# Orthogonal decompositions for generalized stochastic processes with independent values.

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# Orthogonal Decompositions for Generalized Stochastic Processes with Independent Values

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## Abstract

Among all stochastic processes with independent increments, essentially only Brownian motion and Poisson process have a chaotic representation property. In the case of a Lévy process, several approaches have been proposed in order to construct an orthogonal decomposition of the corresponding $L^2$-space. In this dissertation, we deal with orthogonal (chaotic) decompositions for generalized processes with independent values. We do not suppose stationarity of the process, as a result the Lévy measure of the process depends on points of the space. We first construct, in Chapter 3, a unitary isomorphism between a certain symmetric Fock space and the space $L^2(\mathcal{D}', \mu)$. Here $\mathcal{D}'$ is a co-nuclear space of generalized functions (distributions), and $\mu$ is a generalized stochastic process with independent values. This isomorphism is constructed by employing the projection spectral theorem for an (uncountable) family of commuting self-adjoint operators. We next derive, in Chapter 4, a counterpart of the Nualart–Schoutens decomposition for generalized stochastic process with independent values. Our results here extend those in the papers of Nualart–Schoutens and Lytvynov. In Chapter 5, we construct an orthogonal decomposition of the space $L^2(\mathcal{D}', \mu)$ in terms of orthogonal polynomials on $\mathcal{D}'$. We observe a deep relation between this decomposition and the results of the two previous chapters. Finally, in Chapter 6, we briefly discuss the situation of the generalized stochastic processes of Meixner's type.

**DECLARATION**

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

Signed.......................................... Date 04/06/13

**STATEMENT 1**

This dissertation is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended.

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I hereby give my consent for my dissertation, if accepted, to be available for photocopying and for inter-library loan, and for the title and summary to be made available to outside organizations.

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# Contents

# Chapter 1

# Introduction

A chaotic decomposition of the $L^2$-space of functionals of Brownian motion plays a fundamental role in Gaussian analysis, see e.g. [10, 35]. In a parallel way, a chaotic expansion of the $L^2$-space of functionals of Poisson process has also been derived, see e.g. [21,43]. In fact, among all stochastic processes with independent increments, essentially only Brownian motion and Poisson process have a chaotic representation property, see [16]. In fact, for a general stochastic process with independent increments, the space generated by the multiple stochastic integrals with respect to this (centered) process is a proper subspace of the corresponding $L^2$-space. In the case of a Lévy process, several approaches have been proposed in order to construct an orthogonal decomposition of the corresponding $L^2$-space.

The first approach is due to Itô [21] and uses the Itô decomposition of a Lévy process as a Gaussian process and an integral with respect to Poisson random measure over a two-dimensional Euclidean space. As a result, one gets a certain unitary isomorphism between the $L^2$-space and a certain symmetric Fock space. This point of view has been, in particular, used to develop white noise analysis for Lévy processes [17, 18, 33], see also [26–28, 41].

Another approach has been developed by Nualart and Schoutens [36], who used a system of polynomials which are orthogonal with respect to the Lévy measure of the Lévy process, see also [40] and [18].

If one realizes a Lévy process through a probability measure on a space of generalized functions, then, under appropriate assumptions on a Lévy process, polynomials (of infinitely many variables) form a dense subset of the $L^2$-space. Hence following the idea of Skorohod [42], one can orthogonalize these polynomials. A remarkable feature of Lévy processes, proven in [31], is that these orthogonal polynomials can be found explicitly, say in terms of the Nualart-Schoutens decomposition. As a result one can explicitly calculate the scalar product between orthogonal polynomials (see also see also [20]). In fact, it was shown in [31] that there exists a deep relation between the Nualart Schoutens decomposition and orthogonalization of polynomials with respect to a Lévy white noise measure.

Orthogonal polynomials with respect to a Lévy white noise measure have many additional, nice features in the case of a Meixner-type Lévy process, in particular, for Gamma white noise measure, see [23, 24, 30, 31], see also [1, 2, 22, 32, 38] and the references therein.

We should also mention the fundamental paper by Vershik and Tsilevich [44], which proposes an alternative way of constructing an isometry between the $L^2$-space of a Lévy process and the $L^2$-space of a vector-valued Gaussian white noise.

In this dissertation, we deal with generalized processes with independent values, in the sense of [19]. We do not suppose stationarity of the process, as a result the Lévy measure of the process depends on points of the space. It should be noted that majority of the above cited papers, including Itô's fundamental result [21], do assume stationarity. Additionally, many results of the dissertation are new even for Lévy processes. We also mention that, due to our assumptions on the Lévy measures, the corresponding generalized stochastic process is a probability measure on a space of generalized functions whose Laplace transform is analytic in a neighborhood of zero, cf. [25]. It should be however noted that, in this dissertation, we assume that the Lévy measures have an infinite number of points in their support. The case where Lévy measures may have a finite number of points in their support will be treated elsewhere.

The dissertation is organized as follows.

Chapter 2 contains some preliminary information, which is required for our further studies.

In Chapter 3, we employ the projection spectral theorem for an (uncountable) family of commuting self-adjoint operators [10], see also [8, 9, 11, 29, 30, 38] for further applications of this theorem in infinite-dimensional analysis. We construct a certain family of commuting self-adjoint operators $(A(\varphi))_{\varphi\in\mathcal{D}}$ in a symmetric Fock space $\mathcal{F}$. Here $\mathcal{D}$ is the nuclear space of all smooth, compactly supported functions on $\mathbb{R}^d$. We prove that the family $(A(\varphi))_{\varphi\in\mathcal{D}}$ satisfies the assumptions of the projection spectral theorem. As a result, we derive the spectral measure of the family $(A(\varphi))_{\varphi\in\mathcal{D}}$ at the vacuum state $\Omega$ – a probability measure $\mu$ on the space $\mathcal{D}'$, the dual space of $\mathcal{D}$ with respect to the zero space $L^2(\mathbb{R}^d, dx)$. Furthermore, we get a unitary isomorphism $I : \mathcal{F} \to L^2(\mathcal{D}', \mu)$ such that $I\Omega$ is the function identically equal to 1, and the image of each operator $A(\varphi)$ under $I$ is the operator of multiplication by the monomial $\langle\omega, \varphi\rangle$. Here for $\omega \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$, $\langle\omega, \varphi\rangle$ denotes the dual pairing between $\omega$ and $\varphi$. As a by-product of our considerations, we have an explicitly described subset $\Psi$ of the symmetric Fock space $\mathcal{F}$ such that each operator $A(\varphi)$ maps the set $\Psi$ into itself and $A(\varphi)$ is essentially self-adjoint on $\Psi$. We next derive the Fourier transform of the measure $\mu$, which has the form as in the Lévy–Khintchine formula. This, in particular, implies that $\mu$ is a generalized stochastic process with independent values. It should be stressed that all results in this chapter are new even for Lévy processes, i.e., when the Lévy measure of the process does not depend on point $x \in \mathbb{R}^d$.

