Existence of point processes through families of commuting Hermitian operators.

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Abstract

This dissertation is devoted to problems of existence and physical interpretation of some point processes. In the first part of the dissertation, we introduce the notion of the correlation measure of a family of commuting Hermitian operators. Let X be a locally compact, second countable Hausdorff topological space. We consider a family of commuting Hermitian operators $a(\Delta)$ indexed by all measurable, relatively compact sets Δ in X. For such a family, we introduce the notion of a correlation measure and prove that, if this correlation measure exists and satisfies some condition of growth, then there exists a point process over X having the same correlation measure (in the sense of the classical theory of point processes). Furthermore, the operators $a(\Delta)$ can be realised as multiplication operators in the L^2-space with respect to this point process. In particular, our result extends the criterion of existence of a point process from [6, 15], to the case of the topological space X, which is a standard underlying space in the theory of point processes. In the second part of the dissertation, we consider some important applications of our general results. We discuss particle densities of the quasi-free representation of the CAR and CCR, which lead to fermion (determinantal), and boson (permanental) point processes. We also discuss convolutions of these particle densities, which lead to point processes whose correlation functions are given through the Vere–Jones α-determinants.

DECLARATION

This work has not previously been accepted in substance for any degree and is not concurrently submitted in candidature for any degree.

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This dissertation is the result of my own independent work/investigation, except where otherwise stated. Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended.

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Contents

Chapter 1

Introduction

This dissertation is devoted to problems of existence and physical interpretation of some point processes.

First we choose an underlying space. Let X be a locally compact, second countable Hausdorff topological space. (Although in most applications one can choose X to be the Euclidean space $\mathbb{R}^d$, it is important to treat the general case of a topological space.) We denote by Γ_X the space of all locally finite sets (configurations) in X. One usually identifies a configuration with a Radon measure which has atoms at the points of the configuration. Using this identification, one defines the vague topology on Γ_X, and thus gets the Borel σ-algebra $\mathcal{B}(\Gamma_X)$. A point process in X is a probability measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$. From the point of view of classical statistical mechanics, point processes describe infinite (generally speaking, interacting) particle systems in continuum.

In the study of point processes, their correlation measures (respectively correlation functions) play a crucial role. Denote by $\Gamma_{X,0}$ the subset of Γ_X consisting of all finite configurations in X. Following Lenard [17] and Kondratiev and Kuna [15], we introduce the so-called $\mathcal{K}$-transform as follows:

Let $G : \Gamma_{X,0} \to \mathbb{R}, \quad G \geqslant 0$, then we set

$$(\mathcal{K}G)(\gamma) := \sum_{\eta \Subset \gamma} G(\eta),$$

where $\eta \Subset \gamma$ denotes that η is a finite subset of γ. Furthermore, for an arbitrary function $G : \Gamma_{X,0} \to \mathbb{R}$ we set

$$(\mathcal{K}G)(\gamma) := (\mathcal{K}G^{+})(\gamma) - (\mathcal{K}G^{-})(\gamma),$$

where $G^{+} := \max\{G, 0\}$, $G^{-} := \max\{-G, 0\}$, and at least one of the values $(\mathcal{K}G^{+})(\gamma)$ and $(\mathcal{K}G^{-})(\gamma)$ is finite. Thus, the $\mathcal{K}$-transform maps functions on the space of finite configurations, $\Gamma_{X,0}$, into functions on the space of (infinite) configurations, Γ_X. This map has many nice propertities, in particular, it maps measurable functions on $\Gamma_{X,0}$ into measurable functions on Γ_X. One says that a measure ρ on $\Gamma_{X,0}$ is the correlation measure of a point process μ if, for each $G : \Gamma_{X,0} \to \mathbb{R}, \quad G \geqslant 0$, measurable, we have

$$\int_{\Gamma_{X,0}} G(\eta)\rho(d\eta) = \int_{\Gamma_X} (\mathcal{K}G)(\gamma)\mu(d\gamma). \tag{1.1}$$

Equation (1.1) may be interpreted as

$$\rho = \mathcal{K}^{*}\mu.$$

Note that the space $\Gamma_{X,0}$ has a much simpler structure than Γ_X, since $\Gamma_{X,0}$ is, in fact, an (infinite) union of finite–dimensional spaces:

$$\Gamma_{X,0} = \bigsqcup_{n=0}^{\infty} \Gamma_X^{(n)},$$

where $\Gamma_X^{(n)}$ is the space of all n-point configurations in X. Therefore, a measure on $\Gamma_{X,0}$ is a much simpler object than a measure on Γ_X.

Let σ be a Radon, non-atomic measure on X. One can naturally define the measure $\sigma(dx_1)\cdots\sigma(dx_n)$ on $\Gamma_X^{(n)}$, and then one defines the Lebesgue—Poisson measure λ on $\Gamma_{X,0}$ (with intensity σ) as follows:

$$\lambda = \varepsilon_\varnothing + \sum_{n=1}^{\infty} \frac{1}{n!} \chi_{\Gamma_X^{(n)}} \sigma(dx_1)\cdots\sigma(dx_n).$$

Assume that a correlation measure ρ on $\Gamma_{X,0}$ has density κ with respect to the Lebesgue–Poisson measure λ. Then, $\kappa(\varnothing) = 1$ and the restriction of κ to $\bigsqcup_{n=1}^{\infty} \Gamma_X^{(n)}$ may be identified with a sequence of functions $(\kappa^{(n)})_{n=1}^{\infty}$, where each $\kappa^{(n)}$ is a symmetric function on X^n (defined a.e.). The functions $(\kappa^{(n)})_{n=1}^{\infty}$ are called the correlation functions of the point process. (Let us remark that, although we will initially deal with correlation measures, in any reasonable application, the point process has correlation functions, i.e., the correlation measure is absolutely continuous with respect to the Lebesgue–Poisson measure.)

It was shown by Lenard [17] that, under a very mild assumption on the correlation measure, it uniquely characterises a point process. Furthermore, Lenard [17] and Macchi [21] proposed conditions which are sufficient for a given measure ρ on $\Gamma_{X,0}$ to be the correlation measure of a point process, i.e., they gave a solution to the problem of existence of point processes through a correlation measure.

Kondratiev and Kuna [15] treated the $\mathcal{K}$-transform as an analogue of the Fourier transform over the configuration space. They defined a $\star$-convolution of functions on $\Gamma_{X,0}$ so that

$$\mathcal{K}(G_1 \star G_2) = \mathcal{K}G_1 \cdot \mathcal{K}G_2.$$

If ρ is the correlation measure of a point processes μ, then

$$\int_{\Gamma_{X,0}} (G \star G)(\eta)\rho(d\eta) = \int_{\Gamma_X} \mathcal{K}(G \star G)(\gamma)\mu(d\gamma)$$

$$= \int_{\Gamma_X} (\mathcal{K}G)^2(\gamma)\mu(d\gamma) \geqslant 0,$$

and therefore the measure ρ is $\star$-positive definite:

$$\int_{\Gamma_{X,0}} (G \star G)(\eta)\rho(d\eta) \geqslant 0. \tag{1.2}$$

By using the condition of $\star$-positive definiteness (which is weaker than Lenard's and Macchi's conditions), an analogue of the Bochner theorem for point processes was proved by Kondratiev and Kuna [15], in the case where X is a smooth Riemannian manifold. A spectral approach to this construction, together with an important refinement of the local bound satisfied by a measure ρ, was proved by Berezansky *et al.* [6].

We should stress that, in both papers [15] and [6], the assumption that X be a smooth Riemannian manifold was crucial, due to the use of the Minlos theorem in [15], and the projection spectral theorem in [6].

The main difficulty about the existence of a point process through the results of [15, 6] is that, given a measure ρ on $\Gamma_{X,0}$ which is to be shown to be a correlation measure, it is very hard to check that ρ indeed satisfies the condition of $\star$-positive definiteness (1.2).

So, the main idea of the present work is to start with a special family of commuting Hermitian operators, rather than with a measure ρ.

In the first part of this dissertation we introduce the notion of the correlation measure of a family of commuting Hermitian operators. It should be stressed that *not every* family of commuting Hermitian operators possesses the correlation measure. However, if a family of such operator does possess a correlation measure ρ, then under a weak additional assumption, this measure ρ is the correlation measure of a point process. It appears in our approach that ρ automatically satisfies the condition of $\star$-positive definiteness.

It may appear that we have formally replaced the hard problem of checking the condition of $\star$-positivity by the hard problem of proving that a given family of commuting Hermitian operators possesses a correlation measure. But we show in the second part of this dissertation that this is not always so, and that there are families of commuting Hermitian operators for which the existence of correlation measure is physically motivated. So, what one really needs to do in this case is to show that the heuristic physical arguing may be given a rigorous mathematical meaning.

The dissertation is organised as follows. In Chapter 2 we present some preliminaries from functional analysis which are used in the work. In Chapter 3, we discuss in detail the spaces of infinite and finite configurations, the $\mathcal{K}$-transform, correlation measures, and $\star$-convolution. We also recall Lenard's results on the uniqueness and existence of point processes.

In Chapter 4, we discuss a general theorem on the existence of a point process through a family of commuting Hermitian operators. More precisely, we consider a family of commuting Hermitian operators $\mathcal{A} = (a(\Delta))_{\Delta \in \mathcal{B}_0(X)}$ indexed by all measurable, relatively compact sets Δ in X. We define a class $\mathcal{S}$ of "simple" functions on $\Gamma_{X,0}$ and introduce (corresponding to $\mathcal{A}$) Hermitian operators $(\mathcal{Q}(G))_{G\in\mathcal{S}}$ so that

$$(\mathcal{Q}(G_1 \star G_2) = \mathcal{Q}(G_1)\mathcal{Q}(G_2).$$

We fix a vector Ω, and assume that there exists a measure ρ on $\Gamma_{X,0}$ such that

$$(\mathcal{Q}(G)\Omega, \Omega) = \int_{\Gamma_{X,0}} G(\eta)\, \rho(d\eta).$$

We then call ρ the correlation measure of the family $\mathcal{A}$. We have

$$\int_{\Gamma_{X,0}} (G \star G)\rho(d\eta) = (\mathcal{Q}(G \star G)\Omega, \Omega)$$

$$= (\mathcal{Q}(G)\mathcal{Q}(G)\Omega, \Omega)$$
$$= (\mathcal{Q}(G)\Omega, \mathcal{Q}(G)\Omega) \geqslant 0,$$

so that the measure ρ is $\star$-positive definite. We prove that, if the family $\mathcal{A}$ possesses a correlation measure that satisfies some condition of growth, then there exists a point process μ, whose correlation measure is ρ. Furthermore, the operators $a(\Delta)$ can be realised as multiplication operators in $L^2(\Gamma_X, \mu)$. Thus, μ can be thought of as the spectral measure of the family $\mathcal{A}$ [5]. As a corollary, we extend the criterion of existence of a point process proved in [15, 6] to the case of a general topological space X, which is a standard underlying space in the theory of point processes (see e.g. [14]).

In Chapter 5, we consider an important application of the result from Chapter 4. This application has its origin in mathematical physics.

Recall that the nonrelativistic quantum mechanics of many identical particles may be described by means of a field $\Psi(x)$, $x \in \mathbb{R}^d$, satisfying either canonical commutation relations (CCR) and describing bosons:

$$[\Psi(x), \Psi(y)]_- = [\Psi^*(x), \Psi^*(y)]_- = 0,$$
$$[\Psi^*(x), \Psi(y)]_- = \delta(x-y)\mathbf{1}, \tag{1.3}$$

or satisfying canonical anticommutation relations (CAR) and describing fermions:

$$[\Psi(x), \Psi(y)]_+ = [\Psi^*(x), \Psi^*(y)]_+ = 0,$$
$$[\Psi^*(x), \Psi(y)]_+ = \delta(x-y)\mathbf{1}, \tag{1.4}$$

Here, $[A, B]_\mp = AB \mp BA$ is the commutator (anticommmutator respectively). The statictics of the system is thus determined by the algebra which is to be represented.

In the formulation of nonrelativistic quantum mechanics in terms of particle densities and currents, one defines

$$
\begin{aligned}
a(x) &:= \Psi^*(x)\Psi(x), \\
J(x) &:= (2i)^{-1}(\Psi^*(x)\nabla\Psi(x) - (\nabla\Psi^*(x))\Psi(x)).
\end{aligned} \tag{1.5}
$$

Using CCR or CAR, one can formally compute the commutation relations satisfied by the smeared operators $a(f) := \int_{\mathbb{R}^d} dx\, f(x)a(x)$ and $J(v) := \int_{\mathbb{R}^d} dx\, v(x)\cdot J(x)$. These turn out to be

$$
\begin{aligned}
[a(f_1), a(f_2)]_- &= 0, \\
[a(f), J(v)]_- &= ia(v\cdot\nabla f), \\
[J(v_1), J(v_2)]_- &= -iJ(v_1\cdot\nabla v_2 - v_2\cdot\nabla v_1),
\end{aligned} \tag{1.6}
$$

independently of whether one starts with CCR or CAR.

Thus, in a nonrelativistic current theory, the statistics of particles is not determined by a choice of algebra, but instead may be determined by a choice of a representation of the algebra, see e.g. [8, 12, 13] and the references therein.

It follows from (1.6) that the operators $a(f)$ form a family of commuting Hermitian operators. Note also that one can consider a more general case of field operators $\Psi(x)$, where $x \in X$ with X being a topological space as above. Then the CCR, respectively CAR, may be easily generalised, and again the particle densities $a(x) := \Psi^*(x)\Psi(x)$ lead to a family of commuting Hermitian operators

$$
a(\Delta) := \int_\Delta a(x)\sigma(dx), \quad \Delta \in \mathcal{B}_0(X).
$$

Let $G^{(n)}$ be a function from $\mathcal{S}$ such that $G^{(n)}(\eta) = 0$ if the number of points in the configuration η is not n. Then one can heuristically prove that

$$
\mathcal{Q}(G^{(n)}) = \frac{1}{n!}\int_{X^n} \sigma(dx_1)\cdots\sigma(dx_n)\, G^{(n)}(\{x_1,\dots,x_n\})
$$

$$\times\, \Psi^*(x_n)\cdots\Psi^*(x_1)\Psi(x_1)\cdots\Psi(x_n). \tag{1.7}$$

The product

$$\Psi^*(x_n)\cdots\Psi^*(x_1)\Psi(x_1)\cdots\Psi(x_n)$$

is usually called a normal product. Thus, by (1.7), the family $(a(\Delta))_{\Delta\in\mathcal{B}_0(X)}$ has a correlation measure and the corresponding correlation functions are given by

$$\begin{aligned}\kappa^{(n)}(x_1,\dots,x_n) &= (\Psi^*(x_n)\cdots\Psi^*(x_1)\Psi(x_1)\cdots\Psi(x_n)\Omega,\Omega),\\ &= \|\Psi(x_1)\cdots\Psi(x_n)\Omega\|^2.\end{aligned}$$

To the best of our knowledge, heuristic arguments of such kind were first given by Menikoff in [22], see also [23].

So, in Section 5.1, we mathematically realise this idea in the case of fermion (determinantal), and boson (permanental) point processes. These processes were introduced by Girard [11], Menikoff [23], and Macchi [21], and have been actively studied during the past years, see e.g. [10, 27, 25, 28] and the references therein.

So, we start with a quasi-free representation of the CAR (CCR respectively), see e.g. [1, 2, 9]. Such a representation is completely characterised by a linear, bounded, Hermitian operator K in $L^2(X,\sigma)$ which satisfies $\mathbf{0}\le K\le\mathbf{1}$ in the fermion case, and $K\ge\mathbf{0}$ in the boson case. In the case where $X=\mathbb{R}^d$ and K is a convolution operator, it has been already shown by Lytvynov in [19] that the corresponding particle density has a fermion (boson, respectively) point process as its spectral measure.

We treat the most general case of the space X and the operator K. The latter operator is only assumed to be locally of trace class, which is a necessary assumption for finite correlations. The main mathematical (as well as physical) challenge here is to show that all heuristic arguments coming from physics indeed have a precise mathematical meaning. We observe that K automatically appears to be an integral operator, and furthermore, with our approach, we do not even have to additionally discuss the problem of the choice of a version of the kernel $k(x, y)$ of the operator K, compare with [27, Lemma 1] and [10, Lemma A4]. Thus, we, in particular, show that any fermion process corresponding to a Hermitian operator K can be thought of as the spectral measure of the family of operators which represent the particle density of a quasi-free representation of the CAR. Though all our results hold in the case where the operator K acts in the complex Hilbert space $L^2(X \to \mathbb{C}, \sigma)$, for simplicity of presentation we only deal with the case where K acts in the real space $L^2(X, \sigma)$.

Finally in Section 5.2, we consider an l-fold convolution ($l \geqslant 2$) of particle densities $\Psi^*(x)\Psi(x)$ from Section 5.1. The corresponding smeared operators $(a^{(l)}(\Delta))_{\Delta \in \mathcal{B}_0(X)}$ form a family of commuting Hermitian operators. The corresponding correlation functions are represented through the Vere–Jones α-determinants, which generalise usual determinants and permanents. Thus, the family $(a^{(l)}(\Delta))_{\Delta \in \mathcal{B}_0(X)}$ is shown to satisfy the assumptions of our main theorem of Chapter 4 and leads to a corresponding point process. These processes appear to be from the class of point processes discussed by Shirai and Takahashi in [25]. Recall also that it was shown by Tamura and Ito in [28, 29] that, in the case $X = \mathbb{R}^d$, the point processes derived in Section 5.2 describe para-fermions (para-bosons respectively), where the number of convolution, l, corresponds to the order of these particles.

The main results of this dissertation are published in [20].

Chapter 2

Some preliminaries from functional analysis

In this chapter, we will briefly recall some definitions and results of Functional Analysis which we will need in our work. For details, we refer e.g. to [5, 7].

2.1 Bounded operators

For any Hilbert spaces H_1 and H_2, we denote by $\mathcal{B}(H_1, H_2)$ the set of all bounded linear operators from H_1 into H_2. As usual we denote $\mathcal{B}(H) := \mathcal{B}(H, H)$.

We will always assume that all the Hilbert spaces we consider are separable, i.e., they possess a countable dense subset. Also, if it is not explicitly stated, we will deal with complex Hilbert spaces.

Integral operators: Let $(X, \mathcal{A}, \sigma)$ be a measure space with σ-finite measure σ. An operator $K \in \mathcal{B}(L^2(X \to \mathbb{C}, \sigma))$ is called an integral operator if there exists a measurable function $k : X^2 \to \mathbb{C}$ such that

$$(KF)(x) = \int_X k(x,y) f(y) \sigma(dy), \quad f \in L^2(X \to \mathbb{C}, \sigma).$$

The function $k(x,y)$ is called the kernel of the operator K.

Hilbert–Schmidt operators: An operator $T \in \mathcal{B}(H)$ is a Hilbert–Schmidt operator if there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of H such that

$$\sum_{n=1}^{\infty} \|Te_n\|^2 < \infty. \tag{2.1}$$

In the latter case, the inequality (2.1) holds for any orthonormal basis $\{e_n\}_{n=1}^{\infty}$ in H and furthermore, the value $\sum_{n=1}^{\infty} \|Te_n\|^2$ is independent of the choice of an orthonormal basis $\{e_n\}_{n=1}^{\infty}$.

For $T \in \mathcal{B}(H)$, let T^* denote the adjoint operator of T. Then, T is a Hilbert–Schmidt operator if and only if T^* is a Hilbert–Schmidt operator and

$$\sum_{n=1}^{\infty} \|Te_n\|^2 = \sum_{n=1}^{\infty} \|T^* f_n\|^2$$

for any orthonormal bases $\{e_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ of H.

In the case where $H = L^2(X \to \mathbb{C}, \sigma)$, an operator K is Hilbert–Schmidt if and only if K is an integral operator and $k \in L^2(X^2 \to \mathbb{C}, \sigma^{\otimes 2})$, where k is the kernel of K. In fact, one has

$$\sum_{n=1}^{\infty} \|Te_n\|^2_{L^2(X,\sigma)} = \int_X \int_X |k(x,y)|^2 \sigma(dx)\sigma(dy)$$

for any orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of $L^2(X \to \mathbb{C}, \sigma)$.

An operator $T \in \mathcal{B}(H)$ is called a trace class operator if it can be represented as $T = \sum_{k=1}^{n} A_k B_k$, where $n \in \mathbb{N}$ and $A_1, \ldots, A_n$, $B_1, \ldots, B_n$ are Hilbert–Schmidt operators. If T is a trace class operator and $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis in H, then the series $\sum_{n=1}^{\infty} (Te_n, e_n)_H$ converges absolutely and its value, called the trace of the operator T, is independent of the choice of orthonormal basis.

2.2 Bochner integration

Let $(X, \mathcal{A}, \sigma)$ be a measure space with a σ-finite measure. We want to define an integral $\int_X f(x)\sigma(dx)$, where $f : X \to E$ and E is a Banach space.

Bochner's idea of construction of such an integral was to generalise the construction of the Lebesgue integral, i.e., the case where E is either $\mathbb{R}$ or $\mathbb{C}$.

A function $f : X \to E$ is called simple if

$$f(x) = \sum_{k=1}^{n} c_k \chi_{\Delta_k}(x), \tag{2.2}$$

where $c_k \in E$, $\Delta_1, \dots, \Delta_n \in \mathcal{A}$ mutually disjoint, and

$$\max\{\sigma(\Delta_1), \dots, \sigma(\Delta_n)\} < \infty.$$

Then

$$\|f(x)\| = \sum_{k=1}^{n} \|c_k\| \chi_{\Delta_k}(x)$$

is a simple, real-valued function on X.

A function $f : X \to E$ is called strongly measurable if there exists a sequence of simple functions, $(f_n)_{n=1}^{\infty}$, such that

$$\lim_{n\to\infty} \|f_n(x) - f(x)\| = 0 \quad \sigma\text{-a.e.} \tag{2.3}$$

For a simple function $f : X \to E$ of the form (2.2), the Bochner integral is defined by

$$\int_X f(x)\sigma(dx) = \sum_{k=1}^{n} c_k \sigma(\Delta_k).$$

Next, let f be an arbitrary strongly measurable function and let $(f_n)_{n=1}^{\infty}$ be a sequence of simple functions satisfying (2.3). Then $\|f_n - f\|$ is a non-negative measurable function on X, and so the integral

$$\int_X \|f_n(x) - f(x)\| \sigma(dx)$$

is well-defined. Assume that

$$\int_X \|f_n(x) - f(x)\|\sigma(dx) \to 0 \quad \text{as} \quad n \to \infty.$$

Then, one can show that the sequence $\left(\int_X f_n(x)\sigma(dx)\right)_{n=1}^{\infty}$ is a Cauchy sequence in E. Therefore, it has a limiting element in E.

We then say that f is Bochner-integrable and $\int_X f(x)\sigma(dx)$ is the Bochner integral of f.

One can show that $\int_X f(x)\sigma(dx)$ does not depend on the choice of approximating sequence. In fact, the following theorem holds, see e.g. [7].

Theorem 2.1 *A strongly measurable function $f : X \to E$ is Bochner-integrable if and only if $\int_X \|f(x)\|\sigma(dx) < \infty$.*

2.3 Unbounded operators

The aim of this section is to recall the reader some notions connected with unbounded operators. Let us note that we do not aim to recall all the definitions and constructions we are using, but we rather refer to textbooks on functional analysis, like e.g. [7]. So let H be a Hilbert space. A linear operator A with domain $D(A)$, usually denoted by $(A, D(A))$, is called symmetric if, for any $f, g \in D(A)$

$$(Af, g)_H = (f, Ag)_H.$$

If additionally, the domain $D(A)$ is dense in H, the operator A is called Hermitian.

If $(A, D(A))$ is a linear operator with dense domain $D(A)$ in H, then we define $D(A^*)$ as the set of those $g \in H$ for which there exists $g^* \in H$ such that

$$(Af, g)_H = (f, g^*)_H, \quad \text{for all } f \in D(A).$$

In this case we call $D(A^*)$ the domain of the adjoint operator A^* and we set $A^*g = g^*$.

An operator $(A, D(A))$ is called self-adjoint if $(A, D(A)) = (A^*, D(A^*))$, i.e., the operator A coincides with its adjoint.

For an operator $(A, D(A))$, the set

$$\Gamma_A := \{(f, Af) \mid f \in D(A)\} \subset H \times H$$

is called the graph of the operator A.

If Γ_A is a closed subset of $H \times H$, then the operator A is called closed. If this is not the case, then one may take the closure $\bar{\Gamma}_A$ of Γ_A in $H \times H$. However, $\bar{\Gamma}_A$ may happen not to be a graph of a linear operator, i.e., there may exist vectors (f, g_1) and (f, g_2) in $\bar{\Gamma}_A$ such that $g_1 \neq g_2$. If this is not the case, i.e., if $\bar{\Gamma}_A$ is a graph of a linear operator, then we call $(A, D(A))$ a closable operator and the corresponding operator defined by $\bar{\Gamma}_A$ is called the closure of $(A, D(A))$, denoted by $(\tilde{A}, D(\tilde{A}))$.

One may show that any Hermitian operator is closable. However, the closure of such an operator is not necessarily a self-adjoint operator.

If this closure is self-adjoint, then we say that $(A, D(A))$ is an essentially self-adjoint operator.

In applications we are mostly given not self-adjoint operators, but Hermitian operators. Then, if one is able to prove that such an operator is essentially self-adjoint, then, by closing the operator $(A, D(A))$, one derives a self-adjoint operator.

