The Klebanov Polyakov correspondence on a squashed three sphere and Wilson loops in non relativistic AdS/CFT.

Yonge, Mark D

The Klebanov Polyakov Correspondence on a Squashed Three Sphere and Wilson Loops in Non Relativistic AdS/CFT

Mark D. Yonge

Submitted to the University of Wales in fulfilment of the requirements for the degree of Doctor of Physics.

Swansea University

March 19, 2010

Abstract

Motivated by the Klebanov-Polyakov conjecture we investigate the $O(N)$ vector model at large N on a squashed three-sphere and its holographic relation to bulk gravity on asymptotically locally AdS_4 space. We present analytical results for the action of the field theory as the squashing parameter , $\alpha \to -1$, when the boundary becomes effectively one dimensional. In this limit we solve the theory exactly and show that the action of the boundary theory scales as $\ln(1+\alpha)/(1+\alpha)^2$ which is to be compared and contrasted with the $-1/(1+\alpha)^2$ scaling of gravity in AdS-Taub-NUT space. These results are consistent with the numerical evidence presented in hep-th/0503238, and the soft logarithmic departure is interpreted as a prediction for the contribution due to higher spin fields in the bulk AdS_4 geometry.

We then give an introduction to non relativistic AdS/CFT and numerically compute the inter quark potential in the non relativistic theory obtained by taking the DLCQ of $\mathcal{N} = 4$ SYM theory by considering Wilson loops in the dual string theory via the AdS/CFT correspondence.

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Acknowledgements

Mark Yonge would like to thank the STFC for their financial support during this work.

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1 A Brief Overview of the AdS/CFT correspondence

The AdS/CFT correspondence is a set of dualities between string theories and field theories existing in lower dimensional space times. This was originally proposed [1, 2, 3] in the context of a duality between Type IIB String Theory in $AdS_5 \times S^5$ and four dimensional $\mathcal{N} = 4$ $SU(N)$ Super Yang Mills Theory in the large N limit. This duality can be extended to finite temperature by embedding a black hole in AdS_5. A few of their results are reviewed below.

This duality was discovered by taking the low energy limit of D3 branes in $AdS_5 \times S^5$ where the field theory on the brane effectively decouples from the bulk. This means that the behaviour of the D branes is effectively governed by the field theory that lives on them. Alternatively the geometry generated by N coincident D3 branes in the near horizon limit consists of $AdS_5 \times S^5$ near the branes surrounded by a horizon which is connected to the rest of the geometry by an infinite throat. As the horizon can never be reached from outside the infinite throat the physics of the D branes has again effectively decoupled from the bulk. As the same system can be described by both $\mathcal{N} = 4$ $SU(N)$, and by Type IIB string theory in $AdS_5 \times S^5$ it was proposed that these two theories were duals. Additional evidence for this proposal was the matching of the symmetry groups of the two theories.

$\mathcal{N} = 4$ SYM is a supersymmetric theory which is conformally invariant and has two parameters, the 't Hooft coupling, λ, and the rank of the gauge group, N. They are related by

$$\lambda = g_{YM}^2 N. \tag{1}$$

According to the AdS/CFT correspondence they are related to the string tension $\frac{1}{2\pi\alpha'}$, the string coupling constant g_s, and the radius of curvature of the metric, R by

$$\sqrt{\lambda} = \frac{R^2}{\alpha'}, \tag{2}$$

and

$$g_{YM}^2 = \frac{\lambda}{N} = 4\pi g_s. \tag{3}$$

If we take the large N limit with λ fixed, and only then take $\lambda \to \infty$, both α' and g_s will go to zero on the string side. Hence this duality is a strong-weak duality. This is important because it allows quantities which could not be calculated in perturbation theory at strong coupling on the field theory side to be calculated perturbatively in string theory, and vice versa. This has enabled the computation of quantities such as the free energy [2, 4], shear viscosity [5, 6], R charge diffusion constant [6], and the Chern-Simons diffusion rate [7] for the $\mathcal{N} = 4$ theory via the AdS/CFT correspondence. These results differ from those calculated via perturbation theory at weak coupling, which is expected since supersymmetry is broken at finite temperature and there is therefore no non-renormalization theorem is expected to hold [8].

There is a one to one correspondence between every operator, $\mathcal{O}$ in the field theory, and every bulk field, ϕ given by

$$\left\langle e^{\int \phi_0 \mathcal{O}} \right\rangle_{CFT} = \mathcal{Z}_{string}\Big|_{\phi=\phi_0}. \tag{4}$$

$Z_{string}|_{\phi=\phi_0}$ is the string partition function where ϕ is evaluated with the boundary condition $\phi|_{r=\Lambda} = \phi_0$. Equation (4) can be used as a generating functional for correlation functions, for example the two point Greens function for $\mathcal{O}$ can be calculated by differentiating (4) twice with respect to the boundary field

$$G(x-y) = -i\langle T\mathcal{O}(x)\mathcal{O}(y)\rangle = \left.\frac{\delta^2 \mathcal{Z}_{string}[\phi]}{\delta\phi(x)\delta\phi(y)}\right|_{\phi=\phi_0}. \tag{5}$$

The relationship between the dimension of the operator $\mathcal{O}$ and the mass of the corresponding bulk field, ϕ is given by

$$\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + (mR)^2}. \tag{6}$$

The positive root of this equation is normally the one that needs to be used.

Since then it has been extended to other dualities between (non)asymptotically AdS space times and (non)conformal field theories. Two cases which will be of interest to us for the rest of this thesis are the Klebanov Polyakov correspondence between the $O(N)$ vector model and a higher spin gauge theory [9] which we consider in section two , and gravity duals to non relativistic field theories [10, 11, 12] which we consider in sections three and four.

2 The Klebanov Polyakov Correspondence on a Squashed Three Sphere in the Limit $\alpha \to -1$

2.1 Introduction to the Klebanov Polyakov Correspondence

The AdS/CFT correspondence states that string theories in asymptotically AdS space times with d dimensions are dual to certain conformal field theories in $d-1$ dimensions [1, 2, 3]. Testing these dualities is in general difficult because the theories involved are very complicated and are only tractable in different limiting regions of parameter space. However in [9] Klebanov and Polyakov suggested that a simpler duality exists between the large N limit of the singlet sector of the critical $O(N)$ vector model in three dimensions and the minimal bosonic higher spin gauge theory in four dimensional Anti de Sitter space.

Theories with an infinite number of massless high spin gauge fields in AdS_4 were first formulated in [13, 14]. In [9] it was noted that the spectrum of conserved singlet currents

$$J_{\mu_1 \dots \mu_s} = \phi^a \partial_{(\mu_1} \dots \partial_{\mu_s)} \phi^a + \dots \tag{7}$$

for the simplest $O(N)$ invariant theory at large N,

$$S = \frac{1}{2} \int d^3x \sum_{a=1}^{N} (\partial_\mu \phi^a)^2 \tag{8}$$

was precisely what was needed to match the spectrum of massless high spin gauge fields in the minimal bosonic theory in AdS_4 governed by the $ho(1;0|4)$ algebra [15, 16]. Note that in (7) s is any even integer. This lead the authors of [9] to conjecture that the correlation functions for the singlet currents of the free $O(N)$ model in three dimensions could be obtained as in the usual AdS/CFT correspondence via

$$\langle \exp \int d^3x h_0^{(\mu_1 \dots \mu_s)} J_{(\mu_1 \dots \mu_s)} \rangle = e^{S[h_0]}. \tag{9}$$

Here $S[h_0]$ is the action of the high spin theory evaluated at the boundary, and $h_0^{(\mu_1 \dots \mu_s)}$ are the sources of the fields in the field theory.

The dimension of the correlation functions of $\phi^a\phi^a$ is one. Because this lies below $d/2$ it is necessary to use the negative solution for the equation relating the dimension of the operator to the mass of the dual gauge field in the bulk,

$$\Delta_- = \frac{d}{2} - \sqrt{\frac{d^2}{4} + (mL)^2}. \tag{10}$$

We can see that in $d = 3$ to obtain $\Delta_- = 1$ we need $m^2 = -2/L^2$. This corresponds to a conformally coupled scalar in AdS_4 with the action, up to quadratic order in the field h given by

$$S = \frac{N}{2}\int d^4x\sqrt{g}\left[(\partial_\mu h)^2 + \frac{1}{6}Rh^2 + ...\right], \tag{11}$$

where the Ricci scalar, $R = -12/L^2$ in AdS_4 [9]. In [17] it was argued that when we are using the branch Δ_- the correct procedure for extracting the correlation functions is to calculate the generating functional $W[h_0...]$ with the normal dimension Δ_+, and then to perform a Legendre transformation with respect to the source h_0.

This lead to the question, what is the dual field theory for the case where the operator to have the more usual dimension, Δ_+? In [9] it was conjectured that the answer is the interacting critical $O(N)$ model with the action

$$S = \int d^3x\left[\frac{1}{2}(\partial_\mu\phi^a)^2 + \frac{\lambda}{2N}(\phi^a\phi^a)^2\right]. \tag{12}$$

In this theory the dimension of the operator $\phi^a\phi^a$ is $2+O(1/N)$, consistent with $\Delta_+ = 2$ in the large N limit. The anomalous dimension for the currents in (7) with spins greater than zero scale as $1/N$ [18] in this theory, so in the large N limit they should correspond to massless gauge fields in the bulk.

It was also noted in [9] that if a Legendre transformation is applied to the generating functional of the interacting $O(N)$ theory with respect to the source h_0, the result was the generating functional for the singlet current generators of the free theory. This is consistent with the general rule found in [17], that the generating functional for correlation functions in the theory with one choice of operator dimension is a Legendre transform of the one with the other choice.

Further evidence for this conjecture was found in [19] where the authors calculated the tree level three point functions of the minimal bosonic theory, and found that they were in agreement with the correlation functions of the $O(N)$ singlet sector. Also in [20, 21] it was shown that the cubic self coupling of the scalar field in the bulk was zero, which suggests that for the choice Δ_+ the three point function of the scalar operators in any dual CFT should vanish to leading order in $1/N$. This is the case for the critical $O(N)$ model in three dimensions.

2.2 The Klebanov Polyakov correspondence and The Squashed Three Sphere

In [22] an extension of this duality was proposed between the $O(N)$ model on a squashed three sphere and the higher spin gauge theory on AdS Taub-Nut and AdS Taub-Bolt geometries with a phase transition occurring between the two on the gravitational side. In this case they considered the $O(N)$ model with a conformal mass term so the theory would approach a free conformal fixed point in the UV, which corresponds to the asymptotically AdS part of the bulk. The squashed three sphere is an S^1 bundle over S^2 with metric

$$ds^2 = \frac{a^2}{4}\left(\sigma_1^2 + \sigma_2^2 + \frac{\sigma_3^2}{1+\alpha}\right). \tag{13}$$

Where the σ_i are defined by:

$$\sigma_1 + i\sigma_2 = e^{-i\psi}(d\theta + i\sin\theta d\phi) \tag{14}$$

and

$$\sigma_3 = d\psi + \cos\theta d\phi. \tag{15}$$

The Ads Taub Nut metric may be written in terms of these co-ordinates as follows [22]

$$ds^2 = \frac{1}{k^2(1-r^2)^2}\left[\frac{4(1+\alpha r^2)}{1+\alpha r^4}dr^2 + r^2(1+\alpha r^2)(\sigma_1^2+\sigma_2^2) + r^2\frac{1+\alpha r^4}{1+\alpha r^2}\sigma_3^2\right], \tag{16}$$

where the radial co-ordinate runs from zero to one. k is related to the cosmological constant via $\Lambda = -3k^2$.

The scalar curvature of this space is

$$\frac{2(3+4\alpha)}{a^2(1+\alpha)}, \tag{17}$$

and the volume is

$$\mathrm{Vol(M)} = \frac{2\mathrm{a}^3\pi^2}{(1+\alpha)^{\frac{1}{2}}}. \tag{18}$$

The squashing parameter α lies in the range

$$-1 \leq \alpha < \infty \tag{19}$$

with $\alpha = 0$ corresponding to the round three sphere. In the large α limit the squashed sphere approaches the direct product space $S^2 \times S^1$, and the periodicity of the S^1 fibre can be thought of as an inverse temperature. The limit $\alpha \to -1$ is the limit of extreme squashing which was not accessible analytically before, and this will be the main focus of this work. In this limit one of the dimensions becomes very large compared to the others and the field theory becomes effectively one dimensional.

This duality has the advantage compared to the usual string/gauge theory dualities in that the QFT is exactly solvable and can be compared to the semiclassical properties of Einstein's gravity in the absence of a proper formulation of the higher spin gauge theories in AdS Taub-Nut and AdS Taub-Bolt space times. It is useful to solve the $O(N)$ model on a squashed three sphere because it provides a one parameter family of field theory/ gravity dualities, whose free energies exhibit a non monotonic behaviour as a function of the squashing parameter as argued in [22].For other related works on the $O(N)$ model and the Klebanov-Polyakov duality, see [20, 23, 24].

