# A characterisation of optimal strategies to deal with extreme events in insurance.

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# A Characterisation of Optimal Strategies to Deal with Extreme Events in Insurance

by

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in candidature for the degree of

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## Declaration

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

ADAM SHORE
22 January 2008

## Statement

This thesis is the result of my own investigation, except where acknowledgement of other sources is given.

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# Summary

This thesis looks at the Actuarial area of risk, and more specifically Ruin Theory. In the ruin model the stability of an insurer is studied. Starting from capital u at time t=0, his capital is assumed to increase linearly in time by fixed annual premiums, but it decreases with a jump whenever a claim occurs. Ruin occurs when the capital is negative at some point in time. The probability that this ever happens, under the assumption that the premium, as well as the claim generating process remains unchanged, is a good indication of whether the insurer's assets are matched to his liabilities sufficiently well. If not, the insurer has a number of options available to him such as reinsuring the risk, raising the premiums or increasing the initial capital. Analytical methods to compute ruin probabilities exist only for claims distributions that are mixtures and combinations of exponential distributions. Algorithms exist for discrete distributions with few mass points. Also, tight upper and lower bounds can be derived in most cases.

This thesis explores a topic of particular practical interest in queuing and insurance mathematics, namely the analysis of extreme events leading to the financial ruin of an insurance company. The phrase 'extreme events' here, means an unusually high number of claims and/or unexpectedly high claim sizes. However, similar problems also appear naturally in the context of communication networks, where extreme events are responsible for delays to messages. The proper mathematical framework for this analysis is the theory of Large Deviations, one of the most active and dynamic branches of modern applied probability. This framework provides powerful tools for computing the probability of extreme events when the more conventional approaches like the law of large numbers and the central limit theorem fail. The overall objective of this thesis is to study the linking of Large Deviation techniques with elements of control and optimisation theory.

After covering the background theory required for the exploration of the ruin model, and the application of Large Deviations, we explore previous work, with a strong emphasis on methods used to calculate the ruin probability for more realistic distributions. Next, we start to explore some of the options available to the insurer should he wish to reduce his risk (but ultimately retain high profits). The first option we cover is that of taking on new business with the aim of increasing premium income to offset immediate liabilities. In doing so, we produce a simulation package that is able to compute ruin probabilities for many complicated and more realistic situations.

The claims on an insurance company must be met in full, but to protect itself from large claims the company itself may take out an insurance policy. We study a combination of both proportional and excess of loss reinsurance in a Large Deviations Regime and examine the results for both the popular exponential distribution and the more realistic 'heavy tailed' gamma distribution.

Finally, we discuss the findings of our work, and how our results could be beneficial to the Actuarial profession. Our investigations, although based on limited parameter values, illustrate useful conclusions on the use of alternative distributions and, consequently, are of potential value to a practitioner who, prior to making a decision about his risk, would like to know what type of new business to take on, or how much business to reinsure in order to minimise his probability of ruin, whilst maximising profit.

After summarising our results and conclusions, some ideas for future research are detailed.

To Donna.

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# Chapter 1

# Introduction

## 1.1 Modelling Insurance Business

Insurance is a newsworthy and important business, and seems likely to remain so for the foreseeable future. People like to protect what they have, and an easy commodity that can represent the value of most things is money. The basic principles are straightforward, the idea is to protect oneself or someone or something else that one holds dear. What is one protecting against? This can be different for each person, but mainly covers anything from damage to theft, and can be regarded as an event that would alter what one wants to protect. The theory of probability can be applied to any concept in life, including the chance that an event should affect what one holds dear. Essentially, the protection is from one thing - risk. Each coin has two sides, and the other side to this business is the provider of the insurance contract. For a premium, the insurer vows to pay an agreed amount to rectify the loss the insured may have incurred. It is this side of the business that we consider in this thesis, where we discuss various models of the insurance business, and some of the properties of these models. A model is a (usually simplified) view of real world system or process, and can be used to try out different scenarios on our process to investigate the possible consequences. The effect of changing certain input parameters can be studied before a decision is made to implement the plans in the real world. To build this model, a set of mathematical or logical assumptions about how it works in practice needs to be developed. The complexity of the model is thus determined by the complexity of the relationships between the various model parameters. For example, in modelling a life office, consideration must be given to issues such as regulations, taxation and cancellation terms. Future events affecting investment returns, inflation, new business, lapses, mortality and expenses also affect these relationships.

In order to produce the model and determine suitable parameters, data needs to be considered and judgements made as to the relevance of the observed data to the future environment. Such data may result from past observations, from current observations (such as the rate of inflation) or from expectations of future changes. Where observed data is considered to be suitable for producing the parameters for a chosen model, statistical models can be used.

Before finalising the choice of model and parameters, it is important to consider the objectives for creation and use of the model. In this case it is not necessary to take too much time over fitting the distributions or parameters. We simply want to see how the model behaves upon changing such parameters. We hope that the model can be used to make generic statements of optimal strategies, especially when dealing with Large Deviations. (See Chapter 2). Our model will be based on a generic insurance business with the logical assumptions set out below.

## 1.2 The Collective Risk Model

First, we consider a short term insurance contract covering a risk. A risk includes either a single policy or a specified group of policies. For ease of terminology, the term of the contract is assumed to be one year, but it could equally well be any other short period, for example six months. The random variable $S$ denotes the aggregate claims paid by the insurer in the year in respect of this risk. Models will be constructed for this random variable $S$. In this thesis collective risk models will be studied. A first step in the construction of a collective risk model is to write $S$ in terms of the number of claims arising in the year, denoted by the random variable $N$, and the amount of each individual claim. Let the random variable $X_i$ denote the amount of the $i^{\text{th}}$ claim; then

$$S = \sum_{i=1}^{N} X_i \tag{1.1}$$

where the summation is taken to be zero if $N$ is zero. This decomposition of $S$ allows separate consideration of claim numbers and claim amounts, with the practical advantage that the factors affecting claim numbers and claim amounts may well be different.

### 1.2.1 Notation and Assumptions

Throughout this thesis the following three important and realistic assumptions will be made:

- The random variables $\{X_i\}_{i=1}^{N}$ are independent and identically distributed.
- The random variable $N$ is independent of $\{X_i\}_{i=1}^{N}$.
- All claims are for non-negative amounts, so that $\Pr(X_i \leq x) = 0$ for $x < 0$.

Many of the formulae in this thesis will be derived using the moment generating functions (from now on abbreviated to MGFs) of $S$, $N$ and $X_i$. These MGFs will be denoted $M_S(r)$, $M_N(r)$ and $M_X(r)$, respectively, and will be assumed to exist for some positive values of the variable $r$. The existence of the MGF of a non-negative random variable for positive values of $r$ cannot generally be taken for granted; for example the MGFs of the Pareto and of the lognormal distributions do not exist for any positive value of $r$. However, all the formulae derived in this thesis with the help of MGFs can be derived, although less easily, without assuming the MGFs exist for positive values of $r$.

We use $G(x)$, and $F(x)$ to denote the cumulative distribution functions of $S$ and $X_i$, respectively, so that

$$G(x) = \Pr(S \leq x) \text{ and } F(x) = \Pr(X_i \leq x). \tag{1.2}$$

For convenience, it will often be assumed that the probability density function corresponding to $F(x)$ exists, where it will be denoted $f(x)$. In cases where this density does not exist, so that $X_i$ has a discrete or a mixed continuous/discrete distribution, expressions such as

$$\int_0^\infty x f(x)\, dx \tag{1.3}$$

should be interpreted appropriately. The meaning should always be clear from the context.

Note that it is the number of claims, $N$, from the risk as a collective (as opposed to counting the number of claims from individual policies) that is being considered and this gives the name "Collective Risk Model".

## 1.3 Definitions of Distributions and Functions

This section describes some typical probability distributions and other useful functions that are used throughout the thesis.

First, we define the expectation of $X$ when $X$ has probability density function $f(x)$ as

$$\mathbf{E}[X] = \int_{-\infty}^\infty x f(x)\, dx. \tag{1.4}$$

Sometimes $\mathbf{E}[X]$ is written as $\mathbf{E}X$ and is also known as the first moment of $X$. The basic idea generalises: we have

$$\mathbf{E}\left[X^2\right] = \int_{-\infty}^\infty x^2 f(x)\, dx \qquad \text{second moment of } X. \tag{1.5}$$

$$\mathbf{E}\left[X^k\right] = \int_{-\infty}^\infty x^k f(x)\, dx \qquad k^{\text{th}} \text{ moment of } X. \tag{1.6}$$

These expectations help to summarise the distribution of $X$ and appear in the definition of other quantities.

The $k^{\text{th}}$ moment of $X_i$ about zero, $k = 1, 2, 3, \ldots$, will be denoted $m_k$.

The variance of $X$ is defined as:

$$Var(X) = \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2 \tag{1.7}$$

and measures the spread or dispersion in the distribution of $X$. Sometimes $Var\,(X)$ is written as $V\,[X]$.

A special expectation is the *moment generating function* or MGF:

$$M_X\,(r) = \mathbf{E}\,[e^{rx}] = \int_{-\infty}^{\infty} e^{rx} f\,(x)\,dx \tag{1.8}$$

### 1.3.1 Binomial Distribution

$$\begin{aligned} B\,(x;n,p) &= \binom{n}{x} p^x q^{n-x}; x = 0,1,2,\ldots,n, 0 \le p \le 1 \text{ with } q = 1-p \\ \mathbf{E}[X] &= np \\ V[X] &= npq \\ M_X\,(r) &= (q+pe^r)^n \end{aligned}$$

### 1.3.2 Poisson Distribution

$$\begin{aligned} P(x;\lambda) &= \frac{\exp\{-\lambda\}\lambda^x}{\lambda!}; x = 0,1,2,\ldots,\lambda > 0 \\ \mathbf{E}[X] &= \lambda \\ V[X] &= \lambda \\ M_X\,(r) &= \exp\{\lambda\,(e^r - 1)\} \end{aligned}$$

### 1.3.3 Discrete Uniform

$$\begin{aligned} P(x;N) &= \frac{1}{N+1}, x = 0,1,2,\ldots,N, N \text{ a nonnegative integer.} \\ \mathbf{E}[X] &= \frac{N}{2} \\ V[X] &= \frac{N(N+2)}{12} \end{aligned}$$

### 1.3.4 Exponential Distribution

$$\begin{aligned} f(x;\alpha) &= \alpha \exp\{-\alpha x\}; x \ge 0, \alpha > 0 \\ \mathbf{E}[X] &= \frac{1}{\alpha} \\ V[X] &= \frac{1}{\alpha^2} \\ M_X\,(r) &= \left(1 - \frac{r}{\alpha}\right)^{-1} \end{aligned}$$

### 1.3.5 Gamma Distribution

$$\begin{aligned}
f(x;\alpha,\beta) &= \frac{\alpha^{\beta}\exp\{-\alpha x\}x^{\beta-1}}{\Gamma(\beta)}; x \geq 0, \alpha > 0, \beta > 0 \\
\Gamma(\beta) &= \int_0^{\infty} t^{\beta-1}e^{-t}dt, \quad \beta > 0 \\
\mathbf{E}[X] &= \frac{\beta}{\alpha} \\
V[X] &= \frac{\beta}{\alpha^2} \\
M_X(r) &= \left(1-\frac{r}{\alpha}\right)^{-\beta}, r < \alpha
\end{aligned}$$

### 1.3.6 Normal Distribution

$$\begin{aligned}
f(x;\mu,\sigma) &= \frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}; -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0 \\
\mathbf{E}[X] &= \mu \\
V[X] &= \sigma^2 \\
M_X(r) &= \exp\left\{\mu r + \frac{1}{2}\sigma^2 r^2\right\}
\end{aligned}$$

### 1.3.7 Lognormal Distribution

$$\begin{aligned}
f(x;\mu,\sigma) &= \frac{1}{\sqrt{2\pi}\sigma x}e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}, \sigma > 0 \\
E[X] &= \exp\left\{\mu + \frac{1}{2}\sigma^2\right\} \\
V[X] &= \exp\left\{2\mu + \sigma^2\right\}.\left[\exp\left\{\sigma^2\right\} - 1\right]
\end{aligned}$$

### 1.3.8 Inverse Gaussian Distribution

$$\begin{aligned}
f(x;\lambda,\mu) &= \sqrt{\frac{\lambda}{2\pi x^3}}\exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\}; x > 0, \lambda > 0, \mu > 0 \\
\mathbf{E}[X] &= \mu \\
V[X] &= \frac{\mu^3}{\lambda} \\
M_X(r) &= \left(\frac{\lambda}{\mu}\right)\left[1-\sqrt{1-\frac{2\mu^2 r}{\lambda}}\right]
\end{aligned}$$

### 1.3.9 Convex Functions

A function whose value at the midpoint of every closed interval in its domain does not exceed the average of its values at the ends of the interval is called a convex function. In other words, a function $f(x)$ is convex on an interval $[a,b]$ if, for any two points $x_1$ and $x_2$ in $[a,b]$,

$$f\left[\frac{1}{2}(x_1+x_2)\right] \le \frac{1}{2}\left[f(x_1)+f(x_2)\right];$$

see, for example, Gradshteyn & Ryzhik (2000). If $f(x)$ has a second derivative in $[a,b]$, then a sufficient condition for it to be convex on that interval is that the second derivative $f''(x) > 0$ for all $x$ in $[a,b]$.

*If the inequality above is strict for all $x_1$ and $x_2$, then $f(x)$ is called strictly convex.* Examples of convex functions include $x^p$ for $p \ge 1$, $x \ln x$ for $x > 0$, and $|x|$ for all $x$. If the sign of the inequality is reversed, the function is called concave.

### 1.3.10 The compound Poisson distribution

First consider aggregate claims when $N$ has a Poisson distribution with mean $\lambda$, denoted $N \sim P(\lambda)$. $S$ then has a compound Poisson distribution with parameters $\lambda$ and $F(x)$. We refer back to section 1.3.2, where the results required for this distribution for $N$ are

$$\mathbf{E}[N] = V[N] = \lambda \tag{1.9}$$

$$M_N(r) = \exp\left\{\lambda\left(e^r - 1\right)\right\}. \tag{1.10}$$

Note that these results are given in of Actuaries & of Actuaries (2002).

These results then yield:

$$\mathbf{E}[S] = \lambda m_1 \tag{1.11}$$

$$V[S] = \lambda m_2, \tag{1.12}$$

where $m_1$ and $m_2$ are the first and second moments, respectively, of $F(x)$, as defined in (1.6).

$$M_S(r) = \exp\left\{\lambda\left(M_X(r) - 1\right)\right\}. \tag{1.13}$$

### 1.3.11 The compound negative binomial distribution

An alternative choice of distribution for $N$ is the negative binomial distribution, which has probability function

$$\Pr(N=n) = \binom{k+n-1}{n} p^k q^n \quad \text{for } n = 0,1,2,\dots .$$

The parameters of the distribution are $k\,(>0)$ and $p$, where $p+q=1$ and $0<p<1$. This distribution is denoted by $NB\,(k,p)$. When $N\char`~NB\,(k,p)$

$$\begin{aligned} \mathrm{E}\,[N] &= kq/p \\ V\,[N] &= kq/p^2 \\ M_N\,(r) &= p^k\,(1-qe^r)^{-k}\,. \end{aligned}$$

The special case $k=1$ leads to the Geometric distribution.

### 1.3.12 Mixture distributions

The exponential distribution is one of the simplest for insurance losses. Suppose that each individual in a large insurance portfolio incurs losses according to an exponential distribution. Practical knowledge of almost any insurance portfolio reveals that the means of these various distributions will differ among the policyholders. Thus the description of the losses in the portfolio is that which loss follows its own exponential distributions, i.e. the exponential distributions have means which differ from individual to individual.

A description of the variation among the individual means must now be found. One way to do this is to assume that the exponential means themselves follow a distribution. In the exponential case is convenient to make the following assumption. Let $\lambda_i = 1/\theta_i$ be the reciprocal of the mean loss for the $i$-th policyholder. Assume that the variation among the $\lambda_i$ can be described by a known gamma distribution $G(\alpha,\delta)$ where

$$f(\lambda) = \frac{\delta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}\exp\{-\delta\lambda\}\,,\lambda>0. \tag{1.14}$$

Take particular note that this is a PDF in $\lambda$ with known values of $\alpha$ and $\delta$.

This formulations has much in common with that used in Bayesian estimation. Indeed, the fundamental idea in Bayesian estimation is that the parameter of interest (here $\lambda$) can be treated as a random variable with a known distribution. Notice, however, that the purpose here is not to estimate the individual $\lambda_i$, but to describe the aggregate losses over the whole portfolio. Estimation of the individual $\lambda_i$ can be treated by Bayesian estimation, when the $G(\alpha,\delta)$ distribution would be referred to as a prior distribution. In this problem of describing the losses over the whole portfolio, the $G(\alpha,\delta)$ distribution is used to average the exponential distributions; the $G(\alpha,\delta)$ distribution is referred to as the mixing distribution and the resulting loss distribution as a mixture distribution.

The marginal distribution of $X$ is

$$\begin{aligned} f_X(x) &= \int f_{X,\lambda}(x,\lambda)d\lambda &(1.15)\\ &= \int f_\lambda(\lambda) f_{X|\lambda}(x|\lambda)d\lambda \\ &= \int_0^\infty \frac{\delta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}\exp\{-\delta\lambda\}.\lambda\exp\{-\lambda x\}\,d\lambda \\ &= \frac{\delta^\alpha}{\Gamma(\alpha)}.\frac{\Gamma(\alpha+1)}{(x+\delta)^{\alpha+1}} \quad (\text{a } G(\alpha+1, x+\delta) \text{ integral}) \\ &= \frac{\alpha\delta^\alpha}{(x+\delta)^{\alpha+1}},\ x>0 \end{aligned}$$

which can be recognised as a Pareto distribution $\sim Pa(\alpha,\delta)$. This gives a nice interpretation of the Pareto distribution; $Pa(\alpha,\delta)$ arises when exponentially distributed losses are averaged using a $G(\alpha,\delta)$ mixing distribution.

## 1.4 Ruin Theory

### 1.4.1 Introduction

One technical point needed later is:

A function $f(x)$ is described as being $o(x)$ as $x$ goes to zero, if

$$\lim_{x\to 0}\frac{f(x)}{x} = 0. \tag{1.16}$$

### 1.4.2 Notation

In Section 1.2 the aggregate claims generated by a portfolio of policies over a single time period were described. In the Actuarial literature, the word 'risk' is often used instead of the phrase 'portfolio of policies'. In this thesis both terms will be used, so that by a 'risk' will be meant either a single policy or a collection of policies. In this section the model in Section 1.2 will be taken a stage further by considering the claims generated by a portfolio over successive time periods.

Some general notation will be used throughout this thesis. The notation chosen is in line with that used in the Actuarial Core Reading.

$N(t)$ the number of claims generated by the portfolio in the time interval $[0,t]$, for all $t \geq 0$

$X_i$ the amount of the $i$-th claim, $i = 1,2,3\ldots$

$S(t)$ the aggregate claims in the time interval $[0,t]$, for all $t \geq 0$.

$\{X_i\}_{i=1}^{\infty}$ is a sequence of random variables. $\{N(t)\}_{t\geq 0}$ and $\{S(t)\}_{t\geq 0}$ are stochastic processes.

It can be seen that

$$S(t) = \sum_{i=1}^{N(t)} X_i, \tag{1.17}$$

with the understanding that $S(t)$ is zero if $N(t)$ is zero. The stochastic process $\{S(t)\}_{t\geq 0}$ as defined above is known as the aggregate claims process for the risk. The random variables $N(1)$ and $S(1)$ represent the number of claims and the aggregate claims respectively from the portfolio in the first unit of time. These two random variables correspond to the random variables $N$ and $S$, respectively, introduced in Section 1.2.

The insurer of this portfolio will receive premiums from the policy holders. It is convenient at this stage to assume, as will be assumed throughout this thesis, that the premium income is received continuously and at a constant rate. Here is some more notation:

$c$ the rate of premium income per unit time

so that the total premium received in the time interval $[0, t]$ is $ct$. It will also be assumed that $c$ is strictly positive.

### 1.4.3 The Surplus Process

Suppose that at time 0 the insurer has an amount of money set aside for this portfolio. This amount of money is called the initial surplus and is denoted $s$. It will always be assumed that $s \geq 0$. The insurer's surplus at any time $t\,(> 0)$ is a random variable since its value depends on the claims experience up to time $t$. The insurer's surplus at any time $t$ is defined by

$$U(t) = s + ct - S(t)\,,\ U(0) = s. \tag{1.18}$$

The model being used for the insurer's surplus incorporates many simplifications, as will any model of a complex real-life operation. Some important simplifications are that it is assumed that claims are settled as soon as they occur and that no interest is earned on the insurer's surplus.

The classical infinite time ruin probability is thus defined as

$$\psi(s) = \Pr\left(U(t) < 0 \text{ for some } t, 0 < t < \infty\right). \tag{1.19}$$

In this simplified model, the insurer will want to keep the probability of ruin as small as possible, or at least below a predetermined bound. The aim of this thesis is to study some possible ways in which the insurer can achieve this goal.

In some subsequent chapters this process is also called the Lundberg Process or Cramér-Lundberg Process.

### 1.4.4 The Poisson and Compound Poisson Process

In this subsection some assumptions will be made about the claim number process, $\{N(t)\}_{t\geq 0}$, and the claim amounts, $\{X_i\}_{i=1}^{\infty}$. The claim number process will be assumed to be a Poisson process, leading to a compound Poisson process $\{S(t)\}_{t\geq 0}$ for aggregate claims. The assumptions made in this subsection will hold for the remainder of the thesis.

#### 1.4.4.1 The Poisson Process

The Poisson process is an example of a counting process. Here the number of claims arising from a risk is of interest. Since the number of claims is being counted over time, the claim number process $\{N(t)\}_{t\geq 0}$ must satisfy the following conditions:

1. $N(0) = 0$, i.e. there are no claims at time 0.

2. for any $t > 0$, $N(t)$ must be integer valued

3. when $t_1 < t_2$, $N(t_1) \leq N(t_2)$, i.e. the number of claims over the time period is non-decreasing

4. when $t_1 < t_2$, $N(t_2) - N(t_1)$ represents the number of claims occurring in the time interval $(t_1, t_2)$.

The claim number process $\{N(t)\}_{t\geq 0}$ is defined to be a Poisson process with parameter $\lambda$ if the following conditions are satisfied:

5. $N(0) = 0$, and $N(t_1) \leq N(t_2)$ when $t_1 < t_2$

6. $\Pr(N(t+h) = r \mid N(t) = r) = 1 - \lambda h + o(h)$

7. when $t_1 < t_2$, the number of claims in the time interval $(t_1, t_2]$ is independent of the number of claims up to time $t_1$.

The reason why a process satisfying conditions 5. to 7. is called a Poisson process is that for a fixed value of $t$, the random variable $N(t)$ has a Poisson distribution with parameter $\lambda t$.

It is also worth noting that the distribution of the time to the first claim and the times between consecutive claims is exponential with parameter $\lambda$.

#### 1.4.4.2 The Compound Poisson Process

Here the Poisson process for the number of claims will be combined with a claim amount distribution to give a compound Poisson process for the aggregate claims process.

The following three important assumptions are made:

- the random variables $\{X_i\}_{i=1}^{\infty}$ are independent and identically distributed

- the random variables $\{X_i\}_{i=1}^{\infty}$ are independent of $N(t)$ for all $t \geq 0$
- the stochastic process $\{N(t)\}_{t\geq 0}$ is a Poisson process whose parameter is denoted $\lambda$

It can be shown that this last assumption means that for any $t \geq 0$, the random variable $N(t)$ has a Poisson distribution with parameter $\lambda t$.

With these assumptions the aggregate claims process, $\{S(t)\}_{t\geq 0}$, is called a compound Poisson process with Poisson parameter $\lambda$.

Since for a fixed value of $t$, $S(t)$ has a compound Poisson distribution, it follows from Section 1.3.10 that the process $\{S(t)\}_{t\geq 0}$ has mean $\lambda t m_1$, variance $\lambda t m_2$, and MGF $M_S(r,t)$, where

$$M_S(r,t) = \exp\{\lambda t (M_X(r) - 1)\}. \tag{1.20}$$

For the rest of this thesis (unless stated otherwise) the following (intuitively reasonable) assumption will be made concerning the rate of premium income:

$$c > \lambda m_1 \tag{1.21}$$

so that the insurer's premium income (per unit time) is greater than the expected claims outgo (per unit time). Sometimes $c$ will be written as

$$c = (1+\rho)\lambda m_1 \tag{1.22}$$

where $\rho\,(> 0)$ is the premium loading factor.

### 1.4.5 A Technicality

In the next subsection (and again in Chapter 4) a technical result will be needed concerning $M_X(r)$ (the moment generating function of the individual claim amount distribution), which for convenience will be presented here.

Let the distribution of $X$ follow an exponential distribution with parameter $\alpha$, so that $M_X(r)$ is finite for all $r < \gamma$ and

$$\lim_{r\to\gamma-} M_X(r) = \infty. \tag{1.23}$$

(So, if the $X_i$s have an exponential distribution with parameter $\alpha$, then $\gamma$ will be equal to $\alpha$.)

If $\gamma$ is finite, then we obtain

$$\lim_{r\to\gamma-} (\lambda M_X(r) - cr) = \infty. \tag{1.24}$$

Now it will be shown that (1.24) holds when $\gamma$ is infinite. This requires a little more care.

First note that there is a positive number, $\varepsilon$ say, such that

$$\Pr(X_i > \varepsilon) > 0. \tag{1.25}$$

The reason for this is that claim amounts are positive. This probability will be denoted by $\pi$. Then

$$M_X(r) \geq e^{r\varepsilon}\pi. \tag{1.26}$$

Hence

$$\lim_{r\to\infty}(\lambda M_X(r) - cr) \geq \lim_{r\to\infty}(\lambda e^{r\varepsilon}\pi - cr) = \infty. \tag{1.27}$$

### 1.4.6 The Cramér Exponent - Compound Poisson Processes

Lundberg's inequality (see Lundberg (1932)) states that

$$\psi(s) \leq \exp\{-Rs\}. \tag{1.28}$$

$R$, is a parameter associated with a surplus process known as the Cramér exponent. Its value depends upon the distribution of aggregate claims and on the rate of premium income. The Cramér exponent gives a measure of risk for a surplus process (from now on a Lundberg process). When aggregate claims are a compound Poisson process, the Cramér exponent is defined in terms of the Poisson parameter, the moment generating function of individual claim amounts and the premium income per unit time. More precisely, the Cramér exponent, is defined to be the unique positive root of

$$\lambda M_X(r) - \lambda - cr = 0. \tag{1.29}$$

So, $R$ is given by

$$\lambda M_X(R) = \lambda + cR. \tag{1.30}$$

If assumption (1.21) holds, it can be shown that there is indeed only one positive root of (1.29) as follows. Define $g(r) = \lambda M_X(r) - \lambda - cr$ and consider $g(r)$ over the interval $[0, \gamma]$. Note first that $g(0) = 0$. Further $g(r)$ is a decreasing function at $r = 0$, since

$$\frac{dg}{dr} = \lambda\frac{d}{dr}M_X(r) - c \tag{1.31}$$

so that the derivative of $g(r)$ at $r = 0$ is $\lambda\mu - c$ which is less than zero under the assumption (1.21). The second derivative is

$$\frac{d^2g}{dr^2} = \lambda\frac{d^2}{dr^2}M_X(r) \tag{1.32}$$

which is always strictly positive, hence, if the function $g(r)$ has a turning point it must be at the minimum of the function. Since $g(r)$ is a decreasing function at $r = 0$, and from

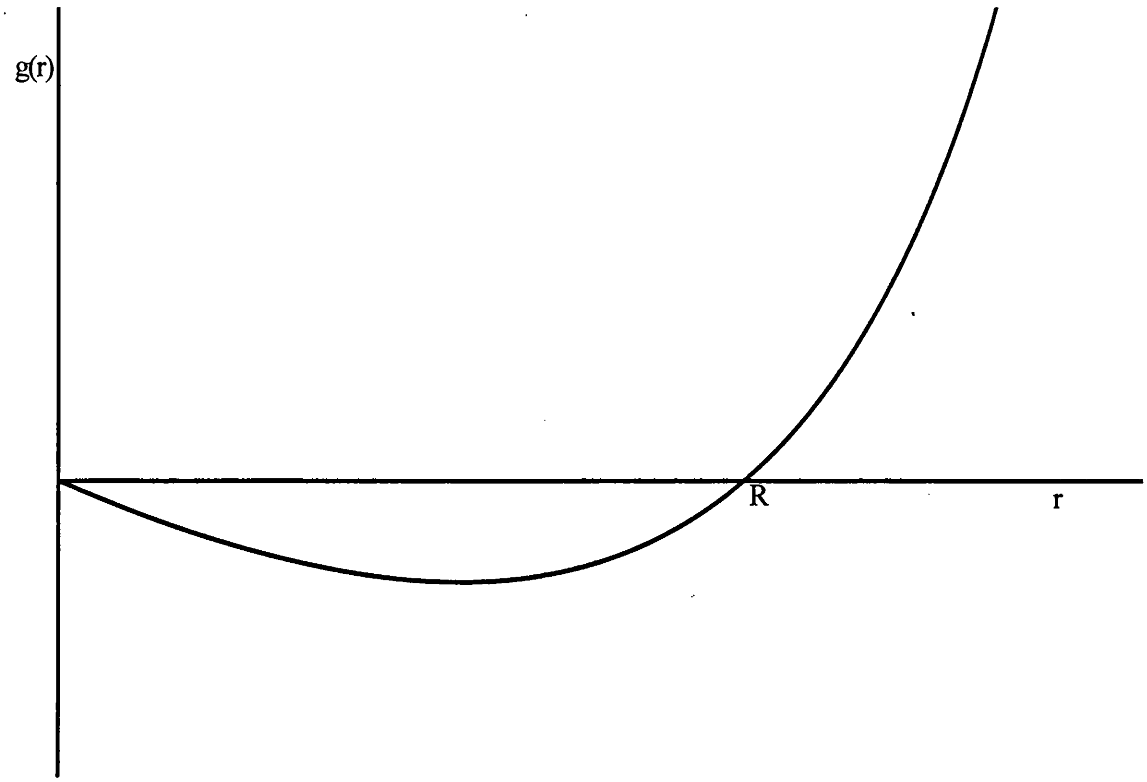


Figure 1.1: Graph of g(r).

(1.24) $\lim_{r \to \gamma -}(\lambda M_X(r) - cr) = \infty$, it must have a minimum turning point, so that there is at most one turning point in $[0, \gamma]$, and so the graph of $g(r)$ is as shown in Figure 1.1. Thus there is a unique positive number $R$ satisfying equation (1.29).

Equation (1.29) is an implicit equation for $R$. For some forms of $F(x)$ it is possible to solve explicitly for $R$; otherwise the equation has to be solved numerically.

## 1.5 Computational Issues

Algorithms for fitting models, which are outlined in later chapters and given explicitly in the appendix, were written using the mathematical software packages MATLAB, Mathematica and $C^{++}$. These algorithms usually involve nested procedures, requiring runs of trials within runs. Consequently, to obtain 10,000 replications to assess agreement with asymptotic results, each simulation - using a single set of parameter values - took up to 8 hours to run.

In chapter six, we use a range of numerical methods to find roots of equations. The problem is magnified when these roots are used in the further calculation of roots of other equations. The main difficulty we had is how to select an appropriate initial value that was accurate every time for changing parameter values. For one or two values, the problem is simplified, as we can take the time to look at the graphs and simply select a root by reading off the $x$-axis. Unfortunately, to produce the graphs required, we needed up to

1000 points for each case, so reading from a graph was not practical. To get around the problem, we came up with estimator equations, involving the changing parameters, that gave a close approximation to the root. This method was not foolproof and did require some error checking, but for the most part was successful.

## 1.6 Outline of Future Chapters

In this chapter, we illustrated the need for risk analysis, and the role it plays in our daily lives. We then established a simple model to define the risk business and defined all necessary functions needed to develop the theory in later chapters. In the next chapter we review the background theory necessary for analysing data arising from a Large Deviations Regime and conclude by highlighting an application of this to risk theory. In chapter three we take an in-depth review of the literature before commencing with the overall objective of the thesis. In chapter four we consider taking on new business to avert impending ruin. Chapter five takes a small deviation into hitting probabilities as a by-product of the work done in chapter four. Finally chapter six explores the linking of Large Deviation techniques with optimal control of reinsurance to minimise the probability of ruin.

# Chapter 2

# Large Deviation Theory

## 2.1 Introduction

The following chapter is taken from "An Introduction to Large Deviations for Teletraffic Engineers" by John T. Lewis and Raymond Russell. All examples have been replicated by myself, and the results can be found in other texts (see for example Schwartz & Weiss (1995)).

Roughly speaking, Large Deviations is a theory of rare events. It is probably the most active field in probability theory at present and one that has many surprising implications. One of its applications is to the analysis of the tails of probability distributions, and in recent years this aspect of the theory has been widely used in queuing theory.

## 2.2 Basic Ideas Underlying Large Deviations

Imagine a coin tossing experiment where we toss a coin $n$ times and record each result. There are 2 possible outcomes for each toss, giving $2^n$ possible outcomes in all. What can we say about the total number of heads? First, there are $n+1$ possible values for the total, ranging from 0 heads to $n$ heads. Secondly, of the $2^n$ possible outcomes, $\binom{n}{r} = \frac{n!}{(n-r)!r!}$ result in $r$ heads. If the coin is fair, every outcome is equally likely, and so the probability of getting $r$ heads is $\frac{\binom{n}{r}}{2^n}$. Thus, the average number of heads per toss has $n+1$ possible values, $0, \frac{1}{n}, \frac{2}{n}, \ldots 1$ and the value $\frac{r}{n}$ has weight $\frac{\binom{n}{r}}{2^n}$. To calculate the probability of the average number of heads per toss being in a particular interval, we sum the weights for values inside that interval. Thus

$$\Pr(x < M_n < y) = \sum_{x<\frac{r}{n}<y} \frac{\binom{n}{r}}{2^n}, \text{ where } \frac{1}{n}\sum_{i=1}^{n} X_i, \text{ is denoted by } M_n$$

We can thus take an integer $n$ and numbers, $x$ and $y$, and compute the value of the expression above. These probabilities then form the distribution of $M_n$; Figure 2.1 shows this distribution for $n = 16, 32, 64$ and 128. We can see clearly the Law of Large Numbers

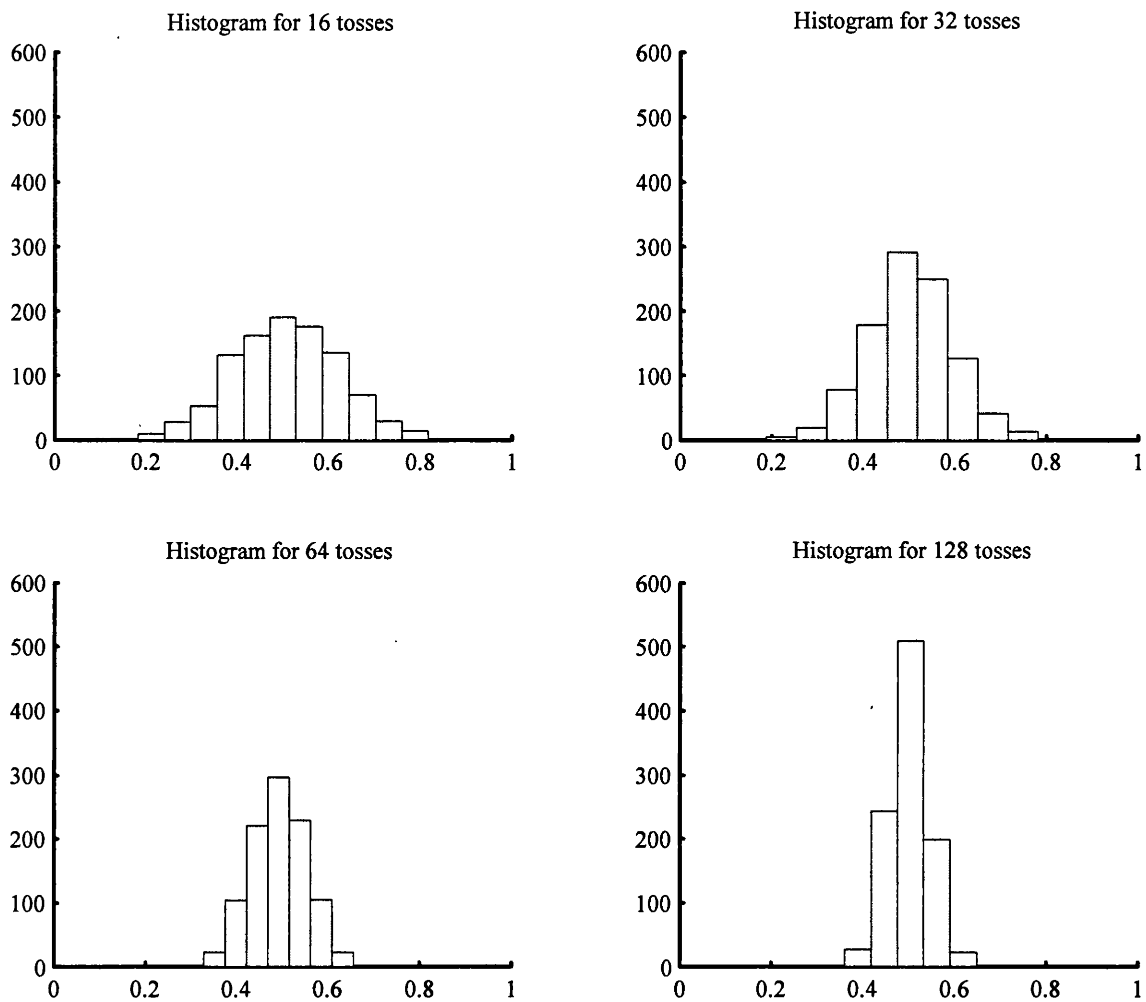


Figure 2.1: Distribution of $M_n$.

at work; as $n$ increases, the distribution becomes more and more sharply peaked about the mean, $\frac{1}{2}$, and the tails become smaller and smaller.

It is possible to pick some point $x$, greater than $\frac{1}{2}$ and calculate, for a range of values of $n$, the logarithm of the probability of $M_n$ exceeding $x$. For instance, with $x = 0.6$, Figure 2.2 shows a plot of a simulation of $\ln P(M_n \geq x)$ against $n$ for $n$ up to 100.

It is clear that, although initially a little jumpy, the plot of functions becomes linear for large $n$. The calculations are then repeated for different values of $x$. Figure 2.3 shows that the same thing happens; no matter what value of $x$, $\left(x > \frac{1}{2}\right)$ is selected, the functions will always be (overall) linear for $n$ large. The value of $x$ affects how quickly it becomes linear, and the asymptotic slope, denoted by $I(x)$. For values of $x$ from $\frac{1}{2}$ to 1, the asymptotic slope is measured (by taking a line of best fit through the data) in each case, and the values of $I(x)$ are plotted against $x$. The same is done for $\ln \Pr(M_n \leq x)$ for a range of values of

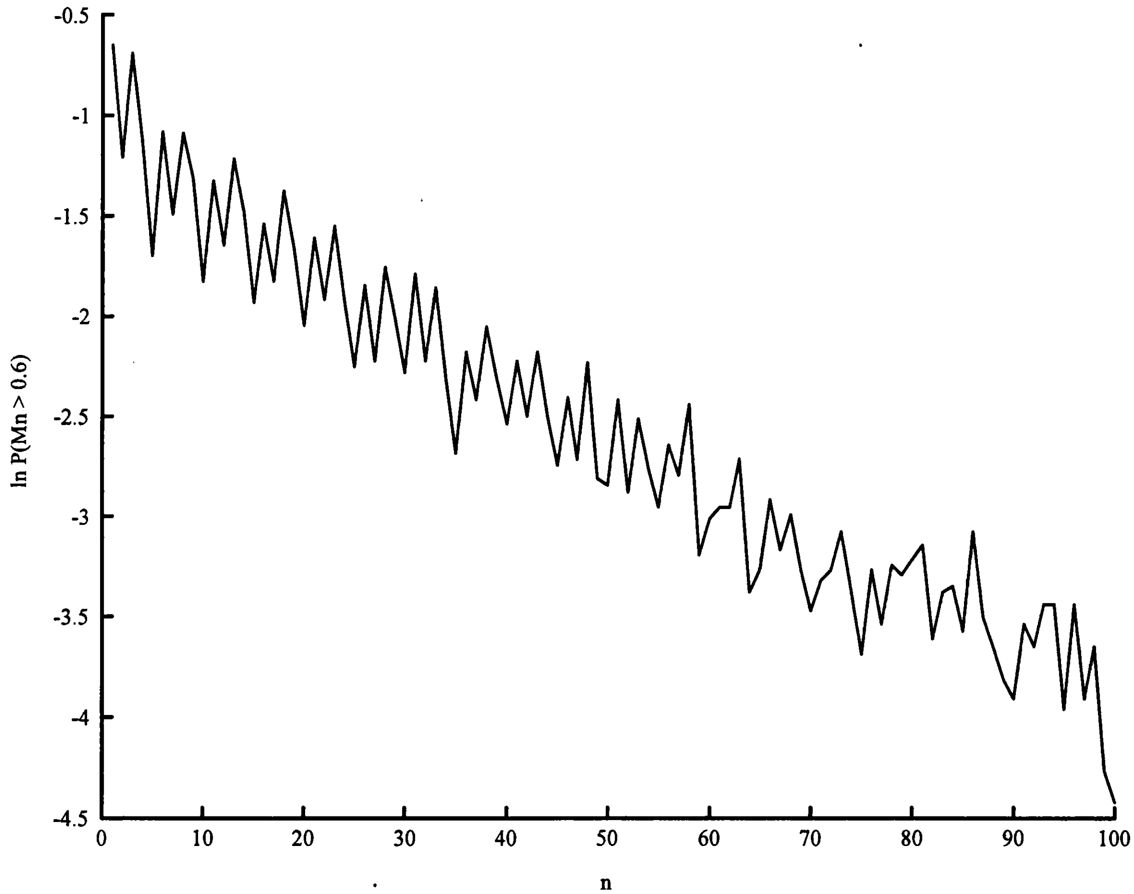


Figure 2.2: $\ln \Pr(M_n > 0.6)$ against $n$.

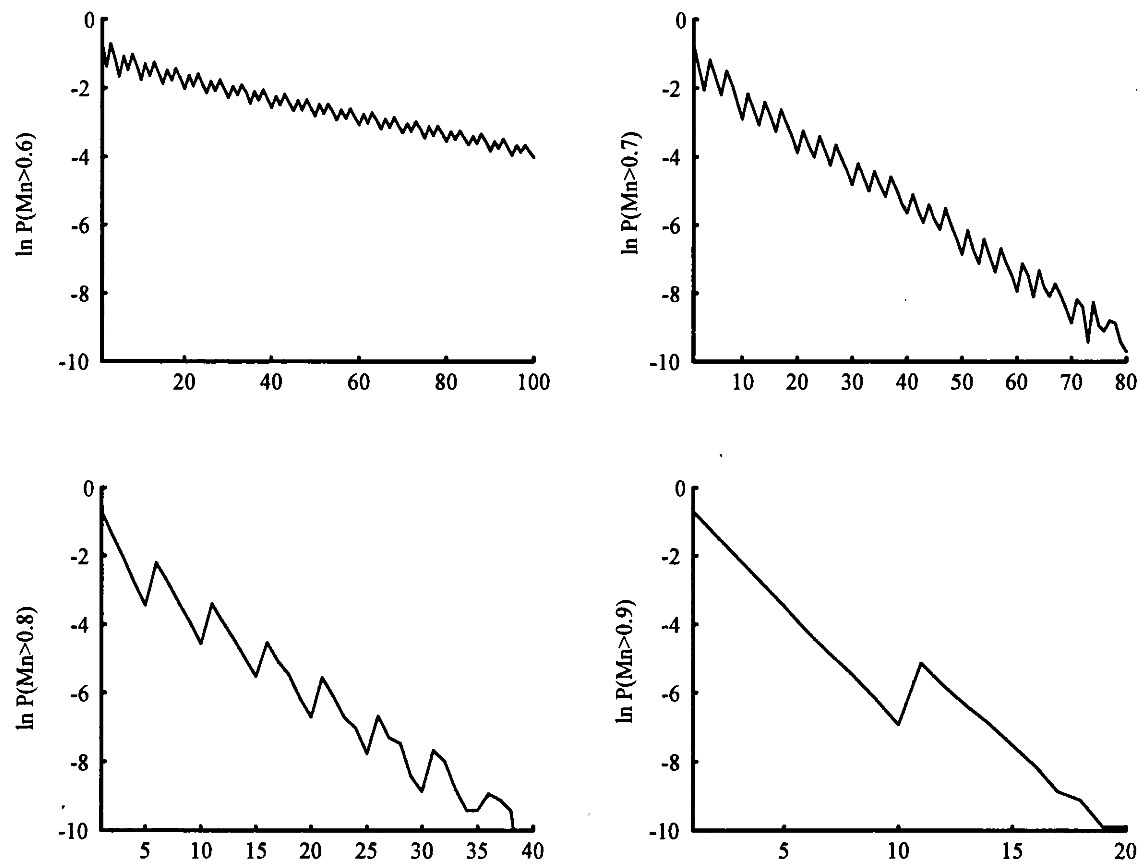


Figure 2.3: $\ln \Pr(M_n > x)$ against $n$ for several values of $x$.