Theorem 2.2 (Nelson's analytic vector criterion) *Let $(A, D(A))$ be a Hermitian operator in H. A vector $f \in D(A)$ is called analytic (for A) if, for each $n \in \mathbb{N}$, $f \in D(A^n)$, and*

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \|A^n f\|_H < \infty$$

for some $t > 0$. *If there is a subset* $\mathfrak{D} \subset D(A)$ *such that* $\mathfrak{D}$ *is dense in* H *and each* $f \in \mathfrak{D}$ *is analytic for* A, *then the operator* $(A, D(A))$ *is essentially self-adjoint.*

Let $(X, \mathcal{A})$ be a measurable space. A mapping

$$\mathcal{A} \ni \alpha \mapsto E(\alpha) \in \mathcal{B}(H)$$

is called a resolution of the identity if the following conditions are satisfied:

- For each $\alpha \in \mathcal{A}$, $E(\alpha)$ is an orthogonal projection in H.
- $E(\varnothing) = 0$, $E(X) = \mathbf{1}$.
- If $\{\alpha_n\}_{n=1}^{\infty}$, $\alpha_n \in \mathcal{A}$, $n \in \mathbb{N}$, α_n are mutually disjoint, then for each $f \in H$
$$E\Big(\bigcup_{n=1}^{\infty} \alpha_n\Big) f = \sum_{n=1}^{\infty} E(\alpha_n) f,$$
where the series converges in H.

It follows from the definition of resolution of the identity that for any vectors $f, g \in H$, the mapping

$$\mathcal{A} \ni \alpha \mapsto (E(\alpha) f, g)_H$$

is a signed measure on $(X, \mathcal{A})$.

To any self-adjoint operator $(A, D(A))$, there corresponds a unique resolution of the identity over $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$A = \int_{\mathbb{R}} \lambda \, dE(\lambda). \tag{2.4}$$

The equality (2.4) should be understood as follows:

$$D(A) := \{ f \in H \mid \int_{\mathbb{R}} \lambda^2 \, d(E(\lambda) f, f)_H < \infty \} \tag{2.5}$$

and for any $f \in D(A)$ and $g \in H$

$$(Af, g)_H = \int_{\mathbb{R}} \lambda \, d(E(\lambda)f, g)_H. \tag{2.6}$$

Furthermore, the inverse statement holds. If E is a resolution of the identity over $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then E determines a self-adjoint operator in H through the formulas (2.5) and (2.6).

Formulas (2.4)–(2.6) are called the spectral decomposition of a self-adjoint operator (in fact, the resolution of the identity is concentrated on the spectrum of A). Using the spectral decomposition (2.5) and (2.6), one easily defines a function of the operator A, $f(A)$, through the formula

$$f(A) := \int_{\mathbb{R}} f(\lambda) \, dE(\lambda)$$

(which has a natural meaning through smearing).

Let us now briefly discuss the commutation of linear operators. In the case where A_1 and A_2 are bounded linear operators, their commutation is defined straightforward:

$$A_1 A_2 f = A_2 A_1 f \quad \text{for each } f \in H.$$

However, in the case where A_1 and A_2 are unbounded operators, the operators $A_1 A_2$ or $A_2 A_1$ may only be well-defined at zero. So, in the case where A_1 and A_2 are additionally self-adjoint operators, one defines their commutation through the commutation of their resolutions of the identity. So we say that self-adjoint operators $(A_1, D(A_1))$ and $(A_2, D(A_2))$ commute in the sense of their resolutions of the identity if, for any $\alpha_1, \alpha_2 \in \mathcal{B}(\mathbb{R})$, the operators $E_1(\alpha_1)$ and $E_2(\alpha_2)$ commute, where E_1, and E_2, denote the resolution of the identity of A_1, and A_2, respectively.

The following theorem (see [5]) allows one to check that two given self-adjoint operators indeed commute in the sense of their resolutions of the identity.

Theorem 2.3 *Let $(A_1, D(A_1))$, $(A_2, D(A_2))$ be two Hermitian operators in H. Let $\mathfrak{D}$ be a dense linear subset of H such that $\mathfrak{D} \subset D(A_1) \cap D(A_2)$, $A_1\mathfrak{D} \subset \mathfrak{D}$, $A_2\mathfrak{D} \subset \mathfrak{D}$ and A_1, A_2 commute on $\mathfrak{D}$ in the usual sense:*

$$A_1 A_2 f = A_2 A_1 f \quad \text{for all } f \in \mathfrak{D}.$$

Assume that each vector in $\mathfrak{D}$ is analytic for both operators A_1 and A_2. Then the operators $(A_1, D(A_1))$ and $(A_2, D(A_2))$ are essentially self-adjoint and their closures $(\tilde{A}_1, D(\tilde{A}_1))$ and $(\tilde{A}_2, D(\tilde{A}_2))$ commute in the sense of their resolutions of the identity.

Let us now consider n self-adjoint operators $(A_1, D(A_1)), \dots, (A_n, D(A_n))$ in H and let us assume that these operators commute in the sense of their resolutions of the identity. Denote by E_i the resolution of identity of the operator $(A_i, D(A_i))$. One can construct the joint resolution of the identity of these operators as a resolution of the identity on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. This resolution of the identity denoted by E, is defined as

$$E(\alpha_1 \times \alpha_2 \times \cdots \times \alpha_n) = E_1(\alpha_1)E_2(\alpha_2)\cdots E_n(\alpha_n), \quad \alpha_1, \dots, \alpha_n \in \mathcal{B}(\mathbb{R}),$$

and then one uniquely extends this to a resolution of the identity on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ (just as in the case of product-measures).

Then we have, for each $i \in \{1, \dots, n\}$,

$$D(A_i) = \{f \in H : \int_{\mathbb{R}^n} \lambda_i^2 \, d(E(\lambda)f, f)_H < \infty\}$$

and

$$A_i = \int_{\mathbb{R}^n} \lambda_i \, dE(\lambda),$$

i.e., for any $f \in D(A_i)$ and $g \in H$

$$(A_i f, g)_H = \int_{\mathbb{R}^n} \lambda_i \, d(E(\lambda)f, g)_H.$$

Chapter 3

Configuration spaces

3.1 The space of finite configurations

Let X be a locally compact, second countable Hausdorff topological space. This means that every point in X has a compact neighbourhood, that X has a countable base and that distinct points may be separated by disjoint neighbourhoods. It is known that X is then a Polish space, i.e., there exists a metrization ρ of X such that, in this metric, X becomes a complete separable metric space (see e.g. [26]).

A set $B \subset X$ is said to be topologically bounded, or relatively compact if its closure $\bar{B}$ is compact. Note that, if we fix a metric ρ in X which generates its topology, an open ball $B(x, r)$ with centre at $x \in X$ and radius $r > 0$ need not be bounded. However, for each fixed $x \in X$, one can always find $\varepsilon > 0$ small enough such that the open ball $B(x, \varepsilon)$ is bounded.

We denote by $\mathcal{B}(X)$ the Borel σ-algebra in X, and by $\mathcal{B}_0(X)$ the collection of all bounded sets from $\mathcal{B}(X)$.

We define the space of *finite multiple configurations* in X as follows:

$$\ddot{\Gamma}_{X,0} := \bigsqcup_{n\in\mathbb{N}_0} \ddot{\Gamma}_X^{(n)}.$$

Here, $\mathbb{N}_0 = 0, 1, 2, \ldots$, $\ddot{\Gamma}_X^{(0)} := \{\varnothing\}$, $\ddot{\Gamma}_X^{(n)}$ is the factor space X^n/S_n, where S_n is the group of all permutations of $\{1, 2, \ldots, n\}$, which naturally acts on X^n:

$$\xi(x_1, \ldots, x_n) = (x_{\xi(1)}, \ldots, x_{\xi(n)}), \quad \xi \in S_n. \tag{3.1}$$

We denote by $[x_1, \ldots, x_n]$ the equivalence class in $\ddot{\Gamma}_X^{(n)}$ corresponding to $(x_1, \ldots, x_n) \in X^n$.

Let $\mathcal{B}(\ddot{\Gamma}_X^{(n)})$ denote the image of the Borel σ-algebra $\mathcal{B}(X^n)$ under the mapping

$$X^n \ni (x_1, \ldots, x_n) \mapsto [x_1, \ldots, x_n] \in \ddot{\Gamma}_X^{(n)}.$$

Then, the real-valued measurable functions on $\ddot{\Gamma}_X^{(n)}$ may be identified with the real-valued $\mathcal{B}_{\mathrm{sym}}(X^n)$-measurable functions on X^n. Here, $\mathcal{B}_{\mathrm{sym}}(X^n)$ denotes the σ-algebra of all sets in $\mathcal{B}(X^n)$ which are symmetric, i.e., which remain invariant under the action (3.1).

We define a σ-algebra $\mathcal{B}(\ddot{\Gamma}_{X,0})$ on $\ddot{\Gamma}_{X,0}$, so that the trace σ-algebra of $\mathcal{B}(\ddot{\Gamma}_{X,0})$ on each $\ddot{\Gamma}_X^{(n)}$ coincides with $\mathcal{B}(\ddot{\Gamma}_X^{(n)})$ for $n \in \mathbb{N}$, and $\{\varnothing\} \in \mathcal{B}(\ddot{\Gamma}_{X,0})$.

Next, we introduce the space of *finite configurations* in X, denoted by $\Gamma_{X,0}$. By definition, $\Gamma_{X,0}$ is the subset of $\ddot{\Gamma}_{X,0}$, given by

$$\Gamma_{X,0} := \bigsqcup_{n\in\mathbb{N}_0} \Gamma_X^{(n)},$$

where $\Gamma_X^{(0)} := \ddot{\Gamma}_X^{(0)}$ and for $n \in \mathbb{N}$, $\Gamma_X^{(n)}$ consists of all $[x_1, \ldots, x_n] \in \ddot{\Gamma}_X^{(n)}$ such that $x_1, \ldots, x_n$ are different points in X. Thus, each element $[x_1, \ldots, x_n] \in \Gamma_X^{(n)}$ can be identified with the set $\{x_1, \ldots, x_n\}$. We denote by $\mathcal{B}(\Gamma_X^{(n)})$ the trace σ-algebra of $\mathcal{B}(\ddot{\Gamma}_X^{(n)})$ on $\Gamma_X^{(n)}$. Let also $\mathcal{B}(\Gamma_{X,0})$ be the trace σ-algebra of $\mathcal{B}(\ddot{\Gamma}_{X,0})$ on $\Gamma_{X,0}$.

Completely analogously, for any $\Delta \in \mathcal{B}_0(X)$, we define $\ddot{\Gamma}_\Delta := \ddot{\Gamma}_{\Delta,0}$ and $\Gamma_\Delta := \Gamma_{\Delta,0}$ as well as the corresponding σ-algebras $\mathcal{B}(\ddot{\Gamma}_\Delta)$, $\mathcal{B}(\Gamma_\Delta)$, $\mathcal{B}(\ddot{\Gamma}_\Delta^{(n)})$, $\mathcal{B}(\Gamma_\Delta^{(n)})$, $n \in \mathbb{N}$.

Let σ be a Radon non-atomic measure on $(X, \mathcal{B}(X))$. Recall that a Radon measure is characterised by the property that $\sigma(\Delta) < \infty$ for each $\Delta \in \mathcal{B}_0(X)$, and "non-atomic" means that $\sigma(\{x\}) = 0$ for each $x \in X$.

Then we can construct the measure $\sigma^{\otimes n}$ on $(X^n, \mathcal{B}(X^n))$. Since σ is non-atomic, we have

$$\sigma^{\otimes n}(D_n) = 0,$$

where

$$D_n := \{(x_1, \dots, x_n) \in X^n : x_i = x_j \quad \text{for some } i \neq j\}.$$

So, we consider $\sigma^{\otimes n}$ as a measure on $(\tilde{X}^n, \mathcal{B}(\tilde{X}^n))$, where $\tilde{X}^n := X^n \setminus D_n$.

Next, we restrict this measure to the σ-algebra $\mathcal{B}_{\mathrm{sym}}(\tilde{X}^n)$, consisting of all symmetric set in $\mathcal{B}(\tilde{X}^n)$. Therefore, we can identify $\sigma^{\otimes n}$ with a measure on $(\Gamma_X^{(n)}, \mathcal{B}(\Gamma_X^{(n)}))$.

We now define the Lebesgue–Poisson measure λ_σ on $(\Gamma_{X,0}, \mathcal{B}(\Gamma_{X,0}))$ as follows:

$$\lambda_\sigma(\{\varnothing\}) = 1,$$

and

$$\lambda_\sigma \restriction \Gamma_X^{(n)} = \frac{1}{n!} \sigma^{\otimes n}.$$

Informally, we may write

$$\lambda_\sigma = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{\otimes n}.$$

3.2 The space of infinite configurations

The configuration space Γ_X over X is defined as

$$\Gamma_X := \{\gamma \subset X : |\gamma \cap \Delta| < \infty \text{ for each } \Delta \in \mathcal{B}_0(X)\}.$$

Here, $|A|$ denotes the cardinality of a set A. We identify each $\gamma \in \Gamma_X$ with the Radon measure

$$\gamma = \sum_{x \in \gamma} \varepsilon_x,$$

where ε_x is the Dirac measure with mass at x. Thus, Γ_X becomes a subset of $\mathcal{M}_X$—the space of all (non-negative) Radon measures on $(X, \mathcal{B}(X))$.

The space $\mathcal{M}_X$ is usually endowed with the vague topology, i.e., the weakest topology with respect to which all mappings of the form

$$\mathcal{M}_X \ni \sigma \mapsto \langle \sigma, f \rangle := \int f d\sigma \in \mathbb{R}, \quad f \in C_0(X),$$

become continuous. Here, $C_0(X)$ denotes the set of all continuous functions on X with compact support.

So, we endow Γ_X with relative topology as a subset of $\mathcal{M}_X$. Hence, the vague topology on Γ_X is the weakest topology with respect to which all mapping of the form

$$\Gamma_X \ni \gamma \mapsto \langle \gamma, f \rangle := \int_X f(x)\gamma(dx) = \sum_{x \in \gamma} f(x) \in \mathbb{R}, \quad f \in C_0(X),$$

become continuous. We denote by $\mathcal{B}(\Gamma_X)$ the Borel σ-algebra on Γ_X.

There is another way of introducing a σ-algebra on Γ_X. Denote by $\mathcal{A}$ the smallest σ-algebra on Γ_X with respect to which all mappings of the form

$$\Gamma_X \ni \gamma \mapsto \gamma(\Delta) = |\gamma \cap \Delta|, \quad \Delta \in \mathcal{B}_0(X),$$

are measurable. It can be proved (see e.g. [14]) that $\mathcal{A}$ coincides with $\mathcal{B}(\Gamma_X)$. A probability measure μ on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ is called a point process (in X).

Let $\Delta \in \mathcal{B}_0(X)$. We call a set $A \subset \Gamma_X$ local with respect to Δ if there exists a set $\tilde{A} \subset \Gamma_\Delta$ (the space of finite configurations in Δ) such that

$$A = \{\gamma \in \Gamma_X : \gamma \cap \Delta \in \tilde{A}\}. \tag{3.2}$$

We denote by $\mathcal{B}_\Delta(\Gamma_X)$ the σ-algebra of sets from $\mathcal{B}(\Gamma_X)$ which are local with respect to Δ. By identifying a local set A as in (3.2) with the set $\tilde{A}$, we may identify the σ-algebra $\mathcal{B}_\Delta(\Gamma_X)$ with a σ-algebra in Γ_Δ. It is not hard to see (cf. [14]) that the latter σ-algebra in Γ_Δ is, in fact, $\mathcal{B}(\Gamma_\Delta)$. Thus, we have identified the σ-algebras $\mathcal{B}_\Delta(\Gamma_X)$ and $\mathcal{B}(\Gamma_\Delta)$.

3.3 $\mathcal{K}$-transform and correlation measure

Following [15, 16, 17], we now introduce the following mapping between functions on $\Gamma_{X,0}$ and functions on Γ_X.

Let $G : \Gamma_{X,0} \to [0, +\infty]$, we define a function $\mathcal{K}G : \Gamma_X \to [0, +\infty]$ by

$$(\mathcal{K}G)(\gamma) = \sum_{\eta \Subset \gamma} G(\eta), \quad \gamma \in \Gamma_X, \tag{3.3}$$

where $\eta \Subset \gamma$ denotes that η is a finite subset of γ.

By [15], if G is a $\mathcal{B}(\Gamma_{X,0})$-measurable function, then $\mathcal{K}G$ is Γ_X-measurable. We call $\mathcal{K}G$ the $\mathcal{K}$-transform of G.

In what follows, we will also need the $\mathcal{K}$-transform of functions of an arbitrary sign. So, if $G : \Gamma_{X,0} \mapsto \mathbb{R}$ and if $(\mathcal{K}G^+)(\gamma) < \infty$ and $(\mathcal{K}G^-)(\gamma) < \infty$, then we set

$$(\mathcal{K}G)(\gamma) := (\mathcal{K}G^+)(\gamma) - (\mathcal{K}G^-)(\gamma).$$

Here, as usual we denoted

$$G^+ = \max\{0, G\}, \quad G^- = \max\{0, -G\}.$$

Using the $\mathcal{K}$-transform, we will now introduce the notion of a correlation measure of a point process.

Let μ be a probability measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$. We define a measure ρ_μ on $(\Gamma_{X,0}, \mathcal{B}(\Gamma_{X,0}))$ as follows: for each $A \in \mathcal{B}(\Gamma_{X,0})$, we set

$$\rho_\mu(A) := \int_{\Gamma_X} (\mathcal{K}\mathbf{1}_A)(\gamma)\mu(d\gamma),$$

where $\mathbf{1}_A$ denotes the indicator of the set A. One easily checks that ρ_μ is, indeed, a measure, and therefore, for any $\mathcal{B}(\Gamma_{X,0})$-measurable function $G : \Gamma_{X,0} \to [0, +\infty]$, we have

$$\int_{\Gamma_{X,0}} G(\eta)\rho_\mu(d\eta) = \int_{\Gamma_X} (\mathcal{K}G)(\gamma)\mu(d\gamma). \tag{3.4}$$

We call ρ_μ the correlation measure of μ. As we have just seen any point process has a correlation measure.

It follows from the definition of the correlation measure that, if $G \in L^1(\Gamma_{X,0}, d\rho_\mu)$, then $\mathcal{K}G$ is μ-a.s. well-defined, $\mathcal{K}G \in L^1(\Gamma_X, \mu)$, and

$$\int_{\Gamma_{X,0}} G(\eta)\rho_\mu(d\eta) = \int_{\Gamma_X} (\mathcal{K}G)(\gamma)\mu(d\gamma).$$

Next, recall the definition of the Lebesgue–Poisson measure λ_σ. Assume that ρ_μ is absolutely continuous with respect to λ_σ. We denote by

$$\kappa_\mu(\eta) := \frac{d\rho_\mu(\eta)}{d\lambda_\sigma(\eta)}, \quad \eta \in \Gamma_{X,0}.$$

Denote

$$\kappa_\mu^{(n)} := \kappa_\mu \restriction \Gamma_X^n, \quad n \in \mathbb{N}.$$

We can identify $\kappa_\mu^{(n)}$ with a function on $\tilde{X}^n$ which is $\mathcal{B}_{\mathrm{sym}}(\tilde{X}^n)$-measurable. Usually, $\kappa_\mu^{(n)}$ is considered as a symmetric function on X^n. The functions $\kappa_\mu^{(n)}$, $n \in \mathbb{N}$, are called the correlation functions of μ.

Note that, unlike the correlation measure, the correlation functions may, generally speaking, not exist.

Since a point process is a probability measure on an "infinite- dimensional space" and the correlation measure is defined on a union of finite-dimensional spaces, it is clear that one has tried to study point processes through their correlation measures.

So a natural problem appears: given a measure ρ on $\Gamma_{X,0}$, whether there exists a point process having ρ as its correlation measure (existence problem) and if such a point process exists, whether it is unique (uniqueness problem).

A very satisfactory solution of the uniqueness problem was given by Lenard in [18].

Theorem 3.1 *Let ρ_μ be the correlation measure of a probability measure μ on $(\Gamma_X, \mathcal{B}(\Gamma_X))$. For each $\Delta \in \mathcal{B}_0(X)$, denote*

$$m_k^\Delta := \frac{1}{k!}\rho_\mu(\Gamma_\Delta^{(k)}), \quad k \in \mathbb{N}.$$

Then, if the series

$$\sum_{k=0}^{\infty}(m_{k+j}^\Delta)^{-1/k}$$

diverges for each $\Delta \in \mathcal{B}_0(X)$ and each $j \in \mathbb{N}_0$, then the measure μ is a unique point process which has ρ_μ as its correlation measure.

Remark 3.1 We note the condition of Theorem 3.1 is satisfied if, for each $\Delta \in \mathcal{B}_0(X)$,

$$\rho(\Gamma_\Delta^{(n)}) \leqslant c^n n! \quad \text{for all } n \in \mathbb{N},$$

where $c = c_\Delta > 0$.

A solution of the existence problem was also given by Lenard in [17].

Theorem 3.2 *Let ρ is a measure on $(\Gamma_{X,0}, \mathcal{B}(\Gamma_{X,0}))$. We suppose that for each $\Delta \in \mathcal{B}_0(X)$ and $n \in \mathbb{N}$*

$$\rho\Big(\bigcup_{i=1}^{n}\Gamma_\Delta^{(i)}\Big) < \infty.$$

We also assume that

$$\rho(\{\varnothing\}) = 1,$$

and the following positivity condition holds:

For each measurable $G : \Gamma_{X,0} \to \mathbb{R}$ *which is bounded and vanishes outside a set* $\bigcup_{i=1}^{N} \Gamma_{\Delta}^{(i)}$ *for some* $N \in \mathbb{N}$ *and* $\Delta \in \mathcal{B}_0(X)$*, and such that*

$$(\mathcal{K}G)(\gamma) \geqslant 0, \quad \text{for all } \gamma \in \Gamma_X,$$

we have

$$\int_{\Gamma_{X,0}} G(\eta) d\rho(\eta) \geqslant 0.$$

Then there exist a probability measure μ *on* $\big(\Gamma_X, \mathcal{B}(\Gamma_X)\big)$ *which has* ρ *as its correlation measure.*

Another solution of the existence problem, which is, in fact, quite analogous to Theorem 3.2, was given by Macchi in [21].

A third way of finding sufficient conditions for existence was proposed by Kondratiev and Kuna in [15] and extended by Berezansky *et al.* [6].

This approach uses the so-called $\star$-convolution, which we will now discuss.

3.4 $\star$-convolution and existence of point processes

A $\star$-convolution is a convolution of two functions on the space of finite configurations whose $\mathcal{K}$-transform becomes the product of the $\mathcal{K}$-transforms of the given functions, i.e.,

$$\mathcal{K}(G_1 \star G_2) = \mathcal{K}G_1 \cdot \mathcal{K}G_2. \tag{3.5}$$

So, let us find an explicit form of $\star$-convolution. We have:

$$
\begin{aligned}
(\mathcal{K}G_1)(\gamma)(\mathcal{K}G_2)(\gamma) &= \Big(\sum_{\eta_1 \Subset \gamma} G_1(\eta_1)\Big)\Big(\sum_{\eta_2 \Subset \gamma} G_2(\eta_2)\Big) \\
&= \sum_{\eta \Subset \gamma} \sum_{\eta_1 \subset \gamma, \eta_2 \subset \gamma : \eta_1 \cup \eta_2 = \eta} G_1(\eta_1) G_2(\eta_2) \\
&= \sum_{\eta \Subset \gamma} \Big(\sum_{(\eta_1, \eta_2, \eta_3) \in P_3(\eta)} G_1(\eta_1 \cup \eta_2) G_2(\eta_2 \cup \eta_3) \Big)
\end{aligned}
$$

where $P_3(\eta)$ denotes the set of all ordered partitions of η into three parts.

Thus, we define

$$
(G_1 \star G_2)(\eta) = \sum_{(\eta_1, \eta_2, \eta_3) \in P_3(\eta)} G_1(\eta_1 \cup \eta_2) G_2(\eta_2 \cup \eta_3), \tag{3.6}
$$

and we indeed have (3.5).

If ρ is the correlation measure of a point process μ, then by (3.3) and (3.5)

$$
\int_{\Gamma_{X,0}} (G_1 \star G_2)(\eta) \rho(d\eta) = \int_{\Gamma_X} (\mathcal{K}G)^2(\gamma) \mu(d\gamma),
$$

and therefore ρ is $\star$-positive definite:

$$
\int_{\Gamma_{X,0}} (G_1 \star G_2)(\eta) \rho(d\eta) \geq 0.
$$

Theorem 3.3 ([6]) *Let X be a connected, oriented C^∞ Riemannian manifold. Let ρ be a measure on $\big(\Gamma_{X,0}, \mathcal{B}(\Gamma_{X,0})\big)$ which satisfy the following assumptions*

1. $\rho(\Gamma_X^{(0)}) = 1$

2. *For each $\Delta \in \mathcal{B}_0(X)$, there exists a constant $C_\Delta > 0$ such that*

$$
\rho(\Gamma_\Delta^{(n)}) \leqslant C_\Delta^n, \quad \text{for all } n \in \mathbb{N}
$$

3. Every compact $\Delta \subset X$ *can be covered by a finite union of open sets* $\Delta_1, \ldots, \Delta_k$, $k \in \mathbb{N}$, *which have compact closures and satisfy the estimate*

$$\rho(\Gamma_{\Delta}^{(n)}) \leq (2+\varepsilon)^{-n}, \quad \text{for all } i = 1, 2, \ldots, k \text{ and } n \in \mathbb{N},$$

where $\varepsilon = \varepsilon(\Delta) > 0$

4. $\star$*-Positive definiteness: For each* G *which is bounded and vanishes outside a set* $\bigcup_{i=1}^{N} \Gamma_{\Delta}^{(i)}$ *for some* $N \in \mathbb{N}$ *and* $\Delta \in \mathcal{B}_0(X)$, *we have*

$$\int_{\Gamma_{X,0}} (G_1 \star G_2)(\eta)\rho(d\eta) \geq 0.$$

Then, there exists a point process μ *on* $\big(\Gamma_X, \mathcal{B}(\Gamma_X)\big)$ *which has* ρ *as its correlation measure.*

Chapter 4

Correlation measure of a family of commuting Hermitian operators

The aim of this chapter is to introduce the notion of a correlation measure of a family of commuting Hermitian operators and to show that this measure is concentrated on the space of finite configurations.

