)| \leq C_R \Delta.
\end{equation}
Due to (3.6)-(3.9), we see
\begin{equation}
\sup_{0 \leq u \leq T} E|I(u \wedge \nu_R)|^2 \leq C \int_0^T \sup_{0 \leq u \leq s} E|e(u \wedge \nu_R)|^2 ds + C_R \Delta^\alpha.
\end{equation}
With assumption (A2), one has
\begin{equation}
\sup_{0 \leq u \leq T} E|e(u \wedge \nu_R)|^2 \leq C \sup_{0 \leq u \leq T} E|I(u \wedge \nu_R)|^2 + \kappa \sup_{0 \leq u \leq T} E|e(u \wedge \nu_R - \tau)|^2 + C_R \Delta^2.
\end{equation}
This implies
\begin{equation}
\sup_{0 \leq u \leq T} E|e(u \wedge \nu_R)|^2 \leq C \sup_{0 \leq u \leq T} E|I(u \wedge \nu_R)|^2 + C_R \Delta^2.
\end{equation}
Applying the Gronwall inequality, we derive from (3.10) and (3.11) that
\begin{equation}
\sup_{0 \leq u \leq T} E|e(u \wedge \nu_R)|^2 \leq C R \Delta^\alpha.
\end{equation}
Thus, combining (3.4), (3.5) and (3.12), we see from (3.3) that for any given $\epsilon > 0$, one can choose $\eta$ small enough such that
\[ \frac{2 \eta}{p} C < \frac{\epsilon}{3}, \]
and then $R$ big enough such that
\[ \frac{p - 2}{p \eta^{\frac{p}{2}}} \frac{C}{R^p} < \frac{\epsilon}{3}, \]
finally $\Delta$ small enough to satisfy
\[ C_R \Delta^\alpha < \frac{\epsilon}{3}. \]
Therefore, we arrive at
\[ \sup_{0 \leq u \leq T} E|Y_\Delta(u) - X(u)|^2 \to 0 \text{ as } \Delta \to 0, \]
as required. \qed
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