# Some investigations on Feller processes generated by pseudo-differential operators.

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# Some investigations on Feller processes generated by pseudo-differential operators

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# Summary

Introducing an appropriate symbolic calculus for non-classical real-valued symbols, so-called negative definite symbols, W. Hoh succeeded to prove that such operators generate often Feller semigroups. In a first part of this thesis we extend this result to complex-valued symbols. Further, using ideas due to H. Kumano-go in case of classical pseudo-differential operators, we construct a parametrix for the fundamental solution of the associated evolution equation, and thus arrive at an approximation for the generated Feller semigroup.

Finally, we use this theory to extend models in financial mathematics based on Lévy processes. This is done by using the above mentioned results in situations where parameters in characteristic exponents of Lévy processes are made state-space dependent. Especially Meixner-type processes are discussed in detail.

# Declaration and Statements

## Declaration

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

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## Statement 1

This thesis is the result of my own investigations, except where otherwise stated. Every other sources are acknowledged by references and a bibliography is appended.

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# Introduction

In [5] O. Barndorff-Nielsen and S. Levendorskiǐ suggested to use the characteristic exponent of a Lévy process to construct a pseudo-differential operator which generates a Feller process by making the parameters of the Lévy exponent state space dependent. This was especially advertised for the Normal Inverse Gaussian process. The resulting Feller process was then proposed as a 'better' model for calculating option prices in finance. We are going to follow these ideas within a more general framework.

The first chapter starts with some basic definitions and properties of stochastic processes and Lévy processes. Also a rough overview of examples of Lévy processes is given. Even though we extend Lévy processes later on, most of the examples and corresponding theory of this chapter is not necessary in order to follow the next chapters. It should be seen as possible starting point for further work in the direction of this thesis.

In chapter 2 we recall some parts of the general theory for pseudo-differential operators generating Feller semigroups, as presented in [22] by N. Jacob. We will work with Hoh's class of symbols $S_\rho^{m,\psi}$ (see [18]) which generalises the classical class $S_{1,0}^m$. Starting with complex valued symbols from $S_\rho^{m,\psi}$ we show that under additional, but natural conditions the corresponding pseudo-differential operator can be extended to a generator of a Feller semigroup. We will follow closely the book [22] of N. Jacob, but note that the theory given therein is only proved for real valued symbols of $S_\rho^{m,\psi}$.

In chapter 3 we show that a result by H. Kumano-go [24] can be used to get approximations for the semigroup generated by pseudo-differential operators with symbols of Hoh's class. In other words, we are going to construct a parametrix for the heat equation corresponding to a pseudo-differential operator with symbol in $S_\rho^{m,\psi}$.

In chapter 4 we show that the theory, which we stated and developed in the previous chapters, can be used to extend models based on Lévy processes. The general framework is that we start with the characteristic exponent of a Lévy process, say $\psi(\xi)$. This exponent depends (in modelling) on some parameters $a, b, c, \ldots$, i.e. we have $\psi_{a,b,c,\ldots}(\xi)$. Now we *make the parameters state space dependent,* that is we define a symbol by

$$q(x,\xi) := \psi_{a(x),b(x),c(x),\ldots}(\xi)$$

where $a, b, c, \ldots$ are functions. Then the final step is to show that this symbol is in Hoh's class and satisfies some necessary conditions. Hence this implies that the corresponding pseudo-differential operator can be extended to the generator of a Feller semigroup. We show this in particular for the Meixner process, which was recently introduced as a model in Finance by W. Schoutens [29]. Analogously we give the proof for the Normal Inverse Gaussian process, but note that the result (without published proof) was already used in [5].

# Chapter 1

# Preliminaries

In this chapter we very briefly recall how a stochastic process is constructed starting from a Feller semigroup. We also give the definition of a Lévy process and mention its basic properties. Afterwards we state some examples of Lévy processes.

## 1.1 Notations

$\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ denote the positive integers, the real numbers and the complex numbers, $n \in \mathbb{N}$ will always denote the dimension of the spaces we are working in, as in $\mathbb{R}^n$. $\mathbb{N}_0$ denotes the positive integers including zero.
For a complex number $x \in \mathbb{C}$ the real part and imaginary part are denoted by Re $x$ and Im $x$, and $\overline{x}$ denotes the complex conjugate of $x$. For $a, b \in \mathbb{R}$ the minimum of $a$ and $b$ is denoted by $a \wedge b$ and their maximum by $a \vee b$.

$0^+$ means that we approach 0 in $\mathbb{R}$ as a limit from the right.

Function spaces are denoted by $A(X;Y)$, i.e. $f \in A(X;Y)$ means $f : X \to Y$ and $f$ has certain properties. Often we abbreviate this notation to $A(X)$ or just $A$ if the spaces are obvious by the context. For example $C^\infty(\mathbb{R}^n; \mathbb{C})$ denotes all arbitray often differentiable functions from $\mathbb{R}^n$ to $\mathbb{C}$. $C^m$, $m \in \mathbb{N} \cup \{\infty\}$ are $m$-times continuously differentiable functions, $C_0$ continuous functions with compact support, $C_\infty$ continuous functions vanishing at infinity. We also combine these notations for $C_0^\infty$, the infinitely often differentiable functions with compact support. $S$ is the Schwartz space and $S'$ its topological dual, the space of tempered distributions. $B$ is the space of Borel measurable functions.

The first derivative of a function $f$ of one variable will be denoted by $f'$, in general $f^{(k)}$, $k \in \mathbb{N}$ denotes the $k^{th}$ derivative of $f$. For functions of several independent variables we use the multiindex notation. That is for $\alpha \in \mathbb{N}_0^n$ we have $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!$ and as notation for partial derivatives of a function $f$ we use $\partial^\alpha f = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}}$.

We say $f$ is similar to $g$ and write $f \sim g$ as $x \to x_0$ if $\frac{f}{g} \to 1$ as $x \to x_0$.

The domain of an operator $A$ will be denoted by $\mathcal{D}(A)$. The identity operator will be denoted by id .

$(.,.)_0$ and $\|.\|_0$ denotes the $L^2$-scalar product and norm, and $\|.\|_\infty$ denotes the supremum norm. The Euclidean scalar product is denoted by $\langle x, y \rangle$ or just as $xy$.

$\mathcal{B}$ is the Borel $\sigma$-algebra on $\mathbb{R}$. For a stochastic process with initial distribution $\delta_x$, $x \in \mathbb{R}^n$, we denote the corresponding probability measure and expectation by $\mathbb{P}^x$ and $\mathbb{E}^x$. Here $\delta_x$ denotes the Dirac measure at $x \in \mathbb{R}^n$.

The Fourier transform of a function (or distribution) $u$ is denoted by

$$\hat{u}(\xi) = F(u)(\xi) = F_{x \to \xi}(u(x)) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) dx,$$

and the Laplace transform is denoted by

$$\mathcal{L}(u)(t) = \mathcal{L}_{x \to t}(u(x)) = \int_0^\infty e^{-xt} u(x) dx.$$

## 1.2 Stochastic processes - Lévy processes

We start with the definition of a stochastic process.

**Definition 1.2.1 (Stochastic process).** *A stochastic process with state space* $(\mathbb{R}^n, \mathcal{B}^n)$ *and parameter (time) set* $[0, \infty)$ *is a quadrupel* $(\Omega, \mathcal{A}, \mathbb{P}, (X_t)_{t \geq 0})$ *where* $(\Omega, \mathcal{A}, \mathbb{P})$ *is a probability space and for each* $t \geq 0$ *the mapping* $X_t : \Omega \to \mathbb{R}^n$ *is a random variable.*

For fixed $\omega \in \Omega$ the mapping $X_\bullet(\omega) : [0, \infty) \to \mathbb{R}^n$, $t \mapsto X_t(\omega)$, is called a path of the process. In the following we often write $(X_t)_{t \geq 0}$ for a stochastic process and omit the probability space. In most cases we need to consider families of stochastic processes parameterised by the state space, i.e. $(\Omega, \mathcal{A}, \mathbb{P}^x, (X_t)_{t \geq 0})_{x \in \mathbb{R}^n}$, where for each $x \in \mathbb{R}^n$ fixed $(\Omega, \mathcal{A}, \mathbb{P}^x, (X_t)_{t \geq 0})$ is a stochastic process.

For every process we have a family of transition probabilities

$$P_t(x, B) = \mathbb{P}(X_t \in B | X_0 = x) = \mathbb{P}^x(X_t \in B), x \in \mathbb{R}^n, t \in [0, \infty), B \in \mathcal{B}^n.$$

In words: $P_t(x,B)$ denotes the probability that the process is at time $t$ in $B$ if it started at time 0 in $x$. Using the family of transition probabilities we may associate a family of operators with $(X_t)_{t\geq 0}$. For this note that for $B \in \mathcal{B}^n$ the mapping $x \mapsto P_t(x,B)$ is measurable and for $x \in \mathbb{R}^n$ fixed the mapping $B \mapsto P_t(x,B)$ is a probability measure, i.e. $P_t(.,.)$ is a Markovian kernel. Hence we may consider on the bounded Borel functions $u$ the family of operators

$$T_t u(x) := \int u(y) P_t(x,dy) = \mathbb{E}^x(u(X_t)).$$

We will describe shortly that this procedure can be reversed, i.e. starting with certain families of operators we can construct families of stochastic processes.

An important class of stochastic processes we are interested in are Feller processes, i.e. stochastic processes associated with a Feller semigroup.

**Definition 1.2.2 (Semigroup, strongly continuous, contraction, positivity preserving).** *Let $X$ with norm $\|\cdot\|_X$ be a Banach space.*

*A) A family $(T_t)_{t\geq 0}$ of bounded linear operators $T_t : X \to X$ is called semigroup of operators if*

$$T_0 = \mathrm{id} \quad \textit{and } T_{s+t} = T_s \circ T_t \textit{ hold for all } s,t \geq 0.$$

*B) The semigroup $(T_t)_{t\geq 0}$ is called strongly continuous if*

$$\lim_{t\to 0} \|T_t u - u\|_X = 0 \textit{ for all } u \in X.$$

*C) The semigroup $(T_t)_{t\geq 0}$ is called contraction semigroup if*

$$\|T_t\| \leq 1 \textit{ for all } t \geq 0,$$

*where $\|\cdot\|$ is the operator norm.*

*D) A linear bounded operator $T : X \to X$ is called positivity preserving if*

$$0 \leq u \textit{ implies } 0 \leq Tu.$$

**Definition 1.2.3 (Feller semigroup).** *Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup on $(C_\infty(\mathbb{R}^n;\mathbb{R}), \|.\|_\infty)$ which is positivity preserving. Then $(T_t)_{t\geq 0}$ is called Feller semigroup.*

If $(T_t)_{t\geq 0}$ is a Feller semigroup then there exists a unique family of sub-Markovian kernels $(P_t)_{t\geq 0}$ such that

$$T_t f(x) = \int_{\mathbb{R}^n} f(y) P_t(x, dy)$$

for all $t \geq 0$ and all $f \in C_b(\mathbb{R}^n)$ compare [21] (p. 425 and Theorem 4.8.1). Especially if $(T_t)_{t\geq 0}$ is conservative, i.e. $T_t u(x) = 1$ holds for $u(x) = 1$ for all $x \in \mathbb{R}^n$, then the kernels are Markovian and therefore $(P_t)_{t\geq 0}$ gives rise to a family of transition probabilities. With these we may construct for each probability measure $\mu$ on $(\mathbb{R}^n, \mathcal{B}^n)$ a projective family of probability measures by defining

$$P_J^\mu(B) := \int \cdots \int \chi_B(x_1, \ldots, x_m) P_{t_m - t_{m-1}}(x_{m-1}, dx_m) \cdots P_{t_1}(x_0, dx_1) \mu(dx_0)$$

where $J = \{t_1, \ldots, t_m\}$ is a finite subset of $[0, \infty)$ with $t_1 < \ldots < t_m$ and $B \in \mathcal{B}^m$. Now an application of the Kolmogorov theorem yields the existence of a canonical process $(\Omega, \mathcal{A}, \mathbb{P}^\mu, (X_t)_{t>0})$ associated with the family $P_J^\mu$, hence with the Feller semigroup $(T_t)_{t\geq 0}$.

The process constructed in this way is called Feller process. Formally we have:

**Definition 1.2.4 (Feller process).** *A family of stochastic processes $(\Omega, \mathcal{A}, \mathbb{P}^x, (X_t)_{t\geq 0})_{x\in\mathbb{R}^n}$ is called Feller process if a Feller semigroup is given by*

$$T_t f(x) = \mathbb{E}^x(f(X_t))$$

*for $f \in C_\infty(\mathbb{R}^n; \mathbb{R})$.*

Another important class of stochastic processes are Lévy processes, [6] and [28] are extensive monographs about this subject.

**Definition 1.2.5 (Lévy process).** *A stochastic process $(X_t)_{t\geq 0}$ on $\mathbb{R}^n$ with $X_0 = 0$ a.s. is called Lévy process if it has the following properties:*

*i) it has independent increments,*
*i.e. for $m \in \mathbb{N}$, $0 \leq t_0 < t_1 < \ldots < t_m$ the random variables*

$$X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_m} - X_{t_{m-1}}$$

*are independent.*

*ii) it has stationary increments,*
*i.e. the distribution of $X_{s+t} - X_s$ does not depend on $s$.*

*iii) it is stochastically continuous,*
*i.e. for all $s \geq 0$ and $\varepsilon > 0$*

$$\lim_{t \to s} \mathbb{P}(|X_{t+s} - X_s| > \varepsilon) = 0.$$

Using these properties we see that for a Lévy process $(X_t)_{t\geq 0}$ the decomposition

$$X_1 = X_{\frac{1}{m}} + (X_{\frac{2}{m}} - X_{\frac{1}{m}}) + \ldots + (X_{\frac{m}{m}} - X_{\frac{m-1}{m}})$$

implies the exsistence of a function $\psi : \mathbb{R}^n \to \mathbb{C}$ such that

$$\mathbb{E}(e^{i\xi X_t}) = e^{-t\psi(\xi)}$$

and $\psi$ is called the characteristic exponent of $X_t$. In fact there is a 1-1 correspondens between characteristic exponents and Lévy processes. Furthermore the characteristic exponent of a Lévy process is also known as a continuous negative definite function, see Definition 1.2.8 below.

If the transition probability of a Lévy process admits a density $p_t$ with respect to the Lebesgue measure then we can compute

$$p_t(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ix\xi} e^{-t\psi(\xi)} \, d\xi,$$

where again $\psi$ is the characteristic exponent of the process.

Some of the properties of a characteristic exponent of a Lévy process are stated in the following proposition.

**Proposition 1.2.6.** *Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a characteristic exponent of a Lévy process. Then $\psi$ has the following properties:*

*i) $Re(\psi(\xi)) \geq \psi(0) = 0$ for all $\xi \in \mathbb{R}^n$,*

*ii) $\psi(\xi) = \overline{\psi(-\xi)}$ for all $\xi \in \mathbb{R}^n$,*

*iii) $\psi$ is continuous.*

For reference see [21] (page 123).

**Lemma 1.2.7.** *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a Lévy exponent which satisfies $\psi(0) = 0$, $\lim_{\xi\to\infty} \psi(\xi) = \infty$ and $\psi'(\xi) > 0$ for $\xi > 0$ (ie. $\psi$ is invertible on the positive real axis). Then the density of the transition probability of the Lévy process is given by*

$$\begin{aligned} p_t(x) &= \frac{t}{\pi} \mathcal{L}_{y\to t} \left( \frac{\sin(x\psi^{-1}(y))}{x} \right) \\ &= \frac{t}{\pi} \int_0^\infty \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} (\psi^{-1}(y))^{2n+1} e^{-ty} \, dy. \end{aligned} \tag{1.2.1}$$

*where $\psi^{-1}$ denotes the inverse of $\psi$ on $(0, \infty)$ and we use the convention $0^0 = 1$.*

**Proof.** Let $x \neq 0$. By the symmetry (Proposition 1.2.6 ii)) of $\psi$ we get

$$\begin{aligned} p_t(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} e^{-t\psi(\xi)} \, d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (\cos(x\xi) + i \sin(-x\xi)) \, e^{-t\psi(\xi)} \, d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(x\xi) e^{-t\psi(\xi)} \, d\xi \\ &= \frac{1}{\pi} \int_{0}^{\infty} \cos(x\xi) e^{-t\psi(\xi)} \, d\xi. \end{aligned}$$

Now we use the substitution rule. But note that since $\psi$ may not be invertible at 0 we first consider the integral starting from $\varepsilon$ and let $\varepsilon$ tend to 0 afterwards. This is here omitted.
Afterwards we use the Laplace transform with the formula $\mathcal{L}(f')(t) = t\mathcal{L}(f)(t) - f(0^+)$.

$$\begin{aligned} p_t(x) &= \frac{1}{\pi} \int_{0}^{\infty} \cos(x\psi^{-1}(y))(\psi^{-1}(y))' \, e^{-ty} \, dy \\ &= \frac{1}{\pi} \int_{0}^{\infty} \frac{d}{dy} \left( \frac{\sin(x\psi^{-1}(y))}{x} \right) e^{-ty} \, dy \\ &= \frac{t}{\pi} \mathcal{L}_{y \to t} \left( \frac{\sin(x\psi^{-1}(y))}{x} \right) \\ &= \frac{t}{\pi} \int_{0}^{\infty} \frac{\sin(x\psi^{-1}(y))}{x} e^{-ty} \, dy \\ &= \frac{t}{\pi} \int_{0}^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} (\psi^{-1}(y))^{2n+1} e^{-ty} \, dy. \end{aligned}$$

For $x = 0$ we get

$$\begin{aligned} p_t(x) &= \frac{1}{\pi} \int_{0}^{\infty} \cos(0 \cdot \psi^{-1}(y))(\psi^{-1}(y))' \, e^{-ty} \, dy \\ &= \frac{1}{\pi} \int_{0}^{\infty} (\psi^{-1}(y))' \, e^{-ty} \, dy \\ &= \frac{t}{\pi} \int_{0}^{\infty} \psi^{-1}(y) \, e^{-ty} \, dy \end{aligned}$$

and by using the convention $0^0 = 1$ we see that this case is included in the formula for $x \neq 0$. ∎

In the next chapters the notion of continuous negative definite functions plays a key role.

**Definition 1.2.8 (continuous negative definite function).** *We call $\psi$ continuous negative definite function if $\psi : \mathbb{R}^n \to \mathbb{C}$ is continuous and for any choice of $k \in \mathbb{N}$ and vectors $\xi^1, \dots, \xi^k \in \mathbb{R}^n$ the matrix*

$$\left(\psi(\xi^j) + \overline{\psi(\xi^l)} - \psi(\xi^j - \xi^l)\right)_{j,l=1,\dots,k}$$

*is positive Hermitian.*

The name arises from the positive definite functions, which are the Fourier transforms of measures by Bochner's theorem. For more details and equivalent definitions see [21] (section 3.5-3.6). These functions are connected to Lévy processes in a very simple way.

**Lemma 1.2.9.** *The characteristic exponent of a Lévy process is a continuous negative definite function and vice versa, i.e. to every continuous negative definite function $\psi$ satisfying $\psi(0) = 0$ exists a Lévy process $X_t$ such that*

$$\mathbb{E}(e^{i\xi X_t}) = e^{-t\psi(\xi)}.$$

It is just a different name of the same object.

An important characterisation of continuous negative definite functions, i.e. characteristic exponents of Lévy processes is the Lévy-Khintchine formula.

**Theorem 1.2.10 (Lévy-Khintchine formula).** *The characteristic exponent of a Lévy process on $\mathbb{R}^n$ has always the following form*

$$\psi(\xi) = i\langle m, \xi\rangle + \frac{1}{2}q(\xi) + \int_{\mathbb{R}^n \setminus \{0\}} 1 - e^{i\langle \xi, x\rangle} + i\langle \xi, x\rangle \chi_{\{|x|<1\}}\ \nu(dx)$$

*where $m \in \mathbb{R}^n$ is called the drift, $q$ is a positive semi-definite quadratic form and is called the diffusion component. The measure $\nu$ is defined on $\mathbb{R}^n \setminus \{0\}$ such that $\int (1 \wedge |x|^2)\ \nu(dx) < \infty$. It is called the Lévy measure.*

For a proof see [6] or [28].