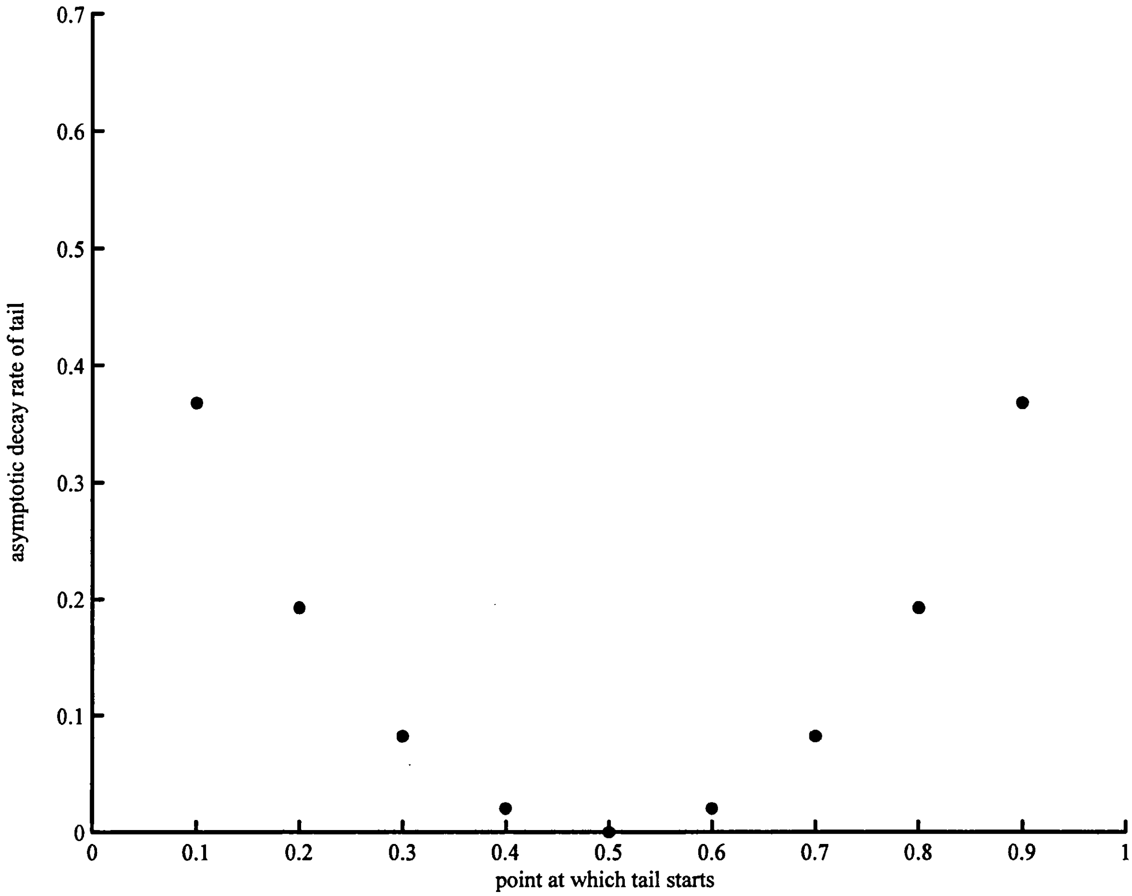


Figure 2.4: Measured decay rates of tails against starting point of tail.

$x$ from 0 to $\frac{1}{2}$; see Figure 2.4

From this we can state the following:

The tail of the distribution of the average number of heads in $n$ tosses decays exponentially as $n$ increases. Figure 2.4 tells us the local rate at which a tail decays as a function of the point from which the tail starts: we have built up a picture of the rate function, $I(x)$.

Next we plot the graph of the function $(x \ln(x) + (1-x)\ln(1-x) + \ln(2))$ against $x$ and compare it with our previous plot. We then superimpose a plot of this function over the function in Figure 2.4. The results are shown in Figure 2.5. We see that the function is a formula for $I(x)$, the rate function for coin tossing.

This is one of the goals of Large Deviation theory; to provide a systematic way of calculating the rate-function.

Here, we have found that, for coin tossing, the tails of the distribution of $M_n$, the average number of heads in $n$ tosses, decay exponentially fast:

$$\Pr(M_n > x) \simeq e^{-nI(x)} \text{ for } x > \frac{1}{2},$$

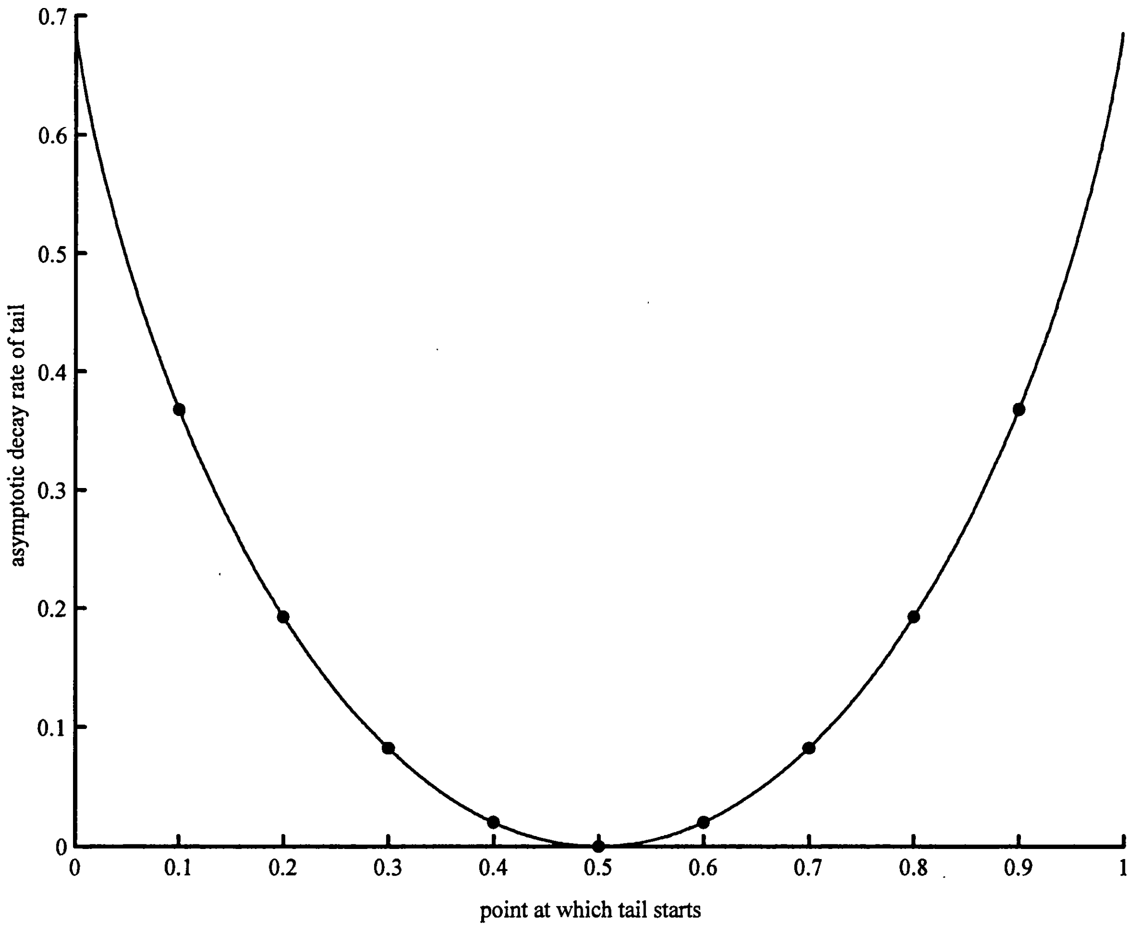


Figure 2.5: Measured decay rates with the rate function superimposed.

$$\Pr(M_n < x) \simeq e^{-nI(x)} \text{ for } x < \frac{1}{2},$$

as $n$ becomes large; in fact, Figure 2.2 shows that this approximation is quite good for surprisingly small values of $n$. The numerical approach can be more formally established using Stirling's Formula;

Proof: Consider $P(M_n < a)$

$$\Pr(M_n < a) = \sum_{k=0}^{[a]-1} \binom{n}{k} \frac{1}{2^n}.$$

If $a < \frac{1}{2}$, then each term in the sum is bounded by ${}^nC_{[na]}$ and so

$$\Pr(M_n < a) \leq [na] \binom{n}{[na]} \frac{1}{2^n} = A_n, \text{ say.}$$

Now consider $\ln A_n$: we can rewrite $\ln^n C_{[na]}$ as

$$-\frac{[na]}{n}\left(\frac{1}{[na]}\ln[na]! - \ln[na]\right) - \frac{[n(1-a)]}{n}\left(\frac{1}{[n(1-a)]}\ln[n(1-a)]! - \ln[n(1-a)]\right)$$
$$+\left(\frac{1}{n}\ln n! - \ln n\right) - \frac{[na]}{n}\ln\left(\frac{[na]}{n}\right) - \frac{[n(1-a)]}{n}\ln\left(\frac{[n(1-a)]}{n}\right)$$

and use Stirling's Formula

$$\lim_{n\longrightarrow\infty}\left(\frac{1}{n}\ln n! - \ln n\right) = -1$$

to get

$$\lim_{n\longrightarrow\infty}\frac{1}{n}\ln A_n = -a\ln a - (1-a)\ln(1-a) - \ln 2.$$

Let us look again at $P(M_n < a)$: not only can we bound it above by something which decays exponentially, but we can also bound it below by something which decays exponentially at the same rate. So long as $a > 0$.

$$\Pr(M_n < a) \geq \binom{n}{[na]-1}\frac{1}{2^n}.$$

Now,

$$\lim_{n\longrightarrow\infty}\frac{1}{n}\ln\binom{n}{[na]-1}\frac{1}{2n} = -a\ln a - (1-a)\ln(1-a) - \ln 2.$$

So we have, for $0 < a < \frac{1}{2}$,

$$\lim_{n\longrightarrow\infty}\frac{1}{n}\ln P(M_n < a) = -a\ln a - (1-a)\ln(1-a) - \ln 2.$$

$$= -I(a)$$

$$= -\inf_{x<a} I(x),$$

our first Large Deviation principle, established using only a little combinatorics.

Notice that a consequence of this result is the Weak Law of Large Numbers for coin tossing. It states that, as $n$ increases, the distribution of $M_n$ becomes more sharply peaked about the mean:

$$\lim_{n\longrightarrow\infty} \Pr\left(\left|M_n - \frac{1}{2}\right| < \varepsilon\right) = 1$$

for each positive number $\varepsilon$. This is equivalent to stating that, as $n$ increases, the tails become smaller and smaller:

$$\lim_{n\longrightarrow\infty} \Pr\left(\left|M_n - \frac{1}{2}\right| > \varepsilon\right) = 0$$

for each positive number $\varepsilon$. This can be proved as follows; first, let us write out $\Pr\left(\left|M_n - \frac{1}{2}\right| > \varepsilon\right)$ in detail and see if we can approximate it or get a bound on it:

$$\Pr\left(\left|M_n - \frac{1}{2}\right| > \varepsilon\right) = \Pr\left(M_n < \frac{1}{2} - \varepsilon\right) + \Pr\left(M_n > \frac{1}{2} + \varepsilon\right).$$

Now we have found that, for coin tossing, the tails of the distribution of $M_n$ decay exponentially fast:

$$\begin{aligned} \Pr(M_n > x) &\simeq e^{-nI(x)} \text{ for } x > \frac{1}{2}, \\ \Pr(M_n < x) &\simeq e^{-nI(x)} \text{ for } x < \frac{1}{2}, \end{aligned}$$

as $n$ becomes large, with $I(x) > 0$; it follows that both terms on the right-hand side of the equation decay to zero so that

$$\lim_{n\longrightarrow\infty} \Pr\left(\left|M_n - \frac{1}{2}\right| > \varepsilon\right) = 0$$

for each positive number $\varepsilon$. This shows that the Weak Law of Large Numbers is a consequence of the Large Deviation Principle.

## 2.3 Why 'Large' in Large Deviations?

Recall what the Central Limit Theorem [See Feller (1968)] tells us: if $X_1$, $X_2$, $X_3\ldots$ is a sequence of independent and identically distributed random variables with mean $\mu$ and variance $\sigma^2 < \infty$, then the average of the first n of them $M_n = \frac{1}{n}(X_1 + \cdots + X_n)$, is approximately normal with mean $\mu$ and variance $\frac{\sigma^2}{n}$. That is, its probability density function is

$$f(x) \simeq \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left[\frac{-n(x-\mu)^2}{2\sigma^2}\right],$$

and the approximation is only valid for $x$ within about $\frac{\sigma}{\sqrt{n}}$ of $\mu$. If we ignore the normalising constant in $f$ and compare the exponential term with the approximation that Cramér's

Theorem gives us, we see that the terms

$$\frac{(x-\mu)^2}{2\sigma^2}$$

occupy a position analogous to that of the rate function.

Let us look at a simple coin tossing experiment: for x close to $\frac{1}{2}$, we can expand our rate function in a Taylor series:

$$x\ln(x) + (1-x)\ln(1-x) + \ln 2 = \frac{\left(x-\frac{1}{2}\right)^2}{2.\frac{1}{4}} + \text{asymptotically small terms.}$$

The mean of each toss of a coin is $\frac{1}{2}$ and the variance of each toss is $\frac{1}{4}$ thus the rate function for coin tossing gives us the Central Limit Theorem. In general, whenever the rate function can be approximated near its maximum by a quadratic form we can expect the Central Limit Theorem to hold.

So much for the similarities between the CLT and Large Deviations; the name Large Deviations arises from the contrast between them. The CLT governs random fluctuations only near the mean - deviations from the mean of the order of $\frac{\sigma}{\sqrt{n}}$. Fluctuations which are of the order of $\sigma$ are, relative to typical fluctuations, much bigger; they are large deviations from the mean. They happen only rarely, and so Large Deviation theory is often described as the theory of rare events - events that take place away from the mean, and out in the tails of the distribution; thus Large Deviation theory can also be described as a theory of extreme events.

## 2.4 An Application to Risk Theory.

**Theorem 2.4.1 (Cramer's Theorem)** *Let $X_1,\ X_2,\ X_3,\ldots$ be a sequence of bounded independent and identically distributed random variables each with mean m, and let*

$$M_n = \frac{1}{n}\left(X_1 + \cdots + X_n\right)$$

*denote the empirical mean; then the tails of the probability distribution of $M_n$ decay exponentially with increasing n at a rate given by a convex rate function $I(x)$ :*

$$\begin{aligned}\Pr\left(M_n > x\right) &\approx e^{-nI(x)} \qquad \textit{for } x > m,\\ \Pr\left(M_n < x\right) &\approx e^{-nI(x)} \qquad \textit{for } x < m.\end{aligned}$$

Large Deviation theory has been applied to sophisticated models in risk theory. To get a flavour of how this is done, consider, again, the classical Lundberg model (1.18). The sizes of the claims are random and there is therefore the risk that at the end of some planning period of length $T$, the total amount paid in settlement of claims will exceed the total income from premium payments over the period. This risk is inevitable, but the company will want to ensure that it is small, either in the interest of its shareholders, or because it is required by its reinsurers, or some regulatory agency. Thus we are interested in the small probabilities concerning the sum of a large number of random variables, this problem lies squarely in the scope of Large Deviations.

If the sizes $X_t$ of claims are independent and identically distributed then we can apply Cramér's Theorem [see Straub (1997)] to approximate the probability of ruin; the probability that the amount $\sum_{x=1}^{T} X_t$ paid out during a period $T$ exceeds the premium income $cT$ received in that period:

$$\Pr\left[\sum_{x=1}^{T} X_t > cT\right] \approx e^{-TI(c)}$$

So if we require that the risk of ruin be small (say $e^{-r}$) for some large positive number $r$, then we can use the rate function $I$ to choose an appropriate value of $c$. From

$$\begin{aligned} \Pr\left[\frac{1}{T}\sum_{x=1}^{T} X_t > c\right] &\approx e^{-r} \\ e^{-TI(c)} &\approx e^{-r} \\ I(c) &\approx \frac{r}{T} \end{aligned}$$

Since $I(x)$ is convex [see section 1.3.9], it is monotonically increasing for $x$ greater than the mean of $X_t$ and so the equation

$$I(x) = \frac{r}{T}$$

has a unique solution for $c$.

Of course, to solve this equation, we must know what $I(x)$ is and that means knowing the statistics of the sizes of the claims. For example if the size of each claim is normally distributed [see section 1.3.6] with mean $\mu$ and variance $\sigma^2$ then the rate function is

$$I(x) = \frac{(x-\mu)^2}{2\sigma^2}$$

It is easy to find the solution to the equation for $c$ in this case: it is

$$c = \mu + \sigma\sqrt{\frac{2r}{T}};$$

thus the premium should be set so that the daily income is the mean claim size plus an additional amount to cover the risk. The ratio

$$(c-\mu)/\mu$$

is called the safety loading; in this case it is given by

$$\frac{\sigma}{\mu}\sqrt{\frac{2r}{T}}.$$

## 2.5 The Large Deviations Principle

The *large deviations principle* (LDP) characterises the limiting behaviour, as $\epsilon \longrightarrow 0$, of a family of probability measures $\{\mu_\epsilon\}$ on $(\mathcal{X}, \mathcal{B})$ in terms of a *rate function.* This characterisation is via asymptotic upper and lower exponential bounds on the values that $\mu_\epsilon$ assigns to measurable subsets of $\mathcal{X}$. Here, $\mathcal{X}$ is a topological space so that open and closed subsets of $\mathcal{X}$ are well defined, and the simplest situation is when elements of $\mathcal{B}_\mathcal{X}$, the Borel $\sigma$-field on $\mathcal{X}$ are of interest.

**Definition 2.5.1** *A rate function $I$ is a lower semicontinuous mapping $I : \mathcal{X} \longrightarrow [0, \infty)$ (such that for all $\alpha \in [0, \infty)$, the level set $\Psi_I(\alpha) \triangleq \{x : I(x) \leq \alpha\}$ is a closed subset of $\mathcal{X}$). A good rate function is a rate function for which all the level sets $\Psi_I(\alpha)$ are compact subsets of $\mathcal{X}$. The effective domain of $I$, denoted $\mathcal{D}_I$, is the set of points in $\mathcal{X}$ of finite rate, namely, $\mathcal{D}_I \triangleq \{x : I(x) < \infty\}$. When no confusion occurs, we refer to $\mathcal{D}_I$ as the domain of $I$.*

Note that if $\mathcal{X}$ is a metric space, the lower semicontinuous property may be checked on sequences, i.e., $I$ is lower semicontinuous if and only if $\liminf_{x_n \longrightarrow x} I(x_n) \geq I(x)$ for all $x \in \mathcal{X}$. A consequence of a rate function being good is that its infimum is achieved over closed sets.

Some notation is required for the next definition. For any set $-$, $\bar{-}$ denotes the closure of $-$, $-^\circ$ the interior of $-$, and $-^c$ the complement of $-$. The infimum of a function over an empty set is interpreted as $\infty$.

**Definition 2.5.2** *$\{\mu_\epsilon\}$ satisfies the large deviation principle with a rate function $I$ if, for all $- \in \mathcal{B}$,*

$$-\inf_{x \in -^\circ} I(x) \leq \liminf_{\epsilon \longrightarrow 0} \epsilon \log \mu_\epsilon(-) \leq \limsup_{\epsilon \longrightarrow 0} \epsilon \log \mu_\epsilon(-) \leq -\inf_{x \in \bar{-}} I(x). \tag{2.1}$$

**Remark 1** *Note that in (2.1) $\mathcal{B}$ need not necessarily be the Borel $\sigma$-field.*

The right- and left-hand sides of (2.1) are referred to as upper and lower bounds, respectively.

It is obvious that if $\{\mu_\epsilon\}$ satisfies the LDP with rate function $I$ and $- \in \mathcal{B}$ is such that

$$\inf_{x \in -^\circ} I(x) = \inf_{x \in \bar{-}} I(x) \triangleq I_-, \tag{2.2}$$

then

$$\lim_{\epsilon \longrightarrow 0} \epsilon \log \mu_\epsilon(-) = -I_-. \tag{2.3}$$

A set – that satisfies (2.2) is called an *I continuity set.* In general, the LDP implies a precise limit in (2.3) only for *I* continuity sets.

### 2.5.1 Sample path large deviations

In this thesis, the interest is actually in rare events that depend on a collection of random variables, or, more generally, on a random process. Whereas some of these questions may be cast in terms of empirical measures, this is not always the most fruitful approach. Our interest in Chapter 6 lies in the probability that a *path* of a random process hits a particular set.

### 2.5.2 The contraction principle

The contraction principle is a theorem that states how a large deviation principle on one space "pushes forward" to a large deviation principle on another space via a continuous function.

**Definition 2.5.3** *Using previous notation, let $\mathcal{Y}$, be another topological space, let $T : \mathcal{X} \rightarrow \mathcal{Y}$ be a continuous function and let $\nu_\epsilon = T(\mu_\epsilon)$ be the push-forward measure of $\mu_\epsilon$ by $T$, i.e., for each measurable set/event $\mathcal{E}$,*

$$\nu_\epsilon(\mathcal{E}) \triangleq \mu_\epsilon\left(T^{-1}(\mathcal{E})\right).$$

*Then $\{\nu_\varepsilon\}_{\varepsilon>0}$ satisfies the large deviation principle on $\mathcal{Y}$ with rate function $J : \mathcal{Y} \rightarrow [0, \infty)$ given by*

$$J(y) \triangleq \inf\{I(x) \,|x \in \mathcal{X} \text{ and } y = T(x)\}.$$

## 2.6 Summary

This chapter outlined the theory behind the study of extreme events. This framework is known as the theory of Large Deviations and is one of the most active and dynamic branches of modern applied probability. This framework provides powerful tools for computing the probability of extreme events when the more conventional approaches like the law of large numbers and the central limit theorem fail.

In the next chapter, we outline previous work with a review of the literature, we then proceed to study the linking of LD techniques with elements of control and optimisation theory.

# Chapter 3

# A Review of the Literature

## 3.1 Introduction

Insurance throws up many mathematical problems in many different areas. As we have already seen, the study of distributions with large tails and how this can be applied to Risk theory is a new and exciting area of research. A good introduction to the area of Risk theory and Ruin can be found in of Actuaries (2002). The key assumption in all the models studied in this review is that the occurrence of a claim and the size of a claim can be studied separately. Thus, a claim occurs according to some simple model for events occurring in time, then the amount of the claim is chosen from a distribution describing the claim amount. A range of statistical techniques can be used to describe the distribution of a random variable. The object is to describe the variation in claim sizes by finding a loss distribution that adequately describes the claims which actually occur. For most of the studies here, the exponential distribution is used to demonstrate the behaviour of the model. In most cases the exponential distribution will allow results to be computed analytically, thus making the presentation of results much easier.

Recall from Chapter 1 that

$$S(t) = \sum_{i=1}^{N(t)} X_i$$

with the understanding that $S(t)$ is zero if $N(t)$ is zero. The stochastic process $\{S(t)\}_{t\geq 0}$ is known as the aggregate claims process for the risk. The random variables $N(1)$ and $S(1)$ represent the number of claims and the aggregate claims respectively from the portfolio in the first unit of time. With a rate of premium income $c$ and an initial surplus $s$, the formula for the classic Cramér-Lundberg process can be written[1]:

$$U(t) = s + ct - S(t),\ U(0) = s,$$

The model being used for the insurer's surplus incorporates many simplifications, as will

[1]This formula is given in Chapter 1, but is repeated here for the reader's benefit.

any model of a complex real-life operation. Some important simplifications are that it is assumed that claims are settled as soon as they occur and that no interest is earned on the insurers surplus. Despite its simplicity this model can give an interesting insight into the mathematics of an insurance operation and is the basis of all the literature in this review.

## 3.2 Methods

The method of selection carried out for this review is a simple systematic process of inclusion. Searches were carried out using online resources such as EBSCO, Kluwer, Science Direct, Synergy and Springer. The search criteria was then split into three main categories: Ruin Probability, Large Deviations and Optimal Control. Ruin Probability, by definition, is associated to insurance. The latter two headings needed filtering for topics relating to insurance. This method gives a relatively broad range of literature, but should give the reader an insight into the area of research, and at the same time allows a broader scope of study for the thesis. These areas of research are directly relevant to the research presented in this thesis.

### 3.2.1 Ruin Probability

In risk theory, the classical risk model is a compound Poisson risk model as described in (1.18). One of the key quantities in the classical risk model is the ruin probability, denoted by $\psi(s)$ in (1.19). In general, it is very difficult to derive explicit and closed expressions for the ruin probability. However, under suitable conditions, we can obtain some approximations to the ruin probability.

The pioneering works on approximations were achieved by Cramér and Lundberg as early as the 1930s under the Cramér-Lundberg condition. The Cramér-Lundberg approximations provide an exponential description of the ruin probability in the classical risk model. They have become two standard results on ruin probabilities in risk theory.

The original proofs of the Cramér-Lundberg approximations were based on Wiener-Hopf methods and can be found in Cramér (1930), Cramér (1955), Lundberg (1926) and Lundberg (1932). However, these two results can be proved in different ways now. For example, the martingale approach of Gerber (1973), Gerber (1979), Wald's identity in Ross (1996), and the induction method in Goovaerts et al. (1990) have been used to prove the Lundberg inequality. Further, the Cramér-Lundberg asymptotic formula can be obtained simply from the key renewal theorem for the solution of a defective renewal equation (see for instance Feller (1971)). All these methods are much simpler than the Wiener-Hopf methods used by Cramér and Lundberg and have been used extensively in risk theory and other disciplines. In particular, the martingale approach is a powerful tool for deriving exponential inequalities for ruin probabilities. See Dassios & Embrechts (1989) for a review of this topic. In addition, the induction method is very effective for us to improve and generalise the Lundberg inequality. The applications of the method for the generalisations and improvements of the Lundberg inequality can be found in Cai & Garrido (1999), Willmot & Lin (1994), Willmot (1996), Willmot (1994) and Willmot & Lin (2001).

Further, the key renewal theorem has become a standard method for deriving exponential asymptotic formulae for ruin probabilities and related ruin quantities, such as the distributions of the surplus just before ruin, the deficit at ruin, and the amount of claim causing ruin; see for example Gerber & Shiu (1998) or Willmot & Lin (2001).

Moreover, the Cramér-Lundberg asymptotic formula is also available for the solution of the defective renewal equation. See for example Gerber (1970) or Schmidli (1997) for details. Also, a generalised Lundberg inequality for the solution of the defective renewal equation can be found in Willmot et al. (2001).

On the other hand, the solution of the defective renewal equations can be expressed as the tail of a compound Geometric distribution, known as Beekman's convolution series.

Thus, the ruin probability in the classical risk model can be characterised as the tail of a compound geometric distribution. Indeed, the Cramér-Lundberg asymptotic formula and the Lundberg inequality can be stated generally for the tail of a compound Geometric distribution. The tail of a compound Geometric distribution is a very useful probability model arising in many applied probability fields such as risk theory, queuing, and reliability. More applications of a compound geometric distribution in risk theory can be found in Kalashnikov (1997), Willmot & Lin (2001) among others.

It is clear that the Cramér-Lundberg condition plays a critical role in the Cramér-Lundberg approximations. However, there are many interesting claim size distributions that do not satisfy the Cramér-Lundberg condition. For example, when the moment generating function of a distribution does not exist or a distribution is heavy-tailed such as Pareto and lognormal distributions, the Cramér-Lundberg condition is not valid. Further, even if the moment generating function of a distribution exists, the Cramér-Lundberg condition may still fail. In fact, there exist some claim distributions, including certain inverse Gaussian and generalised inverse Gaussian distributions, that are said to be medium tailed; see for example Embrechts (1983) for details.

For these medium and heavy-tailed distributions, the Cramér-Lundberg approximations are not applicable. Indeed, the asymptotic behaviours of the ruin probability in these cases are totally different from those where the Cramér-Lundberg condition holds. For instance, if $F$ is a subexponential distribution, then the ruin probability $\psi(s)$ has the following asymptotic form

$$\psi(s) \sim \frac{1}{\rho}\left(1 - F(s)\right) \text{ as } s \to \infty,$$

which implies that ruin is asymptotically determined by a large claim. A review of the asymptotic behaviour of the ruin probability with medium and heavy-tailed claim size distributions can be found in Embrechts et al. (1997), Embrechts & Veraverbeke (1982). For a more general discussion on ruin probabilities with heavy tailed or subexponential distributions, see Ramsay (2003), Konstantinides et al. (2002), Kalashnikov & Konstantinides

(2000).

Dickson (1994) adopted a truncated Lundberg condition and assumed that for any $u > 0$ there exists a constant $\kappa_u > 0$ so that

$$\int_0^u e^{\kappa_u x} dF(x) = 1 + \rho. \tag{3.1}$$

Under the truncated condition (3.1), Dickson (1994) derived an upper bound for the ruin probability, and further Cai & Garrido (1999) gave an improved upper bound and a lower bound for the ruin probability.

The truncated condition (3.1) applies to any positive claim size distribution with a finite mean. In addition, even when the Cramér-Lundberg condition holds, the upper bound may be tighter than the Lundberg upper bound; see Cai & Garrido (1999) for details.

The Cramér-Lundberg approximations are also available for ruin probabilities in some more general risk models. For instance, if the claim number process $N(t)$ in the classical risk model is assumed to be a renewal process, the resulting risk model is called the compound renewal risk model or the Sparre Anderson risk model. In this risk model, interclaim times $\{T_1, T_2, \dots\}$ from a sequence of independent and identically distributed positive random variables with common distribution function $G(t)$ and common mean $\int_0^\infty \bar{G}(t)dt = (1/\alpha) > 0$. The ruin probability in the Sparre Andersen risk model, denoted by $\psi^0(s)$, satisfies the defective renewal equation for $\psi(s)$ and is thus the tail of a compound Geometric distribution (see 1.3.11). However, the underlying distribution in the defective renewal equation in this case is unknown in general; see for example Embrechts & Klüppelberg (1993), Grandell (1991) for details.

Further, if the claim number process $N(t)$ in the classical risk model is assumed to be a stationary renewal process, the resulting risk model is called the compound stationary renewal risk model. In this risk model, interclaim times $\{T_1, T_2, \dots\}$ form a sequence of independent positive random variables; $\{T_2, T_3, \dots\}$ have a common distribution function $G(t)$ as that in the compound risk model; and $T_1$ has an equilibrium distribution function $G_e(t) = \alpha \int_0^t \bar{G}(s)ds$.

The Cramér-Lundberg approximations to the ruin probability in a risk model when the claim number process is a Cox process (see Cox & Isham (1980)) can be found in Björk & Grandell (1988), Grandell (1991), Schmidli (1996). For the Lundberg inequality in the Poisson shot noise delayed-claims risk model, see Brémaud (2000). Moreover, the Cramér-Lundberg approximations to ruin probabilities in dependent risk models can be found in Gerber (1982), Müller & Pflug (2001), Promislow (1991).

In addition, the ruin probability in the perturbed compound Poisson risk model with diffusion also admits the Cramér-Lundberg approximations. In this risk model, the surplus process $X(t)$ satisfies

$$X(t) = s + ct - S(t) + W_t, \quad t \geq 0, \tag{3.2}$$

where $\{W_t, t \geq 0\}$ is a Wiener process, independent of the Poisson process $\{N(t), t \geq 0\}$ and the claim sizes $\{X_1, X_2, \dots\}$, with infinitesimal drift 0 and infinitesimal variance $2D > 0$.

Denote the ruin probability in the perturbed risk model by $\psi_p(s)$ and assume that there exists a constant $R > 0$ so that

$$\lambda \int_0^\infty e^{Rx} dP(x) + DR^2 = \lambda + cR. \tag{3.3}$$

Then Dufresne & Gerber (1991) derived the following Cramér-Lundberg asymptotic formula

$$\psi_p(s) \sim C_p e^{-Rs} \text{ as } s \to \infty,$$

and the following Lundberg upper bound

$$\psi_p(s) \leq e^{-Rs}, \quad s \geq 0, \tag{3.4}$$

where $C_p > 0$ is a known constant. For the Cramér-Lundberg approximations to ruin probabilities in more general perturbed risk models, see Furrer & Schmidli (1994), Schmidli (1995). A review of perturbed risk models and the Cramér-Lundberg approximations to ruin probabilities in these models can be found in Schlegel (1998). For more general references to the perturbed model, we refer to Chiu & Yin (2003), Tsai (2003)

We point out that the Lundberg inequality is also available for ruin models with interest. For example, Sundt & Teugels (1995) derived the Lundberg upper bound for the ruin probability in the classical risk model with a constant force of interest; Cai & Dickson (2003) gave exponential upper bounds for the ruin probability in the Sparre Andersen risk model with a constant force of interest; Yang (1999) obtained exponential upper bounds for the ruin probability in a discrete time risk model with a constant rate of interest; and Cai (2002) derived exponential upper bounds for ruin probabilities in generalised discrete time risk models with dependent rates of interest. A review of risk models with interest and investment and ruin probabilities in these models can be found in Paulsen (1998). For more topics on the Cramér-Lundberg approximations to ruin probabilities, we refer to Asmussen (2000), Embrechts et al. (1997), Gerber (1979), Grandell (1991), Rolski et al. (1999), Willmot & Lin (2001), and references therein.

For various methods of recursive calculation of ruin probabilities, with and without interest, see Cardoso & Waters (2003) Cardoso & Egidio Dos Reis (2002) Brekelmans & de Waegenaere (2001) Dickson & Waters (1991) de Vylder & Goovaerts (1988) de Vylder & Goovaerts (1984).

Other references worth noting include: Embrechts & Schmidli (1994), Asmussen & Højgaard (1999), Cai & Dickson (2002), Valdez & Mo (2002), Albrecher et al. (2001), Dickson & Hipp (2001), Picard & Lefevre (2001), Dickson & Waters (1999), Pitts et al. (1996), Embrechts & Veraverbeke (1982).

### 3.2.2 Large Deviations

We have already discussed Large Deviation (LD) Theory in Chapter 2 and it's relevance and application to risk Theory. Since this is a relatively new area of research, there are not many relevant studies available.

For a definition of the large deviations principle, as well as a few standard introductory results, we refer to Chapter 2 of Schwartz & Weiss (1995). Embrechts et al. (1997) present a comprehensive treatment of extreme value methodology for all standard models that occur in mathematical finance. This is brought up to date in Embrechts et al. (2001). Klüppelberg & Mikosch (1997) and Mikosch & Nagev (1998) look at Large Deviations where the main emphasis is on heavy-tailed distributions. In this paper they indicate the close relationship between large deviation results and the modeling of large insurance claims. Kalashnikov (1997) uses geometric sums to obtain bounds for rare events and shows some applications to insurance; specifically ruin probabilities. Other relevant papers are Vvedenskaya et al. (2000), Gençay et al. (2003), Duffy & Metcalfe (2005) and in particular Aquilina et al. (2004) look at reducing the ruin probability by altering the premium under a large deviations regime.

### 3.2.3 Optimal Control

In a talk given at the Royal Statistical Society of London, Karl Borch in 1967 made the following statement (see Taksar (2000)):

> The theory of control processes seems to be tailor made for the problems which actuaries have struggled to formulate for more than a century. It may be interesting and useful to meditate a little how the theory would have developed if actuaries and engineers had realised that they were studying the same problems and joined forces over 50 years ago. A little reflection should teach us that a highly specialised problem may, when given the proper mathematical formulation, be identical to a series of other, seemingly unrelated, problems.

There are many control variables in insurance which are adjusted dynamically, such as reinsurance, new business or investment. In recent years there has been a large increase in the application of dynamic control theory to these insurance related problems, thus continuing the early works by Gerber (1969), Bühlmann (1970), Dayanada (1970), Martin-Löf (1973), Martin-Löf (1983), Waters (1983), Centeno (1985), Martin-Löf (1994) and Browne (1995). Since then we can see a rapid development of this field with a series of papers written by Soren Asmussen, Christian Hipp, Bjarne Hoejgaard, Hanspeter Schmidli and Michael Taksar amongst others. This area of research can be split into four parts: Optimal Dividend Payout, Optimal Investment, Optimal Reinsurance and New Business and Optimal Premium Control.

#### 3.2.3.1 Optimal Dividend Payout

The idea is that the company wants to pay some of its surplus to the shareholders as dividends, and the problem is to find a payout-scheme that maximises the expected present value of all payouts until ruin occurs, i.e. until the capital is negative for the first time. When ruin occurs, the company is bankrupt and no more dividends can be paid to the existing shareholders. According to Miller Modigliani theory, this approach can be used as a valuation tool for companies, since the value of a company is exactly the expected present value of future dividends. Obviously the problem is of such a general nature that it is of interest to other sectors as well. An alternative interpretation is that of a consumer whose available funds varies, and wants to spend some of those funds for consumption. The objective can then be to maximize expected present value of future consumption up to the time when the funds are exhausted. There are numerous papers dealing with this problem, and a general solution is given in Shreve et al. (1984), who show that under some reasonable assumptions the optimal policy yields a barrier $b^*$, so that whenever the surplus goes above $b^*$ the excess is immediately paid out as dividends. Under such a policy the accumulated dividend process is singular with respect to Lebesgue measure, hence the name 'singular control problem'. Jeanblanc-Picqué & Shiryaev (1995) and Asmussen & Taksar (1997) considered maximising expected value of discounted dividends paid until the time of ruin for a Brownian motion with drift. Paulsen & Gjessing (1997) also allowed for stochastic returns on investments in this model, and in Paulsen (2003) the same problem is studied when dividends paid are subject to solvency constraints. Højgaard & Taksar (1999) extended the work of Asmussen & Taksar (1997) to include control of the proportion of the insurance business that is ceded to reinsurers. then Højgaard & Taksar (2001) solved this problem for the more complicated model used by Paulsen & Gjessing (1997). For this same model, Asmussen et al. (2000) aim to make a dynamic choice of Excess of Loss reinsurance and find the dividend distribution policy, which maximizes the cumulative expected discounted dividend pay-outs. Finally, Taksar (2000) included control of investment strategies as well, which is the subject of the next part.

#### 3.2.3.2 Optimal Investment

For a basic introduction, consider a risky asset in which the insurer can invest, and a riskless asset; a bank account which pays interest $r$. At each point in time $t$ the insurer, with current wealth $R(t)$ will invest an amount $A(t)$ into the risky asset, and what is left over will earn (or cost) interest $r$. For the sake of simplicity, we take the classical Samuelson model (logarithmic Brownian motion) for the dynamics of the asset prices $Z(t)$ :

$$dZ(t) = aZ(t)\,dt + bZ(t)\,dW(t), \quad Z(0) = z_0,$$

where $W(t)$ is a standard Wiener process. If $\theta(t) = A(t)/Z(t)$ is the number of shares held at time $t$, then the total position of the insurer has the following dynamics:

$$dR(t) = rR(t)\,dt + cdt - dS(t) + \theta(t)\,dZ(t) - r\theta(t)\,Z(t)\,dt, \quad R(0) = s,$$

or

$$dR(t) = rR(t)\,dt + cdt - dS(t) + A(t)\left((a-r)\,dt + bW(t)\right), \quad R = s.$$

Browne (1995) was one of the first to look at minimising the probability of ruin by dynamically controlling investment after Merton (1971), Merton (1990) and Karatzas (1989) looked at ordinary investors (those without an external risk process) seeking to maximise utility of terminal wealth. As we have already seen, Taksar (2000) included control of investment strategies as well. All these recent papers work directly with diffusion processes, the argument is that these processes can serve as approximations. In Asmussen et al. (2000) this approximation aspect was part of the model formulation. As in Højgaard & Taksar (1999) they sought to find an optimal dividend payment scheme for a Brownian motion with drift when reinsurance is allowed, but in this case they used a model of excess of loss reinsurance. Using weak convergence arguments, their model again simplified to a diffusion model, but with a drift and diffusion coefficients depending on the retention limit in a non-linear way.

Rather than using a diffusion model, Højgaard & Taksar (2001) use the classical risk process described in 1.18 and let the rate of premium income depend on the size of the business. Hipp & Plum (2000) use the same Poisson process for the risk model and look to minimise the probability of ruin by the choice of a suitable investment strategy for a capital market index using the Bellman equation. They give a more intuitive result using this model than is found in the earlier work of Browne (1995). Gaier & Grandits (2002) extend the work further by looking at claim distributions with regularly varying tails.

For further extensions to this work we refer to Kalashnikov & Norberg (2002), Frovlova et al. (2002), Gaier et al. (2003), Hipp & Plum (2003), and Irgens & Paulsen (2004).

As a bridge to the next section we refer to a paper by Schmidli (2002) which allows investment into a risky asset modelled as a Black-Scholes model, as well as proportional reinsurance.

#### 3.2.3.3 Optimal Reinsurance and New Business

**3.2.3.3.1 Proportional reinsurance** Consider a small insurance company that accepts a fraction $\alpha(t)$ of the incoming claims, where $0 \leq \alpha(t) \leq 1$ for all $t$. The rest of the claim is covered by a reinsurer. In return, the reinsurer gets a fraction of the premium from the customers. The distribution of the number of claims involving the reinsurer is the same as the distribution of the number of claims involving the insurer, as each pays a defined proportion of every claim. The individual claim amounts for the insurer are distributed as

$\alpha X_i$ and for the reinsurer as $(1-\alpha)X_i$. The aggregate claims amounts are distributed as $\alpha S$ and $(1-\alpha)S$ respectively.

**3.2.3.3.2 Excess of loss reinsurance** The amount that an insurer pays on the $i$-th claim under individual excess of loss reinsurance with retention level $M$ is

$$Y_i = \min(X_i, M).$$

The amount that the reinsurer pays is

$$Z_i = \max(0,\ X_i - M).$$

Thus, the insurer's aggregate claims net of reinsurance can be represented as

$$S_I = Y_1 + Y_2 + \cdots + Y_N$$

and the reinsurer's aggregate claims as

$$S_R = Z_1 + Z_2 + \cdots + Z_N.$$

In return, the reinsurer gets a fraction of the premium from the insurer.

After the early works by Dayanada (1970), Waters (1983) and Centeno (1985), a somewhat different approach was taken in Højgaard & Taksar (1997). They sought to choose the optimal proportional reinsurance in order to maximise expected discount average value of assets up to the time of ruin. Again, their model was a Brownian motion with drift. They later extended this work to include transaction costs in Højgaard & Taksar (1998). Centino (1997) uses the classical risk process and a model of excess of loss reinsurance with the objective function being to minimise ruin calculated via an improved Lundberg upper bound (see Gerber (1977)).

Schmidli (2001) considers proportional reinsurance in a diffusion setup and a classical risk model. He makes use of the Bellman equation to find an optimal strategy in the sense of minimising the ruin probability. Hipp & Vogt (2003) extend the results further by using a classical risk model with excess of loss reinsurance, giving numerical examples for exponential, shifted exponential and Pareto claims. Other relevant papers in this area are Kaluszka (2001), Centeno (2002b) and Centeno (2002a) (which extends the earlier work of Centeno (1985) to the Sparre Anderson model), Verlaak & Beirlant (2003) and Markussen & Taksar (2003). Of significant importance is the work by Kelbert et al. (2007), who use the classical risk process under a combination of proportional and excess of loss reinsurance with a grace period to minimise the probability of ruin.

**3.2.3.3.3 New Business** Instead of selling off some part of the insurance risk to a reinsurer, in this problem, we have the option to take on new business (and thus more risk) in return for new premium income.