The squashed three sphere is the conformal boundary of AdS Taub Nut and AdS Taub Bolt geometries [25, 26]. As in the canonical example of the Hawking Page transition [27], only one of these two geometries dominates the partition function. In particular, as a function of α, there is a Hawking-Page transition from AdS Taub-Nut to AdS Taub-Bolt, the latter dominating for large α. In [28, 29] the action of AdS Taub-Nut was found to be:

$$I_{TN} = -\frac{6\pi}{GR}\frac{(1+2\alpha)}{(1+\alpha)^2} \tag{20}$$

where G is Newton‘s constant and R is the Ricci scalar which is negative in these backgrounds. For AdS Taub-Bolt the corresponding result is:

$$I_{TB} = \frac{24\pi}{RG}(1+\alpha)^{-\frac{1}{2}}(m_b + \frac{3}{4}r(1+\alpha)^{-1} - r^3) \tag{21}$$

with

$$m_b = \frac{1}{2}r + \frac{1}{8r}(1+\alpha)^{-1} + \frac{1}{2}(r^3 - \frac{3}{2}r(1+\alpha)^{-1} - \frac{3}{16r}(1+\alpha)^{-2}) \qquad (22)$$

and

$$r = \frac{1}{6}(1+\alpha)^{\frac{1}{2}}(1 + (1 - 12(1+\alpha)^{-1} + 9(1+\alpha)^{-2})^{\frac{1}{2}}) \qquad (23)$$

In the limit of large α the AdS Taub-Bolt action grows linearly:

$$I = \frac{4\pi}{9GR}\alpha \qquad \alpha \to \infty \qquad (24)$$

AdS-Taub-Nut space can be obtained by filling the volume of a squashed three sphere with a hyperbolic metric with negative cosmological constant [30]. In the limit $\alpha \to -1$ the space becomes a Bergmann space which can be described as a coset space $SU(2,1)/U(2)$ which has been studied in [31]. It would be interesting to understand the behaviour of the action from the bulk perspective by considering higher spin gauge fields on this Bergmann space. For a detailed construction of bulk-boundary and bulk-bulk propagators in this space see [32]. Other work in AdS Taub Nut space is contained in [33, 34, 35].

The action of the $O(N)$ model was calculated in [22] for $\alpha > -\frac{8}{9}$ and is shown in Fig 1 below, and the result for the gravitational side is shown in Fig 2, where the action has been normalised so that it agrees with the field theory results at large α and a constant has been added so that the peaks coincide. They found a close agreement between the results for the $O(N)$ model and AdS Taub-Nut space below a critical value of α and with AdS Taub-Bolt above it, but with a smooth crossover between the two which suggests that the higher spin gauge fields have the effect of smoothing out the phase transition.

In [22] the large α behaviour of the QFT action at strong coupling was found to be:

$$I_{a\lambda\gg1} = -\frac{N\zeta_R(3)}{10\pi^2}\alpha \qquad (25)$$

We have extended the work in [22] to the limit $\alpha \to -1$, and in this limit we find from the $O(N)$ model:

$$I_{a\lambda\gg1} = \left(\frac{\ln(1+\alpha)}{3(1+\alpha)^2} + \frac{0.0614093}{(1+\alpha)^2}\right)N \qquad (26)$$

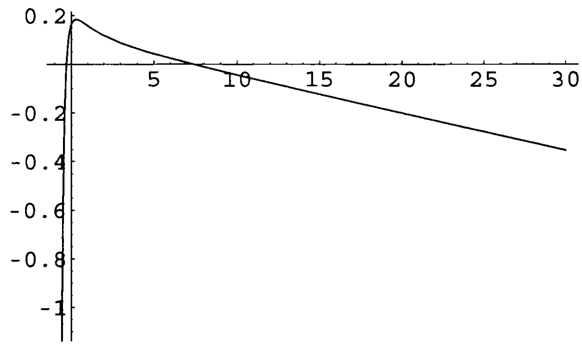

Figure 1: The action $\frac{I}{N}$ of the $O(N)$ model at strong coupling as a function of α.

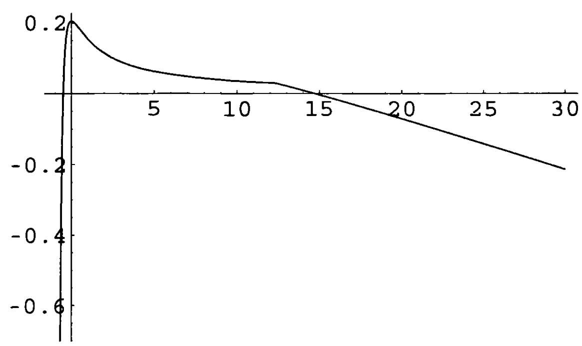

Figure 2: The Gravitational action as a function of α.

The results (25) and (26) are to be compared and contrasted with (24) and (20).

It can be seen that the qualitative behaviour of the free energy of the $O(N)$ model as $\alpha \to -1$ and at large α closely reproduces the results of semiclassical gravity, the logarithmic deviation in the leading order term in the limit $\alpha \to -1$ is a prediction of the effect of including the higher spin gauge fields in addition to gravity. Interestingly there appears to be no a priori reason why the results for the higher spin gauge theory should be so close to the gravitational result, though these results suggest that the effects of the higher spin gauge fields cannot be drastic.

In section 2.3 we summarise some useful results from the $O(N)$ model and in section 2.4 we describe the calculation in more detail. Section 2.5 contains a discussion and summary.

2.3 O(N) Model in the Large N Limit

The $O(N)$ model has been extensively studied in various dimensions e.g. see [22, 36]. The interacting $O(N)$ model is a conformal theory, which in Euclidean space has the classical action,

$$S = \int dx^D \sqrt{g}\Big(\frac{1}{2}\nabla\Phi \cdot \nabla\Phi + \frac{1}{2}m^2\Phi \cdot \Phi + \frac{\lambda}{4N}(\Phi \cdot \Phi)^2\Big). \qquad (27)$$

Φ is a set of N scalar fields that transform as a vector under $O(N)$ rotations. The N dependence of (27) will ensure that this theory has a well defined large N limit. The coupling constant λ flows from a free fixed point in the UV to an interacting fixed point in the IR. This model can be solved exactly in the strictly large N limit by deriving an effective potential. This can be done by defining an auxiliary scalar field

$$\sigma = \frac{\Phi.\Phi}{N}, \qquad (28)$$

and rewriting the action in the form

$$S = \int dx^D \sqrt{g}\Big(\frac{1}{2}\nabla\Phi \cdot \nabla\Phi + \frac{1}{2}(m^2 + \lambda\sigma)\Phi.\Phi - \frac{\lambda N}{4}\sigma^2\Big) \qquad (29)$$

A homogenous background expectation value ϕ is then introduced for the $O(N)$ field which is then split into a VEV and fluctuations as follows:

$$\Phi = (\sqrt{N}\phi + \delta\phi, \pi_1, \pi_2, ..., \pi_{N-1}). \qquad (30)$$

Here ϕ is the homogeneous background and $\delta\phi$ and $\vec{\pi}$ are the fluctuations around it. Normally this would break the $O(N)$ symmetry to $O(N-1)$ resulting in goldstone bosons, however as argued by [22] in these circumstances the symmetry is not broken because the path integral includes an integration over the vacuum manifold which implies that symmetry breaking does not occur in a compact space. By making this choice of fields we have effectively moved over into polar co-ordinates, where $\sqrt{N}\phi$ is the effective radial co-ordinate. When we integrate over the other co-ordinates the partition function picks up an extra term corresponding to the volume of the vacuum manifold $O(N)/O(N-1)$.

$$\frac{2\pi^{\frac{N-1}{2}}}{\Gamma([N-1]/2)}\big(N^{\frac{1}{2}\phi}\big)^{N-1} = \big(\phi\pi^{\frac{1}{2}}e^{\frac{1}{2}}\big)^N = \exp[\Big(\frac{N}{2}[1+\ln\pi+\ln\Big(\frac{\phi^2}{\mu}\Big)]\Big). \qquad (31)$$

We have introduced the arbitrary scale μ here so that the arguments of the logarithms are dimensionless. Integrating over the π fluctuations (the contribution from $\delta\phi$ is subleading in the large N limit), including the above term yields the result

$$\begin{aligned}\frac{V_{\text{eff}}(\phi,\sigma)}{N} =& \frac{1}{2}(m^2+\lambda\sigma)\phi^2 - \frac{\lambda}{4}\sigma^2 + \frac{1}{2\text{Vol}(M)}\ln\det{}'\left(\frac{-\Box+m^2+\lambda\sigma}{\mu^2}\right) \\ & - \frac{1}{2\text{Vol}(M)}\left(1+\ln\pi+\ln\frac{\phi^2}{\mu}\right).\end{aligned} \tag{32}$$

Here Vol(M) is the volume of the manifold on which the field theory is formulated. In the present context M represents the squashed three sphere. μ is a dimensional scale which is like the sliding renormalization scale. The prime in det$'$ indicates that the integration was not done over the constant mode which is then dealt with separately. In the large N limit only the configuration obtained by extremising (32) contributes to the partition function. Minimising the effective potential with respect to ϕ and σ yields the equations:

$$\phi^2(m^2+\lambda\sigma) = \frac{1}{\text{Vol}(M)} \tag{33}$$

and

$$\phi^2 - \sigma + \frac{1}{\text{Vol}(M)}\text{Tr}'\left(\frac{1}{-\Box+m^2+\lambda\sigma}\right) = 0. \tag{34}$$

An "effective pion mass" corresponding to the mass of the π fluctuations can then be defined:

$$m_\pi^2 = m^2 + \lambda\sigma \tag{35}$$

so that equations (33) and (34) can be rewritten as a gap equation for m_π^2

$$m_\pi^2 = m^2 + \frac{\lambda}{\text{Vol}(M)}\text{Tr}\left(\frac{1}{-\Box+m_\pi^2}\right) \tag{36}$$

where the constant mode has been absorbed into the above. If the $O(N)$ model was in a symmetry broken phase we would expect m_π^2 to vanish by Goldstones theorem, and we would have $N-1$ massless goldstone bosons, however it can be seen from equations (33) and (35) that this would require either $Vol(M)$ or ϕ^2 would have to be infinite. If ϕ^2 is infinite then the coupling would have to be set equal to zero, therefore the $O(N)$ model can only be realised on a finite compact manifold, with a non zero coupling in a symmetric phase.

Once (36) has been solved the effective potential can be evaluated at the extremum to give the action:

$$S = \frac{N}{2}\Big(- \frac{\mathrm{Vol}(M)}{2\lambda}(m^2 - m_{\pi^2}^2) + \ln\det\frac{-\Box + m_\pi^2}{\mu^2} + \ln(\mu^3 \mathrm{Vol}(M))\Big). \quad (37)$$

It can be seen that the only effect of the arbitrary scale μ is to add a constant to the action. To evaluate (36) it is necessary to evaluate the trace. This can be done by the method of zeta function regularisation. The zeta function for an elliptic operator A is defined by

$$\zeta(s) = \mathrm{Tr} A^{-s} \quad (38)$$

so that

$$\ln\det\frac{-\Box + m_\pi^2}{\mu^2} = -\lim_{s\to 0}\frac{d}{ds}\mathrm{Tr}\Big(\frac{-\Box + \mathrm{m}_\pi^2}{\mu^2}\Big)^{-\mathrm{s}} = -\zeta'(0), \quad (39)$$

and

$$\mathrm{Tr}\left[\frac{1}{-\Box + \mathrm{m}_\pi^2}\right] = \frac{1}{\mu^2}\lim_{\mathrm{s}\to 0}\frac{\mathrm{d}}{\mathrm{ds}}(\mathrm{s}\zeta(\mathrm{s}+1)). \quad (40)$$

The zeta function on the squashed three sphere can be written in the form[37, 38, 39]

$$\zeta(s) = \sum_{l=1}^{\infty}\sum_{q=0}^{l-1}\frac{l(a\mu)^{2s}}{(l^2 + \alpha(l-1-2q)^2 + a^2 m_\pi^2 - 1)^s.}, \quad (41)$$

The self-consistent gap equation which determines the solution of the model is highly non trivial for two reasons. Firstly, it is a non linear equation for m_π^2. Secondly, it involves a zeta function, namely $\zeta_{m_\pi^2}(1)$ which is a complicated object and needs to be defined via analytic continuation.

In addition to these ingredients, we need to specify the coupling constant of the theory λ which is dimensionful. The relevant dimensionless parameter is the combination $a\lambda$. Since λ is a relevant coupling in three dimensions, $a\lambda << 1$ is the weak coupling limit corresponding to taking the sphere size to be small., thus approaching the UV fixed point.

We will be primarily interested in the strong coupling limit $a\lambda \to \infty$ which corresponds to the IR fixed point of the theory on the squashed sphere. In this limit, a drastic simplification of the gap equation occurs, allowing us to solve the problem analytically in the $\alpha \to -1$ limit. The gap equation at strong coupling (36) determines m_π^2 to be a zero of $\zeta_{m_\pi^2}(1)$. The results of [22] provide evidence that the resulting value of m_π^2 is finite and non-negative for all allowed values of the squashing parameter α.

We will now evaluate the action of the theory in the $\alpha \to -1$ limit, at strong coupling $a\lambda \to \infty$. The fact that m_π^2 has a finite value determined by the zero of $\zeta_{m_\pi^2}(1)$, implies that the first term in (37) is zero at strong coupling. The volume term will finally be found to give a subleading contribution to the action. The dominant contribution in the $\alpha \to -1$ limit therefore is: $-\frac{N}{2}\zeta'(0)$. As this is superficially divergent it needs to be analytically continued by methods described in the section 2.5.