## 1.3 Some examples of Lévy processes

### 1.3.1 $\alpha$-stable processes

A special class of Lévy processes are those with characteristic exponent

$$\psi(\xi) = |\xi|^{\alpha}, \quad 0 < \alpha \leq 2.$$

They are called $\alpha$-stable processes. For $1 < \alpha \leq 2$ the density of the transition probability is given by the following formula.

**Corollary 1.3.1.** *For the characteristic exponents $\psi(\xi) = |\xi|^{\alpha}$, $1 < \alpha \leq 2$ we get*

$$p_t(x) = \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} \Gamma(\frac{2n+1}{\alpha} + 1) t^{-\frac{2n+1}{\alpha}}. \tag{1.3.1}$$

**Proof.** We want to use (1.2.1) and evaluate it by first interchanging integration and summation and then evaluating the integral (Laplace transform) for every element of the series. Obviously we have $\psi^{-1}(\xi) = |\xi|^{\frac{1}{\alpha}}$ and $\psi$ satisfies the condition of lemma 1.2.7. Let

$$S_k(y) := \sum_{n=0}^{k} (-1)^n \frac{(x|y|^{\frac{1}{\alpha}})}{(2n+1)!}$$

and note that $S_k(y) \to \sin(x|y|^{\frac{1}{\alpha}})$ as $k \to \infty$. Addtionally we have

$$|S_k(y)| \leq \sum_{n=0}^{k} \frac{(|x||y|^{\frac{1}{\alpha}})}{(2n+1)!} \leq \sinh(|x||y|^{\frac{1}{\alpha}}) \leq e^{|x||y|^{\frac{1}{\alpha}}}$$

For $1 < \alpha \leq 2$ and $x \in \mathbb{R}$ we find that for all $\varepsilon > 0$ there exists a $M > 0$ such that

$$e^{|x||y|^{\frac{1}{\alpha}}} \leq M e^{\varepsilon |y|} \text{ for all } y \in \mathbb{R}$$

i.e. the sequence $S_k$ is dominated by a function for which the Laplace transform exists. Now we apply the dominated convergence theorem and interchange the integration and the limit. We are left with the Laplace transform of a finite sum, which is simply the sum of the Laplace transforms and using the formula $\mathcal{L}(y^a)(t) = \Gamma(a+1)t^{-(a+1)}$ for $a > 0$, the result follows. ■

This result was known before see for example [32]. But it was not proved in such a way.

Especially it should be no surprise that we get for $\alpha = 2$ the Gaussian semigroup.

$\alpha = 2$: **Brownian Motion**

Brownian Motion is the only continuous Lévy process (except constant drift), i.e for almost all fixed $\omega$ the function $X_t(\omega)$ is continuous in $t$.
It has the characteristic exponent $\psi(\xi) = |\xi|^2$ and with the corollary above we get its transition density as

$$\begin{aligned}
p_t(x) &= \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n+1)!}\Gamma(\frac{2n+1}{2}+1)t^{-\frac{2n+1}{2}} \\
&= \frac{1}{\pi\sqrt{t}}\sum_{n=0}^{\infty}\frac{\Gamma(n+1+\frac{1}{2})}{(2n+1)!}\left(\frac{(ix)^2}{t}\right)^n \\
&= \Gamma(\frac{1}{2})\frac{1}{\pi\sqrt{t}}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\cdot\frac{1}{2n(2n-2)\cdot\ldots\cdot 2}\left(\frac{(ix)^2}{t}\right)^n \\
&= \frac{1}{2\sqrt{\pi t}}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{(ix)^2}{4t}\right)^n \\
&= \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}.
\end{aligned}$$

Obviously this is the density of the well known normal distribution with mean 0 and variance $2t$.

If we interpret $\psi$ as the symbol of a pseudo-differential operator then the corresponding operator is nothing but the Laplace operator $(-\triangle)$. For more details about pseudo-differential operators see chapter 2.

For a simulation of Brownian Motion see Appendix A.1.

$\alpha = 1$: **Cauchy Process**

The Cauchy process has the characteristic exponent $\psi(\xi) = |\xi|$ and its transition density is given by

$$\begin{aligned}
p_t(x) &= \frac{t}{\pi}\mathcal{L}_{y\to t}\left(\frac{\sin(x|y|^{-1})}{x}\right) \\
&= \frac{1}{\pi x}\frac{tx}{t^2+x^2} \\
&= \frac{1}{\pi}\frac{t}{(t^2+x^2)}.
\end{aligned}$$

For a simulation of the Cauchy process see Appendix A.2.

### 1.3.2 Relativistic Hamiltonian

The symbol corresponding to the relativistic Hamiltonian is given by

$$\psi_m(\xi) = \left(|\xi|^2 + m^2\right)^{\frac{1}{2}} - m, \quad m \geq 0.$$

This is a negative definite function, i.e. can be seen as the characteristic exponent of a Lévy process.
From example 3.9.18 in [21] (see also [20]) we know that in $\mathbb{R}^n$ its transition densities are given by

$$h_{t,m}(x) = 2(2\pi)^{-\frac{n+1}{2}} m^{\frac{n+1}{2}} e^{mt} t \left(|x|^2 + t^2\right)^{-\frac{n+1}{4}} K_{\frac{n+1}{2}}\left(m(|x|^2 + t^2)^{\frac{1}{2}}\right),$$

where $K_\nu$ is a modified Bessel function of third kind of order $\nu$.
Obviously $\psi_m \to \psi_{Cauchy}$ for $m \to 0$. Now we want to show that also

$$h_{t,m}(x) \to g_t(x) \text{ as } m \to 0, \text{ for all } x \in \mathbb{R}^n$$

holds, where

$$g_t(x) = \pi^{-\frac{n+1}{2}}\, \Gamma\left(\frac{n+1}{2}\right) \frac{t}{\left(|x|^2 + t^2\right)^{\frac{n+1}{2}}}.$$

**Lemma 1.3.2.** *The density $h_{t,m}$ of the relativistic Hamiltonian converges for fixed $t$ pointwise to the density of the Cauchy process as $m$ tends to 0.*

**Proof.**

We have for $\nu$ fixed (Re $\nu > 0$) and $z \to 0$ the approximation

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)\left(\frac{1}{2}z\right)^{-\nu}$$

compare with 9.6.9. in [1].

Therefore for $m \to 0$ we have

$$\begin{aligned} h_{t,m}(x) &\sim 2(2\pi)^{-\frac{n+1}{2}} m^{\frac{n+1}{2}} e^{mt} t \left(|x|^2 + t^2\right)^{-\frac{n+1}{4}} \\ &\quad \times \frac{1}{2}\, \Gamma\left(\frac{n+1}{2}\right)\left(\frac{1}{2} m(|x|^2 + t^2)^{\frac{1}{2}}\right)^{-\frac{n+1}{2}} \\ &= \pi^{-\frac{n+1}{2}}\, \Gamma\left(\frac{n+1}{2}\right)\, t\left(|x|^2 + t^2\right)^{-\frac{n+1}{2}} e^{mt}. \end{aligned}$$

Taking now the limit $m \to 0$ gives the result.

Actually, due to the series expression of the Bessel function we could get a nicer proof for even dimension. Let $n$ be even. Using [31] (page 80 equation (12)) we find

$$K_{n+\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \sum_{k=0}^{n} a_{n,k}(2z)^{-k} \tag{1.3.2}$$

for some $a_{n,k}$, especially $a_{n,\frac{n}{2}} = 2\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)$.
Using this we get the same result.

■

## 1.4 Lévy processes used in Finance

This section is a compact survey of Lévy processes which are used in finance. Typically the corresponding characteristic exponents will depend on some parameters, some of which we are going to make state space dependent in later chapters. Hence we collect some properties and some details of the parameters used in these processes.

| Model | Characteristic exponent |
|---|---|
| GH (Barndorff-Nielsen)<br>(generalized hyperbolic) | $\psi(\xi) = -i\xi(\mu + \frac{\beta\delta^2 K_{\lambda+1}(\delta\sqrt{\alpha^2-\beta^2})}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}) - \int(e^{i\xi x} - 1 - i\xi x)g(x)\,dx$<br>$g(x) = \frac{e^{\beta x}}{\|x\|}(\int_0^\infty \frac{e^{-\sqrt{2y+\alpha^2}\|x\|}}{\pi^2 y(J^2_{\|\lambda\|}(\delta\sqrt{2y})+Y^2_{\|\lambda\|}(\delta\sqrt{2y}))}\,dy + \lambda e^{-\alpha\|x\|}\chi_{\lambda\geq 0})$ |
| H (Eberlein)<br>(hyperbolic)<br>($\lambda = 1$ in GH) | $\psi(\xi) = -i\xi(\mu + \frac{\beta\delta^2 K_{1+1}(\delta\sqrt{\alpha^2-\beta^2})}{K_1(\delta\sqrt{\alpha^2-\beta^2})}) - \int(e^{i\xi x} - 1 - i\xi x)g(x)\,dx$<br>$g(x) = \frac{e^{\beta x}}{\|x\|}(\int_0^\infty \frac{e^{-\sqrt{2y+\alpha^2}\|x\|}}{\pi^2 y(J^2_1(\delta\sqrt{2y})+Y^2_1(\delta\sqrt{2y}))}\,dy + e^{-\alpha\|x\|})$ |
| NIG (Barndorff-Nielsen)<br>(normal inverse gaussian) | $\psi(\xi) = -i\mu\xi + \delta[\sqrt{\alpha^2-(\beta+i\xi)^2} - \sqrt{\alpha^2-\beta^2}]$ |
| VG (Madan)<br>(variance gamma) | $\psi(\xi) = \frac{1}{\nu}\ln(1 - i\theta\nu\xi + \frac{\sigma^2\nu}{2}\xi^2)$ |
| CGMY<br>(Carr-Geman-Madan-Yor) | $\psi(\xi) = -C\,\Gamma(-Y)\{(M-i\xi)^Y - M^Y + (G+i\xi)^Y - G^Y\}$ |
| TLF (Matacz)<br>(truncated Lévy flights) | $\psi(\xi) = \frac{c^\alpha}{\cos(\pi\alpha/2)}\left((\xi^2+\lambda^2)^{\frac{\alpha}{2}}\cos[\alpha\arctan(\frac{\xi}{\lambda})] - \lambda^\alpha\right)$ |
| TLP (*)<br>(truncated Lévy processes) | $\psi(\xi) = -i\mu\xi - c_+\Gamma(-\nu)[(\lambda_+ + i\xi)^\nu - \lambda_+^\nu]$<br>$-c_-\Gamma(-\nu)[(-\lambda_- + i\xi)^\nu - (-\lambda_-)^\nu]$ |
| Meixner Process<br>(Schoutens) | $\psi(\xi) = -im\xi + 2d\left(\ln\cosh(\frac{a\xi-ib}{2}) - \ln\cos(\frac{b}{2})\right).$ |

(*) Boyarchenko and Levendorskiĭ

Table 1.1: Characteristic exponents of Lévy model

### 1.4.1 Generalized hyperbolic distributions

The class of generalized hyperbolic distributions was used by Barndorff-Nielsen [2] in 1977 for modelling the grain size of wind blown sand. A couple of subclasses are used in financial mathematics, which we will describe in some of the next subsections. Most of the following is taken from [12] and [11].
The density of $X_1$ of a generalized hyperbolic process is given by:

$$d_{GH}(x;\lambda,\alpha,\beta,\delta,\mu) = a(\lambda,\alpha,\beta,\delta)(\delta^2+(x-\mu)^2)^{\frac{\lambda-\frac{1}{2}}{2}} K_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2+(x-\mu)^2})e^{\beta(x-\mu)}$$

where the normalising constant is

$$a(\lambda,\alpha,\beta,\delta) = \frac{(\alpha^2-\beta^2)^{\frac{\lambda}{2}}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^\lambda K_\lambda(\delta\sqrt{\alpha^2-\eta^2})},$$

and $K_\nu$ denotes the modified Bessel function of third kind. [1]
The five parameters have the following restrictions and meanings: $\alpha > 0$ determines the shape and steepness, $\beta : \ 0 \le |\beta| < \alpha$ the skewness and asymmetry, $\mu \in \mathbb{R}$ the location, $\delta > 0$ the scaling and $\lambda \in \mathbb{R}$ the heaviness of the tails.
In [27] Raible calculated the characteristic function of $X_t$ as

$$\phi_{t,(\lambda,\alpha,\beta,\delta,\mu)}(\xi) = e^{i\mu\xi t}\frac{(\delta\sqrt{\alpha^2-\beta^2})^{\lambda t}}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})^t}\frac{K_\lambda(\delta\sqrt{\alpha^2-(\beta+i\xi)^2})^t}{(\delta\sqrt{\alpha^2-(\beta+i\xi)^2})^{\lambda t}}.$$

He also gives the expectation of $X_1$

$$\mathbb{E}(X_1) = \mu + \delta\frac{p}{\sqrt{1-p^2}}\frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)}$$

and its variance

$$\mathrm{Var}(X_1) = \delta^2\left(\frac{K_{\lambda+1}(\zeta)}{\zeta K_\lambda(\zeta)} + \frac{p^2}{1-p^2}\left[\frac{K_{\lambda+2}(\zeta)}{K_\lambda(\zeta)} - (\frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)})^2\right]\right)$$

where $\zeta = \delta\sqrt{\alpha^2-\beta^2}$ and $p = \frac{\beta}{\alpha}$.
The characteristic exponent is given by

$$\psi(\xi) = -i\xi(\mu + \frac{\beta\delta^2 K_{\lambda+1}(\delta\sqrt{\alpha^2-\beta^2})}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}) - \int_{-\infty}^{\infty}(e^{i\xi x}-1-i\xi x)g(x)\,dx$$

where the Lévy density is

$$g(x) = \frac{e^{\beta x}}{|x|}(\int_0^\infty \frac{e^{-\sqrt{2y+\alpha^2}|x|}}{\pi^2 y(J^2_{|\lambda|}(\delta\sqrt{2y})+Y^2_{|\lambda|}(\delta\sqrt{2y}))}\,dy + \lambda e^{-\alpha|x|}\chi_{\lambda\ge 0})$$

and $J, Y$ are Bessel functions, see [11].

[1] $K_\nu(z) = \frac{1}{2}\int_0^\infty y^{\nu-1}\exp(-\frac{1}{2}z(y+y^{-1}))\,dy$

### 1.4.2 Hyperbolic distributions

The class of hyperbolic distributions was introduced as a model in financial mathematics by Eberlein and Keller in 1995 [10].
If we set $\lambda = 1$ in the generalized hyperbolic model then we get the hyperbolic model, $X_1$ has the density:

$$d_H(x) = \frac{\sqrt{\alpha^2-\beta^2}}{2\alpha\delta K_1(\delta\sqrt{\alpha^2-\beta^2})}\exp(-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta(x+\mu)).$$

Where we used the identity: $K_{\frac{1}{2}}(z) = K_{-\frac{1}{2}}(z) = (\frac{\pi}{2z})^{\frac{1}{2}}e^{-z}$.
Again by setting $\lambda = 1$ in the generalized hyperbolic model we obtain the characteristic function of $X_t$

$$\phi_{t(1,\alpha,\beta,\delta,\mu)}(\xi) = e^{i\mu\xi t}\frac{(\delta\sqrt{\alpha^2-\beta^2})^t}{K_1(\delta\sqrt{\alpha^2-\beta^2})^t}\frac{K_1(\delta\sqrt{\alpha^2-(\beta+i\xi)^2})^t}{(\delta\sqrt{\alpha^2-(\beta+i\xi)^2})^t}.$$

The expectation of $X_1$ is given by

$$\mathbb{E}(X_1) = \mu + \delta\frac{p}{\sqrt{1-p^2}}\frac{K_{1+1}(\zeta)}{K_1(\zeta)}$$

and its variance

$$\mathrm{Var}(X_1) = \delta^2\left(\frac{K_2(\zeta)}{\zeta K_1(\zeta)} + \frac{p^2}{1-p^2}\left[\frac{K_3(\zeta)}{K_1(\zeta)} - (\frac{K_{1+1}(\zeta)}{K_1(\zeta)})^2\right]\right)$$

where $\zeta = \delta\sqrt{\alpha^2-\beta^2}$ and $p = \frac{\beta}{\alpha}$.
The characteristic exponent is

$$\psi(\xi) = -i\xi(\mu + \frac{\beta\delta^2K_2(\delta\sqrt{\alpha^2-\beta^2})}{K_1(\delta\sqrt{\alpha^2-\beta^2})}) - \int_{-\infty}^{\infty}(e^{i\xi x}-1-i\xi x)g(x)\,dx$$

with Lévy density

$$g(x) = \frac{e^{\beta x}}{|x|}(\int_0^{\infty}\frac{e^{-\sqrt{2y+\alpha^2}|x|}}{\pi^2 y(J_1^2(\delta\sqrt{2y})+Y_1^2(\delta\sqrt{2y}))}\,dy + e^{-\alpha|x|}).$$

Its moment generating function for $|\beta+u| < \alpha$ is given by

$$M(u) = \frac{e^{\mu u}\sqrt{\alpha^2-\beta^2}}{K_1(\delta\sqrt{\alpha^2-\beta^2})}\frac{K_1(\delta\sqrt{\alpha^2-(\beta+u)^2})}{\sqrt{\alpha^2-(\beta+u)^2}},$$

see [15] for a proof.

To get a feeling for realistic parameters we state that in [12] Eberlein, Keller and Prause calculated the parameters in a stock market application as:

| $\alpha$ | $\beta$ | $\delta$ | $\mu$ |
|---|---|---|---|
| 107.6 | 2.10 | 0.006 | 0.0003 |
| 225.0 | -5.80 | 0.0015 | 0.0006 |

### 1.4.3 Normal Inverse Gaussian distributions

The Normal Inverse Gaussian distribution was first introduced as a Lévy model in finance by Barndorff-Nielsen in 1995 [3]. We get this class of distributions by setting $\lambda = -\frac{1}{2}$ in the generalized hyperbolic distribution.
For the Normal Inverse Gaussian process $(X_t)_{t\geq 0}$ the density of $X_t$ is given by

$$d_{a,b,m,\delta}(x) = \frac{a}{\pi} e^{\delta t\sqrt{a^2-b^2}+b(x-mt)} \frac{K_1(a\delta t\sqrt{1+(\frac{x-mt}{\delta})^2})}{\sqrt{1+(\frac{x-m}{\delta t})^2}}.$$

Where $0 \leq |b| < a, \delta > 0, m \in \mathbb{R}$ and $K_\nu$ denotes a modified Bessel function of third kind.
The characteristic function of $X_t$ is given by

$$\phi_t(\xi) = e^{tim\xi - t\delta[\sqrt{a^2-(b+i\xi)^2}+\sqrt{a^2-b^2}]}.$$

Therefore the following definition is justified.

**Definition 1.4.1 (Normal Inverse Gaussian exponent).** *The Normal Inverse Gaussian exponent is given by*

$$\psi_{NIG}(\xi) = -im\xi + \delta[\sqrt{a^2-(b+i\xi)^2} - \sqrt{a^2-b^2}]$$

*where* $0 \leq |b| < a, \delta > 0, m \in \mathbb{R}$.