In Chapter 4, we derive a counterpart of the Nualart–Schoutens decomposition for generalized stochastic process with independent values. Our results here extend those in [36] and [31], and they have been known in the Lévy process case.

In Chapter 5, we construct an orthogonal decomposition of the space $L^2(\mathcal{D}', \mu)$ in terms of orthogonalized polynomials on $\mathcal{D}'$. We observe a deep relation between this decomposition and the results of the two previous chapters. We get a unitary isomorphism between an extended (symmetric) Fock space $\mathbf{F}$ and $L^2(\mathcal{D}', \mu)$. Here the extended Fock space $\mathbf{F}$ has the form

$$\mathbf{F} = \bigoplus_{n=0}^{\infty} L^2_{\mathrm{sym}}((\mathbb{R}^d)^n, \zeta^{(n)}).$$

where $\zeta^{(n)}$ is an explicitly given measure on $(\mathbb{R}^d)^n$, and $L^2_{\mathrm{sym}}((\mathbb{R}^d)^n, \zeta^{(n)})$ is the space of all symmetric functions on $(\mathbb{R}^d)^n$ which are square integrable with respect to the measure $\zeta^{(n)}$. Such an interpretation of the extended Fock space is new even in the Lévy process case, compare with [31].

Finally, in Chapter 6, we briefly discuss the situation of the generalized stochastic processes of Meixner's type, compare with [30,38]. On the real line, a probability measure of Meixner's type is (almost) completely characterized by two parameters, $\lambda \in \mathbb{R}$ and $\eta \geq 0$. The class of these measures contains, in particular, the Gaussian, Poisson, and Gamma measures. In our infinite dimensional setting, a generalized stochastic process of Meixner's type is characterized by two functions $\lambda(x)$ and $\eta(x)$ for $x \in \mathbb{R}^d$. In our dissertation, we present a sketch of the proof of the following result. Let $\mathfrak{C}(\varphi)$ be the cumulant transform of the probability measure $\mu$ on $\mathcal{D}'$ corresponding to functions $\lambda(x)$ and $\eta(x)$. (Recall that the cumulant transform of a probability measure is the logarithm of its Laplace transform.) Then

$$\mathfrak{C}(\varphi) = \int_{\mathbb{R}^d} \mathfrak{C}_{\lambda(x),\eta(x)}(\varphi(x))\, dx.$$

Here, for a fixed $x \in \mathbb{R}^d$, $\mathfrak{C}_{\lambda(x),\eta(x)}(\varphi(x))$ is the cumulant transform of the probability measure on $\mathbb{R}$ from Meixner's class, corresponding to the parameters $\lambda(x)$, $\eta(x)$, and evaluated at point $\varphi(x)$. A complete proof of this result will be given elsewhere.

# Chapter 2

# Preliminary

We refer our reader to [6, 10, 13, 15] for further details and proofs.

## 2.1 Unbounded operators

The aim of this section is to recall for the reader some notions connected with unbounded operators. So let $H$ be a Hilbert space with inner product $(\cdot,\cdot)_H$ and let $D$ be a linear subset (subspace) of $H$ (thus $D \subset H$). We consider a linear operator $A : D \to H$. We write $D = D(A)$, where $D(A)$ is called the domain of $A$. A linear operator $A$ with domain $D(A)$ is often denoted by $(A, D(A))$ to stress the domain of $A$. If $D$ is dense in $H$, then we say that $A$ is a densely defined linear operator.

A linear operator $A$ is called symmetric if, for any $f, g \in D(A)$,

$$(Af, g)_H = (f, Ag)_H.$$

If, additionally, the domain $D(A)$ is dense in $H$, the operator $A$ is called Hermitian.

If $(A, D(A))$ is a densely defined linear operator in $H$, then we define

$D(A^*)$ as the set of those $g \in H$ for which there exists $g^* \in H$ such that

$$(Af, g)_H = (f, g^*)_H, \quad \text{for all } f \in D(A).$$

$D(A^*)$ is a subspace of $H$. We call $D(A^*)$ the domain of the adjoint operator $A^*$, and we set $A^*g = g^*$. Note that, generally speaking, the domain $D(A^*)$ of the adjoint of a densely defined linear operator $(A, D(A))$ need not be dense in $H$. Furthermore, one can give examples where $D(A^*) = \{0\}$.

Note also that if $(A, D(A))$ is Hermitian, then $D(A) \subset D(A^*)$ and $Af = A^*f$ for each $f \in D(A)$. An operator $(A, D(A))$ is called self-adjoint if $(A, D(A)) = (A^*, D(A^*))$, i.e., the operator $A$ coincides with its adjoint.

For an operator $(A, D(A))$, the set

$$\Gamma_A := \{(f, Af) \mid f \in D(A)\} \subset H \times H$$

is called the graph of the operator $A$.

If $\Gamma_A$ is a closed subset of $H \times H$, then the operator $A$ is called closed. If this is not the case, then one may take the closure $\bar{\Gamma}_A$ of $\Gamma_A$ in $H \times H$. However, $\bar{\Gamma}_A$ may happen not to be a graph of a linear operator, i.e., there may exist vectors $(f, g_1)$ and $(f, g_2)$ in $\bar{\Gamma}_A$ such that $g_1 \neq g_2$. If this is not the case, i.e., if $\bar{\Gamma}_A$ is a graph of a linear operator, then we call $(A, D(A))$ a closable operator and the corresponding operator defined by $\bar{\Gamma}_A$ is called the closure of $(A, D(A))$, denoted by $(\tilde{A}, D(\tilde{A}))$.