Hipp & Taksar (2000) consider the continuous time problem of optimal choice of new business to minimise infinite time ruin probability. They use the simplest risk model to demonstrate how stochastic control tools, such as the Hamilton-Jacobi-Bellman equation can be used to easily solve the optimisation problem. Kelbert et al. (2005) extend this problem with the use of simulation techniques, creating software that can be used to solve any number of more complicated scenarios. They also extend the work of Perry et al. (2002) to show a new application for hitting probabilities.

#### 3.2.3.4 Optimal Premium Control.

An obvious choice of control variable is the premium, $c$. By increasing the price of the insurance contract the insurer will increase his premium income to a point, after which the number of clients willing to pay the higher premium will fall. Conversly, by decreasing the premium, the number of clients wishing to take out such a contract is likely to increase thus giving a higher total income. Of most interest to this thesis is Aquilina et al. (2004). This article studies a risk model where the insurer's profit at a finite time horizon $\tau_1$ can be controlled by making a change of premium at an optimally chosen time $\tau \leq \tau_1$. In the fluid approximation limit (see Chen & Yao (2001)), this probabilistic control problem converges in probability to a deterministic problem, which they solve for specific claim size distributions and a unimodal demand function. This article is also noted for its application of Large Deviations.

Other papers worth noting in this area are: Asmussen & Møller (2003), Cummins (2003), Lin et al. (2003), Verlaak & Beirlant (2003), Centeno (2002b), Højgaard (2002), Pham (2002), Young (1999), Højgaard & Taksar (1997), Kushner (1971) and Dynkin & Yushkevich (1979).

### 3.2.4 Risk Theory

Rather than discuss all the available research that has been done on risk theory in the past forty years, here is a list of the most relevant: Borch (1967), Feller (1968), Feller (1971), Fisz (1963), Patel et al. (1976), Asmussen (1987), Delbaen & Haezendonck (1987), Brockett & Xia (1995), Wang et al. (1997), Straub (1997), Kearns & Pagan (1997), Hellwig (2000) and Norberg (2002). The most used resource for this thesis was of Actuaries (2002) and we refer heavily to chapters 3, 4 and 5.

# Chapter 4

# New business: The basic problem.

## 4.1 Introduction

In the previous chapters we introduced the theory of Large Deviations and discussed how this could be applied to ruin theory. We explained what is meant by the aggregate claim process and the cash-flow process for the risk. In doing so, we defined the probability of ruin in infinite time. In this chapter, we refer to the classical Cramér-Lundberg process discussed in Chapter one, and rely upon the theory therein. We use the adjustment coefficient for a compound Poisson process and calculate it for more interesting cases which are relevant for the study of new business.

The first part of the chapter concentrates on the basic problem of taking on new business and discusses results already obtained by Hipp & Taksar (2000). After this, we touch on some of the more common methods for calculating the probability of ruin and discuss their merits and drawbacks. Having decided on an appropriate method, we extend the results found in Hipp & Taksar (2000) for more complicated situations and discuss an interesting phenomenon relating to the sale of new business. The latter part of the chapter looks at large deviations and how this affects the optimal strategy. Finally we add a brief discussion on hitting probabilities and a practical application to ruin theory. This is not directly related to new business but it was felt that this was the best place to discuss the work.

As already discussed, in insurance there are many control variables which are adjusted dynamically, such as investment, reinsurance or new business, but the use of stochastic control in this context seems relatively new. We refer to a fairly recent paper Hipp & Taksar (2000) which uses standard control tools such as the Hamilton-Jacobi-Bellman equation to characterise and calculate optimal strategies for new business.

In this chapter we shall consider the continuous time problem of optimal choice of new business to minimise infinite time ruin probability. In this problem, at each point of time $t$ the insurer can either: (1) fully invest in a second insurance portfolio (we will call this strategy $b(t) = 1$); or (2) not invest at all in this second portfolio (we will call this strategy $b(t) = 0$). Some alternative models might give an option where a proportion $b(t)$ between 0 and 1 of a certain insurance portfolio can be written, or the intensity $b'(t)$ of acquisition or renewal can be chosen, and this changes the dynamics of the risk process seen in chapter 1. The strategy $b(t)$ is chosen before time $t$ and is based only on the information available at that time. This means that if a claim occurs at time $t$ then $b(t)$ may not depend on the size of the claim, nor the fact that a claim occurred at time $t$. Managing some fixed insurance portfolio, the insurer receives additional premia which are proportional to $b(t)$, and he pays additional claims which occur at a rate proportional to $b(t)$.

We Consider the classical Cramér-Lundberg process, to determine the surplus $s$ at any given time $t$, for insurance business,

$$U_1(t) = s + c_1 t - S_1(t),\ U_1(0) = s. \tag{4.1}$$

Then the classical infinite time ruin probability without new business

$$\psi_0(s) = P\{U_1(t) < 0 \text{ for some } t \geq 0\}$$

is based on the assumption that the insurer uses a static risk management strategy, i.e. risk management decisions are not adjusted in time, such as the decisions on the volume of new business.

For possible new business we consider a second insurance portfolio modelled again by a Cramér-Lundberg process $U_2(t)$ with claims intensity $\lambda_2$ and premium intensity $c_2$ which is independent of $U_1(t)$. The claims in $U_2(t)$ are $Y_1$, $Y_2$, ... and have a common distribution $F_Y$ which will usually differ from the distribution of the $X$'s.

Let us consider the implication of taking on new business and our reasons for doing so. Essentially, we are taking on a new risk to gain premium income. This being the case, it is intuitively reasonable to assume

$$c_2 > \lambda_2 \mu_2 \tag{4.2}$$

so that the insurer's premium income $c_2 t$ is greater than the expected claims outgo $\mathbf{E}[S_2(t)]$ (i.e. the business is profitable). However (as will become clear), close to ruin new business is written irrespective of the mean claim size of new business, as even non profitable business will be written in order to collect premia. The implications of this will be discussed later in this chapter.

For a given strategy $b(t)$ for new business, the risk process of the insurer has the following dynamics:

In a short time interval from $t$ to $t + dt$

- an $X$-claim occurs with probability $\lambda_1 dt + o(dt)$;
- a $Y$-claim occurs with probability $\lambda_2 b(t) dt + o(dt)$;
- no claims occur with probability $1 - \lambda_1 dt - \lambda_2 b(t) dt + o(dt)$;

- the amount $c_1dt + c_2b(t)dt + o(dt)$ is received as premium income in the time interval.

## 4.2 Hamilton-Jacobi-Bellman equation for optimal new business

In the following we deal with functions $\delta(s)$, $\delta_0(s)$ (which represent the survival probabilities with and without new business respectively), which are zero for $s < 0$ and which satisfy equations holding for $s \geq 0$. $\delta'(s)$ and $\delta_0'(s)$ shall represent the derivative of the functions $\delta(s)$ and $\delta_0(s)$ respectively. For $\delta_0(s) = 1 - \psi_0(s)$, the survival probability without new business, we have

$$0 = \lambda_1 \mathbf{E}[\delta_0(s - X) - \delta_0(s)] + c_1\delta_0'(s). \tag{4.3}$$

This follows by considering two distinct cases (see Figure 4.1):

- there is exactly one claim in the interval $[t, t+dt]$ which happens with probability $\lambda_1 dt$ and after this claim of size $X$ we are left with a surplus $s - X + c_1dt$; or
- there is no claim in the interval $[t, t + dt]$ which happens with probability $1 - \lambda_1 dt$, and we are left with a surplus of size $s + c_1dt$.

Averaging over all possible claim sizes we arrive at the equation

$$\delta_0(s) = \lambda_1 dt \mathbf{E}[\delta_0(s - X + c_1dt)] + (1 - \lambda_1 dt)\delta_0(s + c_1dt) + o(dt). \tag{4.4}$$

We see that $\delta_0(s)$ has a right derivative $\delta_0'(s)$, and we obtain Equation (4.3):

$$\begin{aligned}
\delta_0(s) &= (1 - \lambda_1 dt)\ \delta_0(s + c_1dt) + \lambda_1 dt \mathbf{E}\left[\delta_0(s - X + c_1dt)\right] \\
\delta_0(s) &= \delta_0(s + c_1dt) - \lambda_1\delta_0(s)dt + \lambda_1 dt \mathbf{E}\left[\delta_0(s - X + c_1dt)\right] \\
\frac{\delta_0(s) - \delta_0(s + c_1dt)}{dt} &= \lambda_1(\delta_0(s - X + c_1dt) - \delta_0(s)) \\
\delta_0'(s) &\doteq \frac{\lambda_1(\delta_0(s) - \mathbf{E}[\delta_0(s - X + c_1dt)])}{c_1}
\end{aligned}$$

If the distribution of $X$ is continuous, then $\delta_0(s)$ has a continuous derivative, and equation (4.4) holds in the usual sense.

For the survival probability with new business we obtain in exactly the same way; with $b = b(0)$ given, the equation

$$0 = \lambda_1 \mathbf{E}[\delta(s - X + c_1) - \delta(s)] + \lambda_2 b \mathbf{E}[\delta(s - Y + c_2) - \delta(s)] + (c_1 + bc_2)\delta'(s),$$

This follows by considering (see Figure 4.2):

In the short time interval from $t$ to $t + dt$

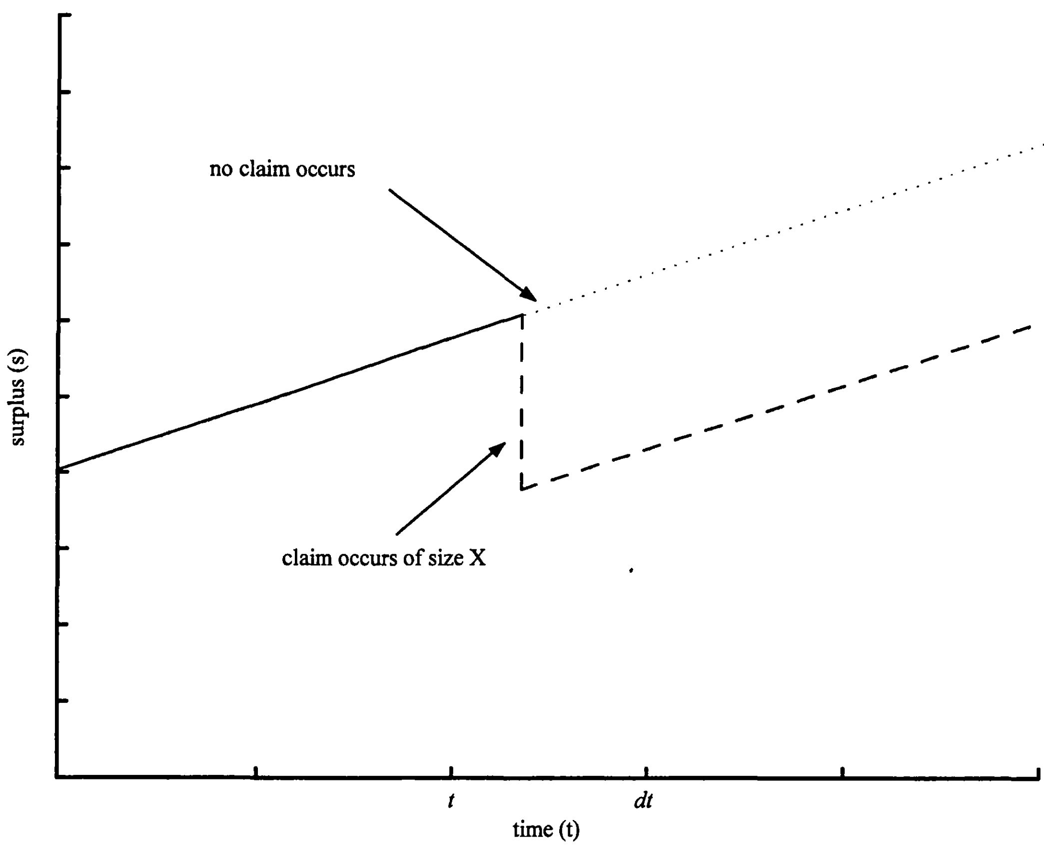


Figure 4.1: A graphical representation of the surplus process without new business.

- an $X$-claim occurs with probability $\lambda_1 dt$ and the surplus left after this claim is $s - X + c_1 dt$
- a $Y$-claim occurs with probability $\lambda_2 dt$ and the surplus remaining is $s - Y + bc_2 dt$
- no claim occurs with probability $1 - \lambda_1 dt - \lambda_2 dt$ and the surplus after this event is $s + (c_1 + bc_2)dt$

so our survival probability is

$$\begin{aligned}
\delta(s) &= (1 - \lambda_1 dt - \lambda_2 dt)\ \delta(s + (c_1 + bc_2)dt) \\
&\quad + \lambda_1 dt \mathbf{E}\left[\delta(s - X + c_1 dt)\right] + \lambda_2 b dt \mathbf{E}\left[\delta(s - Y + c_2 dt)\right] \\
\delta(s) &= \delta(s + (c_1 + bc_2)dt) - \lambda_1 \delta(s) dt - \lambda_2 \delta(s) dt \\
&\quad + \lambda_1 dt \mathbf{E}\left[\delta(s - X + c_1 dt)\right] + \lambda_2 b dt \mathbf{E}\left[\delta(s - Y + c_2 dt)\right] \\
\frac{\delta(s) - \delta(s + (c_1 + bc_2)dt)}{dt} &= \lambda_1 \mathbf{E}\left[(\delta(s - X + c_1 dt) - \delta(s))\right] + \lambda_2 b \mathbf{E}\left[(\delta(s - Y + c_2 dt) - \delta(s))\right] \\
\delta'(s) &= \frac{\lambda_1(\delta(s) - \mathbf{E}[\delta(s - X + c_1 dt)]) + \lambda_2(\delta(s) - b\mathbf{E}[\delta(s - Y + c_2 dt)])}{c_1 + bc_2}
\end{aligned}$$

or, more explicitly

$$\begin{aligned}
\delta(s) &= \delta(s) + dt\{\lambda_1 \mathbf{E}[\delta(s - X + c_1) - \delta(s)] + \lambda_2 b \mathbf{E}[\delta(s - Y + c_2) - \delta(s)] \\
&\quad + (c_1 + bc_2)\delta'(s)\} + o(dt). \qquad (4.5)
\end{aligned}$$

Since $b$ is equal to 0 or 1, then we have either

$$0 = \lambda_1 \mathbf{E}[\delta(s - X + c_1) - \delta(s)] + \lambda_2 \mathbf{E}[\delta(s - Y + c_2) - \delta(s)] + (c_1 + c_2)\delta'(s) \qquad (4.6)$$

or

$$0 = \lambda_1 \mathbf{E}[\delta(s - X + c_1) - \delta(s)] + c_1 \delta'(s). \qquad (4.7)$$

So our HJB equation is equivalent to the equation

$$\delta'(s) = \min\left\{\frac{g_1(s,\delta) + g_2(s,\delta)}{c_1 + c_2}, \frac{g_1(s,\delta)}{c_1}\right\}, \qquad (4.8)$$

$$\begin{aligned}
g_1(s,\delta) &= \lambda_1 \mathbf{E}[\delta(s) - \delta(s - X + c_1)], \\
g_2(s,\delta) &= \lambda_2 \mathbf{E}[\delta(s) - \delta(s - Y + c_2)].
\end{aligned}$$

This equation has a smooth solution:

**Proposition 4.2.1** *If the distributions of $X$ and $Y$ are continuous, then (4.8) has a smooth solution $\delta(s)$ with $\delta(s) \to 1$ for $s \to \infty$. The solution $\delta(s)$ is continuously differentiable and non-decreasing.*

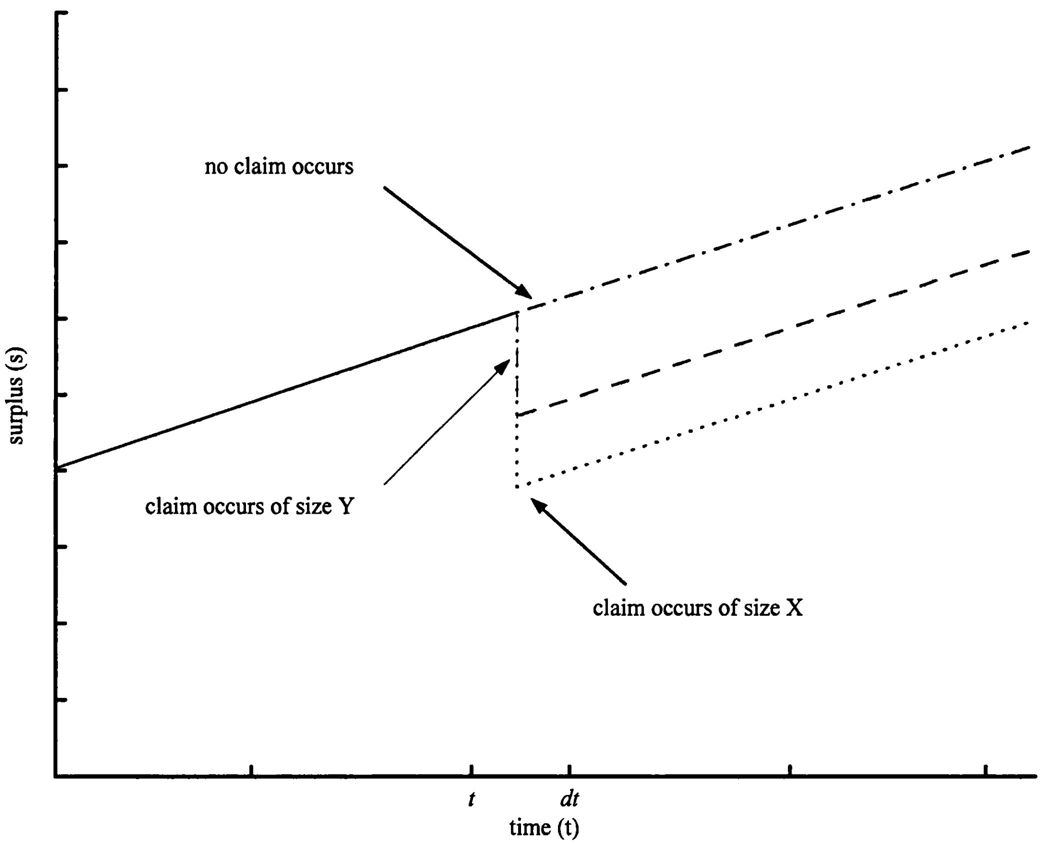


Figure 4.2: A graphical representation of the surplus process with new business.

For a proof, see Hipp & Taksar (2000).

## 4.3 Computation of the optimal strategy

The strategy maximising survival probability will be $b^*(t) = B(U^*(t-))$, where $U^*(t)$ is the risk process resulting from this strategy $b^*(t)$. In Hipp & Taksar (2000), they prove the following theorem and show that this strategy is indeed optimal.

**Theorem 4.3.1** *If the distributions of $X$ and $Y$ are continuous, then there exists a strategy, namely $b^*(t)$, for new business which maximises survival probability.*

Take note that for most HJB equations, the value function has to be convex and smooth. This is not obvious for value functions which are ruin probabilities: the classical ruin probabilities with discrete claim size distributions are neither convex nor smooth, they are not differentiable. Also, our optimal ruin probabilities with new business are not necessarily convex.

The qualitative behaviour of the optimal strategy is best visible at the point $s = 0$, i.e. when the surplus has dropped to zero and the insurer is very close to ruin. The choice $B(0) = 0$ or $B(0) = 1$ depends on the value of $\delta'(0)$ computed with one of the two equations:

$$0 = \lambda_1\delta(0) + \lambda_2\delta(0) - (c_1 + c_2)\delta'(0), \qquad \text{from (4.6)}$$

$$0 = \lambda_1\delta(0) - c_1\delta'(0) \qquad \text{from (4.7).}$$

We have $B(0) = 1$ if, and only if, the first equation leads to a smaller $\delta'(0)$, i.e. if, and only if

$$\frac{\lambda_1\delta(0)}{c_1} > \frac{\lambda_1\delta(0) + \lambda_2\delta(0)}{c_1 + c_2},$$

i.e.

$$\frac{\lambda_1}{c_1} > \frac{\lambda_1 + \lambda_2}{c_1 + c_2}, \text{ if and only if } \frac{\lambda_1}{c_1} > \frac{\lambda_2}{c_2}.$$

This means that close to ruin, new business is written irrespective of the mean claim size of new business (i.e. $\lambda_2 \mathrm{E}Y > c_2$) will be written in order to collect premia, and this money will be used to pay the next claim. If the company survives, then at some large surplus $s$ the (possibly non profitable) new business will need to be sold ($B(s) = 0$), and this may not be possible in real life.

### 4.3.1 Calculating The Cramér Exponent

As was shown in Chapter 1, the Cramér exponent, is defined to be the unique positive root of

$$\lambda M_X(r) - \lambda - cr = 0. \tag{4.9}$$

So, $R$ is given by

$$\lambda M_X(R) = \lambda + cR. \tag{4.10}$$

**4.3.1.0.1 Exponential Distribution** Consider the exponential distribution where $F(x) = 1 - e^{-\alpha x}$.

For this distribution, $M_X(r) = \frac{\alpha}{\alpha - r}$, so

$$\begin{aligned}
\lambda + cR &= \frac{\lambda\alpha}{\alpha - R} \\
&\Rightarrow \lambda\alpha - \lambda R + cR\alpha - cR^2 = \lambda\alpha \\
&\Rightarrow R^2 - (\alpha - \frac{\lambda}{c})R = 0 \\
&\Rightarrow R = \alpha - \frac{\lambda}{c}
\end{aligned} \tag{4.11}$$

Since $R$ is the positive root of (4.9).

If $c = (1+\rho)\frac{\lambda}{\alpha}$, then $R = \frac{\alpha\rho}{1+\rho}$.

#### 4.3.1.1 A Mixture of Exponential Distributions

If we let the claims distribution for the $Y_i$s also follow an exponential distribution, now with parameter $\beta$, then the resulting distribution of the portfolio as a whole (i.e. that which contains both old and new business), will be a mixture of two exponential's.

Let

$$F(u) = \sum_{k \geq 1} p_k(1 - \exp\{-\mu_k u\}), \tag{4.12}$$

where $p_k$ are positive numbers such that their sum is equal to 1 and $\mu_k$ are positive. This can be interpreted as follows. Let all insured claims be divided into groups and the probability for a claim to belong to the $k$th group be $p_k$. If a claim belongs to the $k$th group, then its size is random and has the exponential distribution with parameter $\mu_k$.

For our problem we will set the claims intensities $\lambda_1$ and $\lambda_2$ both equal to 1. This means that the probability of a claim belonging to $U_1(t)$, or $U_2(t)$ is the same for each (ie $= \frac{1}{2}$), hence, our mixture distribution is:

$$\begin{aligned}
F(u) &= \frac{1}{2}(1 - \exp\{-\alpha u\}) + \frac{1}{2}(1 - \exp\{-\beta u\}) \\
&= 1 - \frac{1}{2}(\exp\{-\alpha u\} + \exp\{-\beta u\}).
\end{aligned}$$

For this distribution, $M_X(r) = \frac{1}{2}(\frac{\alpha}{\alpha - r} + \frac{\beta}{\beta - r})$, so

$$\lambda + cR = \frac{\lambda}{2}(\frac{\alpha}{\alpha - R} + \frac{\beta}{\beta - R}) \tag{4.13}$$

$$\Rightarrow R = \frac{\alpha c + \beta c - \lambda - \sqrt{\alpha^2 c^2 - 2\alpha\beta c^2 + \beta^2 c^2 + \lambda^2}}{2c} \tag{4.14}$$

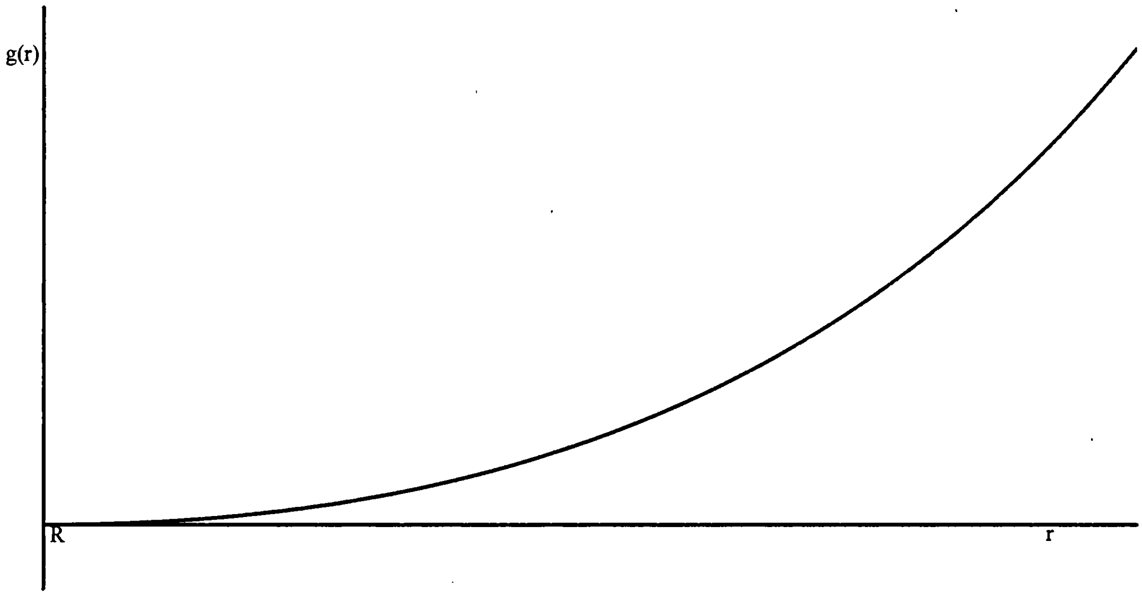


Figure 4.3: g(r) with non-profitable business.

since $R$ is the smallest positive root of (4.9) with limit $\min(\alpha, \beta)$ under (1.23).

### 4.3.2 Non-profitable business

If (as we suggested in section 4.2.1) assumption (4.2) does not hold, then it can be shown that there is no positive root of (4.9) as follows.

Consider once more, the graph of $g(r)$ over the interval $[0, \gamma]$. Note first that $g(0) = 0$. Here $g(r)$ is an increasing function at $r = 0$. since

$$\frac{dg}{dr} = \lambda \frac{d}{dr} M_X(r) - c$$

so that the derivative of $g(r)$ at $r = 0$ is $\lambda\mu - c$ which is now greater than zero. The second derivative is

$$\frac{d^2g}{dr^2} = \lambda \frac{d^2}{dr^2} M_X(r)$$

which is always strictly positive. Hence, if the function $g(r)$ has a turning point it must be at the minimum of the function. Since $g(r)$ is an increasing function at $r = 0$, there can be no turning point and so the graph of $g(r)$ is as shown in Figure 4.3. Thus $R$ in this case is always zero. Note that $R$ is a measure of risk, therefore with $R = 0$, we have infinite risk, which would make sense for non-profitable business.

### 4.3.3 Optimal new business without selling

For a more realistic setup, Hipp & Taksar (2000) consider constraint optimisation modelling, this situation in which written business cannot be sold later. They restrict the strategies $b(t)$ to non-decreasing predictable processes which are bounded by 0 and 1. This problem is harder than before, and a characterisation of the optimal strategy $b(t)$ via the HJB equation is not straightforward. We leave the reader to explore Hipp & Taksar (2000) for greater detail of the complex stochastic calculus involved.

## 4.4 Calculating the Probability of Ruin

Here we present and discuss some algorithms to calculate the probability of ruin for a classical surplus process. Delbaen & Haezendonck (1987) use martingale methods to produce an upper bound for the probability of ruin in finite time cases where the individual claim amount is exponentially bounded. Embrechts & Schmidli (1994) use the theory of piecewise deterministic Markov processes to study the probability of ruin in infinite time. It is interesting to note that the numerical examples in these two papers all assume individual claim amounts are exponentially distributed. Kalashnikov (1996) shows another method, using renewal theory to produce upper and lower bounds for the probability of ruin.

### 4.4.1 Lundberg's Inequality

The method of computing $\psi(s)$ can be a long drawn out process for less than 'simple' distributions. Lundberg's inequality states that

$$\psi(s) \leq \exp\{-Rs\}$$

where $s$ is the insurer's initial surplus. Figure 4.4 shows a graph of both $\exp\{-Rs\}$ and $\psi(s)$ against $s$ when claim amounts are exponentially distributed with mean 1, and the rate of premium income is equal to 2. It can be seen that, for large values of $s$, $\psi(s)$ is very close to the upper bound, so that $\psi(s) \simeq \exp\{-Rs\}$.

In the actuarial literature, $\exp\{-Rs\}$ is often used as an approximation to $\psi(s)$.

The problem we have now, however, is that we are looking at situations close to ruin, i.e. for small values of $s$. This means that the approximation is not so good, as already illustrated.

### 4.4.2 Rossberg-Siegel Bounds

Here we present the method used by Kalashnikov (1996) to calculate upper and lower bounds to the probability of ruin.

**Definition 4.4.1 (Rossberg-Siegel Bounds)** *The upper and lower bounds to the ruin*

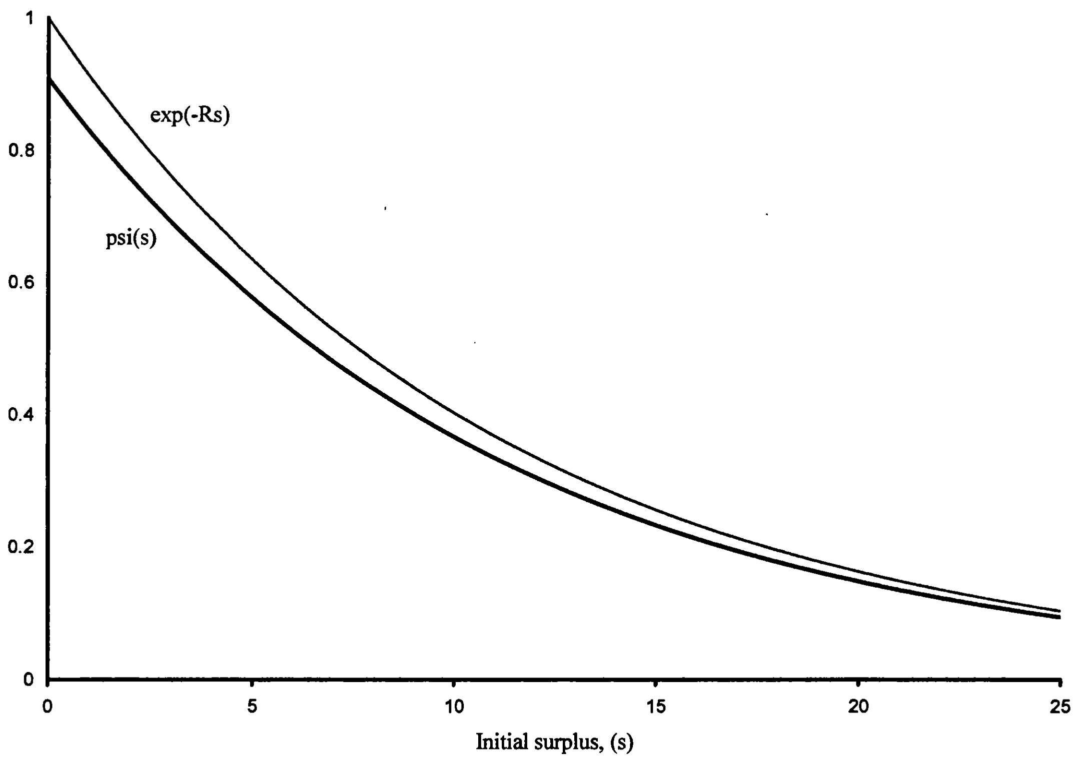


Figure 4.4: Lundberg's upper bound to ruin.

*probability are defined as:*

$$\frac{\exp\{-Rs\}}{f^*} \leq \psi(s) \leq \frac{\exp\{-Rs\}}{f_*} \tag{4.15}$$

*where*

$$\begin{aligned} f_* &= f_*(s) = \inf_{y \geq s} \frac{1}{1-F(y)} \int_y^\infty e^{R(u-y)} dF(x), \\ f^* &= f^*(s) = \sup_{y \geq s} \frac{1}{1-F(y)} \int_y^\infty e^{R(u-y)} dF(x). \end{aligned}$$

*Where $F(x)$ is the distribution function of the size of the claim.*

Let $F(x) = 1 - \exp\{-\alpha x\}$. We know from (4.11) that the Cramér exponent has the form

$$R = \alpha - \frac{\lambda}{c}$$

and

$$f_* = f^* = \frac{c\alpha}{\lambda}.$$

So our upper and lower bounds are equal:

$$\frac{\lambda}{c\alpha} \exp\{-Rs\} \leq \psi(s) \leq \frac{\lambda}{c\alpha} \exp\{-Rs\}.$$

Thus our upper and lower bounds in this case the Rossberg-Siegel bounds give the genuine ruin probability

$$\psi(s) = \frac{\lambda}{c\alpha} \exp\{-(\alpha - \frac{\lambda}{c})s\}.$$

Now let $F(u) = 1 - \frac{1}{2}(\exp\{-\alpha u\} + \exp\{-\beta u\})$. From (4.13), we know that the Cramér exponent has the form

$$R = \frac{\alpha c + \beta c - \lambda - \sqrt{\alpha^2 c^2 - 2\alpha\beta c^2 + \beta^2 c^2 + \lambda^2}}{2c}$$

and

$$\begin{aligned} f_* &= \inf_{y \geq x} \frac{1}{\alpha + \beta e^{(\alpha-\beta)y}} \left( \frac{\alpha}{\alpha - R} + \frac{\beta e^{(\alpha-\beta)y}}{\beta - R} \right), \\ f^* &= \sup_{y \geq x} \frac{1}{\alpha + \beta e^{(\alpha-\beta)y}} \left( \frac{\alpha}{\alpha - R} + \frac{\beta e^{(\alpha-\beta)y}}{\beta - R} \right), \end{aligned}$$

| $c_1$ | $c_2$ | $\lambda_1$ | $\lambda_2$ | $\alpha$ | $\beta$ | $R$ | $f^*$ |
|---|---|---|---|---|---|---|---|
| 2 | 8 | 1.0 | 1.0 | 1.0 | 0.3 | 0.895722 | 9.589762 |
| 2 | 8 | 1.0 | 2.0 | 1.0 | 0.3 | 0.075736 | 4.459029 |
| 2 | 8 | 1.0 | 3.0 | 1.0 | 0.3 | 0.000000 | 3.333333 |
| 2 | 8 | 1.0 | 1.0 | 1.0 | 0.2 | 0.087689 | 8.903882 |
| 2 | 8 | 1.0 | 1.0 | 1.0 | 0.5 | 0.380742 | 8.385165 |
| 2 | 8 | 1.0 | 1.0 | 1.0 | 0.8 | 0.658579 | 7.071068 |
| 2 | 8 | 1.0 | 2.0 | 1.0 | 1.0 | 0.700000 | 3.333333 |
| 2 | 8 | 1.0 | 3.0 | 1.0 | 1.0 | 0.600000 | 2.500000 |
| 2 | 8 | 1.0 | 4.0 | 1.0 | 1.0 | 0.500000 | 2.000000 |
| 2 | 8 | 1.0 | 1.0 | 0.5 | 0.1 | 0.000000 | 10.000000 |
| 2 | 8 | 1.0 | 1.0 | 0.5 | 0.2 | 0.069722 | 7.675919 |
| 2 | 8 | 1.0 | 2.0 | 0.6 | 0.3 | 0.055051 | 4.082483 |
| 2 | 8 | 1.0 | 3.0 | 0.6 | 0.3 | 0.000000 | 3.333333 |

Table 4.1: Relationship between $f^*$ and $R$

This in turn gives

$$\begin{aligned} f_* &= \frac{1}{\alpha + \beta e^{(\alpha-\beta)x}} \left( \frac{\alpha}{\alpha - R} + \frac{\beta e^{(\alpha-\beta)x}}{\beta - R} \right), \\ f^* &= \frac{1}{\beta - R}. \end{aligned} \tag{4.16}$$

#### 4.4.2.1 Numerical examples

Table 4.1 shows the value of the upper bound for varying parameters given.

Section 4.4.5 shows a comparison between the results obtained by this method and our simulation results. It is clear from Figure 4.7 that these bounds are very inaccurate for our purpose.

### 4.4.3 An Euler Scheme; the Hipp and Taksar Method

Since new business is managed dynamically, we cannot use the mixture distribution as described in (4.12). The ruin probability has to be calculated recursively, at each step using the Bellman equation to check if $b(t) = 0$ or $b(t) = 1$.

From equation (??) it is clear that

$$\delta'(s) = \min(q_1, q_2)$$

where

$$q_1 = \frac{\lambda_1(\delta(s) - \mathbf{E}[\delta(s - X)])}{c_1},$$

$$q_2 = \frac{\lambda_1(\delta(s) - \mathbf{E}[\delta(s-X)]) + \lambda_2(\delta(s) - \mathbf{E}[\delta(s-Y)])}{c_1 + c_2}$$

In the case of an exponential $X$ with density

$$\alpha \exp\{-\alpha x\}, \quad x > 0,$$

the problem is essentially a case of solving the differential equation:

$$\begin{aligned} G(s) &= \mathbf{E}[\delta(s-X)] \text{ satisfies the differential equation} \\ G'(s) &= \alpha(\delta(s) - G(s)). \end{aligned}$$

For calculation of the problem we need to find $\mathbf{E}[\delta(s-X)]$ and $\mathbf{E}[\delta(s-Y)]$

Let

$$g_1(s) = \mathbf{E}[\delta(s-X)] = \int_{y=0}^{y=s} \delta(s-y)\alpha e^{-\alpha y} dy$$

then

$$\begin{aligned} -g_1'(s) &= \int_{y=0}^{y=s} d\delta(s-y)\alpha e^{-\alpha y} ds \\ &= \delta(s-y)\alpha e^{-\alpha y}|_0^s + \int_0^s \delta(s-y)\alpha^2 e^{-\alpha y} dy \\ &= \alpha g_1(s) - \alpha\delta(s) \end{aligned}$$

Hence

$$g_1'(s) = \alpha\,[\delta(s) - g_1(s)]$$

Similarly

$$g_2'(s) = \beta\,[\delta(s) - g_2(s)]$$

We now have a system of equations to calculate the survival probability for optimal new business. The probabilities are calculated iteratively with the following initialisation values:

$$\begin{aligned} g_1(0) &= 0 \\ g_2(0) &= 0 \\ \delta(0) &= 1 \text{ (we must normalise at the end).} \end{aligned}$$

Then we update using the following Euler scheme

$$\delta(h) = \delta(0) + h\delta'(0)$$

where $h$, is our (small) step size. Using this value, we obtain $\delta'(h)$, and so on.

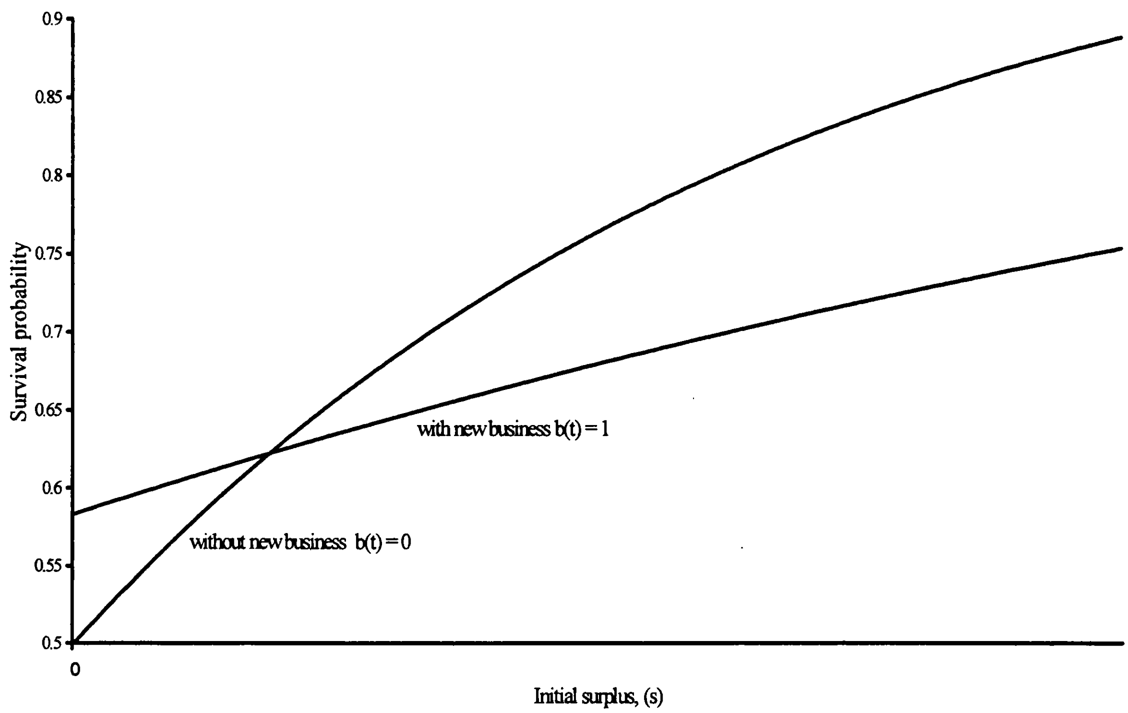


Figure 4.5: Survival probability with and without new business.

When we are at a sufficiently large $s_0$, we can norm the data to achieve

$$\delta(\infty) = 1$$

$$\delta_{new}(s) = \frac{\delta_{old}(s)}{\delta_{old}(s_0)}.$$

Although highly accurate for the exponential distribution, this method becomes more complicated when we start to look at more interesting distributions, since the expectations are computed numerically:

$$\mathbf{E}[\delta(kh - X)]) = h \sum_{i=1}^{k} \delta((k-i)h) f(ih),$$

$$\mathbf{E}[\delta(kh - Y)]) = h \sum_{i=1}^{k} \delta((k-i)h) g(ih),$$

where $f$ and $g$ are the densities of $X$ and $Y$ respectively.

This in itself can cause inaccuracies, and furthermore makes an already lengthy process even more so.

### 4.4.4 Details of the simulations

To benchmark the results of Hipp & Taksar (2000), we created a model that could be used to calculate the survival probabilities via simulations, for both old and new business. The simulations were conducted in MATLAB and C++ and were highly successful. We use the example in Hipp & Taksar (2000) and consider an exponential claim size distribution for both new and old business. We take $\lambda_1 = \lambda_2 = 1, c_1 = 2, c_2 = 8$, and the means of $X$ and $Y$ are 1 and $\frac{10}{3}$,respectively. Figure 4.6 shows the two methods with our simulation results super-imposed on the Hipp & Taksar method.

Taking a close look at the results, it is hard to distinguish between the two graphs. This shows clearly how effective the simulations are.

First we generate a simulated surplus level given by the model parameters and measure the time taken for the business to ruin. This is run a large amount of times, specifically greater than $10^6$, to estimate the probability of ruin.

The simulation runs the two businesses in parallel. The instances at which claims occur are recorded and when the businesses are joined these are rearranged into chronological order to obtain the joint business. The second business is taken on if, and only if, the original business falls bellow a specified level; this level being the optimum.

For new business with selling, (i.e. the [possibly non-profitable] business can be sold off when the surplus has risen above a required level), we simply add a second threshold above which the second business is no longer held. In this case it would seem reasonable to explore transaction costs for selling off this business, i.e. breaking a contract with the insured.

### 4.4.5 Comparing the Rossberg-Siegel Bounds with the simulation results

Figure 4.7 shows the upper and lower bounds $f^*$ and $f_*$ from 4.16, taking parameters $\lambda_1 = \lambda_2 = 1$, $c_1 = 2$, $c_2 = 8$, $\alpha = 1$, $\beta = 0.3$

Where $\alpha$ and $\beta$ are the parameters of a mixture distribution as in (4.12).

## 4.5 Numerical Examples

With the accuracy of the simulations identified we are able to produce examples using many different model parameters. We can study the effect of taking on new business with alternative claims distributions and also the introduction of transaction costs. An area that is of great interest is the case where taking on non-profitable business can improve the position of the company.