For measurable functions $f_1, \dots, f_n : X \to \mathbb{R}$, we denote by $f_1 \widehat{\otimes} \cdots \widehat{\otimes} f_n$ the symmetric tensor product of $f_1, \dots, f_n$. Since $f_1 \widehat{\otimes} \cdots \widehat{\otimes} f_n$ is $\mathcal{B}_{\mathrm{sym}}(X^n)$-measurable, we may consider $f_1 \widehat{\otimes} \cdots \widehat{\otimes} f_n$ as a measurable function on $\ddot{\Gamma}_X^{(n)}$.

For a function $G : \ddot{\Gamma}_{X,0} \to \mathbb{R}$, we denote by $G^{(n)}$ the restriction of G to $\ddot{\Gamma}_X^{(n)}$. Let $\mathcal{S}$ denote the set of all real-valued functions on $\ddot{\Gamma}_{X,0}$ which satisfy the following condition: for each $G \in \mathcal{S}$, there is an $N \in \mathbb{N}$ such that $G^{(n)} = 0$ for all $n > N$ and for each $n \in \{1, \dots, N\}$, $G^{(n)}$ is a finite linear combination of the functions of the form $\chi_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \chi_{\Delta_n}$, where $\Delta_1, \dots, \Delta_n \in \mathcal{B}_0(X)$ and χ_A denotes the indicator of a set A. Note that, by the polarisation identity, in the above definition it suffices to take functions of the form $\chi_\Delta^{\otimes n}$, where

$\Delta \in \mathcal{B}_0(X)$. It is clear that the set $\mathcal{S}$ is sufficiently large, in the sense that $\mathcal{S}$ is an algebra under multiplication, and functions from $\mathcal{S}$ separate any two configurations in $\ddot{\Gamma}_{X,0}$.

Let F be either a real, or complex Hilbert space and let D be a linear subset of F. Let $\big(a(\Delta)\big)_{\Delta \in \mathcal{B}_0(X)}$ be a family of Hermitian operators in F such that:

- for each $\Delta \in \mathcal{B}_0(X)$, $\operatorname{Dom}\big(a(\Delta)\big) = D$ and $a(\Delta)$ maps D into itself;
- for any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$, $a(\Delta_1)a(\Delta_2) = a(\Delta_2)a(\Delta_1)$;
- for any mutually disjoint $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$, we have: $a(\Delta_1 \cup \Delta_2) = a(\Delta_1) + a(\Delta_2)$.

It follows from the definition of the $\mathcal{K}$-transform (see Section 2.3) that, for each $\gamma \in \Gamma_X$ and $\Delta \in \mathcal{B}_0(X)$, $\mathcal{K}(\chi_\Delta)(\gamma) = \gamma(\Delta)$, and that

$$\begin{aligned} \mathcal{K}(\chi_{\Delta_1}\widehat{\otimes}\cdots\widehat{\otimes}\chi_{\Delta_{n+1}})(\gamma) &= \frac{1}{(n+1)^2}\Bigg[\sum_{i=1}^{n+1}\gamma(\Delta_i)\mathcal{K}(\chi_{\Delta_1}\widehat{\otimes}\cdots\widehat{\otimes}\check{\chi}_{\Delta_i}\widehat{\otimes} \\ &\cdots\widehat{\otimes}\chi_{\Delta_{n+1}})(\gamma) - \sum_{i=1}^{n+1}\sum_{j=1,\dots,n+1,\, j\neq i}\mathcal{K}\big((\chi_{\Delta_i\cap\Delta_j})\widehat{\otimes}\chi_{\Delta_1}\widehat{\otimes})\cdots\widehat{\otimes}\check{\chi}_{\Delta_i}\widehat{\otimes} \\ &\cdots\widehat{\otimes}\check{\chi}_{\Delta_j}\widehat{\otimes}\cdots\widehat{\otimes}\chi_{\Delta_{n+1}}\big)(\gamma)\Bigg], \end{aligned} \tag{4.1}$$

where $\Delta_1, \dots, \Delta_{n+1} \in \mathcal{B}_0(X)$, $n \in \mathbb{N}$ and $\check{\chi}_\Delta$ denotes the absense of χ_Δ.

Indeed, let us fix a function $\varphi : X \to \mathbb{R}$ which is bounded and has compact support. Then, we have:

$$\begin{aligned} \langle\varphi,\gamma\rangle\mathcal{K}(\varphi^{\otimes n})(\gamma) &= \Big(\sum_{x\in\gamma}\varphi(x)\Big)\sum_{\{x_1,\dots,x_n\}\subset\gamma}\varphi(x_1)\cdots\varphi(x_n) \\ &= (n+1)\sum_{\{x_1,\dots,x_{n+1}\}\subset\gamma}\varphi(x_1)\cdots\varphi(x_{n+1}) \\ &\quad + \sum_{\{x_1,\dots,x_n\}\subset\gamma}\Big(\varphi^2(x_1)\varphi(x_2)\cdots\varphi(x_n) + \varphi(x_1)\varphi^2(x_2)\cdots\varphi(x_n) \end{aligned}$$

$$+\cdots+\varphi(x_1)\cdots\varphi(x_{n-1})\varphi^2(x_n)\Big)$$
$$=(n+1)\mathcal{K}(\varphi^{\otimes(n+1)})(\gamma)+n\mathcal{K}\big((\varphi^2)\otimes\varphi^{\otimes(n-1)}\big)(\gamma).$$

Therefore, using the polarisation identity, we conclude that, for any functions $\varphi_1,\dots,\varphi_{n+1}: X\to\mathbb{R}$ which are bounded and have compact support

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\langle\varphi_i,\gamma\rangle\mathcal{K}\big(\varphi_1\widehat{\otimes}\cdots\widehat{\otimes}\varphi_{i-1}\widehat{\otimes}\check{\varphi}_i\widehat{\otimes}\varphi_{i+1}\widehat{\otimes}\cdots\widehat{\otimes}\varphi_{n+1}\big)(\gamma)$$
$$=(n+1)\mathcal{K}\big(\varphi_1\widehat{\otimes}\cdots\widehat{\otimes}\varphi_{n+1}\big)(\gamma)$$
$$+\frac{n}{(n+1)n}\sum_{i=1}^{n+1}\sum_{j=1,\dots,n+1,j\neq i}\mathcal{K}\Big((\varphi_i\varphi_j)\widehat{\otimes}\varphi_1\widehat{\otimes}\cdots\widehat{\otimes}\check{\varphi}_i\widehat{\otimes}$$
$$\cdots\widehat{\otimes}\check{\varphi}_j\widehat{\otimes}\cdots\widehat{\otimes}\varphi_{n+1}\Big)(\gamma),$$

where $\check{\varphi}_i$ denotes the absense of φ_i. From here (4.1) follows.

Our aim will be to realise the operators $a(\Delta)$ as operators of multiplication by $\gamma(\Delta)$ in some L^2-space $L^2(\Gamma,\mu)$. Therefore, if we want to have operators $\mathcal{Q}(\chi_{\Delta_1}\widehat{\otimes}\cdots\widehat{\otimes}\chi_{\Delta_n})$ in our initial Hilbert space F, which will be later on realised as operators of multiplication by $\mathcal{K}(\chi_{\Delta_1}\widehat{\otimes}\cdots\widehat{\otimes}\chi_{\Delta_n})(\gamma)$, then these operators must satisfy the following recurrence relation:

$$\mathcal{Q}(\chi_{\Delta_1}\widehat{\otimes}\cdots\widehat{\otimes}\chi_{\Delta_{n+1}})=\frac{1}{(n+1)^2}\Bigg[\sum_{i=1}^{n+1}a(\Delta_i)\mathcal{Q}(\chi_{\Delta_1}\widehat{\otimes}\cdots\widehat{\otimes}\check{\chi}_{\Delta_i}\widehat{\otimes}\cdots\widehat{\otimes}\chi_{\Delta_{n+1}})$$
$$-\sum_{i=1}^{n+1}\sum_{j=1,\dots,n+1,\,j\neq i}\mathcal{Q}\big((\chi_{\Delta_i\cap\Delta_j})\widehat{\otimes}\chi_{\Delta_1}\widehat{\otimes}\cdots\widehat{\otimes}\check{\chi}_{\Delta_i}\widehat{\otimes}\cdots\widehat{\otimes}\check{\chi}_{\Delta_j}\widehat{\otimes}\cdots\widehat{\otimes}\chi_{\Delta_{n+1}}\big)\Bigg],$$
$$\Delta_1,\dots,\Delta_{n+1}\in\mathcal{B}_0(X),\ n\in\mathbb{N},$$
$$\mathcal{Q}(\chi_\Delta)=a(\Delta),\quad \Delta\in\mathcal{B}_0(X). \tag{4.2}$$

Denote by Ξ the function on $\ddot{\Gamma}_{X,0}$ given by $\Xi^{(0)}:=1$, $\Xi^{(n)}:=0$, $n\in\mathbb{N}$. Let

$$\mathcal{Q}(\Xi):=\mathbf{1}. \tag{4.3}$$

We then uniquely define $\mathcal{Q}(G)$ for each $G \in \mathcal{S}$, so that

$$\mathcal{Q}(a_1 G_1 + a_1 G_2) = a_1 \mathcal{Q}(G_1) + a_2 \mathcal{Q}(G_2), \quad a_1, a_2 \in \mathbb{R},\ G_1, G_2 \in \mathcal{S}.$$

Lemma 4.1 *For any $G_1, G_2 \in \mathcal{S}$, we have*

$$\mathcal{Q}(G_1)\mathcal{Q}(G_2) = \mathcal{Q}(G_1 \star G_2). \tag{4.4}$$

Proof. By linearity, it suffices to prove (4.4) in the case where $G_1 = \chi_{\Delta_1}^{\otimes m}$, $G_2 = \chi_{\Delta_2}^{\otimes n}$, $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$, $m, n \in \mathbb{N}$.

Setting $\Delta_1 = \cdots = \Delta_n = \Delta$ in (4.2), we have:

$$\begin{aligned} \mathcal{Q}({\chi_\Delta}^{\otimes(n+1)}) &= \frac{1}{n+1}\Big(a(\Delta)\mathcal{Q}(\chi_\Delta^{\otimes n}) - n\mathcal{Q}(\chi_\Delta^{\otimes n})\Big), \quad n \in \mathbb{N}, \\ \mathcal{Q}(\chi_\Delta) &= a(\Delta). \end{aligned}$$

Therefore, for each $k \in \mathbb{N}$, there exists a polynomial $p_k(x)$ on $\mathbb{R}$, of order k, such that, for each $\Delta \in \mathcal{B}_0(X)$, we have

$$\mathcal{Q}(\chi_\Delta^{\otimes k}) = p_k(a(\Delta)).$$

Therefore, for fixed $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$, we get

$$\mathcal{Q}(\chi_{\Delta_1}^{\otimes m})\mathcal{Q}(\chi_{\Delta_2}^{\otimes n}) = p_m\big(a(\Delta_1)\big)p_n\big(a(\Delta_2)\big).$$

Denote

$$A := \Delta_1 \setminus \Delta_2, \quad B := \Delta_1 \cap \Delta_2, \quad C := \Delta_2 \setminus \Delta_1,$$

so that the sets A, B and C are mutually disjoint and $\Delta_1 = A \cup B$, $\Delta_2 = B \cup C$. Hence

$$\begin{aligned} a(\Delta_1) &= a(A) + a(B), \\ a(\Delta_2) &= a(B) + a(C). \end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{Q}(\chi_{\Delta_1}^{\otimes m})\mathcal{Q}(\chi_{\Delta_2}^{\otimes n}) &= p_m\big(a(A)+a(B)\big)\,p_n\big(a(B)+a(C)\big)\\ &= q_{m+n}\big(a(A),a(B),a(C)\big). \end{aligned} \tag{4.5}$$

Here $q_{m+n}(\cdot,\cdot,\cdot)$ is a polynomial on $\mathbb{R}^3$ of order $m+n$, given by

$$q_{m+n}(x,y,z) = p_m(x+y)\,p_n(y+z).$$

Next, by (4.1), we get

$$\mathcal{K}(\chi_\Delta^{\otimes(n+1)})(\gamma) = \frac{1}{n+1}\Big(\gamma(\Delta)\mathcal{K}(\chi_\Delta^{\otimes n})(\gamma) - n\mathcal{K}(\chi_\Delta^{\otimes n})(\gamma)\Big),$$

and so

$$\mathcal{K}(\chi_\Delta^{\otimes k})(\gamma) = p_k(\gamma_\Delta).$$

Therefore, analogously to the above,

$$\mathcal{K}(\chi_{\Delta_1}^{\otimes m})(\gamma)\mathcal{K}(\chi_{\Delta_2}^{\otimes n})(\gamma) = q_{m+n}\big(\gamma(A),\gamma(B),\gamma(C)\big).$$

Using the definition (3.6) of the $\star$-convolution and formula (4.1) we conclude that there exists a polynomial $\tilde{q}_{m+n}(\cdot,\cdot,\cdot)$ on $\mathbb{R}^3$, of degree $m+n$, such that

$$\mathcal{K}\big(\chi_{\Delta_1}^{\otimes m}\star\chi_{\Delta_2}^{\otimes n}\big)(\gamma) = \tilde{q}_{m+n}\big(\gamma(A),\gamma(B),\gamma(C)\big).$$

Since

$$\mathcal{K}(\chi_{\Delta_1}^{\otimes m})(\gamma)\mathcal{K}(\chi_{\Delta_2}^{\otimes n})(\gamma) = \mathcal{K}\big(\chi_{\Delta_1}^{\otimes m}\star\chi_{\Delta_2}^{\otimes n}\big)(\gamma),$$

we have that, for all $\gamma\in\Gamma$,

$$q_{m+n}\big(\gamma(A),\gamma(B),\gamma(C)\big) = \tilde{q}_{m+n}\big(\gamma(A),\gamma(B),\gamma(C)\big).$$

Since both polynomials q_{m+n} and $\tilde{q}_{m+n}$ are characterized by a finite number of coefficient and since $\gamma(A)$, $\gamma(B)$ and $\gamma(C)$ may take an arbitrary value from $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$, the polynomials q_{m+n} and $\tilde{q}_{m+n}$ coincide.

On the other hand, we evidently have that

$$\mathcal{Q}\big(\chi_{\Delta_1}^{\otimes m} \star \chi_{\Delta_2}^{\otimes n}\big) = \tilde{q}_{m+n}\big(a(A), a(B), a(C)\big).$$

Hence

$$\mathcal{Q}\big(\chi_{\Delta_1}^{\otimes m} \star \chi_{\Delta_2}^{\otimes n}\big) = q_{m+n}\big(a(A), a(B), a(C)\big),$$

and by (4.5)

$$\mathcal{Q}(\chi_{\Delta_1}^{\otimes m})\mathcal{Q}(\chi_{\Delta_2}^{\otimes n}) = \mathcal{Q}\big(\chi_{\Delta_1}^{\otimes m} \star \chi_{\Delta_2}^{\otimes n}\big). \qquad \square$$

We fix any $\Omega \in D$ with $\|\Omega\|_F = 1$. We assume that there exists a (non-negative) measure ρ on $\big(\Gamma_{X,0}, \mathcal{B}_0(\Gamma_{X,0})\big)$ such that, for all $G \in \mathcal{S}$,

$$\int_{\Gamma_{X,0}} G(\eta)\, \rho(d\eta) = (\mathcal{Q}(G)\Omega, \Omega)_F. \tag{4.6}$$

Then, by analogy with (3.4), we call ρ the correlation measure of the family of commuting Hermitian operators $\big(a(\Delta)\big)_{\Delta \in \mathcal{B}_0(X)}$ (with respect to the vector Ω).

Now, we additionally assume that ρ satisfies:

(LB) *Local bound*: for each $\Delta \in \mathcal{B}_0(X)$, there exists $C_\Delta > 0$ such that

$$\rho(\Gamma_\Delta^{(n)}) \leqslant C_\Delta^n, \quad n \in \mathbb{N},$$

where $\Gamma_\Delta^{(n)} := \{\eta \in \Gamma_X^{(n)} \mid \eta \subset \Delta\}$. Furthermore, for any sequence $\{\Delta_n\}_{n\in\mathbb{N}} \in \mathcal{B}_0(X)$ such that $\Delta_n \downarrow \varnothing$ (i.e. $\Delta_1 \supset \Delta_2 \supset \Delta_3 \supset \cdots$ and $\bigcap_{n=1}^{\infty} \Delta_n = \varnothing$), we have $C_{\Delta_n} \to 0$ as $n \to \infty$.

Theorem 4.1 *Assume that a family $\big(a(\Delta)\big)_{\Delta\in\mathcal{B}_0(X)}$ of commuting Hermitian operators possesses a correlation measure which satisfies* (LB). *For each $G \in \mathcal{S}$ denote $Q(G) := \mathcal{Q}(G)\Omega$, and let $\mathfrak{F}$ denote the Hilbert space obtained as the closure of the set $\mathfrak{S} := \{Q(G) \mid G \in \mathcal{S}\}$ in F. For each $\Delta \in \mathcal{B}_0(X)$, consider $a(\Delta)$ as an operator in $\mathfrak{F}$ with domain $\mathfrak{S}$. Then, the operators*

$a(\Delta)$ are essentially self-adjoint and their closures, $\tilde{a}(\Delta)$, commute in the sense of their resolutions of the identity. Furthermore, there exists a unique probability measure μ on $\big(\Gamma_X, \mathcal{B}(\Gamma_X)\big)$ whose correlation measure is ρ, the mapping

$$\mathfrak{S} \ni Q(G) \mapsto \big(\mathcal{I}Q(G)\big)(\gamma) := \sum_{\eta \Subset \gamma} G(\eta) \in L^2(\Gamma, \mu)$$

is well-defined and extends to a unitary operator $\mathcal{I} : \mathfrak{F} \to L^2(\Gamma, \mu)$ such that, under $\mathcal{I}$, $\tilde{a}(\Delta)$ goes over into the operator of multiplication by $\gamma(\Delta)$, i.e.,

$$\mathrm{Dom}(\mathcal{I}\tilde{a}(\Delta)\mathcal{I}^{-1}) = \{f \in L^2(\Gamma, \mu) : \gamma(\Delta)f(\gamma) \in L^2(\Gamma, \mu)\}$$

and

$$(\mathcal{I}\tilde{a}(\Delta)\mathcal{I}^{-1}f)(\gamma) = \gamma(\Delta)f(\gamma), \quad f \in \mathrm{Dom}(\mathcal{I}\tilde{a}(\Delta)\mathcal{I}^{-1}).$$

Remark 4.1 Note that any probability measure μ on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ has a correlation measure. On the other hand, not every family $(a(\Delta))_{\Delta \in \mathcal{B}_0(X)}$ of commuting Hermitian operators possesses a correlation measure. Theorem 4.1 essentially shows that, if a family $(a(\Delta))_{\Delta \in \mathcal{B}_0(X)}$ possesses a correlation measure, then the joint spectrum of this family is concentrated on the configuration space Γ_X.

Proof of Theorem 4.1. Consider the bilinear form

$$\mathcal{S} \times \mathcal{S} \ni (G_1, G_2) \mapsto b_\rho(G_1, G_2) := \int_{\Gamma_{X,0}} (G_1 \star G_2)(\eta)\, \rho(d\eta). \tag{4.7}$$

By (4.4) and (4.6), for each $G \in \mathcal{S}$,

$$\begin{aligned} b_\rho(G, G) &= (\mathcal{Q}(G \star G)\Omega, \Omega)_F \\ &= (\mathcal{Q}(G)\mathcal{Q}(G)\Omega, \Omega)_F \\ &= (\mathcal{Q}(G)\Omega, \mathcal{Q}(G)\Omega)_F \geq 0. \end{aligned}$$

Denote by $\widehat{\mathcal{S}}$ the factorization of $\mathcal{S}$ consisting of factor-classes

$$\widehat{G} = \{G' \in \mathcal{S} : b_\rho(G - G', G - G') = 0\}, \quad G \in \mathcal{S}.$$

Define a Hilbert space $\mathcal{H}_\rho$ as the closure of $\widehat{\mathcal{S}}$ in the norm generated by the scalar product

$$(\widehat{G}_1, \widehat{G}_2)_{\mathcal{H}_\rho} := b_\rho(G_1, G_2). \tag{4.8}$$

Using (4.4) and (4.6), we see that

$$(\widehat{G}_1, \widehat{G}_2)_{\mathcal{H}_\rho} = (Q(G_1), Q(G_2))_F, \tag{4.9}$$

so that we have the unitary isomorphism $U : \mathfrak{F} \to \mathcal{H}_\rho$ defined through $UQ(G) := \widehat{G}$ for $G \in \mathcal{S}$.

For each $\Delta \in \mathcal{B}_0(X)$, we define an operator A_Δ in $\mathcal{H}_\rho$ as the image of the operator $a(\Delta)$ under U. Hence, $\mathrm{Dom}(A_\Delta) = \widehat{\mathcal{S}}$ and since

$$a(\Delta)Q(G) = \mathcal{Q}(\chi_\Delta)\mathcal{Q}(G)\Omega = \mathcal{Q}(\chi_\Delta \star G)\Omega = Q(\chi_\Delta \star G), \quad G \in \mathcal{S},$$

we get:

$$A_\Delta \widehat{G} := \widehat{\chi_\Delta \star G}, \quad G \in \mathcal{S}. \tag{4.10}$$

We will now show that the operators $(A_\Delta)_{\Delta \in \mathcal{B}_0(X)}$ (hence also the operators $(a(\Delta))_{\Delta \in \mathcal{B}_0(X)}$) are essentially self-adjoint, and their closures commute in the sense of their resolutions of the identity.

Lemma 4.2 *Each $\widehat{G} \in \widehat{\mathcal{S}}$ is an analytic vector for any A_Δ, $\Delta \in \mathcal{B}_0(X)$.*

Proof. For any $G_1, G_2 \in \mathcal{S}$, we have by (4.9), Lemma 4.1, and (4.5),

$$\begin{aligned}(\widehat{G}_1, \widehat{G}_2)_{\mathcal{H}_\rho} &= \big(Q(G_1), Q(G_2)\big)_F \\ &= \big(\mathcal{Q}(G_1)\Omega, \mathcal{Q}(G_2)\Omega\big)_F \\ &= \big(\mathcal{Q}(G_1)\mathcal{Q}(G_2)\Omega, \Omega\big)_F\end{aligned}$$

$$= \left(\mathcal{Q}(G_1 \star G_2)\Omega, \Omega\right)_F$$
$$= \int_{\Gamma_{X,0}} (G_1 \star G_2)(\eta)\rho(d\eta).$$

Hence, by (4.10), we get, for each $G \in \mathcal{S}$, $k \in \mathbb{N}$, $\Delta \in \mathcal{B}_0(X)$,

$$\begin{aligned}
\|A_\Delta^k \widehat{G}\|_{\mathcal{H}_\rho}^2 &= \|\widehat{\chi_\Delta^{\star k} \star G}\|_{\mathcal{H}_\rho}^2 \\
&= \int_{\Gamma_{X,0}} (\chi_\Delta^{\star k} \star G) \star (\chi_\Delta^{\star k} \star G)(\eta)\rho(d\eta) \\
&= \int_{\Gamma_{X,0}} (G^{\star 2} \star \chi_\Delta^{\star 2k})(\eta)\rho(d\eta). \qquad (4.11)
\end{aligned}$$

Since any finite linear combination of analytic vectors of a given operator is again an analytic vector of this operator, it suffices to prove that, for any $\Delta, \Delta' \in \mathcal{B}_0(X)$ and any $n \in \mathbb{N}_0$, $\chi_{\Delta'}^{\otimes n}$ is an analytic vector for A_Δ, i.e.,

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \|A_\Delta^k \chi_{\Delta'}^{\otimes n}\|_{\mathcal{H}_\rho} < \infty, \quad \text{for some} \quad t > 0$$

Denote $\Lambda = \Delta \cup \Delta'$. Then, by (4.11)

$$\begin{aligned}
\|A_\Delta^k \chi_{\Delta'}^{\otimes n}\|_{\mathcal{H}_\rho} &= \left(\int_{\Gamma_{X,0}} (\chi_{\Delta'}^{\otimes n})^{\star 2} \star \chi_\Delta^{\star 2k}(\eta)\rho(d\eta)\right)^{1/2} \\
&\leq \left(\int_{\Gamma_{X,0}} (\chi_\Lambda^{\otimes n})^{\star 2} \star \chi_\Lambda^{\star 2k}(\eta)\rho(d\eta)\right)^{1/2}. \qquad (4.12)
\end{aligned}$$

Using the definition of $\star$-convolution, we see that the function $(\chi_\Lambda^{\otimes n})^{\star 2} \star \chi_\Lambda^{\star 2k}$ is a finite linear combination (with non-negative coefficients) of the indicator functions $\chi_\Lambda^{\otimes i}, i = 0, 1, \ldots, 2n + 2k$. Since, by the (LB)

$$\int_{\Gamma_{X,0}} \chi_\Lambda^{\otimes i}(\eta)\rho(d\eta) = \rho(\ddot{\Gamma}_\Lambda^{(i)}) \leqslant C_\Lambda^i,$$

we have from (4.12)

$$\|A_\Delta^k \chi_{\Delta'}^{\otimes n}\|_{\mathcal{H}_\rho} \leqslant \left(\int_{\Gamma_{X,0}} (\chi_\Lambda^{\otimes n})^{\star 2} \star \chi_\Lambda^{\star 2k}(\eta)\tilde{\rho}(d\eta)\right)^{1/2}, \qquad (4.13)$$

where we have set

$$\tilde{\rho}(\ddot{\Gamma}_{\Lambda}^{(i)}) = C_{\Lambda}^{i} \tag{4.14}$$

Note that we understand the expression on the right hand side of (4.13) as if we integrate with respect to some measure $\tilde{\rho}$ satisfying (4.14). In what follows, we will assume that $C_\Lambda \geqslant 1$ (otherwise set $C_\Lambda = 1$).