One may show that any Hermitian operator is closable and any self-adjoined operator is closed. However, the closure of a Hermitian operator is not necessarily a self-adjoint operator. If this closure is self-adjoint, then we say that $(A, D(A))$ is an essentially self-adjoint operator.

In applications we are mostly given Hermitian operators, rather than self-adjoint operators. Then, if one is able to prove that such an operator is essentially self-adjoint, then, by closing the operator $(A, D(A))$, one derives a self-adjoint operator.

**Theorem 2.1** (Nelson's analytic vector criterion). *Let $(A, D(A))$ be a densely defined, Hermitian operator in $H$. A vector $f \in D(A)$ is called analytic (for $A$) if, for each $n \in \mathbb{N}$, $f \in D(A^n)$, and*

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \|A^n f\|_H < \infty$$

*for some $t > 0$. If there is a subset $\mathfrak{D} \subset D(A)$ such that $\mathfrak{D}$ is total in $H$ and each $f \in \mathfrak{D}$ is analytic for $A$, then the operator $(A, D(A))$ is essentially self-adjoint.*

*Remark* 2.2. A linear combination of analytic vectors is an analytic vector, so that we can only demand that $\mathfrak{D}$ be a total set in $H$, i.e., its linear span is dense in $H$.

*Remark* 2.3. Note that for any linear operators $(A, D(A))$ and $(B, D(B))$ in a Hilbert space $H$, one defines

$$D(AB) := \{f \in D(B) : Bf \in D(A)\}$$

and then, for any $f \in D(AB)$,

$$ABf = A(Bf).$$

This, in particular, explains the meaning of the operator $(A^n, D(A^n))$.

Let $(X, \mathcal{A})$ be a measurable space. Let $\mathcal{B}(H)$ denote the Banach space of all bounded linear operators in $H$. A mapping

$$\mathcal{A} \ni \alpha \mapsto E(\alpha) \in \mathcal{B}(H)$$

is called a resolution of the identity if the following conditions are satisfied:

- For each $\alpha \in \mathcal{A}$, $E(\alpha)$ is an orthogonal projection in $H$.
- $E(\varnothing) = \mathbf{0}$, $E(X) = \mathbf{1}$.
- If $\{\alpha_n\}_{n=1}^{\infty}$, $\alpha_n \in \mathcal{A}$, $n \in \mathbb{N}$, $\alpha_n$ are mutually disjoint, then for each $f \in H$

$$E\Big(\bigcup_{n=1}^{\infty} \alpha_n\Big) f = \sum_{n=1}^{\infty} E(\alpha_n) f,$$

  where the series converges in $H$.

It follows from the definition of a resolution of the identity that for any vectors $f, g \in H$, the mapping

$$\mathcal{A} \ni \alpha \mapsto (E(\alpha)f, g)_H$$

is a signed measure on $(X, \mathcal{A})$, and for any $f \in H$,

$$\mathcal{A} \ni \alpha \mapsto (E(\alpha)f, f)_H$$

is a measure on $(X, \mathcal{A})$.

We denote by $\mathcal{B}(\mathbb{R})$ the Borel $\sigma$-algebra on $\mathbb{R}$. To any self-adjoint operator $(A, D(A))$, there corresponds a unique resolution of the identity over $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$A = \int_{\mathbb{R}} \lambda \, dE(\lambda). \tag{2.1}$$

The equality (2.1) should be understood as follows:

$$D(A) := \{f \in H \mid \int_{\mathbb{R}} \lambda^2 \, d(E(\lambda)f, f)_H < \infty\} \tag{2.2}$$

and for any $f \in D(A)$ and $g \in H$

$$(Af, g)_H = \int_{\mathbb{R}} \lambda \, d(E(\lambda)f, g)_H. \tag{2.3}$$

Furthermore, the inverse statement holds: If $E$ is a resolution of the identity over $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then $E$ determines a self-adjoint operator in $H$ through the formulas (2.2) and (2.3).

Formulas (2.1)–(2.3) are called the spectral decomposition of a self-adjoint operator (in fact, the resolution of the identity is concentrated on the spectrum of $A$).

Let us now briefly discuss commutation of linear operators. In the case where $A_1$ and $A_2$ are bounded linear operators, their commutation is defined straightforward:

$$A_1 A_2 f = A_2 A_1 f \quad \text{for each } f \in H. \tag{2.4}$$

However, in the case where $A_1$ and $A_2$ are unbounded operators, the operators $A_1 A_2$ or $A_2 A_1$ may only be well-defined at zero. So, in the case where $A_1$ and $A_2$ are additionally self-adjoint operators, one defines their commutation through the commutation of their resolutions of the identity. So we say that self-adjoint operators $(A_1, D(A_1))$ and $(A_2, D(A_2))$ commute in the sense of their resolutions of the identity if, for any $\alpha_1, \alpha_2 \in \mathcal{B}(\mathbb{R})$, the operators $E_1(\alpha_1)$ and $E_2(\alpha_2)$ commute, where $E_1$, and $E_2$ denote the resolution of the identity of $A_1$, and $A_2$, respectively. It can be shown that this definition of commutation indeed generalizes the definition (2.4) in the case of bounded self-adjoint operator.

The following theorem allows one to check that two given self-adjoint operators indeed commute in the sense of their resolutions of the identity.