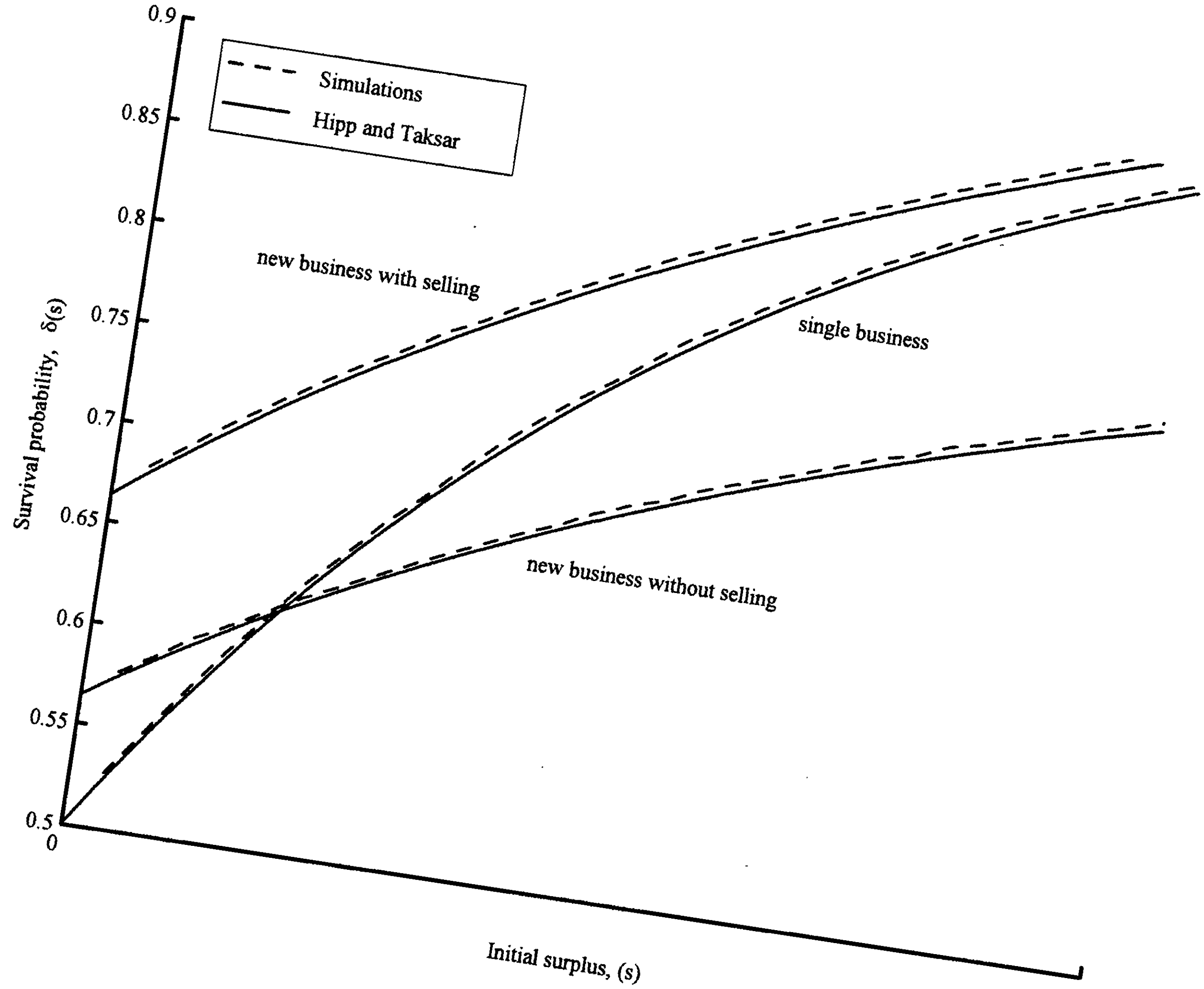


Figure 4.6: Simulation benchmark.

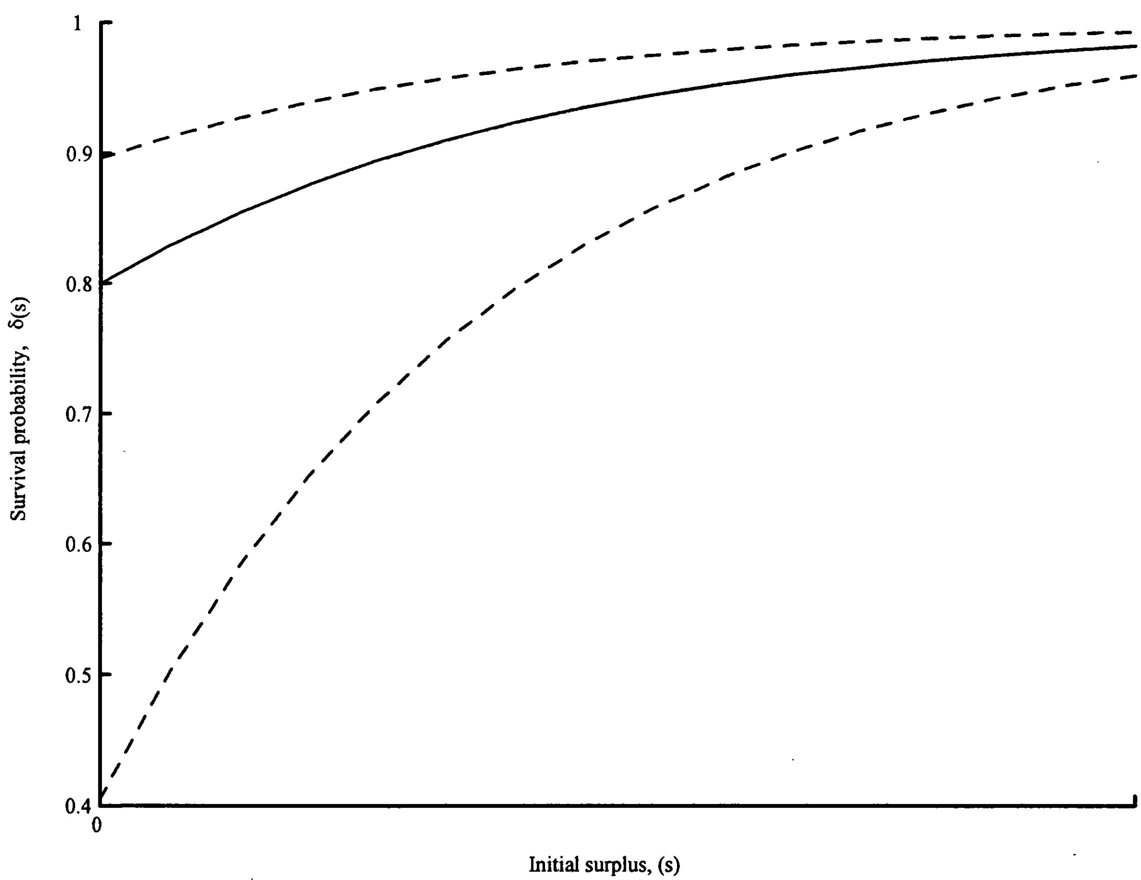


Figure 4.7: Rossberg-Siegel bounds on simulated data.

| $\lambda_1$ | $\lambda_2$ | $c_1$ | $c_2$ | **E**$X$ | **E**$Y$ | $Th$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 4 | 1 | 1.4 | 7.5 |
| 1 | 1 | 2 | 4 | 1 | 1.5 | 3.7 |
| 1 | 1 | 2 | 4 | 1 | 1.6 | 1.6 |
| 1 | 1 | 2 | 4 | 1 | 1.7 | 1 |
| 1 | 1 | 2 | 4 | 1 | 1.8 | 0.7 |
| 1 | 1 | 2 | 4 | 1 | 1.9 | 0.3 |
| 1 | 1 | 2 | 6.2 | 1 | 3 | 0.1 |
| 1 | 1 | 2 | 6.4 | 1 | 3 | 0.2 |
| 1 | 1 | 2 | 6.6 | 1 | 3 | 0.3 |
| 1 | 1 | 2 | 6.8 | 1 | 3 | 0.5 |
| 1 | 1 | 2 | 7 | 1 | 3 | 0.5 |
| 1 | 1 | 2 | 7.2 | 1 | 3 | 0.6 |
| 1 | 1 | 2 | 7.4 | 1 | 3 | 0.7 |
| 1 | 1.3 | 6 | 2 | 1.5 | 3.3333 | 1.3 |
| 1 | 1.31 | 6 | 2 | 1.5 | 3.3333 | 1 |
| 1 | 1.32 | 6 | 2 | 1.5 | 3.3333 | 0.9 |
| 1 | 1.33 | 6 | 2 | 1.5 | 3.3333 | 0.8 |
| 1 | 1.34 | 6 | 2 | 1.5 | 3.3333 | 0.7 |
| 1 | 1.35 | 6 | 2 | 1.5 | 3.3333 | 0.6 |
| 1 | 1.36 | 6 | 2 | 1.5 | 3.3333 | 0.5 |
| 1 | 1.37 | 6 | 2 | 1.5 | 3.3333 | 0.4 |
| 1 | 1.38 | 6 | 2 | 1.5 | 3.3333 | 0.3 |
| 1 | 1.39 | 6 | 2 | 1.5 | 3.3333 | 0.2 |
| 1 | 1.4 | 6 | 2 | 1.5 | 3.3333 | 0.1 |

Table 4.2: Change in threshold - Exponential

### 4.5.1 Exponential

We now explore the effect of changing each of the parameters:

Since the right hand side of (??) is a linear function of $b$, the supremum is always attained at one of the extreme points $b(t) = 1$ or $b(t) = 0$. With this being the case, we can explore how the threshold for new business behaves under parameter restraints. The results can be found in Table 4.2.

The results are clear cut, with no ambiguity. We see that as $EY$ increases, the threshold at which we should take on new business decreases. This is intuitively reasonable, because, as the mean expected claim size of the second business increases, the profitability decreases ($\frac{\lambda_2 \mathbf{E}Y}{c_2} \to 1$), hence we are more reluctant to take on the new business. As we would expect, the threshold increases with $c_2$ (i.e. premium income increases) and falls when $\lambda_2$ is increased. The simplest way to judge how the threshold will behave under parameter

| $Th$ | $\delta(2)$ |
|---|---|
| 1.3 | 0.8351 |
| 1.31 | 0.8349 |
| *1.32 | *0.8355 |
| 1.33 | 0.8350 |
| 1.34 | 0.8352 |
| 1.35 | 0.8353 |
| 1.36 | 0.8349 |
| 1.37 | 0.8349 |
| 1.38 | 0.8349 |
| 1.39 | 0.8352 |
| 1.4 | 0.8348 |

Table 4.3: Optimal Threshold

constraints, is to look at our profitability equation:

$$c_2 - \lambda_2 \mathbf{E}Y = 0 \tag{4.17}$$

With the parameters balanced, there is zero profitability (i.e. premium income = average premium outgo). As the parameters change, the equation is no longer balanced and (4.17) will move in some direction. If premium income increases, (4.17) becomes greater than zero, and our threshold moves up. If premium outgo increases (4.17) becomes less than zero, and our threshold moves down.

#### 4.5.1.1 Selling business

If we have the option to sell business, it is clear that the probability of survival is very much increased. There are two ways of looking at this problem. The most straightforward is to assume that both thresholds (i.e. the level at which we take on new business, and the level at which we sell it) are equal. This gives us just a single value to alter when searching for the optimal strategy. It is clear, when looking at Hipp & Taksar (2000) that, with no transaction costs, this is very much acceptable.

Figure 4.8 shows how dramatic the improvement is when we allow selling of the new business. We take $\lambda_1 = 1$, $c_1 = 2$, $\mathbf{E}X = 1$, $\lambda_2 = 2$, $c_2 = 10$, and $\mathbf{E}Y = 5$. Without selling, there is no threshold at which we should take on the new business, as this would not be beneficial to the insurance company. With the option to sell however, the optimal threshold (from Table 4.3) is 1.32.

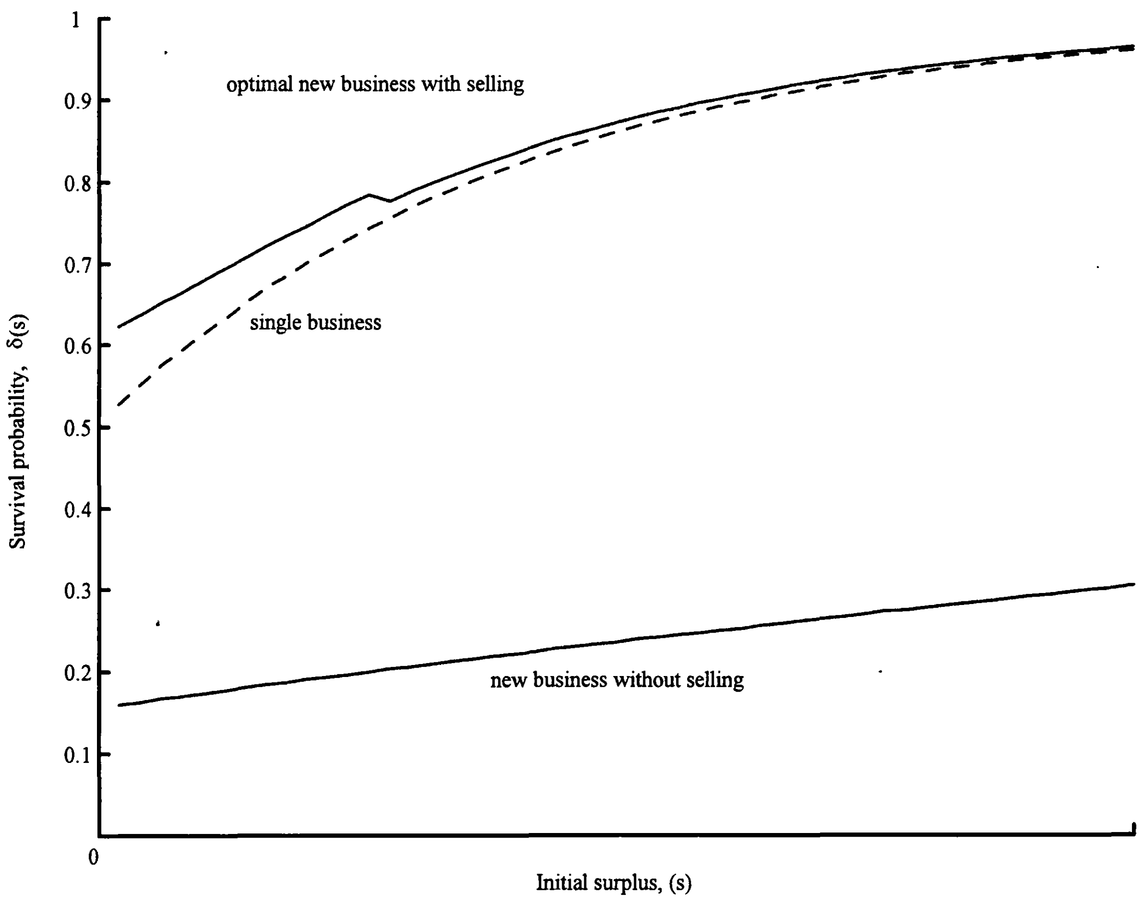


Figure 4.8: Optimal new business with selling.

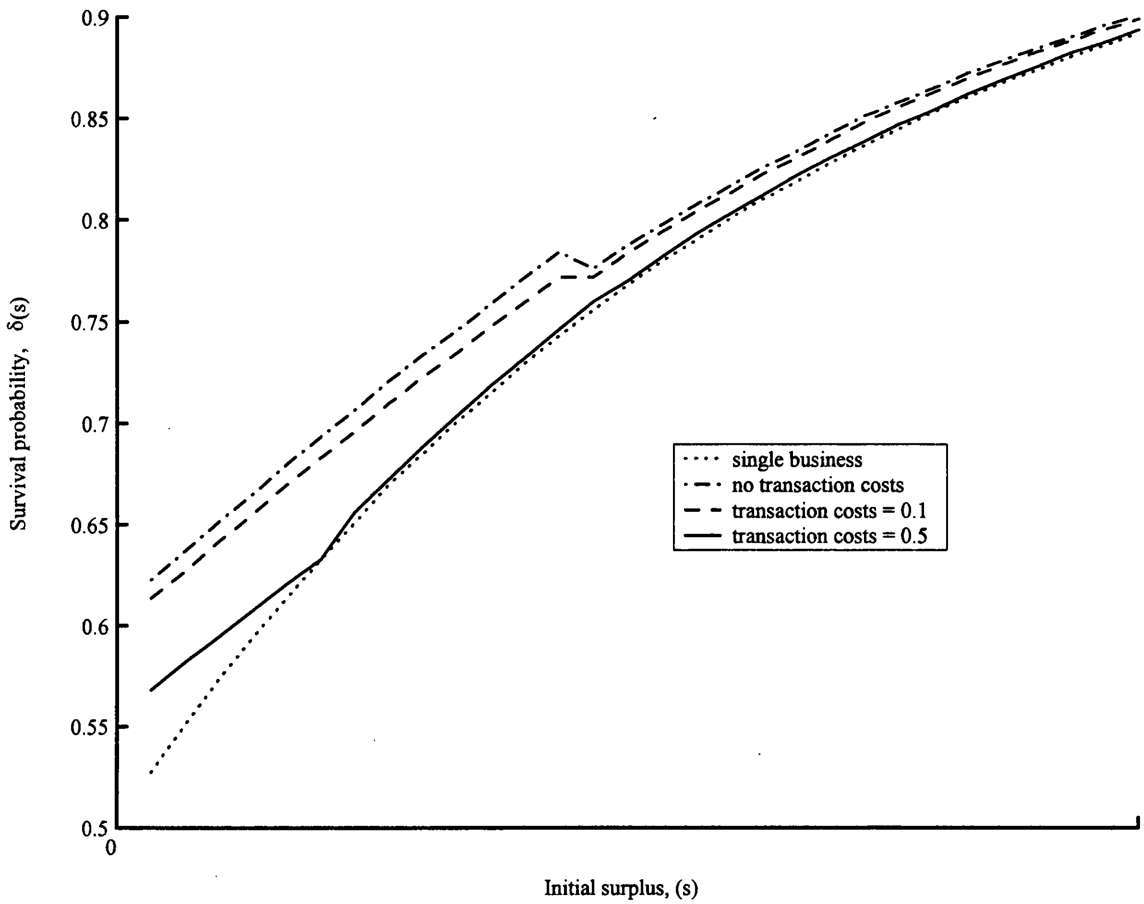


Figure 4.9: Optimal new business with selling (transaction costs included).

#### 4.5.1.2 Taking Account of Transaction Costs

When the cost of selling is taken into account, however, things get much more complicated. Here, we need to search across all possible combinations of the two thresholds within a reasonable range. We take a range that contains the threshold without transaction costs and produce a matrix of upper end values. The optimal strategy is that which improves our position the most after the switch. So we look at a point somewhere above the threshold. We compare this value across all possible combinations of 'buying' and 'selling' to obtain the maximum. This is our optimal strategy.

As in Figure 4.8, we take $\lambda_1 = 1$, $c_1 = 2$, $\mathbf{E}X = 1$, $\lambda_2 = 2$, $c_2 = 10$, and $\mathbf{E}Y = 5$. We omit the results for new business without selling for clarity, as this would not be beneficial to the insurance company. With no transaction costs, when the new business is sold, the optimal threshold is 1.32. Increasing the costs to 0.1 and our optimal strategy becomes (from Table 4.4) $b(1.31, 1.32)$. Increasing the costs to 0.5, and we have the optimal threshold (from Table 4.5) for taking on new business = 0.60, and for selling = 0.70.

| | Buying | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Selling | 1.30 | *1.31 | 1.32 | 1.33 | 1.34 | 1.35 | 1.36 | 1.37 | 1.38 | 1.39 | 1.40 |
| 1.30 | 0.8324 | | | | | | | | | | |
| 1.31 | 0.8323 | 0.8325 | | | | | | | | | |
| *1.32 | 0.8321 | *0.8327 | 0.8323 | | | | | | | | |
| 1.33 | 0.8321 | 0.8321 | 0.8322 | 0.8323 | | | | | | | |
| 1.34 | 0.8316 | 0.8318 | 0.8322 | 0.8319 | 0.8325 | | | | | | |
| 1.35 | 0.8316 | 0.8322 | 0.8320 | 0.8322 | 0.8322 | 0.8324 | | | | | |
| 1.36 | 0.8316 | 0.8318 | 0.8316 | 0.8321 | 0.8320 | 0.8321 | 0.8325 | | | | |
| 1.37 | 0.8317 | 0.8314 | 0.8319 | 0.8318 | 0.8324 | 0.8320 | 0.8322 | 0.8325 | | | |
| 1.38 | 0.8313 | 0.8322 | 0.8316 | 0.8316 | 0.8317 | 0.8316 | 0.8318 | 0.8319 | 0.8316 | | |
| 1.39 | 0.8317 | 0.8316 | 0.8315 | 0.8318 | 0.8319 | 0.8317 | 0.8318 | 0.8320 | 0.8320 | 0.8318 | |
| 1.40 | 0.8319 | 0.8319 | 0.8318 | 0.8315 | 0.8318 | 0.8321 | 0.8321 | 0.8316 | 0.8320 | 0.8318 | 0.8321 |

Table 4.4: Optimal Threshold; Transaction Costs = 0.1

| | Buying | | | | | |
|---|---|---|---|---|---|---|
| Selling | *0.6 | 0.61 | 0.62 | 0.63 | 0.64 | 0.65 |
| 0.6 | 0.8221 | | | | | |
| 0.61 | 0.8223 | 0.8223 | | | | |
| 0.62 | 0.8224 | 0.8223 | 0.8223 | | | |
| 0.63 | 0.8225 | 0.8223 | 0.8225 | 0.822 | | |
| 0.64 | 0.8222 | 0.8225 | 0.822 | 0.822 | 0.8221 | |
| 0.65 | 0.8221 | 0.8221 | 0.8224 | 0.8227 | 0.8223 | 0.8227 |
| 0.66 | 0.8225 | 0.8226 | 0.8229 | 0.822 | 0.8225 | 0.8222 |
| 0.67 | 0.8226 | 0.8227 | 0.8222 | 0.8227 | 0.8225 | 0.8227 |
| 0.68 | 0.8223 | 0.8229 | 0.8222 | 0.8223 | 0.8225 | 0.8224 |
| 0.69 | 0.8228 | 0.8223 | 0.8226 | 0.8221 | 0.8228 | 0.8223 |
| *0.7 | *0.823 | 0.8227 | 0.8225 | 0.8224 | 0.8221 | 0.8226 |
| 0.71 | 0.8223 | 0.8223 | 0.8227 | 0.8226 | 0.8224 | 0.8225 |
| 0.72 | 0.8224 | 0.8224 | 0.8224 | 0.8224 | 0.8226 | 0.8226 |
| 0.73 | 0.8227 | 0.8227 | 0.8223 | 0.8226 | 0.8225 | 0.8222 |
| 0.74 | 0.8222 | 0.8224 | 0.8225 | 0.8224 | 0.8222 | 0.8223 |
| 0.75 | 0.8227 | 0.8221 | 0.8229 | 0.8226 | 0.8225 | 0.8226 |

Table 4.5: Optimal Threshold; Transaction Costs = 0.5

#### 4.5.1.3 A note on heavy tailed distributions

We are interested in ruin probabilities where $S(t)$ has large deviations under the assumption that the distribution function $F$ is heavy-tailed.

A natural class of heavy-tailed distributions are subexponential distributions. By definition, $F$ is subexponential if for every $n \geq 2$ (equivalently, for some $n \geq 2$)

$$\lim_{x\to\infty} \frac{\Pr\left(X_1 + \cdots + X_n > x\right)}{\Pr\left(\max(X_1, \ldots, X_n) > x\right)} = 1.$$

This subexponentiality means that the tail of the sum of $n$ rv's becomes large by a dominating large rv. Examples of subexponential distributions are Pareto, $\alpha$-stable ($\alpha < 2$), loggamma, also Weibull and Benktander distributions for certain parameter values. Subexponential distributions have been recognised as appropriate models for data exhibiting large fluctuations. A textbook treatment of subexponential distributions in the context of insurance and finance can be found in Embrechts et al. (1997). The problem with these heavy tailed distributions, is that by definition, they do not have an MGF. This means that we can not study them in the method that we have used in this thesis. Due to this, we have decided to look at claims with a Gamma distribution, as this gives us more flexibility than the exponential, to vary the shape of the tail of the distribution.

#### 4.5.1.4 Gamma

For claim sizes modeled by a Gamma distribution, simulations have been run and the threshold at which to take on new business have been recorded. These results are presented in Table 4.6.

Obviously what is of interest here is the way the threshold behaves as the shape parameter $\beta$ of the Gamma distribution varies.

As is usual (and expected), when the shape parameter is not varied, and transaction costs are increased, the threshold at which new business is taken on decreases. Interestingly, as the transaction costs are increased the second threshold (that at which the new business is sold) increases. So when transaction costs are higher the optimal threshold at which new business should be taken on moves down, but the optimal threshold at which to sell the new business increases. So it is more profitable to take the new business when closer to zero (and as transaction costs tend upwards, this tends closer and closer to zero) but to keep it for much longer than usual.

Looking at the results for the shape parameter we find an increase has a similar effect to that of an increase in transaction costs.

| $\beta$ | loss | buying | selling |
|---|---|---|---|
| 0.50 | 0.00 | 1.50 | 1.50 |
| 0.50 | 0.10 | 1.40 | 1.40 |
| 0.50 | 0.20 | 1.40 | 1.40 |
| 0.50 | 0.30 | 1.30 | 1.30 |
| 0.50 | 0.40 | 1.00 | 1.60 |
| 0.50 | 0.50 | 0.70 | 1.60 |
| 0.50 | 0.60 | 0.60 | 1.70 |
| 0.50 | 0.70 | 0.40 | 1.80 |
| 0.50 | 0.80 | 0.30 | 2.50 |
| 0.50 | 0.90 | 0.40 | 2.70 |
| 0.50 | 1.00 | 0.20 | 2.80 |
| 0.60 | 0.50 | 0.60 | 2.10 |
| 0.70 | 0.50 | 0.40 | 2.10 |
| 0.80 | 0.50 | 0.40 | 2.20 |
| 0.90 | 0.50 | 0.30 | 2.30 |
| 1.00 | 0.50 | 0.20 | 2.30 |

Table 4.6: Optimal Threshold With Varying Transaction Costs (Gamma)

| $\sigma$ | loss | buying | selling |
|---|---|---|---|
| 0.5 | 0 | 2.17 | 3.96 |
| 0.5 | 0.1 | 2.55 | 4.06 |
| 0.5 | 0.5 | 1.88 | 5.26 |
| 0.6 | 0.1 | 2.10 | 4.12 |
| 0.7 | 0.1 | 2.04 | 3.80 |
| 0.8 | 0.1 | 2.14 | 3.64 |
| 0.9 | 0.1 | 1.71 | 3.88 |
| 1 | 0.1 | 1.76 | 3.26 |
| 1.5 | 0.1 | 1.52 | 2.06 |
| 2 | 0.1 | 1.48 | 1.96 |

Table 4.7: Optimal Threshold With Varying Transaction Costs (Lognormal)

#### 4.5.1.5 Lognormal

As a comparison against the Gamma distribution we vary a single parameter ($\sigma$) which can be seen as the 'shape' parameter (we are not interested in the scale or location parameter). The results are shown in Table 4.7

When we increase the value of the 'shape' parameter for the Lognormal distribution the behaviour of the system differs from the case of the Gamma distribution in the following way: The lower threshold (the threshold at which we take on new business) decreases, similarly to the Gamma distribution, although there is a slight increase at $\sigma = 0.8$. The upper threshold (where the non-profitable new business is sold) also decreases, which is

| $\beta$ | loss | buying | selling |
|---|---|---|---|
| 0.5 | 0 | 2.96 | 3.94 |
| 0.5 | 0.1 | 2.40 | 4.26 |
| 0.5 | 0.5 | 2.01 | 5.11 |
| 0.6 | 0.1 | 2.29 | 3.92 |
| 0.7 | 0.1 | 2.15 | 3.57 |
| 0.8 | 0.1 | 1.80 | 3.27 |
| 0.9 | 0.1 | 1.79 | 3.21 |
| 1 | 0.1 | 1.79 | 3.12 |
| 1.5 | 0.1 | 1.68 | 3.00 |
| 2 | 0.1 | 1.38 | 2.09 |

Table 4.8: Optimal Threshold With Varying Transaction Costs (Weibull)

the opposite behaviour to that of the Gamma distribution. This can be explained by looking at plots of the pdfs for each distribution. As the shape parameter is increased, the Gamma distribution becomes more Normal in its appearance, yet the Lognormal distribution becomes more Exponential in its appearance.

#### 4.5.1.6 Weibull

Under certain circumstances, we are able to obtain results for the Weibull distribution. To compare our results obtained for the Gamma distribution we vary a single parameter ($\beta$) which can be seen as the 'shape' parameter (we are not interested in the scale parameter). The results are shown in Table 4.8

By the properties of the MGFs, we would expect the Weibull distribution to show similar results to that of the Gamma distribution. However, the Weibull actually behaves more like the Lognormal distribution in that both the lower and upper thresholds decrease as the shape parameter is increased.

## 4.6 Conclusions

The results obtained in the study show that stochastic control is a very helpful tool in finance management. In particular, when an insurance business is deemed to be on the path to ruin, one measured action is to take on new business. It is possible to optimise a level at which new business is taken on, determined by minimising the ruin probability associated with the insurance business. The results show that risky business, or even non-profitable business, tend to be more advantageous, given the option of selling the business when the surplus of the insurance business reaches a required level. The level at which new business is sold is also optimised, alongside the first, with the same objective of minimising the ruin probability associated with the insurance business. Of course it would not be possible to sell such business in 'real life' without taking account of some transaction costs. These are included here as a fixed fraction of the premium paid by the policy holder.

The survival probability function is an increasing function of the business surplus, so a greater surplus implies a greater survival probability. The reason why taking on non-profitable business can be viable is simple. The premium income gained is enough to offset incoming claims for a short period. Put simply, in taking on the new business the company's surplus increases, thus increasing its survival probability.

It is interesting to talk about the step down in survival probability caused when the new business is sold. Upon selling the new business, the surplus of the company falls (relative to keeping the new business). As already mentioned, the survival probability is a function that depends on the surplus. As the surplus falls, so does the survival probability. If we look at the parameters involved, we see that the surplus of the company is less than the expected value of a claim. So a single claim could ruin the business. Since we have sold the new business (and therefore effectively lost premium income), there is a small drop in the survival probability.

The numerical methods used allow us to effectively model any conceivable type of policy. This allows us to move away from the usual Exponential example and explore more complicated distributions with ease. The advantages of this are clear, since the application of more general results are likely to be more meaningful.

# Chapter 5

# Hitting Probabilities and the Target Maximum of the Insurance Surplus

Perry et al. (2002) derive a formula for the probability that a compound Poisson process with positive jumps hits a lower straight line before it crosses a parallel upper line. We modify this in order for us to calculate the probability that the surplus of an insurance company reaches a required target $u$ before it is ruined. These hitting probabilities are expressed in terms of the corresponding ruin probabilities.

If $S(t) = \sum_{i=1}^{N(t)} X_i$ as before, then for $s,\ u \geq 0$ let

$$\begin{aligned} T_L(u) &= \inf\{t \geq 0 | t - S(t) = u\} \\ T_U(s) &= \inf\{t \geq 0 | t - S(t) \leq -s\}. \end{aligned}$$

We are interested in the hitting probability

$$\eta(u, s) = \Pr\{T_L(u) < T_U(s)\}.$$

If the premium rate is constantly equal to one and the initial capital is $s$, the probability of eventual ruin is given by

$$\psi(s) = \Pr\{T_U(s) < \infty\}.$$

$\eta(u, s)$ from Perry et al. (2002) is expressed in terms of $\psi(.)$ as

$$\eta(u, s) = \exp\{-\lambda s\} \frac{\psi(s) - \psi(u+s)}{1 - \psi(u+s)} + \frac{1 - \psi(s)}{1 - \psi(u+s)}$$

Using the simulated value of $\delta(s) = 1 - \psi(s)$, we can easily determine the hitting probabilities numerically in various cases. As with the New Business case, we simulate over $10^6$ surplus processes in order to get an estimate for the probability of ruin. These proceses themselves

| $\rho$ | $\eta(3,5)^*$ | $\eta(3,5)$ |
|---|---|---|
| 0.6 | 0.92092 | 0.9333 |
| 0.7 | 0.88767 | 0.8982 |
| 0.8 | 0.84838 | 0.8532 |
| 0.9 | 0.80369 | 0.8005 |
| 0.95 | 0.77960 | 0.7702 |

Table 5.1: Hitting Probabilities for the Lognormal Distribution

are run for a period of $t$x$10^6$ to ensure that each surplus process is given enough time to ruin (should ruin ever occur). These figures were chosen due mostly to time constraints, but also because increasing the order beyond 6 did not significantly alter our results.

We introduce a new parameter $\rho = \lambda \int_0^\infty x dG(x)$, and assume that $\rho < 1$.

Table 5.1 gives some values of $\eta(u, s)$ calculated for the Log-Normal distribution. The results in the column marked with an asterisk are those given in Perry et al. (2002).

A practical application of these hitting probabilities is as follows. If, for example, an investor were to start an insurance company with initial capital $s$, he may want to know what the probability of reaching a desired level $u$ was. Alternatively, were he to specify an amount of acceptable risk, it would be possible to show what maximum surplus $u$ the company would achieve. Upon reaching this required level, the investor could then close the business, knowing that he has reached the maximum surplus. Figure 5.1 shows the probabilities of reaching a maximum surplus $u = £1,000,000$, given an initial surplus, $s$.

## 5.1 Conclusions

The results for the hitting probabilities are significant. The approximation used to calculate $\psi(s)$ for the Log-Normal distribution in Perry et al. (2002) gives an upper bound. It is noted in the paper that this approximation is good for values of $\rho$ close to 1. When we look at the results, we see that the simulated probabilities switch from below the approximation at $\rho = 0.9$ to above the approximation at $\rho \leq 0.8$. What is evident from our results is that, (for $\rho$ close to 1) as $s$ increases, this upper bound increasingly over-estimates $\psi(s)$.

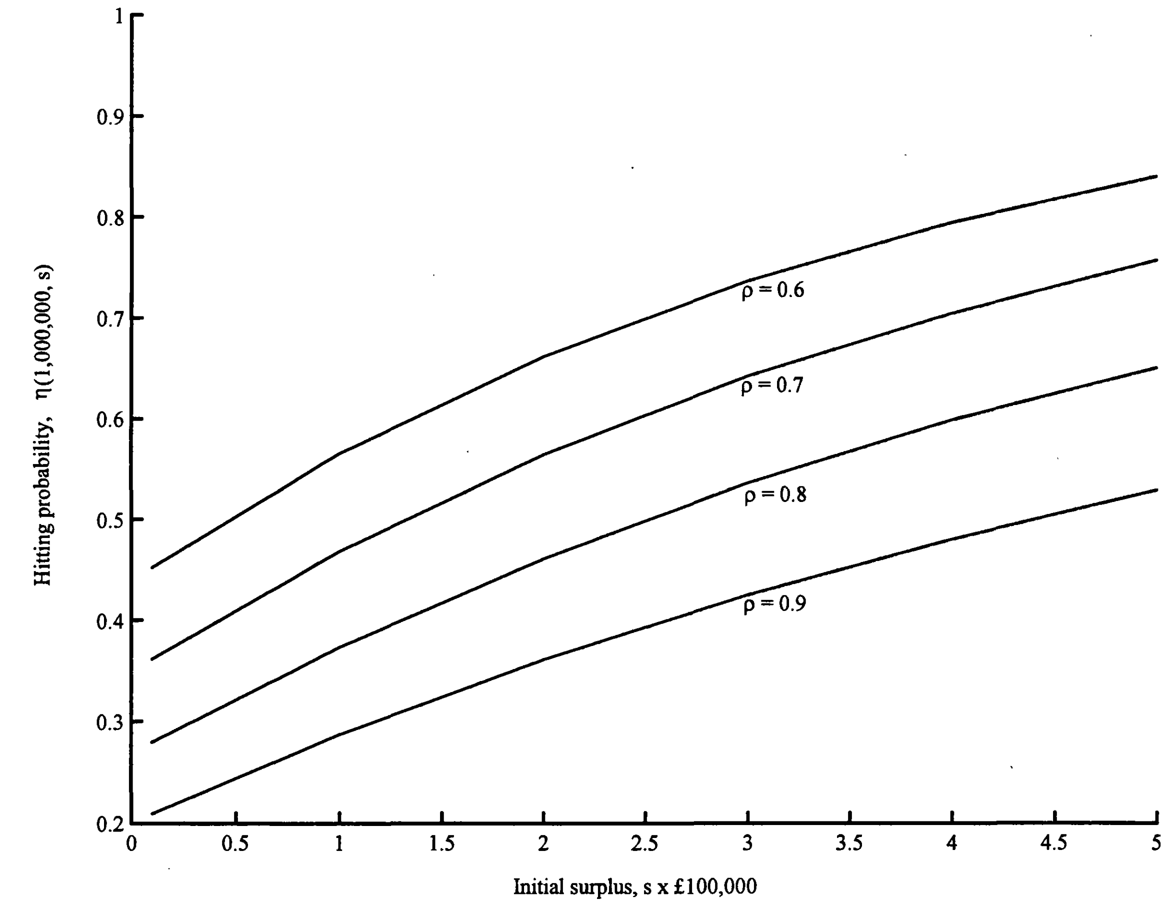


Figure 5.1: Probabilities of hitting the level $u = 10^6$ for initial surplus $s$.

# Chapter 6

# Reinsurance and Ruin

## 6.1 Introduction

In chapter four, we looked at how dynamic control of new business could improve the position of the insurance company by increasing their surplus, and hence increasing their probability of survival. When a company finds themselves on the slope to ruin, this may be an effective strategy to help improve their chances of survival. In this chapter we look at another strategy available to an insurer looking to improve their prospects.

One of the options open to an insurer who wishes to reduce the variability of aggregate claims from a risk is to reinsure their business. A reduction in variability would be expected to increase an insurer's security, and hence reduce the probability of ruin. A reinsurance arrangement could be considered optimal if it minimises the probability of ruin. In this chapter, the effect on the probability of ruin of proportional and of excess of loss reinsurance arrangements will be considered.

The purpose of this chapter is to further investigate Large Deviations regimes, and find the optimal ways of distributing the expenses between the insurer and reinsurer, as well as the optimal premium arrangement. We will use some of the results discussed in chapter 2 to describe the large-scale behaviour of our system. This means that the parameters of the model are such that the ruin probability is extremely small. Here we use the Large Deviations (LD) theory to describe the trajectory along which ruin occurs. Then in accordance with the LD theory all trajectories leading to ruin are concentrated around a deterministic one. Typically, at the level of LDs the behaviour of a properly scaled stochastic process, describing the dynamics of the capital, becomes deterministic; this explains the frequently used expression "rare events happen in the most likely way". The LD theory provides the technique to analyze this asymptotic trajectory and parameters of a ruin process.

We have identified that a practical situation behind our study is one where an insurance company may be interested in minimising its ruin probability by using a reinsurer who accepts excessive claim flow in return for a fraction of the premium. The problem that emerges here is how to optimise parameters of the reinsurance contract to make it beneficial to both parties, in the sense that the reinsurer's income is positive and the time to ruin of the insurer, if ruin occurs, is maximised.

In doing so, we want to incorporate a number of real-life factors in the course of defining and determining optimal policies. We consider both the average and the LD-regimes, and take into account the reinsurer's direct losses. By studying a suitable optimisation problem we explain the behaviour of the system, so as to pick the best possible policy, and draw insightful conclusions.

It has to be said that in reality the reinsurer may not try to maximise his profit. In a competitive market, he may aim to construct a policy which is attractive for the insurers. In other words, the reinsurer may intend to keep his expected profit reasonably large, at the same time allowing the insurer to stay away from bankruptcy for a sufficiently long time. A natural question here is how to find a rule according to which we assign relative weights to the reinsurer's profit on the one hand, and to the length of insurer's ruin time on the other hand, so as to obtain a 'realistically balanced' objective function for the optimisation problem. The lead in to this work was done in Kelbert et al. (2007), where they concluded that in the case of exponentially distributed claims, taking excess of loss reinsurance gave no benefit to the insurer. We extend this work to cover alternative, so called, heavy tailed distributions, in an attempt to see if a more realistic loss distribution effects the behaviour of the system.

## 6.2 The reinsurance model

Consider a small insurance company that agrees to pay any claim in full up to an amount $h$, the retention level. Above this level, the company accepts only a fraction $\alpha$ of the claim, where $0 \leq \alpha \leq 1$. The rest of the claim is covered by a reinsurer. The distribution of the number of claims involving the reinsurer is the same as the distribution of the number of claims involving the insurer, as each pays a defined proportion of every claim.

The amount that an insurer pays on the $i$-th claim is

$$Y_i = \min(X_i, h + \alpha\,(X_i - h)).$$

The amount that the reinsurer pays is

$$Z_i = \max(0,\ (1-\alpha)\,(X_i - h)).$$

In return, the reinsurer gets a fraction of the premium from the insurer. Denote the premium rate for the insurance company after the payment to the reinsurer by $c_1 = \beta c$ and the premium rate received by the reinsurer as $c_2 = (1-\beta)c$ where $0 < \beta \leq 1$ is a constant, $c$ is the premium rate obtained without reinsurance (see (6.3) below).

However, the insurance company will be given a clemency period and stop paying this fraction of the premium to the reinsurer if its total capital $U_1$ falls below an agreed level $P$.

Then the premium rates to both companies, depending on the period, are specified in the following table:

| Period | | Insurer's premium rate | Re-insurer's premium rate |
|---|---|---|---|
| Normal: | $U_1 \geq P$ | $c_1 = \beta c$ | $c_2 = (1-\beta)\, c$ |
| Clemency: | $U_1 < P$ | $c$ | $0$ |

(6.1)

Then the aggregate claims process before reinsurance is

$$S(t) = \sum_{i=1}^{N(t)} X_i, \tag{6.2}$$

where $N(t)$ is a homogeneous Poisson process with constant intensity $\lambda$ that is independent of the claim sizes $X_1, X_2, \ldots$ which are independent and identically distributed with the mean $\mathbf{E}X$. Then the expected total premium paid by the client per unit time can be written as

$$c = (1+\rho)\, \lambda \mathbf{E} X, \tag{6.3}$$

where $\rho$ is the loading factor showing how profitable the combined business is. In a real situation $\rho$ is expected to be reasonably small; otherwise the business will not be attractive for clients.

Here our equations for the risk process, (see Asmussen (2000) and references therein), are:

$$U_1(t) = K + C_1(t) - S_1(t),\ U_1(0) = K, \tag{6.4}$$

$$U_2(t) = C_2(t) - S_2(t) \tag{6.5}$$

where $U_1(t)$ is the capital of the insurer, and $U_2(t)$ is the (positive or negative) increment of the reinsurer's capital generated by his support of the insurer's business. The functions $S_1(t)$ and $S_2(t)$ describe the insurer's and reinsurer's aggregate claims respectively

$$S_1(t) = \sum_{i=1}^{N(t)} Y_i, \quad S_2(t) = \sum_{i=1}^{N(t)} Z_i,$$

The functions $C_1(t)$, $C_2(t)$ describe the insurer's and reinsurer's profits calculated via (6.1) and are the unique solutions to the following integral equations:

$$C_1(t) = \int_0^t \left[c_1 \mathbf{1}\left(U_1(s) > P\right) + c\, \mathbf{1}\left(U_1(s) \leq P\right)\right] ds,$$

$$C_2(t) = \int_0^t c_2 \mathbf{1}\left(U_1(s) > P\right) ds.$$

Here, $\mathbf{1}(\cdot)$ is an indicator function (ie equals 1 if true and 0 otherwise).

Many studies have been made on the kind of distribution that can be used to describe the variation in claim amounts (for a concise text see Hogg & Klugman (1984)). The general conclusion is that claims distributions tend to be positively skewed and long tailed. Two examples of this are the Pareto distribution and the Weibull distribution. However, for these so called heavy-tailed distributions, the asymptotic behaviours of the ruin probability are totally different from those of the Exponential distribution see Embrechts et al. (1997), Embrechts & Veraverbeke (1982), Klüppelberg & Mikosch (1997). Moreover, the analytical results for the probability of ruin cannot be obtained and must be found via numerical methods. In this chapter, we concentrate on the Gamma distribution (whilst using the Exponential distribution to demonstrate analytical forms) as this is seen to be a more realistic distribution for insurance claims than the exponential. The results will not hold if the claim size distribution is heavy-tailed.

## 6.3 Basic formulae for the Large Deviations Regime

Using the theoretical framework laid down in chapter 2, we can investigate the behaviour of our model when the insurer is in a Large Deviations (LD) regime.