Next, using the definition of $\star$-convolution we have:

$$\chi_\Lambda^{\otimes i} \star \chi_\Lambda = (i+1)\chi_\Lambda^{\otimes(i+1)} + i\chi_\Lambda^{\otimes i}.$$

Therefore,

$$\begin{aligned}
\int_{\Gamma_{X,0}} (\chi_\Lambda^{\otimes i} \star \chi_\Lambda)(\eta)\tilde{\rho}(d\eta) &\leqslant \big((i+1)C_\Lambda + i\big) \int_{\Gamma_{X,0}} \chi_\Lambda^{\otimes i}(\eta)\tilde{\rho}(d\eta) \\
&\leqslant C_\Lambda(2i+1) \int_{\Gamma_{X,0}} \chi_\Lambda^{\otimes i}(\eta)\tilde{\rho}(d\eta) \\
&\leqslant 2C_\Lambda(i+1) \int_{\Gamma_{X,0}} \chi_\Lambda^{\otimes i}(\eta)\tilde{\rho}(d\eta).
\end{aligned} \tag{4.15}$$

Hence, by (4.15), we get

$$\begin{aligned}
\int_{\Gamma_{X,0}} (\chi_\Lambda^{\otimes n})^{\star 2} \star \chi_\Lambda^{\star 2k}(\eta)\tilde{\rho}(d\eta) &= \int_{\Gamma_{X,0}} \Big((\chi_\Lambda^{\otimes n})^{\star 2} \star \chi_\Lambda^{\star(2k-1)}\Big) \star \chi_\Lambda(\eta)\tilde{\rho}(d\eta) \\
&\leqslant 2C_\Lambda(2n+2k) \int_{\Gamma_{X,0}} (\chi_\Lambda^{\otimes n})^{\star 2} \star \chi_\Lambda^{\star(2k-1)}(\eta)\tilde{\rho}(d\eta) \\
&= 2C_\Lambda(2n+2k) \int_{\Gamma_{X,0}} \Big((\chi_\Lambda^{\otimes n})^{\star 2} \star \chi_\Lambda^{\star(2k-2)}\Big) \star \chi_\Lambda(\eta)\tilde{\rho}(d\eta) \\
&\leqslant 2C_\Lambda(2n+2k)2C_\Lambda(2n+2k-1) \int_{\Gamma_{X,0}} (\chi_\Lambda^{\otimes n})^{\star 2} \star \chi_\Lambda^{\star(2k-2)}\tilde{\rho}(d\eta) \\
&\leqslant \cdots \leqslant (2C_\Lambda)^{2k}(2n+2k)(2n+2k-1)\cdots(2n+1)\times \\
&\quad \int_{\Gamma_{X,0}} (\chi_\Lambda^{\otimes n})^{\star 2}\tilde{\rho}(d\eta) \\
&= (2C_\Lambda)^{2k}\frac{(2n+2k)!}{(2n)!}\,\alpha,
\end{aligned}$$

where

$$\alpha = \int_{\Gamma_{X,0}} (\chi_\Lambda^{\otimes n})^{\star 2}\tilde{\rho}(d\eta).$$

Hence,

$$\|A^k_\Delta \chi^{\otimes n}_{\Delta'}\|_{\mathcal{H}_\rho} \leqslant (2C_\Lambda)^k \left(\frac{(2n+2k)!}{(2n)!}\right)^{1/2} \sqrt{\alpha}. \tag{4.16}$$

For each $n \in \mathbb{N}$, we have:

$$\begin{aligned}(2n)! &= 1\cdot 2\cdot 3\cdot 4\cdots(2n-1)\cdot 2n\\ &\leqslant 2\cdot 2\cdot 4\cdot 4\cdots 2n\cdot 2n\\ &= (2\cdot 4\cdots 2n)^2\\ &= (2^n n!)^2,\end{aligned} \tag{4.17}$$

which implies by (4.16)

$$\|A^k_\Delta \chi^{\otimes n}_{\Delta'}\|_{\mathcal{H}_\rho} \leqslant (2C_\Lambda)^k \big((2n)!\big)^{-1/2} 2^{n+k}(n+k)!\sqrt{\alpha},$$

and so

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{t^k}{k!}\|A^k_\Delta \chi^{\otimes n}_{\Delta'}\|_{\mathcal{H}_\rho} &\leqslant \big((2n)!\big)^{-1/2} 2^n\sqrt{\alpha}\sum_{k=0}^{\infty}\frac{(4C_\Lambda t)^k}{k!}(k+n)!\\ &= \big((2n)!\big)^{-1/2} 2^n\sqrt{\alpha}\sum_{k=0}^{\infty}(4C_\lambda t)^k(k+1)(k+2)\cdots(k+n).\end{aligned} \tag{4.18}$$

For each $t \in (0, (4C_\Lambda)^{-1})$, we have

$$\begin{aligned}&\Big((4C_\Lambda t)^k(k+1)(k+2)\cdots(k+n)\Big)^{1/k}\\ &\quad = 4C_\Lambda t(k+1)^{1/k}(k+2)^{1/k}\cdots(k+n)^{1/k} \to 4C_\Lambda t < 1 \quad \text{as} \quad k\to\infty\end{aligned}$$

since $(k+i)^{1/k} \to 1$ as $k \to \infty$ for each fixed i. Therefore, for $0 < t < (4C_\Lambda)^{-1}$, the series (4.18) converges. $\square$

By Theorem 3.2 and Lemma 4.2, for each $\Delta \in \mathcal{B}_0(X)$, the closure of A_Δ, denote by $\tilde{A}_\Delta$ is a self-adjoint operator in $\mathcal{H}_\rho$ and for any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$, the operators $\tilde{A}_{\Delta_1}$ and $\tilde{A}_{\Delta_2}$ commute in the sense of their resolutions of the identity.

We will now construct a consistent family of probability measure. For each $\Delta \in \mathcal{B}_0(X)$, denote by E_Δ the resolution of the identity of $\tilde{A}_\Delta$. By the proved above, for any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$, the resolutions of the identity E_{Δ_1} and E_{Δ_2} commute.

So, according to Chapter 2, for any $\Delta_1, \ldots, \Delta_n \in \mathcal{B}_0(X)$ we can construct the joint resolution of the identity

$$E_{\Delta_1,\ldots,\Delta_n} := E_{\Delta_1} \times \cdots \times E_{\Delta_n}. \tag{4.19}$$

Recall the definition of the function Ξ on $\ddot{\Gamma}_{X,0}$. Then

$$\nu_{\Delta_1,\ldots,\Delta_n}(\cdot) := (E_{\Delta_1,\ldots,\Delta_n}(\cdot)\widehat{\Xi}, \widehat{\Xi})_{\mathcal{H}_\rho} \tag{4.20}$$

is a probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Furthermore, it is clear that

$$\{\nu_{\Delta_1,\ldots,\Delta_n} \mid \Delta_1, \ldots, \Delta_n \in \mathcal{B}_0(X),\ n \in \mathbb{N}\} \tag{4.21}$$

is a consistent family of probability measures.

Next, let us show that there exists a point process on X whose "finite-dimensional distributions" are given through (4.21). First, we will prove this result locally.

For any $\Delta \in \mathcal{B}_0(X)$, denote

$$\Gamma_\Delta := \{\eta \in \Gamma_{X,0} \mid \eta \subset \Delta\},$$

and let $\mathcal{B}(\Gamma_\Delta)$ be the trace σ-algebra of $\mathcal{B}(\Gamma_{X,0})$ on Γ_Δ.

Let us introduce an analogue of the operator $\mathcal{K}$ (see Section 2.3) on Γ_Δ. So, we define a mapping $\mathcal{K}_\Delta$, which transforms the set of all (complex-valued) functions on Γ_Δ into itself, as follows:

$$(\mathcal{K}_\Delta G)(\eta) := \sum_{\xi \subset \eta} G(\xi), \quad \eta \in \Gamma_\Delta. \tag{4.22}$$

We evidently have:

$$\big(\mathcal{K}_\Delta(G_1 \star G_2)\big)(\eta) = (\mathcal{K}_\Delta G_1)(\eta)(\mathcal{K}_\Delta G_2)(\eta) \tag{4.23}$$

($G_1 \star G_2$ being given by (3.6)). The inverse of $\mathcal{K}_\Delta$ is then given by

$$(\mathcal{K}_\Delta^{-1} G)(\eta) = \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} G(\xi), \quad \eta \in \Gamma_\Delta \tag{4.24}$$

(the latter being a well-known result, see e.g. [15], which can be checked by direct calculations).

To find the pre-image, under $\mathcal{K}_\Delta$, of an exponential function, we define, for any $f : \Delta \to \mathbb{C}$, a function $\mathrm{Exp}_\Delta(f, \cdot) : \Gamma_\Delta \to \mathbb{C}$ by

$$\begin{aligned} \mathrm{Exp}_\Delta(f, \varnothing) &:= 1, \\ \mathrm{Exp}_\Delta(f, \{x_1, \dots, x_n\}) &:= f(x_1) \cdots f(x_n), \quad \{x_1, \dots, x_n\} \in \Gamma_\Delta,\ n \in \mathbb{N}. \end{aligned}$$

By (4.24), for any $\varphi : \Delta \to \mathbb{C}$, we have:

$$\big(\mathcal{K}_\Delta^{-1} \exp[\langle \varphi, \cdot \rangle]\big)(\eta) = \mathrm{Exp}_\Delta(e^\varphi - 1, \eta), \quad \eta \in \Gamma_\Delta, \tag{4.25}$$

where $\langle \varphi, \eta \rangle := \sum_{x \in \eta} \varphi(x)$.

Let $\Delta \in \mathcal{B}_0(X)$ be so small that

$$C_\Delta \leqslant \frac{1}{16 + \delta}, \quad \delta > 0 \tag{4.26}$$

(see (LB)). We define a set function on $\mathcal{B}(\Gamma_\Delta)$ by

$$\mu^\Delta(A) := \int_{\Gamma_\Delta} (\mathcal{K}_\Delta^{-1} \chi_A)(\eta) \rho(d\eta), \quad A \in \mathcal{B}(\Gamma_\Delta). \tag{4.27}$$

Since

$$\sum_{\xi \subset \eta} 1 = 2^n \quad \text{if } |\eta| = n, \tag{4.28}$$

(LB) and (4.26) imply that μ^Δ is a signed measure of finite variation. Indeed the full variation of μ^Δ on Γ_Δ may be estimate as follows

$$|\mu^\Delta(\Gamma_\Delta)| \leqslant \sum_{n=0}^{\infty} \frac{2^n}{(16+\delta)^n} < \infty.$$

Next, we will show that μ^Δ is, in fact, a probability measure on $(\Gamma_\Delta, \mathcal{B}(\Gamma_\Delta))$.

Let $\Delta_1, \dots, \Delta_n \in \mathcal{B}_0(X)$ be subsets of Δ, $n \in \mathbb{N}$, and for simplicity of notations we assume that these sets are mutually disjoint. Then, by (4.25) and (4.27), for any $(y_1, \dots, y_n) \in \mathbb{R}^n$,

$$\begin{aligned} L(y_1, \dots, y_n) &:= \int_{\Gamma_\Delta} \exp\left[\langle i(y_1\chi_{\Delta_1} + \cdots + y_n\chi_{\Delta_n}), \eta\rangle\right] \mu^\Delta(d\eta) \\ &= \int_{\Gamma_\Delta} \operatorname{Exp}_\Delta\left((e^{iy_1}-1)\chi_{\Delta_1} + \cdots + (e^{iy_n}-1)\chi_{\Delta_n}, \eta\right)\rho(d\eta), \end{aligned} \tag{4.29}$$

where we used the evident formula

$$e^{iy_1\chi_{\Delta_1}+\cdots+iy_n\chi_{\Delta_n}} - 1 = \sum_{i=1}^{n} (e^{iy_i}-1)\chi_{\Delta_i}$$

(recall that the sets Δ_i are mutually disjoint). Note also that

$$|e^{iy_1\chi_{\Delta_1}+\cdots+iy_n\chi_{\Delta_n}} - 1| \leqslant 2,$$

and so by (4.26)

$$\int_{\Gamma_\Delta} \operatorname{Exp}_\Delta(|e^{iy_1\chi_{\Delta_1}+\cdots+iy_n\chi_{\Delta_n}} - 1|, \eta)\rho(d\eta) \leqslant \sum_{n=0}^{\infty} \left(\frac{2}{16+\delta}\right)^n < \infty.$$

Lemma 4.3 *The function $L : \mathbb{R}^n \to \mathbb{C}$ is positive definite in the sense of the Fourier analysis on $\mathbb{R}^n$.*

Proof. Fix any $y_1, \dots, y_m \in \mathbb{R}^n$, $y_k = (y_k^{(1)}, \dots, y_k^{(n)})$, $k = 1, \dots, m$, and fix any $c_1, \dots, c_m \in \mathbb{C}$. We have to prove that

$$\sum_{k,l=1}^{m} c_k \overline{c_l} L(y_k - y_l) \geqslant 0.$$

For each $k \in \{1, \dots, m\}$, denote

$$\varphi_k = y_k^{(1)} \chi_{\Delta_1} + \dots + y_k^{(n)} \chi_{\Delta_n}.$$

Then, by (4.29),

$$\begin{aligned}
\sum_{k,l=1}^{m} c_k \overline{c_l} L(y_k - y_l) &= \sum_{k,l=1}^{m} c_k \overline{c_l} \int_{\Gamma_\Delta} \operatorname{Exp}_\Delta \Big(\exp \Big[i \big((y_k^{(1)} - y_l^{(1)}) \chi_{\Delta_1} + \\
&\qquad \dots + (y_k^{(n)} - y_l^{(n)}) \chi_{\Delta_n} \big) \big) - 1 \Big], \eta \Big) \rho(d\eta) \\
&= \sum_{k,l=1}^{m} c_k \overline{c_l} \int_{\Gamma_\Delta} \operatorname{Exp}_\Delta \Big(\exp[i(\varphi_k - \varphi_l)] - 1, \eta \Big) \rho(d\eta) \\
&= \sum_{k,l=1}^{m} c_k \overline{c_l} \int_{\Gamma_\Delta} \operatorname{Exp}_\Delta \Big((e^{i\varphi_k} - 1) \\
&\qquad + (e^{-i\varphi_l} - 1) + (e^{i\varphi_k} - 1)(e^{-i\varphi_l} - 1), \eta \Big) \rho(d\eta). \quad (4.30)
\end{aligned}$$

By [15, Lemma 5.3], we have:

$$\begin{aligned}
&\operatorname{Exp}_\Delta \Big((e^{i\varphi_k} - 1) \quad + (e^{-i\varphi_l} - 1) + (e^{i\varphi_k} - 1)(e^{-i\varphi_l} - 1), \eta \Big) \\
&\qquad = \operatorname{Exp}_\Delta (e^{i\varphi_k} - 1, \eta) \star \operatorname{Exp}_\Delta (e^{-i\varphi_l} - 1, \eta).
\end{aligned}$$

Hence, we continue (4.30) as follows:

$$\begin{aligned}
&= \sum_{k,l=1}^{m} c_k \overline{c_l} \int_{\Gamma_\Delta} \operatorname{Exp}_\Delta (e^{i\varphi_k} - 1, \eta) \star \operatorname{Exp}_\Delta (e^{-i\varphi_l} - 1, \eta) \rho(d\eta) \\
&= \sum_{k,l=1}^{m} c_k \overline{c_l} \int_{\Gamma_\Delta} \operatorname{Exp}_\Delta (e^{i\varphi_k} - 1, \eta) \star \overline{\operatorname{Exp}_\Delta (e^{i\varphi_l} - 1, \eta)} \rho(d\eta) \\
&= \int_{\Gamma_\Delta} \Big(\sum_{k=1}^{m} c_k \operatorname{Exp}_\Delta (e^{i\varphi_k} - 1, \eta) \Big) \star \overline{\Big(\sum_{l=1}^{m} c_l \operatorname{Exp}_\Delta (e^{i\varphi_l} - 1, \eta) \Big)} \rho(d\eta). \quad (4.31)
\end{aligned}$$

We know that, for each $G \in \mathcal{S}$, we have:

$$\int_{\Gamma_{X,0}} (G \star G)(\eta) \rho(d\eta) \geqslant 0.$$

Denote by $\mathcal{S}_{\mathbb{C}}$ the complexification of $\mathcal{S}$, i.e., all functions of the form G_1+iG_2, where $G_1, G_2 \in \mathcal{S}$. Then, for each $G = G_1 + iG_2 \in \mathcal{S}_{\mathbb{C}}$, we have

$$\begin{aligned}
\int_{\Gamma_{X,0}} (G \star \bar{G})(\eta)\rho(d\rho) &= \int_{\Gamma_{X,0}} \Big((G_1 + iG_2) \star (G_1 - iG_2)\Big)(\eta)\rho(d\eta) \\
&= \int_{\Gamma_{X,0}} \Big((G_1 \star G_1)(\eta) + (G_2 \star G_2)(\eta) \\
&\qquad - i(G_1 \star G_2)(\eta) + i(G_2 \star G_1)\Big)\rho(d\eta) \\
&= \int_{\Gamma_{X,0}} \Big((G_1 \star G_1)(\eta) + (G_2 \star G_2)(\eta) \\
&\qquad - i(G_1 \star G_2)(\eta) + i(G_1 \star G_2)(\eta)\Big)\rho(d\eta) \\
&= \int_{\Gamma_{X,0}} \Big((G_1 \star G_1)(\eta) + (G_2 \star G_2)(\eta)\Big)\rho(d\eta) \geqslant 0.
\end{aligned}$$

In particular, for each $G \in \mathcal{S}_{\mathbb{C}}$ with support in Γ_Δ, we have:

$$\int_{\Gamma_\Delta} (G \star \bar{G})(\eta)\rho(d\eta) \geqslant 0.$$

Let

$$G := \sum_{k=1}^{m} c_k \operatorname{Exp}_\Delta(e^{i\varphi_k} - 1, \eta).$$

This function, of course, does not belong to $\mathcal{S}_{\mathbb{C}}$, however, its restriction to each $\Gamma_\Delta^{(n)}$, denoted by $G^{(n)}$, does. Therefore, for each $N \in \mathbb{N}$,

$$\int_{\Gamma_\Delta} \Big(\big(\sum_{n=1}^{N} G^{(n)}\big) \star \overline{\big(\sum_{n=1}^{N} G^{(n)}\big)}\Big)(\eta)\rho(d\eta) \geqslant 0.$$

Thus, to prove the lemma, it suffices to show that

$$\int_{\Gamma_\Delta} \Big(\big(\sum_{n=1}^{N} G^{(n)}\big) \star \overline{\big(\sum_{n=1}^{N} G^{(n)}\big)}\Big)(\eta)\rho(d\eta) \to \int_{\Gamma_\Delta} (G \star \bar{G})(\eta)\rho(d\eta). \tag{4.32}$$

It is clear that for each fixed $\eta \in \Gamma_\Delta$, we have:

$$\Big(\big(\sum_{n=1}^{N} G^{(n)}\big) \star \overline{\big(\sum_{n=1}^{N} G^{(n)}\big)}\Big)(\eta) = (G \star \bar{G})(\eta)$$

if $N \geqslant |\eta|$. So, we only need to find an integrable function which dominates all $\left|\left(\left(\sum_{n=1}^N G^{(n)}\right) \star \overline{\left(\sum_{n=1}^N G^{(n)}\right)}\right)(\eta)\right|$. To this end, we note that

$$|e^{i\varphi_k} - 1| \leqslant 2,$$

and so

$$|\operatorname{Exp}_\Delta(e^{i\varphi_k} - 1, \eta)| \leqslant 2^{|\eta|}. \tag{4.33}$$

Since

$$\sum_{(\eta_1,\eta_2,\eta_3)\in P_3(\eta)} 1 = \Big(\sum_{\eta_1\subset\eta} 1\Big)^2 = (2^{|\eta|})^2 = 4^{|\eta|},$$

by (4.33), we have

$$\begin{aligned}\left|\left(\Big(\sum_{n=1}^N G^{(n)}\Big) \star \overline{\Big(\sum_{n=1}^N G^{(n)}\Big)}\right)(\eta)\right| &\leqslant \Big(\sum_{k=1}^m |c_k|\Big)^2 (2^{|\eta|})^2 4^{|\eta|} \\ &= \Big(\sum_{k=1}^m |c_k|\Big)^2 16^{|\eta|}.\end{aligned}$$

Setting

$$F(\eta) = \Big(\sum_{k=1}^m |c_k|\Big)^2 16^{|\eta|}, \quad \eta \in \Gamma_\Delta,$$

we get

$$\left|\left(\Big(\sum_{n=1}^N G^{(n)}\Big) \star \overline{\Big(\sum_{n=1}^N G^{(n)}\Big)}\right)(\eta)\right| \leqslant F(\eta).$$

Finally, by (4.26), we get:

$$\int_{\Gamma_\Delta} F(\eta)\rho(d\eta) < \infty.$$

Hence, by the dominated convergence theorem, we get (4.32). □

By Lemma 4.3, L is the Fourier transform of a probability measure on $\mathbb{R}^n$. Therefore, under the mapping

$$\Gamma_\Delta \ni \eta \mapsto \big(\eta(\Delta_1), \ldots, \eta(\Delta_n)\big) \in \mathbb{R}^n,$$

the image of the signed measure μ^Δ is a probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, which we denote by $\mu^\Delta_{\Delta_1,\dots,\Delta_n}$. We also observe that the sets

$$\{\eta \in \Gamma_{X,0} \,|\, \big(\eta(\Delta_1),\dots,\eta(\Delta_n)\big) \in B_n\},$$
$$B_n \in \mathcal{B}(\mathbb{R}^n),\ \Delta_1,\dots,\Delta_n \in \mathcal{B}_0(X),\ \Delta_1 \cup \dots \cup \Delta_n \subset \Delta,\ n \in \mathbb{N}, \tag{4.34}$$

generate the σ-algebra $\mathcal{B}(\Gamma_\Delta)$. Hence, μ^Δ is a probability measure on $\big(\Gamma_\Delta, \mathcal{B}(\Gamma_\Delta)\big)$.

Next, we will prove that for any $\Delta_1,\dots,\Delta_n \in \mathcal{B}_0(X)$ such that

$$\Delta_1 \cup \dots \cup \Delta_n \subset \Delta, \quad n \in \mathbb{N},$$

we have

$$\nu_{\Delta_1,\dots,\Delta_n} = \mu^\Delta_{\Delta_1,\dots,\Delta_n}. \tag{4.35}$$

Using (4.7), (4.8), (4.10), (4.19),(4.20), (4.23), (4.24) and Section 3.2, for any $y^{(1)},\dots,y^{(k)} \in \mathbb{R}^n$, $k \in \mathbb{N}$, we have:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \prod_{i=1}^k (x, y^{(i)})_{\mathbb{R}^n}\, d\nu_{\Delta_1,\dots,\Delta_n}(x) \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^k (x, y^{(i)})_{\mathbb{R}^n}\, d\Big(E_{\Delta_1,\cdots,\Delta_n}(x)\hat{\Xi}, \hat{\Xi}\Big)_{\mathcal{H}_\rho} \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^k \Big(\sum_{j=1}^n x_j y_j^{(i)}\Big) d\Big(E_{\Delta_1} \times \dots \times E_{\Delta_n}(x_1,\dots,x_n)\hat{\Xi}, \hat{\Xi}\Big)_{\mathcal{H}_\rho} \\
&= \Big(\prod_{i=1}^k \Big(\sum_{j=1}^n y_j^{(i)} A_{\Delta_j}\Big)\hat{\Xi}, \hat{\Xi}\Big)_{\mathcal{H}_\rho} \\
&= \Big(\Big(\big(\sum_{j=1}^n y_j^{(1)} \chi_{\Delta_j}\big) \star \dots \star \big(\sum_{j=1}^n y_j^{(k)} \chi_{\Delta_j}\big) \star \Xi\Big)^\wedge, \hat{\Xi}\Big)_{\mathcal{H}_\rho} \\
&= \int_{\Gamma_\Delta} \Big(\sum_{j=1}^n y_j^{(1)} \chi_{\Delta_j}\Big) \star \dots \star \Big(\sum_{j=1}^n y_j^{(k)} \chi_{\Delta_j}\Big) \star \Xi \star \Xi(\eta)\rho(d\eta)
\end{aligned}$$

$$= \int_{\Gamma_\Delta} \Big(\sum_{j=1}^n y_j^{(1)} \chi_{\Delta_j}\Big) \star \cdots \star \Big(\sum_{j=1}^n y_j^{(k)} \chi_{\Delta_j}\Big)(\eta)\rho(d\eta)$$

$$= \int_{\Delta} \mathcal{K}_\Delta^{-1}\Big(\sum_{j=1}^n y_j^{(1)} \chi_{\Delta_j}\Big)(\eta) \cdots \mathcal{K}_\Delta^{-1}\Big(\sum_{j=1}^n y_j^{(k)} \chi_{\Delta_j}\Big)(\eta)\mu_\Delta(d\eta)$$

$$= \int_{\Gamma_\Delta} \Big(\sum_{j=1}^n y_j^{(1)} \eta(\Delta_j)\Big) \cdots \Big(\sum_{j=1}^n y_j^{(k)} \eta(\Delta_j)\Big)\mu_\Delta(d\eta)$$

$$= \int_{\mathbb{R}^n} \Big(\sum_{j=1}^n y_j^{(1)} x_j\Big) \cdots \Big(\sum_{j=1}^n y_j^{(k)} x_j\Big) d\mu^{\Delta}_{\Delta_1,\ldots,\Delta_n}(x_1,\ldots,x_n)$$

$$= \int_{\mathbb{R}^n} (y^{(1)}, x) \cdots (y^{(n)}, x) d\mu^{\Delta}_{\Delta_1,\ldots,\Delta_n}(x). \tag{4.36}$$

By (4.15) and (4.17)

$$\Big| \int_{\Gamma_{X,0}} \Big(\sum_{j=1}^n y_j^{(1)} \chi_{\Delta_j}\Big) \star \cdots \star \Big(\sum_{j=1}^n y_j^{(2k)} \chi_{\Delta_j}\Big)(\eta)\rho(d\eta)\Big|$$

$$\leqslant \int_{\Gamma_{X,0}} \Big(\sum_{j=1}^n |y_j^{(1)}| \chi_{\Delta_j}\Big) \star \cdots \star \Big(\sum_{j=1}^n |y_j^{(2k)}| \chi_{\Delta_j}\Big)(\eta)\rho(d\eta)$$

$$\leqslant \Big(\prod_{i=1}^{2k} \|y_i\|_{\max}\Big) \times \int_{\Gamma_{X,0}} \Big(\sum_{j=1}^n \chi_{\Delta_j}\Big)^{\star(2k)}(\eta)\rho(d\eta)$$

$$\leqslant \Big(\prod_{i=1}^{2k} \|y_i\|_{\max}\Big) \int_{\Gamma_{X,0}} (n\chi_\Delta)^{\star(2k)}(\eta)d(\eta)$$

$$= n^{2k}\Big(\prod_{i=1}^{2k} \|y_i\|_{\max}\Big) \int_{\Gamma_{X,0}} \chi_\Delta^{\star(2k)}(\eta)\rho(d\eta)$$

$$\leqslant n^{2k}\Big(\prod_{i=1}^{2k} \|y_i\|_{\max}\Big)(2C_\Delta)^{2k}(2k)!$$

$$= (2C_\Delta n)^{2k}(2k)!\Big(\prod_{i=1}^{2k} \|y_i\|_{\max}\Big)$$

$$\leqslant (4C_\Delta n)^{2k}(k!)^2\Big(\prod_{i=1}^{2k} \|y_i\|_{\max}\Big),$$

where $\|\cdot\|_{\max}$ denotes the maximum norm on $\mathbb{R}^n$. Therefore, we have

$$\Big| \int_{\mathbb{R}^n} \prod_{i=1}^k (x, y^{(i)})_{\mathbb{R}^n}\, d\nu_{\Delta_1,\ldots,\Delta_n}(x)\Big|$$

$$= \Big| \int_{\mathbb{R}^n} \prod_{i=1}^{k} (x, y^{(i)})_{\mathbb{R}^n} \, d\mu^{\Delta}_{\Delta_1,\dots,\Delta_n}(x) \Big|$$

$$\leqslant (4C_\Delta n)^{2k} (k!)^2 \Big(\prod_{i=1}^{2k} \|y_i\|_{\max} \Big). \tag{4.37}$$

Since $\nu_{\Delta_1,\dots,\Delta_n}$ and $\mu^{\Delta}_{\Delta_1,\dots,\Delta_n}$ are probability measures on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, since these measures have the same moments (see (4.36)), and since these moments satisfy estimate (4.37), we conclude from the theorem on uniqueness of the solution of a moment problem (e.g. [5, Chapter 5, Theorem 2.1 and Remark 3]), that the measures $\nu_{\Delta_1,\dots,\Delta_n}$ and $\mu^{\Delta}_{\Delta_1,\dots,\Delta_n}$ coincide, i.e., (4.35) holds.