2.4 The $O(N)$ model as $\alpha \to -1$

The zeta function (41) is superficially divergent, but a finite value may be obtained by analytically continuing the sum, firstly by applying the Abel-Plana formula, and then by carrying out a Sommerfield Watson transformation which is described in more detail in the next section. We find:

$$\frac{\zeta(s)}{\mu^{2s}} = \frac{a^{2s}A}{(1+\alpha)^s} + a^{2s}\int_0^1 \frac{B(y)dy}{(1+\alpha y^2)^s} - \frac{2ia^{2s}}{(1+\alpha)^s}\int_0^\infty \frac{C(y)dy}{1+\exp(2\pi y)} \tag{42}$$

Where the functions A, B, C are defined in section 2.5. Using this result we find that the zeta function in the limit $\alpha \to -1$, is given by:

$$\begin{aligned}\zeta'(0) = &-\frac{\ln(1+\alpha)}{3(1+\alpha)^2} - \frac{10}{9(1+\alpha)^2} - \frac{2\ln 2}{3(1+\alpha)^2} \\ &+ \int_0^\infty \frac{16y\ln(1+4y^2) - 8(4y^2-1)\tan^{-1}2y}{\exp(2\pi y)-1}dy.\end{aligned} \tag{43}$$

This then evaluates to:

$$I = \frac{N}{2}\Big(\frac{\ln(1+\alpha)}{3(1+\alpha)^2} + \frac{0.0614093}{(1+\alpha)^2}\Big). \tag{44}$$

The above argument relies on $a^2m_\pi^2$ being finite in this limit, though this was not calculable by this method the numerical results in [22] indicate that it approached zero in this limit.

2.5 Analytic continuation of the zeta function

As (41) is divergent if s is set directly to zero it will need to be continued analytically. This was done by converting the sum over q into an integral using the Abel-Plana formula and evaluating the l summation using a Sommerfield-Watson transformation. (41) has branch cuts at

$$q = \frac{l-1}{2} \pm \frac{1}{2}\Big(\frac{1-l^2-a^2m_\pi^2}{\alpha}\Big)^{1/2}, \tag{45}$$

with these branch cuts the Abel-Plana formula of the form (46) may be used to evaluate the l sum.

$$\sum_{i=n}^{m}\phi(x) = \frac{1}{2}(\phi(n)+\phi(m)) + \int_n^m \phi(x)dx$$
$$- i\int_0^\infty \frac{dy}{\exp(2\pi y)-1}\Bigg(\phi(n-iy)-\phi(n+iy)-\phi(m-iy)+\phi(m+iy)\Bigg). \tag{46}$$

Using this we obtain:

$$\frac{\zeta(s)}{\mu^{2s}} = \frac{a^{2s}A}{(1+\alpha)^s} + a^{2s}\int_0^1 \frac{B(y)dy}{(1+\alpha y^2)^s} - \frac{2ia^{2s}}{(1+\alpha)^s}\int_0^\infty \frac{C(y)dy}{\exp(2\pi y)-1}, \tag{47}$$

where:

$$A = \sum_{l=1}^{\infty}\frac{l}{((l+G)^2-h^2)^s}, \tag{48}$$

$$B = \sum_{l=1}^{\infty}\frac{l(l-1)}{((l+I)^2-J^2)^s}, \tag{49}$$

$$C = \sum_{l=1}^{\infty} l(\frac{1}{((l+K)^2-M^2)^s} - \frac{1}{((l+K^*)^2-(M^2)^*)^s}. \tag{50}$$

Here G, H, I, J, K, M are given by:

$$G = \frac{-\alpha}{1+\alpha} \tag{51}$$

$$-H^2 = \frac{a^2(m_\pi)^2(1+\alpha)-1}{(1+\alpha y^2)^2}, \tag{52}$$

$$I = \frac{-\alpha y^2}{(1+\alpha y^2)}, \tag{53}$$

$$-J^2 = \frac{a^2 m_\pi^2 (1+\alpha y^2) - 1}{(1+\alpha y^2)^2}, \tag{54}$$

$$K = \frac{\alpha(-1+2iy)}{1+\alpha}, \tag{55}$$

$$-M^2 = \frac{a^2 m_\pi^2 - 1 + \alpha(-1+2iy)}{1+\alpha} - \frac{\alpha^2(-1+2iy)^2}{(1+\alpha)^2}, \tag{56}$$

The sums over l can be evaluated using a Sommerfield-Watson transformation. Apply this to the A we find

$$A = \frac{i}{2}\int_{C_1} \frac{z \cot \pi z dz}{((z+G)^2 - H^2)^s}, \tag{57}$$

where the contour C_1 is shown in figure three. For $\mathrm{Re}(s) > 2$ this can be deformed into C_2, also shown in figure three.

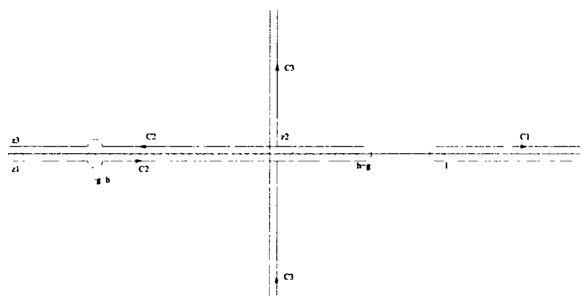

Figure 3: The contours used for the analytic continuation of the zeta function.
untitled folder

It is useful to rewrite $\cot \pi z$ with the identities

$$\cot \pi z = i\Big(1 + \frac{2}{\exp 2i\pi z - 1}\Big), \tag{58}$$

between z_1 and z_2 and

$$\cot \pi z = i\Big(-1 + \frac{2}{1 - \exp(-2i\pi z)}\Big), \tag{59}$$

between z_2 andz_3.

The integrals over the exponential pieces are then manifestly finite and can be evaluated along C_3. The integrals over the constant pieces can be done analytically for $\mathrm{Re}(s) > 2$. These expressions are then evaluated at $s = 0$ to define the analytically continued function. A similar method is used for B and C. In the case of C the branch points are not on the real axis so the integrals are no longer along the real axis but along a tilted contour. Using this method the following results are obtained in the limit α tends to minus one.:

$$A \mid_{s=0} = \frac{1}{(1+\alpha)^2}, \tag{60}$$

$$\frac{d}{ds} A \mid_{s=0} = \frac{2\ln(1+\alpha)}{(1+\alpha)^2} + \frac{3}{(1+\alpha)^2} - \frac{2\ln 2}{(1+\alpha)^2}, \tag{61}$$

$$\int_0^1 B \mid_{s=0} (2\ln a - \ln(1+\alpha y^2)) = -\frac{4\ln a}{3(1+\alpha)^2} + \frac{1}{3(1+\alpha)^2} + \frac{2\ln(1+\alpha)}{3(1+\alpha)^2}, \tag{62}$$

$$\int_0^1 \frac{dB}{ds} \mid_{s=0} = \frac{4\ln(1+\alpha)}{3(1+\alpha)^2} - \frac{10}{9(1+\alpha)^2} - \frac{4\ln 2}{3(1+\alpha)^2} \tag{63}$$

$$C \mid_{s=0} = -\frac{8yi}{(1+\alpha)^2}, \tag{64}$$

$$\begin{aligned} \frac{dC}{ds} \mid_{s=0} = \frac{2i}{(1+\alpha)^2} \Big(& 4y(-7 + 8\ln 2 - 4\ln(1+\alpha) + 2\ln(1+4y^2)) \\ & - 4(4y^2-1)\tan^{-1} 2y \Big). \end{aligned} \tag{65}$$

Putting the above together gives:

$$\begin{aligned} \zeta'(0) = & -\frac{\ln(1+\alpha)}{3(1+\alpha)^2} - \frac{10}{9(1+\alpha)^2} - \frac{2\ln 2}{3(1+\alpha)^2} \\ & + \frac{1}{(1+\alpha)^2} \int_0^\infty \frac{16y\ln(1+4y^2) - 8(4y^2-1)\tan^{-1} 2y}{\exp(2\pi y) - 1} dy. \end{aligned} \tag{66}$$

The integrals can then be evaluated numerically.

2.6 Summary and Discussion

We have solved the strongly coupled $O(N)$ model exactly in the limit $\alpha \to -1$ and found a soft logarithmic deviation from the results of semiclassical gravity in this regime. It is surprising that the singularity structure in this limit is so similar to that of classical gravity on AdS-Taub-Nut space. The anomalous logarithmic deviation can only be explained within the confines of the Klebanov-Polyakov conjecture as being due to the affects of the higher spin gauge fields. There appears to be no simple physical explanation for the behaviour of the action in this limit, but it would be interesting to see if it is because in the $\alpha \to -1$ limit one the field theory becomes effectively one dimensional.

In the other extreme of the allowed range of α, namely at large α, it is evident from (24) and(25) that classical gravity and boundary field theory are qualitatively similar. This is not very surprising, since at large α, the boundary theory can be reinterpreted as being at a finite temperature given by α. The linear scaling of the action with α, and equivalently the free energy scaling as α^2, is what one expects in a field theory in three dimensions. Nevertheless, from this we learn, assuming the validity of the Klebanov-Polyakov conjecture, that the higher spin gauge theory dual to the O(N) model at large squashing, should behave in qualitatively the same fashion as Einstein gravity in AdS-Taub-Bolt space. We remak that the coefficients for the field theory (25) and gravity (24) actions are not expected to match as the higher spin gauge fields were not included in the calculation. In any case, matching of these coefficients only sets up a dictionary between $\frac{1}{N}$ in field theory and the bulk curvature in units of the 4d planck mass. The above discussion compares with the AdS_5/CFT_4 case where doing a strongly coupled field theory calculation is difficult and there is a 3/4 discrepancy factor between the strong and weak t'hooft coupling results due to higher stringy modes becoming light at large string scale curvatures in the string dual of the weakly coupled gauge theory.

The analytic results we have obtained for the strongly coupled field theory near $\alpha \to -1$, when combined with the linear behaviour at large α, reproduce remarkably well the non-monotonic behaviour of the classical bulk gravity action presented in Figure 2. Note that the non-monotonic behaviour in the bulk (without higher spin fields) is due to a Hawking-Page transition which is necessary in order to pass over from the Ads-Taub-Nut to the AdS-Taub-Bolt phase, the latter showing a linear behaviour with α at large α.The message is that even though we don't have a proper formulation of the higher spin theory in these backgrounds, our results suggest that gravity reproduces qualitatively similar results to the higher spin gauge theory dual to the O(N) model.

3 Introduction to Non Relativistic AdS/CFT

The standard AdS/CFT correspondence ia a set of dualities between string theories and relativistic quantum field theories in d dimensions and relativistic quantum field theories in $d-1$ dimensions. However there are many important physical systems which are governed by non relativistic field theories such as fermions at unitarity, superconductors, atomic and nuclear physics. As many of these phenomenon occur at strong coupling they cannot be calculated by perturbation theory in the QFT. It would therefore be desirable to see if it could be calculated in some string theory as is the case for relativistic field theories.

In non relativistic field theories there is an additional conserved quantity in addition to the energy, namely the particle number. As such we would expect a d dimensional string theory to be dual to a $d-2$ dimensional non relativistic field theory, with the extra dimension being dual to the particle number.

One of the most important pieces of evidence for the AdS/CFT correspondence is the matching of the asymptotic symmetry group of the metric and the symmetry group of the field theory. For a non relativistic field theory this is the d dimensional Schrodinger algebra. This contains the group of translations(P_i), rotations(M_{ij}), Galilean invariance(K_i), time translations(H), and scale invariance. The standard Galilean algebra can be extended to a non relativistic conformal group by adding a dilaton operator, D, and a number operator N with the non trivial commutation relations

$$\begin{aligned} [D,P_i] =& iP_i, \quad [D,K_i] = i(1-z)K_i \quad [D,H] = izH \\ & [D,N] = i(2-z) \quad [P_i,K_j] = -\delta_{ij}N \end{aligned} \tag{67}$$

z is the dynamical exponent which defines the scaling between the dimensions of time and length,$time = length^z$. The special case $z=2$ corresponds to the Schrodinger algebra, in this case it is possible to extend the symmetry group by adding another generator, C with the commutation relations

$$[C,P_i] = iK_i \quad [D,C] = 2iC \quad [H,C] = iD \tag{68}$$

Geometries consistent with this symmetry group without a black hole were proposed in [40, 41], with a metric of the form

$$ds^2 = L^2\left(-\frac{dt^2}{r^{2z}} + \frac{d\vec{x}^2 + 2d\xi dt}{r^2} + \frac{dr^2}{r^2}\right) \tag{69}$$

z is the dynamical exponent, the Schrodinger case corresponds to $z = 2$. Note that the ξ co-ordinate is an additional compact dimension. The metric (69) is manifestly invariant under translations and rotations in $d\vec{x}$, as well as translations in time. A form of scale invariance occurs if you make the transformations

$$x' = \lambda x, \quad t' = \lambda^z t, \quad r' = \lambda r, \quad \xi' = \lambda^{2-z}\xi. \tag{70}$$

It is also invariant under a Galilean boost, providing the ξ coordinate transforms as

$$\xi' = \xi + \frac{1}{2}(2\vec{v}.\vec{x} - v^2 t). \tag{71}$$

In [40, 41] it was found that this metric is consistent with coupling Einstein's gravity to a gauge field with a mass $m^2 = \frac{z(z+d)}{L^2}$ in the presence of a cosmological constant $\Lambda = \frac{(d+1)(d+2)}{L^2}$. . In [42, 43] a solution was found with only a cosmological constant term present. This was obtained by taking the DLCQ of AdS_5 and the symmetry group was broken to the Schrodinger subgroup by a periodic identification. This is effectively the degenerate limit of the background considered in [40, 41].

The number operator, N can be identified with $i\partial_\xi$ [44]. Because the number operator is discreet this means that ξ must be compact, however it is also null from the point of view of the bulk geometry. This suggests that the dual field theory will have to undergo a discrete light cone quantisation which is gone into in more detail in the next section. It may look as though compactifying ξ will break boost invariance, however as pointed out in [44] the fact that ξ is null together with the commutator $[N, K_i] = 0$ prevents this from happening. Compactifying ξ might also introduce a conical singularity in the large r limit, however whenever when you go to non zero temperature by adding a black hole it will be found that this singularity will always be hidden by the event horizon, so temperature can act as an effective regulator in the infra red in the field theory.