**Theorem 2.4.** *Let $(A_1, D(A_1))$, $(A_2, D(A_2))$ be two Hermitian operators in $H$. Let $\mathfrak{D}$ be a dense linear subset of $H$ such that $\mathfrak{D} \subset D(A_1) \cap D(A_2)$, $A_1\mathfrak{D} \subset \mathfrak{D}$, $A_2\mathfrak{D} \subset \mathfrak{D}$ and $A_1$, $A_2$ commute on $\mathfrak{D}$ in the usual sense:*

$$A_1 A_2 f = A_2 A_1 f \quad \textit{for all } f \in \mathfrak{D}.$$

*Assume that each vector in $\mathfrak{D}$ is analytic for both operators $A_1$ and $A_2$. Then the operators $(A_1, D(A_1))$ and $(A_2, D(A_2))$ are essentially self-adjoint and their closures $(\tilde{A}_1, D(\tilde{A}_1))$ and $(\tilde{A}_2, D(\tilde{A}_2))$ commute in the sense of their resolutions of the identity.*

## 2.2 Orthogonal polynomials

Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \sigma)$ be a probability space. We assume that the probability measure $\sigma$ has all moments finite, i.e.

$$\int_{\mathbb{R}} |x|^n \sigma(dx) < \infty, \quad \forall n \in \mathbb{N}.$$

Therefore the integrals $\int_{\mathbb{R}} x^n \sigma(dx)$ are well defined. The numbers

$$m_n = \int_{\mathbb{R}} x^n \sigma(dx), \quad n \in \mathbb{N},$$

are called the moments of $\sigma$.

If we take a sequence of monomials $(x^n)_{n=0}^{\infty}$, then according to the Gram-Schmidt procedure they may be orthogonalized. Thus, we get a system of monic orthogonal polynomials:

$$P_n(x) = x^n + \alpha_{n-1}x^{n-1} + \alpha_{n-2}x^{n-2} + \cdots + \alpha_0. \tag{2.5}$$

('Monic' means that the leading coefficient, i.e., the coefficient by $x^n$, is 1.)

Now we have to distinguish the two following cases:

Case 1: Suppose the support of $\sigma$ is infinite. Then $(P_n)_{n=0}^{\infty}$ is an infinite system of orthogonal polynomials.

Case 2: If the support of $\sigma$ consists of $k$ points, $k \in \mathbb{N}$, then we get only $k$ orthogonal polynomials $(P_n)_{n=0}^{k-1}$.

In this dissertation we will deal only with Case 1.

**Theorem 2.5** (Farward's theorem). *Assume that the support of $\sigma$ is infinite. Then, there exist $a_n > 0$, $n = 1, 2, \dots$, and $b_n \in \mathbb{R}$, $n = 0, 1, 2, \dots$, such that*

$$\begin{aligned} xP_n(x) &= P_{n+1}(x) + b_n P_n(x) + a_n P_{n-1}(x), \quad n \geq 1, \\ xP_0(x) &= P_1(x) + b_0. \end{aligned} \tag{2.6}$$

*Furthermore, for any $a_n > 0$, $n = 1, 2, \dots$, and $b_n \in \mathbb{R}$, $n = 0, 1, 2, \dots$, there exists a probability measure $\sigma$ with finite moments such that the corresponding polynomials $(P_n(x))_{n=0}^{\infty}$ defined by (2.6) form a system of monic orthogonal polynomials for measure $\sigma$.*

We note that, generally speaking, there may exist different probability measures which have the same moments. That is, the measure $\sigma$ in the second part of Farward's theorem is, generally speaking, not unique. However, there exist sufficient conditions which guarantee that the measure $\sigma$ is unique. The following theorem is an example of such a condition.

**Theorem 2.6.** *Assume that $\sigma$ is a probability measure on $\mathbb{R}$ which has finite moments. Then the following three conditions are equivalent:*

*(i) There exists $c > 0$ such that, for all $n \in \mathbb{N}$,*

$$m_n \leq c^n n! \tag{2.7}$$

*(ii) There exists $\epsilon > 0$, such that*

$$\int_{\mathbb{R}} e^{\epsilon |x|} \sigma(dx) < \infty.$$

*(iii) There exists $\epsilon > 0$ such that the Laplace transform of $\sigma$,*

$$(-\epsilon, \epsilon) \ni y \mapsto \int_{\mathbb{R}} e^{yx} \sigma(dx) \in \mathbb{R}$$

*is well defined and can be extended to an analytic function*

$$\{z \in \mathbb{C} \mid |z| < \epsilon\} \ni z \mapsto \int_{\mathbb{R}} e^{zx} \sigma(dx) \in \mathbb{C}.$$

*If either (i), or (ii), or (iii) holds, then the measure $\sigma$ is a unique probability measure on $\mathbb{R}$ which has moments $m_n$, so that there is a one-to-one correspondence between $\sigma$ and the system of orthogonal polynomials.*

*Remark* 2.7. Let $\sigma$ be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which satisfies either condition (i), or (ii), or (iii) of Theorem 2.6. Then, if we know the Fourier transform of $\sigma$, $\int_{\mathbb{R}} e^{itx}\,\sigma(dx)$, for $t$ from a neighborhood of zero in $\mathbb{R}$, then we can evaluate the moments $m_n$ of $\sigma$ by differentiating the Fourier transform at zero, and so we can recover the measure $\sigma$. Hence the measure $\sigma$ is uniquely identified by its Fourier transform in a neighborhood of zero.

## 2.3 Rigged Hilbert spaces

Let $H_0$ be a real Hilbert space with scalar product $(\cdot,\cdot)_{H_0}$ and norm $\|\cdot\|_{H_0}$. We suppose that

$$H_+ \subseteq H_0, \tag{2.8}$$

where $H_+$ is a dense subset of $H_0$. We also suppose that $H_+$ is a Hilbert space with respect to another scalar product $(\cdot,\cdot)_{H_+}$ and the norm $\|\cdot\|_{H_+}$ in $H_+$ is such that

$$\|\varphi\|_{H_0} \le \|\varphi\|_{H_+}, \quad \varphi \in H_+. \tag{2.9}$$

(The more general case when $\|\cdot\|_{H_0} \le c\|\cdot\|_{H_+}$ for some constant $c < \infty$ can be reduced to (2.9) by introducing an equivalent norm in $H_+$.) The elements of the set $H_+$ play the role of test functions.