Next, let $\Delta' \in \mathcal{B}_0(X)$ be such that $\Delta' \subset \Delta$. It is clear that $\Gamma_{\Delta'} \in \mathcal{B}(\Gamma_\Delta)$ and $\mathcal{B}(\Gamma_{\Delta'})$ coincides with the trace σ-algebra of $\mathcal{B}(\Gamma_\Delta)$ on $\Gamma_{\Delta'}$. Then it follows from the above that $\mu^{\Delta'}$ is the restriction of μ^{Δ} to $\mathcal{B}(\Gamma_{\Delta'})$.

Now, we will show that there exists a random measure M on X such that, for any $\Delta_1, \dots, \Delta_n \in \mathcal{B}_0(X)$, $n \in \mathbb{N}$, the distribution of $\big(M(\Delta_1), \dots, M(\Delta_n)\big)$ is $\nu_{\Delta_1,\dots,\Delta_n}$ (see e.g. [14] for details on random measures).

Lemma 4.4 i) *For any* $\Delta_1, \dots, \Delta_n \in \mathcal{B}_0(X)$, $n \in \mathbb{N}$,

$$\nu_{\Delta_1,\dots,\Delta_n}\big([0, +\infty)^n\big) = 1.$$

ii) *For any disjoint* $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$,

$$\nu_{\Delta_1,\Delta_2,\Delta_1\cup\Delta_2}\big(\{(x,y,z) \in \mathbb{R}^3 \mid x + y = z\}\big) = 1.$$

iii) *Let* $\Delta_n \in \mathcal{B}_0(X)$, $n \in \mathbb{N}$, *be such that* $\Delta_n \downarrow \varnothing$. *Then* ν_{Δ_n} *weakly converges to* ε_0.

Proof. i) By (LB), for any $x \in X$, there exists an open neighbourhood of x, denoted by $\Delta(x)$, such that $\Delta(x) \in \mathcal{B}_0(X)$ and $C_{\Delta(x)} \leqslant 1/(16+\delta)$. Therefore, for any $\Delta \in \mathcal{B}_0(X)$, there exist mutually disjoint sets $\Delta_1, \dots, \Delta_m \in \mathcal{B}_0(X)$, $m \in \mathbb{N}$, such that $\Delta = \Delta_1 \cup \dots \cup \Delta_m$, $C_{\Delta_i} \leqslant 1/(16+\delta)$, $i = 1, \dots, m$. (We have used the definition of a compact set: from any open covering of the set one can choose a finite covering of the set).

For each $i = 1, \dots, m$, denote

$$C_i = \{x = (x_1 \dots, x_m) \in \mathbb{R}^m : x_i \geqslant 0\}.$$

Then, using the definition of $\nu_{\Delta_1,\dots,\Delta_m}$, we have

$$\nu_{\Delta_1,\dots,\Delta_m}(C_i) = \nu_{\Delta_i}\big([0,+\infty)\big) = 1, \quad i = 1, \dots, m.$$

Since

$$\bigcap_{i=1}^{m} C_i = [0,+\infty)^m,$$

we therefore get

$$\nu_{\Delta_1,\dots,\Delta_m}\big([0,+\infty)^m\big) = 1.$$

Since $\Delta = \Delta_1 \cup \dots \cup \Delta_m$ and $\Delta_1, \dots, \Delta_m$ are mutually disjoint, we have

$$A_\Delta = A_{\Delta_1} + \dots + A_{\Delta_m}.$$

Hence, by Lemma 4.2, for each $B \in \mathcal{B}(\mathbb{R})$, we have

$$E_\Delta(B) = \int_{\mathbb{R}^m} \chi_B(x_1 + \dots + x_m) dE_{\Delta_1,\dots,\Delta_m}(x_1, \dots, x_m).$$

Therefore,

$$\nu_\Delta(B) = \int_{\mathbb{R}^m} \chi_B(x_1 + \dots + x_m) d\nu_{\Delta_1,\dots,\Delta_m}(x_1, \dots, x_m).$$

In particular,

$$\begin{aligned}\nu_\Delta\big([0,+\infty)\big) &= \int_{\mathbb{R}^m} \chi_{[0,+\infty)}(x_1 + \dots + x_m) d\nu_{\Delta_1,\dots,\Delta_m}(x_1, \dots, x_m) \\ &= \int_{[0,+\infty)^m} \chi_{[0,+\infty)}(x_1 + \dots + x_m) d\nu_{\Delta_1,\dots,\Delta_m}(x_1, \dots, x_m)\end{aligned}$$

$$= \int_{[0,+\infty)^m} d\nu_{\Delta_1,\ldots,\Delta_m}(x_1,\ldots,x_m) = 1.$$

ii) Since Δ_1 and Δ_2 are disjoint,

$$A_{\Delta_1 \cup \Delta_2} = A_{\Delta_1} + A_{\Delta_2}.$$

Therefore, by Lemma 4.2, we have, for any $B \in \mathcal{B}(\mathbb{R})$,

$$E_{\Delta_1 \cup \Delta_2}(B) = \int_{\mathbb{R}^2} \chi_B(x_1 + x_2) dE_{\Delta_1,\Delta_2}(x_1, x_2).$$

Therefore, for any $C \in \mathcal{B}(\mathbb{R}^3)$,

$$E_{\Delta_1,\Delta_2,\Delta_1\cup\Delta_2}(C) = \int_{\mathbb{R}^3} \chi_C(x_1, x_2, x_1 + x_2) dE_{\Delta_1,\Delta_2}(x_1, x_2),$$

and so

$$\nu_{\Delta_1,\Delta_2,\Delta_1\cup\Delta_2}(C) = \int_{\mathbb{R}^2} \chi_C(x_1, x_2, x_1 + x_2) d\nu_{\Delta_1,\Delta_2}(x_1, x_2).$$

Setting

$$C = \{(x, y, z) \in \mathbb{R}^3 : x + y = z\},$$

we get

$$\nu_{\Delta_1,\Delta_2,\Delta_1\cup\Delta_2}(C) = \int_{\mathbb{R}^2} d\nu_{\Delta_1,\Delta_2}(x_1, x_2) = 1.$$

iii) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and bounded. We need to prove that

$$\int_{\mathbb{R}} f(x) d\nu_{\Delta_n}(x) \to \int_{\mathbb{R}} f(x) d\varepsilon_0(x) = f(0).$$

By (LB), without loss, we may assume that $C_{\Delta_1} \leqslant 1/(16 + \delta)$. Then, each ν_{Δ_n} is concentrated on the set $\mathbb{N}_0$. Assume that

$$\nu_{\Delta_n}(\mathbb{N}) \to 0 \quad \text{as} \quad n \to \infty. \tag{4.38}$$

Then since $\nu_{\Delta_n}(\mathbb{N}) + \nu_{\Delta_n}(\{0\}) = 1$, we also conclude that $\nu_{\Delta_n}(\{0\}) \to 1$ as $n \to \infty$. Therefore, since f is bounded

$$\int_{\mathbb{R}} f(x) d\nu_{\Delta_n}(x) = \nu_{\Delta_n}(\{0\}) f(0) + \int_{\mathbb{N}} f(x) d\nu_{\Delta_n}(x) \to f(0).$$

So, it suffices to prove (4.38). But ν_{Δ_n} is the distribution of the random variable

$$\Gamma_{\Delta_1} \ni \eta \to \eta(\Delta_n)$$

under μ^{Δ_1}. Hence

$$\nu_{\Delta_n}(\mathbb{N}) = \int_{\Gamma_{\Delta_1}} \chi_{\mathbb{N}}\big(\eta(\Delta_n)\big)\mu^{\Delta_1}(d\eta).$$

For each fixed $\eta \in \Gamma_{\Delta_1}$

$$\eta(\Delta_n) \to 0 \quad \text{as} \quad n \to \infty,$$

and so

$$\chi_{\mathbb{N}}\big(\eta(\Delta_n)\big) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, by the dominated convergence theorem,

$$\int_{\Gamma_{\Delta_1}} \chi_{\mathbb{N}}\big(\eta(\Delta_n)\big)\mu^{\Delta_1}(d\eta) \to 0,$$

which proves (4.38). $\square$

Now, by Lemma 4.4 and [14, Theorem 5.4], there exists a random measure M on X such that, for any $\Delta_1, \dots, \Delta_n \in \mathcal{B}_0(X)$, $n \in \mathbb{N}$, the distribution of $\big(M(\Delta_1), \dots, M(\Delta_n)\big)$ is $\nu_{\Delta_1,\dots,\Delta_n}$. In fact, the random measure M is concentrated on Γ_X. Indeed, we already know that, for any $x \in X$, there exists an open neighbourhood of x, denoted by $\Delta(x)$, such that $\Delta(x) \in \mathcal{B}_0(X)$ and the restriction of M to $\Delta(x)$ is concentrated on $\Gamma_{\Delta(x)}$. Since X is Polish, there exist a countable sequence of compact sets Δ_n in X such that $X = \bigcup_{n=1}^{\infty} \Delta_n$. Now, using the definition of a compact set, we easily see that the restriction of M to each Δ_n is concentrated on Γ_{Δ_n}, and so M is concentrated on Γ.

Letting μ denote the distribution of M on Γ_X, we obtain a unique probability measure on $\big(\Gamma_X, \mathcal{B}(\Gamma_X)\big)$ whose "finite-dimensional distributions" are given through the measures (4.21).

Lemma 4.5 *For any $G_1, G_2 \in \mathcal{S}$,*

$$\int_{\Gamma_{X,0}} (G_1 \star G_2)(\eta)\, \rho(d\eta) = \int_{\Gamma_X} \Big(\sum_{\eta \Subset \gamma} G_1(\eta) \Big) \Big(\sum_{\eta \Subset \gamma} G_2(\eta) \Big) \mu(d\gamma). \qquad (4.39)$$

Proof. Let $\Delta \in \mathcal{B}_0(X)$ be such that (4.26) is satisfied. As usual, we identify $\mathcal{B}(\Gamma_\Delta)$ as a sub-σ-algebra of $\mathcal{B}(\Gamma_{X,0})$. Then, for any $G_1, G_2 \in \mathcal{S}$ which, restricted to $\Gamma_{X,0}$, are $\mathcal{B}(\Gamma_\Delta)$-measurable, we have:

$$\begin{aligned} &\int_{\Gamma_{X,0}} (G_1 \star G_2)(\eta)\, \rho(d\eta) \\ &\quad = \int_{\Gamma_\Delta} (G_1 \star G_2)(\eta) \rho(d\eta) \\ &\quad = \int_{\Gamma_\Delta} (\mathcal{K}_\Delta G_1)(\eta)(\mathcal{K}_\Delta G_2)(\eta)\, \mu^\Delta(d\eta) \\ &\quad = \int_{\Gamma_X} \Big(\sum_{\eta \Subset \gamma} G_1(\eta) \Big) \Big(\sum_{\eta \Subset \gamma} G_2(\eta) \Big)\, \mu(d\gamma). \qquad (4.40) \end{aligned}$$

We will now need the following lemma.

Lemma 4.6 *Fix any $\varepsilon > 0$. Then, any $G \in \mathcal{S}$ can be represented as*

$$G = \sum_{j=1}^{k} G_j,$$

where $k \in \mathbb{N}$, each G_j belongs to $\mathcal{S}$, and restricted to $\Gamma_{X,0}$ is $\mathcal{B}(\Gamma_{\Delta_j})$-measurable with $\Delta_j \in \mathcal{B}_0(X)$, $C_{\Delta_j} \leqslant \varepsilon$.

Proof. It suffices to prove the result of the lemma in the case where $G = \chi_\Delta^{\otimes n}$, $\Delta \in \mathcal{B}_0(X)$. Using the definition of a compact set and (LB), we see that there exist mutually disjoint open sets $\Delta_1, \dots, \Delta_m \in \mathcal{B}_0(X)$ such that $\Delta_1 \cup \dots \cup \Delta_m = \Delta$ and $C_{\Delta_j} \leqslant \frac{\varepsilon}{n}$, $\quad i = 1, \dots, m$. Then

$$\chi_\Delta^{\otimes n} = (\chi_{\Delta_1} + \dots + \chi_{\Delta_m})^{\otimes n}.$$

Therefore, $\chi_\Delta^{\otimes n}$ is a finite linear combination of functions of the form

$$\chi_{\Delta_1}^{\otimes i_1} \hat{\otimes} \chi_{\Delta_2}^{\otimes i_2} \hat{\otimes} \cdots \hat{\otimes} \chi_{\Delta_m}^{\otimes i_m},$$

where $i_1, i_2, \ldots, i_m \in \{0, \ldots, n\}, i_1 + \cdots + i_m = n$.

Evidently, this function is $\mathcal{B}(\Gamma_\alpha)$-measurable, where α is the union of those sets Δ_j from $\Delta_1, \ldots, \Delta_m$ for which the correspondents coefficient $i_j \neq 0$. So, α is a union of maximum n sets from the collection $\Delta_1, \ldots, \Delta_m$. Therefore,

$$C_\alpha \leqslant n\frac{\varepsilon}{n} = \varepsilon.$$

Indeed, to prove this estimate, and so to complete the proof of the lemma, we only need to show that, for disjoint $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$,

$$C_{\Delta_1 \cup \Delta_2} \leqslant C_{\Delta_1} + C_{\Delta_2}.$$

But

$$\Gamma^{(n)}_{\Delta_1 \cup \Delta_2} = \bigcup_{m=0}^{n} \left(\Gamma^{(m)}_{\Delta_1} \cap \Gamma^{(n-m)}_{\Delta_2}\right),$$

so that

$$\begin{aligned}
\rho(\Gamma^{(n)}_{\Delta_1 \cup \Delta_2}) &\leqslant \sum_{m=0}^{n} \rho\left(\Gamma^{(m)}_{\Delta_1} \cap \Gamma^{(n-m)}_{\Delta_2}\right) \\
&\leqslant \sum_{m=0}^{n} \left(\rho(\Gamma^{(m)}_{\Delta_1}) + \rho(\Gamma^{(n-m)}_{\Delta_2})\right) \\
&\leqslant \sum_{m=0}^{n} \left(C^m_{\Delta_1} + C^{n-m}_{\Delta_2}\right) \\
&\leqslant \sum_{m=0}^{n} \binom{n}{m} \left(C^m_{\Delta_1} + C^{n-m}_{\Delta_2}\right) \\
&= (C_{\Delta_1} + C_{\Delta_2})^n. \quad \square
\end{aligned}$$

For each $G \in \mathcal{S}$, choosing $G_1 = G$ and $G_2 = \Xi$ in (4.39), we get

$$\int_{\Gamma_{X,0}} G(\eta)\, \rho(d\eta) = \int_{\Gamma_X} \sum_{\eta \Subset \gamma} G(\eta)\, \mu(d\gamma).$$

Hence ρ is the correlation measure of μ. By Theorem 2.1 and Remark 2.1, we have that μ is the unique measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ whose correlation measure is ρ. (In fact, the uniqueness can also be derived directly from the above arguments.)

Finally, we prove the statement of the theorem concerning the operator $\mathcal{I}$. Define the mapping

$$\widehat{\mathcal{S}} \ni \widehat{G} \mapsto (\mathcal{K}\widehat{G})(\gamma) := \sum_{\eta \Subset \gamma} G(\eta). \tag{4.41}$$

Then, by (4.39), $\mathcal{K}$ extends to an isometry of $\mathcal{H}_\rho$ into $L^2(\Gamma, \mu)$. Furthermore, it is clear that the image of $\mathcal{K}$ is dense in $L^2(\Gamma, \mu)$, and so $\mathcal{K}$ is a unitary operator.

For each $\Delta \in \mathcal{B}_0(X)$,

$$\mathcal{K}(\widehat{\chi_\Delta \star G})(\gamma) = \gamma(\Delta)(\mathcal{K}\widehat{G})(\gamma), \quad G \in \mathcal{S},\ \gamma \in \Gamma_X.$$

Therefore, $\tilde{A}_\Delta$ goes over, under $\mathcal{K}$, into the operator of multiplication by $\gamma(\Delta)$. Recalling the unitary isomorphism $U : \mathfrak{F} \to \mathcal{H}_\rho$, under which each operator $\tilde{a}(\Delta)$ goes over into the operator $\tilde{A}_\Delta$, we finish the proof. $\square$

It is clear that any correlation measure ρ satisfies the following condition:

(N) *Normalization*: $\rho(\Gamma_X^{(0)}) = 1$.

As we discussed in Introduction, any correlation measure ρ also satisfies:

(PD) *$\star$-positive definiteness*: For each $G \in \mathcal{S}$:

$$\int_{\Gamma_{X,0}} (G \star G)(\eta)\, \rho(d\eta) \geqslant 0.$$

We now conclude the following criterion of existence of a point process, which generalises [15, Theorem 6.5] and [6, Theorem 2].

Corollary 4.1 *Let ρ be a measure on $\big(\Gamma_{X,0}, \mathcal{B}(\Gamma_{X,0})\big)$ satisfying* (N), (PD), *and* (LB). *Then, there exists a unique probability measure on $\big(\Gamma_X, \mathcal{B}(\Gamma_X)\big)$ which has ρ as correlation measure.*

Proof. Consider the bilinear form (4.7). Then, by (PD), we have for each $G \in \mathcal{S}$,

$$b_\rho(G, G) \geqslant 0.$$

Denote by $\widehat{\mathcal{S}}$ the factorization of $\mathcal{S}$ consisting of factor-classes

$$\widehat{G} = \{G' \in \mathcal{S} : b_\rho(G - G', G - G') = 0\}, \quad G \in \mathcal{S}.$$

Define a Hilbert space $\mathcal{H}_\rho$ as the closure of $\widehat{\mathcal{S}}$ in the norm generated by the scalar product,

$$(\widehat{G}_1, \widehat{G}_2)_{\mathcal{H}_\rho} := b_\rho(G_1, G_2).$$

For each $\Delta \in \mathcal{B}_0(X)$, define an operator A_Δ in $\mathcal{H}_\rho$ by

$$A_\Delta \widehat{G}_i = \widehat{\chi_\Delta \star G}, \quad G \in \mathcal{S}. \tag{4.42}$$

For any $G_1, G_2 \in \mathcal{S}$, we have

$$\begin{aligned} b_\rho(\chi_\Delta \star G_1, G_2) &= \int_{\Gamma_0} (\chi_\Delta \star G_1) \star G_2(\eta)\rho(d\eta) \\ &= \int_{\Gamma_0} G_1 \star (\chi_\Delta \star G_2)\rho(d\eta) \\ &= b_\rho(G_1, \chi_\Delta \star G_2). \end{aligned}$$

Therefore, by [5, Chapter 5, Section 5, subsection 2], we see that the definition (4.42) makes sense, i.e., the factor class $\widehat{\chi_\Delta \star G}$ is independent of the choice of a representative of the factor-class of $\widehat{G}$.

Now, in order to prove the corollary, one just need to follow the lines if the proof of the Theorem 4.1. $\square$

Chapter 5

Particle densities in quasi-free representations of the CAR and CCR

5.1 Fermion and boson point processes

Let X be a topological space as in Chapter 3. Let σ be a non-atomic Radon measure on $(X, \mathcal{B}(X))$. We denote by H the real space $L^2(X, \sigma)$. Let K be a linear, bounded, symmetric operator in H which satisfies $\mathbf{0} \leqslant K \leqslant \mathbf{1}$.

Let us recall the construction of the quasi-free representation of the CAR corresponding to the operator K [3].

Denote $K_1 := \sqrt{K}$ and $K_2 := \sqrt{\mathbf{1} - K}$. For a real separable Hilbert space $\mathcal{H}$, we denote by $\mathcal{AF}(\mathcal{H})$ the antisymmetric Fock space over $\mathcal{H}$:

$$\mathcal{AF}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{AF}^{(n)}(\mathcal{H}).$$

Here, $\mathcal{AF}^{(0)}(\mathcal{H}) := \mathbb{R}$ and for $n \in \mathbb{N}$ $\mathcal{AF}^{(n)}(\mathcal{H}) := \mathcal{H}^{\wedge n} n!$, where $\wedge$ stands for antisymmetric tensor product and $n!$ is a normalizing factor, so that, for any $f^{(n)} \in \mathcal{AF}^{(n)}(\mathcal{H})$,

$$\|f^{(n)}\|^2_{\mathcal{AF}^{(n)}(\mathcal{H})} = \|f^{(n)}\|^2_{\mathcal{H}^{\wedge n}}\, n!\,.$$

We denote by $\mathcal{AF}_{\mathrm{fin}}(\mathcal{H})$ the subset of $\mathcal{AF}(\mathcal{H})$ consisting of all elements $f = (f^{(n)})_{n=0}^{\infty} \in \mathcal{AF}(\mathcal{H})$ for which $f^{(n)} = 0$, $n \geqslant N$, for some $N \in \mathbb{N}$. We endow $\mathcal{AF}_{\mathrm{fin}}(\mathcal{H})$ with the topology of the topological direct sum of the spaces $\mathcal{AF}^{(n)}(\mathcal{H})$. Thus, the convergence in $\mathcal{AF}_{\mathrm{fin}}(\mathcal{H})$ means uniform boundedness and coordinate-wise convergence.

For $g \in \mathcal{H}$, we denote by $\Phi(g)$ and $\Phi^*(g)$ the annihilation and creation operators in $\mathcal{AF}(\mathcal{H})$, respectively. These are linear continuous operators in $\mathcal{AF}_{\mathrm{fin}}(\mathcal{H})$ defined through the formulas

$$\begin{aligned}
\Phi(g)\, h_1 \wedge \cdots \wedge h_n :=& \sum_{i=1}^{n} (-1)^{i+1} (g, h_i)_{\mathcal{H}}\, h_1 \wedge \cdots \wedge h_{i-1} \wedge \check{h}_i \wedge h_{i+1} \wedge \cdots \wedge h_n,\\
\Phi^*(g)\, h_1 \wedge \cdots \wedge h_n :=& g \wedge h_1 \wedge \cdots \wedge h_n,
\end{aligned}$$

where $h_1, \dots, h_n \in \mathcal{H}$.

We now set $\mathcal{H} := H_1 \oplus H_2$, where H_1 and H_2 are two copies of H. For $f \in H$, we denote

$$\Phi_1(f) := \Phi(f, 0), \quad \Phi_2(f) := \Phi(0, f),$$

and analogously $\Phi_i^*(f)$, $i = 1, 2$. We set, for each $f \in H$,

$$\begin{aligned}
\Psi(f) :=& \Phi_2(K_2 f) + \Phi_1^*(K_1 f),\\
\Psi^*(f) :=& \Phi_2^*(K_2 f) + \Phi_1(K_1 f).
\end{aligned} \tag{5.1}$$

The operators $\{\Psi(f), \Psi^*(f) \mid f \in \mathcal{H}\}$ satisfy the CAR:

$$\begin{aligned}
&[\Psi(f), \Psi(g)]_+ = [\Psi^*(f), \Psi^*(g)]_+ = \mathbf{0},\\
&[\Psi^*(f), \Psi(g)]_+ = (f, g)_H\, \mathbf{1}, \quad f, g \in H,
\end{aligned} \tag{5.2}$$

where $[A,B]_+ := AB + BA$. This representation of the CAR is called quasi-free. The so-called n-point functions of this representation have the structure

$$(\Psi^*(f_n)\cdots\Psi^*(f_1)\Psi(g_1)\cdots\Psi(g_m)\Omega,\Omega)_{\mathcal{AF}(\mathcal{H})} = \delta_{n,m}\det[(Kf_i,g_j)_H]_{i,j=1}^n. \tag{5.3}$$

Here, $\Omega := (1,0,0,\dots)$ is the vacuum vector in $\mathcal{AF}(\mathcal{H})$.