It was found it [45] that the n point Green's function of an operator, O in any non relativistic field theory depends on the scaling dimension of the operator as well as the particle number N_0. This should be true for any non relativistic field theory as this is what is needed to define a representation of the $z = 2$ Schrodinger algebra. The two point greens function for (69) was calculated in [41] for the case $z = 2$. They found

$$\langle O_1(x,t)O_2(0,0)\rangle \propto \frac{\Gamma(1-\nu)}{\Gamma(\nu)}\delta_{\triangle_1\triangle_2}\theta(t)\frac{1}{|\epsilon^2 t|^{\triangle}}e^{-ilx^2/2|t|}, \tag{72}$$

where θ is a step function, l is the momentum in the compact dimension, $\triangle_{i,j}$ are the scaling dimensions of the operators, and

$$\nu = \sqrt{\left(1+\frac{d}{2}\right)^2 + l^2 + m^2}. \tag{73}$$

The number operators spectrum in theories dual to these geometries is the set of integers because it arises from the Kaluza-Klein modes in ξ. The conservation law for ξ momentum

$$l_1 + l_2 + ... = l_1' + l_2' + ... \tag{74}$$

is called "Bargmann's superselection rule on the mass" [46]. from the Schrodinger algebra it follows that the stress tensor commutes with the particle number, therefore the fluctuations of the dual to the stress tensor will have zero ξ momentum.

The geometry (69) is not invariant under time reversal, however it is invariant under

$$t \rightarrow -t, \quad \xi \rightarrow -\xi. \tag{75}$$

As was pointed out in [44] this is consistent with the fact that time reversal is an antiunitary operator that acts like $T\ \ O_l \rightarrow O_{-l}$.

A geometry dual to a non relativistic field theory was found in [44, 12] by applying a Null Melvin twist to a black hole embedded in $AdS_5 \times S^5$ and the results are reviewed in the next section.

3.1 Calculation of a Non Relativistic Metric via a Null Melvin Twist

Several gravity duals to non relativistic field theories have been calculated in [40, 41, 44, 12, 47, 48, 49, 50, 51]. One method outlined in [49] involves selecting an appropriate background of type IIB string theory and applying a boost, γ in a direction dy. A T duality is then applied along the direction dy. A twist is then applied of the form $d\phi \rightarrow d\phi + \alpha dy$, where α is the twist parameter and $d\phi$ is an isometry in the compact dimensions. Another T duality is then applied in the y direction, followed by a boost back to the original frame, $-\gamma$. Finally the limit $\alpha \to 0$, $\gamma \to \infty$ such that $\alpha cosh\gamma = b$. This transformation is called the Null Melvin twist and preserves the event horizon of the black hole whilst altering the asymptotic in such a way that they are consistent with a dual to a non relativistic field theory. The curvature scalars are also left unchanged by this twist [44] which implies that the constraints on when supergravity is a reliable approximation are unchanged.

This procedure will work for any background of the form $AdS_n \times Y$ at the boundary, where Y is a compact space. We will now review this calculation for the case of a black hole in AdS_5 where Y is a S^5, which we will use later to calculate Wilson Loops in these backgrounds in the next section. A version of this calculation was done in [12, 44].

The metric of a black hole in $AdS \times S^5$ is

$$ds^2 = \left(\frac{r}{R}\right)^2(-hdt^2 + dy^2 + dx_2^2) + h^{-1}\left(\frac{R}{r}\right)^2 dr^2 + R^2 d\Omega_5^2, \tag{76}$$

where h is the black hole factor $1 - \left(\frac{r_h}{r}\right)^4$.

To do this calculation it will be necessary to pick out a direction in the S^5 to apply the twist to. A convenient way to do this is to parameterise the S^5 as a Hopf fibration over a P^2

$$ds^2_{S^5} = ds^2_{P^2} + (d\phi + A)^2. \tag{77}$$

Here $A = sin^2(\mu)\sigma_3$, and $d\phi + A$ is the vertical one form along the Hopf fibration.

The left invariant one forms of SU(3) can be parameterised as:

$$\sigma_1 = \frac{1}{2}(d\theta\cos(\psi) + d\rho\sin(\theta)\sin(\psi)), \tag{78}$$

$$\sigma_2 = \frac{1}{2}(d\theta\sin(\psi) - d\rho\cos(\psi)\sin(\theta)), \tag{79}$$

$$\sigma_3 = \frac{1}{2}(d\psi + d\rho\cos(\theta)). \tag{80}$$

In terms of these variables the $ds^2_{P^2}$ becomes

$$ds^2_{P^2} = d\mu^2 + sin^2(\mu)(\sigma_1^2 + \sigma_2^2 + cos^2(\mu)\sigma_3^2), \tag{81}$$

and the metric becomes

$$ds^2 = \left(\frac{r}{R}\right)^2(-hdt^2+dy^2+dx_2^2)+h^{-1}\left(\frac{R}{r}\right)^2dr^2+R^2(ds_{P2}^2+(d\phi+A)^2). \quad (82)$$

In what follows the radial, dx_2^2, and ds_{P2}^2 terms remain unchanged and will be dropped for brevity. The truncated metric we will use is

$$ds^2 = \left(\frac{r}{R}\right)^2(-hdt^2+dy^2)+R^2(d\phi+A)^2. \quad (83)$$

applying a boost in the y direction, $t \to ct - sy$, $y \to cy - st$ with $c = cosh\gamma$, and $c^2 - s^2 = 1$:

$$ds^2 = \left(\frac{r}{R}\right)^2(-dt^2(1-gc^2)+dy^2(1+gs^2)-2dtdy(gcs)+R^2(d\phi+A)^2. \quad (84)$$

Performing a T duality along the y direction involves making the following changes:

$$g'_{yy} = \frac{1}{g_{yy}} \quad g'_{ay} = \frac{B_{ay}}{g_{yy}} \quad g'_{ab} = g_{ab} - \frac{g_{ay}g_{yb}+B_{ay}B_{yb}}{g_{yy}} \quad (85)$$

$$\Phi' = \Phi - \frac{1}{2}\ln g_{yy} \quad B'_{ay} = \frac{g_{ay}}{g_{yy}} \quad B'_{ab} = B_{ab} - \frac{g_{ay}B_{yb}+B_{ay}g_{yb}}{g_{yy}} \quad (86)$$

This gives the new metric

$$ds^2 = -\left(\frac{r}{R}\right)^2\frac{h}{1+s^2g}dt^2+\left(\frac{R}{r}\right)^2\frac{1}{1+s^2g}dy^2+R^2(d\phi+A)^2. \quad (87)$$

A non zero dilaton and B field are also generated

$$B = dy \wedge dt\left[\frac{gcs}{1+gs^2}\right] \quad (88)$$

$$\Phi = \Phi_0 - \frac{1}{2}\ln\left[(1+gs^2)\left(\frac{r}{R}\right)^2\right]. \quad (89)$$

We can now twist this metric by shifting the one form $d\phi \to d\phi + \alpha dy$. The metric then becomes

$$
\begin{aligned}
ds^2 = &-\left(\frac{r}{R}\right)^2 \frac{h}{1+s^2 g} dt^2 + R^2(d\phi + A)^2 + \left(\frac{R}{r}\right)^2 \frac{1+r^2\alpha^2(1+gs^2)}{1+gs^2} dy^2 \\
&+ 2\alpha R^2 dy(d\phi + A).
\end{aligned}
\tag{90}
$$

The dilaton and B fields remain unchanged by this transformation. Note here that the parameter α has dimensions of inverse length. Performing the inverse T duality along dy gives the new fields

$$
\begin{aligned}
ds^2 = &-\left(\frac{r}{R}\right)^2 \frac{hr^2\alpha^2 + 1 - c^2 g}{1+r^2\alpha^2(1+gs^2)} dt^2 - \frac{2gcsr^2}{R^2(1+(1+gs^2)r^2\alpha^2)} dtdy \\
&+ \left(\frac{r}{R}\right)^2 \frac{1+gs^2}{1+r^2\alpha^2(1+gs^2)} dy^2 + R^2 \frac{1}{1+r^2\alpha^2(1+gs^2)} (d\phi + A)^2
\end{aligned}
\tag{91}
$$

$$
B = \frac{r^2\alpha}{1+(1+gs^2)r^2\alpha^2} (d\phi + A) \wedge (gcsdt + (1+gs^2)dy) \tag{92}
$$

$$
\Phi = \Phi_0 - \frac{1}{2} \ln(1 + (1+gs^2)r^2\alpha^2) \tag{93}
$$

Finally we need to boost back by $-\gamma$ to the original frame and take the limit $\alpha \to 0$, $c \to \infty$, with $\alpha c = b$, where b is a fixed finite constant with dimensions of inverse length. Most terms in the fields become zero in this limit and we are left with the string frame metric

$$
\begin{aligned}
ds^2 = &K^{-1}\left(\frac{r}{R}\right)^2 [-(1+b^2r^2)hdt^2 - 2b^2r^2hdtdy + (1-b^2r^2h)dy^2 + Kdx_2^2] \\
&+ \left(\frac{R}{r}\right)^2 h^{-1}dr^2 + K^{-1}R^2\eta^2 + R^2 ds^2_{P2}, \\
\Phi = &-1/2 \ln K, \\
B = &K^{-1}\left(\frac{r}{R}\right)^2 b(hdt + dy) \wedge \eta, \\
C_4 = &\left(\frac{r}{R}\right)^4 dt \wedge dx \wedge dy \wedge dz, \\
K = &1 - (h-1)b^2r^2.
\end{aligned}
\tag{94}
$$

In [44] it was shown that if we change co-ordinates to $t' = (y+t)/\sqrt{2}$ and $\xi = (y-t)\sqrt{2}$, take the near horizon limit, and compactify the S^5 the geometry (69) is recovered for the case $z = 2$.

The five form flux, C_4 is not affected by the twist $d\phi \to d\phi + \alpha dy$, because the T duality sends $d\Omega_5$ to $dy \wedge d\Omega_5$. The T dualities and boosts will therefore cancel each other out and leave us with the initial C_4. In the above η^2 is short for $(d\phi + A)^2$. Note that in the limit $b \to 0$ this metric reduces to the original metric we had before starting the Null Melvin twist.

Following [12] it is possible to introduce light cone co-ordinates of the form

$$x^+ = bR(t+y), \tag{95}$$

and

$$x^- = \frac{1}{2bR}(t-y). \tag{96}$$

The angular part of the metric is unchanged by this and will be dropped in what follows:

$$\begin{aligned} ds^2 =& K^{-1}\left(\frac{r}{R}\right)^2\left[-\left(\frac{h-1}{(2bR)^2}+\left(\frac{r}{R}\right)^2 h\right)dx^{+2} - (1+h)dx^+dx^- \right.\\ &\left. + (bR)^2(1-h)dx^{-2} + Kdx_2^2\right] + \left(\frac{R}{r}\right)^2 h^{-1}dr^2 \\ \Phi =& -\frac{1}{2}\ln K \\ B =& K^{-1}\left(\frac{r}{R}\right)^2 \frac{b}{2}\left(\frac{dx^+}{bR}(1+h) + 2bR(h-1)dx^-\right)\wedge\eta \\ C_4 =& \left(\frac{r}{R}\right)^4 dx^+\wedge dx^-\wedge dx\wedge dz. \end{aligned} \tag{97}$$

Note that this metric is independent of b in the extremal case, $h = 1$ suggesting that this is not a physical parameter in this case. It was suggested in [40] that x^- is a compact direction dual to the particle number in the corresponding field theory. This would give rise to a discreet mass spectrum, as expected in a non relativistic field theory. e.g. for fermions at unitarity the mass of any operator is an integer multiple of the mass of the elementary fermion. The x^+ co-ordinate can be interpreted as a time co-ordinate, except in the extremal case when the g_{--} component vanishes.

The compactification scale is not fixed by the Null Melvin Twist, there is therefor a second distance scale, M_-, the size of the compact dimension.

Because of the non zero g_{+-} co-ordinate this geometry is not a static one. This can be interpreted as a rotation in the compact dimension. It will be convenient to shift to the co-rotating frame by making the coordinate change $dx^- \rightarrow dx^- + vdx^+$ and requiring that the g_{+-} component vanishes at the horizon. For

$$v = \frac{1}{2(bR)^2}, \tag{98}$$

$$\begin{aligned}
ds^2 =& K^{-1}\left(\frac{r}{R}\right)^2\left[-h\left(\frac{1}{(bR)^2}+\left(\frac{r}{R}\right)^2\right)dx^{+2} - 2hdx^+dx^-\right.\\
&\left.+(bR)^2(1-h)dx^{-2}+Kdx_2^2\right]+\left(\frac{R}{r}\right)^2 h^{-1}dr^2,\\
\Phi =& -\frac{1}{2}\ln K,\\
B =& K^{-1}\left(\frac{r}{R}\right)^2\frac{b}{2}\left(\frac{h}{bR}dx^+ + 2bR(h-1)dx^-\right)\wedge\eta.\\
C_4 =& \left(\frac{r}{R}\right)^4 dx^+\wedge dx^-\wedge dx\wedge dz.
\end{aligned} \tag{99}$$

In [12] it was pointed out that the angular momentum and angular velocity are normally identified with the charge and conjugate chemical potential of the system. In this case however, things are slightly more subtle because normally the chemical potential is defined to be the difference between the angular velocities at the horizon and boundary. As the angular velocity diverges at the boundary for this metric they could not make this identification, however they were able to show that defining the chemical potential to be the angular velocity at the horizon yielded a consistent entropy.

The g_{--} component in the extremal case is degenerate, suggesting that g_{++} is not a good choice of time co-ordinate in this case. This pathology is not co-ordinate dependant, for instance it manifests as the g_{yy} component becoming timelike at significantly large r in (94). This anomaly was pointed out in [11] where the authors suggested this was evidence that this geometry was dual to a non relativistic field theory as they should allow action outside the light cone.