Each element $f \in H_0$ generates a linear continuous functional $\langle f, \cdot\rangle$ on $H_+$ according to the formula

$$\langle f, \varphi\rangle := (f, \varphi)_{H_0}. \tag{2.10}$$

We introduce a new norm $\|\cdot\|_{H_-}$ in $H_0$ by taking the norm of $f$ as the norm of the functional $\langle f, \cdot\rangle$ corresponding to it:

$$\|f\|_{H_-} := \|\langle f, \cdot\rangle\| = \sup\left\{\frac{|(f,\varphi)_{H_0}|}{\|\varphi\|_{H_+}} \mid \varphi \in H_+,\ \varphi \neq 0\right\}. \tag{2.11}$$

We complete $H_0$ in the norm (2.11) and obtain a linear normed space $H_-$, which is called the space with negative norm and its elements play the role of generalized functions. Thus we have constructed the chain

$$H_+ \subseteq H_0 \subseteq H_- \tag{2.12}$$

of spaces with positive, zero and negative norms. (The elements of $H_-$, $H_0$, $H_+$ will be called generalized functions, ordinary functions and test functions, respectively). A rigging of the space $H_0$ by the spaces $H_+$ and $H_-$ is given by (2.12).

Each element $\xi \in H_-$ is evidently a linear continuous functional on $H_+$ so that

$$H_- \subseteq (H_+)', \tag{2.13}$$

where $(H_+)'$ denotes the dual space of $H_+$. We will write $(\xi,\varphi)_{H_0}$, or $\langle\xi,\varphi\rangle$ for the action of the functional $\xi$ on an element $\varphi \in H_+$. It is obvious that

$$|(\xi,\varphi)_{H_0}| \leq \|\xi\|_{H_-}\|\varphi\|_{H_+}, \quad \xi \in H_-, \varphi \in H_+ \tag{2.14}$$

which is a generalization of the Cauchy–Schwarz inequality.

Initially, we have defined $H_-$ as a Banach space. However, one may prove that $H_-$ is a Hilbert space, i.e., the norm $\|\cdot\|_{H_-}$ in $H_-$ is generated by some scalar product $(\cdot,\cdot)_{H_-}$. Furthermore, $H_- = (H_+)'$, i.e. $H_-$ can be thought of as the dual space of $H_+$.

A rigging $H_+ \subseteq H_0 \subseteq H_-$ is called quasi-nuclear if the inclusion operator $O : H_+ \to H_0$ is quasi-nuclear (or of the Hilbert–Schmidt class), that is for one (and hence any) orthonormal basis $(e_n)_{n=1}^{\infty}$ of $H_+$, we have

$$\sum_{n=1}^{\infty} \|e_n\|_{H_0}^2 < \infty.$$

In this case, we shall say that the space $H_+$ is imbedded into $H_0$ quasi-nuclearly. The corresponding rigging (or the chain (2.12)) will also be called quasi-nuclear.

*Example* 2.8. Let $H_0 = \ell_2 = \ell_2(\mathbb{R})$ be the Hilbert space of all square summable real sequences $x = (x_k)_{k=0}^{\infty}$ with scalar product

$$(x, y)_{H_0} = \sum_{k=0}^{\infty} x_k y_k.$$

More generally, for each sequence $\tau = (\tau_k)_{k=0}^{\infty}$, $\tau_k > 0$, we define the corresponding Hilbert space

$$H_\tau = \ell_2(\tau) = \{(x_k)_{k=0}^{\infty} \mid x_k \in \mathbb{R},\ \sum_{k=0}^{\infty} x_k^2 \tau_k =: \|x\|_{H_\tau}^2 < \infty\},$$
$$(x, y)_{H_\tau} = \sum_{k=0}^{\infty} x_k y_k \tau_k. \tag{2.15}$$

Evidently, $\ell_2 = \ell_2(\tau)$ with $\tau_k = 1$, $k \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$.

Denote by $T$ the set of all sequences $\tau = (\tau_k)_{k=0}^{\infty}$ with $\tau_k \geq 1$, $k \in \mathbb{Z}_+$. Clearly, for any $\tau, \tau' \in T$ such that $\tau_k' \geq \tau_k$, $k \in \mathbb{Z}_+$, $H_{\tau'} \subset H_\tau$ and $\|\cdot\|_{H_\tau'} \geq \|\cdot\|_{H_\tau}$. Denote by $\mathbb{R}_0^\infty$ the set of all finite real sequences, i.e., all real sequences $(x_k)_{k=0}^{\infty}$ such that $x_k = 0$, $k \geq K$, for some $K \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$. For each $\tau \in T$, $\mathbb{R}_0^\infty \subset \ell_2(\tau)$ with dense inclusion. Therefore, if $\tau$ and $\tau'$ are as above, $\ell_2(\tau')$ is dense in $\ell_2(\tau)$.

For every $\tau = (\tau_k)_{k=0}^{\infty} \in T$, one can take $\tau' = (2^k \tau_k)_{k=0}^{\infty}$ so that the imbedding $O_{\tau',\tau} : H_{\tau'} \to H_\tau$ is quasi-nuclear. Indeed, let $(e_n)_{n=0}^{\infty}$ be the natural orthonormal basis in $\ell_2$, that is

$$e_n = (x_k)_{k=0}^{\infty}, \quad x_k = 0 \text{ for } k \neq n,\ x_n = 1. \tag{2.16}$$

Then the vectors $(\tau_n^{-\frac{1}{2}} e_n)_{n=0}^{\infty}$ form a orthonormal basis in $H_\tau$ and, therefore, for the Hilbert–Schmidt norm of the imbedding operator $O_{\tau',\tau}$, we have

$$\|O_{\tau',\tau}\|_{HS}^2 = \sum_{k=0}^{\infty} \|(\tau_k')^{-\frac{1}{2}} e_k\|_{H_\tau}^2 = \sum_{k=0}^{\infty} 2^{-k} < \infty.$$

For every $\tau \in T$, the Hilbert space $H_{\tau^{-1}} = \ell_2(\tau^{-1})$, $\tau^{-1} := (\tau_k^{-1})_{k=0}^{\infty}$, is dual to $H_\tau = \ell_2(\tau)$ with respect to $H_0 = \ell_2$. The scalar product in $H_0 = \ell_2$ defines a natural pairing of the elements of $\ell_2(\tau^{-1})$ and $\ell_2(\tau)$, namely,

$$(\xi, \varphi)_{H_0} = \sum_{k=0}^{\infty} \xi_k \varphi_k, \quad \xi \in \ell_2(\tau^{-1}),\ \varphi \in \ell_2(\tau).$$