In what follows, we will assume that, for each $\Delta \in \mathcal{B}_0(X)$, the operator $P_\Delta K P_\Delta$ is of trace class. Here, P_Δ denotes the operator of multiplication by χ_Δ.

For an integral operator I in H, we will denote by $\mathcal{N}(I)$ the kernel of I. For each $\Delta \in \mathcal{B}_0(X)$,

$$P_\Delta K_1(P_\Delta K_1)^* = P_\Delta K P_\Delta.$$

Therefore, the operator $P_\Delta K_1$ is of Hilbert–Schmidt class. Hence, $P_\Delta K_1$ is an integral operator, whose kernel $\mathcal{N}(P_\Delta K_1)$ belongs to $L^2(X^2,\sigma^2)$. This implies that K_1 is an integral operator, whose kernel satisfies

$$\int_\Delta\int_X \mathcal{N}(K_1)(x,y)^2\,\sigma(dx)\,\sigma(dy) < \infty, \qquad \Delta \in \mathcal{B}_0(X). \tag{5.4}$$

Note also that the kernel $\mathcal{N}(K_1)$ is symmetric.

Thus, K is an integral operator, whose kernel is given by

$$k(x,y) := \mathcal{N}(K)(x,y) = \int_X \mathcal{N}(K_1)(x,z)\mathcal{N}(K_1)(z,y)\,\sigma(dz).$$

By (5.4), for any $\Delta \in \mathcal{B}_0(X)$, we get:

$$\begin{aligned}\int_\Delta k(x,x)\,\sigma(dx) &= \int_\Delta\int_X \mathcal{N}(K_1)(x,y)\mathcal{N}(K_1)(y,x)\,\sigma(dy)\,\sigma(dx)\\ &= \int_\Delta\int_X \mathcal{N}(K_1)(x,y)^2\,\sigma(dx)\,\sigma(dy) < \infty.\end{aligned}$$

Note that the kernel $\mathcal{N}(K_1)(x,y)$ is defined up to a set of $\sigma^{\otimes 2}$-measure 0 in X^2, but the value $\int_X k(x,x)\,\sigma(dx)$ is independent of the choice of a version of $\mathcal{N}(K_1)$.

Let us now derive the particle density corresponding to the quasi-free representation of the CAR. For a fixed $x \in X$, we define the function $\varkappa_{1,x} : X \to \mathbb{R}$ by

$$\varkappa_{1,x}(y) := \mathcal{N}(K_1)(x,y), \quad y \in X.$$

By (5.4),

$$\varkappa_{1,x} \in L^2(X,\sigma) \quad \text{for } \sigma\text{-a.a. } x \in X. \tag{5.5}$$

Then, for each $f \in H$, we have:

$$\Phi_1(K_1 f) = \int_X \sigma(dx) f(x) \Phi_1(\varkappa_{1,x}), \quad \Phi_1^*(K_1 f) = \int_X \sigma(dx) f(x) \Phi_1^*(\varkappa_{1,x}). \tag{5.6}$$

These equalities are to be understood through the corresponding bilinear forms. For example, the first equality means that, for any $g_1, \dots, g_{n+1}, h_1, \dots, h_n \in \mathcal{H}$,

$$\begin{aligned}&\big(\Phi_1(K_1 f) g_1 \wedge \cdots \wedge g_{n+1}, h_1 \wedge \cdots \wedge h_n\big)_{\mathcal{AF}(\mathcal{H})} \\ &\qquad = \int_X \sigma(dx) f(x) \big(\Phi_1(\varkappa_{1,x}) g_1 \wedge \cdots \wedge g_{n+1}, h_1 \wedge \cdots \wedge h_n\big)_{\mathcal{AF}(\mathcal{H})}.\end{aligned}$$

Next, for each $x \in X$ and any $h_1, \dots, h_n \in \mathcal{H}$, we set

$$\Phi_2(\varkappa_{2,x}) h_1 \wedge \cdots \wedge h_n = \sum_{i=1}^{n} (-1)^{i+1} (K_2 h_i^{(2)})(x) h_1 \wedge \cdots \wedge \check{h}_i \wedge \cdots \wedge h_n.$$

(Here and below, we use the notation $h_i = \big(h_i^{(1)}, h_i^{(2)}\big)$.)

Remark 5.1 Heuristically, $\varkappa_{2,x}(y) = \mathcal{N}(K_2)(x,y)$ and $\Phi_2(\varkappa_{2,x})$ is the corresponding annihilation operator.

Then, analogously to (5.6), we get:

$$\Phi_2(K_2 f) = \int_X \sigma(dx) f(x) \Phi_2(\varkappa_{2,x}), \quad \Phi_2^*(K_2 f) = \int_X \sigma(dx) f(x) \Phi_2^*(\varkappa_{2,x}). \tag{5.7}$$

Note that the second equality in (5.7) means that, for any $g_1, \dots, g_n, h_1, \dots, h_{n+1} \in \mathcal{H}$,

$$\begin{aligned}\big(\Phi_2^*(K_2 f) g_1 \wedge \cdots \wedge g_n, h_1 \wedge \cdots \wedge h_{n+1}\big)_{\mathcal{AF}(\mathcal{H})} \\ = \int_X \sigma(dx) f(x) \big(g_1 \wedge \cdots g_n, \Phi_2(\varkappa_{2,x}) h_1 \wedge \cdots \wedge h_{n+1}\big)_{\mathcal{AF}(\mathcal{H})}\end{aligned}$$

(compare with e.g. [24, Section X.7]).

By (5.1), (5.6), and (5.7), we get the following heuristic operators, for each $x \in X$:

$$\begin{aligned}\Psi(x) &= \Phi_2(\varkappa_{2,x}) + \Phi_1^*(\varkappa_{1,x}), \\ \Psi^*(x) &= \Phi_2^*(\varkappa_{2,x}) + \Phi_1(\varkappa_{1,x}). \end{aligned} \tag{5.8}$$

Therefore, the particle density at point $x \in X$ is heuristically given by

$$a(x) = \Psi^*(x)\Psi(x) = \big(\Phi_2^*(\varkappa_{2,x}) + \Phi_1(\varkappa_{1,x})\big)\big(\Phi_2(\varkappa_{2,x}) + \Phi_1^*(\varkappa_{1,x})\big).$$

Thus, for each $\Delta \in \mathcal{B}_0(X)$, we need to rigorously define an operator

$$\begin{aligned}a(\Delta) &= \int_\Delta \sigma(dx) a(x) \\ &= \int_\Delta \sigma(dx) \Phi_2^*(\varkappa_{2,x}) \Phi_1^*(\varkappa_{1,x}) + \int_\Delta \sigma(dx) \Phi_1(\varkappa_{1,x}) \Phi_2(\varkappa_{2,x}) \\ &\quad + \int_\Delta \sigma(dx) \Phi_2^*(\varkappa_{2,x}) \Phi_2(\varkappa_{2,x}) + \int_\Delta \sigma(dx) \Phi_1(\varkappa_{1,x}) \Phi_1^*(\varkappa_{1,x}). \end{aligned} \tag{5.9}$$

In fact, it is not hard to see that each of the four integrals in (5.9) determines a linear continuous operator in $\mathcal{AF}_{\mathrm{fin}}(\mathcal{H})$ through the corresponding bilinear form. Indeed, for any $g_1, \dots, g_n, h_1, \dots, h_{n+2} \in \mathcal{H}$,

$$\begin{aligned}\left(\int_\Delta \sigma(dx) \Phi_2^*(\varkappa_{2,x}) \Phi^*(\varkappa_{1,x}) g_1 \wedge \cdots \wedge g_n, h_1 \wedge \cdots \wedge h_{n+2}\right)_{\mathcal{AF}(\mathcal{H})} \\ = \int_\Delta \sigma(dx) \big(\Phi^*(\varkappa_{1,x}) g_1 \wedge \cdots \wedge g_n, \Phi_2(\varkappa_{2,x}) h_1 \wedge \cdots \wedge h_{n+2}\big)_{\mathcal{AF}(\mathcal{H})}\end{aligned}$$

$$= \int_\Delta \sigma(dx)(n+1)! \sum_{i=1}^{n+2} (-1)^{i+1} (K_2 h_i^{(2)})(x)$$

$$\times \big(\varkappa_{1,x} \wedge g_1 \wedge \cdots \wedge g_n, h_1 \wedge \cdots \wedge \check{h}_i \wedge \cdots \wedge h_{n+2}\big)_{\mathcal{H}^{\wedge(n+1)}}$$

$$= n! \sum_{i,j \in \{1,\dots,n+2\},\, i \neq j} (-1)^{i+j+\chi_{\{j>i\}}(i,j)} (K_2 h_i^{(2)}, P_\Delta K_1 h_j^{(1)})_H$$

$$\times \big(g_1 \wedge \cdots \wedge g_n, h_1 \wedge \cdots \wedge \check{h}_i \wedge \cdots \wedge \check{h}_j \wedge \cdots \wedge h_{n+2}\big)_{\mathcal{H}^{\wedge n}}. \tag{5.10}$$

Since $P_\Delta K_1$ is of Hilbert–Schmidt class, so is $K_2 P_\Delta K_1$. Therefore,

$$(K_2 h_i^{(2)}, P_\Delta K_1 h_j^{(1)})_H = (\mathcal{N}(K_2 P_\Delta K_1)_{2,1}, h_i \otimes h_j)_{\mathcal{H}^{\otimes 2}}.$$

Here, $\mathcal{N}(K_2 P_\Delta K_1)_{2,1}$ is the element of the space $\mathcal{H}^{\otimes 2}$ which belongs to its subspace $H_2 \otimes H_1$ and coincides in it with $\mathcal{N}(K_2 P_\Delta K_1)$. Let also $\mathcal{N}(K_2 P_\Delta K_1)^\wedge_{2,1}$ denote the orthogonal projection of $\mathcal{N}(K_2 P_\Delta K_1)_{2,1}$ onto $\mathcal{H}^{\wedge 2}$. Hence, we continue (5.10) as follows:

$$= \big(\mathcal{N}(K_2 P_\Delta K_1)^\wedge_{2,1} \wedge g_1 \wedge \cdots \wedge g_n, h_1 \wedge \cdots \wedge h_{n+2}\big)_{\mathcal{AF}(\mathcal{H})}. \tag{5.11}$$

Thus, $\int_\Delta \sigma(dx) \Phi_2^*(\varkappa_{2,x}) \Phi_1^*(\varkappa_{1,x})$ identifies the operator of creation by $\mathcal{N}(K_2 P_\Delta K_1)^\wedge_{2,1}$, which we denote by $a^+(\mathcal{N}(K_2 P_\Delta K_1)^\wedge_{2,1})$:

$$a^+(\mathcal{N}(K_2 P_\Delta K_1)^\wedge_{2,1}) g_1 \wedge \cdots \wedge g_n = \mathcal{N}(K_2 P_\Delta K_1)^\wedge_{2,1} \wedge g_1 \wedge \cdots \wedge g_n.$$

Clearly, $\int_\Delta \sigma(dx) \Phi_1(\varkappa_{1,x}) \Phi_2(\varkappa_{2,x})$ identifies (the restriction to $\mathcal{AF}_{\mathrm{fin}}(\mathcal{H})$ of) the adjoint operator of $a^+(\mathcal{N}(K_2 P_\Delta K_1)^\wedge_{2,1})$, i.e., the annihilation operator by $\mathcal{N}(K_2 P_\Delta K_1)^\wedge_{2,1}$:

$$a^-(\mathcal{N}(K_2 P_\Delta K_1)^\wedge_{2,1}) h_1 \wedge \cdots \wedge h_n = n(n-1)\big(\mathcal{N}(K_2 P_\Delta K_1)_{2,1}, h_1 \wedge \cdots \wedge h_n\big)_{\mathcal{H}^{\otimes 2}}, \tag{5.12}$$

where the scalar product is taken in the first two "variables". Therefore,

$$a^-(\mathcal{N}(K_2 P_\Delta K_1)^\wedge_{2,1}) h_1 \wedge \cdots \wedge h_n$$

$$= \sum_{i,j=1,\dots,n,\, i\neq j} (-1)^{i+j+\chi_{\{i<j\}}(i,j)} \big(\mathcal{N}(K_2 P_\Delta K_1), h_i^{(2)} \otimes h_j^{(1)}\big)_{H^{\otimes 2}}$$
$$\times h_1 \wedge \cdots \wedge \check{h}_i \wedge \cdots \wedge \check{h}_j \wedge \cdots \wedge h_n.$$

Next, for any $g_1, \dots, g_n, h_1, \dots, h_n \in \mathcal{H}$,

$$\left(\int_\Delta \sigma(dx) \Phi_2^*(\varkappa_{2,x}) \Phi_2(\varkappa_{2,x}) g_1 \wedge \cdots \wedge g_n, h_1 \wedge \cdots \wedge h_n \right)_{\mathcal{AF}(\mathcal{H})}$$
$$= \int_\Delta \sigma(dx) \big(\Phi_2(\varkappa_{2,x}) g_1 \wedge \cdots \wedge g_n, \Phi_2(\varkappa_{2,x}) h_1 \wedge \cdots \wedge h_n\big)_{\mathcal{AF}(\mathcal{H})}$$
$$= \sum_{i,j=1,\dots,n} (-1)^{i+j} (K_2 P_\Delta K_2 g_i^{(2)}, h_j^{(2)})_H (n-1)!$$
$$\times (g_1 \wedge \cdots \wedge \check{g}_i \wedge \cdots \wedge g_n, h_1 \wedge \cdots \wedge \check{h}_j \wedge \cdots \wedge h_n)_{\mathcal{H}^{\wedge(n-1)}}. \tag{5.13}$$

For any linear continuous operator A on $\mathcal{H}$, we define the second quantization of A, denoted by $d\Gamma(A)$, as the linear continuous operator on $\mathcal{AF}_{\mathrm{fin}}(\mathcal{H})$ given by

$$d\Gamma(A) \restriction \mathcal{AF}^{(0)}(\mathcal{H}) = \mathbf{0},$$
$$d\Gamma(A) \restriction \mathcal{AF}^{(n)}(\mathcal{H}) = A \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \mathbf{1} \otimes A \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$$
$$+ \cdots + \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes A, \quad n \in \mathbb{N}.$$

Then, we continue (5.13) as follows;

$$= \big(d\Gamma(\mathbf{0} \oplus (K_2 P_\Delta K_2)) g_1 \wedge \cdots \wedge g_n, h_1 \wedge \cdots \wedge h_n\big)_{\mathcal{AF}(\mathcal{H})}.$$

Therefore, $\int_\Delta \sigma(dx) \Phi_2^*(\varkappa_{2,x}) \Phi_2(\varkappa_{2,x})$ identifies the operator $d\Gamma(\mathbf{0} \oplus (K_2 P_\Delta K_2))$. Analogously, $\int_\Delta \sigma(dx) \Phi_1(\varkappa_{1,x}) \Phi_1^*(\varkappa_{1,x})$ identifies the operator

$$\int_\Delta \|\varkappa_{1,x}\|_H^2 \, \sigma(dx) \, \mathbf{1} - d\Gamma(K_1 P_\Delta K_1 \oplus \mathbf{0}).$$

Summing up, we see that, for each $\Delta \in \mathcal{B}_0(X)$, the operator $a(\Delta)$ is given by

$$a(\Delta) = a^+(\mathcal{N}(K_2 P_\Delta K_1)^\wedge_{2,1}) + a^-(\mathcal{N}(K_2 P_\Delta K_1)^\wedge_{2,1})$$

$$+ d\Gamma((-K_1P_\Delta K_1) \oplus (K_2P_\Delta K_2)) + \int_\Delta \|\varkappa_{1,x}\|_H^2\, \sigma(dx)\, \mathbf{1}. \tag{5.14}$$

Lemma 5.1 *The operators $a(\Delta)$, $\Delta \in \mathcal{B}_0(X)$, commute on $\mathcal{AF}_{\mathrm{fin}}(\mathcal{H})$.*

Remark 5.2 Let us first give a heuristic proof of Lemma 5.1. The operators $\Psi^*(x)$, $\Psi(x)$, $x \in X$, satisfy the CAR:

$$\begin{aligned}
&[\Psi(x), \Psi(y)]_+ = [\Psi^*(x), \Psi^*(y)]_+ = \mathbf{0},\\
&[\Psi^*(x), \Psi(y)]_+ = \delta(x,y)\mathbf{1}, \quad x, y \in X,
\end{aligned} \tag{5.15}$$

where

$$\int_{X^2} \delta(x,y) f(x) g(y)\, \sigma(dx)\sigma(dy) := \int_X f(x)g(x)\, \sigma(dx).$$

Therefore, for any $x, y \in X$,

$$\begin{aligned}
a(x)a(y) &= \Psi^*(x)\Psi(x)\Psi^*(y)\Psi(y)\\
&= -\Psi^*(x)\Psi^*(y)\Psi(x)\Psi(y) + \delta(x,y)\Psi^*(x)\Psi(x)\\
&= -\Psi^*(y)\Psi^*(x)\Psi(y)\Psi(x) + \delta(x,y)\Psi^*(x)\Psi(x)\\
&= \Psi^*(y)\Psi(y)\Psi^*(x)\Psi(x)\\
&= a(y)a(x).
\end{aligned} \tag{5.16}$$

Proof of Lemma 5.1. For any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$, we trivially have:

$$\begin{aligned}
&a^+(\mathcal{N}(K_2P_{\Delta_1}K_1)^\wedge_{2,1})a^+(\mathcal{N}(K_2P_{\Delta_2}K_1)^\wedge_{2,1})\\
&\qquad = a^+(\mathcal{N}(K_2P_{\Delta_2}K_1)^\wedge_{2,1})a^+(\mathcal{N}(K_2P_{\Delta_1}K_1)^\wedge_{2,1}).
\end{aligned} \tag{5.17}$$

Next, we evaluate

$$d\Gamma\Big((K_1P_{\Delta_1}K_1) \oplus \mathbf{0}\Big)\mathcal{N}(K_2P_{\Delta_2}K_1)^\wedge_{2,1} = \big((\mathbf{1} \otimes K_1P_{\Delta_1}K_1)\mathcal{N}(K_2P_{\Delta_2}K_1)_{2,1}\big)^\wedge, \tag{5.18}$$

where $\wedge$ denotes antisymmetrization. For any $u_1 \in H_1$ and $u_2 \in H_2$, we get:

$$\big((\mathbf{1} \otimes K_1P_{\Delta_1}K_1)\mathcal{N}(K_2P_{\Delta_2}K_1)_{2,1}, u_2 \otimes u_1\big)_{\mathcal{H}^{\otimes 2}}$$

$$= \big(\mathcal{N}(K_2P_{\Delta_2}K_1)_{2,1}, u_2 \otimes K_1P_{\Delta_1}K_1u_1\big)_{\mathcal{H}^{\otimes 2}}$$
$$= \big(u_2, K_2P_{\Delta_2}K_1K_1P_{\Delta_1}K_1u_1\big)_H$$
$$= \big(u_2, K_2P_{\Delta_2}KP_{\Delta_1}K_1u_1\big)_H.$$

Therefore, $(\mathbf{1} \otimes K_1P_{\Delta_1}K_1)\mathcal{N}(K_2P_{\Delta_2}K_1)_{2,1}$ is the kernel of the operator $K_2P_{\Delta_2}KP_{\Delta_1}K_1$ realized as the element of $H_2 \otimes H_1$. We denote it by $\mathcal{N}(K_2P_{\Delta_2}KP_{\Delta_1}K_1)_{2,1}$. Therefore, by (5.18),

$$d\Gamma\big(K_1P_{\Delta_1}K_1\big)\mathcal{N}(K_2P_{\Delta_2}K_1)^{\wedge}_{2,1} = \mathcal{N}\big(K_2P_{\Delta_2}KP_{\Delta_1}K_1\big)^{\wedge}_{2,1}. \tag{5.19}$$

Analogously, we get, for any $u_1 \in H_1$, $u_2 \in H_2$,

$$\big((K_2P_{\Delta_1}K_2 \otimes \mathbf{1})\mathcal{N}(K_2P_{\Delta_2}K_1)_{2,1}, u_2 \otimes u_1\big)_{\mathcal{H}^{\otimes 2}}$$
$$= \big(u_2, K_2P_{\Delta_1}(\mathbf{1} - K)P_{\Delta_2}K_1u_1\big)_H,$$

and hence,

$$d\Gamma\Big(\mathbf{0} \oplus \big(K_2P_{\Delta_1}K_2\big)\Big)\mathcal{N}(K_2P_{\Delta_2}K_1)^{\wedge}_{2,1} = \mathcal{N}(K_2P_{\Delta_1}(\mathbf{1} - K)P_{\Delta_2}K_1)^{\wedge}_{2,1}. \tag{5.20}$$

By (5.19) and (5.20), a straightforward calculation shows that

$$\begin{aligned}
&d\Gamma((-K_1P_{\Delta_1}K_1) \oplus (K_2P_{\Delta_1}K_2))a^+(\mathcal{N}(K_2P_{\Delta_2}K_1)^{\wedge}_{2,1}) \\
&\quad + a^+(\mathcal{N}(K_2P_{\Delta_1}K_1)^{\wedge}_{2,1})d\Gamma((-K_1P_{\Delta_2}K_1) \oplus (K_2P_{\Delta_2}K_2)) \\
&= d\Gamma((-K_1P_{\Delta_2}K_1) \oplus (K_2P_{\Delta_2}K_2))a^+(\mathcal{N}(K_2P_{\Delta_1}K_1)^{\wedge}_{2,1}) \\
&\quad + a^+(\mathcal{N}(K_2P_{\Delta_2}K_1)^{\wedge}_{2,1})\Gamma((-K_1P_{\Delta_1}K_1) \oplus (K_2P_{\Delta_1}K_2)).
\end{aligned} \tag{5.21}$$

Next, by (5.12), we have:

$$\begin{aligned}
&a^-(\mathcal{N}(K_2P_{\Delta_1}K_1)^{\wedge}_{2,1})a^+(\mathcal{N}(K_2P_{\Delta_2}K_1)^{\wedge}_{2,1})h_1 \wedge \cdots \wedge h_n \\
&\quad = \big((\mathcal{N}(K_2P_{\Delta_1}K_1), \mathcal{N}(K_2P_{\Delta_2}K_1))_{H^{\otimes 2}}\mathbf{1} + a^+(\Delta_2)a^-(\Delta_1)\big)h_1 \wedge \cdots \wedge h_n \\
&\quad - \sum_{i=1}^{n} h_1 \wedge \cdots \wedge h_{i-1}
\end{aligned}$$

$$\wedge \left(\int_X \int_X \mathcal{N}(K_2 P_{\Delta_2} K_1)(x,\cdot) h_i^{(1)}(y) \mathcal{N}(K_2 P_{\Delta_1} K_1)(x,y)\, \sigma(dx)\, \sigma(dy) \right.$$
$$\left. + \int_X \int_X \mathcal{N}(K_2 P_{\Delta_2} K_1)(\cdot,y) h_i^{(2)}(x) \mathcal{N}(K_2 P_{\Delta_1} K_1)(x,y)\, \sigma(dx)\, \sigma(dy) \right)$$
$$\wedge h_{i+1} \wedge \cdots \wedge h_n. \tag{5.22}$$

For any $u \in H$,

$$\left(\int_X \int_X \mathcal{N}(K_2 P_{\Delta_2} K_1)(x,\cdot) h_i^{(1)}(y) \mathcal{N}(K_2 P_{\Delta_1} K_1)(x,y)\, \sigma(dx)\, \sigma(dy), u \right)_H$$
$$= \int_X \int_X \int_X \mathcal{N}(K_2 P_{\Delta_2} K_1)(x,z) h_i^{(1)}(y) \mathcal{N}(K_2 P_{\Delta_1} K_1)(x,y) u(z)$$
$$\sigma(dx)\, \sigma(dy)\, \sigma(dz)$$
$$= \int_X \sigma(dy) \int_X \sigma(dz)\, h_i^{(1)}(y) u(z) \int_X \sigma(dx)\, \mathcal{N}(K_1 P_{\Delta_2} K_2)(z,x)$$
$$\mathcal{N}(K_2 P_{\Delta_1} K_1)(x,y)$$
$$= \int_X \sigma(dy) \int_X \sigma(dz)\, h_i^{(1)}(y) u(z) \mathcal{N}(K_1 P_{\Delta_2} (\mathbf{1} - K) P_{\Delta_1} K_1)(z,y)$$
$$= \left(K_1 P_{\Delta_2} (\mathbf{1} - K) P_{\Delta_1} K_1 h_i^{(1)}, u \right)_H.$$

Therefore,

$$\int_X \int_X \mathcal{N}(K_2 P_{\Delta_2} K_1)(x,\cdot) h_i^{(1)}(y) \mathcal{N}(K_2 P_{\Delta_1} K_1)(x,y)\, \sigma(dx)\, \sigma(dy)$$
$$= K_1 P_{\Delta_2} (\mathbf{1} - K) P_{\Delta_1} K_1 h_i^{(1)}. \tag{5.23}$$

Analogously

$$\int_X \int_X \mathcal{N}(K_2 P_{\Delta_2} K_1)(\cdot,y) h_i^{(2)}(x) \mathcal{N}(K_2 P_{\Delta_1} K_1)(x,y)\, \sigma(dx)\, \sigma(dy)$$
$$= K_2 P_{\Delta_2} K P_{\Delta_1} K_2 h_i^{(2)}. \tag{5.24}$$

By (5.22)–(5.24),

$$a^-(\mathcal{N}(K_2 P_{\Delta_1} K_1)_{2,1}^{\wedge}) a^+(\mathcal{N}(K_2 P_{\Delta_2} K_1)_{2,1}^{\wedge}) h_1 \wedge \cdots \wedge h_n$$

$$= \big((\mathcal{N}(K_2 P_{\Delta_1} K_1), \mathcal{N}(K_2 P_{\Delta_2} K_1))_{H^{\otimes 2}} \mathbf{1} + a^+(\Delta_2) a^-(\Delta_1) \big) h_1 \wedge \cdots \wedge h_n$$
$$- d\Gamma\big((K_1 P_{\Delta_2} (\mathbf{1} - K) P_{\Delta_1} K_1) \oplus (K_2 P_{\Delta_2} K P_{\Delta_1} K_2) \big). \tag{5.25}$$

Using (5.25), we conclude that

$$\begin{aligned}
&a^+(\mathcal{N}(K_2 P_{\Delta_1} K_1)^{\wedge}_{2,1}) a^-(\mathcal{N}(K_2 P_{\Delta_2} K_1)^{\wedge}_{2,1}) \\
&\quad + a^-(\mathcal{N}(K_2 P_{\Delta_1} K_1)^{\wedge}_{2,1}) a^+(\mathcal{N}(K_2 P_{\Delta_2} K_1)^{\wedge}_{2,1}) \\
&\quad + d\Gamma((-K_1 P_{\Delta_1} K_1) \oplus (K_2 P_{\Delta_1} K_2)) d\Gamma((-K_1 P_{\Delta_2} K_1) \oplus (K_2 P_{\Delta_2} K_2)) \\
&= a^+(\mathcal{N}(K_2 P_{\Delta_2} K_1)^{\wedge}_{2,1}) a^-(\mathcal{N}(K_2 P_{\Delta_1} K_1)^{\wedge}_{2,1}) \\
&\quad + a^-(\mathcal{N}(K_2 P_{\Delta_2} K_1)^{\wedge}_{2,1}) a^+(\mathcal{N}(K_2 P_{\Delta_1} K_1)^{\wedge}_{2,1}) \\
&\quad + d\Gamma((-K_1 P_{\Delta_2} K_1) \oplus (K_2 P_{\Delta_2} K_2)) d\Gamma((-K_1 P_{\Delta_1} K_1) \oplus (K_2 P_{\Delta_1} K_2)).
\end{aligned} \tag{5.26}$$

By (5.17), (5.21), (5.26) and the equalities obtained by taking the adjoint operators in (5.17), (5.21), we conclude the statement of the lemma. □

We will now show that the family $(a(\Delta))_{\Delta \in \mathcal{B}_0(X)}$ has a correlation measure ρ with respect to the vacuum vector Ω.