3.2 DCLQ'S

If a discreet light cone quantisation (DLCQ) procedure is applied to a relativistic field theory the result looks like a non relativistic field theory [52]. For example if you go over to light cone coordinates $p_{\pm} = E \mp p_1$ the mass shell condition becomes $p_+ = \frac{\vec{p}^2+m^2}{4p_-}$, if p_- is identified with the mass. Normally p_- would be a continuous variable, however if we compactify the x^- direction x^- $x^- + 2\pi r$ then $p-$ can only take the values $\frac{-N}{r}$, where N is a positive integer. This procedure is called discrete light cone quantisation. It was noted in [53] that this breaks the symmetry group to those that commute with p_-, if the original field theory is conformal then this is the Conformal Schrodinger group. When doing a DLCQ you need to be careful how you treat the zero modes [54] because one of the dimensions has been compactified they are effectively described by a field theory in $d-1$ dimensions. Proposals for applying this procedure to $N = 4$ SYM are contained in [55, 56]. The DLCQ of several other theories was obtained in [52].

The relationship of solutions like (94) to the initial fields is not obvious, however following [44, 47] it is possible to make the relationship clearer. If we consider a DCLQ along the $\xi = (y-t)/\sqrt{2}$ direction, and by requiring all fields to be invariant under translations along ξ

$$\Phi(\xi + L_\xi) = e^{L_\xi J_\xi}\Phi(\xi) = \Phi(\xi). \tag{100}$$

Here L_ξ and J_ξ are the momentum generators in the ξ direction. This will generate a solution corresponding to $\beta = 0$. In the third step of a Null Melvin Twist the symmetry generators were effectively re-diagonalise by mixing the dy and $d\phi$ terms. This has the effect of sending the y momentum, J_y to $J_y - \alpha J_\phi$. Boosting back to the original frame, and taking the limit $\alpha \to 0$, $c \to \infty$, $\alpha c = b$ has the effect of turning the momentum into $J_\xi - \beta J_\phi = J_\beta$.The Null Melvin twist therefore effectively eliminates the modes of the original solution without a fixed light cone energy and momentum. We therefore expect (94) to be dual to the DCLQ of $\mathcal{N} = 4$ SYM.

4 Wilson Loops in Non Relativistic AdS/CFT

4.1 Introduction to Wilson Loops

It is useful to calculate the expectation values of Wilson Loops because they contain gauge invariant information about the nonperturbative physics of non-abelian gauge theories. At high temperatures, where matter is in the quark-gluon plasma phase they can be related to several quantities measurable by heavy ion collision experiments such as the jet quenching parameter and the quark anti quark potential.

Wilson loops are defined as

$$W = \mathcal{P}\mathrm{Tr}\big(\exp[\mathrm{i}\int_{\mathrm{C}} \mathrm{dx}_\mu \mathrm{A}^\mu(\mathrm{x})]\big), \tag{101}$$

where the trace is over a $SU(N)$ matrix in either the fundamental or adjoint representations. The vector potential can be expressed in terms of the generators of the appropriate representation $A_a^\mu(x)T^a$. Here $\mathcal{P}$ indicates that this is a path ordered integral. For the quark anti quark potential this evaluates to

$$W = \exp[-\mathrm{i}\mathcal{T}(\mathrm{E(L)} - \mathrm{E_{ren}})], \tag{102}$$

where L is the inter quark distance, and E_{ren} is an infinite L independent regularisation. $E(L)$ is measured in the $\mathcal{T} \to \infty$ limit, where the end effects may be neglected. The standard relation between the Euclidean Wilson loop and the static potential is recovered if $i\mathcal{T}$ is analytically continued to $\mathcal{T}$ at zero temperature.[57]. When calculations are done in lattice QCD the inter quark potential is usually obtained from the correlation function of two Polyakov loops wrapped around an Euclidean periodic time coordinate [58, 59, 60, 61]. In these calculations the results were regularised by requiring that the zero temperature results match at short distances.

In some cases, such as in the high energy limit of scattering problems the Wilson Loop (101) becomes a real quantity. One such case occurs in the calculation of the jet quenching parameter. We will now give a brief overview of three cases where this parameter occurs in field theory. Other more detailed reviews of this material are contained in [62, 63, 64] .

4.2 Virtual Photoabsorption Cross Section

In deep inelastic scattering processes the virtual photon does not have sufficient time to split into several soft particles. The most important part of it's wave function that interacts with the target nucleus is the $q\bar{q}$ component[65, 66, 67]

$$|\gamma^* >= \int d^2(\mathbf{x}-\mathbf{y})dz\psi(\mathbf{x}-\mathbf{y},z)\frac{1}{\sqrt{N}}\delta_{\alpha\bar{\alpha}}|\alpha(\mathbf{x}),\bar{\alpha}(\mathbf{y}),z> . \qquad (103)$$

$|\alpha(\mathbf{x}),\bar{\alpha}(\mathbf{y}),z>$ is a $q\bar{q}$ state carrying an energy fraction z at transverse position $\mathbf{x}$. $\mathbf{y}$ is the anti quarks transverse position, and it carries an energy fraction $1-z$. From this state the photoabsorption cross section has beeb calculated [62] in the eikonal approximation by squaring the difference between the outgoing and incoming state

$$\sigma^{DIS} = \int d^2\mathbf{x}d^2\mathbf{y}dz\psi(\mathbf{x}-\mathbf{y},z)\psi^*(\mathbf{x}-\mathbf{y},\mathbf{z})P^{q\bar{q}}_{tot}(\mathbf{x},\mathbf{y}), \qquad (104)$$

$$P^{q\bar{q}}_{tot} = \left\langle 2 - \frac{1}{N}\mathrm{Tr}[\mathrm{W^F}(\mathbf{x})\mathrm{W^{F\dagger}}(\mathbf{y})] - \frac{1}{\mathrm{N}}\mathrm{Tr}[\mathrm{W^F}(\mathbf{y})\mathrm{W^{F\dagger}}(\mathbf{x})]\right\rangle. \qquad (105)$$

The cross section was then be expressed in terms of fundamental Wilson loops of the form

$$\frac{1}{N} < \mathrm{Tr}[\mathrm{W^{F\dagger}}(\mathbf{y})\mathrm{W^F}(\mathbf{x})] > \longrightarrow < \mathrm{W^F}(\mathcal{C}_{\mathrm{lightlike}}) >= \exp\left[-\frac{1}{8}\mathrm{Q_s^2L^2}\right] + \mathcal{O}\left(\frac{1}{\mathrm{N^2}}\right). \qquad (106)$$

The arrow indicates that two light like Wilson loops have been connected by two transverse lines located a long way from each other. The resultant rectangular Wilson loop gives us a gauge invariant formulation. Q_s is called the saturation scale and it is a characteristic property of any hadronic target. Roughly this means the photons probability of interacting with a target is almost one if it's transverse size, $L = |\mathbf{x}-\mathbf{y}| > 1/Q_s$ and is negligible otherwise.

It is important to note that the exponential in (106) must have a real component because in the large L limit the interaction probability ought to approach one, this requires that $P^{q\bar{q}}_{tot} \rightarrow 2$ because in $P^{q\bar{q}}_{tot}$(105) is the sum of both the elastic and inelastic scattering probabilities which are each normalised to one. This is impossible if the exponent in (106) is imaginary.

4.3 The Cronin Effect

When the results of proton-proton and proton-nucleus collision are analysed it is found that the transverse momentum is enhanced e.g see[68, 69, 70]. This is called the Cronin effect and is interpreted as being caused by the broadening of the momentum of the partons, before a high p_T parton is produced in a hard interaction. This process has been studied in [71, 72, 73, 74, 75] by calculating the gluon radiation induced by a quark interacting with a nucleus. In the eikonal approximation

$$\Phi^{\alpha}_{out} = W^F_{\alpha\gamma}(\mathbf{0})|\gamma > + \int d\mathbf{x} f(\mathbf{x}) T^b_{\alpha\beta} W^F_{\beta\gamma}(\mathbf{0}) W^A_{bc}(\mathbf{x})|\gamma; c(\mathbf{x}) > . \tag{107}$$

Here $\mathbf{\Phi^{\alpha}_{in}}$ differs from $\mathbf{\Phi^{\alpha}_{out}}$ by a colour rotation with the Wilson lines, $W^F_{\alpha\beta}$ for the quarks, and W^A_{bc} for the gluons. $\mathbf{x}$ and $\mathbf{0}$ represent the transverse positions of the gluon and quark. Greek letters represent the fundamental indices, Latin letters are the adjoint indices.[62]. In order to calculate a quantity that can be measured experimentally such as the gluon production rate $dN/d\mathbf{k}$ we first identify the subspace orthogonal to the incoming state $|\delta\Phi >= [1 - |\Phi_{in} >< \Phi_i n|]|\Phi_{out} >$ and calculate the gluons in this state [63, 76]

$$\begin{aligned} \frac{dN}{d\mathbf{k}} =& \frac{1}{N}\sum_{\alpha,d} < \delta\Phi_\alpha | a^\dagger_d(\mathbf{k}) a_d(\mathbf{k}) | \delta\Phi_\alpha \\ =& \frac{\alpha_s C_F}{2\pi} \int d\mathbf{x} d\mathbf{y} e^{i\mathbf{k}.(\mathbf{x}-\mathbf{y})} \frac{\mathbf{x}.\mathbf{y}}{\mathbf{x^2 y^2}} \frac{1}{N^2-1} \Bigg[< Tr[W^{A\dagger}(\mathbf{y}) W^A \mathbf{0}] > \\ & - < Tr[W^{A\dagger}(\mathbf{x}) W^A(\mathbf{0})] > - < Tr[W^{A\dagger}(\mathbf{y}) W^A(\mathbf{0})] > \\ & + < Tr[W^{A\dagger}(\mathbf{y}) W^A(\mathbf{x})] > \Bigg] . \end{aligned} \tag{108}$$

Where $\mathbf{x}$ and $\mathbf{y}$ are the gluons transverse positions. The only relevant information encoded in (108) is in the transverse dependence of two light like Wilson lines which are closed to form a loop

$$\frac{1}{N^2-1} < Tr[W^{A\dagger}(\mathbf{y}) W^(\mathbf{x})] > \rightarrow < W^A(\mathrm{C_{light}}) >= \exp\left[-\frac{1}{4}\mathrm{Q_s^2 L^2}\right] + \mathrm{O}\left(\frac{1}{\mathrm{N^2}}\right). \tag{109}$$

Note that the factor of two between (106) and (109) is consistent with the expected difference between fundamental and adjoint Wilson lines from the identity $\mathrm{TrW^A = TrW^F TrW^F} - 1$.

If we put this into (108) we find

$$\frac{dN}{d\mathbf{k}} = \frac{4\pi}{Q_s^2}\int d\mathbf{q}\exp\left[-\frac{\mathbf{q^2}}{Q_s^2}\right]\frac{\mathbf{q^2}}{\mathbf{k^2}(\mathbf{q}-\mathbf{k})^2}. \tag{110}$$

The above calculation was done without properly regulating the divergent integrals. A better parameterisations of the saturation scales, Q_s would include a logarithmic dependence on the transverse separation, L which would allow a proper regularisation to be performed [72, 76]. This is more complicated but in the final result Q_s^2 still governs the transverse momentum transferred from the medium to the parton. Q_s^2 depends on the target size, $Q_s^2 \propto A^{\frac{1}{3}}$, where A is the target size. It is convenient to separate this dependence on the path length [77] by defining

$$Q_s^2 = \hat{q}\frac{L^-}{\sqrt{2}}. \tag{111}$$

L^- is the light cone distance which is related to the longitudinal distance, z by $z = \frac{L^-}{\sqrt{2}}$ $\hat{q}$ is the jet quenching parameter which is a measure of the average transverse momentum transferred to the parton per unit length. Unlike L^- we see that the jet quenching parameter is still well defined even in the $L^- \to \infty$ limit. It was pointed out in [64] that the ratio of the jet quenching parameters of two conformal theories was equal to the square route of their central charges, provided that they have a gravity dual.

4.4 BDMPS Energy Loss and the Jet Quenching parameter

Highly energetic partons produced in hard scattering experiments split up into multiple partons before undergoing hadronization. This shower interferes with the radiation produced by interactions with the medium which can resolve longitudinal distances in the target[78, 79, 80].

In the limit

$$E >> \omega >> |\mathbf{k}|, |q| = |\sum \mathbf{q_i}| >> T, \Lambda_{QCD} \tag{112}$$

the amplitude for a quark splitting into a quark and a gluon and a gluon splitting into two gluons was calculated in the Baier-Dokshitzer-Mueller-Peigne-Schiff calculation [78]. Here ω is the energy of the radiated gluon, $\mathbf{k}$ is the transverse momentum of the gluon, and $\mathbf{q}$ is the total transverse momentum acquired by scattering.

Alternatively in the BDMPS formalism $1/E$ corrections are kept to allow us to calculate interference effects. In order to do this Wilson lines are replaced by retarded Green's functions[79, 80, 81] of the form

$$
\begin{aligned}
G(x_2^-, \mathbf{r_2}, \mathbf{r_1}|p) = \int_{\mathbf{r}(x_1^-)=\mathbf{r_1}}^{\mathbf{r}(x_2^-)=\mathbf{r_2}} \mathcal{D}\mathbf{r}(x^-) \exp\Bigg[\frac{ip}{2} \int_{x_1^-}^{x_2^-} dx^- \left(\frac{d\mathbf{r}(x^-)^2}{dx^-} \right)^2 \\
- i \int_{x_1^-}^{x_2^-} dx^- dx^- A^+(x^-, \mathbf{r}(x)) \Bigg].
\end{aligned}
\tag{113}
$$

Here p is the momentum of the gluon and A^+ is the colour field, and the integration is done over all possible paths within the light cone. In the limit of an infinite boost this equation reduces to the Wilson loop of the form(101).