In [4] Barndorff-Nielsen gives its Lévy density as

$$\nu_{NIG}(x) = \frac{\delta a}{\pi} e^{bx} \frac{K_1(a|x|)}{|x|}$$

where $K_1$ is again a modified Bessel function of third kind of order 1. The diffusion component is 0 and the drift is given by

$$m_{NIG} = m + \frac{2\delta a}{\pi} \int_0^1 \sinh(bx) K_1(a|x|)\, dx.$$

The parameters for models calculated by Raible [27] (page 104) are in the following range

| | $\alpha$ | $\beta$ | $\delta$ | $\mu$ |
|---|---|---|---|---|
| min | 133 | -26.4 | 0.000384 | 0.000017 |
| max | 1030 | -2.15 | 0.00378 | 0.000214 |

### 1.4.4 Variance gamma process

The variance gamma process was introduced by Madan. The following is taken from the survey of Geman [15].
This process can be considered as a time changed Brownian motion $W$

$$X(t;\sigma,\nu,\theta) = \theta G(t;\nu) + \sigma W(G(t;\nu))$$

where $\theta$ is the drift, $\sigma$ the volatility and G is an increasing gamma process with variance $\nu$. Recall that the probability density of a gamma process with mean $t$ and variance $\nu t$ is given by

$$d(x) = \frac{x^{\frac{t}{\nu}-1}e^{-\frac{x}{\nu}}}{\nu^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu})}.$$

The characteristic function of the variance gamma process is given by

$$\phi_{t,VG}(\xi) = \left(\frac{1}{1 - i\theta\nu\xi + \frac{\sigma^2\nu}{2}\xi^2}\right)^{\frac{t}{\nu}}$$

therefore the characteristic exponent is

$$\psi(\xi) = \frac{1}{\nu}\ln(1 - i\theta\nu\xi + \frac{\sigma^2\nu}{2}\xi^2).$$

The Lévy density can be calculated as

$$k_{VG}(x) = \begin{cases} \frac{Ce^{-Mx}}{x} & x > 0 \\ \frac{Ce^{-G|x|}}{|x|} & x < 0 \end{cases}$$

where $C = \frac{1}{\nu}; G = \sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} + \frac{\theta\nu}{2}; M = \sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} - \frac{\theta\nu}{2}.$

### 1.4.5 The Carr-Geman-Madan-Yor process

The CGMY process is a generalization of the variance gamma model. Introduced in [9]. The Lévy density gets the following form:

$$k_{VG}(x) = \begin{cases} \frac{Ce^{-Mx}}{x^{1+Y}} & x > 0 \\ \frac{Ce^{-G|x|}}{|x|^{1+Y}} & x < 0 \end{cases}$$

where $C > 0, M \geq 0, G \geq 0, Y < 2, Y \notin \mathbb{Z}$ are the parameter.
The characteristic function is given by

$$\phi_{t,CGMY}(\xi) = \exp[tC\ \Gamma(-Y)\{(M - i\xi)^Y - M^Y + (G + i\xi)^Y - G^Y\}]$$

and therefore the characteristic exponent is given by:

$$\psi(\xi) = -C\ \Gamma(-Y)\{(M - i\xi)^Y - M^Y + (G + i\xi)^Y - G^Y\}.$$

### 1.4.6 Truncated Lévy process

Boyarchenko and Levendorskiǐ suggested to use Truncated Lévy Processes for option pricing in [7]. This process is directly constructed via the characteristic exponent. They set: $c > 0, \lambda > 0$

$$\psi_(\nu, \lambda, c)(\xi) = \begin{cases} \frac{c[\ln(\lambda \pm i\xi) - \ln\lambda]}{2} & , \text{ if } \nu = 0 \\ -\frac{c}{2\cos(\frac{\pi\nu}{2})}(\lambda^\nu - (\lambda \pm i\xi)^\nu) & , \text{ if } \nu \in (0,1) \cup (1,2) \\ -\frac{c}{\pi}[(\lambda \pm i\xi)\ln(\lambda \pm i\xi) - \lambda \ln \lambda] & , \text{ if } \nu = 1 \end{cases}$$

Then for $c_\pm > 0, \lambda_- < 0 < \lambda_+, \nu_\pm \in [0, 2)$ we get the characteristic exponent by

$$\psi(\xi) = \psi_{(\nu_+, \lambda_+, c_+)}(\xi) + \psi_{(\nu_-, -\lambda_-, c_-)}(-\xi).$$

In [8] the characteristic exponent for $\nu \in (0,1) \cup (1,2)$ is given by

$$\psi(\xi) = -i\mu\xi - c_+\Gamma(-\nu)[(\lambda_+ + i\xi)^\nu - \lambda_+^\nu] - c_-\Gamma(-\nu)[(-\lambda_- + i\xi)^\nu - (-\lambda_-)^\nu].$$

This is the same as the CGMY exponent with drift and the constant $C$ is splited into $c_+, c_-$.

### 1.4.7 Truncated Lévy Flights

In [25] Truncated Lévy Flights are "advertised" for financial models. See also [26]. The distribution of a TLF has the following form.

$$P(x) = \begin{cases} 0 & x > l \\ cP_L(x) & -l \leq x \leq l \\ 0 & x < -l \end{cases}.$$

Where $P_L$ is a $\alpha$-stable distribution of index $0 < \alpha < 2$ and scale factor $\gamma$:

$$P_L(x) = \frac{1}{\pi}\int_0^\infty e^{-\gamma|q|^\alpha}\cos(qx)dq$$

This distribution is cut off outside $l$. The following is the characteristic function of a TLF with smooth (exponential) cutoff ([23]).

$$\phi_t(\xi) = \exp\{t\left(c_0 - c_1 \frac{(\xi^2 + \frac{1}{l^2})^{\frac{\alpha}{2}}}{\cos(\pi\frac{\alpha}{2})} \cos[\alpha \arctan(l|\xi|)]\right)\}$$

where $c_1$ is a scaling parameter and $c_0 = \frac{l^{-\alpha}}{\cos(\pi\frac{\alpha}{2})}$.
Equivalently Matacz derived ([26] page 156) the following characteristic function.

$$\phi_t(\xi) = \exp\{-\frac{c^\alpha t}{\cos(\pi\alpha/2)}\left((\xi^2 + \lambda^2)^{\frac{\alpha}{2}} \cos[\alpha \arctan(\frac{\xi}{\lambda})] - \lambda^\alpha\right)\}, \alpha \neq 1.$$

Therefore we get the characteristic exponent

$$\psi(\xi) = \frac{c^\alpha}{\cos(\pi\alpha/2)}\left((\xi^2 + \lambda^2)^{\frac{\alpha}{2}} \cos[\alpha \arctan(\frac{\xi}{\lambda})] - \lambda^\alpha\right).$$

He also calculated (page 149/50) in an example the parameters
$\alpha = 1.2,\ c = 1.1,\ \lambda = \frac{1}{80}$.

### 1.4.8 Meixner process

The Meixner process is a Lévy process which was introduced as a model in finance by Schoutens in [29].

A stochastic process $X = \{X_t, t \geq 0\}$ is called Meixner process if for all $t \geq 0$ and $\xi \in \mathbb{R}$ we have

$$\mathbb{E}(e^{i\xi X_t}) = \left(\frac{\cos(\frac{b}{2})}{\cosh(\frac{a\xi - ib}{2})}\right)^{2st} e^{imt\xi},$$

where $a > 0, -\pi < b < \pi, s > 0, m \in \mathbb{R}$.

Equivalently we can characterise the process by its exponent.

**Definition 1.4.2 (Meixner exponent).** *The charactereistic exponent corresponding to the Meixner process is defined as*

$$\psi_M(\xi) = -im\xi + 2s\left(\ln\cosh(\frac{a\xi - ib}{2}) - \ln\cos(\frac{b}{2})\right).$$

It can be shown that the density of $X_t$ is given by

$$d_{(a,b,mt,st)}(x) = \frac{(2\cos(\frac{b}{2}))^{2st}}{2a\pi\Gamma(2st)} \exp\left(\frac{b(x - mt)}{a}\right)\left|\Gamma\left(st + \frac{i(x - mt)}{a}\right)\right|^2, \quad (1.4.1)$$

for a proof and more details about the Meixner process see [16], [29] and [30]. Therein some moments of the Meixner Process are calculated:

| | |
|---|---|
| mean | $m + as\tan(\frac{b}{2})$ |
| variance | $\frac{a^2 s}{2\cos^2(\frac{b}{2})}$ |
| third moment | $3 + \frac{3-2\cos^2(\frac{b}{2})}{s}$ |

The Lévy measure is

$$\nu(dx) = s\frac{\exp(\frac{bx}{a})}{x\sinh(\frac{\pi x}{a})}\,dx,$$

the drift is

$$m_{Meixner} = -as\tan(\frac{b}{2}) + 2s\int_1^\infty \frac{\sinh(\frac{by}{a})}{\sinh(\frac{\pi y}{a})}dy - m$$

and there is no diffusion component.
In modelling by Schoutens [29] used parameter values are

| a | b | s | m |
|---|---|---|---|
| 0.02982825 | 0.12716244 | 0.57295483 | -0.00112426 |
| 0.1277 | - 1.8742 | 2.2603 | |
| 0.0279 | -0.1708 | 22.0914 | |

For a simulation of the meixner process see Appendix A.3.

### 1.4.9 Real Meixner Processes

In this section we look at the real part of the Meixner exponent and the properties of the corresponding distribution. The real part of the Meixner exponent is given by

$$\psi_{ReM}(\xi) := \text{Re }\psi(\xi) = -2s\ln\cos\left(\frac{b}{2}\right) + s\ln\left(\cosh^2\frac{a\xi}{2} - \sin^2\frac{b}{2}\right).$$

We get the characteristic function

$$\phi_t(\xi) := e^{-t\psi_{ReM}(\xi)} = \frac{\cos^{2st}(\frac{b}{2})}{(\cosh^2(\frac{a\xi}{2}) - \sin^2(\frac{b}{2}))^{st}}.$$

The moments of the corresponding distribution can be easily calculated by the formula

$$\mu_n = (-i)^n\partial_\xi^n\phi(\xi)|_{\xi=0},$$

where $\mu_n$ is the n$^{th}$ moment. In our case we get:

$\mu_n = 0$ for $n$ odd,

$\mu_2 = (-1) \times (-\frac{1}{2})\frac{sta^2}{\cos^2(\frac{b}{2})}$,

$\mu_4 = (1) \times (-\frac{1}{4})\frac{sta^4(-3+2\cos^2(\frac{b}{2})-3st)}{\cos^2(\frac{b}{2})}$.

All even moments do exist but the higher ones are more involved.

**Theorem 1.4.3.** *The density of the transition probability of the real Meixner process is given by*

$$p_t(x) = \frac{4^{st}}{2\pi a\Gamma(2st)}|\Gamma(st + i\frac{x}{a})|^2$$

*for $b = 0$ and otherwise ($b \neq 0$) by*

$$p_t(x) = \frac{\cos^{2st}(\frac{b}{2})4^{st}}{(2\pi)^2\Gamma(2st)a^2}\sum_{k=0}^{\infty}\frac{st(st+1)\cdot\ldots\cdot(st+k-1)}{k!}$$
$$\times \sin^{2k}(\frac{b}{2})\frac{4^k}{\Gamma(2k)}\int_{-\infty}^{\infty}|\Gamma(st + i\frac{(x-v)}{a})|^2 \cdot |\Gamma(k + i\frac{v}{a})|^2\ dv.$$

**Proof.** If its transition density exists then we have for a stochastic process $X_t$ the equation

$$\int_{-\infty}^{\infty} e^{ix\xi}p_t(x)\ dx = \mathbb{E}(e^{i\xi X_t}) = \phi_t(\xi),$$

where $p_t$ is the density function.
With our definition of the Fourier transform we have

$$F^{-1}_{x\to\xi}(p_t(x)) = \frac{1}{\sqrt{2\pi}}\phi_t(\xi).$$

Therefore we get the density (if existent) as

$$p_t(x) = \frac{1}{\sqrt{2\pi}}F_{\xi\to x}\phi_t(\xi).$$

In the case $b = 0, m = 0$ the characteristic exponent $\psi_M$ is already real and obviously $\psi_M = \psi_{ReM}$. We get the density by equation (1.4.1).
For $b \neq 0$ we do not find such a neat expression and we need some auxiliary results.

We know that

$$\int_{-\infty}^{\infty}|\Gamma(\gamma + ix)|^2 e^{ix\xi}\ dx = 2\pi\Gamma(2\gamma)\left(\frac{1}{2\cosh\frac{\xi}{2}}\right)^{2\gamma}$$

holds for $\gamma > 0,\ \xi \in \mathbb{C},\ -\pi < \mathrm{Im}\ \xi < \pi$, see [16] and the references given therein. Therefore we can derive

$$F_{\xi\to x}(\cosh^{-2\gamma}\frac{a\xi}{2}) = \frac{4^\gamma}{\sqrt{2\pi}\Gamma(2\gamma)a}|\Gamma(\gamma + i\frac{x}{a})|^2.$$

Now we can calculate the density, where $*$ denotes the convolution.

$$\begin{aligned}
&F_{\xi\to x}\left(\phi_t(\xi)\right)\\
&= F_{\xi\to x}\left(\frac{\cos^{2st}(\frac{b}{2})}{(\cosh^2(\frac{a\xi}{2}) - \sin^2(\frac{b}{2}))^{st}}\right)\\
&= \frac{\cos^{2st}(\frac{b}{2})}{\sqrt{2\pi}} F_{\xi\to x}\left(\frac{1}{\cosh^{2st}\frac{a\xi}{2}}\right) * F_{\xi\to x}\left(\left(\frac{1}{1-\frac{\sin^2\frac{b}{2}}{\cosh^2\frac{a\xi}{2}}}\right)^{st}\right)\\
&= \frac{\cos^{2st}(\frac{b}{2})}{\sqrt{2\pi}} F_{\xi\to x}\left(\frac{1}{\cosh^{2st}\frac{a\xi}{2}}\right)\\
&\quad * F_{\xi\to x}\left(\sum_{k=0}^{\infty}\frac{st(st+1)\cdot\ldots\cdot(st+k-1)}{k!}\frac{\sin^{2k}\frac{b}{2}}{\cosh^{2k}\frac{a\xi}{2}}\right)\\
&= \frac{\cos^{2st}(\frac{b}{2})}{2\pi}\left(\frac{4^\gamma}{\Gamma(2\gamma)a}|\Gamma(\gamma+i\frac{x}{a})|^2\right)\\
&\quad * \left(\sum_{k=0}^{\infty}\frac{st(st+1)\cdot\ldots\cdot(st+k-1)}{k!}\sin^{2k}(\frac{b}{2})\cdot\frac{4^k}{\sqrt{2\pi}\Gamma(2k)a}|\Gamma(k+i\frac{x}{a})|^2\right)\\
&\quad = \frac{\cos^{2st}(\frac{b}{2})4^{st}}{(2\pi)^{\frac{3}{2}}\Gamma(2st)a^2}\sum_{k=0}^{\infty}\frac{st(st+1)\cdot\ldots\cdot(st+k-1)}{k!}\\
&\qquad\quad \times \sin^{2k}(\frac{b}{2})\cdot\frac{4^k}{\Gamma(2k)}\int_{-\infty}^{\infty}|\Gamma(st+i\frac{(x-v)}{a})|^2\cdot|\Gamma(k+i\frac{v}{a})|^2\,dv
\end{aligned}$$

We used that $0 \le \frac{\sin^2\frac{b}{2}}{\cosh^2\frac{a\xi}{2}} < 1$ since $|b| < \pi$, which implies the convergence of the series.
In addition we used the formula

$$\left(\sum_{k=0}^{\infty}p^k\right)^r = \sum_{k=0}^{\infty}\frac{r(r+1)\cdots(r+k-1)}{k!}p^k$$

which holds for $|p| < 1, r \in \mathbb{R}$. ■

## 1.5 Orthogonal polynomials and Lévy processes

In this section we state the relations of Lévy processes and polynomials as presented by Schoutens and Teugels [30]. This also shows how the Meixner process was first developed.

**Definition 1.5.1 (Lévy-Sheffer system).** *A set $\{Q_m(x,t), m \geq 0, m \in \mathbb{N}, t \geq 0, t \in \mathbb{R}\}$ of polynomials is called a Lévy-Scheffer system if its generating function is of the form*

$$\sum_{m=0}^{\infty} Q_m(x,t)\frac{z^m}{m!} = (f(z))^t e^{xu(z)} \tag{1.5.1}$$

*where*

*i) $f(z)$ and $u(z)$ are analytic in a neighborhood of $z = 0$,*

*ii) $u(0) = 0$, $f(0) = 1$ and $u'(0) \neq 0$,*

*iii) $\phi(\xi) := \frac{1}{f(\tau(i\xi))}$ is the characteristic function of a Lévy process. Here $\tau$ is defined as the inverse function of $u$, i.e. $\tau(u(z)) = z$.*

If the above system of polynomials is orthogonal it is called Lévy-Meixner system. This name is due to the fact that Meixner determined all families of orthogonal polynomials defined by (1.5.1) with $t = 1$. The characteristic exponents of all Lévy processes corresponding to Lévy-Meixner systems are given by

$$\psi(\xi) = \log\phi(\xi) = \begin{cases} i\frac{\alpha+\beta}{\alpha\beta}\xi + \frac{1}{\alpha\beta}\log\left(\frac{\alpha-\beta}{\alpha\exp(i\alpha\xi)-\exp(i\beta\xi)}\right) & \text{if } 0 \neq \alpha \neq \beta \neq 0 \\ i\frac{\xi}{\alpha} - \frac{1}{\alpha^2}\log(1+i\alpha\xi) & \text{if } \alpha = \beta \neq 0 \\ i\frac{\xi}{\alpha} - \frac{1}{\alpha^2}(1-\exp(-i\alpha\xi)) & \text{if } \alpha \neq \beta = 0 \\ -\frac{\xi^2}{2} & \text{if } \alpha = \beta = 0 \end{cases}$$

where $\alpha\beta \geq 0$.

By the different choices of $\alpha$ and $\beta$ we get the following correspondence of certain Lévy processes and families of orthogonal polynomials.

| Lévy-Meixner parameter | Polynomial | Lévy Process |
|---|---|---|
| $\alpha = \beta = 0$ | Hermite | Brownian motion |
| $\alpha \neq \beta = 0$ | Charlier | Poisson process |
| $\alpha = \beta \neq 0$ | Laguerre | Gamma process |
| $0 \neq \alpha \neq \beta \neq 0$<br>$\alpha, \beta \in \text{Re}\ ,\ \alpha\beta > 0$ | Meixner | Negative-binomial process<br>(Pascal process) |
| $0 \neq \alpha,\ \beta = \overline{\alpha}$ | Meixner-Pollaczek | Meixner process |

Table 1.2: Polynomials and Processes

# Chapter 2

# From a pseudo-differential operator to a Feller semigroup

A pseudo-differential operator has the form

$$q(x,D)u = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} q(x,\xi)\hat{u}(\xi)\, d\xi$$

where $\hat{u}$ denotes the Fourier transform of $u$. The function $q(x,\xi)$ is called the symbol of the operator. In the following we restrict ourselves to symbols which are in the second component continuous negative definite functions.

**Definition 2.0.2 (continuous negative definite symbol).** *We call $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ a continuous negative definite symbol, if $q(x,\xi)$ is for any fixed $x$ a continuous negative definite function in its second component.*

In order to extend a given pseudo-differential operator to a generator of a Feller semigroup we want to use the Hille-Yosida-Ray Theorem. But first recall the definition of a generator of a semigroup.

**Definition 2.0.3 (Generator of a Semigroup).** *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup of operators on a Banach space $(X, \|.\|_X)$. The generator $A$ of $(T_t)_{t\geq 0}$ is defined by*

$$Au := \lim_{t\to 0} \frac{T_t u - u}{t}$$

*in the sense of a strong limit with domain*

$$\mathcal{D}(A) := \left\{ u \in X \,|\, \lim_{t\to 0} \frac{T_t u - u}{t} \text{ exists as strong limit} \right\}.$$

**Theorem 2.0.4 (Hille-Yosida-Ray).** *A linear operator $(A, \mathcal{D}(A))$, $\mathcal{D}(A) \subset C_\infty(\mathbb{R}^n;\mathbb{R})$, on $C_\infty(\mathbb{R}^n;\mathbb{R})$ is closable and its closure is the generator of a Feller semigroup if and only if the three following conditions hold:*

*i)* $\mathcal{D}(A) \subset C_\infty(\mathbb{R}^n;\mathbb{R})$ *is dense,*

*ii)* $(A, \mathcal{D}(A))$ *satisfies the positive maximum principle,*

*iii) the range of* $(\lambda - A)$ *is dense in* $C_\infty(\mathbb{R}^n;\mathbb{R})$ *for some* $\lambda > 0$.