**Definition 6.3.1 (The LD Slope)** *The LD slope is given by*

$$D = \lambda \frac{dM_X(R)}{dR}$$

*where $M_X(R)$ is the MGF of the claims distribution, and $R$ is the maximising $r$, given by the unique positive solution in $(0, r_\infty)$ (given that we have positive safety loading) to the Cramér-Lundberg equation (1.29).*

**Definition 6.3.2 (The Rate Function)** *The rate function is given by*

$$I_0^T = t_n \left[D_{1n}R_{1n} + D_{2n}R_{2n} - \lambda \left(M\left(R_{1n}, R_{2n}\right) - 1\right)\right] + t_c \left[D_{1c}R_{1c} + D_{2c}R_{2c} - \lambda \left(M\left(R_{1c}, R_{2c}\right) - 1\right)\right]$$

*where the vectors $(R_{1\text{n}}, R_{2\text{n}})$ and $(R_{1\text{c}}, R_{2\text{c}})$ are the values maximising the instantaneous rate function during the normal and clemency periods respectively, with the two expressions being independent of time.*

The expression for $I_0^T$ gives us the necessary information to work out the minimising values for $D_{1\text{n}}$, $D_{1\text{c}}$, $D_{2\text{n}}$, $D_{2\text{c}}$ which will represent the rate of expenses on average of the insurer and the reinsurer in the LD-regime (their LD slope), with the subscripts $_\text{n}$ and $_\text{c}$ representing the normal and clemancy periods respectively. These will be used later when maximising the expected profit of the reinsurer (see (6.7) below).

Here $t_\text{n}$ is the duration of the payment period, and $t_\text{c}$ is the duration of the clemency period, before the ruin of the insurance company. The time of ruin of the company is defined as $t_r = t_n + t_c$. The values $D_{1n}$ and $D_{1c}$ (geometrically, the slopes of the piecewise LD trajectory) can be easily found by geometric arguments.

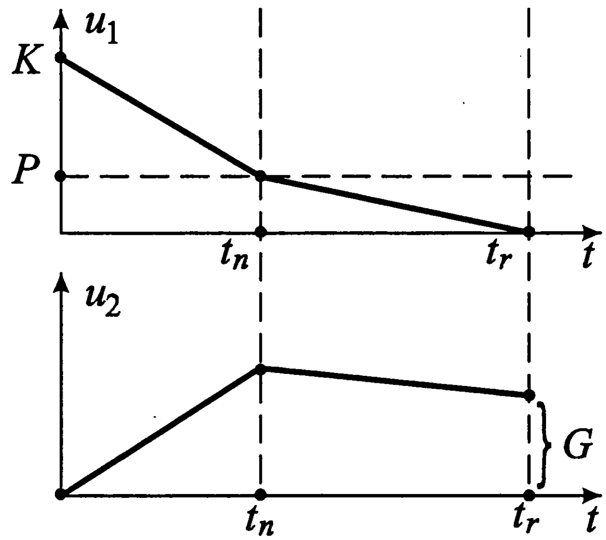


Figure 6.1: Deterministic trajectory minimising the LD rate function

The insurer has rate of income $c_1 = c\beta$ and rate of expenses $D_{1n}$ from 0 to $t_n$, and his income falls from $K$ to $P$. Similarly for the second period. So the slopes $\dot{u}_1$ in Figure (6.1) are $D_{1n} - c_1$ and $D_{1c} - c$ (the rate of loss). Hence, the time in the normal period is:

$$t_n = \frac{K - P}{D_{1n} - c_1},$$

and the time in the clemency period is:

$$t_c = \frac{P}{D_{1c} - c}.$$

i.e.

$$\begin{aligned} D_{1n} &= \frac{(K-P)}{t_n} + c\beta, \\ D_{1c} &= \frac{P}{t_c} + c \end{aligned} \tag{6.6}$$

The values $D_{2n}$, and $D_{2c}$ can be considered as fixed constants and values $R_{1n}$, $R_{2n}$, $R_{1c}$, $R_{2c}$ can be found from the following partial derivatives:

$$\begin{aligned} D_{1n} &= \lambda \frac{\partial}{\partial R_1} M(R_{1n}, R_{2n}), \ D_{1c} = \lambda \frac{\partial}{\partial R_1} M(R_{1c}, R_{2c}), \\ D_{2n} &= \lambda \frac{\partial}{\partial R_2} M(R_{1n}, R_{2n}), \ D_{2c} = \lambda \frac{\partial}{\partial R_2} M(R_{1c}, R_{2c}). \end{aligned} \tag{6.7}$$

Next we minimise the rate function $I_0^T$ with respect to $D_{2n}$ and $D_{2c}$ for fixed values of $t_n$ and $t_c$. A simple computation yields

$$R_{2n} = R_{2c} = 0. \tag{6.8}$$

Hence, we can rewrite the rate function in the form

$$I_{\min} = t_n \left(D_{1n}R_{1n} - \lambda \left(M \left(R_{1n}, 0\right) - 1\right)\right) + t_c \left(D_{1c}R_{1c} - \lambda \left(M \left(R_{1c}, 0\right) - 1\right)\right). \tag{6.9}$$

Without varying the parameters $K$, $P$ and $c_1$, we find the optimal values of $t_n$ and $t_c$. Observe that $R_{1c}$ does not depend on $t_n$ and $R_{1n}$ does not depend on $t_c$. Differentiating with respect to $t_n$ and $t_c$ yields for $R_{1n}$ and $R_{1c}$, respectively,

$$c_1 R_{1n} = \lambda \left(M \left(R_{1n}, 0\right) - 1\right) \tag{6.10}$$

and

$$cR_{1c} = \lambda \left(M \left(R_{1c}, 0\right) - 1\right). \tag{6.11}$$

The minimised LD rate function (6.9) becomes

$$I_{\min} = (K - P)\, R_{1n} + PR_{1c} \tag{6.12}$$

This allows us to estimate the ruin probability $\psi(U)$ in the zero approximation:

$$\ln \psi(U) = -I_{\min} + \text{asymptotically small terms.}$$

The expected ruin time in the LD regime can now be obtained from (6.6):

$$t_r = t_n + t_c = \frac{K - P}{D_{1n} - c_1} + \frac{P}{D_{1c} - c}. \tag{6.13}$$

The slopes $\dot{u}_2$ in Figure (6.1) are $c_2 - D_{2n}$ and $-D_{2c}$ for the normal and the clemency period, respectively. The expected profit $G = u_2(t_r)$ generated by the reinsurer by his support of the insurer's business is

$$G = \frac{c_2 - D_{2n}}{D_{1n} - c_1}(K - P) + \frac{-D_{2c}}{D_{1c} - c}P. \tag{6.14}$$

Note that the results of this section use only the fact that $M_X(\mathbf{R})$ is a convex function. Hence, they are valid for a wide class of claim distributions (including the Gamma distribution, which we use as our example).

## 6.4 General Behaviour

In this section, we restrict the problem by setting $c = 1$.

| K | P | $\mu$ | $\gamma$ | $\lambda$ | $\alpha$ | G* | c* |
|---|---|---|---|---|---|---|---|
| 2 | 1 | 2 | 1 | 1 | 0.6 | 0.0476 | 0.4243 |
| 2 | 1 | 4 | 1 | 2 | 0.6 | 0.0476 | 0.4243 |
| 2 | 1 | 5 | 1 | 1 | 0.6 | 0.2424 | 0.2683 |
| 2 | 1 | 10 | 1 | 2 | 0.6 | 0.2424 | 0.2683 |
| 2 | 1 | 10 | 1 | 1 | 0.6 | 0.2908 | 0.1897 |
| 2 | 1 | 100 | 1 | 1 | 0.6 | 0.3292 | 0.060 |
| 2 | 1 | 5 | 1 | 1 | 0.3 | -1.4824 | 0.1342 |
| 2 | 1 | 5 | 1 | 1 | 0.4 | -0.6305 | 0.1789 |
| 2 | 1 | 5 | 1 | 1 | 0.5 | -0.1111 | 0.2236 |
| 2 | 1 | 5 | 1 | 1 | 0.7 | 0.5016 | 0.3131 |
| 2 | 1 | 5 | 1 | 1 | 0.8 | 0.7024 | 0.3578 |
| 2 | 1 | 5 | 1 | 1 | 0.9 | 0.8645 | 0.4025 |
| 10 | 1 | 2 | 1 | 1 | 0.6 | 8.0474 | 0.4243 |
| 100 | 1 | 2 | 1 | 1 | 0.6 | 98.045 | 0.4243 |
| 500 | 1 | 2 | 1 | 1 | 0.6 | 498.0342 | 0.4243 |
| 2 | 1 | 2 | 1 | 0.1 | 0.6 | 0.3127 | 0.1342 |
| 2 | 1 | 2 | 1 | 0.5 | 0.6 | 0.2157 | 0.3000 |
| 2 | 1 | 2 | 1 | 1.5 | 0.6 | -0.2122 | 0.5196 |
| 2 | 1 | 2 | 0.5 | 1 | 0.6 | 1.3002 | 0.3000 |
| 2 | 1 | 5 | 2 | 1 | 0.6 | 0.2034 | 0.3795 |
| 2 | 1 | 5 | 3 | 1 | 0.6 | 0.0142 | 0.4648 |
| 2 | 1 | 5 | 3 | 1 | 0.4 | -0.879 | 0.3098 |
| 2 | 1 | 5 | 3 | 1 | 0.5 | -0.3576 | 0.3873 |
| 2 | 1 | 5 | 3 | 1 | 0.7 | 0.3074 | 0.5422 |

Table 6.1: Proportional Reinsurance

### 6.4.1 Proportional Reinsurance

Initially our objective is to maximise the profit $G$, as if viewing the contract from the reinsurers point. From our formula for $G$, we see that for a fixed $\alpha$, the smaller $c_1$, the longer the length of the normal period, and at the same time, the larger the income of the reinsurer and the smaller the LD slope. So we expect the value of $c_1$ which will maximise the profit of the reinsurer to be the smallest possible $c_1$. Table 6.1 shows how the problem varies when claims have a Gamma distribution. We alter the values of $\gamma$, to see how the behaviour of the system changes as the tail of the distribution becomes heavier.

Looking at the results from Table 6.1, we see that in all cases, we pick the smallest $c_1$ possible. This is equivalent to maximising the length of the normal period, and at the same time minimising the expenses of the reinsurer. It is not possible to show analytically the equation for $G(\alpha)$ in the general case due to the numerical methods required in finding $R$. Substituting our optimal value for $c_1$, we obtain, after some manipulation (for the exponential case)

$$G(\alpha) = K - P - \frac{(1-\alpha)P}{\alpha(1-\mu\lambda\alpha)}$$

| K | P | $\mu$ | $\gamma$ | $\lambda$ | $\alpha$ | G* | c* |
|---|---|---|---|---|---|---|---|
| 2 | 1 | 2 | 1 | 1 | 0.56 | -0.0916 | 0.396 |
| 2 | 1 | 2 | 1 | 1 | 0.58 | -0.0202 | 0.4101 |
| 2 | 1 | 2 | 1 | 1 | 0.59 | 0.014 | 0.4172 |
| 2 | 1 | 2 | 1 | 1 | 0.6 | 0.0474 | 0.4243 |
| 2 | 1 | 5 | 1 | 1 | 0.5 | -0.1114 | 0.2236 |
| 2 | 1 | 5 | 1 | 1 | 0.52 | -0.0305 | 0.2326 |
| 2 | 1 | 5 | 1 | 1 | 0.53 | 0.0078 | 0.237 |
| 2 | 1 | 5 | 1 | 1 | 0.54 | 0.0447 | 0.2415 |
| 2 | 1 | 5 | 1 | 1 | 0.3 | -1.4832 | 0.1342 |
| 3 | 1 | 5 | 1 | 1 | 0.3 | -0.4841 | 0.1342 |
| 4 | 1 | 5 | 1 | 1 | 0.3 | 0.515 | 0.1342 |
| 5 | 1 | 5 | 1 | 1 | 0.3 | 1.5141 | 0.1342 |
| 2 | 1 | 5 | 1 | 1 | 0.3 | -0.4388 | 0.1342 |
| 2 | 1 | 5 | 1 | 1 | 0.3 | -0.0906 | 0.1342 |
| 2 | 1 | 5 | 1 | 1 | 0.3 | 0.2575 | 0.1342 |
| 2 | 1 | 5 | 1 | 1 | 0.3 | 0.6057 | 0.1342 |

Table 6.2: Proportional Reinsurance - Turning points for G to become positive

So the profit of the reinsurer during the normal period depends only on $\alpha$ and $c_1$ through their ratio, which in the optimal case is independent of these two parameters. So to calculate the optimal value, we only consider the contribution from the grace period. Moreover, the total profit of the reinsurer depends only on $\lambda$, $\mu$ through their ratio. This is also verified by our results, as seen from the first four lines of output.

In some cases we cannot start a profitable business. This is because the LD slope for the reinsurer dominates his income because of the contribution of $\alpha^2$ in the denominator. We can find some critical values for $\alpha$, $K$, $P$, below which (or above accordingly), the reinsurer will definitely make a loss (see Table 6.2).

Clearly, the value of $c_1^*$ does not change with $K$ and $P$. However, the profit of the reinsurer increases with $K$ and decreases with $P$. This makes sense; the reinsurer prefers a small grace period and a larger normal period. A large $P$ will mean a larger grace period. On the other hand, a large $K$ will mean a larger normal period, since the length of the grace period doesn't depend on $K$.

So, we have established that once all the other parameters are fixed, the insurer and the reinsurer will want to pick the same value for $c_1$. However, we need to examine what happens when we let two values vary freely, namely $\alpha$ and $c_1$. See Table 6.3 for some examples.

This is a trivial case, obvious from the expression for $G$; the reinsurer will want to set $\alpha = 1$, whilst making $c_1 < 1$. Then the reinsurer will be receiving premium but not paying any fraction of the claims, hence making a profit without even being at any risk.

This is not a sensible objective to take. In a competitive market, our insurer will be seeking a more appealing policy. So the objective of the reinsurer needs to take account of increasing the insurers survival probability, whilst at the same time making his profit large.

| K | P | $\mu$ | $\gamma$ | $\lambda$ | $\alpha$ | G* | c* |
|---|---|---|---|---|---|---|---|
| 2 | 1 | 2 | 1 | 1 | 1 | 0.9085 | 0.7171 |
| 2 | 1 | 4 | 1 | 2 | 1 | 0.9085 | 0.7171 |
| 2 | 1 | 5 | 1 | 1 | 1 | 0.9231 | 0.4572 |
| 2 | 1 | 10 | 1 | 2 | 1 | 0.9231 | 0.4572 |
| 2 | 1 | 10 | 1 | 1 | 1 | 0.9129 | 0.3262 |
| 2 | 1 | 100 | 1 | 1 | 1 | 0.8091 | 0.1100 |
| 10 | 1 | 2 | 1 | 1 | 1 | 8.1767 | 0.7171 |
| 100 | 1 | 2 | 1 | 1 | 1 | 89.9437 | 0.7171 |
| 500 | 1 | 2 | 1 | 1 | 1 | 453.3523 | 0.7171 |
| 2 | 1 | 2 | 1 | 0.1 | 1 | 0.8934 | 0.2336 |
| 2 | 1 | 2 | 1 | 0.5 | 1 | 0.9238 | 0.5100 |
| 2 | 1 | 2 | 1 | 1.5 | 1 | 0.8422 | 0.8760 |
| 2 | 1 | 5 | 1.1 | 1 | 1 | 0.9399 | 0.4790 |
| 2 | 1 | 5 | 1.5 | 1 | 1 | 0.9741 | 0.5577 |
| 2 | 1 | 5 | 1.7 | 1 | 1 | 0.9792 | 0.5931 |
| 2 | 1 | 5 | 2 | 1 | 1 | 0.9787 | 0.6425 |
| 2 | 1 | 5 | 3 | 1 | 1 | 0.9378 | 0.7846 |

Table 6.3: Proportional Reinsurance alpha = 1

We want to find some function according to $t$, which we will assign weights to the profit of the reinsurer on one hand, and the length of the survival time of the insurer on the other, so as to obtain a total value function for our problem.

#### 6.4.1.1 Introducing a penalty $\Pi$

We have calculated all the quantities and optimum values for our parameters in the case where a LD regime rules. In reality things are different; a LD regime begins and lasts some finite amount of time, say $T$ Having found our optimal parameters, we know that in a LD regime, the insurer will go bust after time $t_{\mathrm{n}}$, and this will happen before or after time $T$ depending on the relative sizes of the two stopping times.

Allocating a fixed cost $\Pi$ to the ruin of the insurance company, we want to minimise the total cost incurred $\Pi \Pr(T \geq t_{\mathrm{n}}) - G$. Having found the rate function $I = (K-P) R_{1\mathrm{n}} + PR_{1\mathrm{c}}$, we are able to compute the probability of being in a LD regime in the interval $[0, t_{\mathrm{n}}]$. This is equivalent to the probability that the LD regime lasts at least $t_{\mathrm{n}}$. Chernoff's Theorem gives us that this probability is at most

$$e^{-I} = e^{(P-K)R_{1\mathrm{n}} - PR_{1\mathrm{c}}}.$$

Formally, we define:

$$\tau_n \equiv \inf\{t : U_n(t) = 0\}$$

where $U_n$ is the capital of the insurer, and then use the following result:

**Proposition 6.4.1**

$$\begin{aligned}\frac{1}{n}\ln\Pr(\tau_n \leq T) &= \frac{1}{n}\ln\Pr\left(\inf\{\zeta_n(t) : 0 \leq t \leq T\} \leq 0\right) \\ &\leq -\inf_{s\in\mathcal{A}} I_0^T(s) + o(1)\end{aligned}$$

A proof for the above may be found in Schwartz & Weiss (1995). It makes use of $\zeta_n(t)$, the free $M/M/1$ process, which is a $M/M/1$ queue without boundary.

Intuitively, this means that the larger the initial capital, the less time a LD regime will last. Also, a larger safety loading means a longer time period before the LD regime sets in, but the slope $D$ will be steeper.

We are now in a position to construct a value function for our problem

$$\begin{aligned}V &= G - \Pi\Pr(T > t_{\mathrm{n}}) = G - \Pi\exp\{(P-K)R_{1\mathrm{n}} - PR_{1\mathrm{c}}\} \\ &= \left(1 - c_1 - \frac{c_1^2(1-\alpha)}{\lambda\mu\alpha^2}\right)\frac{(K-P)\lambda\mu\alpha}{c_1(c_1-\lambda\mu\alpha)} - \frac{(1-\alpha)P}{\alpha(1-\lambda\mu\alpha)} \\ &\quad -\Pi\exp\left\{(P-K)\frac{c_1-\lambda\mu\alpha}{c_1\alpha\mu} - P\frac{1-\lambda\mu\alpha}{\alpha\mu}\right\}.\end{aligned}$$

Our task now, is to choose an appropriate value for $\Pi$. Clearly a large $\Pi$ will mean that we will try to make $t_{\mathrm{n}}$ larger, whereas a small $\Pi$ makes us concentrate around making the reinsurer's profit maximum. The value of $\Pi$ will depend on a number of different factors. A large probability of the insurer getting ruined will make the specific reinsurance company less appealing to our insurer. The ruin of the insurer might also incur a cost in that the reinsurer will stop making profit through him and will need to replace him with a new insurance company, and during the period of seeking a new insurer will still have to pay for maintenance costs, or the reinsurer might have to pay compensation to the insurer in the event of ruin.

Moreover, the cost it will have to the reinsurer if the insurer ruins will also depend on the value of that specific insurer. i.e. the profit the reinsurer could have made out of him had he survived. A natural parameter to consider here is the average profit of the reinsurer during a non-LD period. This will be $1 - c_1 - \lambda(1-\alpha)\mathbf{E}X$, and we might choose to set $\Pi = B + A(1 - c_1 - \lambda(1-\alpha)\mathbf{E}X)$ the cost proportional to the size of the lost insurer.

At the first stage we will consider $\Pi$ to be a constant, independent of the other variables. This only takes account of fixed costs related to the ruin of an insurer. We choose a number of different $\Pi s$ and the results are shown in table 6.4. We are interested in the cases where $\Pi$ is large enough for the insurer to survive longer.

Remember that we established that, given $\alpha$, the optimal $c_1$ is the same for both the reinsurer and insurer, i.e. a $c_1$ which maximises the survival time of the insurer and the profit of the reinsurer. This is shown clearly through our results, since all the $c_1^*$ are equal

| $\Pi$ | $K$ | $P$ | $\mu$ | $\gamma$ | $\lambda$ | $\alpha$ | $c^*$ | $V^*$ | $t_n^*$ |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 1 | 2 | 1 | 1 | 1.00 | 0.7071 | 0.5904 | 3.4142 |
| 2 | 2 | 1 | 4 | 1 | 2 | 1.00 | 0.7071 | 0.9161 | 3.4142 |
| 2 | 2 | 1 | 5 | 1 | 1 | 1.00 | 0.4472 | 0.9977 | 1.8090 |
| 2 | 2 | 1 | 10 | 1 | 2 | 1.00 | 0.4472 | 1.0000 | 1.8090 |
| 2 | 2 | 1 | 10 | 1 | 1 | 1.00 | 0.3162 | 1.0000 | 1.4625 |
| 2 | 2 | 1 | 100 | 1 | 1 | 1.00 | 0.1000 | 1.0000 | 1.1111 |
| 5 | 2 | 1 | 5 | 1 | 1 | 1.00 | 0.4472 | 0.9942 | 1.8090 |
| 10 | 2 | 1 | 5 | 1 | 1 | 1.00 | 0.4472 | 0.9885 | 1.8090 |
| 20 | 2 | 1 | 5 | 1 | 1 | 1.00 | 0.4472 | 0.9769 | 1.8090 |
| 100 | 2 | 1 | 5 | 1 | 1 | 1.00 | 0.4472 | 0.8845 | 1.8090 |
| 500 | 2 | 1 | 5 | 1 | 1 | 0.85. | 0.3801 | 0.6407 | 2.1283 |
| 1000 | 2 | 1 | 5 | 1 | 1 | 0.79 | 0.3533 | 0.5377 | 2.2899 |
| 1000 | 10 | 1 | 2 | 1 | 1 | 0.76 | 0.5374 | 8.3007 | 40.4315 |
| 1000 | 100 | 1 | 2 | 1 | 1 | 1.00 | 0.7071 | 99.0000 | 338.0071 |
| 1000 | 500 | 1 | 2 | 1 | 1 | 1.00 | 0.7071 | 499.0000 | 1703.693 |
| 5 | 2 | 1 | 2 | 1 | 0.1 | 1.00 | 0.2236 | 0.8417 | 1.2880 |
| 5 | 2 | 1 | 2 | 1 | 0.5 | 1.00 | 0.5000 | 0.5896 | 2.0000 |
| 5 | 2 | 1 | 2 | 1 | 1.5 | 0.61 | 0.5283 | -0.7227 | 12.2362 |
| 5 | 2 | 1 | 5 | 1 | 1 | 1.00 | 0.4472 | 0.9942 | 1.8090 |
| 5 | 2 | 1 | 5 | 2 | 1 | 1.00 | 0.6325 | 0.9119 | 2.8930 |
| 5 | 2 | 1 | 5 | 3 | 1 | 0.83 | 0.6429 | 0.1811 | 5.6028 |
| 10 | 2 | 1 | 5 | 2 | 1 | 1.00 | 0.6325 | 0.7606 | 2.8930 |
| 11 | 2 | 1 | 5 | 2 | 1 | 1.00 | 0.6325 | 0.7303 | 2.8930 |
| 12 | 2 | 1 | 5 | 2 | 1 | 0.97 | 0.6135 | 0.7025 | 2.9825 |

Table 6.4: Proportional Reinsurance, fixed penalty

to $\alpha\sqrt{\lambda \mathbf{E}X}$. Hence we may substitute for $c_1 = c_1^*$ and obtain a simpler expression:

$$V = (K-P) - \frac{(1-\alpha)\,P}{\alpha\,(1-\lambda\mu\alpha)} - \Pi \exp\left\{(P-K)\,\frac{1-\sqrt{\lambda}}{\alpha} - P\frac{1-\lambda\mu\alpha}{\alpha\mu}\right\}$$

Immediately we see that in order for the penalty $\Pi$ to have an effect, it needs to be exponentially larger than the function $G$, because it is multiplied by an exponentially small probability and will be negligible if it is small. In addition, we saw that the duration of the normal period $v$ is inversely proportional to $\alpha$, hence the insurer will want to pick the smallest $\alpha$ possible. This is confirmed from our output, since the larger the penalty $\Pi$, which implies that the insurers benefit is very important, the smaller the optimum $\alpha$ becomes.

Now we change our assumptions about $\Pi$, and let $\Pi$ vary depending on the value of each insurer. For a simple model, we consider the penalty to be

$$\Pi = B + A\,(1 - c_1 - \lambda\,(1-\alpha)\,\mathbf{E}X)$$

Of course this is a very crude approximation, however, it will serve our purpose to obtain a qualitative picture of the behaviour of $\alpha^*$. For each set of parameters, we try a range of different $A$s and investigate the behaviour.

We have only considered values for $A$ and $B$ which give us a positive $V^*$ for $\alpha \neq 1$. The cases where $V > 0$ for $a = 1$ are irrelevant, since clearly no insurer would accept that policy and hence the total profit of the reinsurer would be zero. Thus when these cases arise, they indicate that we have chosen too small a value for the fixed cost $B$, as no insurance company of any size would accept such a policy.

### 6.4.2 Excess of Loss Reinsurance

The computation required in this case is more complicated due to the numerical techniques required. We start by using a Newton Raphson method to find the routes to equation 6.10. We can then go on to calculate the remaining values. Essentially, the results should be the same as before when $h = 0$. However, as $h$ is increased, the insurer is exposed to more of the claim, more of the time, hence his expected ruin probability probability increases and this is indicated by a decrease in the time to ruin $t_{\mathrm{n}}$. Table 6.6 shows how the reinsurers profit is affected by the increase in $h$. We can see that there is a limiting $h$, above which the probability of a claim occurring whose size is greater than $h$ is so small that $h$ can be considered to be infinite. To put it another way, the reinsurer takes no risk but still takes profit. In this case, the insurer would not accept such a policy in real life.

#### 6.4.2.1 Introducing a Penalty $\Pi$

Now we carry out the same analysis as before in order to explore the retention level $h$, taking into account the cost it has for the reinsurer to have a policy which is bad for the

| A | B | K | P | $\mu$ | $\gamma$ | $\lambda$ | $\alpha$ | c* | V* |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 10 | 1 | 2 | 1 | 1 | 1.000 | 0.7072 | 9.0002 |
| 4000 | 0 | 10 | 1 | 2 | 1 | 1 | 0.739 | 0.5202 | 8.2391 |
| 10000 | 0 | 10 | 1 | 2 | 1 | 1 | 0.666 | 0.4709 | 8.0699 |
| 100000 | 0 | 10 | 1 | 2 | 1 | 1 | 0.543 | 0.3838 | 7.6839 |
| 0 | 2000 | 10 | 1 | 2 | 1 | 1 | 0.704 | 0.4979 | 8.1736 |
| 0 | 6000 | 10 | 1 | 2 | 1 | 1 | 0.633 | 0.4475 | 7.9846 |
| 0 | 10000 | 10 | 1 | 2 | 1 | 1 | 0.605 | 0.4277 | 7.9002 |
| 0 | 100000 | 10 | 1 | 2 | 1 | 1 | 0.505 | 0.3574 | 7.5372 |
| 100000 | 2000 | 10 | 1 | 2 | 1 | 1 | 0.541 | 0.3823 | 7.6759 |
| 10000 | 2000 | 10 | 1 | 2 | 1 | 1 | 0.637 | 0.4506 | 7.9937 |
| 10000 | 6000 | 10 | 1 | 2 | 1 | 1 | 0.607 | 0.4291 | 7.9045 |
| 10000 | 10000 | 10 | 1 | 2 | 1 | 1 | 0.589 | 0.4165 | 7.8482 |
| 2000 | 10000 | 10 | 1 | 2 | 1 | 1 | 0.601 | 0.4251 | 7.8884 |
| 6000 | 10000 | 10 | 1 | 2 | 1 | 1 | 0.595 | 0.4205 | 7.8671 |
| 6000 | 100000 | 10 | 1 | 2 | 1 | 1 | 0.505 | 0.3568 | 7.5336 |
| 0 | 0 | 10 | 1 | 5 | 2 | 1 | 1.000 | 0.6325 | 9.5699 |
| 4000 | 0 | 10 | 1 | 5 | 2 | 1 | 1.000 | 0.6325 | 9.5684 |
| 10000 | 0 | 10 | 1 | 5 | 2 | 1 | 1.000 | 0.6325 | 9.5662 |
| 100000 | 0 | 10 | 1 | 5 | 2 | 1 | 1.000 | 0.6325 | 9.5328 |
| 0 | 2000 | 10 | 1 | 5 | 2 | 1 | 1.000 | 0.6325 | 9.5679 |
| 0 | 6000 | 10 | 1 | 5 | 2 | 1 | 1.000 | 0.6325 | 9.5639 |
| 0 | 10000 | 10 | 1 | 5 | 2 | 1 | 1.000 | 0.6325 | 9.5598 |
| 0 | 100000 | 10 | 1 | 5 | 2 | 1 | 0.980 | 0.6201 | 9.4724 |
| 100000 | 2000 | 10 | 1 | 5 | 2 | 1 | 1.000 | 0.6325 | 9.5308 |

Table 6.5: Proportional Reinsurance, linear penalty

| K | P | $\mu$ | $\lambda$ | $h$ | $\alpha$ | G* | c* | $t_n^*$ |
|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 4 | 1 | 2 | 0.6 | 0.1397 | 0.8565 | 0.5723 |
| 2 | 1 | 4 | 1 | 3 | 0.6 | 0.1254 | 0.8554 | 0.5218 |
| 2 | 1 | 4 | 1 | 4 | 0.6 | 0.1185 | 0.8546 | 0.4999 |
| 2 | 1 | 4 | 1 | 5 | 0.6 | 0.1154 | 0.8541 | 0.4909 |
| 2 | 1 | 4 | 1 | 10 | 0.6 | 0.1128 | 0.8536 | 0.4853 |
| 2 | 1 | 4 | 1 | 20 | 0.6 | 0.1128 | 0.8536 | 0.4852 |
| 2 | 1 | 4 | 1 | 50 | 0.6 | 0.1128 | 0.8536 | 0.4852 |
| 2 | 1 | 4 | 0.5 | 5 | 0.6 | 0.0109 | 0.9331 | 0.0783 |
| 2 | 1 | 4 | 0.6 | 5 | 0.6 | 0.0200 | 0.9188 | 0.1223 |
| 2 | 1 | 4 | 0.7 | 5 | 0.6 | 0.0330 | 0.9038 | 0.1793 |
| 2 | 1 | 5 | 1 | 2 | 0.6 | 0.0836 | 0.8893 | 0.3986 |
| 2 | 1 | 6 | 1 | 2 | 0.6 | 0.0557 | 0.9097 | 0.3064 |
| 2 | 1 | 10 | 1 | 2 | 0.6 | 0.0186 | 0.9478 | 0.1597 |
| 5 | 1 | 4 | 1 | 2 | 0.6 | 0.2447 | 0.8565 | 2.2893 |
| 10 | 1 | 4 | 1 | 2 | 0.6 | 0.4196 | 0.8565 | 5.1509 |
| 50 | 1 | 4 | 1 | 2 | 0.6 | 1.8191 | 0.8565 | 28.044 |

Table 6.6: Excess of Loss Reinsurance

| Π | K | P | $\mu$ | $\lambda$ | $h$ | $\alpha$ | V* | c* | $t_n^*$ |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 1 | 4 | 1 | 2 | 0.6 | 0.1357 | 0.8579 | 0.5703 |
| 2 | 2 | 1 | 4 | 1 | 3 | 0.6 | 0.1201 | 0.8573 | 0.5193 |
| 2 | 2 | 1 | 4 | 1 | 4 | 0.6 | 0.1129 | 0.8566 | 0.4973 |
| 2 | 2 | 1 | 4 | 1 | 5 | 0.6 | 0.1096 | 0.8561 | 0.4882 |
| 2 | 2 | 1 | 4 | 1 | 10 | 0.6 | 0.1069 | 0.8556 | 0.4825 |
| 2 | 2 | 1 | 4 | 1 | 20 | 0.6 | 0.1069 | 0.8556 | 0.4825 |
| 10 | 2 | 1 | 4 | 1 | 5 | 0.6 | 0.0865 | 0.8642 | 0.4776 |
| 20 | 2 | 1 | 4 | 1 | 5 | 0.6 | 0.058 | 0.8744 | 0.4646 |
| 100 | 2 | 1 | 4 | 1 | 5 | 0.6 | -0.157 | 0.9596 | 0.3754 |
| 1000 | 2 | 1 | 5 | 1 | 2 | 0.6 | -0.1595 | 0.9933 | 0.3233 |
| 1000 | 2 | 1 | 6 | 1 | 2 | 0.6 | 0.0216 | 0.9242 | 0.2976 |
| 1000 | 2 | 1 | 10 | 1 | 2 | 0.6 | 0.0186 | 0.9478 | 0.1597 |
| 1000 | 5 | 1 | 4 | 1 | 2 | 0.6 | 0.2444 | 0.8565 | 2.2891 |
| 1000 | 10 | 1 | 4 | 1 | 2 | 0.6 | 0.4196 | 0.8564 | 5.1518 |
| 1000 | 50 | 1 | 4 | 1 | 2 | 0.6 | 1.8191 | 0.8564 | 28.0485 |

Table 6.7: Excess of Loss Reinsurance, fixed penalty

insurer. This time we use that the equation defining $R$ is

$$\lambda\left(\frac{\left(e^{M\left(R-\frac{1}{\mu}\right)}-1\right)}{\mu R-1}+e^{M\left(R-\frac{1}{\mu}\right)}-1\right)-c_1R=0$$

and we introduce

$$V(h)=G(h)-\Pi e^{-I(D)}.$$

Again, the computation is complicated by the introduction of the rate function, where the routes have to be found using Newton Raphson's method.

To begin with, we apply only a fixed penalty for the ruin of the insurer. Table 6.7 shows the results. Here, we see that the general pattern is similar to that of Table 6.6, but the profit achieved by the reinsurer is much lower because of the higher penalty associated with the ruin of the insurer.

Next, we use a linear penalty

$$\Pi=A\left(1-c_1-\exp\left[-\frac{h}{\mu}\right](h+\lambda\mu)\right)+B$$

Table 6.8 shows the results. As expected, high values of $A$ and $B$ greatly reduce the reinsurers profit due to the low time to ruin of the insurer. In order to increase $t_n$ the reinsurer would have to accept more of the claims above $h$ more of the time, i.e. an attractive policy for the insurer is that which has high values of $A$ and $B$ with low values of $h$ and $\alpha$. Of course, if these values are too extreme, then the policy would not be viable for the

| A | B | K | P | $\mu$ | $\lambda$ | $h$ | $\alpha$ | V* | c* | $t_n^*$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 1 | 4 | 1 | 2 | 0.6 | 0.1397 | 0.8564 | 0.5724 |
| 10 | 0 | 2 | 1 | 4 | 1 | 2 | 0.6 | 0.1369 | 0.8619 | 0.5649 |
| 20 | 0 | 2 | 1 | 4 | 1 | 2 | 0.6 | 0.1342 | 0.8647 | 0.5575 |
| 40 | 0 | 2 | 1 | 4 | 1 | 2 | 0.6 | 0.1291 | 0.8785 | 0.5431 |
| 10 | 10 | 2 | 1 | 4 | 1 | 2 | 0.6 | 0.1167 | 0.8695 | 0.5547 |
| 10 | 20 | 2 | 1 | 4 | 1 | 2 | 0.6 | 0.0968 | 0.8771 | 0.5448 |
| 10 | 40 | 2 | 1 | 4 | 1 | 2 | 0.6 | 0.0576 | 0.8922 | 0.5261 |
| 20 | 10 | 2 | 1 | 4 | 1 | 2 | 0.6 | 0.1142 | 0.875 | 0.5475 |
| 20 | 10 | 2 | 1 | 5 | 1 | 2 | 0.6 | 0.0804 | 0.8919 | 0.3964 |
| 20 | 10 | 2 | 1 | 6 | 1 | 2 | 0.6 | 0.0553 | 0.9101 | 0.3061 |
| 20 | 10 | 2 | 1 | 10 | 1 | 2 | 0.6 | 0.0186 | 0.9478 | 0.1597 |
| 10 | 10 | 3 | 1 | 4 | 1 | 2 | 0.6 | 0.1736 | 0.857 | 1.1432 |
| 10 | 10 | 4 | 1 | 4 | 1 | 2 | 0.6 | 0.2097 | 0.8564 | 1.7173 |
| 10 | 10 | 5 | 1 | 4 | 1 | 2 | 0.6 | 0.2447 | 0.8564 | 2.2897 |
| 50 | 20 | 2 | 1 | 4 | 1 | 2 | 0.6 | 0.0881 | 0.8993 | 0.5177 |
| 50 | 20 | 2 | 1 | 4 | 1 | 4 | 0.6 | 0.0485 | 0.9201 | 0.4231 |
| 50 | 20 | 2 | 1 | 4 | 1 | 6 | 0.6 | 0.0419 | 0.9229 | 0.4057 |
| 50 | 20 | 2 | 1 | 4 | 1 | 50 | 0.6 | 0.0405 | 0.9232 | 0.4023 |

Table 6.8: Excess of Loss Reinsurance, linear penalty

reinsurer, so we need to find a way of optimising these quantities.

## 6.5 The optimisation problem

The goal of this section is to analyze the optimisation problem of minimising the associated ruin probability of the insurance company, $\psi(U) \approx \exp(-I_{\min})$, whilst maximising the reinsurers expected profit.

The expected profit rate of the insurer in the absence of re-insurance is

$$\Delta_1^s \equiv \Delta_1|_{\alpha=\beta=1} = c - \lambda \mathbf{E}X.$$

The ruin probability becomes (taking into account its main asymptotic term only):

$$\ln \psi(U) \cong -I_{\min}^s = -K\ R_{1n}|_{\alpha=\beta=1}. \tag{6.15}$$

If we reinsure the insurer's business, we expect to have a smaller profit rate (still positive)

$$\Delta_1 = c_1 - \lambda\, \mathbf{E}Y > 0. \tag{6.16}$$

However, we expect to get a lower probability of ruin for the insurer, i.e. a greater value $I_{\min}$ (it makes sense to reinsure only when $I_{\min} > I_{\min}^s$).

The goal of the reinsurer is: having a positive expected profit rate, i.e.

$$\Delta_2 = c_2 - \lambda \mathbf{E} Z > 0, \tag{6.17}$$

minimise the expected loss in the case where insurer's business ruins. That is, the reinsurer wants to maximise objective $G$ in (6.14) and wishes to have the optimal value as large as possible.

Therefore, the problem of the optimisation could be formalised as follows. Given the expected profit rate

$$\Delta_1 = A\Delta_1^s \tag{6.18}$$

(i.e. given $0 < A < 1$), find parameters $\alpha, c_1, P, h$ such that

$$I_{\min} \to \max, \qquad G \to \max$$

It is convenient to introduce the dimension-free parameters

$$B = \frac{I_{\min}}{I^s_{\min}} \tag{6.19}$$

which must be greater than 1 (otherwise the joint business is worthless). Then, disregarding asymptotically smaller terms, we have

$$\psi(U) \cong \exp\{-I^s_{\min} B\}.$$

By using the dimension-free reinsurer's profit

$$\overline{G} = \frac{G}{K},$$

the dimension-free optimisation problem will be formalised as

$$\text{Given } A : B, \overline{G} \to \max. \tag{6.20}$$

## 6.5.1 Proportional Reinsurance: $h = 0$

### 6.5.1.1 Exponential Case

First we set $h = 0$, and let claims follow an exponential distribution with parameter $\frac{1}{\mu}$. The distribution of the insurer's individual claims, net of reinsurance, is exponential with parameter $\frac{1}{\mu\alpha}$. This can be seen by noting that if $Y = \alpha X$, then

$$\Pr[Y \le y] = \Pr\left[X \le \frac{y}{\alpha}\right] = 1 - e^{-\frac{1}{\mu\alpha}y} \tag{6.21}$$

Then the joint moment-generating function takes the form

$$M_X(R_1, R_2) = \frac{1}{1 - \alpha\mu R_1 - (1-\alpha)\,\mu R_2}. \tag{6.22}$$

Solving (6.10 and 6.11) for this function and selecting non-trivial solutions, we have

$$R_{1\text{n}} = \frac{1}{\mu}\left(\frac{1}{\alpha} - \frac{q}{\beta}\right), \qquad R_{1\text{c}} = \frac{1}{\mu}\left(\frac{1}{\alpha} - q\right).$$

Here

$$q = \frac{\lambda\mu}{c} = \frac{1}{1+\rho}, \qquad 0 < q < 1,$$

is a dimension-free parameter defined in terms of the loading factor $\rho$.

Substituting $R_{1\text{n}}$ and $R_{1\text{c}}$ into (6.12), we immediately obtain

$$I_{\min} = \frac{K}{\mu}\left[\left(\frac{1}{\alpha} - \frac{q}{\beta}\right)(1-p) + \left(\frac{1}{\alpha} - q\right)p\right], \tag{6.23}$$

where

$$p = \frac{P}{K}.$$

Then we calculate the $D$s in (6.7):

$$\begin{aligned} D_{1\text{n}} &= c\frac{\beta^2}{\alpha\, q}, & D_{2\text{n}} &= c\frac{\beta^2(1-\alpha)}{\alpha^2 q}, \\ D_{1\text{c}} &= c\frac{1}{\alpha\, q}, & D_{2\text{c}} &= c\frac{1-\alpha}{\alpha^2 q}. \end{aligned} \tag{6.24}$$

Now we can calculate the ruin time

$$t_{\text{r}} = \frac{K}{c}\,\alpha q\left(\frac{1-p}{\beta(\beta - \alpha q)} + \frac{p}{1-\alpha q}\right).$$

and the reinsurers expected LD profit

$$\overline{G} = \left(1 - \frac{\beta^2 - \alpha^2 q}{\alpha\beta(\beta - \alpha q)}\right)(1-p) - \frac{1-\alpha}{\alpha(1-\alpha q)}\,p. \tag{6.25}$$

Analysing the second term in the brackets, we can see that the smaller $\beta$ is, the greater $\overline{G}$. This is natural: the greater the share of the reinsurer in the profit, $1-\beta$, the greater his profit.

#### 6.5.1.2 Gamma Case

The distribution of the insurer's individual claims is Gamma with parameters $\left(\frac{1}{\mu},\gamma\right)$. So the joint mgf is:

$$\begin{aligned} M_X\left(R_1,R_2\right) &= \int_0^\infty e^{R_1\alpha x+R_2(1-\alpha)x}\frac{x^{\gamma-1}e^{-\frac{1}{\mu}x}}{\mu^\gamma\Gamma\left(\gamma\right)}dx \\ &= \frac{1}{\mu^\gamma\Gamma\left(\gamma\right)}\int_0^\infty x^{\gamma-1}e^{-x\left(\frac{1}{\mu}-R_1\alpha-(1-\alpha)R_2\right)}dx \end{aligned}$$

let

$$u = x\left(\frac{1}{\mu}-R_1\alpha-\left(1-\alpha\right)R_2\right)$$

then

$$\begin{aligned} x &= \frac{u}{\left(\frac{1}{\mu}-R_1\alpha-\left(1-\alpha\right)R_2\right)} \\ dx &= \frac{du}{\left(\frac{1}{\mu}-R_1\alpha-\left(1-\alpha\right)R_2\right)} \end{aligned}$$

then we have

$$\frac{1}{\Gamma\left(\gamma\right)\left(1-R_1\alpha\mu-\left(1-\alpha\right)\mu R_2\right)^\gamma}\int_0^\infty u^{\gamma-1}e^{-u}du$$

and since

$$\int_0^\infty u^{\gamma-1}e^{-u}du = \Gamma\left(\gamma\right)$$

$$M_X\left(R_1,R_2\right) = \left(\frac{1}{1-R_1\alpha\mu-\left(1-\alpha\right)\mu R_2}\right)^\gamma \tag{6.26}$$

Solving (6.10) and (6.11) for this function, setting $\gamma = 2$, and selecting the smallest positive, non-trivial solutions, we have

$$R_{1n} = \frac{4\beta-\alpha q-\sqrt{\alpha q\left(\alpha q+8\beta\right)}}{4\mu\alpha\beta},$$

$$R_{1c} = \frac{4-\alpha q-\sqrt{\alpha q\left(\alpha q+8\right)}}{4\mu\alpha}$$

Here

$$q = \frac{\lambda\gamma\mu}{c} = \frac{1}{\left(1+\rho\right)}, \qquad 0<q<1, \tag{6.27}$$

is a dimension-free parameter defined in terms of the loading factor $\rho$.