In the BDMPS formalism the energy distribution of the gluonic radiation emitted by a parton inside a medium may be expressed as the expectation values of pairs of green's functions of the form (113). After a highly technical calculation they may be written in the form [80]

$$
\begin{aligned}
\omega \frac{dI}{d\omega d\mathbf{k}} = & \frac{\alpha_s C_R}{(2\pi\omega)^2} 2Re \int_{\xi_0}^{\infty} dy_l \int_{y_l}^{\infty} d\bar{y}_l \int d\mathbf{u} e^{-i\mathbf{k}.\mathbf{u}} exp\Big[-\frac{1}{4} \int_{\bar{y}_l}^{\infty} d\xi \hat{q}(\xi) \mathbf{u^2} \Big] \\
& \times \frac{\partial}{\partial \mathbf{y}} . \frac{\partial}{\partial \mathbf{u}} \int_{y_l}^{\mathbf{u}=\mathbf{r}(\bar{y}_l)} d\mathcal{D}\mathbf{r} \exp\left[\int_{y_l}^{\bar{y}_l} d\xi \Big(\frac{i\omega}{2} \dot{\mathbf{r}}^2 - \frac{1}{4} \hat{q}(\xi) \mathbf{r^2} \Big) \right].
\end{aligned}
\tag{114}
$$

C_R is the Casimir operator in the representation of the parton, ξ_0 is the position where the parton is created, y_l and $\bar{y}_l$ are the longitudinal positions at which where a gluon is emitted in the amplitude and complex conjugate amplitude. See [63, 80] for more details. $\hat{q}(\xi)$ is the value of the jet quenching parameter at longitudinal position ξ. In our case of a static plasma this is just a constant. In [63] the authors were able to express this in terms of the jet quenching parameter defined in (111) by using similar identities to (110) where the Wilson loop is now evaluated in a hot plasma instead of a cold nucleus:

$$
< W^A(\mathcal{C}_{light}) >= \exp\left[-\frac{1}{4\sqrt{2}} \hat{q} L^- L^2 \right] + \mathcal{O}\Big(\frac{1}{N^2} \Big) \tag{115}
$$

In the limit of large L^- (114) becomes [78, 82]

$$
\omega \frac{dI}{d\omega} = \frac{\alpha_s C_R}{\pi} 2\Re \ln\left[\cos\left((1+i) \sqrt{\frac{\hat{q} L^{-2}}{8\omega}} \right) \right] \tag{116}
$$

This can then be integrated to give

$$\Delta E = \frac{1}{4}\alpha_s C_R \hat{q}\frac{L^{-2}}{2}. \tag{117}$$

This tells us that the medium induced energy loss does not depend on E, but scales with the square of the path length. As the energy loss due to collisions is proportional to the path length this tells us that radiative energy loss is most important in the high energy limit. The BDMPS procedure is well defined, however the calculation of $\hat{q}$ can only be done perturbatively in field theories if T is big enough for the physics to be perturbative at that scale. This makes it worthwhile to look for a method that can evaluate this quantity at strong coupling. For field theories with a gravity dual one such method involves doing a calculation in string theory and relating it to $\hat{q}$ via the AdS/CFT correspondence.

4.5 Calculation of Wilson Loops in AdS/CFT via string theory

In string theory Wilson loops are evaluated by finding the minimal area of a string world sheet which intersects the boundary in a contour of the same shape as the contour used to evaluate the integral in the field theory[83, 84, 85, 86, 87]. The large N, strong coupling limit(i.e taking N to be large at fixed λ, and then taking $\lambda \to \infty$) of $N = 4$ SYM corresponds to taking the limit $g_s \to 0$ and $\alpha' \to 0$ in the dual string theory. String world sheet fluctuations can be neglected in this limit as the string tension is infinite. Interactions can also be neglected as the coupling constant is zero, so the problem reduces to finding the classical minimal surface bound by a contour.

This is done by introducing a probe $D3$ brane at the boundary to allow for strings to hang down from it. The Nambu-Goto action for the string world sheet is

$$S = -\frac{1}{2\pi\alpha'}\int d\sigma d\tau\sqrt{-det g_{\alpha\beta}}, \tag{118}$$

where

$$g_{\alpha\beta} = G_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu. \tag{119}$$

$G_{\mu\nu}$ is the normal space time metric and $g_{\alpha\beta}$ is the induced metric on the string world sheet. The integral is over the string world sheet coordinates. The location of the string world sheet is then parameterised by

$$x^\mu = x^\mu(\tau, \sigma)\ , \tag{120}$$

where τ and σ are the string world sheet coordinates.

4.6 Calculation of Wilson Loops in a Background Dual to a Non relativistic Field Theory

We have previously shown that a background dual to a non relativistic field theory is:

$$\begin{aligned} ds^2 =& K^{-1}\left(\frac{r}{R}\right)^2[-(1+b^2r^2)hdt^2 - 2b^2r^2hdtdy + (1-b^2r^2h)dy^2 + Kdx_2^2] \\ &+ \left(\frac{R}{r}\right)^2 h^{-1}dr^2 + K^{-1}R^2\eta^2 + R^2ds_{P2}^2. \end{aligned} \tag{121}$$

Note that in these co-ordinates the system is already in the co-rotating frame. It can be seen from (121) that a static Wilson loop with a boundary in the $x-z$ plane will be the same as the one calculated in the original theory.

We will now proceed to calculate the Wilson Loop corresponding to a quark anti quark pair moving through a plasma whilst separated by a distance L with this background; for the conventional $AdS_5 \times S^5$ black hole this calculation was done in [64]. The appropriate contour is an infinitely long rectangle in the temporal direction, with ends small compared to the length, so edge effects can be neglected. To do this it will be convenient to boost to the rest frame of the quarks. Without loss of generality we can assume that they are moving in the z direction. In this frame (121) becomes

$$\begin{aligned} ds^2 =& K^{-1}\left(\frac{r}{R}\right)^2\Bigg[-\left(c^2(1+b^2r^2)h - s^2K\right)dt^{+2} - 2b^2r^2chdtdy \\ &+ (1-b^2r^2h)dy^2 + 2cs((1+b^2r^2)h - K)dtdz + 2b^2r^2shdydz \\ &+ (Kc^2 - s^2(1+b^2r^2)h)dz^2 + Kdx^2\Bigg] + \left(\frac{R}{r}\right)^2 h^{-1}dr^2 \\ &+ K^{-1}R^2\eta^2 + R^2ds_{P2}^2, \end{aligned} \tag{122}$$

where c is the usual lorentz boost factor.

The $B_{\mu\nu}$ field can be neglected because it's contribution to the action is of the form

$$S = \int d\tau d\sigma B_{\mu\nu}\left(\frac{\partial X^{\mu}}{\partial \tau}\frac{\partial X^{\nu}}{\partial \sigma} - \frac{\partial X^{\nu}}{\partial \tau}\frac{\partial X^{\mu}}{\partial \sigma}\right). \tag{123}$$

As one part of the $B_{\mu\nu}$ field is always directed into the S^5, and the string will only exist at a point in the S^5 one of these derivatives is always zero. The five form flux does not appear in the action and can therefore also be ignored. The S^5 will play no part in the following calculations, so we will drop it for brevity.

For simplicity we will first consider case where the quarks are separated in a direction orthogonal to the flow of the plasma. We can choose to parameterise our coordinates by

$$\tau = t, \tag{124}$$

$$\sigma = x \in [-\frac{L}{2}, \frac{L}{2}]. \tag{125}$$

The z coordinate, and the coordinates in the S^5 can be set to zero without loss of generality, and by symmetry

$$r = r(\sigma). \tag{126}$$

This yields the action

$$S = -\frac{\mathcal{T}}{2\pi\alpha'}\int_{-\frac{L}{2}}^{\frac{L}{2}} d\sigma \sqrt{\left[-g_{tt}\left(g_{xx} + g_{rr}\left(\frac{dr}{d\sigma}\right)^2\right)\right]} \tag{127}$$

where $\mathcal{T}$ is the infinite length of the rectangle in the time direction. As the Lagrangian in (127) has no explicit σ dependence, the Hamiltonian

$$\mathcal{H} = \mathcal{L} - r'\frac{\partial \mathcal{L}}{\partial r'} = q \tag{128}$$

is conserved. Here q is the constant of integration, which is related to the separation between the quarks. Using this equation we find

$$\mathcal{L} = -\frac{g_{tt}g_{xx}}{q}\ . \tag{129}$$

It is then possible to rewrite (127) as

$$S = -\frac{\mathcal{T}}{2\pi\alpha'}\int_{-\frac{L}{2}}^{\frac{L}{2}} d\sigma \frac{-g_{tt}g_{xx}}{q}\ . \tag{130}$$

In order to evaluate this action it will be convenient to switch to the dimensionless variables

$$r = r_h u, \quad b = \frac{\beta}{r_h}. \tag{131}$$

In terms of the new variables $K = 1 + \frac{\beta^2}{u^2}$ and $h = 1 - \frac{1}{u^4}$. The metric becomes

$$\begin{aligned} ds^2 =& K^{-1}\left(\frac{ur_h}{R}\right)^2\Big[-\Big(c^2(1+\beta^2u^2)h - s^2K\Big)dt^2 - 2\beta^2u^2chdtdy \\ &+ (1-\beta^2u^2h)dy^2 + 2cs((1+\beta^2u^2)h - K)dtdz + 2\beta^2u^2shdydz \\ &+ (Kc^2 - s^2(1+\beta^2u^2)h)dz^2 + Kdx^2\Big] + \left(\frac{R}{u}\right)^2 h^{-1}du^2 \\ &+ K^{-1}R^2\eta^2 + R^2 ds^2_{P^2}, \end{aligned} \tag{132}$$

and the action

$$S = -\frac{\mathcal{T}}{\pi\alpha'}\int_0^{\frac{L}{2}}\frac{d\sigma}{q}\left(\frac{ur_h}{R}\right)^4\Big(c^2(1+\beta^2u^2)hK^{-1} - s^2\Big), \tag{133}$$

where we have taken advantage of the fact that $r = r(\sigma)$ is an even function to rewrite the integral as one over the range $0 - \frac{L}{2}$. Using (129) we find

$$\frac{du}{d\sigma} = \frac{\sqrt{g_{xx}(-g_{tt}g_{xx} - q^2)}}{q\sqrt{g_{uu}}}, \tag{134}$$

and

$$\begin{aligned} S &= -\frac{\mathcal{T}T\sqrt{\lambda}}{r_h}\int_{u_{min}}^{\Lambda} du \frac{\sqrt{-g_{tt}g_{uu}}}{\sqrt{1 + \frac{q^2}{g_{tt}g_{xx}}}} \\ &= -\mathcal{T}T\sqrt{\lambda}\int_{u_{min}}^{\Lambda} du \sqrt{\frac{K^{-1}h^{-1}(c^2(1+\beta^2u^2)h - s^2K)}{1 - \frac{\tilde{q}^2Kh}{(c^2(1+\beta^2u^2)h - s^2K)}}}. \end{aligned} \tag{135}$$

Here we have rewritten α' in terms of the field theory quantity,

$$\sqrt{\lambda} = \frac{R^2}{\alpha'}. \tag{136}$$

T is the Hawking temperature of the black hole,

$$T = \frac{r_h}{\pi R^2}. \tag{137}$$

When the metric is written in this form it is manifest that the action will only depend on the temperature and not either r_h or R separately. To do this we needed to fold a factor of R^4 in to the integration constant,

$$\tilde{q} = q\frac{R^2}{r_h^2}. \tag{138}$$

This action is too complicated to evaluate analytically so we will have to resort to numerical integration. In order to do this we will need to calculate q^2 as a function of u_{min} by using (134), and requiring that the derivative vanishes. This gives the result,

$$q^2 = -g_{tt}^0 g_{xx}^0, \tag{139}$$

where by g_{tt}^0 and g_{xx}^0 we mean the fields evaluated at $u = u_{min}$. In this way we can think of u_{min} as an alternative integration constant to q.

We can then determine the distance separating the quark anti quark pair corresponding to this value of u_{min} using the equation

$$\frac{L}{2} = \int_{u_{min}}^{\Lambda} d\sigma. \tag{140}$$

Using (134) this becomes

$$L = 2\sqrt{-g_{tt}^0 g_{xx}^0} \int_{u_{min}}^{\Lambda} du \frac{\sqrt{g_{uu}}}{\sqrt{g_{xx}(-g_{tt}g_{xx} + g_{tt}^0 g_{xx}^0)}}, \tag{141}$$

and the action is

$$S = -\frac{\mathcal{T}T\sqrt{\lambda}}{r_h} \int_{u_{min}}^{\Lambda} du \frac{\sqrt{-g_{tt}g_{uu}}}{\sqrt{1 - \frac{g_{tt}^0 g_{xx}^0}{g_{tt}g_{xx}}}}. \tag{142}$$

Because $g_{tt}^0 g_{xx}^0$ is always greater than or equal to $g_{tt}g_{xx}$ over the integration range the action is guaranteed to be real and the string world sheet is timelike.

There is a singularity in these integrals at $u = u_{min}$, but because it is of the form $(u - u_{min})^{-0.5}$, L and the action remain finite. For any given value of c there is a maximum value of L for which this equation has a solution, which can be interpreted as the point where a bound quark anti quark pair can no longer exist under any circumstances and the only available state is that of two separate strings stretching from the boundary to the horizon. On the field theory side it is interpretation as the screening length. In [64, 88] the authors found that in AdS_5 L scaled as

$$L_{max}(v) = \frac{L_{max}(0)}{\sqrt{\gamma}}, \tag{143}$$

in the limit $\gamma \to \infty$. In our case this limit was not accessible analytically, but extensive numerical calculations indicated that it approached a constant in this case. For $\beta = 1$, $L_{max} = 0.448$. If β is also sent to infinity, $L_{max} \to 0.416$. As β approaches zero this constant term also approaches zero and the leading term is then given by (143).