A proof of this fundamental result can be found in [21].
Note that $\mathcal{D}(A)$ denotes the domain of the operator $A$. Our final aim of this chapter is to apply this theorem to the operator $-q(x, D)$. Part *i)* of this theorem is just the question of finding a suitable domain. Part *ii)* is satisfied for every pseudo-differential operator with continuous negative definite symbol, as we will see shortly. The last part is the hardest problem, which we will solve using Hoh's calculus for pseudo-differential operators with continuous negative definite symbols.

## 2.1 Positive maximum principle

The positive maximum principle has strong relations to pseudo-differential operators due to a representation result of Courrége, compare Theorem 4.5.21 in [21]. We will state the results we need without proofs.

**Definition 2.1.1 (positive maximum principle).** *Let* $A : \mathcal{D}(A) \to B(\mathbb{R}^n;\mathbb{R})$ *be a linear operator,* $\mathcal{D}(A) \subset B(\mathbb{R}^n;\mathbb{R})$*. We say that* $(A, \mathcal{D}(A))$ *satisfies the positive maximum principle if for any* $u \in \mathcal{D}(A)$ *and some* $x_0 \in \mathbb{R}^n$ *the fact* $u(x_0) = \sup_{x\in\mathbb{R}^n} u(x) \geq 0$ *implies that* $Au(x_0) \leq 0$.

The space $B(\mathbb{R}^n;\mathbb{R})$ is the space of the Borel measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$.
We have for a pseudo-differential operator with a continuous negative definite symbol on $C_0^\infty$ the following theorem.

**Theorem 2.1.2.** *Let* $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ *be a locally bounded function such that for any* $x \in \mathbb{R}^n$ *the function* $q(x, .) : \mathbb{R}^n \to \mathbb{C}$ *is a continuous negative definite function. Define on* $C_0^\infty(\mathbb{R}^n;\mathbb{R})$ *the operator*

$$-q(x, D)u(x) := -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} q(x, \xi)\hat{u}(\xi)\, d\xi.$$

*Then the operator* $(-q(x, D), C_0^\infty(\mathbb{R}^n;\mathbb{R}))$ *satisfies the positive maximum principle.*

A proof can be found in [21] Theorem 4.5.6.
The domain $C_0^\infty(\mathbb{R}^n)$ turns out to be too small to prove part *iii)* of the Hille-Yosida-Ray Theorem for the corresponding operator, but we are able to extend the domain of the operator.

**Theorem 2.1.3.** *Let $\mathcal{D}(A) \subset C_\infty(\mathbb{R}^n;\mathbb{R})$ and suppose that $A : \mathcal{D}(A) \to C_\infty(\mathbb{R}^n;\mathbb{R})$ is a linear operator. In addition assume that $C_0^\infty(\mathbb{R}^n;\mathbb{R}) \subset \mathcal{D}(A)$ is an operator core of $A$, i.e. to every $u \in \mathcal{D}(A)$ there exists a sequence $(\varphi_k)_{k\in\mathbb{N}}$, $\varphi_k \in C_0^\infty(\mathbb{R}^n;\mathbb{R})$, such that*

$$\lim_{k\to\infty} \|\varphi_k - u\|_\infty = \lim_{k\to\infty} \|A\varphi_k - Au\|_\infty = 0.$$

*If $A|_{C_0^\infty}$ satisfies the positive maximum principle on $C_0^\infty(\mathbb{R}^n;\mathbb{R})$, then it satisfies the positive maximum principle also on $\mathcal{D}(A)$.*

The proof of this result can be found in [22] Theorem 2.6.1.
Therefore we have that a pseudo-differential operator with continuous negative definite symbol, for which $C_0^\infty(\mathbb{R}^n;\mathbb{R})$ is an operator core, satisfies the positive maximum principle on its domain.

## 2.2 Hoh's calculus

To satisfy the condition *iii)* of the Hille-Yosida-Ray Theorem, we need to show that the range of $\lambda + q(x, D)$ is dense in $C_\infty(\mathbb{R}^n;\mathbb{R})$ for some $\lambda > 0$. This is equivalent to solving the equation

$$(\lambda + q(x, D))u = f \tag{2.2.1}$$

for sufficiently many $f$ i.e. all functions $f$ from a dense subset of $C_\infty(\mathbb{R}^n;\mathbb{R})$. Since this is too hard to be solved directly, we are going to solve the equation in a $L^2$-sense and then use some regularity results. We follow closely Hoh's calculus and the estimates based on it as presented in section 2.4 and 2.5 of [22], see also [18] and [17].

First we are going to define some function spaces and classes of operators. The spaces are going to be certain anisotropic Sobolev spaces defined with the help of a continuous negative definite function. These spaces will be used as a scale for the domain and range of the operators.

**Definition 2.2.1 (space $H^{\psi,s}$).** *Let $s \in \mathbb{R}$ and $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function. The space $H^{\psi,s}$ is defined by*

$$H^{\psi,s}(\mathbb{R}^n) := \{u \in S'(\mathbb{R}^n); \|u\|_{\psi,s} < \infty\}$$

*where the norm is*

$$\|u\|_{\psi,s} := \|(1 + \psi(D))^{s/2}u(.)\|_0 < \infty,$$

*and $\|.\|_0$ denotes the $L^2$ norm.*

Here $S(\mathbb{R}^n)$ denotes the Schwartz space and its topological dual space is $S'(\mathbb{R}^n)$.

**Remark 2.2.2.** *A) The spaces $H^{\psi,s}$ are Hilbert spaces. In addition, if $\psi$ satisfies*

$$\psi(\xi) \geq c_0|\xi|^{r_0} \tag{2.2.2}$$

*for some $c_0 > 0,\ r_0 > 0$ and all $\xi, |\xi| \geq R$, and if $s > \frac{n}{r_0}$ then*

$$H^{\psi,s}(\mathbb{R}^n) \hookrightarrow C_\infty(\mathbb{R}^n),$$

*in the sense of a continuous embedding.*
*For a more general result and its proof see Theorem 3.10.12 in [21], note that the space $H^{\psi,s}$ equals the space $B^s_{\psi,2}$ in [21].*

*B) For any continuous negative definite function and any $s \in \mathbb{R}$ we know that the space $C_0^\infty(\mathbb{R}^n)$ is dense in $H^{\psi,s}$.*
*For a proof see Theorem 3.10.3 in [21].*

**Definition 2.2.3 (class $\Lambda$).** *We say that a continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ belongs to the class $\Lambda$ if for all $\alpha \in \mathbb{N}_0^n$ there exists a constant $c_{|\alpha|} \geq 0$ such that*

$$|\partial_\xi^\alpha(1 + \psi(\xi))| \leq c_{|\alpha|}(1 + \psi(\xi))^{\frac{2-\rho(|\alpha|)}{2}}$$

*where $\rho : \mathbb{N}_0 \to \mathbb{N}_0, k \mapsto \rho(k) := k \wedge 2$.*

**Definition 2.2.4 (symbol class $S_\rho^{m,\psi}$ and $S_0^{m,\psi}$).** *Let $m \in \mathbb{R}$ and $\psi \in \Lambda$. We call a $C^\infty$-function $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ a symbol in the class $S_\rho^{m,\psi}(\mathbb{R}^n)$ if for all $\alpha, \beta \in \mathbb{N}_0^n$ there are constants $c_{\alpha,\beta} \leq 0$ such that*

$$|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)| \leq c_{\alpha,\beta}(1 + \psi(\xi))^{\frac{m-\rho(|\alpha|)}{2}} \tag{2.2.3}$$

*holds for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$. We call $m \in \mathbb{R}$ the order of the symbol.*
*If we replace $\rho(|\alpha|)$ in (2.2.3) by 0 then we say that the symbol is in the class $S_0^{m,\psi}(\mathbb{R}^n)$.*

**Definition 2.2.5 (pseudo-differential operator class $\Psi_\rho^{m,\psi}(\mathbb{R}^n)$ and $\Psi_0^{m,\psi}(\mathbb{R}^n)$).** *For $q \in S_\rho^{m,\psi}$ or $q \in S_0^{m,\psi}$, resp., we define on $S(\mathbb{R}^n)$ the pseudo-differential operator $q(x,D)$ by*

$$q(x,D)u(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} q(x,\xi)\hat{u}(\xi)\ d\xi.$$

*The class of these operators is denoted by $\Psi_\rho^{m,\psi}(\mathbb{R}^n)$ and $\Psi_0^{m,\psi}(\mathbb{R}^n)$ respectively.*

The operator is well defined since the symbol is polynomial bounded in its second component. Furthermore we can show that any operator of these classes maps $S(\mathbb{R}^n)$ continuously into itself. For a proof of this see Theorem 2.4.11 in [22].

For convenience we will use the following notation

$$B(u, v) := (q(x, D)u, v)_0,$$

$$q_\lambda(x, D)u := q(x, D)u + \lambda u$$

and

$$B_\lambda(u, v) := (q_\lambda(x, D)u, v)_0,$$

where $(.,.)_0$ denotes the $L^2$ scalar product.

In order to solve (2.2.1) we will use the Lax-Milgram Theorem, which is stated here to help the reader understand the purpose of the following estimates. But it won't be used before Theorem 2.2.17.

**Theorem 2.2.6 (Lax-Milgram).** *Let $B$ be a sesquilinear form on a complex Hilbert space $(H, (.,.)_H)$. Suppose that*

$$|B(u, v)| \leq c\|u\|_H\|v\|_H \tag{2.2.4}$$

*and*

$$|B(u, u)| \geq \gamma\|u\|_H^2 \tag{2.2.5}$$

*hold for all $u,\ v \in H$ with some $\gamma > 0$. In addition, let $l : H \to \mathbb{C}$ be a continuous linear functional. Then there exist unique elements $v,\ w \in H$ such that*

$$l(u) = B(u, v) = \overline{B(w, u)}$$

*holds for all $u \in H$.*

A proof can be found in [13].

Now we want to show (2.2.4) and (2.2.5) for $B(u, v) = (q(x, D)u, v)_0$. Therefor we need the following estimate.

**Theorem 2.2.7.** *Let $q \in S_0^{m,\psi}(\mathbb{R}^n)$ and let $q(x, D)$ be the corresponding pseudo-differential operator. For all $s \in \mathbb{R}$ the operator $q(x, D)$ maps the space $H^{\psi,m+s}(\mathbb{R}^n)$ continuously into the space $H^{\psi,s}(\mathbb{R}^n)$ and for all $u \in H^{\psi,m+s}(\mathbb{R}^n)$ we have the estimate*

$$\|q(x, D)u\|_{\psi,s} \leq c\|u\|_{\psi,m+s}.$$

For a proof see Theorem 2.5.4 in [22].

This theorem is already sufficient to show (2.2.4) for the sesquilinear form $B$ as we will see in Theorem 2.2.14. But first we concentrate on the lower estimate.

We start with proving a sharp Gårding inequality for symbols in $S_\rho^{m,\psi}$. Then we show a Gårding inequality which will imply (2.2.5). In [22] (respectively [18]) this is only proved for real valued symbols, but with a slight variation we can extend the proofs to a class of complex valued symbols. In order to give proofs we have to quote some more definitions and theorems of Hoh's Calculus. The main elements of this calculus are expressions for the symbols of composed and adjoint operators (see Theorem 2.2.18), as well as a variant of Friedrichs symmetrisation. The expressions for the symbols are derived with the help of double symbols and the corresponding operators.

**Definition 2.2.8 (double symbol).** *Let $\psi \in \Lambda$ and $m, m' \in \mathbb{R}$. The class $S_0^{m,m',\psi}(\mathbb{R}^n)$ of double symbols of order $m$ and $m'$ consists of all $C^\infty$-functions $q : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ satisfying*

$$|\partial_\xi^\alpha \partial_x^\beta \partial_{\xi'}^{\alpha'} \partial_{x'}^{\beta'} q(x,\xi;x',\xi')| \leq c_{\alpha,\beta,\alpha',\beta'}(1+\psi(\xi))^{\frac{m}{2}}(1+\psi(\xi'))^{\frac{m'}{2}}$$

*for all $\alpha, \beta, \alpha', \beta' \in \mathbb{N}_0^n$.*
*For $q \in S_0^{m,m',\psi}(\mathbb{R}^n)$ we define on $S(\mathbb{R}^n)$ the operator*

$$\begin{aligned} q(x,&D_x;x',D_{x'})u(x) \\ &= (2\pi)^{\frac{-3n}{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-x')\xi + ix'\xi'} q(x,\xi;x',\xi')\hat{u}(\xi')\,d\xi'\,dx'\,d\xi. \end{aligned}$$

*This operator can also be expressed by a so called simplified operator $q_L(x,D)u$ with a corresponding simplified symbol $q_L(x,\xi)$.*

For more details about the simplified symbol see [22] Theorem 2.4.17 pp.

**Definition 2.2.9 (Friedrichs symmetrisation).** *Let $q \in S_0^{m,\psi}$. Its Friedrichs symmetrization is the double symbol*

$$q_F(\xi;x',\xi') := \int_{\mathbb{R}^n} F(\xi,\zeta)q(x',\zeta)F(\xi',\zeta)\,d\zeta$$

*where*

$$F(\xi,\zeta) := (1+\psi(\xi))^{-n/8} r\left((\zeta-\xi)(1+\psi(\xi))^{-1/4}\right)$$

*and $\psi \in \Lambda$, $r \in C_0^\infty(\mathbb{R}^n)$ is a fixed non-negative function which is even, supported in the unit ball and satisfies $\int_{\mathbb{R}^n} r^2(\xi)\,d\xi = 1$.*

**Theorem 2.2.10.** *For the Friedrichs symmetrization of $q \in S_0^{m,\psi}(\mathbb{R}^n)$, $\psi \in \Lambda$, we have the estimate*

$$|\partial_\xi^\alpha \partial_{\xi'}^{\alpha'} \partial_{x'}^{\beta'} q_F(\xi;x',\xi')| \leq c_{\alpha,\alpha',\beta'}(1+\psi(\xi))^{\frac{m-\frac{1}{2}\rho(\beta)}{2}}(1+\psi(\xi'))^{\frac{-\rho(\beta')}{2}}. \tag{2.2.6}$$

*In particular $q_F$ belongs $S_0^{m,0,\psi}(\mathbb{R}^n)$ and the simplified symbol $q_{F,L}$ belongs to $S_0^{m,\psi}(\mathbb{R}^n)$. Moreover, if $q \in S_\rho^{m,\psi}(\mathbb{R}^n)$, then*

$$q - q_{F,L} \in S_0^{m-1,\psi}(\mathbb{R}^n)$$

*holds.*

A proof can be found in [22] Theorem 2.4.25, but is originally due to W. Hoh [18].

**Remark 2.2.11.** *The $\rho$ in (2.2.6) originates from the definition of $\Lambda$.*

**Theorem 2.2.12.** *Let $q \in S_0^{m,\psi}(\mathbb{R}^n)$ with* Re $q(x,\xi) \geq 0$ *for all $x, \xi \in \mathbb{R}^n$. Then* Re $(q_F(D_x; x', D_{x'})u, u)_0 \geq 0$ *for all $u \in S(\mathbb{R}^n)$.*

**Proof.** This proof is an extension to complex symbols of the proof of Theorem 2.4.28 in [22].
We have

$$\begin{aligned}
&(q_F(D_x; x', D_{x'})u, v)_0 \\
&= c \int_{\mathbb{R}^n} F^{-1}_{\xi \to x} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix'\xi + ix'\xi'} q_F(\xi; x', \xi') \hat{u}(\xi')\, d\xi'\, dx' \right)(x) \overline{v(x)}\, dx \\
&= c \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix'\xi + ix'\xi'} \int_{\mathbb{R}^n} F(\xi, \eta) q(x', \eta) \\
&\quad \times F(\xi', \eta)\, d\eta \hat{u}(\xi')\, d\xi'\, dx' \overline{\hat{v}(\xi)}\, d\xi \\
&= c \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} q(x', \eta) \left( \int_{\mathbb{R}^n} e^{ix'\xi'} F(\xi', \eta) \hat{u}(\xi')\, d\xi' \right) \\
&\quad \times \overline{\left( \int_{\mathbb{R}^n} e^{ix'\xi} F(\xi, \eta) \hat{v}(\xi)\, d\xi \right)}\, d\eta\, dx'.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\text{Re } (q_F(D_x; x', D_{x'})u, u)_0 \\
&\quad = c \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\text{Re } q(x', \eta)) \left( \int_{\mathbb{R}^n} e^{ix'\xi'} F(\xi', \eta) \hat{u}(\xi')\, d\xi' \right) \\
&\qquad \times \overline{\left( \int_{\mathbb{R}^n} e^{ix'\xi} F(\xi, \eta) \hat{u}(\xi)\, d\xi \right)}\, d\eta\, dx' \geq 0.
\end{aligned}$$

proves the Theorem. ■

Now we can prove a sharp Gårding inequality.

**Theorem 2.2.13.** *Let $q \in S_\rho^{m,\psi}(\mathbb{R}^n)$ with* Re $q(x,\xi) \geq 0$ *for all $x, \xi \in \mathbb{R}^n$. Then there exists a constant $K \geq 0$ such that*

$$\text{Re } (q(x,D)u,u)_0 \geq -K\|u\|^2_{\psi,\frac{m-1}{2}}$$

*holds for all $u \in S(\mathbb{R}^n)$.*

**Proof.** This proof is an extension of the proof given by W. Hoh [18]. We follow the presentation in [22], Theorem 2.5.5.
From Theorem 2.2.10 we know that $q - q_{F,L} \in S_0^{m-1,\psi}(\mathbb{R}^n)$, hence by Theorem 2.2.7 the operator $q(x,D) - q_F(D_x;x',D_{x'})$ maps $H^{\psi,m-1+s}(\mathbb{R}^n)$ continuously into $H^{\psi,s}(\mathbb{R}^n)$, $s \in \mathbb{R}$.
Moreover, by Theorem 2.2.12 Re $(q_F(D_x;x',D_{x'})u,u)_0 \geq 0$ holds. This implies

$$\begin{aligned}
\text{Re } (q(x,D)u,u)_0 &= \text{Re } (q_F(D_x;x',D_{x'})u,u)_0 \\
&\quad + \text{Re } ((q(x,D) - q_F(D_x;x',D_{x'}))u,u)_0 \\
&\geq \text{Re } \big((1+\psi(D))^{-\frac{m-1}{4}}(q(x,D) \\
&\quad - q_F(D_x;x',D_{x'}))u, (1+\psi(D))^{\frac{m-1}{4}}u\big)_0 \\
&\geq -\|(q(x,D) - q_F(D_x;x',D_{x'}))u\|_{\psi,-\frac{m-1}{2}} \|u\|_{\psi,\frac{m-1}{2}} \\
&\geq -K\|u\|^2_{\psi,\frac{m-1}{2}}.
\end{aligned}$$

Where we used again Theorem 2.2.7 for the last inequality. ■

Now we come back to the sesquilinear form $B(u,v) = (q(x,D)u,v)_0$.