*Example* 2.9. Given $l \in \mathbb{Z}_+$ and a weight function $p : \mathbb{R}^d \to \mathbb{R}$, $p(x) \geq 1$, $x \in \mathbb{R}^d$, $p \in C(\mathbb{R}^d)$ (the space of all continious functions on $\mathbb{R}^d$), the Sobolev space $W_2^l(\mathbb{R}^d, p(x)\, dx)$ is defined as the completion of $C_0^\infty(\mathbb{R}^d)$ (the space of all infinitely differentiable functions on $\mathbb{R}^d$ with bounded support) with respect to the scalar product

$$(\varphi, \psi)_{W_2^l(\mathbb{R}^d, p(x)\, dx)} = \sum_{|\alpha| \leq l} (D^\alpha \varphi, D^\alpha \psi)_{L^2(\mathbb{R}^d, p(x)\, dx)}, \quad \varphi, \psi \in C_0^\infty(\mathbb{R}^d). \quad (2.17)$$

The summation in (2.17) is over all indices $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_1, \ldots, \alpha_d \in \mathbb{Z}_+$, $|\alpha| = \alpha_1 + \cdots + \alpha_d \leq l$, and $D^\alpha$ denotes the corresponding partial derivative.

We set $H_0 = L^2(\mathbb{R}^d, dx) = W_2^0(\mathbb{R}^d, dx)$ and $H_+ = W_2^l(\mathbb{R}^d, p(x)\, dx)$, $l \in \mathbb{N}$. Clearly, $H_+$ is densely and continuously embedded into $H_0$. (Note that each $W_2^l(\mathbb{R}^d, p(x)\, dx)$ contains the set $C_0^\infty(\mathbb{R}^d)$, which is dense in every Sobolev space, by its construction.) Then $H_-$ is called a Sobolev space with negative index $-l$ and is denoted by $W_2^{-l}(\mathbb{R}^d, p^{-1}(x)\, dx)$.

If $l > d/2$, then the Sobolev space $W_2^l(\mathbb{R}^d, p(x)\, dx)$ consists of continuous functions, i.e., $W_2^l(\mathbb{R}^d, p(x)\, dx) \subset C(\mathbb{R}^d)$. Furthermore, if $m \in \mathbb{Z}_+$, $l > d/2$, and if $1 \leq p_1(x) \leq p_2(x)$ with

$$\int_{\mathbb{R}^d} \frac{p_1^2(x)}{p_2^2(x)}\, dx < \infty,$$

then the inclusion $W_2^{m+l}(\mathbb{R}^d, p_2(x)\,dx) \subset W_2^m(\mathbb{R}^d, p_1(x)\,dx)$ is quasi-nuclear.

## 2.4 Rigging of a Hilbert space by a nuclear space

Let $\Phi$ be a linear topological space that is topologically (i.e., densely and continuously) imbedded into a Hilbert space $H_0$. Just as in Section 2.3, each element $f \in H_0$ generates the linear continuous functional on $\Phi$ according to the formula

$$l_f(\varphi) = (f, \varphi)_{H_0}, \quad \varphi \in \Phi.$$

Let $\Phi'$ denote the dual space of $\Phi$ (i.e., the space of all linear continuous functionals on $\Phi$). Identifying $f$ with $l_f$, we obtain the imbedding of $H_0$ into the space $\Phi'$. Hence, we have constructed the chain

$$\Phi \subseteq H_0 \subseteq \Phi'. \tag{2.18}$$

We also say that (2.18) is a rigging of $H_0$ by the spaces $\Phi$ and $\Phi'$, and $\Phi'$ is the dual space of $H_0$ with respect to zero space $H_0$.

In what follows, we will only consider the case where $\Phi$ is a nuclear space. So, let us define such a space. Let $(H_\tau)_{\tau \in T}$ be a family of Hilbert spaces. We assume that the set $\Phi := \bigcap_{\tau \in T} H_\tau$ is dense in each $H_\tau$, and that the family $(II_\tau)_{\tau \in T}$ is directed by imbedding, i.e.,

$$\forall \tau', \tau'' \in T \; \exists \tau''' \in T : \; H_{\tau'''} \subset H_{\tau'}, \; H_{\tau'''} \subset H_{\tau''}, \tag{2.19}$$

where the imbeddings are continuous. On $\Phi$, we introduce a projective limit topology with respect to the family of Hilbert spaces $(H_\tau)_{\tau \in T}$ and natural imbeddings $O_\tau : \Phi \to H_\tau$. By definition, this means that we consider the weakest topology on $\Phi$ for which all the mappings $O_\tau$, $\tau \in T$, are continuous. One can show that the collection of all possible open balls

$$\mathcal{U}(\varphi;\tau;\epsilon) = \{\psi \in \Phi \mid \|\varphi - \psi\|_{H_\tau} < \epsilon\}, \quad \varphi \in \Phi,\ \tau \in T,\ \epsilon > 0. \tag{2.20}$$

may be taken as a system of base neighbourhoods of this topology. This space $\Phi$ constructed as above is called the projective limit of the family $(H_\tau)_{\tau\in T}$ and is denoted by

$$\Phi = \operatorname*{proj\,lim}_{\tau\in T} H_\tau. \tag{2.21}$$

If, additionally, for each $\tau \in T$, there exists $\tau' \in T$ such that $H_{\tau'} \subset H_\tau$ and the inclusion operator $O_{\tau',\tau}, H_{\tau'} \to H_\tau$ is quasi-nuclear, then the space $\Phi$ is called a nuclear space.