Remark 5.3 We will first make heuristic calculations. We will write, for any $\Delta_1 \dots, \Delta_n \in \mathcal{B}_0(X)$,

$$\mathcal{Q}(\chi_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \chi_{\Delta_n}) = \frac{1}{n!} \int_{\Delta_1} \sigma(dx_1) \cdots \int_{\Delta_n} \sigma(dx_n) \, {:}a(x_1) \cdots a(x_n){:} \, .$$

Then, the recursion relation (4.2) takes the form:

$$\begin{aligned}
{:}a(x_1) \cdots a(x_{n+1}){:} = \frac{1}{n+1} \Bigg[&\sum_{i=1}^{n+1} a(x_i) \, {:}a(x_1) \cdots \check{a}(x_i) \cdots a(x_{n+1}){:} \\
- \sum_{i=1}^{n+1} \sum_{j=1,\dots,n+1,\, j \neq i} &\delta(x_i, x_j) \, {:}a(x_1) \cdots \check{a}(x_i) \cdots a(x_{n+1}){:} \Bigg], \quad n \in \mathbb{N},
\end{aligned}$$
$${:}a(x){:} = a(x), \tag{5.27}$$

Using (5.27) and the CAR (5.15), it is easy to show by induction that

$$:a(x_1)\cdots a(x_n): = \Psi^*(x_n)\cdots\Psi^*(x_1)\Psi(x_1)\cdots\Psi(x_n).$$

Hence, by (5.8),

$$:a(x_1)\cdots a(x_n):\Omega = \Psi^*(x_n)\cdots\Psi^*(x_1)\Phi_1^*(\varkappa_{1,x_1})\cdots\Phi_1^*(\varkappa_{1,x_n})\Omega. \tag{5.28}$$

Thus, $:a(x_1)\cdots a(x_n):$ is the Wick product of the operators $a(x_1),\dots,a(x_n)$, i.e., one first takes the formal product

$$a(x_1)\cdots a(x_n) = \Psi^*(x_1)\Psi(x_1)\cdots\Psi^*(x_n)\Psi(x_n),$$

and then one rearranges the right hand side, so that one first has all the operators $\Psi^*(x_1),\dots,\Psi^*(x_n)$ and then all the operators $\Psi(x_1),\dots,\Psi(x_n)$. From here

$$\begin{aligned}
&(:a(x_1)\cdots a(x_n):\Omega,\Omega)_{\mathcal{AF}(\mathcal{H})}\\
&\quad= (\Phi_1(\varkappa_{1,x_n})\cdots\Phi_1(\varkappa_{1,x_1})\Phi_1^*(\varkappa_{1,x_1})\cdots\Phi_1^*(\varkappa_{1,x_n})\Omega,\Omega)_{\mathcal{AF}(\mathcal{H})}\\
&\quad= \|\varkappa_{1,x_1}\wedge\cdots\wedge\varkappa_{1,x_n}\|^2_{H^{\wedge n}}\\
&\quad= \det[k(x_i,x_j)]_{i,j=1}^n,
\end{aligned}$$

which is the n-th correlation function.

We will now show that the calculations in Remark 5.3 can be given a rigorous meaning. Taking into account (5.28), let us make sense out of the following operators:

$$\begin{aligned}
&\mathcal{T}(\Delta_1,\dots,\Delta_n)\\
&\quad= \int_{\Delta_n}\sigma(dx_n)\,\Psi^*(x_n)\biggl(\int_{\Delta_{n-1}}\sigma(dx_{n-1})\,\Psi^*(x_{n-1})\biggl(\cdots\\
&\qquad\cdots\biggl(\int_{\Delta_1}\sigma(dx_1)\,\Psi^*(x_1)\Phi_1^*(\varkappa_{1,x_1})\biggr)\cdots\Phi_1^*(\varkappa_{1,x_{n-1}})\biggr)\Phi_1^*(\varkappa_{1,x_n}),
\end{aligned}$$

$$\Delta_1, \ldots, \Delta_n \in \mathcal{B}_0(X), \ n \geq 2. \tag{5.29}$$

For Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, we denote by $\mathcal{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ the space of linear continuous operators from $\mathfrak{H}_1$ into $\mathfrak{H}_2$. We also denote by $\mathcal{L}(\mathcal{AF}_{\mathrm{fin}}(\mathcal{H}))$ the space of all linear continuous operators acting in $\mathcal{AF}_{\mathrm{fin}}(\mathcal{H})$.

Let $\Delta \in \mathcal{B}_0(X)$ and let $R_{k,n} \in \mathcal{L}(\mathcal{H}^{\wedge k}, \mathcal{H}^{\wedge n})$. Analogously to (5.10), (5.11), we conclude that the integral

$$\int_\Delta \sigma(dx)\, \Phi_2^*(\varkappa_{2,x}) R_{k,n} \Phi_1^*(\varkappa_{1,x}) \tag{5.30}$$

identifies, though the corresponding bilinear form, the operator in $\mathcal{L}(\mathcal{H}^{\wedge(k-1)}, \mathcal{H}^{\wedge(n+1)})$ which is given by

$$\int_\Delta \sigma(dx)\, \Phi_2^*(\varkappa_{2,x}) R_{k,n} \Phi_1^*(\varkappa_{1,x}) f^{(k-1)} := \mathcal{P}_{n+1}(\mathbf{1} \otimes (R_{k,n}\mathcal{P}_k))$$
$$(\mathcal{N}(K_2 P_\Delta K_1)_{2,1} \otimes f^{(k-1)}), f^{(k-1)} \in \mathcal{H}^{\wedge(k-1)}.$$

Here, $\mathcal{P}_i$ denotes the orthogonal projection of $\mathcal{H}^{\otimes i}$ onto $\mathcal{H}^{\wedge i}$.

Next, using (5.4) and Theorem 2.1, we easily conclude that the integral

$$\int_\Delta \sigma(dx)\, \Phi_1(\varkappa_{1,x}) R_{k,n} \Phi_1^*(\varkappa_{1,x}) \tag{5.31}$$

identifies an operator in $\mathcal{L}(\mathcal{H}^{\wedge(k-1)}, \mathcal{H}^{\wedge(n-1)})$ even in the sense of Bochner integration.

Hence, for each $R \in \mathcal{L}(\mathcal{AF}_{\mathrm{fin}}(\mathcal{H}))$, the integrals (5.30) and (5.31), in which $R_{k,n}$ is replaced by R, identify operators in $\mathcal{L}(\mathcal{AF}_{\mathrm{fin}}(\mathcal{H}))$. So, by induction, the operator (5.29) is well defined.

Lemma 5.2 *For each $n \in \mathbb{N}$ and any $\Delta_1, \ldots, \Delta_n \in \mathcal{B}_0(X)$, we have:*

$$Q(\chi_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \chi_{\Delta_n}) = \frac{1}{n!}\, \mathcal{T}(\Delta_1, \ldots, \Delta_n)\Omega.$$

Proof. We first state that, for any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$ and any $R \in \mathcal{L}(\mathcal{AF}_{\mathrm{fin}}(\mathcal{H}))$, we have

$$a(\Delta_1)\int_{\Delta_2}\sigma(dx)\,\Psi^*(x)R\Phi_1^*(\varkappa_{1,x}) = \int_{\Delta_2}\sigma(dx)\,\Psi^*(x)a(\Delta_1)R\Phi_1^*(\varkappa_{1,x}) + \int_{\Delta_1\cap\Delta_2}\sigma(dx)\,\Psi^*(x)R\Phi_1^*(\varkappa_{1,x}). \quad (5.32)$$

Intuitively, equality (5.32) follows from the CAR (5.15). Indeed,

$$\begin{aligned}
&a(\Delta_1)\int_{\Delta_2}\sigma(dx)\,\Psi^*(x)R\Phi_1^*(\varkappa_{1,x})\\
&\quad= \int_{\Delta_1}\sigma(dx)\Psi^*(x)\Psi(x)\int_{\Delta_2}\sigma(dy)\Psi^*(y)R\Phi_1^*(\varkappa_{1,y})\\
&\quad= \int_{\Delta_1}\sigma(dx)\int_{\Delta_2}\sigma(dy)\Psi^*(x)\Psi(x)\Psi^*(y)R\Phi_1^*(\varkappa_{1,y})\\
&\quad= -\int_{\Delta_1}\sigma(dx)\int_{\Delta_2}\sigma(dy)\Psi^*(x)\Psi^*(y)\Psi(x)R\Phi_1^*(\varkappa_{1,y})\\
&\qquad+\int_{\Delta_1\cap\Delta_2}\sigma(dx)\Psi^*(x)R\Phi_1^*(\varkappa_{1,x})\\
&\quad= \int_{\Delta_1}\sigma(dx)\int_{\Delta_2}\sigma(dy)\Psi^*(y)\Psi^*(x)\Psi(x)R\Phi_1^*(\varkappa_{1,y})\\
&\qquad+\int_{\Delta_1\cap\Delta_2}\sigma(dx)\Psi^*(x)R\Phi_1^*(\varkappa_{1,x})\\
&\quad= \int_{\Delta_2}\sigma(dy)\Psi^*(y)\Big(\int_{\Delta_1}\sigma(dx)\Psi^*(x)\Psi(x)\Big)R\Phi_1^*(\varkappa_{1,y})\\
&\qquad+\int_{\Delta_1\cap\Delta_2}\sigma(dx)\Psi^*(x)R\Phi_1^*(\varkappa_{1,x})\\
&\quad= \int_{\Delta_2}\sigma(dy)\Psi^*(y)a(\Delta_1)R\Phi_1^*(\varkappa_{1,y}) + \int_{\Delta_1\cap\Delta_2}\sigma(dx)\Psi^*(x)R\Phi_1^*(\varkappa_{1,x}).
\end{aligned}$$

In fact, a rigorous proof of (5.32) can be carried out analogously to the proof of Lemma 5.1. Indeed, let us fix an arbitrary orthonormal basis $(e_n)_{n=1}^{\infty}$ of $L^2(X,\sigma)$. Then, by continuity, it is easy to see that it suffices to prove (5.32) in the case where $R \in \mathcal{L}(\mathcal{H}^{\wedge k}, \mathcal{H}^{\wedge n})$ and

$$R\; e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_k} = e_{l_1}\wedge e_{l_2}\wedge\cdots\wedge e_{l_n}$$

for some $1 \leqslant i_1 < i_2 < \cdots < i_k < \infty$ and some $1 \leqslant l_1 < l_2 < \cdots < l_n < \infty$, and

$$R\, e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k} = 0$$

for any $1 \leqslant j_1 < j_2 < \cdots < j_k < \infty$ such that $(j_1, j_2, \dots, j_k) \neq (i_1, i_2, \dots, i_k)$. Then, for any $h_1, h_2, \dots, h_{k-1} \in \mathcal{H}$,

$$\begin{aligned} R\,\Phi_1^*(\varkappa_{1,x})h_1 \wedge \cdots \wedge h_{k-1} & \\ &= R\ \varkappa_{1,x} \wedge h_1 \wedge \cdots \wedge h_{k-1} \\ &= \big(\varkappa_{1,x} \wedge h_1 \wedge \cdots \wedge h_{k-1}, e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}\big) R\ e_{i_2} \wedge \cdots \wedge e_{i_k} \\ &= \big(\varkappa_{1,x} \wedge h_1 \wedge \cdots \wedge h_{k-1}, e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}\big)\, e_{l_1} \wedge e_{l_2} \wedge \cdots \wedge e_{l_n}. \end{aligned}$$

Now, the rest of calculations follows on complete analogy with the proof of Lemma 5.1. Thus, (5.32) is proven.

Next, analogously to the above, we see that, for any $\Delta_1.\Delta_2 \in \mathcal{B}_0(X)$ and any $R \in \mathcal{L}(\mathcal{AF}_{\mathrm{fin}}(\mathcal{H}))$, we have:

$$\begin{aligned} &\int_{\Delta_1} \sigma(dx)\Psi^*(x)\Big(\int_{\Delta_2} \sigma(dy)\Psi^*(y)R\Phi_1^*(\varkappa_{1,y})\Big)\Phi_1^*(\varkappa_{1,x}) \\ &\quad = \int_{\Delta_2} \sigma(dy)\Psi^*(y)\Big(\Big(\int_{\Delta_1} \sigma(dx)\Psi^*(x)\Big)R\Phi_1^*(\varkappa_{1,x})\Big)\Phi_1^*(\varkappa_{1,y}). \end{aligned}$$

(The above equality is heuristically clear and follows from the CAR). Therefore, for any $\Delta_1, \dots, \Delta_n \in \mathcal{B}_0(X)$ and each $i \in \{1, 2, \dots, n-1\}$,

$$\mathcal{T}(\Delta_1, \Delta_2, \dots, \Delta_n) = \mathcal{T}(\Delta_1, \dots, \Delta_{i-1}, \Delta_{i+1}, \Delta_i, \Delta_{i+2}, \dots, \Delta_n).$$

From here it follows that the operator $\mathcal{T}(\Delta_1, \dots, \Delta_n)$ does not depend on the order of the sets $\Delta_1, \dots, \Delta_n \in \mathcal{B}_0(X)$.

Next, for each $\Delta \in \mathcal{B}_0(X)$, we have by (5.8),

$$\begin{aligned} \mathcal{Q}(\chi_\Delta)\Omega &= a(\Delta)\Omega \\ &= \int_\Delta \sigma(dx)\Psi^*(x)\Psi(x)\Omega \end{aligned}$$

$$\begin{aligned}
&= \int_\Delta \sigma(dx)\Psi^*(x)(\Phi_2(\varkappa_{2,x}) + \Phi_1^*(\varkappa_{1,x}))\Omega \\
&= \int_\Delta \sigma(dx)\Psi^*(x)\Phi_1^*(\varkappa_{1,x})\Omega \\
&= \mathcal{T}(\Delta)\Omega.
\end{aligned}$$

Thus, by (4.2), to prove the lemma, it suffices to show that, for each $n \in \mathbb{N}$, and any $\Delta_1, \dots, \Delta_{n+1} \in \mathcal{B}_0(X)$, we have:

$$\sum_{i=1}^{n+1}\Big(a(\Delta_i)\mathcal{T}(\Delta_1, \dots, \check{\Delta}_i, \dots, \Delta_{n+1}) \\ - \sum_{j=1,\dots,n+1,\, j\neq i} \mathcal{T}(\Delta_i \cap \Delta_j, \Delta_1, \dots, \check{\Delta}_i, \dots, \check{\Delta}_j, \dots, \Delta_{n+1})\Big) \\ = (n+1)\mathcal{T}(\Delta_1, \dots, \Delta_{n+1}).$$

So, it suffices to prove that, for each $i \in \{1, \dots, n\}$

$$a(\Delta_i)\mathcal{T}(\Delta_1, \dots, \check{\Delta}_i, \dots, \Delta_{n+1}) \\ - \sum_{j=1,\dots,n+1,\, j\neq i} \mathcal{T}(\Delta_i \cap \Delta_j, \Delta_1, \dots, \check{\Delta}_i, \dots, \check{\Delta}_j, \dots, \Delta_{n+1}) \\ = \mathcal{T}(\Delta_1, \dots, \Delta_{n+1}). \tag{5.33}$$

Since the operators $\mathcal{T}(\cdot)$ do not depend on the order of the sets which index them, (5.33) is equivalent to

$$a(\Delta_1)\mathcal{T}(\Delta_2, \dots, \Delta_{n+1})\Omega - \sum_{j=2}^{n+1} \mathcal{T}(\Delta_2, \dots, \Delta_{j-1}, \\ \Delta_j \cap \Delta_1, \Delta_{j+1}, \dots, \Delta_{n+1})\Omega \\ = \mathcal{T}(\Delta_1, \dots, \Delta_{n+1})\Omega.$$

Now, by (5.32), we have:

$$a(\Delta_1)\mathcal{T}(\Delta_2, \dots, \Delta_{n+1})\Omega - \sum_{j=2}^{n+1} \mathcal{T}(\Delta_2, \dots, \Delta_{j-1},$$

$$
\begin{aligned}
&\Delta_j \cap \Delta_1, \Delta_{j+1}, \ldots, \Delta_{n+1})\Omega \\
&= a(\Delta_1) \int_{\Delta_{n+1}} \sigma(dx_{n+1}) \Psi^*(x_{n+1}) \mathcal{T}(\Delta_2, \ldots, \Delta_n) \Phi_1^*(\varkappa_{1,x_{n+1}})\Omega \\
&\quad - \sum_{j=2}^{n+1} \mathcal{T}(\Delta_2, \ldots, \Delta_{j-1}, \Delta_j \cap \Delta_1, \Delta_{j+1}, \ldots, \Delta_{n+1})\Omega \\
&= \int_{\Delta_{n+1}} \sigma(dx_{n+1}) \Psi^*(x_{n+1}) \Big(a(\Delta_1) \mathcal{T}(\Delta_2, \ldots, \Delta_n) \Big) \Phi_1^*(\varkappa_{1,x_{n+1}})\Omega \\
&\quad + \int_{\Delta_1 \cap \Delta_{n+1}} \sigma(dx) \Psi^*(x) \mathcal{T}(\Delta_2, \ldots, \Delta_n) \Phi_1^*(\varkappa_{1,x})\Omega \\
&\quad - \sum_{j=2}^{n+1} \mathcal{T}(\Delta_2, \ldots, \Delta_{j-1}, \Delta_j \cap \Delta_1, \Delta_{j+1}, \ldots, \Delta_{n+1})\Omega \\
&= \int_{\Delta_{n+1}} \sigma(dx_{n+1}) \Psi^*(x_{n+1}) \Big(a(\Delta_1) \mathcal{T}(\Delta_2, \ldots, \Delta_n) \Big) \Phi_1^*(\varkappa_{1,x_{n+1}})\Omega \\
&\quad - \sum_{j=2}^{n} \mathcal{T}(\Delta_2, \ldots, \Delta_{j-1}, \Delta_j \cap \Delta_1, \Delta_{j+1}, \ldots, \Delta_n)\Omega \\
&= \int_{\Delta_{n+1}} \sigma(dx_{n+1}) \Psi^*(x_{n+1}) \Big(\int_{\Delta_n} \sigma(dx_n) \Psi^*(x_n) a(\Delta_1) \\
&\qquad \mathcal{T}(\Delta_2, \ldots, \Delta_{n-1}) \Phi_1^*(\varkappa_{1,x_n}) \Big) \Phi_1^*(\varkappa_{1,x_{n+1}})\Omega \\
&\quad + \int_{\Delta_{n+1}} \sigma(dx_{n+1}) \Psi^*(x_{n+1}) \Big(\int_{\Delta_1 \cap \Delta_n} \sigma(dx) \Psi^*(x) \mathcal{T}(\Delta_2, \ldots, \Delta_{n-1}) \\
&\qquad \Phi_1^*(\varkappa_{1,x}) \Big) \Phi_1^*(\varkappa_{1,x_{n+1}})\Omega \\
&\quad - \sum_{j=2}^{n} \mathcal{T}(\Delta_2, \ldots, \Delta_{j-1}, \Delta_j \cap \Delta_1, \Delta_{j+1}, \ldots, \Delta_n)\Omega \\
&= \int_{\Delta_{n+1}} \sigma(dx_{n+1}) \Psi^*(x_{n+1}) \Big(\int_{\Delta_n} \sigma(dx_n) \Psi^*(x_n) a(\Delta_1) \\
&\qquad \mathcal{T}(\Delta_2, \ldots, \Delta_{n-1}) \Phi_1^*(\varkappa_{1,x_n}) \Big) \Phi_1^*(\varkappa_{1,x_{n+1}})\Omega \\
&\quad - \sum_{j=2}^{n-1} \mathcal{T}(\Delta_2, \ldots, \Delta_{j-1}, \Delta_j \cap \Delta_1, \Delta_{j+1}, \ldots, \Delta_n)\Omega \\
&= \cdots = \int_{\Delta_{n+1}} \sigma(dx_{n+1}) \Psi^*(x_{n+1}) \Big(\int_{\Delta_n} \sigma(dx_n) \Psi^*(x_n) \Big(\cdots
\end{aligned}
$$

$$\Big(\int_{\Delta_1}\sigma(dx_1)\Psi^*(x_1)\Phi_1^*(\varkappa_{1,x_1})\Big)\cdots\Big)\Phi_1^*(\varkappa_{1,x_n})\Big)\Phi_1^*(\varkappa_{1,x_{n+1}})\Omega$$
$$=\mathcal{T}(\Delta_1,\dots,\Delta_{n+1})\Omega. \quad \square$$

Lemma 5.3 *The family of operators* $(a(\Delta))_{\Delta\in\mathcal{B}_0(X)}$ *has a correlation measure* ρ *with respect to* Ω, *and the restriction of* ρ *to* $(\Gamma_X^{(n)},\mathcal{B}(\Gamma_X^{(n)}))$ *is given by*

$$\rho^{(n)}(dx_1,\dots,dx_n)=\frac{1}{n!}\det[k(x_i,x_j)]_{i,j=1}^n\,\sigma(dx_1)\cdots\sigma(dx_n) \tag{5.34}$$

(*recall that we have identified* $\mathcal{B}(\ddot{\Gamma}_X^{(n)})$ *with* $\mathcal{B}_{\mathrm{sym}}(X^n)$, *and* $\mathcal{B}(\Gamma_X^{(n)})\subset\mathcal{B}(\ddot{\Gamma}_X^{(n)})$).

Proof. By (5.29) and Lemma 5.2, for each $n\in\mathbb{N}$ and any $\Delta_1,\dots,\Delta_n\in\mathcal{B}_0(X)$, we have

$$\begin{aligned}&(Q(\chi_{\Delta_1}\widehat{\otimes}\cdots\widehat{\otimes}\chi_{\Delta_n}),\Omega)_{\mathcal{AF}(\mathcal{H})}\\ &\quad=\frac{1}{n!}\Big(\Big(\int_{\Delta_n}\sigma(dx_n)\,\Phi_1(\varkappa_{1,x_n})\Big(\int_{\Delta_{n-1}}\sigma(dx_{n-1})\,\Phi_1(\varkappa_{1,x_{n-1}})\Big(\cdots\\ &\quad\cdots\Big(\int_{\Delta_1}\sigma(dx_1)\,\Phi_1(\varkappa_{1,x_1})\Phi_1^*(\varkappa_{1,x_1})\Big)\cdots\Phi_1^*(\varkappa_{1,x_{n-1}})\Big)\Phi_1^*(\varkappa_{1,x_n})\Omega,\Omega\Big)_{\mathcal{AF}(\mathcal{H})}\end{aligned}$$

(note that all the integrals involving $\Phi_2^*(\varkappa_{2,x_i})$ vanish). Therefore,

$$\begin{aligned}&(Q(\chi_{\Delta_1}\widehat{\otimes}\cdots\widehat{\otimes}\chi_{\Delta_n}),\Omega)_{\mathcal{AF}(\mathcal{H})}\\ &\quad=\int_{\Delta_n}\sigma(dx_n)\cdots\int_{\Delta_1}\sigma(dx_1)\big\|\varkappa_{1,x_1}\wedge\cdots\wedge\varkappa_{1,x_n}\big\|^2_{H^{\wedge n}} &(5.35)\\ &\quad=\frac{1}{n!}\int_{\Delta_n}\sigma(dx_n)\cdots\int_{\Delta_1}\sigma(dx_1)\det[k(x_i,x_j)]_{i,j=1}^n. &(5.36)\end{aligned}$$

Note that, by (5.35), the right hand side of (5.34) indeed defines a measure. Hence, the statement of the lemma follows from (5.36) $\square$

Lemma 5.4 *The correlation measure given in* (5.34) *satisfies* (LB).