It is interesting to see how the values $R_{1n}$, $R_{1c}$ alter with $\gamma$, and so we show this in Figure 6.2.

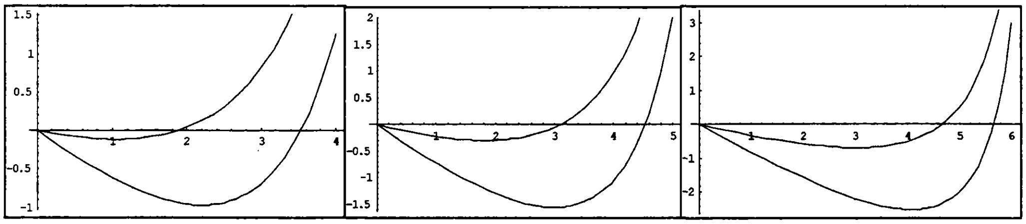


Figure 6.2: Graph of $g(r) = \lambda(M(r,0)-1) - cr$ for $\gamma = 1, 1.5, 2$. ($r = R_{1n}$ for upper line and $r = R_{1c}$ for lower line).

Substituting $R_{1n}$ and $R_{1c}$ into (6.12), we immediately obtain

$$I_{\min} = \frac{K}{4\mu\alpha}\left[\begin{array}{r} \frac{1}{\beta}\left(4\beta - \alpha q - \sqrt{\alpha q\,(\alpha q + 8\beta)}\right)(1-p) \\ +\left(4 - \alpha q - \sqrt{\alpha q\,(\alpha q + 8)}\right)p \end{array}\right], \tag{6.28}$$

where

$$p = \frac{P}{K}.$$

Then we calculate the $D$s in (6.7):

$$\begin{aligned} D_{1\mathrm{n}} &= \frac{16c\beta^3}{\alpha^2q^2 + 6\alpha q\beta + (\alpha q + 2\beta)\sqrt{\alpha q\,(\alpha q + 8\beta)}}, \\ D_{1\mathrm{c}} &= \frac{16c}{\alpha^2q^2 + 6\alpha q + (\alpha q + 2)\sqrt{\alpha q\,(\alpha q + 8)}}, \\ D_{2\mathrm{n}} &= \frac{(1-\alpha)16c\beta^3}{\alpha\left(\alpha^2q^2 + 6\alpha q\beta + (\alpha q + 2\beta)\sqrt{\alpha q\,(\alpha q + 8\beta)}\right)}, \\ D_{2\mathrm{c}} &= \frac{(1-\alpha)16c}{\alpha\left(\alpha^2q^2 + 6\alpha q + (\alpha q + 2)\sqrt{\alpha q\,(\alpha q + 8)}\right)}. \end{aligned} \tag{6.29}$$

Now we can calculate the ruin time

$$t_r = \frac{K}{c}\left(\frac{(1-p)\,N_n}{16\beta^b - \beta N_n} + \frac{p\alpha N_c}{16 - \alpha N_c}\right),$$

where

$$\begin{aligned} N_n &= \alpha^2q^2 + 6\alpha q\beta + (\alpha q + 2\beta)\sqrt{\alpha q\,(\alpha q + 8\beta)}, \\ N_c &= \alpha^2q^2 + 6\alpha q + (\alpha q + 2)\sqrt{\alpha q\,(\alpha q + 8)}, \end{aligned}$$

are the numerators of the LD slopes in the normal and clemency periods respectively.

Now, the expected profit of the reinsurer is:

$$\overline{G} = \left(1 - \frac{16\beta^3 - \alpha N_n}{\alpha\beta\left(16\beta^2 - N_n\right)}\right)(1-p) - \frac{16\,(1-\alpha)}{\alpha\,(16 - N_c)}p.$$

### 6.5.2 The optimisation problem for $h = 0$ and $P = 0$

#### 6.5.2.1 Exponential Formulae

The expected insurer's profit rate $\Delta_1$ in case of successful joint business (see (6.16)) is

$$\Delta_1 = c\,(\beta - \alpha q). \tag{6.30}$$

Substituting $\alpha = \beta = 1$ into (6.23) and (6.30), we obtain $I^s_{\min}$ and $\Delta^s_1$ for the single business

$$I^s_{\min} = \frac{K}{\mu}\,(1-q), \qquad \Delta^s_1 = c\,(1-q).$$

Now we can calculate parameters $A$ and $B$ in (6.18) and (6.19):

$$A = \frac{\beta - \alpha q}{1-q}, \tag{6.31}$$

$$B = \frac{\beta - \alpha q}{\alpha\beta\,(1-q)} \tag{6.32}$$

We also write down $\overline{G}$ for $P = 0$:

$$\overline{G} = 1 - \frac{\beta^2 - \alpha^2 q}{\alpha\beta\,(\beta - \alpha q)}. \tag{6.33}$$

Now we have

- controlling parameters to be specified: $\alpha$, $\beta$ (and also: $p$, $\sigma = h/\mu$ used later);
- given fixed parameters of the single business: $q$ (and also $K/\mu$ used later);
- output parameters that must be optimised: $A$, $B$ and $\overline{G}$.

Solving (6.31) and (6.32) for $\alpha$ and $\beta$, we obtain

$$\alpha = \frac{\sqrt{A^2B^2(1-q)^2 + 4\,q\,AB} - AB\,(1-q)}{2\,B\,q}, \tag{6.34}$$

$$\beta = \frac{\sqrt{A^2B^2(1-q)^2 + 4\,q\,AB} + AB\,(1-q)}{2\,B}. \tag{6.35}$$

Substituting these values into (6.33) yields

$$\overline{G}\Big|_{p=0} = \frac{A\left[B\,(1-q)^2 + 2\,q\right] + 2\,q - (1+q)\,\sqrt{A^2B^2\,(1-q)^2 + 4\,q\,AB}}{2\,A\,q}.$$

#### 6.5.2.2 Gamma Formulae

The expected insurer's profit rate $\Delta_1$ in case of successful joint business (see (6.16)) is

$$\Delta_1 = c\,(\beta - \alpha q). \tag{6.36}$$

Substituting $\alpha = \beta = 1$ into (6.28) and (6.36), we obtain $I^s_{\min}$ and $\Delta^s_1$ for the single business

$$I^s_{\min} = \frac{K}{4\mu}\,(4 - q - \sqrt{q^2 + 8q}), \qquad \Delta^s_1 = c\,(1 - 2q).$$

Now we can calculate parameters $A$ and $B$ in (6.18) and (6.19):

$$A = \frac{\beta - \alpha q}{1 - 2q}, \tag{6.37}$$

$$B = \frac{4\beta - \alpha q - \sqrt{\alpha q\,(\alpha q + 8\beta)}}{\alpha\beta\left(4 - q - \sqrt{q^2 + 8q}\right)} \tag{6.38}$$

We also write down $\overline{G}$ for $P = 0$:

$$\overline{G} = 1 - \frac{16\beta^3 - \alpha N_n}{\alpha\beta\,(16\beta^2 - N_n)}. \tag{6.39}$$

Now we have

- controlling parameters to be specified: $\alpha$, $\beta$ (and also $p$ used later);
- given fixed parameters of the single business: $q$ (and also $K/\mu$ used later);
- output parameters that must be optimised: $A$, $B$ and $\overline{G}$.

To solve the optimisation problem, we use numerical methods and a Mathematica routine.

#### 6.5.2.3 Exponential Results

Level curves of function $G$ in the $\{A, B^{-1}\}$ plane are shown in Figure 6.3. Here we can see the curves slightly change with $q$.

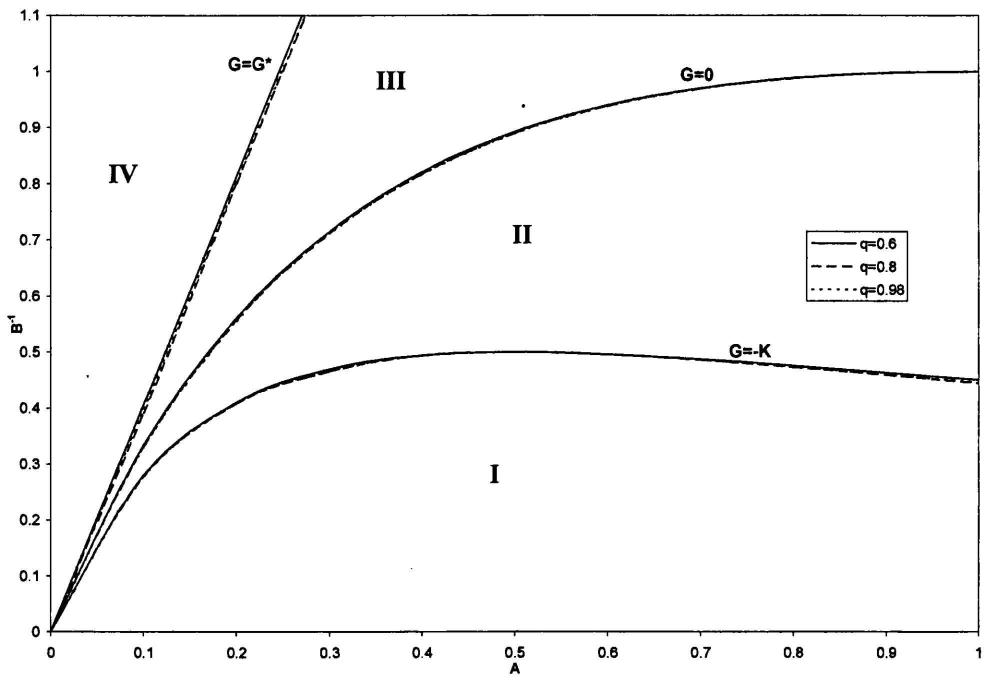


Figure 6.3: Level curves of $G$ with step $K$ for $q = 0.99,\ 0.8,\ 0.6$.

The line $G = K$ coincides with the line $G = G^* \Leftrightarrow \beta = \beta^*$ for the case where $P = 0$ (and $h = 0$).

The lines $G = 0, K$ split the feasible region into four zones (parameter domains) labelled by Roman numerals.

**Zone I:** $G < -K$. Here the re-insurer loses more than the insurer in the case of insurer's LD ruin. This is a zone clearly unacceptable for the reinsurer.

**Zone II:** $-K < G < 0$. This is most likely to be the zone of choice: the insurer and reinsurer must decide between themselves what is appropriate for them.

**Zone III:** $0 < G < G^*$ the region is evidently acceptable for the reinsurer: he has a positive profit in the LD regime even in the case of insurer's ruin.

**Zone IV:** $G > G^*$. Here the LD theory fails.

Thus, only zones II and III are of prime interest.

We suggest that the line of reasonable compromise between the insurer and the re-insurer is where $G = 0$: here the reinsurer has no risk in the case of ruin in the LD limit, and at the same time the insurer is able to reduce his exposure to risk.

Now let us explore the notion of penalties as we previously suggested in Section 6.4. Let us first consider the option that the insurer is prepared to share, say, 20% of the expected profit rate ($A = 0.8$). Also, assume that the insurer offers to reinsure his business with

$G = 0$ or $G = -0.2\,K$ (so that the reinsurer makes a 20% loss if the insurer goes bankrupt). Finally, assume that $q = 0.9$ (i.e. $\rho \cong 0.11$).

We substitute (6.34) into (6.25) and solve (6.25) with respect to $B$ for a constant $\overline{G}$. For the Exponential case, we obtain the explicit formula:

$$B = \frac{A\,(1-\overline{G})\,(1-q)^2 - 4\,q + (1+q)\sqrt{A^2\,(1-\overline{G})^2\,(1-q)^2 + 4\,q}}{2\,A\,(1-q)^2}. \tag{6.40}$$

It gives $B \simeq 1.012$, $\alpha \simeq 0.894$, $\beta \simeq 0.884$ for $\overline{G} = 0$, and $B \simeq 1.201, \alpha \simeq 0.817, \beta \simeq 0.815$ for $\overline{G} = -0.2$. The ruin probability $\psi(U)$ (see (6.15)) in both cases is reduced.

Suppose that $B$ and $q$ are close to 1. Recall that the LD approximation gives an asymptotic result when $n \to \infty$. Thus, the ratio $K/\mu$ must be large, as after scaling (??), it becomes $nK/\mu$. For example, if $K/\mu = 100$ and $q = 0.9$, then ruin probability $\psi(U)$ is reduced by 1.13 when $G = 0$ and 7.4 when $G = -0.2\,K$. If $K/\mu = 1000$, and $q = 0.9$, the ruin probability is reduced by 3.5 when $G = 0$ and by a massive $5\text{x}10^8$ when $G = -0.2\,K$.

Another way of looking at the problem is to say that by taking out reinsurance, the insurer wishes to reduce his probability of ruin by a factor $N = 1000$. In this case, we can calculate how much business (and potential profit) he must give away to achieve this. Again, let us take $q = 0.9$ and $K/\mu = 1000$. Then, calculating $B$ yields

$$B = 1 + \frac{\log(N)}{I^s_{\min}};$$

it gives $B \cong 1.069$.

Next, solving (6.33) for $A$, we obtain:

$$A_{1,2} = \frac{q\,(2B + \overline{G} - 1) \pm (1+q)\sqrt{qB\,(B + \overline{G} - 1)}}{[Bq - (B + \overline{G} - 1)][B - q(B + \overline{G} - 1)]}$$

We see that problem (6.20) will have no solution if $\overline{G} < 1 - B$

On the contrary, the problem has two solutions when $\overline{G}$ takes a negative value in the range:

$$\begin{aligned} &\overline{G}^- \le \overline{G} \le \overline{G}^+, \\ \text{where}\quad &\overline{G}^- = 1 - B \\ &\overline{G}^+ = 1 - B + \frac{2\,q - (1+q)\,\sqrt{B^2\,(1-q)^2 + 4\,B\,q} + B\,\left(1+q^2\right)}{2\,q}. \end{aligned}$$

Hence, when $\overline{G} \notin \left[\overline{G}^-, \overline{G}^+\right]$, the only proper solution will be $A_2$, since $A_1$ will be $> 1$.

For $B = 1.069$, we obtain $\overline{G}^- = -0.069$ and $\overline{G}^+ = -0.068$. Thus, if $\overline{G} = 0$, then the solution is $A = 0.59, \alpha = 0.754, \beta = 0.738$.

Furthermore, if, for example, $\overline{G} = -0.0685$,then $A = 0.98, \alpha = 0.956, \beta = 0.959$, and

$A = 0.89, \alpha = 0.915, \beta = 0.913$. We see that the same ruin probability and the same reinsurer's LD loss can be obtained when the insurer sacrifices 2% or 11% of his expected profit.

#### 6.5.2.4 Gamma Results

Figure 6.4 shows how the problem varies as $\gamma$ increases. We can see that there is almost no change in the optimal position for $p = 0$.

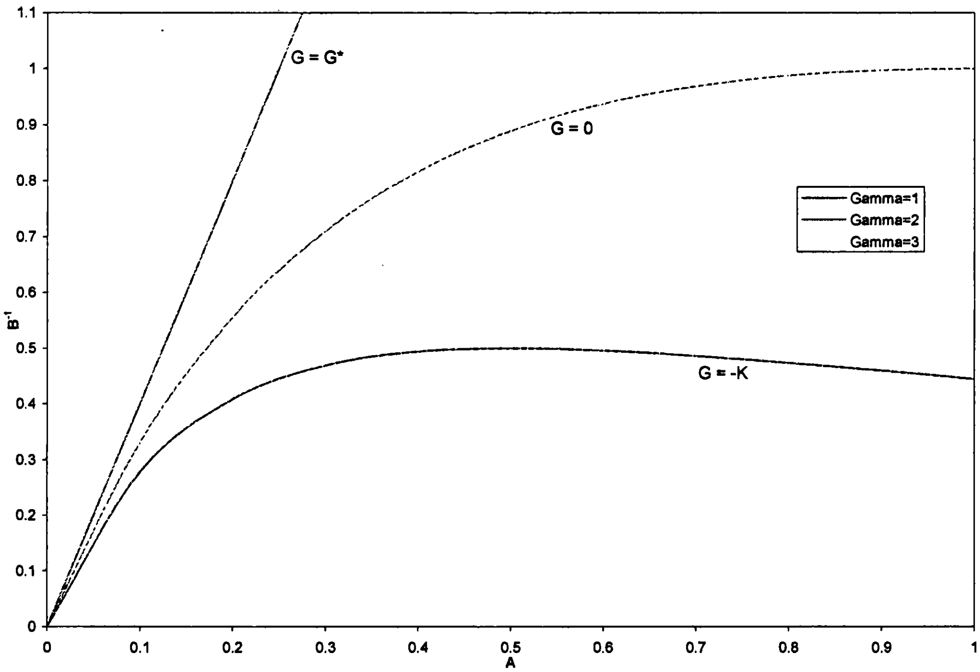


Figure 6.4: Level curves of $G$ on the $\{A, B^{-1}\}$-plane for $\gamma = 1, 2, 3,\ q = 0.9,\ p = 0$.

It's worth taking a moment to discuss the method for calculating the results where $\gamma = 3$. In this case, finding an initial value for the $R$s can only be done via graphical representations. Similarly, the root of the equation $\overline{G} = 0$, can also only be found graphically, so each point on the plot needs to be obtained manually. Due to the amount of time involved in producing results for $\gamma = 3$, it made sense to only consider $\gamma = 1, 2$.

Upon further examination for $\gamma = 1, 2$, we see that almost any change in $q$ makes little difference to the final position (see Figures 6.5 and 6.6).

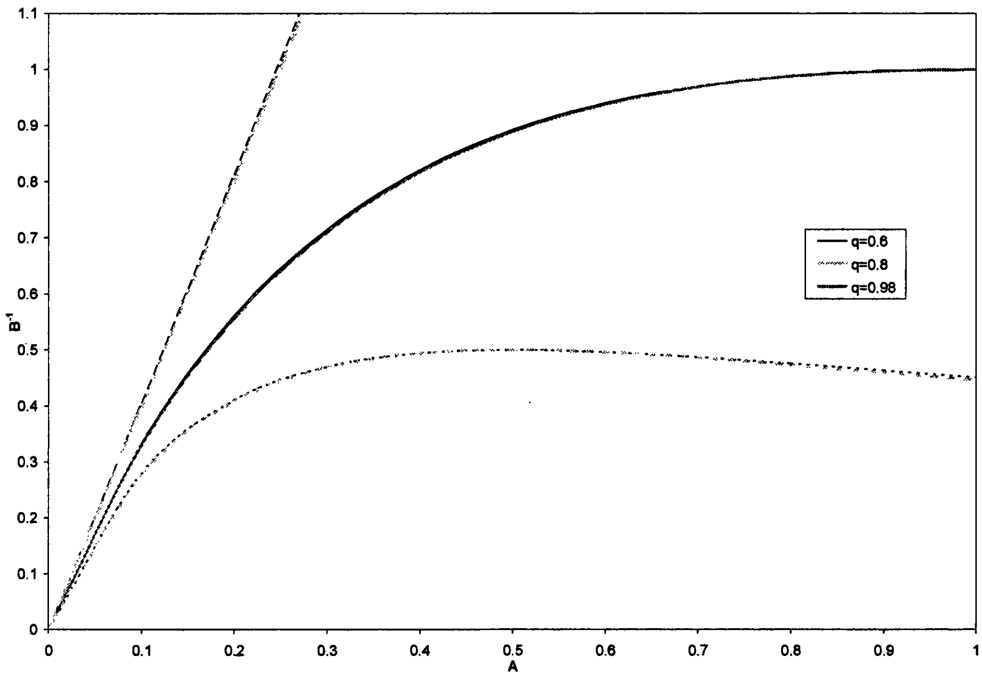


Figure 6.5: Level curves of $G = G^*$ (dashed), $G = 0$ (solid) and $G = -K$ (dotted), on the $\{A, B^{-1}\}$-plane for $\gamma = 1,\ p = 0$ with varying $q$.

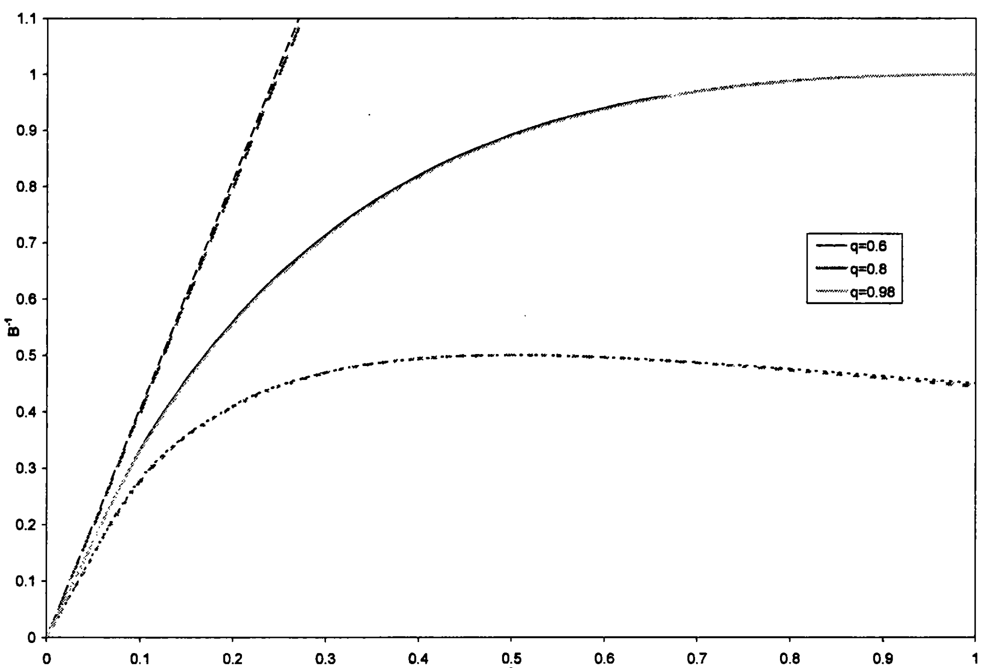


Figure 6.6: Level curves of $G = G^*$ (dashed), $G = 0$ (solid) and $G = -K$ (dotted), on the $\{A, B^{-1}\}$-plane for $\gamma = 1,\ p = 0$ with varying $q$.

It is easy for us to conclude that for $P = 0$, there is no significant difference in our results for the exponential or the gamma distribution. Given the ease of analytical results found with the exponential distribution, it would make sense that this would be the distribution of choice for this particular problem.

#### 6.5.2.5 The optimisation problem for $h = 0$ and $P > 0$

**6.5.2.5.1 Exponential Formula** In this section we assume that $P > 0$.

Substituting (6.23), into (6.19) we obtain

$$B = \frac{\beta - \alpha q}{\alpha\beta(1-q)}(1-p) + \frac{1-\alpha q}{\alpha(1-q)}p, \quad p = \frac{P}{K}. \tag{6.41}$$

Parameter $A$ will be independent of $p$. Solving (6.31) and (6.41) and selecting the appropriate root, we obtain

$$\begin{aligned} \alpha &= \frac{\sqrt{(\overline{A}\,\overline{B} - pq)^2 + 4q\overline{A}\,\overline{B}} - \overline{A}\,\overline{B} + pq}{2\overline{B}q}, \\ \beta &= \frac{\sqrt{(\overline{A}\,\overline{B} - pq)^2 + 4q\overline{A}\,\overline{B}} + \overline{A}\,\overline{B} + pq}{2\overline{B}}, \end{aligned} \tag{6.42}$$

where $\overline{A} = A(1-q)$, $\overline{B} = B(1-q) + pq$.

The function $\overline{G}$ can now be re-written in the form:

$$\overline{G} = (1-p) - \frac{(\beta^2 - \alpha^2 q)(\alpha\overline{B} - p)}{\alpha\overline{A}^2} - p\frac{1-\alpha}{\alpha(1-\alpha q)}.$$

#### 6.5.2.6 Gamma Formula

Now assume that $P > 0$.

Substituting (6.28), into (6.19) we obtain

$$B = \frac{4\beta - \alpha q - \sqrt{\alpha q(\alpha q + 8\beta)}}{\alpha\beta\left(4 - q - \sqrt{q^2 + 8q}\right)}(1-p) + \frac{4 - \alpha q - \sqrt{\alpha q(\alpha q + 8)}}{\alpha\left(4 - q - \sqrt{q^2 + 8q}\right)}p, \quad p = \frac{P}{K}. \tag{6.43}$$

Parameter $A$ will be independent of $p$. Solving (6.37) and (6.43) and selecting the appropriate root we can solve the optimisation problem numerically.

#### 6.5.2.7 Exponential Results

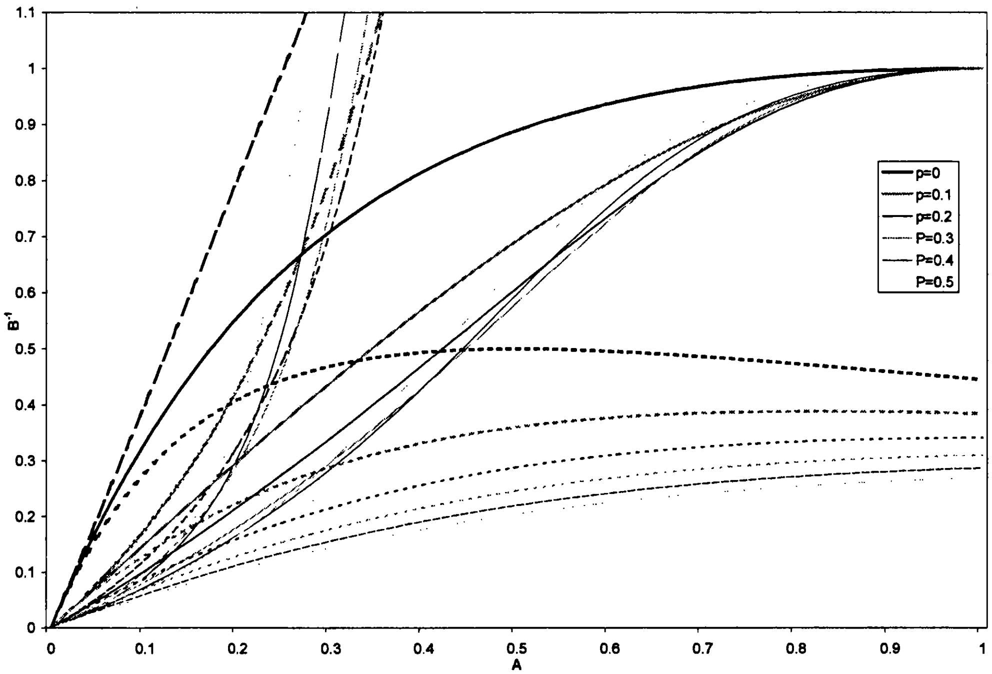


Figure 6.7: The lines $G = 0$ (solid), $G = G^*$ (dashed) and $G = -K$ (dotted) on the $\{A, B^{-1}\}$plane for $q = 0.9$ and $p = 0, 0.1, 0.2, 0.3, 0.4, 0.5$.

Level curves of $G$ for different values of $p$ are plotted in Figure 6.7. We can see that increasing $p$ 'pushes' the lines right and down, i.e. towards better parameters $A$ and $B$ for the same $G$. But the curve $G = 0$ in its upper part starts to move left when $p$ is large enough. This picture confirms that introducing parameter $P > 0$ allows the participants to improve business characteristics.

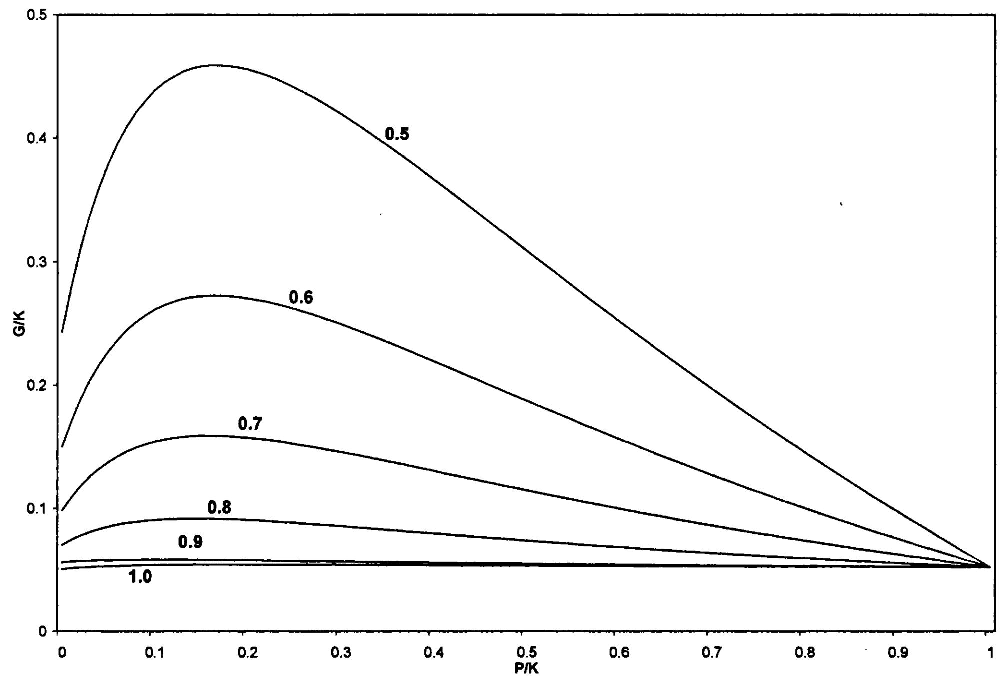


Figure 6.8: The curve $G(P)$ for $q = 0.9$, $B = 0.95$ and various values of $A$ (indicated on each curve).

Lines $G$ versus $P$ for selected values of $A$ and $B$ are shown in Figure 6.8. The existence of optimal $P$ for fixed $A$ and $B$ is apparent.

#### 6.5.2.8 Gamma Results

From the previous section, we found that the most interesting differences occur when $q$ is high (this is also relevant, because the higher $q$ is, the lower $\rho$ is, and if $\rho$ is too high then clients will be put off from taking out the policy in the first place). We look at examples comparing $\gamma = 1$ with $\gamma = 2$ for $q = 0.9$ and varying values of $p = 0, 0.1, 0.2, 0.3, 0.4$ and 0.5. We compare the lines $G = 0$, $G = G^*$, and $G = -K$ separately for clarity.

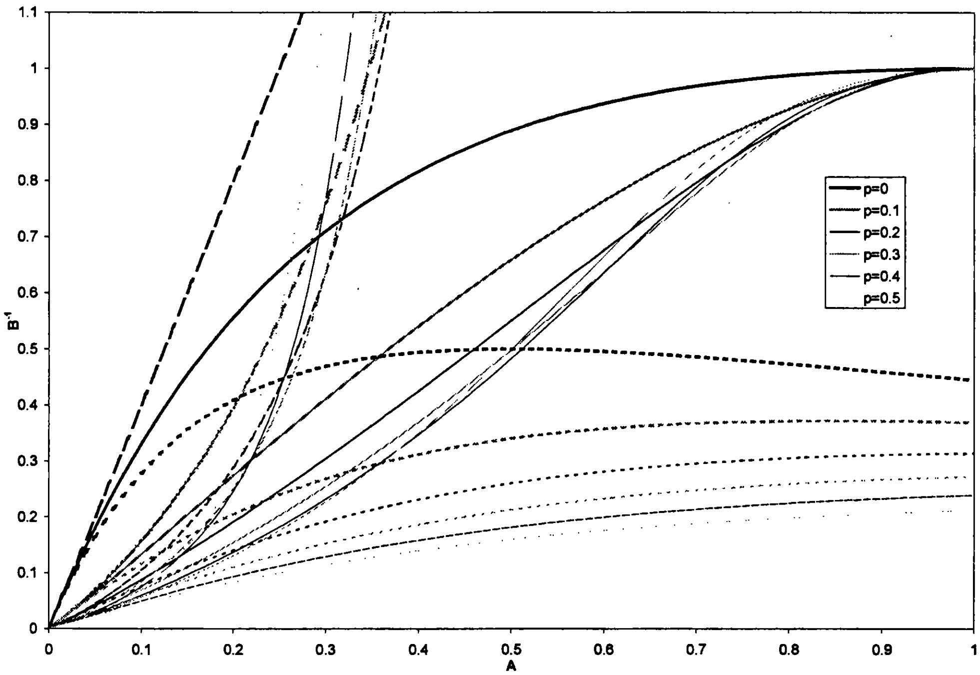


Figure 6.9: Level curves of $G = G^*$ (dashed), $G = 0$ (solid) and $G = -K$ (dotted), on the $\{A, B^{-1}\}$-plane for $\gamma = 2$, $q = 0.9$, $p = (0, 0.5)$.

Figure 6.9 show us how our optimal position improves with $p$, up to about 0.3, after this point, for higher values of $A$, the curves move back up and left towards the original position of $p = 0$.

To see how the plots alter with $\gamma$, we compare for each value of $p$, separately for clarity.

Figures 6.10 to 6.14 also show that as $\gamma$ increases, so does the improvement gained with increasing $p$. It is our belief that as $\gamma$ increases further the benefit gained reduces exponentially to a limiting curve.

The insurer has to give away so much business in the case where $G = G^*$ for values of $A > 0.4$, that to maintain the reinsurers profit, the value of $\alpha$ exceeds 1, hence the reason that the curves above this level are not shown. As $p$ increases beyond 0.2 we begin to see a regression in the insurers optimum position. As the insurer tries to reduce his loss of profit, his ruin probability decreases, beyond that of lower values of $p$. This effect increases with $p$.

If the reinsurer accepts no profit ($G = 0$), then the degenerating improvement of increasing $p$, seen when $G = G^*$, is less obvious and occurs in the case when $p = 0.5$, for $A > 0.35$, and $p = 0.4$ for $A > 0.55$. If we compare this to the case when $\gamma = 1$, the degeneration of improvement with increasing $p$ is more apparent and occurs with lower values of $p$ (and even lower values of A). For example, we see that when $p = 0.5$, the degeneration starts to occur at $A = 0.2$.

For the curve $G = -K$, we see no degeneration of improvement and we see that for both

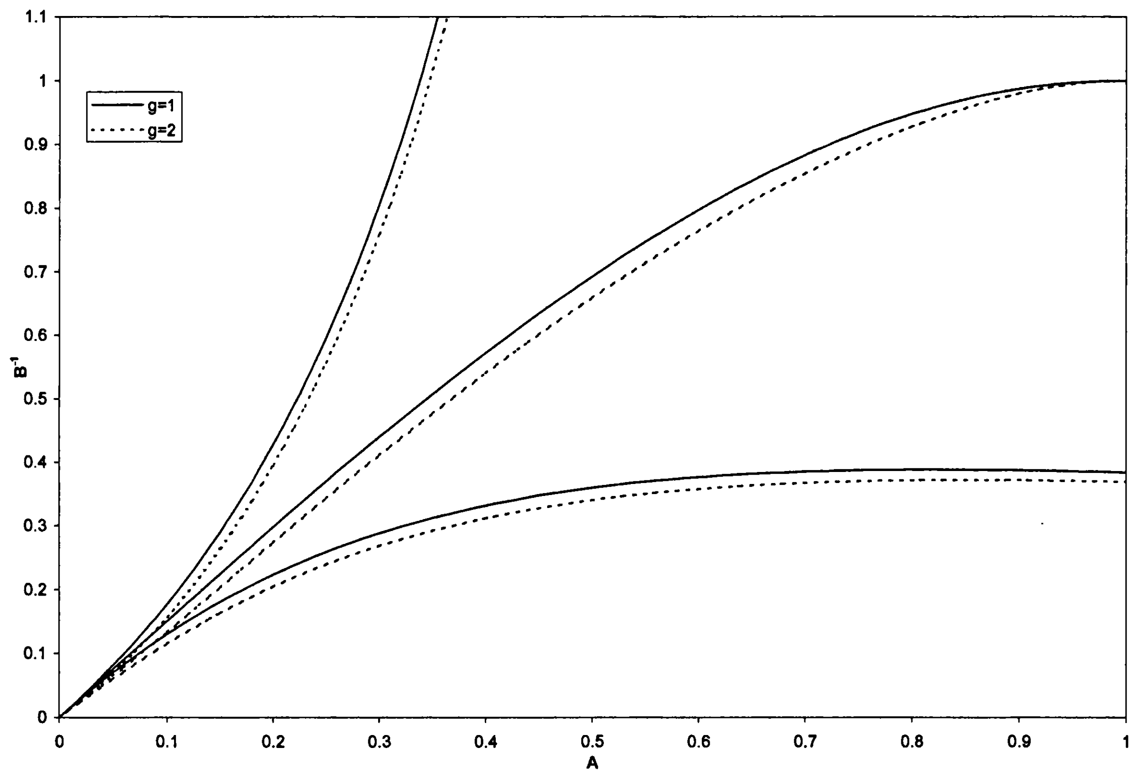


Figure 6.10: Level curves of $G = G^*$ (dashed), $G = 0$ (solid) and $G = -K$ (dotted), on the $\{A, B^{-1}\}$ plane for $\gamma = 1$, and $\gamma = 2$, $q = 0.9$, $p = 0.1$.

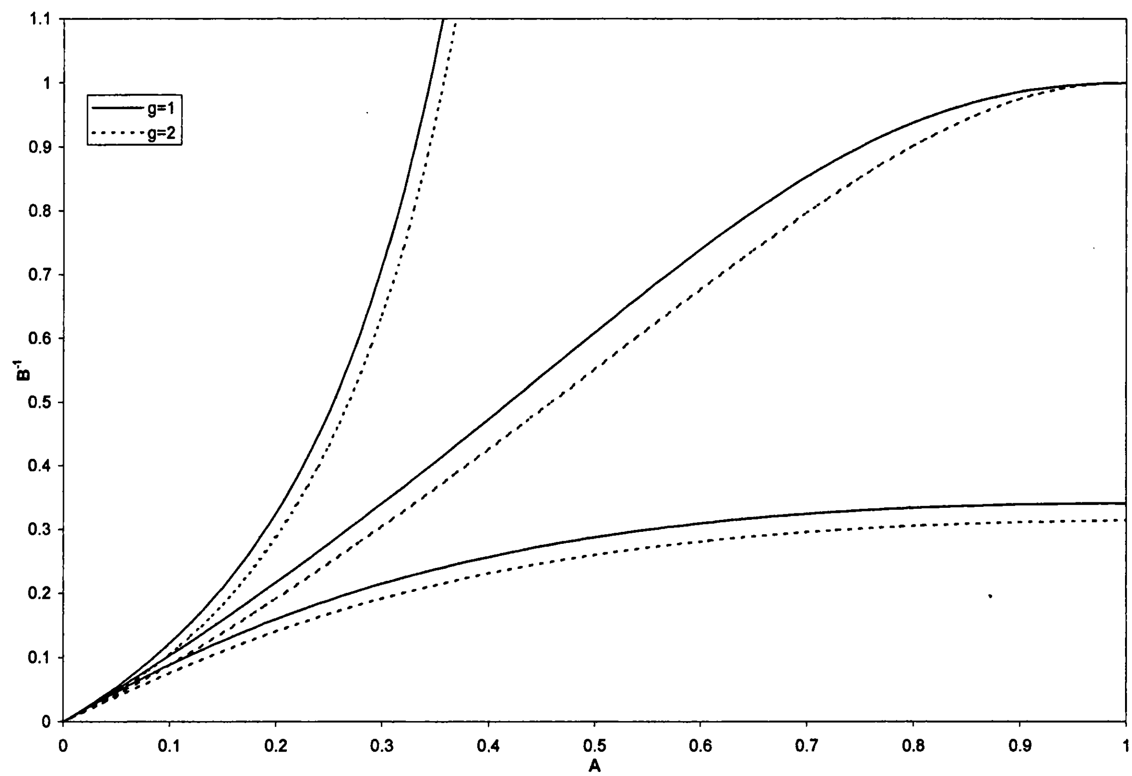


Figure 6.11: Level curves of $G = G^*$ (dashed), $G = 0$ (solid) and $G = -K$ (dotted), on the $\{A, B^{-1}\}$ plane for $\gamma = 1$, and $\gamma = 2$, $q = 0.9$, $p = 0.2$

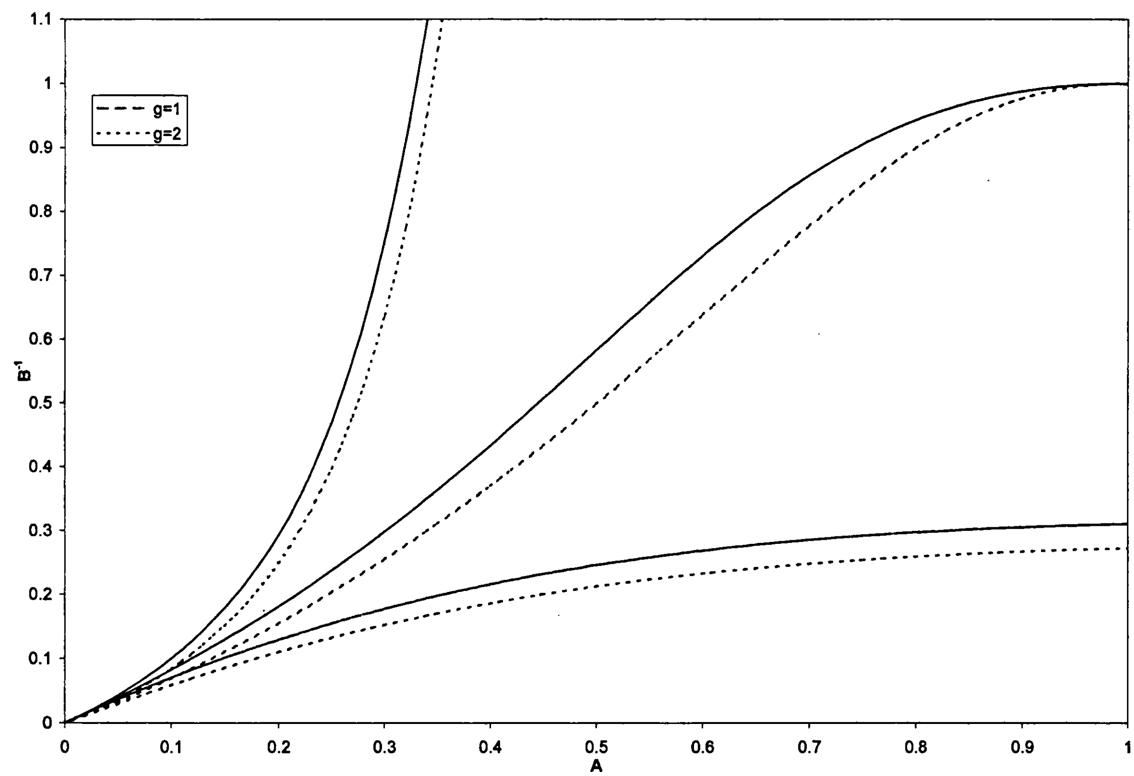


Figure 6.12: Level curves of $G = G^*$ (dashed), $G = 0$ (solid) and $G = -K$ (dotted), on the $\{A, B^{-1}\}$ plane for $\gamma = 1$, and $\gamma = 2$, $q = 0.9$, $p = 0.3$.