**Theorem 2.2.14.** *Let $q \in S_\rho^{m,\psi}(\mathbb{R}^n)$ and $m > 0$.*

*A) For all $u, v \in S(\mathbb{R}^n)$*

$$|B(u,v)| \leq c\|u\|_{\psi,\frac{m}{2}}\|v\|_{\psi,\frac{m}{2}}$$

*holds. Hence the sesquilinear form $B$ has a continuous extension onto $H^{\psi,\frac{m}{2}}(\mathbb{R}^n)$.*

*B) If in addition there exists $\gamma_0 > 0$ and $R \geq 0$ such that*

$$\text{Re } q(x,\xi) \geq \gamma_0(1+\psi(\xi))^{m/2} \text{ for } |\xi| \geq R \tag{2.2.7}$$

*holds for all $x \in \mathbb{R}^n$ and*

$$\lim_{|\xi|\to\infty} \psi(\xi) = \infty \tag{2.2.8}$$

*holds, then we have for all $u \in H^{\psi,\frac{m}{2}}(\mathbb{R}^n)$ the Gårding inequality*

$$\text{Re } B(u,u) \geq \frac{\gamma_0}{2}\|u\|^2_{\psi,\frac{m}{2}} - \lambda_0\|u\|_0^2. \tag{2.2.9}$$

**Proof.** The proof follows the proof of Theorem 2.5.6 in [22] and is originally due to W. Hoh [18]. Part *A)* is identical, but for part *B)* we need a modification:
For $\lambda$ sufficiently large we have

$$\text{Re } q(x,\xi) + \lambda \geq \gamma_0(1+\psi(\xi))^{m/2}.$$

Hence the symbol

$$r(x,\xi) = q(x,\xi) + \lambda - \gamma_0(1+\psi(\xi))^{m/2}$$

satisfies the conditions for Theorem 2.2.13 and we get

$$\text{Re } B(u,u) - \gamma_0\|u\|^2_{\psi,\frac{m}{2}} + \lambda\|u\|_0^2 = \text{Re } (r(x,D)u,u)_0 \geq -K\|u\|^2_{\psi,\frac{m-1}{2}}$$

or

$$\text{Re } B(u,u) \geq \gamma_0\|u\|^2_{\psi,\frac{m}{2}} - \lambda\|u\|_0^2 - K\|u\|^2_{\psi,\frac{m-1}{2}}.$$

For $m-1 \leq 0$ we have $\|u\|_{\psi,\frac{m-1}{2}} \leq \|u\|_0$ which yields

$$\text{Re } B(u,u) \geq \gamma_0\|u\|^2_{\psi,\frac{m}{2}} - (K+\lambda)\|u\|_0^2.$$

Otherwise for $m-1>0$ it follows from (2.2.8) that for any $\varepsilon > 0$ we have

$$(1+\psi(\xi))^{\frac{m-1}{2}} \leq \varepsilon^2(1+\psi(\xi))^{\frac{m}{2}} + c_\varepsilon^2$$

which leads to

$$\|u\|_{\psi,\frac{m-1}{2}} \leq \varepsilon\|u\|_{\psi,\frac{m}{2}} + c_\varepsilon\|u\|_0.$$

Taking $\varepsilon = \frac{\gamma_0}{2K}$ we arrive at

$$\text{Re } B(u,u) \geq \frac{\gamma_0}{2}\|u\|^2_{\psi,\frac{m}{2}} - (Kc_{\frac{\gamma_0}{2K}} + \lambda)\|u\|_0^2.$$

■

**Remark 2.2.15.** *The proof above also yields the estimate*

$$\text{Re } B(u,u) \geq \frac{\gamma_0}{2}\|u\|^2_{\psi,\frac{m}{2}} - \lambda_1\|u\|^2_{\psi,\frac{m-1}{2}},$$

*since for $m-1 \geq 0$ we have $\|u\|_0 \leq \|u\|_{\psi,\frac{m-1}{2}}$ and for $-1 < m-1 < 0$ we get with* (2.2.8)

$$1 \leq \varepsilon^2(1+\psi(\xi))^{\frac{m}{2}} + \frac{1}{4\varepsilon^2}(1+\psi(\xi))^{\frac{m-1}{2}}.$$

*(This estimate is needed in Theorem 2.2.20.)*

Now we define what we mean by a solution to (2.2.1) and prove the existence and uniqueness of such a solution.

**Definition 2.2.16 (variational solution).** *Let $q \in S_\rho^{m,\psi}(\mathbb{R}^n)$ and assume that (2.2.7) and (2.2.8) are satisfied. Then we call $u \in H^{\psi,\frac{m}{2}}(\mathbb{R}^n)$ a variational solution to the equation*

$$q_\lambda(x, D)u = q(x, D)u + \lambda u = f \tag{2.2.10}$$

*for $\lambda \in \mathbb{R}$ and $f \in L^2(\mathbb{R}^n)$ if*

$$B_\lambda(u, \varphi) = (\varphi, f)_0 = \overline{(f, \varphi)_0}$$

*holds for all $\varphi \in C_0^\infty(\mathbb{R}^n)$.*

**Theorem 2.2.17.** *Suppose $\psi \in \Lambda$ satisfies (2.2.8) and $q \in S_\rho^{m,\psi}(\mathbb{R}^n)$ satisfies (2.2.7). Then for all $\lambda \geq \lambda_0, \lambda_0$ taken from (2.2.9), there exists for all $f \in L^2(\mathbb{R}^n)$ a unique variational solution $u \in H^{\psi,\frac{m}{2}}(\mathbb{R}^n)$ to (2.2.10).*

**Proof.** This proof is analogously to the proof of Theorem 2.5.12 in [22].
For $f \in L^2(\mathbb{R}^n)$ a continuous linear functional on $H^{\psi,\frac{m}{2}}(\mathbb{R}^n)$ is given by $\varphi \mapsto (\varphi, f)_0$, since we have

$$|(\varphi, f)_0| \leq \|\varphi\|_0 \|f\|_0 \leq \|f\|_0 \|\varphi\|_{\psi,\frac{m}{2}}.$$

By part A of Theorem 2.2.14 we know that we can extend $B(u, v) = (q(x, D)u, v)_0$ onto $H^{\psi,\frac{m}{2}}(\mathbb{R}^n)$ and it satisfies inequality (2.2.4). Part B of Theorem 2.2.14 gives a the Gårding inequality

$$\text{Re } B_\lambda(u, u) \geq \frac{\gamma_0}{2} \|u\|^2_{\psi,\frac{m}{2}} - \lambda_0 \|u\|_0^2.$$

Which implies

$$\text{Re } B(u, u) \geq \frac{\gamma_0}{2} \|u\|^2_{\psi,\frac{m}{2}}$$

for $\lambda \geq \lambda_0$. This implies (2.2.5) since $|B(u, u)| \geq \text{Re } B(u, u)$. Hence the conditions of the Lax-Milgram Theorem 2.2.6 are satisfied and it gives us the existence of a unique solution. ■

The final step is to 'increase regularity' which means that we want to show that a variational solution $u \in H^{\psi,\frac{m}{2}}$ already belongs to $H^{\psi,m+s}$ if $f \in H^{\psi,s}$. Then we can conclude that $q_\lambda(x, D)u(x) = f(x)$ for all $x \in \mathbb{R}^n$, since the solution is unique and our operator maps $H^{\psi,m+s}$ to $H^{\psi,s}$ by Theorem 2.2.7.
To proceed, we need the following main result of Hoh's symbolic calculus as well as the Friedrichs mollifier, which is a smoothing technique.

**Theorem 2.2.18.** *Let $\psi \in \Lambda$.*

*A)* *If $q_1 \in S_0^{m_1,\psi}(\mathbb{R}^n)$ and $q_2 \in S_0^{m_2,\psi}(\mathbb{R}^n)$, then $q_1(x,D) \circ q_2(x,D) \in \Psi^{m_1+m_2,\psi}$.*

*B)* *For $q_1 \in S_\rho^{m_1,\psi}(\mathbb{R}^n)$ and $q_2 \in S_0^{m_2,\psi}(\mathbb{R}^n)$ the symbol $q$ of the operator $q(x,D) := q_1(x,D) \circ q_2(x,D)$ is given by*

$$q(x,\xi) = q_1(x,\xi) \cdot q_2(x,\xi) + \sum_{j=1}^{n} \partial_{\xi_j} q_1(x,\xi) D_{x_j} q_2(x,\xi) + q_{r_1}(x,\xi)$$

*with $q_{r_1} \in S_0^{m_1+m_2-2,\psi}(\mathbb{R}^n)$.*

*C)* *For any $q \in S_0^{m,\psi}(\mathbb{R}^n)$ there exists $q^* \in S_0^{m,\psi}(\mathbb{R}^n)$, such that*

$$(q(x,D)u, v)_0 = (u, q^*(x,D)v)_0$$

*holds for all $u, v \in S(\mathbb{R}^n)$.*

*D)* *For $q \in S_\rho^{m,\psi}(\mathbb{R}^n)$ the symbol of $q^*(x,D)$ is given by*

$$q^*(x,\xi) = \overline{q(x,\xi)} + \sum_{j=1}^{n} \partial_{\xi_j} D_{x_j} q(x,\xi) + q_{r_2}(x,\xi)$$

*with $q_{r_2} \in S_0^{m-2,\psi}(\mathbb{R}^n)$.*

A proof can be found in [18] Corollary 3.5, 3.6 and 3.11.

**Remark 2.2.19.** *i) In Corollary 3.11 in [18] it is stated that (for our case B)) $q_2$ has to come from class $S_\rho^{m,\psi}$ but for the proof given therein it is sufficient that $q_2 \in S_0^{m,\psi}$.*

*ii) The number of terms in the expansion and their orders depend on the choice of $\rho(|\alpha|) = |\alpha| \wedge 2$. For $\rho(|\alpha|) = |\alpha| \wedge j$ ($j \in \mathbb{N}, j \geq 2$) we get expansions up to order $j$. This statement corresponds to Remark 3.12 in [18]*

*iii) For pseudo-differential operators $\psi_j(D)$, $j = 1,2$ with continuous negative definite symbols of the form $\psi_j(\xi)$ (i.e. not $x$ dependent) we have that*

- *$\psi_1(D) \circ \psi_2(D)$ has the symbol $\psi_1(\xi)\psi_2(\xi)$;*
- *$\psi^*(D)$ has the symbol $\overline{\psi(\xi)}$.*

In chapter 3 we will see the importance of this remark.
Now we prove a lower bound for the operators.

**Theorem 2.2.20.** *Let $q \in S_\rho^{m,\psi}(\mathbb{R}^n)$ and assume (2.2.7) and (2.2.8). Then for $s > -m$ we have*

$$\frac{\gamma_0^2}{4}\|u\|^2_{\psi,m+s} \le \|q(x,D)u\|^2_{\psi,s} + c\|u\|^2_{\psi,m+s-\frac{1}{2}}$$

*for all $u \in H^{\psi,s+m}(\mathbb{R}^n)$.*

**Proof.** This is a modification of the proof of Theorem 2.5.8 in [22], see also W. Hoh [18], to suit complex valued symbols.
We set

$$r_s(x,\xi) := q(x,\xi)\overline{q(x,\xi)}(1+\psi(\xi))^s$$

and observe that

$$\text{Re } r_s(x,\xi) \ge \gamma_0^2(1+\psi(\xi))^{m+s} \text{ for } |\xi| \ge R.$$

From Theorem 2.2.18 part D) we know that the leading term in the expansion of the symbol $q^*(x,\xi)$ is given by $\overline{q(x,\xi)}$. Thus we get

$$\begin{aligned}\|q(x,D)u\|^2_{\psi,s} &= \left((1+\psi(D))^{\frac{s}{2}}q(x,D)u, (1+\psi(D))^{\frac{s}{2}}q(x,D)u\right)_0 \\ &= (q^*(x,D)(1+\psi(D))^s q(x,D)u, u)_0 \\ &= \text{Re } (r_s(x,D)u,u)_0 + \text{Re } (\tilde{r}(x,D)u,u)_0\end{aligned}$$

with $\tilde{r}(x,D) \in \Psi_0^{2(s+m)-1,\psi}(\mathbb{R}^n)$. By the same method we used in the proof of Theorem 2.2.14 A) we get that

$$|\text{Re } (\tilde{r}(x,D)u,u)_0| \le c\|u\|^2_{\psi,s+m-\frac{1}{2}}$$

Applying Theorem 2.2.14 in the form of Remark 2.2.15 we get

$$\|q(x,D)u\|^2_{\psi,s} \ge \frac{\gamma_0^2}{4}\|u\|^2_{\psi,m+s} - c\|u\|^2_{\psi,m+s-\frac{1}{2}} - c'\|u\|^2_{\psi,m+s-\frac{1}{2}}.$$

■

The following is quoted for completeness from [22] and [18]. The proofs do not need explicit modifications to suit complex valued symbols.

**Definition 2.2.21 (Friedrichs mollifier).** *Let $j : \mathbb{R}^n \to \mathbb{R}$ be defined by*

$$j(x) := \begin{cases} c_0 e^{(|x|^2-1)^{-1}} & , |x| < 1 \\ 0 & , |x| \ge 1 \end{cases}$$

*where* $c_0^{-1} = \int_{|x|<1} e^{(|x|^2-1)^{-1}}\, dx$. *Now we set for* $\varepsilon > 0$

$$j_\varepsilon(x) := \varepsilon^{-n} j(\frac{x}{\varepsilon}).$$

*Then the Friedrichs mollifier is given by*

$$J_\varepsilon(u)(x) = (j_\varepsilon * u)(x) = (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} \hat{j}(\varepsilon\xi)\hat{u}(\xi)\, d\xi.$$

**Proposition 2.2.22.** *Let* $J_\varepsilon$ *be defined as above. For any* $s \geq 0$ *and* $u \in H^{\psi,s}(\mathbb{R}^n)$ *we have*

$$J_\varepsilon(u) \in \bigcap_{t\geq 0} H^{\psi,t}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$$

*and*

$$\|J_\varepsilon\|_{\psi,s} \leq \|u\|_{\psi,s}.$$

*In addition, if for* $\varepsilon \in (0,\vartheta)$, $\vartheta > 0$, *we have for some* $u \in L^2(\mathbb{R}^n)$

$$\|J_\varepsilon(u)\|_{\psi,s} \leq c_{u,s}$$

*with a constant independent of* $\varepsilon$, *it follows that* $u \in H^{\psi,s}(\mathbb{R}^n)$.

For a proof see Proposition 2.3.15 in [22].
In the following $[.,.]$ denotes the commutator, i.e. $[A, B] = AB - BA$.

**Theorem 2.2.23.** *For* $s \in S_\rho^{m,\psi}(\mathbb{R}^n)$ *and* $s \in \mathbb{R}$ *there is a constant* $c$ *independent of* $\varepsilon$, $0 < \varepsilon \leq 1$, *such that*

$$\|[q(x,D), J_\varepsilon]u\|_{\psi,s} \leq c\|u\|_{\psi,m+s-1}$$

*holds for all* $u \in H^{\psi,m-1+s}(\mathbb{R}^n)$.

For a proof see Theorem 2.5.11 in [22].

**Theorem 2.2.24.** *Let* $\psi \in \Lambda$ *satisfying* (2.2.8) *and* $q \in S_\rho^{m,\psi}(\mathbb{R}^n)$ *satisfying* (2.2.7), $m \geq 1$. *Then for* $f \in H^{\psi,s}(\mathbb{R}^n)$, $s \geq 0$, *any variational solution* $u \in H^{\psi,\frac{m}{2}}(\mathbb{R}^n)$ *to* (2.2.10) *belongs to* $H^{\psi,m+s}(\mathbb{R}^n)$.

**Proof.** This proof is the same as the proof of Theorem 2.5.13 in [22] only the references have to point to the modified theorems above. ■

Therefore we get the regularity and conclude all of the above in the following theorem.

**Theorem 2.2.25.** *Let $\psi \in \Lambda$ and suppose*

$$\psi(\xi) \geq c_0 |\xi|^{r_0}$$

*holds for some $c_0 > 0$ and $r_0 > 0$. If $q(x,\xi)$ is a continuous negative definite symbol belonging to $S_\rho^{2,\psi}(\mathbb{R}^n)$ and satisfies*

$$\text{Re } q(x,\xi) \geq \delta(1 + \psi(\xi))$$

*for some $\delta > 0$ and all $\xi \in \mathbb{R}^n$, $|\xi|$ sufficiently large, then $-q(x, D)$ defined on $C_0^\infty(\mathbb{R}^n;\mathbb{R})$ is closable in $C_\infty(\mathbb{R}^n;\mathbb{R})$ and its closure is a generator of a Feller semigroup.*

**Proof.** The proof is the same as the proof of Theorem 2.6.9 in [22] only the references have to point to the modified Theorems above. ■

# Chapter 3

# Approximation of the semigroup

In order to get an approximation for the Feller semigroup generated by $q \in S_\rho^{m,\psi}$ we use a result by Kumano-go Chapter 7 §4 in [24]. Therein the result is given for pseudo-differential operators in the class $S_{\rho,\delta}^m$. We show that it is also applicable to Hoh's class of operators.
To increase the readability we use in this chapter the notation $S_\rho^{m,\psi}$ instead of $S_\rho^{m,\psi}(\mathbb{R}^n)$.

We approximate the symbol of the semigroup $T_t = e^{-tq(x,D)}$, using the fact that this operator gives a solution to the equation

$$\frac{du}{dt} + q(x,D)u = 0.$$

Formally we have the following initial-value problem:

$$Lu(t) := \frac{\partial u}{\partial t} + q(x,D)u = f \text{ in } (0,T)\ (T > 0), \tag{3.0.1}$$

$$\lim_{t \downarrow 0} u(.,t) = u_0 \text{ in } L^2(\mathbb{R}^n),$$

for which we are going to construct a fundamental solution.

**Definition 3.0.26 (fundamental solution).** *We call an operator $U(t,s;x,D_x)$ defined on $L^2(\mathbb{R}^n) \times C([0,T])$ a fundamental solution to*

$$\frac{\partial u}{\partial t}(x,t) + q(x,D)u(x,t) = 0 \tag{3.0.2}$$

*if for fixed $s$ $(0 \leq s < t \leq T)$ and $g \in L^2(\mathbb{R}^n) \times C([0,T])$*

*i)* $u(x,t) = U(t,s;x,D_x)g(x,t)$ *solves* (3.0.2),

*ii)* $\lim_{t \downarrow 0} U(t,0;x,D_x)g(x,t) = g(x,0).$

We need a little modification to Hoh's Calculus, which is possible as mentioned in Remark 2.2.19. We use instead of $\rho(|\alpha|) = |\alpha| \wedge 2$ the functions

$$\rho_j(|\alpha|) = |\alpha| \wedge j \quad , \text{ for } j = 1, 2$$

as 'order cutoff' for the symbol classes.

For the construction we need that our symbol is elliptic.

**Definition 3.0.27 (elliptic symbol).** *We call a symbol $q \in S^{m,\psi}_{\rho_j}$ elliptic if there exists $c > 0$ such that*

$$\text{Re } q(x,\xi) \geq c(1+\psi(\xi))^{\frac{m}{2}} \textit{ for all } x, \xi \in \mathbb{R}^n. \tag{3.0.3}$$

Note that for $q \in S^{m,\psi}_{\rho_j}$ and $q$ elliptic it follows that for all $\alpha, \beta$ there exist constants $c_{\alpha,\beta}$ such that

$$\left| \frac{\partial_x^\beta \partial_\xi^\alpha q(x,\xi)}{\text{Re } q(x,\xi)} \right| \leq c_{\alpha,\beta}(1+\psi(\xi))^{\frac{-\rho_j(|\alpha|)}{2}}. \tag{3.0.4}$$

**Remark 3.0.28.** *If* (3.0.3) *is only satisfied for large $|\xi|$ then there exists a $\lambda$ such that $q+\lambda$ is elliptic. And we can apply the following results to $q+\lambda$ instead of $q$. Especially for the symbols corresponding to the Meixner-type process and the Normal Inverse Gaussian-type process we have to use $q+1$.*

To construct the solution to (3.0.1) we need some estimates. But first note that $S^{m,\psi}_{\rho_j}$ is a Fréchet space with the semi-norms

$$|q|_l^{(m)} := \max_{|\alpha+\beta|\leq l} \sup_{\mathbb{R}^{2n}} \{ |\partial_\xi^\alpha \partial_x^\beta q(x,\xi)| (1+\psi(\xi))^{\frac{-m+\rho_j(|\alpha|)}{2}} \}$$

with $q \in S^{m,\psi}_{\rho_j}$ and $l \in \mathbb{N}_0$.
We call $B \subset S^{m,\psi}_{\rho_j}$ bounded in $S^{m,\psi}_{\rho_j}$ if

$$\sup_{q\in B} \{ |q|_l^{(m)} \} < \infty \quad \text{for all } l = 0, 1, \ldots$$

and we write $u(t) \in \mathcal{B}_t^m(V)$ if $u$ is $m$-times continuously differentiable in the topology of $V$ with bounded derivatives.
Furthermore we say that $q_k$ converges weakly to $q$ in $S^{m,\psi}_{\rho_j}$ $(k \to \infty)$ if $\{q_j\}$ is bounded in $S^{m,\psi}_{\rho_j}$ and $q_k \to q$ on every compact subset of $\mathbb{R}^{2n}$.