Let $\Phi = \operatorname{proj\,lim}_{\tau\in T} H_\tau$ be a nuclear space, and let $H_0$ be a Hilbert space. Assume that, for each $\tau \in T$, $H_\tau \subset H_0$ with continuous imbedding, and that $\Phi$ (and therefore each $H_\tau$) is a dense subset of $H_0$. We can now construct the riggings

$$\begin{aligned} H_\tau \subseteq \quad & H_0 \quad \subseteq H_{-\tau}, \quad \tau \in T,\\ \Phi \subseteq \quad & H_0 \quad \subseteq \Phi'. \end{aligned}$$

Notice that if $H_{\tau'} \subseteq H_\tau$, we have

$$H_{-\tau} \subseteq H_{-\tau'}. \tag{2.22}$$

**Theorem 2.10** (Schwartz). *We have*

$$\Phi' = \bigcup_{\tau\in T} H'_\tau = \bigcup_{\tau\in T} H_{-\tau}.$$

*This equality should be understood as follows: for each $l \in \Phi'$ there exists $\tau \in T$ such that $l$ may be extended by continuity from $\Phi$ to a linear continuous functional on $H_\tau$, and vice versa if $l \in H_{-\tau}$ for some $\tau \in T$, then $l \upharpoonright \Phi \in \Phi'$.*

Since $\Phi' = \bigcup_{\tau\in T} H_{-\tau}$, one can introduce on $\Phi'$ the topology of the inductive limit of Hilbert spaces $(H_{-\tau})_{\tau\in T}$. This topology is defined by basic open sets

$$\mathcal{U}(\xi,\varepsilon(\cdot)) = \text{c.l.s.}\Big(\bigcup_{\tau\in T}\{\|\xi-\xi'\|_{H_{-\tau}} < \varepsilon(\tau) \mid \xi'\in\Phi',\ \xi-\xi'\in H_{-\tau'}\}\Big),$$

where c.l.s. denotes 'the convex linear span', $\xi\in\Phi'$ and $T\ni\tau\mapsto\varepsilon(\tau)>0$. One writes $\Phi' = \operatorname{ind\,lim}_{\tau\in T} H_{-\tau}$. Thus, we have

$$\Phi = \operatorname*{proj\,lim}_{\tau\in T} H_\tau \subseteq H_0 \subseteq \operatorname*{ind\,lim}_{\tau\in T} H_{-\tau} = \Phi', \tag{2.23}$$

which is called a Gel'fand (standard) triple. The dual space $\Phi'$ is often called a co-nuclear space.

*Example* 2.11. Recall Example 2.8. Clearly, $\bigcap_{\tau\in T}\ell_2(\tau) = \mathbb{R}_0^\infty$. Indeed, the inclusion $\mathbb{R}_0^\infty\subset\bigcap_{\tau\in T}\ell_2(\tau)$ is evident, whereas for any sequence of real numbers $(x_k)_{k=0}^\infty$ which has an infinite number of non-zero elements, one can always find $\tau\in T$ such that $(x_k)_{k=0}^\infty\notin\ell_2(\tau)$.

Setting $\Phi = \mathbb{R}_0^\infty = \operatorname{proj\,lim}_{\tau\in T}\ell_2(\tau)$, we get a nuclear space. In fact, convergence in this space means uniform finiteness of all sequences and coordinate-wise convergence. That is, a sequence $(x^{(n)})_{n=1}^\infty$ converges to $x$ in $\mathbb{R}_0^\infty$ if and only if there exists $N\in\mathbb{Z}_+$ such that $x_k^{(n)}=0$, $x_k=0$ for all $k\ge N$ and all $n\in\mathbb{N}$, and $x_k^{(n)}\to x_k$ as $n\to\infty$ for each $k\in\{0,1,2,\dots,N-1\}$.

By Theorem 2.10 and Example 2.8,

$$\Phi' = \operatorname*{ind\,lim}_{\tau\in T}\ell_2(\tau^{-1}).$$

Note that

$$\{\tau^{-1}\mid\tau\in T\} = \{(\tau_k)_{k=0}^\infty \mid 0<\tau_k<1,\ k\in\mathbb{N}\}.$$

Denote by $\mathbb{R}^\infty = \mathbb{R}\times\mathbb{R}\times\cdots$. Then, clearly, $\Phi'\subset\mathbb{R}^\infty$, since each $\ell_2(\tau^{-1})\subset\mathbb{R}^\infty$. On the other hand, for each sequence $(x_k)_{k=0}^\infty\in\mathbb{R}^\infty$ one can always find

$\tau \in T$ such that $(x_k)_{k=0}^{\infty} \in \ell_2(\tau^{-1})$. Therefore, $\Phi' = \mathbb{R}^{\infty}$. Thus, we get the Gel'fand triple

$$\mathbb{R}_0^{\infty} \subset \ell_2 \subset \mathbb{R}^{\infty}. \tag{2.24}$$

In fact for each $\xi = (\xi_k)_{k=0}^{\infty} \in \mathbb{R}^{\infty}$ and $x = (x_k)_{k=0}^{\infty} \in \mathbb{R}_0^{\infty}$, the dual pairing between $\xi$ and $x$ with respect to the zero space $\ell_2$ is given by

$$\langle \xi, x \rangle = (\xi, x)_{H_0} = \sum_{k=0}^{\infty} \xi_k x_k. \tag{2.25}$$

(Note that since $x \in \mathbb{R}_0^{\infty}$, the series in (2.25) contains only a finite number of non-zero terms, and hence it is well-defined).

*Example* 2.12. Recall Example 2.9. Denote $\mathcal{D} = \mathcal{D}(\mathbb{R}^d) := C_0^{\infty}(\mathbb{R}^d)$. As mentioned in Example 2.9, $\mathcal{D}$ is a dense subset of each Sobolev space $W_2^l(\mathbb{R}^d, p(x)dx)$, $l \in \mathbb{Z}_+$, $p(x) \geq 1$. Therefore

$$\mathcal{D} \subset \bigcap_{l \in \mathbb{Z}_+, p(x) \geq 1} W_2^l(\mathbb{R}^d, p(x)dx).$$

In fact, one can show that

$$\mathcal{D} = \bigcap_{l \in \mathbb{Z}_+, p(x) \geq 1} W_2^l(\mathbb{R}^d, p(x)dx).$$

Furthermore, using the fact stated in the end of Example 2.9, one can show $\mathcal{D}$ is a nuclear space. The convergence in $\mathcal{D}$ may be described as follows: If $(f_n)_{n=1}^{\infty} \subset \mathcal{D}$, $\quad f \in \mathcal{D}$, then $f_n \to f$ on $\mathcal{D}$ if and only if

$$\bigcup_{n \in \mathbb{N}} \operatorname{supp}(f_n)$$

is a bounded set in $\mathbb{R}^d$ (i.e., the functions $f_n$ are uniformly finite), and for each index $(\alpha_1, \alpha_2, \ldots, \alpha_d)$, $\alpha_i \in \mathbb{Z}_+$, $i = 1, 2, \ldots, d$ ,

$$(D^{\alpha} f_n)(x) \to (D^{\alpha} f)(x) \quad \text{as } n \to \infty$$

uniformly on $\mathbb{R}^d$. Here $D^\alpha f$ denotes the corresponding partial derivative of $f$. By Theorem 2.10,

$$\mathcal{D}' = \operatorname*{ind\,lim}_{l\in\mathbb{Z}_+,\, p(x)\geq 0} W_2^{-l}(\mathbb{R}^d, p^{-1}(x)dx).$$