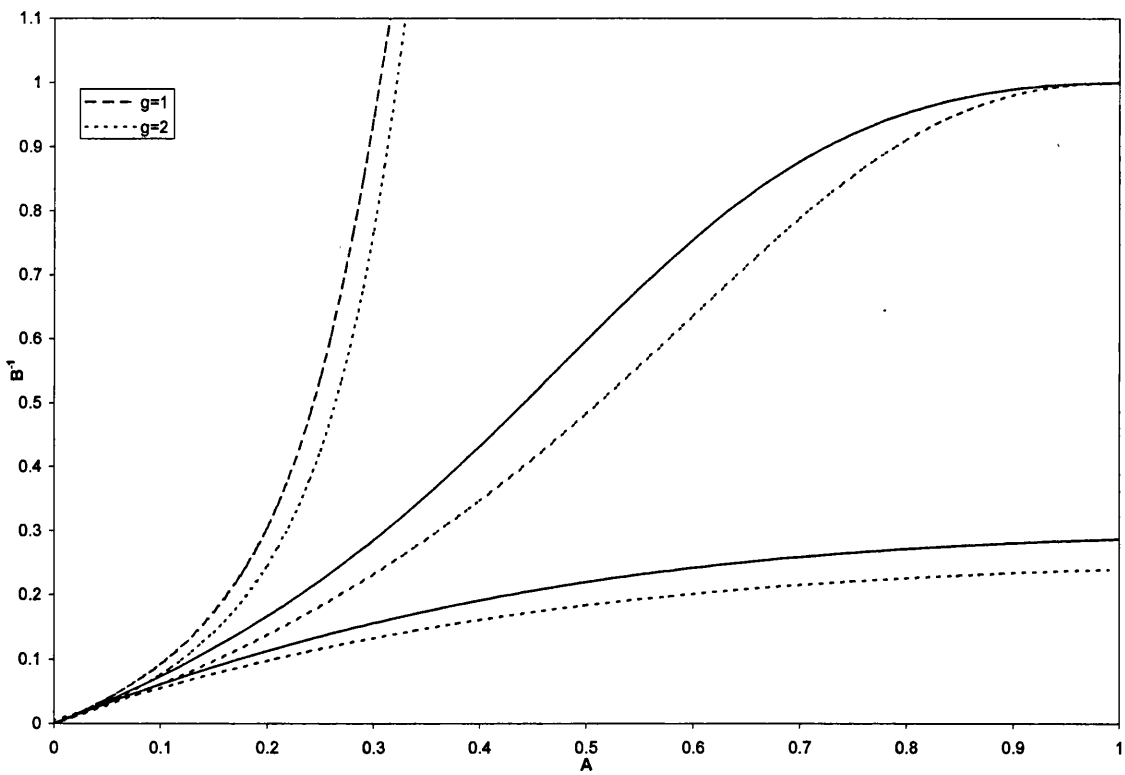


Figure 6.13: Level curves of $G = G^*$ (dashed), $G = 0$ (solid) and $G = -K$ (dotted), on the $\{A, B^{-1}\}$ plane for $\gamma = 1$, and $\gamma = 2$, $q = 0.9$, $p = 0.4$.

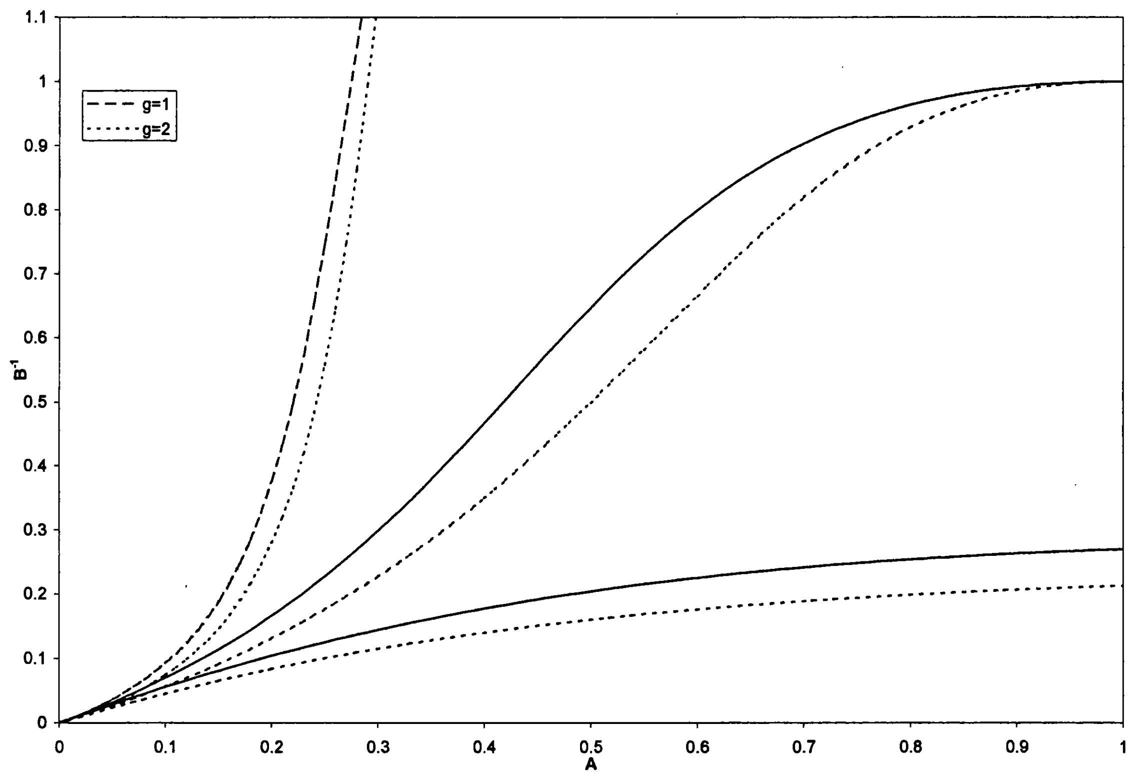


Figure 6.14: Level curves of $G = G^*$ (dashed), $G = 0$ (solid) and $G = -K$ (dotted), on the $\{A, B^{-1}\}$ plane for $\gamma = 1$, and $\gamma = 2$, $q = 0.9$, $p = 0.5$.

$\gamma = 1$ and $\gamma = 2$, the ruin probability decreases with $p$, this increase further increases with $\gamma$.

### 6.5.3 Excess of Loss Reinsurance

#### 6.5.3.1 The general case $h > 0$, $\alpha > 0$

From the general model we consider the general case $h > 0$. The moment generating function takes the form

$$M_X(R_1, R_2) = \frac{e^{(R_1\mu\alpha + R_2\mu(1-\alpha)-1)\,(h/\mu)}}{1 - R_1\mu\alpha - R_2\mu(1-\alpha)} + \frac{\left[1 - e^{(R_1\mu-1)\,(h/\mu)}\right]}{1 - R_1\mu}. \tag{6.44}$$

The second addend in (6.44) is an analytic function of $R_1$, including the point $R_1 = 1/\mu$. The first addend tends to infinity if $R_1\mu\alpha + R_2\mu(1-\alpha) \to 1$. We consider $M_X(R_1, R_2)$ in the half plane $R_1\alpha + R_2(1-\alpha) < 1/\mu$. The minimising values are given by (6.8) and (6.10), (6.11).

In the absence of explicit answers, we have to compute the roots of (6.8) numerically. First, introducing the dimension-free variables $i = R_1\mu, j = R_2\mu, \sigma = h/\mu$, we rewrite (6.44) in the following form:

$$M_X(i,j) = \frac{e^{\sigma(\alpha i + (1-\alpha)j - 1)}}{1 - \alpha i - (1-\alpha)j} + J(\sigma(i-1)).$$

Here $J(j) = [\exp{(j)} - 1]/j$ has no singularities in any finite part of the complex plane.

Equations (6.10) and (6.11) in terms of $i$ and $j$ are, respectively,

$$\frac{\beta}{q} i^n = M_X(i^n, 0) - 1, \tag{6.45}$$

$$\frac{1}{q} i^c = M_X(i^c, 0) - 1. \tag{6.46}$$

Here

$$M_X(i,0) - 1 = \frac{\alpha i}{1 - \alpha i} + \sigma \left\{ J\left[\sigma(i-1)\right] - J\left[\sigma(\alpha i - 1)\right] \right\}.$$

The function $M_X(i,0) - 1$ vanishes at $i = 0$ and has a simple pole at $i = 1/\alpha$. It is monotonically increasing and convex for $i \in [0, 1/\alpha)$ provided that $\sigma \geq 0$ and $\alpha \in (0,1]$. Therefore, equation (6.45) has a single non-zero solution in the interval $(0, 1/\alpha)$ if $\beta/q > \phi'_x(0,0)$.

Hence, the condition of existence of the LD regime in the normal period is

$$q\left[1 - (1-\alpha)(1+\sigma)\exp(-\sigma)\right] < \beta.$$

The similar condition for the clemency period is:

$$q\left[1 - (1-\alpha)(1+\sigma)\exp(-\sigma)\right] < 1.$$

Finally we consider problem (6.20) for the case in hand. Calculating $I_{\min}, \Delta_1, G, \beta^*$, we obtain the following formulas

$$\begin{aligned} A &= \frac{\beta - q\,[\alpha\, e^{-\sigma} + (1 - e^{-\sigma})]}{1-q}, \\ B &= \frac{(1-p)\, i^n + p\, i^c}{1-q}, \\ \overline{G} &= \frac{(1-\beta) - qM_j^n}{qM_h^n - \beta}(1-p) + \frac{-qM_j^c}{qM_h^c - 1}\, p, \\ \beta &> \beta^* \Rightarrow M_h^n + M_j^n > \frac{1}{q}, \end{aligned}$$

where $M_h^{n,c} = \frac{\partial}{\partial h} M_X(h^{n,c}, 0), M_j^{n,c} = \frac{\partial}{\partial j} M_X(j^{n,c}, 0)$,

The lines $G = 0$ for the case $p = 0$ calculated numerically on the $\{A, B^{-1}\}$-plane and shown in Figure 6.15. We can see that the greater the value of $h$ the more the lines are displaced upwards and to the left. In Figure 6.16, we can see that the lines tend to a limiting curve. A detailed study shows that in contrast with the normal regime, increasing the retention level $h$ results in worse output parameters for both businesses when the process is considered in the LD regime.

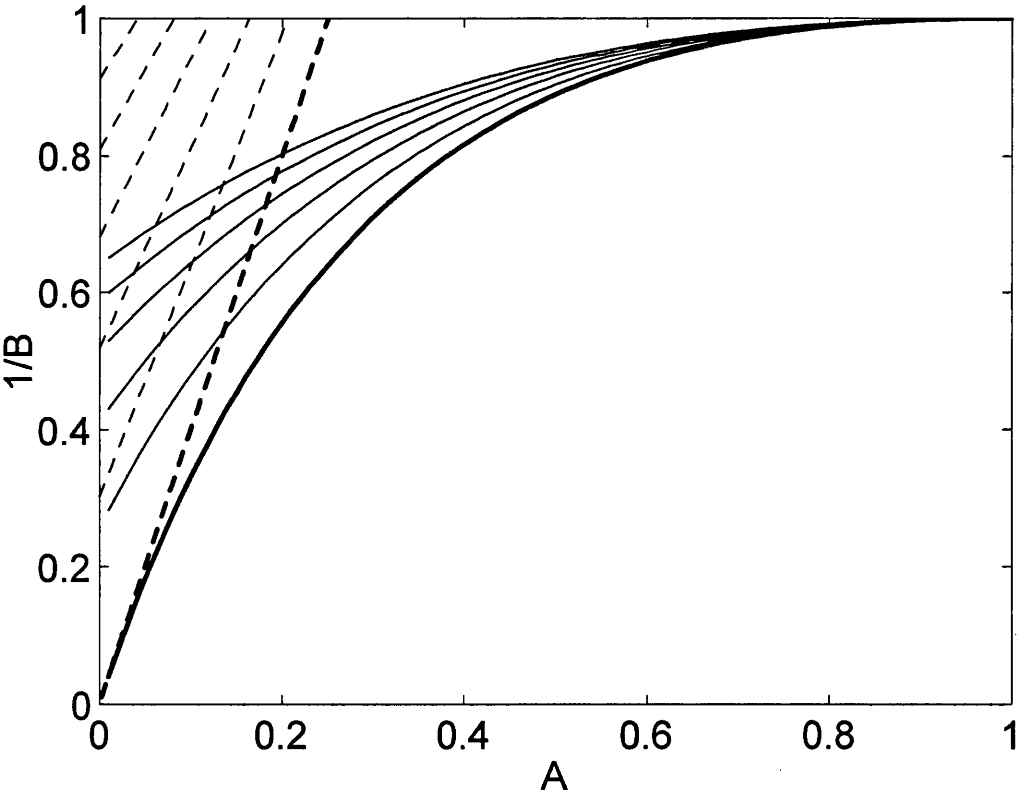


Figure 6.15: The curves $G(A, B) = 0$ (solid) and $\beta(A, B) = \beta^*$ (dashed) for $q = 0.9$, $P = 0$ and $\frac{h}{\mu} = 0, 0.01, 0.02, 0.03, 0.04, 0.05$ (the thinner the line, the bigger the value of $\frac{h}{\mu}$).

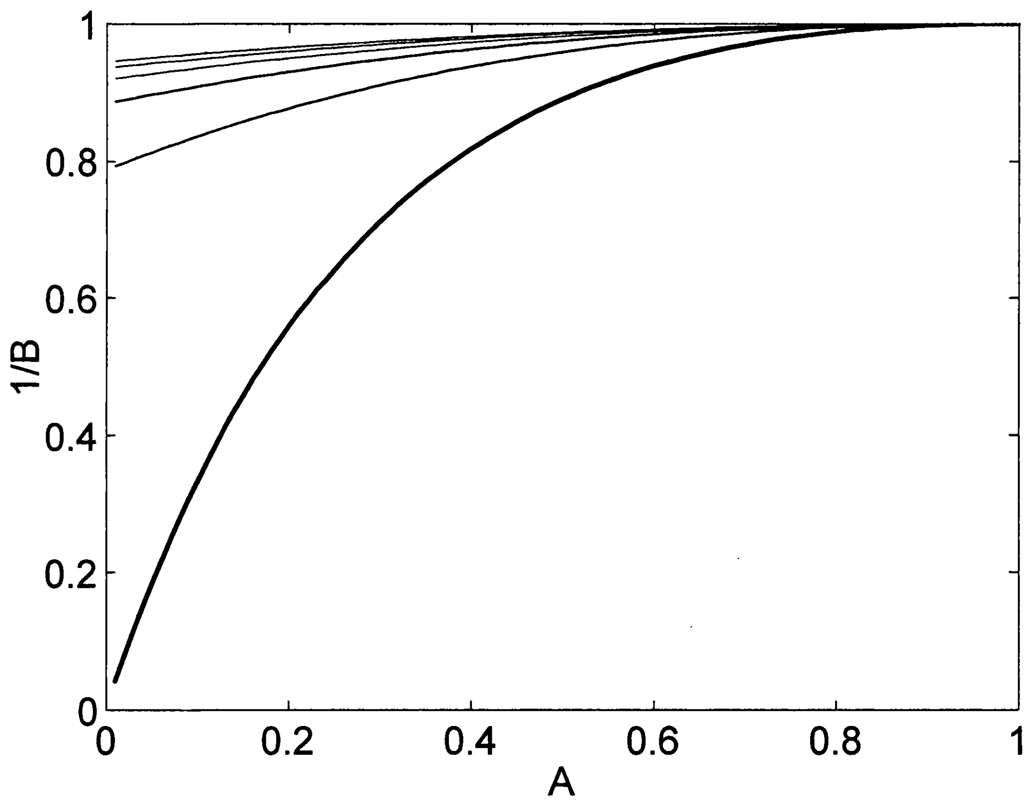


Figure 6.16: The curves $G(A, B) = 0$ for $q = 0.9$, $P = 0$ and $\frac{h}{\mu} = 0, 0.1, 0.2, 0.3, 0.4, 0.5$ (the thinner the line, the bigger the value of $\frac{h}{\mu}$).

## 6.6 Conclusions

In this chapter we study a model of proportional and excess of loss reinsurance under a Large Deviations framework. By exploring the LD rate function we are able to find expressions for the ruin time and the reinsurers expected profit. We have the contradictory aims of the insurer of minimising his probability of ruin, whilst maximising his profit (by giving away as little of his business as possible). We also take account of the aims of the reinsurer, which are also contradictory, again looking to maximise profit, and maximise the time to ruin of the insurer (thus making his offer attractive to the insurer in the first place).

The aim of this chapter was to study the optimisation problem for claims governed by a $\text{Gam}\left(\gamma, \frac{1}{\mu}\right)$, distribution. From the literature available, it was expected that using this distribution would distort the results previously found in Kelbert et al. (2007). However, despite this prediction, it turns out to be very stable. Because of the computational involvement of producing results for the $\text{Gam}\left(\gamma, \frac{1}{\mu}\right)$ distribution, we concentrated our study on $\gamma = 2$. In this case it is possible to make some analytical progress, thus reducing the complexity of the problem.

For the case where $P = 0$, i.e. there is no clemency period, we find that for values of $\gamma = 1, 2$ and 3, there is almost no difference in the value function and the optimum position is therefore left unchanged.

However, for the case where $P > 0$, we find that there are some notable changes in the optimal position between $\gamma = 1$ and $\gamma = 2$. In general, increasing $P$ increases the benefit of the optimum position; the insurer wants the value of $1/B$ to decrease, as this indicates an improvement in his probability of ruin. At the same time, he wants $A$ to increase, as this indicates he is giving away less profit. From our results, it is clear that the benefit gained by increasing $P$ is increased further with increasing $\gamma$. In particular, we show that there exists and optimal value of $P$ to maximising the insurer's profit.

Comparing proportional and excess of loss reinsurance, we demonstrate that introducing a clemency period level $P$ can produce a noticeable increase in the reinsurer's profit. Since $P$ is a free variable (not fixed in the model), the alteration of $P$ can also be beneficial. In particular, we show that there exists an optimal value of $P$ to maximising the insurer's profit.

In the case of excess of loss reinsurance, a retention level $h$, though allowing a considerable growth of the reinsurer's profit in the normal regime, does not improve the behaviour of the system in the LD-regime when claims are distributed exponentially. It would be interesting to find out whether this phenomenon is preserved for other claim distributions (e.g. with heavy tails). In reality, of course, parameters of claim process are not precisely known and need to be estimated from data. In the case of excess of loss reinsurance, the reinsurer may only be informed about claims that affect him. In most cases, if a claim is below the level $h$ then the reinsurer will not even know that they have occured. In other words, the claims distribution for the reinsurer is truncated as he does not have information about the claims below level $h$; for that reason it is hard to draw conclusions about the true distribution of the claim size. In addition, most reinsurance companies will reinsure more than one insurer, in which case we should consider joint distributions.

# Chapter 7

# Summary and Conclusions

In this final chapter, we provide an overview of our work and present our conclusions. We begin by summarising each chapter in turn and then move on to discuss the aims of our work and the extent to which these were met. We then present an overall conclusion and finish by considering further areas of investigation.

## 7.1 Chapter Summary

This thesis considered the popular Actuarial subject of Risk, specifically looking at a non-life insurance company with an aim of reducing the probability of ruin. More specifically we took interest in the lesser explored area of Large Deviations, and how this could be applied to Risk theory and our examples. The study was restricted to a specific area of research defined in Hipp (2003) as optimal reinsurance and new business. The main reason for this was that new business is one of the least studied areas and Reinsurance was a subject area that was already familiar to me from my interest in the subject during my undergraduate degree.

### 7.1.1 Chapter One

Chapter one introduced the basic idea of the collective risk model, specifically the Cramér Lundberg risk model. In doing so we looked at some examples of loss distributions and introduced the compound Poisson process. This then led us into looking at ruin theory and defining the probability of ruin via the adjustment coefficient, or Cramér exponent, and Lundberg's inequality. Having laid down the theoretical groundwork for the thesis we then moved on to describing the subject of Large Deviations.

### 7.1.2 Chapter Two

Since part of our study was to be the application of large deviations to risk theory, chapter two was set aside to introduce the required theory and explain why it was applicable to this area of research. We began by looking at the underlying ideas of Large Deviations with a simple coin tossing experiment. We defined one of the goals of Large Deviation theory; to provide a systematic way of calculating the rate function, and through some simple combinatorics were able to find an approximation to the ruin probability and hence and application to risk theory.

### 7.1.3 Chapter Three

Chapter three set out a time line review of the literature that was applicable to the restricted area of study contained in this thesis. Having defined a methodology for our investigation, we set out the principle research papers in the subject areas of Ruin Probability, Large Deviations and Optimal Control. The latter area was dissected further into optimal dividend payout, optimal investment, optimal reinsurance and new business and optimal premium control. As has already been discussed, we decided to contain our investigation to the area of optimal reinsurance and new business.

The final part of chapter three reviewed any general literature for risk theory to complete our investigation.

### 7.1.4 Chapter Four

Having laid out what our research will entail, chapter four gets down to the business of exploring new work. First we explain the theory of optimal new business and discuss the results obtained by previous work. We then go on to use numerical methods to produce a simulation package that allows us to model more complicated situations. The simulation is used to benchmark the solutions of the Hamilton-Jacobi-Bellman equation and to obtain results in the case of distributions where solutions cannot be obtained analytically.

This chapter explains why taking on new business is a possible strategy to increase the probability of ruin. Our results found that it is possible to optimise a level at which new business is taken on, determined by minimising the ruin probability associated with the insurance business. Moreover, the results show that risky business, or even non-profitable business, can be advantageous when the option of selling this business at a future time is given.

### 7.1.5 Chapter Five

A by-product of the research in chapter four was a simulation program that could be used for other applications. Chapter five discusses and application of this program to the problem of estimating the probability of achieving a target maximum capital before ruin. Through our experimentation it is found that the current upper bound used to approximate the ruin probability increasing over estimates this probability as the surplus increases.

### 7.1.6 Chapter Six

Chapter six investigated a few aspects of Large Deviations in reinsurance. Having first described the reinsurance model, we set about constructing methods and algorithms by which to find optimal arrangements for the conflicting interests of the insurer and reinsurer. Continuing the same trend throughout the thesis we use the standard Cramér Lundberg risk model and look at claims that are independent and identically distributed by an exponential distribution in the first instance. We then extend the work to look at a Gamma distribution; one of the so called 'heavy tailed' distributions. The model also assumes a clemency period, where the insurer receives a respite from paying the reinsurer, should his total premium fall below a pre-determined level $P$.

Our investigations looked first at a model of proportional reinsurance with no clemency period, where we found that many optimal quantities can be evaluated explicitly. In this case we found that varying the claims distribution had very little effect on the results. When implementing a clemency period, we found that this difference between the exponential distribution and the gamma distribution increased with both the level of $P$ and the parameter of the gamma distribution. In general, it is demonstrated that introducing a clemency period can produce a notable increase in the reinsurer's profit. It is also shown that there is an optimal value of $P$ to maximise the insurer's profit.

Finally our investigation looked at the more complicated example of excess of loss reinsurance, where it was found that, with a retention level $h$, though allowing a considerable growth in the reinsurers profit in the normal regime, does not improve the behaviour of the system in the Large Deviations regime when claims are governed by an exponential distribution.

## 7.2 Conclusions

Risk theory is a very popular area of research at the moment, as our every day lives are fraught with many individual risks, and we as people are inclined to want to insure against this risk. In particular, an important area of interest for modern day life, is the analysis of extreme events, and how these may lead to the financial ruin of an insurance company. The phrase 'extreme events' here means an unusually high number of claims and/or unexpectedly high claim sizes. The proper mathematical framework for this analysis is the theory of Large Deviations, one of the most active and dynamic branches of modern applied probability. This framework provides powerful tools for computing the probability of extreme events when the more conventional approaches like the law of large numbers and the central limiting theorem fail. The overall objective of this thesis was to study the linking of Large Deviation techniques with elements of control theory.

Of course it would be wholly impractical to look at all areas of control theory, so we chose one that was of personal interest; this being the area of optimal reinsurance and new business. First we spent the time looking at optimal new business which, having already explored the theory of ruin, was the first case where we needed to calculate the ruin probability. As it turns out, this is only possible analytically for very simple situations, so we took the time to produce a program that could simulate both a single and joint business. Simulating Large Deviations however, is an extremely complicated process and one that we decided not to explore at this time. Keeping our investigations in the normal regime still allowed us new areas of study as we were able to look at more interesting claim distributions, where analytical results cannot be obtained.

The development of this program realised further potential and we looked at applying the package to hitting probabilities. The program allows us to compute the probability of reaching a desired profit target before ruin, and this application could have obvious beneficial uses in the market place.

Finally, our investigation led us to explore the area of optimal reinsurance, where we were able to achieve our objective of linking Large Deviation techniques with control theory.

By studying the LD rate function and specifying the limiting deterministic trajectory of the insurer's and reinsurer's capital in the unlikely event of ruin, we were able to find expression for the ruin time and the reinsurers expected profit in the Large Deviation regime. In the case of the exponential distribution for proportional reinsurance we were able to derive simple formulas by which we can evaluate the total profit/loss, using not just specific numerical data such as the average claim size of the Poisson rate of intensity, but also more qualitative parameters, such as advertising reinsurance and/or offering appealing deals. We develop algorithms to find optimal parameters in order to create the best possible business for both the insurer and the reinsurer. The aims of both parties are somewhat contradictory: the insurer wants to minimise his probability of ruin, whilst giving away as little portion of the expected profit as possible. The reinsurer wants to maximise his profit, but in order to make his business appealing to an insurer, must maximise the predicted time to ruin of the insurer.

We find that by introducing a clemency period, both parties can benefit, and the reinsurer's profit can increase. We find that there is an optimum value of $P$, the level at which the insurer stops paying the reinsurer should his capital fall below this point. We further extend our investigation to look at a more 'heavy tailed' example; the Gamma distribution. In this case, it is expected that the change in claims distribution will significantly distort the results. However, it is found to be very stable, and for the case where $P = 0$, there is almost no difference. As we increase, $P$, we find that the effect of increasing the Gamma parameter exaggerates this difference.

In the case of excess of loss reinsurance, with retention level $h$, we find that even though in the normal regime there is a considerable growth in the reinsurers profit, under the Large Deviation regime, there is no improvement in the behaviour of the system for exponentially distributed claims. In this case, the reinsurer may only have a record of claims that are greater than $h$. For claims below this level, it is conceivable that the reinsurer may not even know that a claim has occurred. Thus it is difficult to estimate the underlying claims distribution, as the reinsurer effectively observes claims from a truncated distribution.

## 7.3 Areas for Further Research

The most obvious area for immediate further study is to explore the effect of heavy tailed claims distributions under the model of excess of loss reinsurance. It would be of interest to discover if the phenomenon of no system improvement in the Large Deviations regime is preserved for other claims distributions and hence is stable, as was the case with proportional reinsurance, or if these heavy tailed distributions distort the results in such a way that the system shows an improvement. Further more, it would be of interest to explore pure excess of loss reinsurance, where the reinsurer pays all of the claim (rather than just a proportion $(1-\alpha)$ of it) above the retention level $h$.

Further progress could also be made with optimal new business and the linking of Large Deviations theory. However, in our simulations, we found that the switching of unprofitable business benefits the company only when the current capital falls below a small level. After this, the company needs to sell the unprofitable business at a higher level of current capital. Our computer simulations clearly demonstrate that this upper level, divided by $n$, the parameter of the Large Deviation limit, tends to 0. In other words, in the Large Deviation regime, the second business should not be taken on at all. On the other hand, if the business is profitable, it will reduce the probability of ruin in the Large Deviation regime, and the graph of $\ln P_{\text{ruin}}$ is a linear function with the slope predicted by the Large Deviation theory.

Other areas of research would involve exploring other control parameters as the objective function as detailed in chapter three. These include optimal dividend payout, optimal investment and optimal premium control.

To take the study further still, we could alter the underlying risk process from a compound Poisson process to a compound Binomial process, or even use a risk process that is perturbed by diffusion. Of course this then causes the scope of the research available to balloon, as the combination of alternative models is quite vast.

# Bibliography


of Actuaries, F. & of Actuaries, I. (2002), *Formulae and Tables for Examinations of the Faculty of Actuaries and the Institute of Actuaries.* Institute of Actuaries.

of Actuaries, I. (2002), *Subject 106 Actuarial Mathematics Core Reading.* The Faculty of Actuaries and Institute of Actuaries.

Albrecher, H., Teugels, J. L. & Tichy, R. F. (2001), On a Gamma Series Expansion for the Time-Dependent Probability of Collective Ruin. *Insurance: Mathematics and Economics*, **29**: 345–355.

Aquilina, J., Kelbert, M. & Suhov, Y. M. (2004), On Optimal Strategies to Deal with Extreme Regimes in Insurance. *Quality Technology and Quantitative Management*, **1**(1): 161–172.

Asmussen, S. (1987), *Applied Probability and Queues.* John Wiley and Sons, Inc.

Asmussen, S. (2000), *Ruin Probabilities.* World Scientific, Singapore.

Asmussen, S. & Højgaard, B. (1999), Approximatins for Finite Horizon Ruin Probabilities in the Renewal Model. *Scandinavian Actuarial Journal*, **1999**(2): 106–119.

Asmussen, S., Højgaard, B. & Taksar, M. (2000), Optimal Risk Control and Dividend Distribution Policies. Example of Excess-of Loss Reinsurance for an Insurance Corporation. *Finance and Stochastics*, **4**: 299–324.

Asmussen, S. & Møller, J. R. (2003), Risk Comparisons of Premium Rules: Optimality and a Life Insurance Study. *Insurance: Mathematics and Economics*, **32**: 331–344.

Asmussen, S. & Taksar, M. (1997), Controlled Diffusion Models for Optimal Dividend Pay-Out. *Insurance: Mathematics and Ecenomics*, **20**: 1–15.

Björk, T. & Grandell, J. (1988), Exponential Inequalities for Ruin in the Cox Case. *Scandinavian Actuarial Journal*: 77–111.

Borch, K. (1967), The Theory of Risk. *Journal of the Royal Statistical Society, Series B (Methodological)*, **29**(3): 432–467.

Brekelmans, R. & de Waegenaere, A. (2001), Approximating the Finite-Time Ruin Probability under Interest Force. *Insurance: Mathematics and Economics*, **29**: 217–299.

Brémaud, P. (2000), An Insesitivity Property of Lundberg's Estimate for Delayed Claims. *Journal of Applied Probability*, **37**: 914–917.

Brockett, P. L. & Xia, X. (1995), Operations Research in Insurance: A Review. *Transactions of Society of Actuaries*, **47**: 7–87.

Browne, S. (1995), Optimal Investment Policies for a Firm with Random Risk Process: Exponential Utility and Minimising the Probability of Ruin. *Mathematics of Oprations Research*, **4**: 937–958.

Bühlmann, H. (1970), *Mathematical Models in Risk Theory.* Springer, Berlin.

Cai, J. (2002), Ruin Probabilities under Dependent Rates of Interest. *Journal of Applied Probability*, **39**: 312–323.

Cai, J. & Dickson, D. (2003), Upper Bounds for Ultimate Ruin in the Sparre Andersen Model with Interest. *Insurance: Mathematics and Economics*, **32**: 61–71.

Cai, J. & Dickson, D. C. M. (2002), On the expected discounted penalty function at ruin of a surplus process with interest. *Insurance: Mathematics and Economics*, **30**(3): 389–404, available at http://ideas.repec.org/a/eee/insuma/v30y2002i3p389-404.html.

Cai, J. & Garrido, J. (1999), Two-Sided Bounds for Ruin Probabilities When the Adjustment Coefficient Does Not Exist. *Scandinavian Actuarial Journal*, **1999**(1): 80–92.

Cardoso, R. M. & Egidio Dos Reis, A. D. (2002), Recursive Calculation of Time to Ruin Distributions. *Insurance: Mathematics and Economics*, **30**: 219–230.

Cardoso, R. M. R. & Waters, H. R. (2003), Recursive Calculation of Finite Time Ruin Probabilities under Interest Force. *Insurance: Mathematics and Economics*, **33**: 659–676.

Centeno, L. (1985), Measuring the Effects of Reinsurance by the Adjustment Coefficient. *Insurance: Mathematics and Economics*, **5**(2): 169–182.

Centeno, M. d. L. (2002a), Excess of Loss Reinsurance and Gerber's Inequality in the Sparre Anderson Model. *Insurance: Mathematics and Economics*, **31**(3): 415–427, available at http://ideas.repec.org/a/eee/insuma/v31y2002i3p415-427.html.

Centeno, M. d. L. (2002b), Measuring the Effects of Reinsurance by the Adjustment Coefficient in the Sparre Anderson Model. *Insurance: Mathematics and Economics*, **30**(1): 37–49, available at http://ideas.repec.org/a/eee/insuma/v30y2002i1p37-49.html.

Centino, M. d. L. (1997), Excess of Loss Reinsurance and the Probability of Ruin in Finite Horizon. *Astin Bulletin*, **27**(1): 59–70.

Chen, H. & Yao, D. D. (2001), *Fundamentals of Queueing Networks: Performance, Asymptotics and Optimization.* Springer.

Chiu, S. N. & Yin, C. C. (2003), The Time of Ruin, the Surplus Prior to Ruin and the Deficit at Ruin for The Classical Risk Process Perturbed by Diffusion. *Insurance: Mathematics and Economics*, **33**: 59–66.

Cox, D. R. & Isham, V. (1980), *Point Processes.* Chapman Hall, London.

Cramér, H. (1930), On the Mathematical Theory of Risk. Technical report, Skandia Jubilee Volume, Stokholm.

Cramér, H. (1955), Collective Risk Theory. Technical report, Skandia Jubilee Volume, Stockholm.

Cummins, J. D. (2003), Optimal Insurance with Divergent Beliefs About Insurer Total Default Risk. *The Journal of Risk and Uncertainty*, **27**(2): 121–138.

Dassios, A. & Embrechts, P. (1989), Martingales and Insurance Risk. *Stochastic Models*, **5**: 181–217.

Dayanada, P. (1970), Optimal Reinsurance. *Journal of Applied Probability*, **7**: 134–156.

Delbaen, F. & Haezendonck, J. (1987), Classical Risk Theory in an Economic Environment. *Insurance: Mathematics and Economics*, **6**(2): 85–116.

Dickson, D. (1994), An Upper Bound for the Probability of Ultimate Ruin. *Scandinavian Actuarial Journal*: 131–138.

Dickson, D. C. M. & Hipp, C. (2001), On the Time to Ruin for Erlang(2) Risk Processes. *Insurance: Mathematics and Economics*, **29**: 333–344.

Dickson, D. C. M. & Waters, H. R. (1999), Ruin Probabilities with Compounding Assets. *Insurance: Mathematics and Economics*, **25**: 49–62.

Dickson, D. C. M. & Waters, R. H. (1991), Recursive Calculation of Survival Probabilities. *Astin Bulletin*, **21**(2): 199–221.

Duffy, K. & Metcalfe, A. P. (2005), The Large Deviations of Estimating Rate-Funtions. *Journal of Applied Probability*, **42**(1): 267–274.

Dufresne, F. & Gerber, H. (1991), Risk Theory for the Compound Poisson Process That is Perturbed by Diffusion. *Insurance: Mathematics and Economics*, **10**: 51–59.

Dynkin, E. B. & Yushkevich, A. A. (1979), *Controlled Markov Processes.* Springer-Verlag.

Embrechts, P. (1983), A Property of Teh Generalised Inverse Gaussian Distribution with some Applications. *Journal of Applied Probability*, **20**: 537–544.

Embrechts, P. & Klüppelberg, C. (1993), Some Aspects of Insurance Mathematics. *Theory of Probability and its Applications*, **38**: 262–295.

Embrechts, P., Klüppelberg, C. & Mikosch, T. (1997), *Modelling Extremal Events for Insurance and Finance.* Springer, Berlin.

Embrechts, P., Klüppelberg, C. & Mikosch, T. (2001), *Modelling Extremal Events for Insurance and Finance.* Springer, Berlin, 3rd edition.

Embrechts, P. & Schmidli, H. (1994), Ruin Estimation for a General Insurance Risk Model. *Advances in Applied Probability*, **26**: 404–422.

Embrechts, P. & Veraverbeke, N. (1982), Estimates for the Probability of Ruin with Special Emphasis on the Possibility of Large Claims. *Insurance: Mathematics and Economics*, **1**: 55–72.

Feller, W. (1968), *An Introduction to Probability Theory and its Applications*, volume 1. John Wiley and Sons, Inc.

Feller, W. (1971), *An Introduction to Probability Theory and its Applications*, volume 2. John Wiley and Sons, Inc.

Fisz, M. (1963), *Probability Theory and Mathematical Statistics.* John Wiley and Sons. Inc.

Frovlova, A., Kabanov, Y. & Pergamenshchikov, S. (2002), In the Insurance Business Risky Investments are Dangerous. *Finance and Stochastics*, **6**: 227–235.

Furrer, H. & Schmidli, H. (1994), Exponential Inequalities for Ruin Probabilities of Risk Processes Perturbed by Diffusion. *Insurance: Mathematics and Economics*, **15**: 23–36.

Gaier, J. & Grandits, P. (2002), Ruin Probabilities in the Presence of Regularly Varying Tails and Optimal Investment. *Insurance: Mathematics and Economics*, **30**(2): 211–217, available at http://ideas.repec.org/a/eee/insuma/v30y2002i2p211-217.html.

Gaier, J., Grandits, P. & Schachermayer, W. (2003), Asymptotic Ruin Probabilities and Optimal Investment. *Annals of Applied Probability*, **13**: 1054–1076.

Gençay, R., Selçuk, F. & Ulugülyagci (2003), High Volatility, Thick Tails and Extreme Value Theory in Value-at-Risk Estimation. *Insurance: Mathematics and Economics*, **33**: 337–356.

Gerber, H. (1969), Entscheidigungskriterien Für Den Zusammengesetzen Poisson Prozess. *Mittelungen der Vereinigung Schweizer Versicherungsmathematiker*, **19**: 185–228.

Gerber, H. (1970), An Extension to the Renewal Equation and its Application in the Collective Theory of Risk. *Skandinavisk Aktuarietidskrift*: 205–210.

Gerber, H. (1973), Martingales in Risk Theory. *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker*, **73**: 205–210.

Gerber, H. (1979), *An Introduction to Mathematical Risk Theory.* S.S. Heubner Foundation Monograph Series 8, University of Pennsylvania, Philadelphia.

Gerber, H. (1982), Ruin Theory in the Linear Model. *Insurance: Mathematics and Economics*, **1**: 177–184.

Gerber, H. & Shiu, F. (1998), On the Time Value of Ruin. *North Americal Actuarial Journal*, **2**: 48–78.

Gerber, H. U. (1977), On Optimal Cancellation of Policies. *Astin Bulletin*, **9**(2): 125–138.

Goovaerts, M., Kass, R., Van Heerwaarden, A. & Bauwelinckx, T. (1990), *Effective Actuarial Methods.* North Holland, Amsterdam.

Gradshteyn, I. S. & Ryzhik, I. M. (2000), *Tables of Integrals, Series, and Products, 6th Ed.* Academic Press.

Grandell, J. (1991), *Aspects of Risk Theory.* Springer-Verlag, New York.

Hellwig, M. (2000), Financial Intermediation with Risk Aversion. *Review of Economic Studies*, **67**: 719–742.

Hipp, C. (2003), Stochastic Control with Application in Insurance. Technical report, University of Karlsruhe.

Hipp, C. & Plum, M. (2000), Optimal Investment for Insurers. *Insurance: Mathematics and Economics*, **27**(2): 215–228.

Hipp, C. & Plum, M. (2003), Optimal Investment for Investors with State Dependent Income, and for Insurers. *Finance and Stochastics*, **7**: 299–321.

Hipp, C. & Taksar, M. (2000), Stochastic Control for Optimal New Business. *Insurance Mathematics and Economics*, **26**: 185–192.

Hipp, C. & Vogt, M. (2003), Optimal Dynamic XL Reinsurance. *ASTIN Bulletin*, **33**: 193–208.

Hogg, R. V. & Klugman, S. A. (1984), *Loss Distributions.* Wiley.

Højgaard, B. (2002), Optimal Dynamic Premium Control in Non-Life Insurance. Maximising Dividend Pay-Outs. *Scandinavian Actuarial Journal*, **2002**(4): 225–245.

Højgaard, B. & Taksar, M. (1997), Optimal Proportional Reinsurance Policies for Diffusion Models. *Scandinavian Actuarial Journal*, **81**(2): 166–180.

Højgaard, B. & Taksar, M. (1998), Optimal Propportional Reinsurance Policies for Diffusion Models with Transaction Costs. *Insurance: Mathematics and Economics*, **22**: 41–51.

Højgaard, B. & Taksar, M. (1999), Controlling Risk Exposure and Dividends Paypout Schemes: Insurance Company Example. *Mathematical Finance*, **9**(2): 153–182.

Højgaard, B. & Taksar, M. (2001), Optimal Risk Control for a Large Corporation in the Presence of Returns on Investments. *Finance and Stochastics*, **5**(4): 527–547.

Irgens, C. & Paulsen, J. (2004), Optimal Control of Risk Exposure, Reinsurance and Investments for Insurance Portfolios. *Insurance: Mathematics and Economics*, **35**: 21–51.

Jeanblanc-Picqué, M. & Shiryaev, A. (1995), Optimization of the Flow of Dividends. *Russian Matematical Surveys*, **50**: 257–277.

Kalashnikov, V. (1996), Two-Sided Bounds for Ruin Probabilities. *Scandinavian Actuarial Journal*, **1996**(1): 1–18.

Kalashnikov, V. (1997), *Geometric Sums: Bounds for Rare Events with Applications.* Kluwer Publishers, Dordrecht.

Kalashnikov, V. & Konstantinides, D. (2000), Ruin under Interest Force and Subexponential Claims: A Simple Treatment. *Insurance: Mathematics and Economics*, **27**: 145–149.

Kalashnikov, V. & Norberg, R. (2002), Power Tailed Ruin Probabilities in the Presence of Small Claims and Risky Investments. *Stochastic Processes and their Applications*, **98**: 211–228.

Kaluszka, M. (2001), Opimal Reinsurance under Mean-Variance Premuim Principles. *Insurance: Mathematics and Economics*, **28**: 61–67.

Karatzas, I. (1989), Optimisation Problems in the Theory of Continuous Trading. *SIAM Journal on Control and Optimisation*, **7**: 1221–1259.

Kearns, P. & Pagan, A. (1997), Estimating the Density Tail Index for Financial Time Series. *The Review of Economics and Statistics*, **79**(2): 171–175.

Kelbert, M., Manolopoulou, I., Sazonov, I. & Suhov, Y. M. (2007), Large Deviations for a Model of Excess of Loss Reinsurance. *Special Issue of Markov Processes and Related Fields*, **13**(1-2): 137–158.

Kelbert, M., Sazonov, I. & Shore, A. P. (2005), Simulating the Continuous Time Problem of Optimal Choice of New Business. *Quality Technology and Quantitative Management*, **2**(2): 189–200.

Klüppelberg, C. & Mikosch, T. (1997), Large Deviations of Heavy-Tailed Random Sums with Applications in Insurance and Finance. *Journal of Applied Probability*, **34**: 293–308.

Konstantinides, D., Tang, Q. & Tsitsiashvili, G. (2002), Estimates for the Ruin Probability in the Classical Risk Model with Constant Interest Force in the Presence of Heavy Tails. *Insurance: Mathematics and Economics*, **31**: 477–460.

Kushner, H. (1971), *Introductian to Stochastic Control.* Holt, Rinehart and Winston, Inc.

Lin, S. X., Willmot, G. E. & Drekic, S. (2003), The Classical Risk Model with a Constant Dividend Barrier: Analysis of the Gerber-Shiu Discounted Penalty Function. *Insurance: Mathematics and Economics*, **33**: 551–566.

Lundberg, F. (1926), *Försäkringsteknisk Riskutjämning.* F. Englunds, A.B. Bobtryckeri, Stockholm.

Lundberg, F. (1932), Some Supplementary Researches on the Collective Risk Theory. *Skandinavisk Aktuarietidskrift*, **15**: 137–158.

Markussen, C. & Taksar, M. I. (2003), Optimal Dynamic Reinsurance Policies for Large Insurance Portfolios. *Finance and Stochastics*, **7**(1): 97–121, available at http://ideas.repec.org/a/spr/finsto/v7y2003i1p97-121.html.

Martin-Löf, A. (1973), A Method for Finding the Optimal Decision Rule for a Policy Holder of an Insurance with a Bonus System. *Scandinavian Actuarial Journal*: 23–39.

Martin-Löf, A. (1983), Premium Control in an Insurance System, an Approach Using Linear Control Theory. *Scandinavian Actuarial Journal*: 1–27.

Martin-Löf, A. (1994), Lectures on the Use of Control Theory in Insurance. *Scandinavian Actuarial Journal*, **1**: 1–25.

Merton, R. (1971), Optimum Consumption and Portfolio Rules in a Continuous-Time Model. *Journal of Economic Theory*, **3**: 373–413.

Merton, R. (1990), *Continuous-Time Finance.* Blackwell, Massachusetts.

Mikosch, T. & Nagev, A. V. (1998), Large Deviations of Heavy-Tailed Sums with Applications in Insurance. *Extremes*, **1**(1): 81–110.

Müller, A. & Pflug, G. (2001), Asymptotic Ruin Probabilities for Risk Processes with Dependent Increments. *Insurance: Mathematics and Economics*, **28**: 381–392.

Norberg, R. (2002), Anomalous PDEs in Markov Chains: Domains of Validity and Numerical Solutions. *Draft version.*

Patel, J. K., Kapadia, C. H. & Owen, D. B. (1976), *Handbook of Statistical Distributions.* Marcel Dekker, Inc., New York.

Paulsen, J. (1998), Ruin Theory with Compounding Assets - a Survey. *Insurance: Mathematics and Economics*, **22**(1): 3–16.