Proof. For each $\Delta \in \mathcal{B}_0(X)$ and $n \in \mathbb{N}$, we evidently have

$$\begin{aligned}\rho(\Gamma_\Delta^{(n)}) &\le \left(\int_\Delta \|\varkappa_{1,x}\|_H^2 \, \sigma(dx)\right)^n \\ &= \left(\int_\Delta \int_X \mathcal{N}(K_1)(x,y)^2 \, \sigma(dx)\, \sigma(dy)\right)^n,\end{aligned}$$

from where the statement follows. □

By Lemmas 5.1, 5.3, 5.4 and Theorem 4.1, we get

Theorem 5.1 *For the family* $(a(\Delta))_{\Delta\in\mathcal{B}_0(X)}$ *defined by* (5.14), *the statement of Theorem* 4.1 *holds with the correlation measure given by* (5.34).

Let us now briefly mention the boson case. About the operator K we make the same assumptions as in the fermion case, apart from the assumption that $K \le \mathbf{1}$. We set $K_1 := \sqrt{K}$ (just as above) and $K_2 := (\mathbf{1} + K)^{1/2}$. We then essentially repeat the fermion case, using however the symmetric Fock space $\mathcal{SF}(\mathcal{H})$ instead of the antysymmetric Fock space $\mathcal{AF}(\mathcal{H})$. The operators $\Psi(f)$, $\Psi^*(f)$ (see (5.1)) now satisfy the CCR (use the commutator $[A,B]_- := AB - BA$ instead of the anticommutator in (5.2)). The counterpart of formulas (5.35), (5.36) reads as folllows:

$$\begin{aligned}&(Q(\chi_{\Delta_1}\widehat{\otimes}\cdots\widehat{\otimes}\chi_{\Delta_n}),\Omega)_{\mathcal{AF}(\mathcal{H})}\\ &\quad= \int_{\Delta_n}\sigma(dx_n)\cdots\int_{\Delta_1}\sigma(dx_1)\left\|\varkappa_{1,x_1}\widehat{\otimes}\cdots\widehat{\otimes}\varkappa_{1,x_n}\right\|^2_{\mathcal{H}^{\widehat{\otimes} n}}\\ &\quad= \frac{1}{n!}\int_{\Delta_n}\sigma(dx_n)\cdots\int_{\Delta_1}\sigma(dx_1)\,\mathrm{per}[k(x_i,x_j)]_{i,j=1,\dots,n}\,\sigma(dx_1)\cdots\sigma(dx_n).\end{aligned}$$

Thus the corresponding correlation measure is given by (5.34) in which the determinant is replaced by the permanent:

$$\mathrm{per}\left[k(x_{i,x_j})\right]_{i,j=1,\dots,n} = \sum_{\sigma\in S_n}\prod_{i=1}^n k(x_i, x_{\sigma(i)}).$$

5.2 Fermion-like and boson-like particle densities

Let the operators K, K_1, K_2 be as in the fermion part of Section 5. Let $l \in \mathbb{N}$, $l \geq 2$, and we now take $2l$ copies of the Hilbert space $H = L^2(X, \sigma)$: $H_{1,i}$ and $H_{2,i}$, $i = 1, \dots, l$. We denote $\mathcal{H}^{(l)} := \bigoplus_{i=1}^{l}(H_{1,i} \oplus H_{2,i})$.

For each $i \in \{1, 2, \dots, l\}$, let us consider the following heuristic operators, for each $x \in X$,

$$\Psi_i(x) = \Phi_2(\varkappa_{2,i,x}) + \Phi_1^*(\varkappa_{1,i,x}),$$
$$\Psi_i^*(x) = \Phi_2^*(\varkappa_{2,i,x}) + \Phi_1(\varkappa_{1,i,x}),$$

where $\varkappa_{1,i,x}, \varkappa_{2,i,x}$ are the corresponding elements of $H_{1,i} \oplus H_{2,i}$.

Define, for each $x \in X$,

$$a^{(l)}(x) = \sum_{i=1}^{l} \Psi_i^*(x)\Psi_i(x).$$

Thus, we consider the l-fold convolution of particle densities $\Psi^*(x)\Psi(x)$.

Then, for each $\Delta \in \mathcal{B}_0(X)$, we set

$$a^{(l)}(\Delta) = \int_\Delta a^{(l)}(x)\sigma(dx).$$

If we denote

$$a_i^{(l)}(\Delta) := \int_\Delta \Psi_i^*(x)\Psi_i(x)\sigma(dx),$$

then

$$a^{(l)}(\Delta) = \sum_{i=1}^{l} a_i^{(l)}(\Delta).$$

Each of the operators $a_i^{(l)}(\Delta)$ is clearly well defined and Hermitian in $\mathcal{AF}(\mathcal{H}^{(l)})$, and hence so is the operator $a^{(l)}(\Delta)$.

Next, for each $i \in \{1, 2, \dots, l\}$ and for any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$, the operators $a_i^{(l)}(\Delta_1)$ and $a_i^{(l)}(\Delta_2)$ commute, and hence so do the operators $a^{(l)}(\Delta_1)$

and $a^{(l)}(\Delta_2)$. Hence, $\left(a^{(l)}(\Delta)\right)_{\Delta\in\mathcal{B}_0(X)}$ is a family of commuting Hermitian operators in $\mathcal{AF}(\mathcal{H}^{(l)})$.

Completely analogously to the proof of Lemma 5.2, we get, for each $n\in\mathbb{N}$ and any $\Delta_1,\dots,\Delta_n\in\mathcal{B}_0(X)$,

$$Q(\chi_{\Delta_1}\widehat{\otimes}\cdots\widehat{\otimes}\chi_{\Delta_n})\Omega := Q(\chi_{\Delta_1}\widehat{\otimes}\cdots\widehat{\otimes}\chi_{\Delta_n}) = \frac{1}{n!}\sum_{i_1=1}^{l}\cdots\sum_{i_n=1}^{l}$$
$$\int_{\Delta_n}\sigma(dx_n)\,\Psi^*_{i_n}(x_n)\biggl(\int_{\Delta_{n-1}}\sigma(dx_{n-1})\,\Psi^*_{i_{n-1}}(x_{n-1})\biggl(\cdots$$
$$\cdots\biggl(\int_{\Delta_1}\sigma(dx_1)\,\Psi^*_{i_1}(x_1)\Phi^*_1(\varkappa_{1,i_1,x_1})\biggr)\cdots\biggr)\Phi^*_1(\varkappa_{1,i_{n-1},x_{n-1}})\biggr)\Phi^*_1(\varkappa_{1,i_n,x_n})\Omega.$$

Hence, analogously to the proof of Lemma 5.3, we have

$$\begin{aligned}
&\Bigl(Q(\chi_{\Delta_1}\widehat{\otimes}\cdots\widehat{\otimes}\chi_{\Delta_n}),\Omega\Bigr)_{\mathcal{AF}(\mathcal{H}^{(l)})}\\
&=\frac{1}{n!}\sum_{i_1=1}^{l}\cdots\sum_{i_n=1}^{l}\biggl(\int_{\Delta_n}\sigma(dx_n)\Phi_1(\varkappa_{1,i_n,x_n})\biggl(\int_{\Delta_{n-1}}\sigma(dx_{n-1})\Phi_1(\varkappa_{1,i_{n-1},x_{n-1}})\\
&\quad\biggl(\cdots\biggl(\int_{\Delta_1}(dx_1)\Phi_1(\varkappa_{1,i_1,x_1})\Phi^*_1(\varkappa_{1,i_1,x_1})\biggr)\cdots\biggr)\Phi^*_1(\varkappa_{1,i_{n-1},x_{n-1}})\biggr)\\
&\quad\times\Phi^*_1(\varkappa_{1,i_n},x_n)\Omega,\Omega\biggr)\\
&=\sum_{i_1=1}^{l}\cdots\sum_{i_n=1}^{l}\int_{\Delta_n}\sigma(dx_n)\cdots\int_{\Delta_1}\sigma(dx_1)\|\varkappa_{1,i_1,x_1}\wedge\cdots\wedge\varkappa_{1,i_n,x_n}\|^2_{(\mathcal{H}^{(l)})^{\wedge n}}.
\end{aligned}$$

Hence, the correlation measure ρ is given through

$$\begin{aligned}
&\rho^{(n)}(dx_1,\dots,dx_n)\\
&=\biggl(\sum_{i_1=1}^{l}\cdots\sum_{i_n=1}^{l}\|\varkappa_{1,i_1,x_1}\wedge\cdots\wedge\varkappa_{1,i_n,x_n}\|^2_{(\mathcal{H}^{(l)})^{\wedge n}}\biggr)\sigma(dx_1)\cdots\sigma(dx_n).
\end{aligned}\tag{5.37}$$

Analogously to Lemma 5.4, we conclude that the correlation measure given in (5.37) satisfies (LB).

Following [30] and [25], we introduce the notion of α-determinant. So let $\alpha \in \mathbb{R}$ be fixed. For a square matrix $A = (a_{ij})_{i,j=1}^n$, we define its α-determinant as follows:

$$\det_\alpha A = \sum_{\xi \in S_n} \alpha^{n-\nu(\xi)} \prod_{i=1}^n a_{i\xi(i)}. \tag{5.38}$$

Here $\nu(\xi)$ denotes the number of cycles in ξ. Since for each $\xi \in S_n$

$$(-1)^{n-\nu(\xi)} = \operatorname{sign}(\xi), \tag{5.39}$$

$\det_{-1}$A is the usual determinant $\det A$, whereas $\det_1$A is clearly the usual permanent $\operatorname{per} A$.

Lemma 5.5 *For any $x_1, \dots, x_n \in X$, we have*

$$\sum_{i_1=1}^l \cdots \sum_{i_n=1}^l \|\varkappa_{1,i_1,x_1} \wedge \cdots \wedge \varkappa_{1,i_n,x_n}\|^2_{(\mathcal{H}^{(l)})^{\wedge n}} = \frac{1}{n!}\det_{-1/l}\left[lk(x_i,x_j)\right]_{i,j=1}^n.$$

Proof. By (5.38) and (5.39), we have

$$\begin{aligned}
&\frac{1}{n!}\mathrm{d}\,et_{-1/l}\left[lk(x_i,x_j)\right]_{i,j=1}^n \\
&\quad = \frac{1}{n!}\sum_{\xi \in S_n}\Big(-\frac{1}{l}\Big)^{n-\nu(\xi)} l^n k(x_1, x_{\xi(1)}) \cdots k(x_n, x_{\xi(n)}) \\
&\quad = \frac{1}{n!}\sum_{\xi \in S_n}(-1)^{n-\nu(\xi)} l^{\nu(\xi)} k(x_1, x_{\xi(1)}) \cdots k(x_n, x_{\xi(n)}) \\
&\quad = \frac{1}{n!}\sum_{\xi \in S_n}\operatorname{sign}(\xi) l^{\nu(\xi)} k(x_1, x_{\xi(1)}) \cdots k(x_n, x_{\xi(n)}).
\end{aligned} \tag{5.40}$$

On the other hand,

$$\begin{aligned}
&\sum_{i_1=1}^l \cdots \sum_{i_n=1}^l \|\varkappa_{1,i_1,x_1} \wedge \cdots \wedge \varkappa_{1,i_n,x_n}\|^2_{(\mathcal{H}^{(l)})^{\wedge n}} \\
&\quad = \sum_{i_1=1}^l \cdots \sum_{i_n=1}^l (\varkappa_{1,i_1,x_1} \wedge \cdots \wedge \varkappa_{1,i_n,x_n}, \varkappa_{1,i_1,x_1} \wedge \cdots \wedge \varkappa_{1,i_n,x_n})_{(\mathcal{H}^{(l)})^{\wedge n}} \\
&\quad = \sum_{i_1=1}^l \cdots \sum_{i_n=1}^l (\varkappa_{1,i_1,x_1} \otimes \cdots \otimes \varkappa_{1,i_n,x_n}, \varkappa_{1,i_1,x_1} \wedge \cdots \wedge \varkappa_{1,i_n,x_n})_{(\mathcal{H}^{(l)})^{\wedge n}}
\end{aligned}$$

$$= \frac{1}{n!}\sum_{i_1=1}^{l}\cdots\sum_{i_n=1}^{l}\sum_{\xi\in S_n}\operatorname{sign}(\xi)\Big(\varkappa_{1,i_1,x_1}\otimes\cdots\otimes\varkappa_{1,i_n,x_n},\varkappa_{1,i_{\xi(1)},x_{\xi_1}}\otimes$$
$$\cdots\otimes\varkappa_{1,i_{\xi(n)},x_{\xi(n)}}\Big)$$
$$= \frac{1}{n!}\sum_{\xi\in S_n}\operatorname{sign}(\xi)\sum_{i_1=1}^{l}\cdots\sum_{i_n=1}^{l}\Big(\varkappa_{1,i_1,x_1},\varkappa_{1,i_{\xi(1)},x_{\xi(1)}}\Big)_{\mathcal{H}^{(l)}}$$
$$\cdots\Big(\varkappa_{1,i_n,x_n},\varkappa_{1,i_{\xi(n)},x_{\xi(n)}}\Big)_{\mathcal{H}^{(l)}}$$
$$= \frac{1}{n!}\sum_{\xi\in S_n}\operatorname{sign}(\xi)\sum_{i_1=1}^{l}\cdots\sum_{i_n=1}^{l}\delta_{i_1,i_{\xi(1)}}k(x_1,x_{\xi(1)})\cdots\delta_{i_n,i_{\xi(n)}}k(x_n,x_{\xi(n)})$$
$$= \frac{1}{n!}\sum_{\xi\in S_n}\operatorname{sign}(\xi)\Big(\sum_{i_1=1}^{l}\cdots\sum_{i_n=1}^{l}\delta_{i_1,i_{\xi(1)}}\cdots\delta_{i_n,i_{\xi(n)}}\Big)$$
$$\times k(x_1,x_{\xi(1)})\cdots k(x_n,x_{\xi(n)}) \tag{5.41}$$

Comparing (5.40) and (5.41), we see that, in order to prove the lemma, we need to prove that, for any fixed $\xi\in S_n$,

$$\sum_{i_1=1}^{l}\cdots\sum_{i_n=1}^{l}\delta_{i_1,i_{\xi(1)}}\cdots\delta_{i_n,i_{\xi(n)}} = l^{\nu(\xi)}. \tag{5.42}$$

So, let us prove (5.42). Let $\xi\in S_n$ be fixed and denote $m:=\nu(\xi)$. The cycles of ξ divide the set $\{1,2,\dots,n\}$ into m disjoint subsets, which we denote by

$$\{u_1,\dots,u_{j_1}\},\{u_{j_1+1},\dots,u_{j_1+j_2}\},\{u_{j_1+j_2+1},\dots,u_{j_1+j_2+j_3}\},$$
$$\dots,\{u_{j_1+\dots+j_{m-1}+1},\dots,u_n\}.$$

Then

$$\delta_{i_1,i_{\xi(1)}}\cdots\delta_{i_n,i_{\xi(n)}}$$
$$=\delta(i_{u_1}=i_{u_2}=\dots=i_{u_{j_1}})\delta(i_{u_{j_1+1}}=i_{u_{j_1+2}}=\dots=i_{u_{j_1+j_2}})$$
$$\cdots\delta(i_{u_{j_1+\dots+j_{m-1}}}=i_{u_{j_1+\dots+j_{m-1}+1}}=\dots=i_{u_n}),$$

where, for some condition A, $\delta(A)$ is equal to 1 if condition A is satisfied and 0 otherwise. Therefore

$$\begin{aligned}
&\sum_{i_1=1}^{l}\cdots\sum_{i_n=1}^{l}\delta_{i_1,i_{\xi(1)}}\cdots\delta_{i_n,i_{\xi(n)}}\\
&=\sum_{i_1=1}^{l}\cdots\sum_{i_n=1}^{l}\delta(i_{u_1}=i_{u_2}=\cdots=i_{u_{j_1}})\delta(i_{u_{j_1+1}}=i_{u_{j_1+2}}=\cdots=i_{u_{j_1+j_2}})\\
&\quad\cdots\delta(i_{u_{j_1+\cdots+j_{m-1}}}=i_{u_{j_1+\cdots+j_{m-1}+1}}=\cdots=i_{u_n})\\
&=\sum_{i_{u_1}=1}^{l}\sum_{i_{u_{j_1+1}}=1}^{l}\cdots\sum_{i_{u_{j_1+\cdots+j_{m-1}+1}}=1}^{l}1\\
&=l^m. \quad \square
\end{aligned}$$

Thus, we have proved the following theorem (compare with [25]).

Theorem 5.2 *For the family* $\left(a^{(l)}(\Delta)\right)_{\Delta\in\mathcal{B}_0(X)}$, *the statement of Theorem 4.1 holds with the correlation measure given by*

$$\rho^{(n)}(dx_1,\ldots,dx_n)=\frac{1}{n!}\det\nolimits_{-1/l}\left[lk(x_i,x_j)\right]_{i,j=1}^{n}.$$

Let us show that the measure derived in Theorem 5.2, which we denote by $\mu^{(l)}$, is indeed an l-fold convolution of fermion measure μ with kernel $k(x,y)$. We will give a direct proof of this result using correlation functions and without appealing to the formula of the Laplace transform of this measure given in [25].

Let us first consider a probability measure μ on $(\Gamma,\mathcal{B}(\Gamma))$ whose correlation measure ρ satisfies (LB) and such that

$$\rho^{(n)}(dx_1,\ldots,dx_n)=\frac{1}{n!}\kappa^{(n)}(x_1,\ldots,x_n)\sigma(dx_1)\cdots\sigma(dx_n),$$

(recall that $\kappa^{(n)}$ are called correlation functions of μ). The l-fold convolution of μ is defined by

$$\mu^{*l}(A)=\int_{\Gamma^l}\chi_A(\gamma_1\cup\gamma_2\cdots\cup\gamma_l)\ \mu(d\gamma_1)\cdots\mu(d\gamma_l),$$

where $A \in \mathcal{B}(\Gamma)$. (It is easy to see that μ^{*l} is well-defined). Let us find the correlation functions of μ^{*l}. Let $\varphi \in C_0(X)$. Then

$$
\begin{aligned}
&\int_\Gamma \sum_{\{x_1,\dots,x_n\}\subset\gamma} \varphi^{\otimes n}(x_1,\dots,x_n)\mu^{*l}(d\gamma)\\
&= \int_{\Gamma^l} \sum_{\{x_1,\dots,x_n\}\subset\gamma_1\cup\dots\cup\gamma_l} \varphi^{\otimes n}(x_1,\dots,x_n)\mu(d\gamma_1)\cdots\mu(d\gamma_l)\\
&= \int_{\Gamma^l} \sum_{\substack{k_1,\dots,k_l=0,\dots,n\\ k_1+\dots+k_l=n}} \sum_{\{x_1,\dots,x_{k_1}\}\subset\gamma_1} \sum_{\{x_{k_1+1},\dots,x_{k_1+k_2}\}\subset\gamma_2}\\
&\quad\cdots \sum_{\{x_{k_1+\dots+k_{l-1}+1},\dots,x_n\}\subset\gamma_l} \varphi^{\otimes n}(x_1,\dots,x_n)\mu(d\gamma_1)\cdots\mu(d\gamma_l)\\
&= \sum_{\substack{k_1,\dots,k_l=0,\dots,n\\ k_1+\dots+k_l=n}} \int_\Gamma \sum_{\{x_1,\dots,x_{k_1}\}\subset\gamma_1} \varphi^{\otimes k_1}(x_1,\dots,x_{k_1})\mu(d\gamma_1)\\
&\quad\times \int_\Gamma \sum_{\{x_{k_1+1},\dots,x_{k_1+k_2}\}\subset\gamma_2} \varphi^{\otimes k_2}(x_{k_1+1},\dots,x_{k_1+k_2})\mu(d\gamma_2)\dots\\
&\quad\dots \int_\Gamma \sum_{\{x_{k_1+\dots+k_{l-1}+1},\dots,x_n\}\subset\gamma_l} \varphi^{\otimes k_l}(x_{k_1+\dots+k_{n-1}+1},\dots,x_n)\mu(d\gamma_l)\\
&= \sum_{\substack{k_1,\dots,k_l=0,\dots,n\\ k_1+\dots+k_l=n}} \frac{1}{k_1!}\int_{X^{k_1}} \varphi^{\otimes k_1}(x_1,\dots,x_{k_1})\kappa^{(k_1)}(x_1,\dots,x_{k_1})\sigma(dx_1)\cdots\sigma(dx_{k_1})\\
&\quad\times \frac{1}{k_2!}\int_{X^{k_2}} \varphi^{\otimes k_2}(x_{k_1+1},\dots,x_{k_1+k_2})\kappa^{(k_2)}(x_{k_1+1},\dots,x_{k_1+k_2})\\
&\qquad\qquad \sigma(dx_{k_1+1})\cdots\sigma(dx_{k_1+k_2})\\
&\quad\dots\times \frac{1}{k_l!}\int_{X^{k_l}} \varphi^{\otimes k_l}(x_{k_1+\dots+k_{l-1}+1},\dots,x_n)\kappa^{(k_l)}(x_{k_1+\dots+k_{l-1}+1},\dots,x_n)\\
&\qquad\qquad \sigma(dx_{k_1+\dots+k_{l-1}+1})\cdots\sigma(dx_n)\\
&= \frac{1}{n!}\int_{X^n} \varphi^{\otimes n}(x_1,\dots,x_n)\Bigg(\sum_{\substack{k_1,\dots,k_l=0,\dots,n\\ k_1+\dots+k_l=n}} \frac{n!}{k_1!k_2!\cdots k_l!}\kappa^{(k_1)}(x_1,\dots,x_{k_1})\\
&\quad \kappa^{(k_2)}(x_{k_1+1},\dots,x_{k_1+k_2})\cdots\kappa^{(k_l)}(x_{k_1+\dots+k_{l-1}+1},\dots,x_n)\Bigg)\sigma(dx_1)\cdots\sigma(dx_n).
\end{aligned}
$$

Therefore, the correlation functions of the measure μ^{*l}, which we denote by $\kappa^{(n)}_{\mu^{*l}}$, are given by

$$\kappa^{(n)}_{\mu^{*l}}(x_1,\dots,x_n)$$
$$= \sum_{\substack{k_1,\dots,k_l=0,\dots,n\\ k_1+\dots+k_l=n}} \frac{n!}{k_1!k_2!\cdots k_l!}\Big(\kappa^{(k_1)}(x_1,\dots,x_{k_1})$$
$$\times\, \kappa^{(k_2)}(x_{k_1+1},\dots,x_{k_1+k_2})\cdots\kappa^{(k_l)}(x_{k_1+\dots+k_{l-1}+1},\dots,x_n)\Big)^{\sim}, \qquad (5.43)$$

where $(\cdot)^{\sim}$, as above, denotes the symmetrization of a function.

Since the correlation functions $\kappa^{(n)}$ are symmetric functions, we may treat the sequence of correlation functions $(\kappa^{(n)})_{n=0}^{\infty}$ (where $\kappa^{(0)}=1$) as a function κ on $\Gamma_{X,0}$ (compare with Chapter 3). Then, formula (5.43) takes the following form

$$\kappa_{\mu^{*l}}(\eta) = \sum_{(\eta_1,\eta_2,\dots,\eta_l)\in P_l(\eta)} \kappa(\eta_1)\kappa(\eta_2)\cdots\kappa(\eta_l), \qquad (5.44)$$

where $P_l(\eta)$ denotes the set of all ordered partition of η into l parts (again compare with Chapter 3).

Proposition 5.1 *Let $\mu^{(l)}$ be the probability measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ which has correlation function given by*

$$\kappa^{(n)}_{\mu^{(l)}}(x_1,\dots,x_n) = \det\nolimits_{-1/l}\left[lk(x_i,x_j)\right]_{i,j=1}^{n}.$$

Then $\mu^{(l)}$ is the l-fold convolution of fermion point processes μ corresponding to the kernel $k(x,y)$ as in Section 5.1.

Proof. By (5.40), we have

$$\kappa^{(n)}_{\mu^{(l)}}(x_1,\dots,x_n) = \sum_{\xi\in S_n} \operatorname{sign}(\xi) l^{\nu(\xi)}\kappa(x_1,x_{\xi(1)})\dots\kappa(x_n,x_{\xi(n)}). \qquad (5.45)$$

On the other hand, by (5.44),

$$\kappa^{(n)}_{\mu^{*l}}(x_1,\dots,x_n) = \sum_{(\eta_1,\eta_2,\dots,\eta_l)\in P_l(\{x_1,\dots,x_n\})} \kappa_\mu(\eta_1)\kappa_\mu(\eta_2)\cdots\kappa_\mu(\eta_l),$$

where

$$\kappa_\mu(\{y_1,\dots,y_m\}) = \sum_{\xi\in S_m}(-1)^{m-\nu(\xi)}\kappa(y_1,y_{\xi(1)})\dots\kappa(y_m,y_{\xi(m)}).$$

Therefore,

$$\kappa^{(n)}_{\mu^{(l)}}(x_1,\dots,x_n) = \sum_{\xi\in S_m}(-1)^{m-\nu(\xi)}N_l\big(\nu(\xi)\big)\kappa(x_1,x_{\xi(1)})\dots\kappa(x_n,x_{\xi(n)}), \tag{5.46}$$

where, for $m\in\{0,1,2,\dots\}$, $N_l(m)$ denotes the number of all possible ordered partitions of a set consisting of m elements into l parts. Thus,

$$\begin{aligned} N_l(m) &= \sum_{(\eta_1,\eta_2,\dots,\eta_m)\in P_l(\{x_1,\dots,x_m\})} 1 \\ &= \sum_{\substack{k_1,\dots,k_l=0,\dots,n \\ k_1+\dots+k_l=m}} \binom{n}{k_1 k_2\dots k_l} \\ &= l^m. \end{aligned}$$

Hence, by (5.46)

$$\kappa^{(n)}_{\mu^{*l}}(x_1,\dots,x_n) = \sum_{\xi\in S_n}\operatorname{sign}(\xi) l^{\nu(\xi)}\kappa(x_1,x_{\xi(1)})\cdots\kappa(x_n,x_{\xi(n)}). \tag{5.47}$$

Comparing (5.47) with (5.45), we conclude from Theorem 3.1 (see also Remark 3.1) that the proposition is proven. □

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