**Lemma 3.0.29.** *Let $q \in S^{m,\psi}_{\rho_2}$ and let* (3.0.3) *be satisfied. Set*

$$e_0(t,s;x,\xi) := e^{-(t-s)q(x,\xi)} \ (0 \leq s \leq t \leq T),$$

$$q_1(t,s;x,\xi) := \sum_{|\alpha|=1} \partial_\xi^\alpha q(x,\xi)\partial_x^\alpha e_0(t,s;x,\xi)$$

*and define $e_1$ by*

$$\left(\frac{d}{dt} + q(x,\xi)\right) e_1(s,t;x,\xi) = -q_1(t,s;x,\xi)$$

*where $e_1(s,t;x,\xi)|_{t=s} = 0$.*
*(Note that this is just an inhomogeneous linear differential equation with continuous coefficients depending on some parameters, therefore $e_1$ is well defined.)*
*Then for all $\alpha, \beta$ there are constants $c_{\alpha,\beta}, c'_{\alpha,\beta} > 0$ such that for $j = 0,1$ we have*

$$|\partial_\xi^\alpha \partial_x^\beta e_j(t,s;x,\xi)| \le \begin{cases} c_{\alpha,\beta}(1+\psi(\xi))^{\frac{-\rho_{2-j}(|\alpha|)-j}{2}} & \text{for } j \ge 0, \\ c'_{\alpha,\beta}(t-s)(1+\psi(\xi))^{\frac{m-\rho_{2-j}(|\alpha|)-j}{2}} & \text{for } j+|\alpha+\beta| \ge 1. \end{cases}$$

**Proof.** There exist $A_k \in \mathbb{R}$ such that

$$s^k e^{-s} \le A_k \ \text{ for all } s \in \mathbb{R}^+. \tag{3.0.5}$$

We have for a sufficiently smooth function $w$ the following formula

$$\partial^\alpha(e^w) = e^w \sum_{\substack{\alpha^1+\ldots+\alpha^l=\alpha \\ l=1,\ldots,|\alpha|}} c_{\{\alpha^k\}} \prod_{j=1}^{l} \partial^{\alpha^j} w$$

see [21] 14pp. Using the above and (3.0.4) we get

$$|\partial_\xi^\alpha \partial_x^\beta e_0(t,s;x,\xi)| \le \begin{cases} c_{\alpha,\beta}(1+\psi(\xi))^{\frac{-\rho_2(|\alpha|)}{2}} \\ c'_{\alpha,\beta}(t-s)(1+\psi(\xi))^{\frac{m-\rho_2(|\alpha|)}{2}} & \text{for } |\alpha+\beta| \ge 1 \end{cases},$$

i.e. $e_0(t,s) \in S_{\rho_2}^{0,\psi}$ and $\{\frac{e_0}{t-s}\}_{0\le s<t\le T}$ is bounded in $S_{\rho_2}^{m,\psi}$. In the following we often write $e_0$ or $e_0(t,s)$ when we mean $e_0(t,s;x,\xi)$, the same applies for $e_1$ and $q_1$.
Note that $e_1$ is given by

$$e_1(t,s;x,\xi) = -\left(\int_s^t \frac{q_1(\tau,s)}{e_0(\tau,s)} d\tau\right) e_0(t,s)$$

and that

$$
\begin{aligned}
|\partial_\xi^\alpha \partial_x^\beta (\frac{q_1}{e_0})| &= |\partial_\xi^\alpha \partial_x^\beta \left( \frac{\sum_{|\gamma|=1} \partial_\xi^\gamma q(x,\xi) \partial_x^\gamma e_0}{e_0} \right) | \\
&= |\partial_\xi^\alpha \partial_x^\beta \left( \sum_{|\gamma|=1} \partial_\xi^\gamma q(x,\xi)(t-s) \partial_x^\gamma q(x,\xi) \right) | \\
&= |\partial_x^\beta \left( \sum_{|\gamma|=1} \sum_{\alpha' \le \alpha} \binom{\alpha}{\alpha'} \partial_\xi^{\gamma+\alpha'} q(x,\xi)(t-s) \partial_\xi^{\alpha-\alpha'} \partial_x^\gamma q(x,\xi) \right) | \\
&\le \sum_{|\gamma|=1} \sum_{\alpha' \le \alpha} \binom{\alpha}{\alpha'} c_{\alpha,\alpha',\beta,\gamma} \text{Re } q(x,\xi)(t-s) \text{Re } q(x,\xi)(1+\psi(\xi))^{\frac{-\rho_2(|\gamma+\alpha'|)-\rho_2(|\alpha-\alpha'|)}{2}} \\
&\le c_{\alpha,\beta} \text{Re } q(x,\xi)(t-s) \text{Re } q(x,\xi)(1+\psi(\xi))^{\frac{-1-\rho_1(|\alpha|)}{2}}
\end{aligned}
$$

holds. Now we can either estimate $(t-s)(\text{Re } q(x,\xi))^2 e_0$ only with (3.0.5) or with (3.0.5) and the fact that $\text{Re } q \in S_{\rho_2}^{m,\psi}$. This gives us the estimates. ■

**Remark 3.0.30.** *The construction and result above yield that $e_j(t,s) \in \mathcal{B}_t^0(S_{\rho_{2-j}}^{-j,\psi})$. We also observe that $q_1 \in \mathcal{B}_t^0(S_0^{m,\psi})$ and it follows that $e_j(t,s) \in \mathcal{B}_t^1(S_0^{m,\psi})$.*

**Lemma 3.0.31.** *Suppose the conditions and definitions of Lemma 3.0.29 hold. Let*

$$U_2(t,s;x,D) := e_0(t,s;x,D) + e_1(t,s;x,D)$$

*and*

$$R_2(t,s;x,D) := \left( \frac{\partial}{\partial t} + q(x,D) \right) U_2(t,s;x,D).$$

*Then*

$$R_2(t,s;x,\xi) \in \mathcal{B}_t^0(S_0^{m-2,\psi}) \tag{3.0.6}$$

*and*

$$\{\frac{R_2(t,s;x,\xi)}{t-s}\}_{0\le s<t\le T} \text{ is a bounded set in } S_0^{2m-2,\psi}. \tag{3.0.7}$$

**Proof.** Using Theorem 2.2.18 we get

$$
\begin{aligned}
&\sigma(q(x,D)e_0(t,s;x,D))(x,\xi) \\
&\qquad = q(x,\xi)e_0(t,s;x,\xi) \\
&\qquad\quad + \sum_{|\gamma|=1} \partial_\xi^\gamma q(x,\xi) \partial_x^\gamma e_0(t,s;x,\xi) + r_{2,0}(t,s;x,\xi)
\end{aligned}
\tag{3.0.8}
$$

and

$$\sigma\left(q(x,D)e_1(t,s;x,D)\right)(x,\xi) = q(x,\xi)e_1(t,s;x,\xi) + r_{2,1}(t,s;x,\xi), \tag{3.0.9}$$

where $r_{2,0}(t,s), r_{2,1}(t,s) \in S_0^{m-2,\psi}$, since $q \in S_{\rho 2}^{m,\psi}$ and $e_j \in S_{\rho 2-j}^{-j,\psi}$.

It follows that

$$\begin{aligned}\sigma\left(R_2(x,D)\right) &= \sigma\left(\left(\frac{\partial}{\partial t} + q(x,D)\right)(e_0(x,D) + e_1(x,D))\right) \\ &= \sum_{i=0}^{1}\left(\frac{\partial}{\partial t} + q(x,\xi)\right)e_i + \sum_{|\gamma|=1} \partial_\xi^\gamma q(x,\xi)\partial_x^\gamma e_0 + \sum_{i=0}^{1} r_{2,i} \\ &= \left(\frac{\partial}{\partial t} + q(x,\xi)\right)e_1 + q_1 + \sum_{i=0}^{1} r_{2,i} \\ &= \sum_{i=0}^{1} r_{2,i},\end{aligned} \tag{3.0.10}$$

which implies (3.0.6). Analogously follows (3.0.7) by using the second estimate of Lemma 3.0.29. ∎

The following lemma and its proof are almost identical with Kumano-go [24], Lemma 4.5.

**Lemma 3.0.32.** *Let $m \leq 2$. We use $R_2(t,s)$ of Lemma 3.0.31 to define the sequence $\{W_\nu(t,s;x,D)\}_{\nu=1}^{\infty}$ inductively by*

$$W_1(t,s;x,D) := -R_2(t,s;x,D),$$

$$W_\nu(t,s;x,D) := \int_s^t W_1(t,s_1;x,D)W_{\nu-1}(s_1,s;x,D)\; ds_1.$$

*Then*

$$\sum_{\nu=1}^{l} W_\nu(t,s) = -R_2(t,s) - \int_s^t R_2(t,\tau)\sum_{\nu=1}^{l-1} W_\nu(\tau,s)\; d\tau \tag{3.0.11}$$

*holds and for any $\alpha, \beta$ there exist constants $A_{\alpha,\beta}, A'_{\alpha,\beta}$ such that*

$$|\partial_\xi^\alpha \partial_x^\beta W_\nu(t,s;x,\xi)| \leq \begin{cases} (A_{\alpha,\beta})^\nu \frac{(t-s)^{\nu-1}}{(\nu-1)!}(1+\psi(\xi))^{m-2} \\ (A'_{\alpha,\beta})^\nu \frac{(t-s)^{\nu-1}}{(\nu-1)!}(t-s)(1+\psi(\xi))^{2m-2} \end{cases}. \tag{3.0.12}$$

**Proof.** Equation (3.0.11) follows from the Definition. Now note that for $\nu \geq 2$ we have

$$W_\nu(t,s) = \int_s^t \int_s^{s_1} \cdots \int_s^{s_{\nu-2}} W_1(t,s_1)W_1(s_1,s_2)\cdots W_1(s_{\nu-1},s)\ ds_{\nu-1}\cdots ds_2 ds_1,$$

with $W_1(t,s) \in S_0^{m-2,\psi} \subset S_0^{0,\psi}$ and $\{\frac{W_1(t,s)}{t-s}\}$ is bounded in $S_0^{2m-2,\psi}$. Therefore we get with Theorem 2.2.18 that $W_1(t,s_1)W_1(s_1,s_2)\cdots W_1(s_{\nu-1},s) \in S_0^{m-2,\psi}$ and $\{\frac{W_1(t,s_1)W_1(s_1,s_2)\cdots W_1(s_{\nu-1},s)}{t-s}\}$ is bounded in $S_0^{2m-2,\psi}$, i.e.

$$\begin{aligned} &|\partial_\xi^\alpha \partial_x^\beta \sigma(W_1(t,s_1)W_1(s_1,s_2)\cdots W_1(s_{\nu-1},s))(x,\xi)| \\ &\qquad \leq \begin{cases} (A_{1,\alpha,\beta})^\nu (1+\psi(\xi))^{m-2} \\ (A'_{1,\alpha,\beta})^\nu (t-s)(1+\psi(\xi))^{2m-2} \end{cases}, \end{aligned} \tag{3.0.13}$$

(3.0.12) follows. ■

**Theorem 3.0.33.** *Let $q \in S_\rho^{m,\psi}$ $(0 < m \leq 2)$ satisfy (3.0.3). Then there exists a fundamental solution $U(t,s;x,D)$ to (3.0.1). In addition we have:*

*i)* $U(t,s;x,\xi) \in \mathcal{B}_t^0(S_0^{0,\psi}) \cap \mathcal{B}_t^1(S_0^{m,\psi})$

*ii) The symbol $U(t,s;x,\xi)$ satisfies $U(t,s;x,\xi) \to 1$ in $S_0^{0,\psi}$ weakly $(t \downarrow s)$.*

*iii) Writing $U$ as*

$$U(t,s;x,\xi) = e^{-(t-s)q(x,\xi)} + r_0(t,s;x,\xi),$$

*the symbol $r_0(t,s;x,\xi)$ satisfies*

$$r_0(t,s;x,\xi) \to 0 \text{ in } S_0^{max\{-1,m-2\},\psi} \text{ weakly } (t \downarrow s)$$

*and*

$$\{\frac{r_0(t,s;x,\xi)}{t-s}\}_{0 \leq s < t \leq T} \text{ is a bounded set in } S_0^{m-1,\psi}.$$

**Proof.** By Lemma 3.0.32 we see that $W(t,s) = \sum_{\nu=1}^\infty W_\nu(t,s)$ converges in the topology of $\mathcal{B}_t^0(S_0^{m-2,\psi})$. Using $U_2(t,s)$ of Lemma 3.0.31 we set

$$U(t,s) = U_2(t,s) + \int_s^t U_2(t,\tau)W(\tau,s)\ d\tau. \tag{3.0.14}$$

Now we get

$$\begin{aligned} LU(t,s) &= LU_2(t,s) + W(s,t) + \int_s^t LU_2(t,\tau)W(\tau,s)\ d\tau \\ &= R_2(t,s) + W(t,s) + \int_s^t R_2(t,\tau)W(r,s)\ d\tau. \end{aligned} \tag{3.0.15}$$

For $l \to \infty$ in (3.0.11) we get

$$W(t,s) = -R_2(t,s) - \int_s^t R_2(t,\tau)W(\tau,s)\ d\tau.$$

Putting the two equations above together we get $LU(t,s) = 0$.
If we write (3.0.14) as

$$U(t,s) = e_0(t,s) + e_1(t,s) + \int_s^t U_2(t,\tau)W(\tau,s)\ d\tau$$

and note that

$$\begin{aligned} W(t,x) &\in \mathcal{B}_t^0(S_0^{m-2,\psi}) \subset \mathcal{B}_t^0(S_0^{0,\psi}), \\ e_j &\in \mathcal{B}_t^0(S_{\rho_{2-j}}^{-j,\psi}), \\ e_0 &= e^{-(t-s)q(x,\xi)}, \end{aligned}$$

also recall Remark 3.0.30 and the fact that

$$\left\{ \frac{e_1(t,s)}{t-s} \right\}_{0\leq s<t\leq T} \quad \text{is bounded in } S_{\rho_1}^{m-1,\psi},$$

then *i)*, *ii)* and *iii)* follow. ∎

Finally we get a solution to the initial value problem.

**Corollary 3.0.34.** *For* $u_0 \in L_2$ *and* $f \in L^2(\mathbb{R}^n) \times C([0,T])$ *the solution to* (3.0.1) *is given by*

$$u(t) = U(t,0)u_0 + \int_0^t U(t,\tau)f(\tau)\ d\tau.$$

# Chapter 4

# Extending some Lévy processes to Lévy-type processes

In this chapter we show how the previous results can be used to extend a Lévy process to a new process which is still a Feller process but not anymore a Lévy process. We call the new process a Lévy-type process.

Recall that a Lévy process $(X_t)_{t\geq 0}$ is completely characterised by the equation

$$\mathbb{E}(e^{i\xi X_t}) = e^{-t\psi(\xi)}$$

where $\psi(\xi)$ is the characteristic exponent of $X_t$.

In practical modeling, compare with section 1.4, the characteristic exponent often depends on parameters, i.e.

$$\psi(\xi) = \psi^{a,b,c,\dots}(\xi).$$

Following the ideas of Barndorff-Nielsen and Levendorskiĭ [5] we can define

$$q(x,\xi) := \psi^{a(x),b(x),c(x),\dots}(\xi)$$

for some functions $a, b, c, \dots$. We say, *the parameters become state space dependent.* For a sufficient choice of the functions $a, b, c, \dots$ we may have that $q \in S_\rho^{2,\psi}$. In this case we can apply the theory from the second chapter to get a Feller semigroup. The corresponding process is called Lévy-type process. The results of chapter 3 can be used to get an approximation of the transition probability of the Feller process.

If $(T_t)_{t\geq 0}$ is a Feller semigroup then we have

$$T_t u(x) = \left(e^{-tq(x,D)}\right) u(x) = \mathbb{E}^x(u(X_t)).$$

This equation is essentially the link between semigroups, pseudo-differential operators and the corresponding process, i.e. between analysis and probability theory.

In particular we get the transition probability for starting at 0 in $x$ and being at $t$ in $A$ by

$$p_t(x,A) = T_t\chi_A(x) \text{ where } \chi_A(x) = \begin{cases} 1 & ,x \in A \\ 0 & ,x \notin A \end{cases}.$$

We will apply this framework to two examples: The Meixner process and the Normal Inverse Gaussian process.

## 4.1 Meixner-type Processes

In the following we are going to show that the characteristic exponent of the Meixner process with state space dependent parameters can be used as a symbol for a generator of a Feller semigroup. This means we want to show that we can construct an operator corresponding to the Meixner process, which satisfies the conditions of Theorem 2.2.25.

Recall that the Meixner process has the characteristic exponent

$$\psi_M(\xi) = -im\xi + 2s\left(\ln\cosh(\frac{a\xi - ib}{2}) - \ln\cos(\frac{b}{2})\right),$$

where $a > 0, -\pi < b < \pi, s > 0, m \in \mathbb{R}$. We will call this the Meixner exponent and it will be denoted by $\psi_M$.

We are going to show that this exponent is in the classical symbol class $S^1$. Recall that $S^1$ is defined as the class of all symbols $q \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all $\alpha, \beta \in \mathbb{N}^n$ there exists $c_{\alpha,\beta} > 0$ satisfying

$$|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)| \leq c_{\alpha\beta}(1 + |\xi|)^{1-|\alpha|}$$

for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$. For more details see for example section 18 in [19]. In the following we restrict ourselves to one dimension, i.e. $n = 1$.

**Lemma 4.1.1.** *We have*

$$\text{Re } \psi_M(\xi) = -2s\ln\cos(\frac{b}{2}) + s\ln(\cosh^2\frac{a\xi}{2} - \sin^2\frac{b}{2})$$

*and*

$$\text{Im } \psi_M(\xi) = -m\xi + 2s\tan^{-1}(-\tan\frac{b}{2}\tanh\frac{a\xi}{2}).$$

**Proof.** Using

$$\cosh(x+iy) = \cos(y)\cosh(x) + i\sin(y)\sinh(x),$$

$$|\cosh(x+iy)| = \sqrt{\cosh^2 x - \sin^2 y}$$

and the main branch of the complex logarithm, i.e.

$$\ln(x+iy) = \ln|x+iy| + i\tan^{-1}(\frac{y}{x}),$$

the result follows. ∎

**Theorem 4.1.2.** *Let $R = \frac{\ln 8}{a}$. Then*

$$\text{Re } \psi_M(\xi) \geq \frac{as}{2}|\xi|$$

*holds for all $\xi \in \mathbb{R}$, $|\xi| > R$.*

**Proof.** Since $-\pi < b < \pi$ we have $\cos(\frac{b}{2}) \in (0,1]$ which leads to $-2s\ln\cos(\frac{b}{2}) \geq 0$ and $\sin^2\frac{b}{2} \leq 1$. In addition we know that $\cosh x > \frac{1}{2}e^{|x|}$. Hence we get

$$\text{Re } \psi_M(\xi) \geq s\ln(\frac{1}{4}e^{|a\xi|} - 1) \geq \frac{as}{2}|\xi|$$

for $|\xi| > \frac{\ln 8}{a}$.