Paulsen, J. (2003), Optimal Dividend Payouts for Diffusions with Solvency Constraints. *Finance and Stochastics*, **7**(4): 457–474.

Paulsen, J. & Gjessing, H. K. (1997), Optimal Choice of Dividend Barriers for a Risk Process with Stochastic Return on Investments. *Insurance: Mathematics and Economics*, **20**: 215–223.

Perry, D., Stadje, W. & Zacks, S. (2002), Hitting and Ruin Probabilities for Compound Poisson Processes and the Cycle Maximum of the M/G/1 Queue. *Stochastic Models*, **18**(4): 553–564.

Pham, H. (2002), Minimising Shortfall Risk and Applications to Finance and Insurance Problems. *The Annals of Applied Probability*, **12**(1): 143–172.

Picard, P. & Lefevre, C. (2001), On the Probability of (Non-) Ruin in Infinite Time. *Scandinavian Actuarial Journal*, **2001**(2): 148–161.

Pitts, S. M., Grübel, R. & Embrechts, P. (1996), Confidence Bounds for the Adjustment Coefficient. *Advances in Applied Probability*, **28**: 802–827.

Promislow, S. (1991), The Probability of Ruin in a Process with Dependent Increments. *Insurance: Mathematics and Economics*, **10**: 99–107.

Ramsay, C. M. (2003), A Solution to the Ruin Problem for Pareto Distributions. *Insurance: Mathematics and Economics*, **33**: 109–116.

Rolski, T., Schmidli, H., Schmidt, V. & Teugels, J. (1999), *Stochastic Processes for Insurance and Finance.* John Wiley and Sons, Chichester.

Ross, S. (1996), *Stochastic Processes.* Wiley, New York, 2nd edition edition.

Schlegel, S. (1998), Ruin Probabilities in Perturbed Risk Models. *Insurance: Mathematics and Economics*, **22**: 93–104.

Schmidli, H. (1995), Cramér-Lundberg Approximations for Ruin Probabilities of Risk Processes Perturbed by Diffusion. *Insurance: Mathematics and Economics*, **16**: 135–149.

Schmidli, H. (1996), Lundberg Inequalities for a Cox Models with a Piecewise Constant Intensity. *Journal of Applied Probability*, **33**: 196–210.

Schmidli, H. (1997), An Extension to the Renewal Theorem and an Application to Risk Theory. *Annals of Applied Probability*, **7**: 121–133.

Schmidli, H. (2001), Optimal Proportional Reinsurance Policies in a Dynamic Setting. *Scandinavian Actuarial Journal*, **2001**(1).

Schmidli, H. (2002), On Minimising the Ruin Probability by Investment and Reinsurance. *The Annals of Applied Probability*, **12**(3): 890–907.

Schwartz, A. & Weiss, A. (1995), *Large Deviations for Performance Analysis.* Chapman and Hall.

Shreve, S., Lehoczky, J. & Gaver, D. (1984), Optimal Consumption for General Diffusions with Absorbing and Reflecting Barriers. *SIAM Journal on Control and Optimisation.*, **22**: 55–75.

Straub, E. (1997), *Non-Life Insurance Mathematics.* Springer-Verlag.

Sundt, B. & Teugels, J. (1995), Ruin Estimates under Interest Force. *Insurance: Mathematics and Economics*, **16**: 7–22.

Taksar, M. (2000), Optimal Risk and Dividend Distribution Control Models for an Insurance Company. *Mathematical Methods of Operations Research*, **51**(1): 1–41.

Tsai, C. C. (2003), On the Expectations of the Present Values of the Time of Ruin Perturbed by Diffusion. *Insurance: Mathematics and Economics*, **32**: 413–429.

Valdez, A. & Mo, K. (2002), *Ruin Probabilities with Dependent Claims.* Master's thesis, School of Actuarial Studies, Faculty of Commerce and Economics, The University of New South Wales.

Verlaak, R. & Beirlant, J. (2003), Optimal Reinsurance Programs. An Optimal Combination of Several Reinsurance Protections on a Heterogeneous Insurance Portfolio. *Insurance: Mathematics and Economics*, **33**: 381–403.

Vvedenskaya, N. D., Pecherskii, E. A. & Suhov, Y. (2000), Large Deviations in some Queueing Systems. *Problems of Information Transmission*, **36**(1): 48–59.

de Vylder, F. & Goovaerts, M. J. (1984), A Stable Recursive Algorithm for Evaluation of Ultimate Ruin Probabilities. *Astin Bulletin*, **14**(1): 53–59.

de Vylder, F. & Goovaerts, M. J. (1988), Recursive Calculation of Finite-Time Ruin Probabilities. *Insurance: Mathematics and Ecenomics*, **7**: 1–7.

Wang, S. S., Young, V. R. & Panjer, H. H. (1997), Axiomatic Characterization of Insurance Policies. *Insurance: Mathematics and Economics*, **21**: 173–183.

Waters, H. (1983), Some Mathematical Aspects of Reinsurance. *Insurance: Mathematics and Economics*, **2**: 17–26.

Willmot, G. (1994), Refinements and Distributional Generalisiations of Lundberg's Inequality. *Insurance: Mathematics and Economics*, **15**: 49–63.

Willmot, G. (1996), A Non-Exponential Generalisation of an Inequality Arising in Queuing and Insurance Risk. *Journal of Applied Probability*, **33**: 176–183.

Willmot, G., Cai, J. & Lin, X. (2001), Lundberg Inequalities for Renewal Equations. *Advances of Applied Probability*, **33**: 674–689.

Willmot, G. & Lin, X. (1994), Lundberg Bounds on the Tails of Compound Distributions. *Journal of Applied Probability*, **31**: 743–756.

Willmot, G. & Lin, X. (2001), *Lundberg Approximations for Compound Distributions with Insurance Applications.* Springer-Verlag, New York.

Yang, H. (1999), Non-Exponential Bounds for Ruin Probability with Interest Effect Included. *Scandinavian Actuarial Journal*: 66–79.

Young, V. R. (1999), Optimal Insurance under Wang's Premium Principle. *Insurance: Mathematics and Economics*, **25**: 109–122.

# Appendix A : Pure excess of loss reisurance

In this Appendix we show the derivation of the MGF of an insurer who is looking to purchase a pure excess of loss reinsurance policy, i.e. $h \neq 0$, $\alpha = 0$.

The insurer's premium income (before reinsurance) per unit time is

$$(1+\rho)\,\lambda m_1$$

The premium received by the reinsurer is

$$(1+\xi)\,\lambda \mathbf{E} Z$$

where $\xi\,(\geq \rho)$ is the reinsurer's premium loading factor and $Z = \max(0, X-h)$. Note here, that we introduce the possibility of the reinsurers profit loading factor differing to that of the insurers.

During a normal period, the insurer's individual claim payments are distributed as $Y = \min(X, h)$, and the premium received by the insurer, net of reinsurance, is

$$c_1 = (1+\rho)\,\lambda m_1 - (1+\xi)\,\lambda \mathbf{E} Z$$

When $\rho < \xi$, there is a minimum retention level for the same reason as in the previous section. Here, the constraint on the minimum retention level is

$$c_1 > \lambda \mathbf{E} X - \lambda \mathbf{E} Z.$$

As a result of the excess of loss reinsurance, the distribution of the insurer's individual claims, net of reinsurance, is truncated, and thus has the following MGF

$$\begin{aligned} M_x(R) &= \int_0^h e^{Rx}\mu e^{-\mu x}dx + e^{Rh}\Pr[x>h] \\ &= \frac{\mu\left(e^{M(R-\mu)}-1\right)}{R-\mu} + e^{M(R-\mu)} \end{aligned}$$

Thus the LD slope of the insurer is given by

$$D_1 = \lambda\frac{dM_x(R)}{dR} = \frac{\lambda\mu\,(R-\mu)\left(he^{M(R-\mu)}\right) - \mu\left(e^{M(R-\mu)}-1\right)}{(R-\mu)^2} + \lambda he^{M(R-\mu)}$$

Where $R$ is the unique positive solution to the equation

$$0 = \lambda\left(\mu\left(e^{M(R-\mu)}-1\right) + (R-\mu)\,e^{M(R-\mu)}\right) - (R-\mu)\,(\lambda + c_1R)$$

which in this case has to be found using numerical methods.

Now, the MGF of the reinsurer is

$$\begin{aligned} M_X(R) &= \int_h^\infty e^{R(x-h)} f(x)\,dx \\ &= e^{-Rh}\int_h^\infty \mu e^{x(R-\mu)}dx \\ &= \frac{\mu e^{-h\mu}}{\mu - R} \end{aligned}$$

Thus the LD slope of the reinsurer is

$$D_2 = \lambda\frac{dM_x(R)}{dR} = \frac{\lambda\mu e^{-h\mu}}{(\mu-R)^2}$$

Where $R$ is given by

$$\frac{\lambda\mu e^{-h\mu}}{\mu - R} = \lambda + (1-c_1)\,R$$

We work out $t_n$, $t_c$

$$t_n = \frac{K-P}{\lambda M_Y'(r_1^1) - c_1}$$

similarly

$$t_c = \frac{P}{\lambda M_Y'(r_1^2) - 1}$$

Hence we can calculate $G$ which in this case is

$$G = (1-c_1)\,t_n - (D_{2n}t_n + D_{2c}t_c)$$

# Appendix B : Programming code used for optimal new business

First, we show the Euler scheme used in chapter 4 to benchmark the results with Hipp & Taksar (2000).

```
lambda1=1          (*Poisson parameter for 1st business*);
a=1          (*parameter of 1st claims distribution (exponential)*);
c1=2          (*premium income for first business*);
s=0.0          (*initial starting capital*);
h = 0.001          (*stepsize*);
f[0]=1          (*initialisation value of delta);
g1[0]=0          (*initialisation value for expectation w.r.t X*);
g1prime[i_]:=a(f[i]-g1[i])          (*using the differential equation*);
g1[i_]:=g1[i-1]+h g1Prime[i-1]          (*expectation w.r.t X*);
q1[i_]:=λ1(f[i]-g1[i])/c1          (*from the HJB equation*);
fPrime[i_]:=q1[i];
f[i_]:=f[i-1+h fPrime[i-1];
w[i_]:=1-λ1/(c1 a) Exp[-(a-λ1/c1)(i h)];          (*the updating equation*);
x={f[0]}
y={W[0]}          (*to define f[0] as a 1x1 matrix*);
i=1          (*the first itteration*);
```

```
Do[g1[i]=g1[i];
g1Prime[i]=g1Prime[i];
q1[i]=q1[i];
fPrime[i]=fPrime[i];
f[i]=f[i];
x=Append[x,f[i]];
y=Append[y,W[i]];
(*running the itteration appending each step onto the previous matrix*)
i=i+1;,{30000}];
x=x/f[i] ;
(* with sufficiently high surplus, we can divide the entire vector x by the
highest value to normalise so that δ(∞)=1 *)
```

Here, we present a shortened version of the C++ and MATLAB code used for the simulations. We omit the underlying MATLAB functions:

**MAIN MATLAB FUNCTION:**

```
%function gojoint()
pfile = fopen('goini.dat','r');
% default values
N = 1e5; umax = 10; du = 1;
x = 1; c = 2; lambda = 1; pdf = 0; alpha = 1;
x2 = 1; c2 = 2; lambda2 = 1; pdf2 = 0; alpha2 = 1;
th = 0.5; th2 = 2.5; dur = 2; loss = 0.1;
while 1
txt = fgetl(pfile);
if ~ischar(txt), break, end
k = findstr(txt,'N:'); if length(k), N = str2num(txt(k+2:end)); end
k = findstr(txt,'umax:'); if length(k), umax = str2num(txt(k+5:end)); end
k = findstr(txt,'du:'); if length(k), du = str2num(txt(k+3:end)); end
k = findstr(txt,'c:'); if length(k), c = str2num(txt(k+2:end)); end
k = findstr(txt,'lambda:'); if length(k), lambda = str2num(txt(k+7:end)); end
k = findstr(txt,'pdf:'); if length(k), pdf = str2num(txt(k+4:end)); end
k = findstr(txt,'x:'); if length(k), x = str2num(txt(k+2:end)); end
if pdf>0
k = findstr(txt,'alpha:'); if length(k), alpha = str2num(txt(k+6:end)); end
end
k = findstr(txt,'c2:'); if length(k), c2 = str2num(txt(k+3:end)); end
k = findstr(txt,'lambda2:'); if length(k), lambda2 = str2num(txt(k+8:end)); end
k = findstr(txt,'pdf2:'); if length(k), pdf2 = str2num(txt(k+5:end)); end
if pdf2>0
k = findstr(txt,'alpha2:'); if length(k), alpha2 = str2num(txt(k+7:end)); end
end
k = findstr(txt,'x2:'); if length(k), x2 = str2num(txt(k+3:end)); end
k = findstr(txt,'th:'); if length(k), th = str2num(txt(k+3:end)); end
k = findstr(txt,'th2:'); if length(k), th2 = str2num(txt(k+4:end)); end
k = findstr(txt,'loss:'); if length(k), loss = str2num(txt(k+5:end)); end
k = findstr(txt,'dur:'); if length(k), dur = str2num(txt(k+4:end)); end
end
fclose(pfile);
y = input(sprintf('input new N [%g] : ',N),'s'); if length(y), N = str2num(y); end
y = input(sprintf('input new umax [%g] : ',umax),'s'); if length(y), umax = str2num(y);
end
```

```
y = input(sprintf('input new du [%g] : ',du),'s'); if length(y), du = str2num(y); end
y = input(sprintf('input new c [%g] : ',c),'s'); if length(y), c = str2num(y); end
y = input(sprintf('input new lambda [%g] : ',lambda),'s'); if length(y), lambda = str2num(y); end
y = input(sprintf('input new pdf [%g] : ',pdf),'s'); if length(y), pdf = str2num(y); end
y = input(sprintf('input new x [%g] : ',x),'s'); if length(y), x = str2num(y); end
if pdf>0
y = input(sprintf('input new alpha [%g] : ',alpha),'s'); if length(y), alpha = str2num(y); end
end
y = input(sprintf('input new c2 [%g] : ',c2),'s'); if length(y), c2 = str2num(y); end
y = input(sprintf('input new lambda2 [%g] : ',lambda2),'s'); if length(y), lambda2 = str2num(y); end
y = input(sprintf('input new pdf2 [%g] : ',pdf2),'s'); if length(y), pdf2 = str2num(y); end
y = input(sprintf('input new x2 [%g] : ',x2),'s'); if length(y), x2 = str2num(y); end
if pdf2>0
y = input(sprintf('input new alpha2 [%g] : ',alpha2),'s'); if length(y), alpha2 = str2num(y); end
end
y = input(sprintf('input new th [%g] : ',th),'s'); if length(y), th = str2num(y); end
y = input(sprintf('input new th2 [%g] : ',th2),'s'); if length(y), th2 = str2num(y); end
y = input(sprintf('input new loss [%g] : ',loss),'s'); if length(y), loss = str2num(y); end
%y = input(sprintf('input new dur [%g] : ',dur),'s'); if length(y), dur = str2num(y); end
pfile = fopen('goini.dat','w');
fprintf(pfile,'Number of avaregings, N: %.0f\n',N);
fprintf(pfile,'Maximal initial capital, umax: %g\n',umax);
fprintf(pfile,'Step of initial capital, du: %g\n',du);
fprintf(pfile,'Premium income rate, c: %g\n',c);
fprintf(pfile,'Mean expected claim, x: %g\n',x);
fprintf(pfile,'Claim distribution: pdf: %i\n',pdf);
fprintf(pfile,'Clain parameter, alpha: %g\n',alpha);
fprintf(pfile,'Poisson process parameter, lambda: %g\n',lambda);
fprintf(pfile,'Premium income rate, c2: %g\n',c2);
fprintf(pfile,'Mean expected claim, x2: %g\n',x2);
fprintf(pfile,'Claim distribution: pdf2: %i\n',pdf2);
fprintf(pfile,'Claim parameter: alpha2: %g\n',alpha2);
fprintf(pfile,'Poisson process parameter, lambda2: %g\n',lambda2);
fprintf(pfile,'Threshold open, th: %g\n',th);
fprintf(pfile,'Threshold close, th2: %g\n',th2);
```

```
fprintf(pfile,'Duration of 2nd business, dur: %g\n',dur);
fprintf(pfile,'Losses (absolute), loss: %g\n',loss);
fclose(pfile);
disp('Please wait')
!del nonruin.dat
tic
!go.exe
toc
while 1
if exist('nonruin.dat')==2, break; end
pause(1)
end
u = du:du:umax;
P = load('nonruin.dat','ascii');
figure(2);
%plot(u,P(:,2),'-g');return
plot(u,P(:,1),'.-b',u,P(:,2),'.-r')
legend('Single','Joint')
axis([0 umax 0 1])
grid on
xlabel('initial capital, u')
ylabel('nonrouin probability, u')
switch pdf
case 0, pd = 'Exp';
case 1, pd = 'Gamma';
case 2, pd = 'Lognorm';
case 3, pd = 'Weibull';
case 4, pd = 'Pareto';
otherwise, pd = 'Unkown';
end
switch pdf2
case 0, pd2 = 'Exp';
case 1, pd2 = 'Gamma';
case 2, pd2 = 'Lognorm';
case 3, pd2 = 'Weibull';
case 4, pd2 = 'Pareto';
otherwise, pd2 = 'Unkown';
end
txt1 = ['N=',num2str(N),'; c_1=',num2str(c),' ',pd,': \lambda_1=',num2str(lambda),'
x_1=',num2str(x)]
```

```
if pdf>0, txt1 = [txt1,' \alpha=',num2str(alpha)]; end
txt1 = [txt1,' \theta_1=',num2str(th)];
txt2 = [' c_2=',num2str(c2),' ',pd2,': \lambda_2=',num2str(lambda2),' x_2=',num2str(x2)]
if pdf2>0, txt2 = [txt2,' \alpha_2=',num2str(alpha2)]; end
txt2 = [txt2,' \theta_2=',num2str(th2)];
txt = cell(2,1);
txt{1} = txt1; txt{2} = txt2;
title(txt)
type goini.dat
```

**GO.cpp C++ code refered to in MATLAB function above:**

```
#include <stdio.h>
#include <iostream.h>
#include <math.h>
#include <stdlib.h>
#include <string.h>
#include <time.h>
#define MIN(A,B) ( ((A)<(B)) ? (A) : (B) )
/*————————————————————————————————*/
/* generator of random numbers with uniform distribution in the interval [0,1] */
/*————————————————————————————————*/
const double RAND_MAX_DOUBLE = double(RAND_MAX);
const double pi = 4.0*atan(1.0), e = exp(1.0);
double seed() {return double(rand()) / RAND_MAX_DOUBLE;}
static double seed_pos();
double Poisson(double lambda){
    // next instant of Poisson process with the density lambda
    return -log(seed_pos()) / lambda;
}
double ExpDist(double a, double b = 0.0) {
    // next point of a process with exponential distribution:
    //          p(x) dx = exp(-x/a) dx/a, x >= 0
    // mean = a: [ a = x; b does not matter]
    return -log(seed_pos()) * a;
}
double GammaDist(double a, double b = 1.0); // next point is a random with the
Gamma distribution
    //The Gamma distribution of order a>0 is defined by:
    //          p(x) dx = {1 / \Gamma(a) b^a } x^{a-1} e^{-x/b} dx for x>0.
    //          mean = a*b: [ a = x; b = alpha ];
```

```
double LognormDist(double zeta, double sigma); // next point is a random with the Gamma distribution
    //    p(x) dx = 1/(x * sqrt(2 pi sigma^2)) exp(-(ln(x) - zeta)^2/2 sigma^2) dx
    // mean = exp(zeta + sigma^2/2):    [zeta = log(x) - alpha^2/2; sigma = alpha];
double WeibullDist(double a, double b = 1.0) {
    // next point is a random with the Weibull distribution
    // The Weibull distribution has the form:
    // p(x) dx = (b/a) (x/a)^(b-1) exp(-(x/a)^b) dx
    // mean = a * Gamma(1/b + 1):    [ a = x/Gamma(1+1/alpha); b = alpha ];
return a * pow (-log(seed_pos()), 1.0 / b);
}
double ParetoDist(double a, double b) {
    // next point is a random with the Pareto distribution
    // The Pareto distribution has the form,
    // p(x) dx = (b/a) / (x/a)^(b+1) dx for x >= a
    // mean = Inf
    return a * pow (seed_pos(), -1.0/b);
}
double Gamma(const double);
int time_ruined(double* t, double u, double c, double lambda,
                double (*pfun)(double, double), double x, double alpha,
                double tmax, double tini);
int time_ruined_joint(double* t, double u, double c, double lambda,
                      double (*pfun)(double, double), double x, double alpha,
                      double tmax, double tini,
                      double c2, double lambda2,
                      double (*pfun2)(double, double), double x2, double alpha2,
                      double th, double th2, double loss, double dur);
void main() {
    //----------------//
    // 1.0: Default parameters: //
    //----------------//
    double umax = 3.0, du = 0.1, tmax = 10; // common parameters
    double dN = 1e5;
    long N = long(dN), count;                        // number of averging
    double c = 2.0, x = 1.0, lambda = 1.0;           // for the first business
    double c2 = 2.0, x2 = 10., lambda2 = 1.0;        // for the second business
    double th = 0.5, th2 = 2.5, dur = 2.0, loss = 0.5;    // th is threshold, dur is duration, loss is losses aftre closing the second business
```

```
int pdf = 0, pdf2 = 0; double alpha = 1, alpha2 = 1;
char *pdf_ch, *pdf2_ch;
double a, a2, b, b2;
//------------------------//
// 2.0: Read the input file:
//------------------------//

FILE *pfile; int i;
if(!(pfile=fopen("goini.dat","r"))) {printf("Cannot open 'goini.dat'\n"); return;}
char txt[100];
int line = 0;
while(fgets(txt,100,pfile)!=NULL) {line++;
    for(i=0;i<100 && txt[i]!='\0';i++) {
        if(!strncmp("N:" ,txt+i, 2)) dN = atof(txt+i+2);
        if(!strncmp("umax:" ,txt+i, 5))     umax = atof(txt+i+5);
        if(!strncmp("du:" ,txt+i, 3))     du = atof(txt+i+3);

        if(!strncmp("x:" ,txt+i, 2))     x = atof(txt+i+2);
        if(!strncmp("pdf:" ,txt+i, 4)) pdf = atoi(txt+i+4);
        if(!strncmp("alpha:" ,txt+i, 6))     alpha = atof(txt+i+6);
        if(!strncmp("c:" ,txt+i, 2))     c = atof(txt+i+2);
        if(!strncmp("lambda:",txt+i, 7)) lambda = atof(txt+i+7);

        if(!strncmp("x2:" ,txt+i, 3)) x2 = atof(txt+i+3);
        if(!strncmp("pdf2:" ,txt+i, 5)) pdf2 = atoi(txt+i+5);
        if(!strncmp("alpha2:" ,txt+i, 7)) alpha2 = atof(txt+i+7);
        if(!strncmp("c2:" ,txt+i, 3)) c2 = atof(txt+i+3);
        if(!strncmp("lambda2:",txt+i, 8)) lambda2 = atof(txt+i+8);

        if(!strncmp("th:" ,txt+i, 3)) th = atof(txt+i+3);
        if(!strncmp("th2:" ,txt+i, 4)) th2 = atof(txt+i+4);
        if(!strncmp("dur:" ,txt+i, 4)) dur = atof(txt+i+4);
        if(!strncmp("loss:" ,txt+i, 5)) loss = atof(txt+i+5);
    }
}
fclose(pfile);
N = long(dN);

double (*pfun)(double,double), (*pfun2)(double,double);
switch (pdf) {
```

```
    case 0: {pdf_ch = "Exponent"; pfun = ExpDist; a = x; b = 0; break;}
    case 1: {pdf_ch = "Gamma"; pfun = GammaDist; a = x/alpha; b = alpha; break;}
    case 2: {pdf_ch = "Weibull"; pfun = WeibullDist; a = x/Gamma(1.0+1.0/alpha); b =
alpha; break;}
    case 3: {pdf_ch = "Lognormal"; pfun = LognormDist; a = log(x) - alpha*alpha*0.5; b
= alpha; break;}
    case 4: {pdf_ch = "Pareto"; pfun = ParetoDist; a = x*(alpha-1)/alpha; b = alpha;
break;}
    default: {pdf_ch = "Unknown distribution"; return;}
    }
    switch (pdf2) {
    case 0: {pdf2_ch = "Exponent"; pfun2 = ExpDist; a2 = x2; b2 = 0; break;}
    case 1: {pdf2_ch = "Gamma"; pfun2 = GammaDist; a2 = x2/alpha2; b2 = alpha2;
break;}
    case 2: {pdf2_ch = "Weibull"; pfun2 = WeibullDist; a2 = x2/Gamma(1.0+1.0/alpha2);
b2 = alpha2; break;}
    case 3: {pdf2_ch = "Lognormal"; pfun2 = LognormDist; a2 = log(x2) - alpha2*alpha2*0.5;
b2 = alpha2; break;}
    case 4: {pdf2_ch = "Pareto"; pfun2 = ParetoDist; a2 = x2*(alpha2-1)/alpha2; b2 =
alpha2; break;}
    default: {pdf2_ch = "Unknown distribution"; return;}
    }
    // Check input
    printf("N=%d\tumax = %g\tdu = %g\n",N,umax,du);
    printf("%s: x = %g\talpha = %g\tc = %g\t lambda = %g\n",pdf_ch,x,alpha,c,lambda);
    printf("%s: x2 = %g\talpha2 = %g\tc2 = %g\t lambda2 = %g\n",pdf2_ch,x2,alpha2,c2,lambda2
    printf("th=%g\tth2=%g\tdur=%g\tloss=%g\n",th,th2,dur,loss);


        double u, truin; // u - current initial captital, tm - time of ruin


        // event[i] - to store the number of non-ruine events for current u
        int M = int(umax/du), j;
        long* event = new long[M];        for(j=0;j<M;j++) event[j] = 0;
        long* event2 = new long[M];        for(j=0;j<M;j++) event2[j] = 0;


        srand( (unsigned)time( NULL ) ); // launch of generator from timer
```

```
        // non-ruin probability for given u //
        for (j=0;j<M;j++) {
            u = du*(j+1);
            for(count=0;count<N;count++) {
                int iq = time_ruined(&truin, u, c, lambda, pfun, a, b, tmax, 0.0);
                if (iq==0) event[j]++;
                iq = time_ruined_joint(&truin, u, c, lambda, pfun, a, b, tmax, 0.0,
                        c2, lambda2, pfun2, a2, b2, th, th2, loss, dur);
                if (iq==0) event2[j]++;
            }
        }
    /* time_ruined_joint(double* truin, double u, double c1, double lambda1,
                        double (*pfun1)(double, double), double x1, double alpha1,
                        double tmax, double tini,
                        double c2, double lambda2,
                        double (*pfun2)(double, double), double x2, double alpha2,
                        double th, double th2, double loss, double dur) */
        if(!(pfile=fopen("nonruin.dat","w"))) {printf("Cannot open 'poisson.dat'\n"); re-
turn;}

        for(j=0;j<M;j++) {
            printf("u=%g:\t n=%i\tn2=%i\n",du*(j+1),event[j],event2[j]);
            fprintf(pfile,"%g\t%g\n",double(event[j])/double(N),double(event2[j])/double(N));
        }
        fclose(pfile);
    }

    int time_ruined(double* truin, double u, double c, double lambda,
                    double (*pfun)(double, double), double x, double alpha,
                    double tmax, double tini) {
        double R = u; // initial capital
        double t = tini;
        //double (*pfun)(double,double)
        while(t<tmax) {
            // next instant:
            double dt = Poisson(lambda); // later to write separate function
            t += dt;
            if (t>tmax) break;
            R += c*dt;      // capital just before the claim
```

```
            double X = pfun(x,alpha);       //claim in this instant; // later to write separate
function
            R -= X;

            if(R<0) {
                *truin = t;
                return 1;
            }
        };
        *truin = tmax;
        return 0;
    }
    int time_ruined_joint(double* truin, double u, double c1, double lambda1,
                          double (*pfun1)(double, double), double x1, double alpha1,
                          double tmax, double tini,
                          double c2, double lambda2,
                          double (*pfun2)(double, double), double x2, double alpha2,
                          double th, double th2, double loss, double dur) {
        double R = u; // initial capital          //, R2 = 0
        double t = tini, dt, dt1, dt2, X1, X2; // t_open, t_close;
        int flag = 0; // is 1 if the second business is opened

        if (R<th) {
            flag = 1;
            dt1 = 0.0;
            dt2 = 0.0;
        }
        while(t<tmax) {
            // next instant:
            if (flag) {
                // dt1 unterval from previous claim to claim in the buisness #1
                // dt2 unterval from previous claim to claim in the buisness #2
                if (dt1==0.0) {
                    dt1 = Poisson(lambda1);
                    X1 = pfun1(x1, alpha1);
                }
                if (dt2==0.0) {
                    dt2 = Poisson(lambda2);
                    X2 = pfun2(x2, alpha2);
                }
```

```
// temprary swithch off the closing of the second business:
dt = MIN(dt1,dt2); // dt = min(dt1,dt2)
if (t+dt > tmax) break;
if (R+dt*(c1+c2) > th2) { // then switch off the business
    flag = 0;
    dt = (th2 - R) / (c1+c2);      // intersection must be less then min(d1,d2)
    t += dt1;
    dt1 -= dt;

    //R += dt*(c1+c2);      // growth of capital before closing
    //R -= loss;             // losses of selling
    //R += dt1*c1;          //     growth of single business before claim
    //R -= X1;              // claim in the first busness
    R = th2 - loss + dt1*c1 - X1;

    dt2 = 0;
    X2 = 0;
    continue;
}
if (dt1 < dt2) { // then claim1 is earlier
    t += dt1;
    R += (c1+c2*flag)*dt1;
    R -= X1;
    dt2 -= dt1;
    dt1 = 0.0;
}
else if (dt1 > dt2) {// then claim2 is earlier
    t += dt2;
    R += (c1+c2*flag)*dt2;
    R -= X2;
    dt1 -= dt2;
    dt2 = 0.0;
}
else {// both climes at the same instant
    t += dt1;
    R += (c1+c2*flag)*dt1;
    R -= (X1 + X2);
    dt1 = 0.0;
    dt2 = 0.0;
}
```

```
            }
            else { // now consider case when the 2nd busines is not opened

                dt = Poisson(lambda1); // later to write separate function
                t += dt;
                if (t>tmax) break;
                X1 = pfun1(x1,alpha1);      //claim it this instant; // later to write sepa-
rate function
                R += c1*dt;      // capital just before the claim
                R -= X1;
                if(R<th) { // open new business
                    flag = 1;
                    dt1 = 0.0;
                    dt2 = 0.0;
                }
            } //if (flag)
            if(R<0) {
                *truin = t;
                return 1;
            }
        };
        *truin = tmax;
        return 0;
    }
    static double seed_pos() {
        int i;
        do {
            i = rand();
        } while(i==0);
        return double(i) / RAND_MAX_DOUBLE;
    }
    static double gamma_dist_int (const int alpha);
    static double gamma_dist_large (const double alpha);
    static double gamma_dist_frac (const double alpha);
    double GammaDist(const double x, const double alpha) {
        //The Gamma distribution of order a>0 is defined by:
        //          p(x) dx = {1 / \Gamma(a) b^a } x^{a-1} e^{-x/b} dx for x>0.
        // If X and Y are independent gamma-distributed random
        // variables of order alpha1 and a2 with the same scale parameter b, then
        // X+Y has gamma distribution of order a1+a2.
```

```
    // The algorithms below are from Knuth, vol 2, 2nd ed, p. 129.

    int na = int(floor(alpha));
    if (alpha==na)      return x * gamma_dist_int (na);
    else if (na==0)     return x * gamma_dist_frac (alpha);
    else          return x * (gamma_dist_int (na) + gamma_dist_frac (alpha - na)) ;
}
static double gamma_dist_int (const int alpha) {
    double prod = 1.; int i;
    if (alpha < 12) {
        for (i = 0; i < alpha; i++) {
            prod *= seed_pos();
        }
        return -log (prod);
    }
    else {
        return gamma_dist_large ((double) alpha);
}
}
static double gamma_dist_large (const double alpha) {
    //Works only if alpha > 1, and is most efficient if alpha is large
    //     This algorithm, reported in Knuth, is attributed to Ahrens. A
// faster one, we are told, can be found in: J. H. Ahrens and
// U. Dieter, Computing 12 (1974) 223-246.
    double sqa, x, y, v, u;
    sqa = sqrt (2 * alpha - 1);
    do {
        do {
            y = tan (pi * seed());
            x = sqa * y + alpha - 1;
} while (x <= 0);
        v = seed();
        u = (1 + y * y) * exp ((alpha - 1) * log (x / (alpha - 1)) - sqa * y);
}    while (v > u);
    return x;
}
static double gamma_dist_frac (const double alpha) {
/* This is exercise 16 from Knuth; see page 135, and the solution is
on page 551. */
double p, q, x, u, v, w;
```

```
p = e / (alpha + e);
do {
u = seed();
v = seed_pos();
if (u < p) {
x = exp ((1 / alpha) * log (v));
q = exp (-x);
    }
else {
x = 1 - log (v);
q = exp ((alpha - 1) * log (x));
        w = seed();
    }
} while ( w >= q);
return x;
}
//==============================================//
//                                              //
//                Lognormal                      //
//                                              //
//==============================================//
double LognormDist (double sigma, double alpha) {
    //    The lognormal distribution has the form:
    //
//        p(x) dx = 1/(x * sqrt(2 pi sigma^2)) exp(-(ln(x) - alpha)^2/2 sigma^2) dx
    //
//    for x > 0. Lognormal random numbers are the exponentials of Gaussian random
numbers
    // mean is exp (sigma^2/2)
    // if x is mean then sigma = sqrt(log(
double u, v, r2, normal, z; //, sigma = sqrt(2.0*log(alpha));
do {
    // choose x,y in uniform square (-1,-1) to (+1,+1):
u = -1.0 + 2.0 * seed();
v = -1.0 + 2.0 * seed();
// see if it is in the unit circle
r2 = u * u + v * v;
} while (r2 > 1.0 || r2==0.);
normal = u * sqrt (-2.0 * log (r2) / r2);
z = exp (sigma * normal + alpha);
```

```
return z;
}
//==========================================================//
//                                                          //
//                    Weibull                               //
//                                                          //
//==========================================================//
double Gamma(const double xi) {
    //GAMMA Gamma function.
    // Y = GAMMA(X) evaluates the gamma function for each element of X.
    // X must be real. The gamma function is defined as:
    //
    // gamma(x) = integral from 0 to inf of t^(x-1) exp(-t) dt.
    //
    // The gamma function interpolates the factorial function. For
    // integer n, gamma(n+1) = n! (n factorial) = prod(1:n).
    //
    // See also GAMMALN, GAMMAINC.
    // C. B. Moler, 5-7-91, 11-4-92.
    // Ref: Abramowitz & Stegun, Handbook of Mathematical Functions, sec. 6.1.
    // Copyright 1984-2001 The MathWorks, Inc.
    // $Revision: 5.15 $ $Date: 2001/04/15 12:01:40 $
    // This is based on a FORTRAN program by W. J. Cody,
    // Argonne National Laboratory, NETLIB/SPECFUN, October 12, 1989.
    //
    // References: "An Overview of Software Development for Special
    // Functions", W. J. Cody, Lecture Notes in Mathematics,
    // 506, Numerical Analysis Dundee, 1975, G. A. Watson
    // (ed.), Springer Verlag, Berlin, 1976.
    //
    // Computer Approximations, Hart, Et. Al., Wiley and
    // sons, New York, 1968.
    //
    double P[] = {-1.71618513886549492533811e+0, 2.47656508055759199108314e+1,
    -3.79804256470945635097577e+2, 6.29331155312818442661052e+2,
    8.66966202790413211295064e+2, -3.14512729688483675254357e+4,
    -3.61444134186911729807069e+4, 6.64561438202405440627855e+4};
    double Q[] = {-3.08402300119738975254353e+1, 3.15350626979604161529144e+2,
    -1.01515636749021914166146e+3, -3.10777167157231109440444e+3,
    2.25381184209801510330112e+4, 4.75584627752788110767815e+3,
```

```
-1.34659959864969306392456e+5, -1.15132259675553483497211e+5};
double C[] = {-1.910444077728e-03, 8.4171387781295e-04,
-5.952379913043012e-04, 7.93650793500350248e-04,
-2.777777777777681622553e-03, 8.333333333333333331554247e-02,
5.7083835261e-03};
double x = xi;
double pi = 4.0*atan(1.0), spi = 0.9189385332046727417803297;
double y, res = 0, fact, xnum = 0.0, xden = 1.0, z, sum, x1;
int m, i, kneg = 0, k1 = 0, k = 0;
// Catch negative x.
if (x <= 0) {
    kneg = 1;
    y = -x;
    m = int(y);
    res = y - m;
    fact = -pi / sin(pi*res) * (1 - 2 * (m%2) );
    x = y + 1;
}
// x is now positive.
// Map x in interval [0,1] to [1,2]
if(x < 1) {
    k1 = 1;
    x1 = x;
    x++;
}
// Map x in interval [1,12] to [1,2]
if(x < 12.0){// Evaluate approximation for 1 < x < 2
    k = 1;
    m = int(x) - 1;
    x = x - m;
z = x - 1.0;
for(i=0;i<8;i++) {
    xnum = (xnum + P[i]) * z;
    xden = xden * z + Q[i];
    }
res = xnum / xden + 1;
}
// Adjust result for case 0.0 < x < 1.0
if (k1) {
    res /= x1;
```

```
    }
    // Adjust result for case 2.0 < x < 12.0
    for(i=0;i<m;i++) {
res = res * x;
    x++;
    }

    // Evaluate approximation for x >= 12
    if (x>=12.0) {
y = x * x;
sum = C[6];
for(i=0;i<6;i++) {
sum = sum / y + C[i];
}
sum = sum / x - x + spi;
sum = sum + (x-0.5) *log(x);
res = exp(sum);
    }
    // if x is negative
if (kneg) res = fact / res;
    return res;
}
    /* Block for the histogram:
    double t[10] ={1, 2, 3, 4, 5, 6, 7, 8, 9, 10}; int n[10] = {0.0};
    for(i=0;i<N;i++) {
        int iq = time_ruined(&tm, u, c, x, lambda, tmax, 0.0);
        if (iq==1) {
            j = int(tm);
            if(j>=0 && j<10) n[j]++;
        }
    }
    if(!(pfile=fopen("hist.dat","w"))) {printf("Cannot open 'poisson.dat'\n"); return;}
    for(j=0;j<10;j++) {
        printf("t<%g n=%i\n",t[j],n[j]);
        fprintf(pfile,"%g\n",double(n[j])/double(N));
    }
    fclose(pfile);
    */
```

# Appendix C : Programming code used for optimal reinsurance

In this appendix, we present the Mathematica code used for the optimisation problem under proportional reinsurance. The MATLAB code is omitted as there are too many different functions making it even more difficult for any reader wishing to understand the programs.

First we show the numerical optimisation routine:

Initially, the parameter values are set out.

```
$MaxExtraPrecision = 300000000;
capA = 0.1;
α = 0.5;
μ = 0.45;
c = 1;
γ = 2;
λ = 1;
K = 200;
P = 100;
p = P / K;
q = λμγ / c ;
```

Then we begin the feedback routine. Note that for $\gamma > 2$, $R_{1n}, R_{1c}$ have to be found using the function Plot[].

```
β[α_] := capA (1 - q) + α q;
R1n[α_] := Solve[λ (1/(1 - R α μ))^γ == λ + β[α] c R, R][[2]][[1]][[2]]
R1c[α_] := Solve[λ (1/(1 - R α μ))^γ == λ + c R, R][[2]][[1]][[2]]
R1s := Solve[λ (1/(1 - R μ))^γ == λ + c R, R][[2]][[1]][[2]]
Imin[α_] := K ((1 - p) R1n[α] + p R1c[α])
ISmin := K ((1 - p) R1s + p R1s)

LDSlopeN[α_] := λ α μ γ (1 - α μ R1n[α])^(-γ-1)
LDSlopeC[α_] := λ α μ γ (1 - α μ R1c[α])^(-γ-1)
truin[α_] := K ((1 - p)/(LDSlopeN[α] - β[α] c) + p/(LDSlopeC[α] - c));

ReinsurerProfit[α_] :=
  K (((1 - β[α]) c - (1 - α)/α LDSlopeN[α]) (1 - p) / (LDSlopeN[α] - β[α] c) +
     (- (1-α)/α LDSlopeC[α])/(LDSlopeC[α] - c) p);

Delta[α_] := c (β[α] - α q);
DeltaS := c (1 - q);

G[α_] := ReinsurerProfit[α] / K;
```

```
α = FindRoot[G[a] == 0, {a, 0.1}][[1]][[2]];

N[G[α]];

A[α_] := Delta[α] / DeltaS ;

B[α_] := Imin[α] / ISmin ;
Print["Truin = ", truin[α], "\nG = ", G[α], "\nA = ", A[α],
 "\nB = ", B[α], "\n1/B = ", N[1/B[α]], "\nα = ", α, "\nβ = ",
 β[α], "\nR1n = ", R1n[α], "\nR1c = ", R1c[α], "\nR1s = ", R1s,
 "\nc = ", c, "\nq = ", q, "\np = ", N[p]]
Print["Pruin before = ", e^(-ISmin), "\nPruin After = ", e^(-Imin[α])]
```

Next, we show the routines for solving $G = 0$, for $B$, first when $\gamma = 1$, then when $\gamma = 2$.

```
$MaxExtraPrecision = 1000000000000;
Reinsurance[A_, q_, p_] := (
  β := A (1 - q) + α q;

  B := (β - α q)/(α β (1 - q)) (1 - p) + (1 - α q)/(α (1 - q)) p;
  G := (1 - (β^2 - α^2 q)/(α β (β - α q))) (1 - p) - (1 - α)/(α (1 - α q)) p;
  α = FindRoot[G == -1, {α, A}][[1]][[2]];
  N[1/B]
 )
```

Table[Reinsurance[i,0.9,j],{i,0.01,1,0.01},{j,0,0.5,0.1}]

```
$MaxExtraPrecision = 30000000;

Reinsurance[A_, q_, p_] := (

  β := A (1 - q) + q α;

  B := (4 β - α q - Sqrt[α q (α q + 8 β)]) / (α β (4 - q - Sqrt[q (q + 8)])) (1 - p) +
       (4 - α q - Sqrt[α q (α q + 8)]) / (α (4 - q - Sqrt[q (q + 8)])) p;
  G :=
   (1 + (α (q^2 α^2 + 2 β Sqrt[q α (q α + 8 β)] + q α (6 β + Sqrt[q α (q α + 8 β)])) -
         16 β^3) /
       (α β
         (16 β^2 - (q^2 α^2 + 2 β Sqrt[q α (q α + 8 β)] +
              q α (6 β + Sqrt[q α (q α + 8 β)]))))) (1 - p) -
    ((16 (1 - α)) /
       (α
         (16 - (q^2 α^2 + 2 β Sqrt[q α (q α + 8 β)] +
              q α (6 β + Sqrt[q α (q α + 8 β)]))))) p;
  α = FindRoot[G == -1, {α, A}][[1]][[2]];
  N[1 / B]

)

Table[Reinsurance[i,0.9,j],{i,0.01,1,0.01},{j,0,0.5,0.1}]
```

Finally, we show the program used to plot the curves $G(P)$ in Figure 6.8:

```
B = 0.95;
q = 0.9;
α[A_] :=
 (AB - 2ABq - pq + Apq + ABq^2 - Apq^2 -
    √(-4 (A - Aq) (-Bq + Bq^2 - pq^2) +
       (-AB + 2ABq + pq - Apq - ABq^2 + Apq^2)^2)) /
  (2 (-Bq + Bq^2 - pq^2))

β[A_] :=
  A (1 - q) +
   (AB - 2ABq - pq + Apq + ABq^2 - Apq^2 -
      √(-4 (A - Aq) (-Bq + Bq^2 - pq^2) +
         (-AB + 2ABq + pq - Apq - ABq^2 + Apq^2)^2)) /
     (2 (-Bq + Bq^2 - pq^2)) q;

G[p_, A_] := (1 - (β[A]^2 - α[A]^2 q) / (α[A] β[A] (β[A] - α[A] q))) (1 - p) -
    (1 - α[A]) / (α[A] (1 - α[A] q)) p;

Table[G[p, A], {p, 0, 1, 0.01}, {A, 0.5, 1, 0.1}]
```