It can be seen from (135) that the action has a quadratic and logarithmic divergence in u as $r \to \infty$ of the form

$$-\frac{\mathcal{T}T}{2}\sqrt{\lambda}\left(\beta c\Lambda^2 + \frac{(1-c^2\beta^4)\log\Lambda}{\beta c}\right). \tag{144}$$

To obtain a finite answer from the numerical integral it will be necessary to regulate this integral.

It was argued in [89], that because the strings satisfy Neumann boundary conditions in the radial co-ordinate and the S^5, the correct action was the Legendre transformation of the action we have been considering. We can accomplish this by adding terms to the action of the form

$$-\int d\tau P_i Y^i \tag{145}$$

where Y^i are the co-ordinates in which the string has Neumann boundary conditions and

$$P_i = \left.\frac{\delta\mathcal{L}}{\delta\partial_\sigma Y^i}\right|_\Lambda. \tag{146}$$

In our case the only relevant term is

$$r\left.\frac{\delta\mathcal{L}}{\delta\partial_\sigma r}\right|_\Lambda, \tag{147}$$

which evaluates to

$$\frac{\mathcal{T}T}{2}\sqrt{\lambda}\left(\beta c\Lambda^2 + \frac{1-c^2\beta^4}{2\beta c} + O(\Lambda^{-2})\right). \tag{148}$$

Whilst this will always cancel out the quadratically divergent term in the action, in our case there is also a logarithmic divergence which did not occur in the cases considered in [89]. This procedure will never ba able to remove a logarithmic divergence because the action and hence $\partial_\sigma r$ can be expressed as polynomials in Λ at large r and differentiating this cannot give rise to a logarithm. Interestingly the coefficient of the logarithmic divergence is precisely half of the constant term in (148). In [64, 90, 91, 92] an alternative regularisation procedure was used which consisted of subtracting the action of two separate strings stretching from the boundary to a probe $D3$ brane located at u_0, the root of the equation $g_{tt} = 0$. In the case we are considering this occurs when the integration constant, $q = 0$. This gives a corrected action of the form

$$S = \frac{\mathcal{T}T\sqrt{\lambda}}{r_h} \int_{u_{min}}^{\Lambda} du \sqrt{-g_{tt}g_{uu}} \left(1 - \frac{1}{\sqrt{1 - \frac{g_{tt}^0 g_{xx}^0}{g_{tt}g_{xx}}}} \right) + \frac{\mathcal{T}T\sqrt{\lambda}}{r_h} \int_{u_0}^{u_{min}} du \sqrt{-g_{tt}g_{uu}} \tag{149}$$

When we apply this regularisation procedure to the action the results are finite as $\Lambda \to \infty$. It was found in [64] that after this regularisation procedure had been applied that the small distance behaviour of the inter quark potential is independent of velocity in normal AdS_5 space. On the field theory side this is effectively requiring that the potential is independent of medium at short distances, which is a method often used in Lattice Gauge Theory calculations [59]. A few of our results are shown in figures (4), (5) and (6) below.

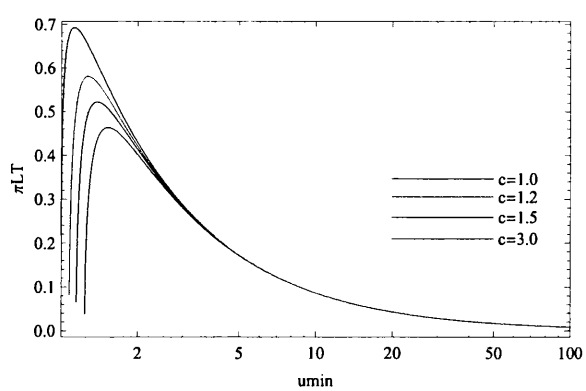

Figure 4: Graph of L v u_{min} for $\beta = 1$

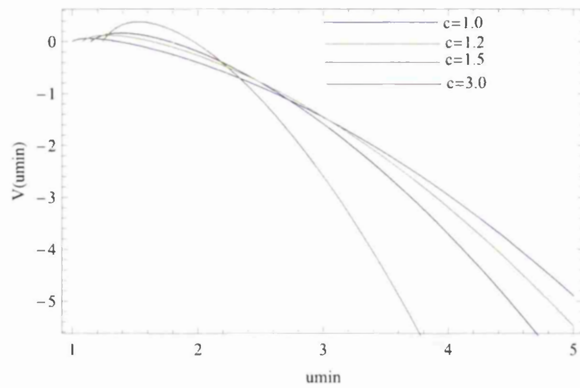

Figure 5: Graph of $V(u_{min})$ v u_{min} for $\beta = 1$

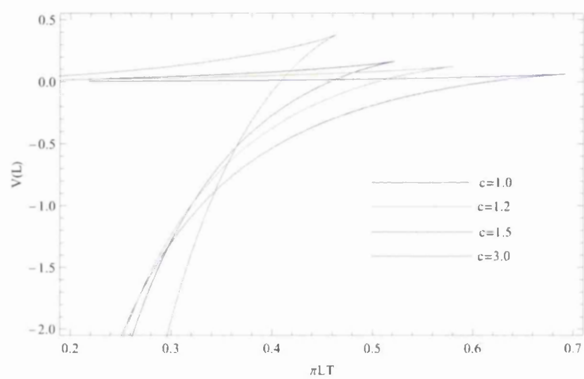

Figure 6: Graph of $V(L)$ v L for $\beta = 1$

A positive V indicates that the configuration has more energy than two isolated strings stretching from the boundary to u_0. It can be seen from Fig (6) that for all c there is a critical value of L, which decreases with increasing c for which V does become positive for this configuration. Beyond this point we would therefore expect any string to decay into two isolated strings.

It can be seen from figure (4) that for a given L there are two values of u_{min}, and hence q associated with it. The one with the minimal energy is the one with the largest value of u_{min}. The other solution is presumably an unstable configuration which will either decay to the solution with largest u_{min}, or to two isolated strings.

Note that at large u_{min}, which is also at small L, L's dependence on velocity approaches zero for the upper branch. This is not however true of the action. This contrasts with the results of [64] where these Wilson loops were calculated in normal AdS space and it was found that the short distance potential was independent of velocity at small L for the upper branch.

It was noted in [92] that the subtraction was unique when the string was stationary in AdS_5, corresponding to two stationary straight strings stretching from the boundary to u_0. However for moving strings there were an infinite number of possible subtraction terms corresponding to string configurations which stretched from the boundary to u_0, satisfied the equations of motion, and would yield a finite action. In [64] the authors subtracted the action of two stationary strings of the form

$$S = \sqrt{\lambda} \mathcal{T} T \int_1^\Lambda d\sigma. \tag{150}$$

This is a velocity independent subtraction, and can be derived from the co-ordinate choice

$$t = \tau \quad r = \sigma \quad x = z = 0. \tag{151}$$

This yielded an action whose small L dependence was independent of velocity, and hence medium independent, which is the regularisation procedure used in lattice calculations[59]. We have used a different velocity dependent subtraction. This was necessary because the divergent terms in the action depended on the velocity. In [64] the authors noted that if a similar velocity dependent subtraction was considered in AdS_5 space then the small L behaviour of the potential for the upper branch is no longer velocity independent, however for the lower branch is, $V(L) \to 0$ as $L \to 0$. We can see from Fig (6) that we have a similar behaviour in our case.

4.7 The Light Like Limit

Previously we have taken the limit $\Lambda \to \infty$ whilst c remains finite. However some quantities such as the jet quenching parameter are calculated in string theory by taking the $c \to \infty$, whilst the probe D3 brane stays fixed, and only then taking the limit $\Lambda \to \infty$ [64]. As can be seen from (134) if this limit was consistent with a quark anti quark pair coming in from the boundary, reaching some minimal value of u, u_{min} in the bulk and then returning the minimal point would occur when u_{min} satisfied

$$q^2 = K^{-1}(u_{min})\left(\frac{u_{min} r_h}{R}\right)^4 \left(c^2(1+\beta^2 u_{min}^2)h(u_{min}) - s^2 K(u_{min})\right). \tag{152}$$

However this is only consistent with a finite integration constant if $u_{min} \to 0$, or if it approaches the positive root of the equation

$$c^2(1+\beta^2 u_{min}^2)h(u_{min}) - s^2 K(u_{min}) = 0, \tag{153}$$

which to leading order in c becomes

$$\beta^2 u_{min}^6 - 2\beta^2 u_{min}^2 - 1 = 0. \tag{154}$$

The only other positive root of (134) occurs at $u = 1$, however as the Lagrangian becomes imaginary when u is below the root of (153) this is inconsistent with the equation of motion (129), except in a special limit we will discuss later. This means the string must descend all the way to the root of (153), before heading out again. This contrasts with the behaviour found in [64], where in AdS_5 the only valid solution was a string stretching to the horizon and then returning to the boundary. The relationship between the integration constant can then be determined from

$$L = 2q \int_{u_{min}}^{\Lambda} du \frac{\sqrt{g_{uu}}}{\sqrt{g_{xx}(-g_{tt}g_{xx} - q^2)}}, \tag{155}$$

and the action is

$$S = -\frac{\mathcal{T}T\sqrt{\lambda}}{r_h} \int_{u_{min}}^{\Lambda} du \frac{\sqrt{-g_{tt}g_{uu}}}{\sqrt{1 + \frac{q^2}{g_{tt}g_{xx}}}}. \tag{156}$$

To extract the jet quenching parameter as in [64] we need to consider the small L(which is also the small q limit), we will therefore expand (155) about $q = 0$ to give:

$$L = \frac{2qR^4}{cr_h^3} \int_{u_\beta}^{\Lambda} du \frac{\sqrt{K}}{\sqrt{u^4-1}\sqrt{\beta^2u^6 - 2\beta^2u^2 - 1}}(1 + \mathcal{O}(q^2, c^{-2})), \tag{157}$$

in the limit $c \to \infty$. u_β is the solution to (154). Note that this expansion is not valid in the case $\beta = 0$.

This integral cannot be evaluated analytically but it can be evaluated numerically to give

$$L = \frac{2qR^4}{cr_h^3} A(\beta), \tag{158}$$

where A is a number whose β dependance is shown in Fig 7.

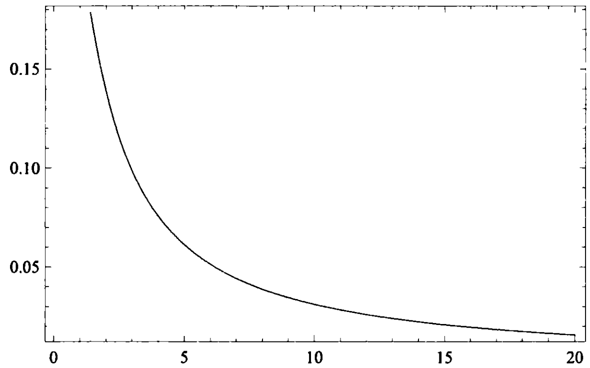

Figure 7: $A(\beta)$ as a function of β.

Note that $A(\beta)$ approaches zero as β approaches infinity.
If we look at the action in the same limit we get the result

$$S(q) = S_0 + q^2 S_1 + \mathcal{O}(q^4), \tag{159}$$

where,

$$S_0 = -\frac{\mathcal{T}T\sqrt{\lambda}}{r_h} \int_{u_\beta}^{\Lambda} du \sqrt{-g_{tt} g_{uu}}, \tag{160}$$

and

$$S_1 = -\frac{\mathcal{T}T\sqrt{\lambda}}{2r_h} \int_{u_\beta}^{\Lambda} du \frac{\sqrt{g_{uu}}}{g_{xx}\sqrt{-g_{tt}}}. \tag{161}$$

This action will also need regularising when we take the limit $\Lambda \to \infty$. In the previous section we did this by subtracting a term of the form (160) from the action, and we can see that this will again produce a finite answer as it simply removes S_0. To first order in c and q^2 the action then becomes

$$S = -\frac{\mathcal{T}T\tilde{q}^2\sqrt{\lambda}A(\beta)}{2c} \tag{162}$$

For the original metric, before the Null Melvin twist is applied it was found in [64] that the string world sheet became space like due to the g_{tt} component becoming positive in this limit, however in our case g_{tt} will remain negative due to the $-b^2c^2u^2$ term so unlike the case considered in [64] our action will remain real in this limit and the string world sheet remains timelike. This means that the exponential of the action, ,e^{iS} will be purely imaginary and as has been remarked after (106) it is necessary for it to have a real component in order to get sensible results from scattering experiments.