∎

**Proposition 4.1.3.** *There exists a constant $c > 0$ such that*

$$|\text{Im } \psi_M(\xi)| \leq c(1 + \text{Re } \psi_M(\xi))$$

*for all $\xi \in \mathbb{R}$.*

**Proof.** We have $\tanh x \in (-1,1)$. Therefore

$$-\tan\frac{b}{2}\tanh\frac{a\xi}{2} \in (-\tan\frac{b}{2}, \tan\frac{b}{2}),$$

equivalently we have

$$\tan^{-1}(-\tan\frac{b}{2}\tanh\frac{a\xi}{2}) \in (-\frac{b}{2}, \frac{b}{2}).$$

This leads to

$$|\text{Im } \psi_M(\xi)| \leq |m||\xi| + sb$$

and with the previous Theorem we get

$$|\text{Im } \psi_M(\xi)| \leq \begin{cases} \frac{|m| \ln 8}{a} + sb & \text{if } |\xi| \leq \frac{\ln 8}{a} \\ \text{Re } \psi_M(\xi) \frac{|m| 2}{as} + sb & \text{if } |\xi| > \frac{\ln 8}{a} \end{cases},$$

i.e. with $c = \max\{\frac{|m| \ln 8}{a} + sb, \frac{2|m|}{as}\}$ we get

$$|\text{Im } \psi_M(\xi)| \leq c(1 + \text{Re } \psi_M(\xi)).$$

■

Now we can show that $\psi_M(\xi) \in S^1$. We are going to split the proof into several steps.

**Proposition 4.1.4.** *There exists a constant $c > 0$ such that*

$$|\psi_M(\xi)| \leq c(1 + |\xi|) \tag{4.1.1}$$

*holds for all $\xi \in \mathbb{R}$.*

**Proof.** We have

$$\begin{aligned} \text{Re } \psi_M(\xi) &= -2s \ln \cos(\frac{b}{2}) + s \ln(\cosh^2 \frac{a\xi}{2} - \sin^2 \frac{b}{2}) \\ &\leq -2s \ln \cos(\frac{b}{2}) + s \ln(\cosh^2 \frac{a\xi}{2}) \\ &= -2s \ln \cos(\frac{b}{2}) + s \ln(\left(\frac{e^{\frac{a\xi}{2}} + e^{-\frac{a\xi}{2}}}{2}\right)^2) \\ &\leq -2s \ln \cos(\frac{b}{2}) + s \ln(e^{a|\xi|}) = -2s \ln \cos(\frac{b}{2}) + sa|\xi|. \end{aligned}$$

The result follows with the help of Proposition 4.1.3. ■

**Lemma 4.1.5.** *For $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$*

$$|\tanh(x + iy)| \leq \frac{\sqrt{2}}{1 - \sin^2(y)}$$

*holds for all $x \in \mathbb{R}$.*

**Proof.** We have the formula

$$\tanh(x+iy) = \frac{\sinh x \cosh x}{\cosh^2 x - \sin^2 y} + i\frac{\sin y \cos y}{\cosh^2 x - \sin^2 y}.$$

Additionally we know $|\sinh x| \leq \cosh x$ and $\cosh x \geq 1$ by which we get

$$|\text{Re } \tanh(x+iy)| \leq \frac{1}{1-\sin^2(y)}$$

and

$$|\text{Im } \tanh(x+iy)| \leq \frac{1}{1-\sin^2(y)},$$

giving the result. ■

It follows immediately that

$$|\psi'_M(\xi)| = |-im + sa\tanh(\frac{a\xi - ib}{2})| \leq c \tag{4.1.2}$$

holds.

**Lemma 4.1.6.** *For $y \in (-\frac{\pi}{2}, \frac{\pi}{2}), r \geq 0$ there exists $c > 0$ such that*

$$|\operatorname{sech}^2(x+iy)| \leq c(1+|x|)^{-r}$$

*for all $x \in \mathbb{R}$.*

**Proof.** Note that

$$\cosh(x+iy) = \cos y \sinh x + i \sin y \sinh x.$$

We fix $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$. For $\tilde{r} \in \mathbb{R}$ it follows that

$$\begin{aligned} |\cosh(x+iy)| &= \sqrt{\cosh^2 x - \sin^2 y} \\ &= \sqrt{\cosh x - |\sin y|}\sqrt{\cosh x + |\sin y|} \\ &\geq \sqrt{1-|\sin y|}\sqrt{\cosh x} \\ &\geq \frac{\sqrt{1-|\sin y|}}{\sqrt{2}} e^{\frac{|x|}{2}} \\ &\geq \tilde{c}(1+|x|)^{\tilde{r}}, \end{aligned}$$

for a suitable $\tilde{c} > 0$, since the exponential function growths faster than any power. Therefore

$$|\operatorname{sech}^2(x+iy)| \leq c(1+|\xi|)^{-r}$$

holds where $c = \frac{1}{\tilde{c}}^2$ and $r = 2\tilde{r}$. ■

Now it follows for any fixed $k \in \mathbb{N}$ that

$$|\psi''_M(\xi)| = |\frac{1}{2}a^2 s \operatorname{sech}^2\left(\frac{a\xi - ib}{2}\right)| \leq c(1+|\xi|)^{-k} \tag{4.1.3}$$

holds and we can show that $\psi_M$ is in $S^1$.

**Theorem 4.1.7.** *The Meixner exponent $\psi_M$ is in the symbol class $S^1$, i.e. for all $\alpha \in \mathbb{N}_0$ there exists $c_\alpha > 0$ such that*

$$|\psi_M^{(\alpha)}(\xi)| \leq c_\alpha(1+|\xi|)^{1-|\alpha|}$$

*holds for all $\xi \in \mathbb{R}$.*

**Proof.** By (4.1.1), (4.1.2) and (4.1.3) it is clear that the result holds for $\alpha = 0, 1, 2$.

For $\alpha = 3$ we find

$$\psi_M^{(3)}(\xi) = -\frac{1}{2}a^3 s \operatorname{sech}^2\left(\frac{a\xi}{2} - \sin^2\frac{b}{2}\right)\tanh\left(\frac{a\xi}{2} - \sin^2\frac{b}{2}\right).$$

But this is nothing but

$$\psi_M^{(3)}(\xi) = -\frac{1}{s}\psi'_{M,0}(\xi)\psi''_M(\xi),$$

where $\psi_{M,0}$ denotes $\psi_M$ with zero drift ($m = 0$). By this recursion we get that the $\alpha^{th}$ derivative is a linear combination of the product of powers of $\psi'_{M,0}(\xi)$ and $\psi''_M(\xi)$. Therefore with repeated use of (4.1.2) and (4.1.3) its absolute value is bounded by $c_\alpha(1+|\xi|)^{1-\alpha}$ for a suitable $c_\alpha$. ■

We have just shown that $\psi_M \in S^1$. Next we look at symbols with state space dependent parameter.

**Definition 4.1.8 (Meixner symbol).** *The Meixner symbol is defined by*

$$q_M(x,\xi) = -im(x)\xi + 2s(x)\left(\ln\cosh\left(\frac{a(x)\xi - ib(x)}{2}\right) - \ln\cos\left(\frac{b(x)}{2}\right)\right),$$

*with $a, b, s, m \in C^\infty$ such that for all $x \in \mathbb{R}$ and all $k \in \mathbb{N}_0$*

$$\begin{aligned} 0 \leq & a^{(k)}(x) \leq a_k^+ < \infty \\ \text{and } 0 < a_0^- \leq & a(x) \\ -\pi < b_k^- \leq & b^{(k)}(x) \leq b_k^+ < \pi \\ 0 \leq & s^{(k)}(x) \leq s_k^+ < \infty \\ \text{and } 0 < s_0^- \leq & s(x) \\ |m^{(k)}(x)| \leq & m_k \end{aligned} \tag{4.1.4}$$

*where* $a_k^+, a_0^-, b_k^\pm, s_k^+, s_0^-$ *and* $m_k$ *are real constants. This symbol is denoted by* $q_M(x,\xi)$.

We are going to show that $q_M$ is in $S^1$. The proof splits into several steps. We already know that for all $\alpha \in \mathbb{N}_0$ there exists a $c_\alpha > 0$ such that

$$|\partial_\xi^\alpha q_M(x,\xi)| \le c_\alpha (1+|\xi|)^{1-\alpha}$$

holds for all $x \in \mathbb{R}$, $\xi \in \mathbb{R}$.

**Proposition 4.1.9.** *For* $\beta \in \mathbb{N}_0$ *there exists* $c_{0,\beta} > 0$ *such that*

$$|\partial_x^\beta q_M(x,\xi)| \le c_{0,\beta}(1+|\xi|)^1$$

*holds for all* $x \in \mathbb{R}$ *and* $\xi \in \mathbb{R}$.

**Proof.** The case $\beta = 0$ is proved by Theorem 4.1.7.

$\beta = 1$:

$$\begin{aligned}\partial_x^1 q_M(x,\xi) =& -im'(x)\xi + 2s'(x)\left(\ln\cosh(\frac{a(x)\xi - ib(x)}{2}) - \ln\cos(\frac{b(x)}{2})\right)\\ &+ s(x)\tan(\frac{b(x)}{2})\frac{b'(x)}{2}\\ &+ 2s(x)\tanh(\frac{a(x)\xi - ib(x)}{2})(\frac{a'(x)\xi - ib'(x)}{2})\end{aligned}$$

The first part of the right hand side is a Meixner exponent $\tilde{\psi}_M$ (just with different parameters) and therefore it is estimated by $c(1+|\xi|)$ with the help of (4.1.1), the second part is bounded by a constant by the choice of our parameter (and taking the maximum) and the third part is by Lemma 4.1.5 (and taking the maximum) also bounded by a constant times $1+|\xi|$.

$\beta = 2$:

$$\begin{aligned}\partial_x^2 q_M(x,\xi) =& -im''(x)\xi + 2s''(x)\left(\ln\cosh(\frac{a(x)\xi - ib(x)}{2}) - \ln\cos(\frac{b(x)}{2})\right)\\ &+ 2s'(x)\tan(\frac{b(x)}{2})b'(x)\\ &+ 2s'(x)\tanh(\frac{a(x)\xi - ib(x)}{2})(\frac{a'(x)\xi - ib'(x)}{2})\\ &+ 2s(x)\tan(\frac{b(x)}{2})\frac{b''(x)}{2} + s(x)\sec^2(\frac{b(x)}{2})\frac{b'^2(x)}{2}\\ &+ 2s(x)\tanh(\frac{a(x)\xi - ib(x)}{2})(\frac{a''(x)\xi - ib''(x)}{2})\\ &+ 2s(x)\operatorname{sech}^2(\frac{a(x)\xi - ib(x)}{2})(\frac{a'(x)\xi - ib'(x)}{2})^2\end{aligned}$$

The terms in the first three lines can be estimated by $c(1 + |\xi|)$ using the case $\beta = 1$, the term in the next line is less than a constant by taking the maximum and the following term is again smaller than a constant times $1+|\xi|$ by Lemma 4.1.5 (and taking the maximum). For the last term we use Lemma 4.1.6 to get the result.

For $\beta \geq 3$ we can apply a recursion argument:
We have

$$(\operatorname{sech}^2 z)' = - \operatorname{sech}^2 z \tanh z.$$

It follows that $\partial_x^\beta q_M(x, \xi)$ is the sum of terms as in the case $\beta = 2$ plus derivatives of the form "$\operatorname{sech}^2(z) z^n$" but these can be estimated, with the help of Lemma 4.1.6 (and taking the maximum) by a constant. Therefore the result follows. ■

**Proposition 4.1.10.** *For all $\beta \in \mathbb{N}_0$ there exists $c_{1,\beta} > 0$ such that*

$$|\partial_\xi^1 \partial_x^\beta q_M(x, \xi)| \leq c_{1,\beta}$$

*holds for all $x \in \mathbb{R}$ and $\xi \in \mathbb{R}$.*

**Proof.** Recall

$$\partial_\xi^1 q_M(x, \xi) = -im(x) + s(x)a(x) \tanh(\frac{a(x)\xi - ib(x)}{2}).$$

For $\beta = 0$ the result follows by Theorem 4.1.7.

$\beta = 1$:

$$\begin{aligned} \partial_\xi^1 \partial_x^1 q_M(x, \xi) = & - im'(x) + s'(x)a(x) \tanh(\frac{a(x)\xi - ib(x)}{2}) \\ & + s(x)a'(x) \tanh(\frac{a(x)\xi - ib(x)}{2}) \\ & + s(x)a(x) \operatorname{sech}^2(\frac{a(x)\xi - ib(x)}{2})(\frac{a'(x)\xi - ib'(x)}{2}) \end{aligned}$$

The right hand side is less than a constant if we use Lemma 4.1.5 and 4.1.6 (and take the maximum).

The higher derivatives contain again terms of the same type and those of the form "$\operatorname{sech}^2(z) z^n$" which are less than a constant by Lemma 4.1.6. ■

We finally get

**Theorem 4.1.11.** *The Meixner symbol $q_M$ is in the symbol class $S^1$, i.e. for all $\alpha, \beta \in \mathbb{N}$ there exists $c_{\alpha,\beta} > 0$ such that*

$$|\partial_\xi^\alpha \partial_x^\beta q_M(x,\xi)| \leq c_{\alpha,\beta}(1+|\xi|)^{1-\alpha}$$

*holds for all $x \in \mathbb{R}$ and $\xi \in \mathbb{R}$.*

**Proof.** For $\alpha = 0, 1$ this holds by Proposition 4.1.9 and 4.1.10.
For $\alpha = 2$ we have

$$\partial_\xi^2 q_M(x,\xi) = 2a^2(x)s(x)\,\mathrm{sech}^2(\frac{a(x)\xi - ib(x)}{2}).$$

Hence by Lemma 4.1.6 and the same argument as in the previous Propositions we have that for any $r \in \mathbb{R}$ there exists a constant $c_{2,\beta}$ such that

$$|\partial_x^\beta \partial_\xi^2 q_M(x,\xi)| \leq c_{2,\beta}(1+|\xi|)^r$$

holds for all $\beta \in \mathbb{N}_0$ by .

For $\alpha \geq 3$ we use the same recursion argument as in Theorem 4.1.7 and get the desired result. ∎

Now it follows that $-q_M(x,D)$ can be extended to a generator of a Feller semigroup.

**Theorem 4.1.12.** *The pseudo-differential operator corresponding to the symbol*

$$q_M(x,\xi) = -im(x)\xi + 2s(x)\left(\ln\cosh(\frac{a(x)\xi - ib(x)}{2}) - \ln\cos(\frac{b(x)}{2})\right)$$

*with the restrictions* (4.1.4) *defined on $C_0^\infty(\mathbb{R};\mathbb{R})$ is closable in $C_\infty(\mathbb{R};\mathbb{R})$ and its closure is the generator of a Feller semigroup.*

**Proof.** So far all our estimates have been with respect to powers of $|.|$ but this is not in class $\Lambda$. Therefore we use $\psi_{|.|}(\xi) := \sqrt{1+\xi^2} - 1$ which is in $\Lambda$. We have that

$$\frac{1}{\sqrt{2}}(1+|\xi|) \leq 1 + \psi_{|.|}(\xi) \leq (1+|\xi|)$$

holds for all $\xi$ and therefore all the estimates above hold also with respect to $\psi_{|.|}$ (with respectively modified constants).
Obviously for $r_0 = 1$ there exists a $c_0 > 0$

$$\psi_{|.|}(\xi) \geq c_0|\xi|^{r_0}$$

holds for all $\xi \in \mathbb{R}$.
Further $q_M(x,.) : \mathbb{R} \to \mathbb{C}$ is a continuous negative definite function by Theorem 1.2.9. We have by Theorem 4.1.11 that

$$q_M(x,\xi) \in S_\rho^{2,\psi_{|.|}}.$$

By Theorem 4.1.2 and the parameter restrictions it follows that for some $\gamma > 0$ we have

$$\text{Re } q_M(x,\xi) \geq \gamma(1 + \psi_{|.|}(\xi))$$

for all $x \in \mathbb{R}$ and $\xi \in \mathbb{R}$, $|\xi|$ sufficiently large.
Hence every condition of Theorem 2.2.25 is satisfied and the result is proved. ■

## 4.2 Real Meixner-type Processes

The pseudo-differential operator corresponding to the real Meixner process is also the generator of a Feller semigroup.

**Theorem 4.2.1.** *$\psi_{ReM}(\xi)$ is in $S^1$ and the corresponding pseudo-differential operator defined on $C_0^\infty(\mathbb{R};\mathbb{R})$ is closable in $C_\infty(\mathbb{R};\mathbb{R})$ and its closure is the generator of a Feller semigroup.*

**Proof.** This is immediately clear by Theorem 4.1.7 and Theorem 4.1.12. We just have to note that

$$\partial_\xi^k \psi_{ReM}(\xi) = \text{Re } (\partial_\xi^k \psi_M(\xi)) \text{ for all } k \in \mathbb{N}$$

and

$$|\text{Re } z| \leq |z| \text{ for all } z \in \mathbb{C}.$$

■

## 4.3 Normal Inverse Gaussian-type processes

Similar to the Meixner process we can also apply the theory to the Normal Inverse Gaussian process. Especially we show that the corresponding symbol is in $S^1$. This has already been implicitly used in [5] by Barndorff-Nielsen and Levendorskiǐ. But we will give some explicit argument for it. Additionally we will show that the pseudo-differential operator corresponding to the symbol of the Normal Inverse Gaussian process with state space dependent parameters can be extended to a generator of a Feller semigroup.