This suggests that for the background we are considering the $c \to \infty$, followed by $\Lambda \to \infty$ is not a good limit from which to extract the jet quenching parameter. If b was set equal to zero the problem term would disappear, however if we do this directly we recover the original AdS space. If however we take the limit $c \to \infty$, $\beta \to 0$ such that the product $c\beta = d$ remains constant the string world sheet will become imaginary. In this limit the metric becomes

$$\begin{aligned} ds^2 =& K^{-1}\left(\frac{ur_h}{R}\right)^2\Bigg[-\left((c^2+d^2u^2)h - s^2K\right)dt^2 - 2\frac{d^2}{c}u^2hdtdy \\ &+ (1-\frac{d^2}{c^2}u^2h)dy^2 + 2cs((1+\frac{d^2}{c^2}u^2)h-K)dtdz + 2\frac{d^2}{c^2}u^2shdydz \\ &+ (Kc^2 - s^2(1+\frac{d^2}{c^2}u^2)h)dz^2 + Kdx^2\Bigg] + \left(\frac{R}{u}\right)^2 h^{-1}du^2. \end{aligned} \tag{163}$$

The action is now space like in this limit,

$$S = -i\frac{\mathcal{T}T\sqrt{\lambda}}{r_h}\int_{u_{min}}^{\Lambda} du \frac{\sqrt{g_{tt}g_{uu}}}{\sqrt{1+\frac{q^2}{g_{tt}g_{xx}}}}. \tag{164}$$

This action can also be expanded in the form

$$S(q) = S_0 + q^2 S_1 + \mathcal{O}(q^4), \tag{165}$$

S_0 is removed once again by the regularisation procedure and we are left with

$$S_1 = \frac{i\mathcal{T}T\sqrt{\lambda}}{2r_h}\int_{u_{min}}^{\Lambda} du \frac{\sqrt{g_{uu}}}{g_{xx}\sqrt{g_{tt}}}, \tag{166}$$

where

$$u_{min} = \left(\frac{c}{d}\right)^{\frac{1}{3}} - \frac{1}{6d^{5/3}}\left(\frac{1}{c}\right)^{\frac{1}{3}} + \frac{(1+8d^4)}{24d^3}\frac{1}{c} - \frac{7+72d^4}{1296d^{13/3}}\left(\frac{1}{c}\right)^{\frac{5}{3}} + \mathcal{O}(c^{-7/3}). \tag{167}$$

As we are taking the limits $c \to \infty$, first followed by the $\Lambda \to \infty$, the lower limit of the integral in (166) will actually be larger than the upper one, this is clearly unphysical. This does however mean that a string can stretch from the boundary to the horizon before returning to the boundary in this limit. To leading order in c this gives us the same as the action in normal AdS_5 space, and yields the results [64]

$$q^2 S_1 = i\frac{\Gamma(\frac{3}{4})\sqrt{\lambda}\pi^{\frac{3}{2}}T^3}{8\Gamma(\frac{5}{4})}\mathcal{T}cL^2, \tag{168}$$

which lead to the identification of the jet quenching parameter as

$$\hat{q} = \frac{\pi^{\frac{3}{2}}\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})}\sqrt{\lambda}T^3. \tag{169}$$

One might think that the integral in (167) would make sense if we took Λ and c to infinity together in such a way that $c^{1/3}d^{-1/3} < \Lambda$. Whilst this is possible the integral is actually zero in this limit.

This means that the only limit in which a sensible non zero value can be obtained for the jet quenching parameter is in the limit where β is so small the theory is effectively the relativistic one we started with before applying the Null Melvin twist. If these results are interpreted via the standard AdS/CFT dictionary we are forced to conclude that the quark anti quark pair does not lose energy to the medium via jet quenching in the dual non relativistic field theory.

4.8 The Static Potential at arbitrary angles to the Plasma

We will now extend the previous calculation where the quark anti quark pair was orthogonal to the direction it was travelling in. The appropriate metric to use is still (132), however the string world sheet will now be parameterised as

$$t = \tau \quad z = \sigma; \quad x = x(\sigma) \quad u = u(\sigma). \tag{170}$$

If L is still defined to be the length of the quark anti quark pair, and the pair lie at an angle θ to the direction of the boost then σ will run between $\pm L/2\cos\theta$. We then have the boundary conditions

$$u\Big(\pm\frac{L}{2}\cos\theta\Big) = \Lambda, \tag{171}$$

and

$$x\left(\pm\frac{L}{2}\cos\theta\right)=\pm\frac{L}{2}\sin\theta. \tag{172}$$

As the boundary conditions for $u(\sigma)$are symmetric under $\sigma\rightarrow-\sigma$ it will be an even function which descends to some point, u_{min} at $\sigma=0$ before returning to the boundary. Similarly because $x'(\sigma)$ depends only on the even function $u(\sigma)$ and the boundary condition for $x(\sigma)$ is antisymmetric under $\sigma\rightarrow-\sigma$, $x(\sigma)$ will be an odd function. Note that this parameterisation is not a good one for $\theta=\pi/2$ because then all parts of the string would have the same σ coordinate, however as the Nambu Goto action is reparameterisation invariant, and the action depends smoothly on L, c, and θ we would expect to recover the results of the previous section in the limit $\theta\rightarrow\pi/2$. In terms of this parameteristaion the Nambu Goto action becomes

$$S=-\frac{\mathcal{T}}{2\pi\alpha'}\int_{-\frac{L\cos\theta}{2}}^{\frac{L\cos\theta}{2}}d\sigma\sqrt{-g_{tt}\left(g_{zz}+g_{uu}\left(\frac{\partial u}{\partial\sigma}\right)^2+g_{xx}\left(\frac{\partial x}{\partial\sigma}\right)^2\right)+g_{tz}^2} \tag{173}$$

As there is no explicit σ dependence in the action the Hamiltonian is once again conserved:

$$\mathcal{H}=\mathcal{L}-u'\frac{\partial\mathcal{L}}{\partial u'}-x'\frac{\partial\mathcal{L}}{\partial x'}=q. \tag{174}$$

The Lagrangian is also independent of x, therefore it's conjugate momentum

$$p=\frac{\partial\mathcal{L}}{\partial x'} \tag{175}$$

is also conserved. For a timelike string worldsheet and a real action both p and q must be positive. These yield the equations

$$\mathcal{L}=\frac{-}{q}[g_{tz}^2-g_{tt}g_{zz}]=\frac{-g_{tt}g_{xx}x'}{p}, \tag{176}$$

$$\left(\frac{\partial u}{\partial\sigma}\right)^2=\frac{1}{q^2(-g_{tt}g_{uu})}[-g_{tt}g_{zz}+g_{tz}^2][(-g_{tt}g_{zz}+g_{tz}^2)(1+\frac{p^2}{g_{tt}g_{xx}})-q^2], \tag{177}$$

and

$$\left(\frac{\partial x}{\partial\sigma}\right)^2=\frac{p^2}{q^2}\left(\frac{-g_{tt}g_{zz}+g_{tz}^2}{-g_{tt}g_{xx}}\right)^2. \tag{178}$$

This leads to the action

$$S = -\frac{\mathcal{T}T\sqrt{\lambda}}{r_h}\int_{u_{min}}^{\Lambda} du\, \frac{\sqrt{-g_{tt}g_{uu}}\sqrt{-g_{tt}g_{zz}+g_{tz}^2}}{\sqrt{(-g_{tt}g_{zz}+g_{tz}^2)(1+\frac{p^2}{g_{tt}g_{xx}})-q^2}} \tag{179}$$

The constants of integration p, q determine the angle θ and the length, L. They are given by the equations

$$\begin{aligned}\tan\theta &= \frac{\int_{u_{min}}^{\Lambda} du \frac{dx}{d\sigma}\frac{d\sigma}{du}}{\int_{u_{min}}^{\Lambda} du \frac{d\sigma}{du}}\\ &= \frac{p}{q}\frac{\int_{u_{min}}^{\Lambda} du \frac{-g_{tt}g_{zz}+g_{tz}^2}{-g_{tt}g_{xx}} \frac{\sqrt{-g_{tt}g_{uu}}}{\sqrt{[-g_{tt}g_{zz}+g_{tz}^2][(-g_{tt}g_{zz}+g_{tz}^2)(1+\frac{p^2}{g_{tt}g_{xx}})-q^2]}}}{\int_{u_{min}}^{\Lambda} du \frac{\sqrt{-g_{tt}g_{uu}}}{\sqrt{[-g_{tt}g_{zz}+g_{tz}^2][(-g_{tt}g_{zz}+g_{tz}^2)(1+\frac{p^2}{g_{tt}g_{xx}})-q^2]}}}\end{aligned} \tag{180}$$

and

$$\begin{aligned}L^2 &= 4\left[\left(\int_{u_{min}}^{\Lambda} du\frac{d\sigma}{du}\right)^2 + \left(\int_{u_{min}}^{\Lambda} du\frac{dx}{d\sigma}\frac{d\sigma}{du}\right)^2\right]\\ &= 4\left[\left(\int_{u_{min}}^{\Lambda} du \frac{q\sqrt{-g_{tt}g_{uu}}}{\sqrt{[-g_{tt}g_{zz}+g_{tz}^2][(-g_{tt}g_{zz}+g_{tz}^2)(1+\frac{p^2}{g_{tt}g_{xx}})-q^2]}}\right)^2\right.\\ &\quad + \left(\int_{u_{min}}^{\Lambda} du \frac{-g_{tt}g_{zz}+g_{tz}^2}{-g_{tt}g_{xx}} \times\right.\\ &\quad \left.\left.\frac{p\sqrt{-g_{tt}g_{uu}}}{\sqrt{[-g_{tt}g_{zz}+g_{tz}^2][(-g_{tt}g_{zz}+g_{tz}^2)(1+\frac{p^2}{g_{tt}g_{xx}})-q^2]}}\right)^2\right].\end{aligned} \tag{181}$$

Once again the action has a logarithmically divergent term as well as a quadratically divergent term, so imposing Neumann boundary conditions will not be an appropriate regularisation procedure, so once again we will subtract the action of two isolated strings travelling at $\theta = \pi/2$ stretching from the boundary to a probe D3 brane located at u_0, the root of $g_{tt} = 0$.

$$S = -\frac{1}{\pi\alpha'}\int_{u_{min}}^{\Lambda} du\,\sqrt{-g_{tt}g_{uu}}\left(\frac{\sqrt{-g_{tt}g_{zz}+g_{tz}^2}}{\sqrt{(-g_{tt}g_{zz}+g_{tz}^2)(1+\frac{p^2}{g_{tt}g_{xx}})-q^2}}-1\right)$$
$$+\frac{1}{\pi\alpha'}\int_{u_0}^{u_{min}} du\,\sqrt{-g_{tt}g_{uu}} \tag{182}$$

The results are once again finite in the $\Lambda \to \infty$ limit.

Using (177) we can eliminate the integration constant q by writing it in terms of p and u_{min},

$$q^2 = (-g_{tt}^0 g_{zz}^0 + (g_{tz}^0)^2)(1+\frac{p^2}{g_{tt}^0 g_{xx}^0}), \tag{183}$$

again the fields g_{tt}^0, g_{tz}^0, g_{zz}^0 and g_{xx}^0 are evaluated at $u = u_{min}$ here. Note that in (183), while $q^2 > 0$ for large u_{min}, at some critical value it will become negative. This means there is a minimum value of u_{min} below which no solutions of the form of a string descending to u_{min} and heading back out exist.

We can now calculate these quantities numerically and a few of our results are shown below.

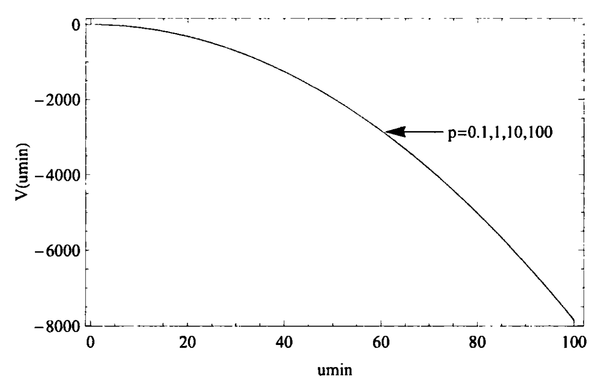

Figure 8: Graph of $V(u_{min})$ v u_{min} for $\beta = 1$, $c = 1$.

As can be seen from Fig (8) the potential is nearly independent of the integration constant p. We find a similar behaviour for all values of c. The only noticeable difference is that the smallest possible value of u_{min} decreases with increasing p, which is shown in Fig (9).

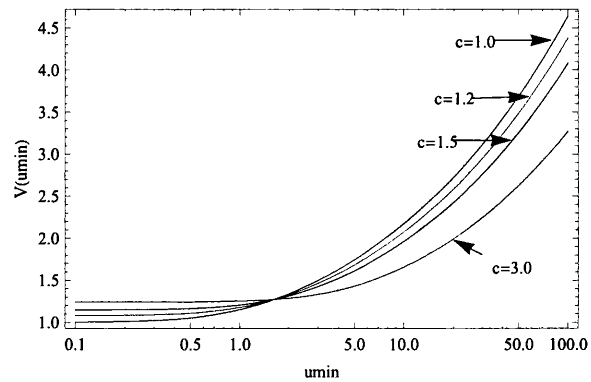

Figure 9: Graph of the minimum value of u_{min} against p.

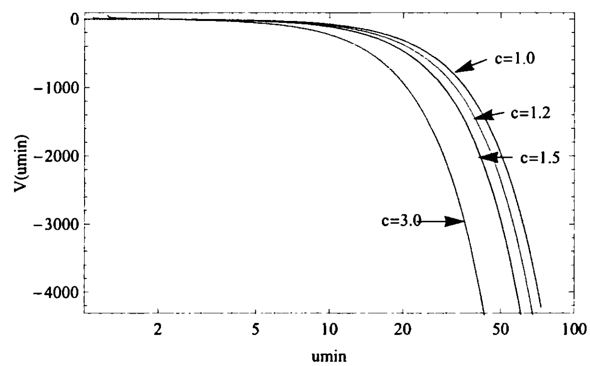

Figure 10: Graph of $V(u_{min})$ against u_{min} for $b = 1$.

For all c the minimal value of u_{min} approaches u_0 as p approaches zero, but grows without bound as $p \to \infty$. This means that unless p is small the unstable upper branch solution found when the quark anti quark pair was orthogonal to the medium no longer exists.

We can also see that are regularisation procedure has again yielded a potential $V(u_{min}) \to 0$ as u_{min}, $L \to 0$ for the unstable upper branch where it exists.

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