Recall that the Normal Inverse Gaussian exponent is given by

$$\psi_{NIG}(\xi) = -im\xi + \delta[\sqrt{a^2 - (b+i\xi)^2} - \sqrt{a^2 - b^2}]$$

where $0 \leq |b| < a, \delta > 0, m \in \mathbb{R}$.
Now we are going to show that $\psi_{NIG} \in S^1$. We have the following representations of the real and imaginary part of $\psi_{NIG}$:

$$\begin{aligned}
&\text{Re } \psi_{NIG}(\xi) \\
&= \pm\delta[\frac{\sqrt{2}}{2}\sqrt{\sqrt{(a^2-b^2+\xi^2)^2 + (-\xi 2b)^2} + (a^2-b^2+\xi^2)} - \sqrt{a^2-b^2}] \\
&= \pm\delta[\sqrt[4]{(a^2-b^2+\xi^2)^2 + (-\xi 2b)^2}\cos[\frac{1}{2}\tan^{-1}\left(\frac{-\xi b}{a^2-b^2+\xi^2}\right)] - \sqrt{a^2-b^2}]
\end{aligned}$$

and

$$\begin{aligned}
&\text{Im } \psi_{NIG}(\xi) \\
&= m\xi \pm \text{sgn}((-\xi b))\delta\frac{\sqrt{2}}{2}\sqrt{\sqrt{(a^2-b^2+\xi^2)^2 + (-\xi b)^2} - (a^2-b^2+\xi^2)} \\
&= m\xi \pm \delta\sqrt[4]{(a^2-b^2+\xi^2)^2 + (-\xi b)^2}\sin[\frac{1}{2}\tan^{-1}\left(\frac{-\xi b}{a^2-b^2+\xi^2}\right)].
\end{aligned}$$

**Theorem 4.3.1.** *For all $\xi \in \mathbb{R}$ with $|\xi| > 2a$*

$$\text{Re } \psi_{NIG}(\xi) \geq \frac{\delta}{2}|\xi|$$

*holds.*

**Proof.** Since $0 \leq |b| < a$ we find

$$\begin{aligned}
&\text{Re } \psi_{NIG}(\xi) \\
&= \delta[\frac{\sqrt{2}}{2}\sqrt{\sqrt{(a^2-b^2+\xi^2)^2 + (-\xi 2b)^2} + (a^2-b^2+\xi^2)} - \sqrt{a^2-b^2}] \\
&\geq \delta[\frac{\sqrt{2}}{2}\sqrt{\sqrt{\xi^4 + 4\xi^2 b^2} + \xi^2} - \sqrt{a^2-b^2}] \\
&\geq \delta[\frac{\sqrt{2}}{2}\sqrt{2\xi^2} - \sqrt{a^2}] \\
&= \delta(1 - \frac{a}{|\xi|})|\xi| \\
&\geq \frac{\delta}{2}|\xi|,
\end{aligned}$$

where the last inequality holds for $|\xi| > 2a$. ■

**Proposition 4.3.2.** *There exists a constant $c > 0$ such that*

$$|\text{Im } \psi_{NIG}(\xi)| \leq c(1 + \text{Re } \psi_{NIG}(\xi))$$

*holds for all $\xi \in \mathbb{R}$.*

**Proof.** We find

$$\begin{aligned}&|\text{Im } \psi_{NIG}(\xi)| \\ &\leq |m\xi| + |\delta \frac{1}{\sqrt{2}}| \sqrt{\sqrt{(a^2 - b^2 + \xi^2)^2 + (-\xi b)^2} - (a^2 - b^2 + \xi^2)}.\end{aligned}$$

The previous proposition together with the obvious fact that

$$\sqrt{\sqrt{(a^2 - b^2 + \xi^2)^2 + (-\xi b)^2} - (a^2 - b^2 + \xi^2)} \leq \text{Re } \psi_{NIG}(\xi)$$

for $\xi$ large enough implies the result. ■

**Theorem 4.3.3.** *The Normal Inverse Gaussian exponent $\psi_{NIG}$ is in the symbol class $S^1$.*

**Proof.** Recall that

$$\psi_{NIG}(\xi) = -im\xi + \delta[\sqrt{a^2 - (b + i\xi)^2} - \sqrt{a^2 - b^2}].$$

We have for $h \in \mathbb{N}_0$ the following formula

$$\partial_z^h \sqrt{a^2 - z^2} = \begin{cases} \sum_{k=1}^{\frac{h+1}{2}} c_{h,k} \frac{z^{2k-1}}{(a^2-z^2)^{\frac{h-1+(2k-1)}{2}}} & h \text{ odd,} \\ \sum_{k=1}^{\frac{h+2}{2}} c_{h,k} \frac{z^{2k-2}}{(a^2-z^2)^{\frac{h-1+(2k-2)}{2}}} & h \text{ even.} \end{cases}$$

Now for $h$ odd with $z = b + i\xi$ we find

$$\left| \frac{(b + i\xi)^{2k-1}}{(a^2 - (b + i\xi)^2)^{\frac{h-1+(2k-1)}{2}}} \right| \leq c(1 + |\xi|)^{1-h}$$

for some constant $c$, since

$$|b + i\xi| = \sqrt{b^2 + \xi^2}$$

and

$$|a^2 - (b + i\xi)^2| = \sqrt{(a^2 - b^2 + \xi^2)^2 + 4b^2\xi^2} \geq |b||\xi|^2.$$

For $h$ even it follows analogous. Therefore the series representation yields the result. ■

**Definition 4.3.4 (Normal Inverse Gaussian symbol).** *The Normal Inverse Gaussian symbol is given by*

$$q_{NIG}(x,\xi) = -im(x)\xi + \delta(x)[\sqrt{a(x)^2 - (b(x)+i\xi)^2} - \sqrt{a(x)^2 - b(x)^2}]$$

*with $a, b, \delta, m \in C^\infty(\mathbb{R})$ satisfying the following restrictions: for all $x \in \mathbb{R}$ and $l \in \mathbb{N}_0$*

$$\begin{aligned} 0 \le |b^{(l)}(x)| < a^{(l)}(x) &\le a_l \\ 0 < \delta^{(0)}(x) &\le \delta_0 \\ 0 \le \delta^{(l)}(x) &\le \delta_l \\ m_l^- \le m^{(l)}(x) &\le m_l^+ \end{aligned} \tag{4.3.1}$$

*where $a_l, \delta_l, m_l^{\mp}$ are real constants.*
*We denote the symbol by $q_{NIG}$.*

Now we can show the following Theorem.

**Theorem 4.3.5.** *The Normal Inverse Gaussian symbol as defined in 4.3.4 is an element of the symbol class $S^1$.*

**Proof.** Using the same idea as in Theorem 4.3.3, we have to show that for all $l \in \mathbb{N}_0$ the inequality

$$\left| \partial_x^l \left( \frac{(b(x)+i\xi)^{2k-1}}{(a(x)^2 - (b(x)+i\xi)^2)^{\frac{h-1+(2k-1)}{2}}} \right) \right| \le c_{h,k,l}(1+|\xi|)^{1-h} \tag{4.3.2}$$

holds. Again we just look at the case where $h$ is odd. For $l = 0$ the inequality is shown in Theorem 4.3.3. For $l \ge 0$ we need some more work.

By the Leibniz rule we get

$$\begin{aligned} &\partial_x^l \left( \frac{(b(x)+i\xi)^{2k-1}}{(a(x)^2 - (b(x)+i\xi)^2)^{\frac{h-1+(2k-1)}{2}}} \right) \\ &= \sum_{l' \le l} \binom{l}{l'} \frac{\partial_x^{l'}(b(x)+i\xi)^{2k-1}}{\partial_x^{l-l'}(a(x)^2 - (b(x)+i\xi)^2)^{\frac{h-1+(2k-1)}{2}}}. \end{aligned} \tag{4.3.3}$$

And now we apply the following formula for derivatives of the composition of real and smooth functions $u, v$, (see [21] 14pp.)

$$\begin{aligned} \frac{d^l}{dx^l} u(v(x)) = \sum & \frac{l!}{j_1! \cdot j_2! \cdot \ldots \cdot j_h!} \frac{d^m u(y)}{dy^m}\Big|_{y=v(x)} \\ & \cdot \left(\frac{v'(x)}{1!}\right)^{j_1} \cdot \left(\frac{v''(x)}{2!}\right)^{j_2} \cdot \ldots \cdot \left(\frac{v^{j_h}(x)}{h!}\right)^{j_h} \end{aligned} \tag{4.3.4}$$

where

$$j_0 + 2j_1 + \ldots + hj_h = l$$
$$j_0 + j_1 + \ldots + j_h = m.$$

For the numerator in (4.3.3) we get

$$\partial_x^{l'}(b(x)+i\xi)^{2k-1} = \sum \frac{l'!}{j_1! \cdot j_2! \cdot \ldots \cdot j_h!} \cdot (2k-1)\cdot(2k-2)\cdot\ldots\cdot(2k-1-m)(b(x)+i\xi)^{2k-1-m} \cdot \left(\frac{b'(x)}{1!}\right)^{j_1} \cdot \left(\frac{b''(x)}{2!}\right)^{j_2} \cdot \ldots \cdot \left(\frac{b^{j_h}(x)}{h!}\right)^{j_h}$$

for $2k-1 > m$ otherwise it is 0. Therefore we have

$$|\partial_x^{l'}(b(x)+i\xi)^{2k-1}| \leq c(1+|\xi|)^{2k-1}$$

if $|b(x)+i\xi| > 1$, i.e. for large $\xi$. But we can take a suitable constant such that the inequality holds for for all $\xi$.
For the denominator we use again (4.3.4) to get

$$\partial_x^{l-l'}(a(x)^2-(b(x)+i\xi)^2)^{\frac{h-1+(2k-1)}{2}} = \sum \frac{(l-l')!}{j_1!\cdot j_2! \cdot \ldots \cdot j_h!} \frac{d^m}{dy^m}(y^{\frac{h-1+(2k-1)}{2}})\big|_{a(x)^2-(b(x)+i\xi)^2} \cdot \left(\frac{\partial_x^1(a(x)^2-(b(x)+i\xi)^2)}{1!}\right)^{j_1} \cdot \left(\frac{\partial_x^2(a(x)^2-(b(x)+i\xi)^2)}{2!}\right)^{j_2} \cdot \ldots \cdot \left(\frac{\partial_x^{j_h}(a(x)^2-(b(x)+i\xi)^2)}{h!}\right)^{j_h}.$$

Now note for $j \geq 1$ we get

$$\partial_x^j(a(x)^2-(b(x)+i\xi)^2) = \partial_x^j(a(x)^2-b^2(x)) - 2ib^{(j)}(x)\xi.$$

Therefore with 4.3.1 and the calculation for the numerator we get

$$|\partial_x^{l-l'}(a(x)^2-(b(x)+i\xi)^2)^{\frac{h-1+(2k-1)}{2}}| \geq |\xi|^{h-1+(2k-1)}.$$

This finally leads us to (4.3.2) and the result follows. ■

Analog to Theorem 4.1.12 we have

**Theorem 4.3.6.** *The pseudo-differential operator corresponding to the symbol*

$$q_{NIG}(x,\xi) = -im(x)\xi + \delta(x)[\sqrt{a(x)^2-(b(x)+i\xi)^2} - \sqrt{a(x)^2-b(x)^2}]$$

*with the restrictions* (4.3.1) *defined on* $C_0^\infty(\mathbb{R};\mathbb{R})$ *is closable in* $C_\infty(\mathbb{R};\mathbb{R})$ *and its closure is the generator of a Feller semigroup.*

**Proof.** We have that $\psi_{|.|}(\xi) := \sqrt{1+\xi^2}-1$ is in $\Lambda$ and obviously for $r_0 = 1$ there exists a $c_0 > 0$

$$\psi_{|.|}(\xi) \geq c_0|\xi|^{r_0}$$

holds for all $\xi \in \mathbb{R}$.
$q_{NIG}(x,.) : \mathbb{R} \to \mathbb{C}$ is a continuous negative definite function by Theorem 1.2.9. We have by Theorem 4.3.5 that

$$q_{NIG}(x,\xi) \in S_\rho^{2,\psi_{|.|}}.$$

By Theorem 4.3.1 and the parameter restrictions it follows that for some $\gamma > 0$ we have

$$\text{Re } q_{NIG}(x,\xi) \geq \gamma(1+\psi_{|.|}(\xi))$$

for all $x \in \mathbb{R}$ and $\xi \in \mathbb{R}$, $|\xi|$ sufficiently large.
Hence every condition of Theorem 2.2.25 is satisfied and the result follows. ∎

## 4.4 Further Lévy-type processes

The fundamental properties a continuous negative definite function has to satisfy, in order to be considered as a defining function for a symbol class, are:

a) $\psi(0) = 0$,

b) Re $\psi(\xi) \geq c_2|\xi|^{r_0}$ for large $|\xi|$ and $c_2, r_0 > 0$,

c) $|\text{Im } \psi(\xi)| \leq c_1(1 + \text{Re } \psi(\xi))$ for some $c_1 > 0$.

Property a) is generally trivial. All exponents listed in the table on page 19 satisfy a). We have shown in Theorem 4.3.1 and Proposition 4.3.2 resp. Theorem 4.1.2 and Proposition 4.1.3 that b) and c) hold for the Normal Inverse Gaussian exponent resp. the Meixner exponent. For the CGMY and therefore also for the exponent of the Truncated Lévy processes it certainly possible to show b) and c) for a restricted parameter range. The same seems likely for the General Hyperbolic model and therefore also the Hyperbolic model. The problem in this case is to estimate the Bessel function.

For the variance gamma model we have

$$\psi_V G(\xi) = \frac{1}{\nu}\ln(1 - i\theta\nu\xi + \frac{\sigma^2\nu}{2}\xi^2)$$
$$= \frac{1}{\nu}\ln(|1 - i\theta\nu\xi + \frac{\sigma^2\nu}{2}\xi^2|) + i\frac{1}{\nu}\tan^{-1}\left(\frac{1 + \frac{\sigma^2\nu}{2}\xi^2}{-\theta\nu\xi}\right).$$

Therefore for large $|\xi|$ we get Re $\psi(\xi) \sim c\ln\xi$ and this is not greater than $c|\xi|^{r_0}$ for any $c$ and $r_0$ constant, i.e. the exponent of the variance gamma model does not satisfy the second condition.

# Appendix A

# Path simulation

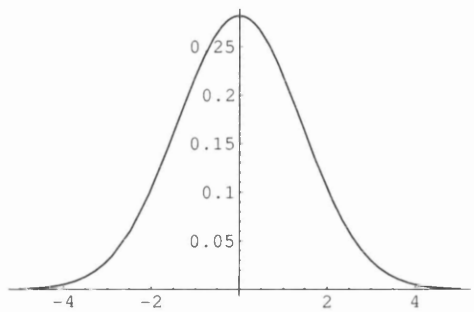
This section is just to give a further feeling for the behavior of a stochastic process. The Simulations are done with Mathematica. And the random variables involved are generated with the help of the Acceptance-Rejection Technique as described in chapter 3 of [14].

## A.1 Brownian Motion

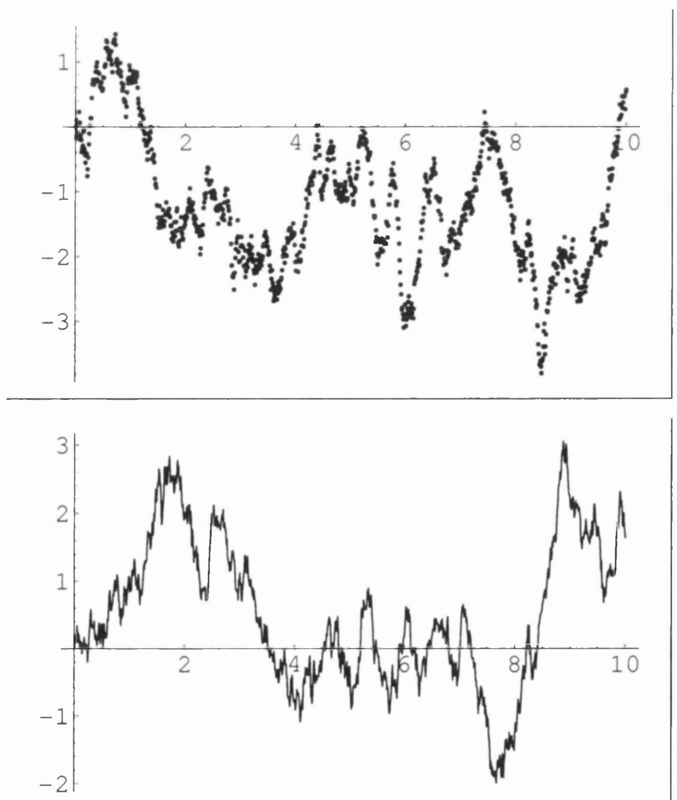
The transition density is given by

$$p_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}}$$

For $t = 1$ it looks like:




Here are two paths of Brownian motion, which we obtained by simulating Brownian motion on $[0, 10]$ as the sum of independent increments with density $p_t(x)$ and step size $t = 0.01$.

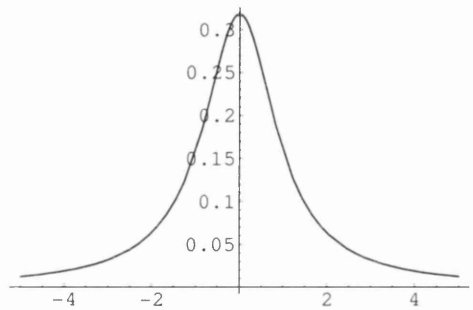


## A.2 Cauchy process

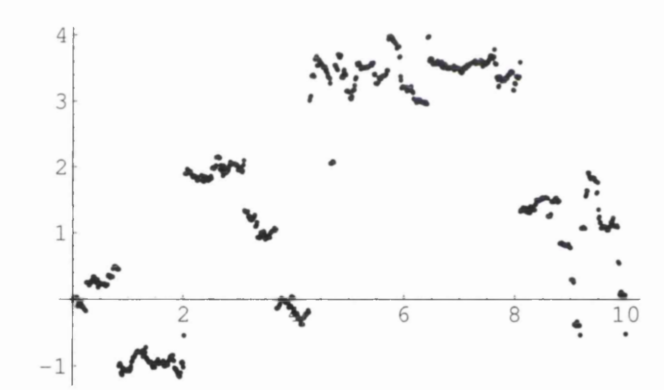
The transition density is given by

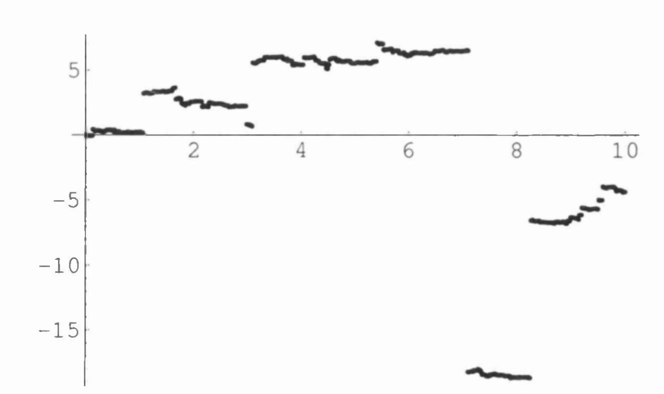
$$p_t(x) = \frac{t}{\pi} \frac{1}{t^2 + x^2}$$

For $t = 1$ it looks like:




Here are two paths of the Cauchy process, which we obtained by simulating the Cauchy process on $[0, 10]$ as the sum of independent increments with density $p_t(x)$ and step size $t = 0.01$.



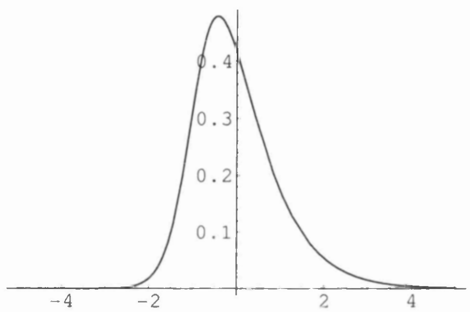



## A.3 Meixner process

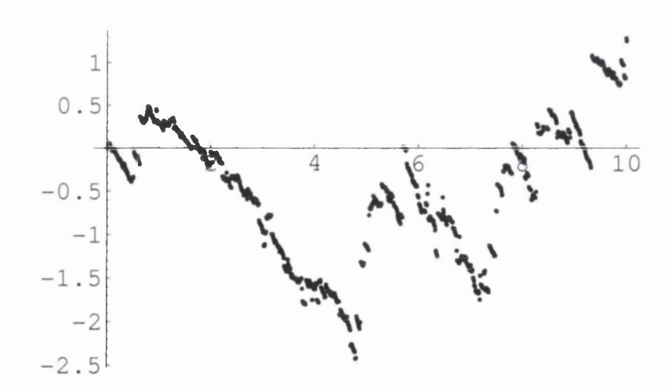
The transition density is given by ($a = 1,\ b = \frac{\pi}{2},\ s = 1,\ m = -as\tan\frac{b}{2}$)

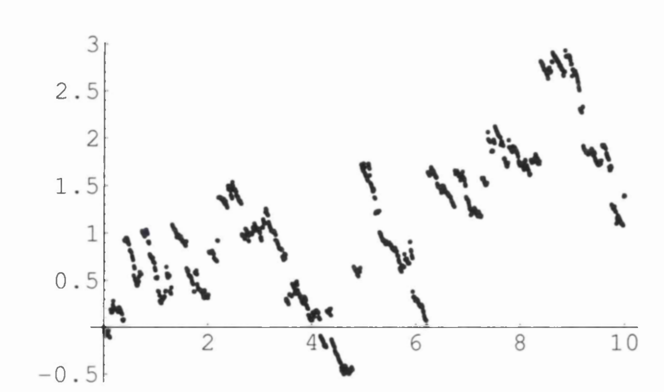
$$p_t(x) = \frac{(2\cos(\frac{\pi}{4}))^{2t}}{2\pi\Gamma(2t)} e^{\frac{\pi}{2}(x+\tan(\frac{\pi}{4})t)} \left|\Gamma(t + i(x + \tan(\frac{\pi}{4})t))\right|^2.$$

For $t = 1$ it looks like:




Here are two paths of the Meixner process, which we obtained by simulating the Meixner process on $[0, 10]$ as the sum of independent increments with density $p_t(x)$ and step size $t = 0.01$.



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