*Example* 2.13. Let $\sigma$ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which satisfies condition (i) (or equivalently condition (ii)) of Theorem 2.6 and has an infinite support. Recall Example 2.11. Denote by $\mathcal{P}$ the set of all polynomials on $\mathbb{R}$. Recall the vectors $e_n$ in $\ell_2$ defined by (2.16). They form an orthonormal basis in $\ell_2$. Consider the linear mapping $I$ defined by

$$Ie_n = x^n,$$

and extended by linearity to the linear span of the $e_n$ vectors, i.e., to $\mathbb{R}_0^\infty$. Thus, $I$ is a bijective mapping between $\mathbb{R}_0^\infty$ and $\mathcal{P}$. Through it we define a nuclear space topology on $\mathcal{P}$. Note that $p_k \to p$ in $\mathcal{P}$ as $k \to \infty$, means that there exists $N \in \mathbb{N}$ such that

$$p_k = \sum_{i=0}^{N} a_{ik} x^i, \quad p = \sum_{i=0}^{N} a_i x^i,$$

and for each $i = 0, 1, \dots, N$, $a_{ik} \to a_i$ as $k \to \infty$.

As easily seen, the nuclear space $\mathcal{P}$ is densely and continuously embedded into $L^2(\mathbb{R}, \sigma)$. [Note that, if the measure $\sigma$ had finite support, the latter statement would fail, since in $L^2(\mathbb{R}, \sigma)$ we would find non-zero polynomials on $\mathbb{R}$ which would be zero elements of $L^2(\mathbb{R}, \sigma)$.]

## 2.5 Probability measures on co-nuclear spaces, Minlos theorem

Let

$$\Phi \subseteq H_0 \subseteq \Phi'$$

be a Gel'fand triple. We first need a $\sigma$-algebra on $\Phi'$. For each $\varphi \in \Phi$, we define a mapping as follows

$$\Phi' \ni \omega \mapsto \langle \omega, \varphi \rangle \in \mathbb{R}. \tag{2.26}$$

Then $\mathcal{C}(\Phi')$ is the minimal $\sigma$-algebra on $\Phi'$ with respect to which all mappings (2.26) are measurable. In particular, if $\varphi_1, \varphi_2, \ldots, \varphi_n \in \Phi, n \in \mathbb{N}$ and $g := \mathbb{R}^n \to \mathbb{R}$ measurable, then

$$F(\omega) = g(\langle \omega, \varphi_1 \rangle, \langle \omega, \varphi_2 \rangle, \ldots, \langle \omega, \varphi_n \rangle) \tag{2.27}$$

is a measurable function on $\Phi'$. A function of the form (2.27) is called a cylinder function, and $\mathcal{C}(\Phi')$ is called the cylinder $\sigma$-algebra on $\Phi'$.

Now, if $\mu$ is a probability measure on $(\Phi', \mathcal{C}(\Phi'))$, then we call $\mu$ a generalized stochastic process.

Let $\mu$ be a probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Then the Fourier transform of $\mu$ is defined by

$$F_\mu(x) = \int_{\mathbb{R}^n} e^{i\langle x, y \rangle} \mu(dy), \quad x \in \mathbb{R}^n,$$

where $\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$. The classical Bochner theorem states that a function $F : \mathbb{R}^n \to \mathbb{C}$ is the Fourier transform of a unique probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, i.e., $F = F_\mu$, if and only if $F(0) = 1$ and $F$ is positive definite, i.e., for all $c_1, \ldots, c_n \in \mathbb{C}$, $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathbb{R}^d$, we have

$$\sum_{j,k=1}^{n} c_j \overline{c_k} F(x_j - x_k) \geq 0.$$

Let now $\mu$ be a probability measure on $(\Phi', \mathcal{C})$. Analogously to the finite-dimensional case, we define the Fourier transform of $\mu$ by

$$F_\mu(\varphi) = \int_{\Phi'} e^{i\langle \varphi, \omega \rangle} \mu(d\omega), \quad \varphi \in \Phi.$$

The following theorem is an infinite-dimensional generalization of the Bochner theorem.

**Theorem 2.14** (Minlos). *Suppose $F : \Phi \to \mathbb{C}$. Then $F$ is the Fourier transform of a unique probability measure $\mu$ on $(\Phi', \mathcal{C}(\Phi'))$ if and only if*

- $F(0) = 1$;

- *$F$ is positive definite, i.e., for all $c_1, \dots, c_n \in \mathbb{C}$, $n \in \mathbb{N}$, $\varphi_1, \dots, \varphi_n \in \Phi$ we have*

$$\sum_{j,k=1}^{n} c_j \overline{c_k} F(\varphi_j - \varphi_k) \geq 0;$$

- *$F$ is continuous on $\Phi$, i.e., $F_\mu$ is continuous on each $H_\tau$, $\tau \in T$.*

*Remark:* The third condition of the Minlos theorem is a new condition compared with the Bochner theorem (in the finite-dimensional case, one automatically gets the continuity of the Fourier transform).

## 2.6 Projection spectral theorem

As we see from the previous section, one way of defining a probability measure on a co-nuclear space is through the Minlos theorem. Another possible way of construction of such a measure is given through the projection spectral theorem, which we will discuss below.

Let us first recall the spectral theorem in the case of one self-adjoint operator. Let $H$ be a real, separable Hilbert space and let $(A, D(A))$ be a self-adjoint operator. Let $\Omega \in H$ and assume that $\Omega$ is cyclic for $A$, i.e., $\Omega \in D(A^n)$, $n \in \mathbb{N}$, and the set $\{\Omega, A\Omega, A^2\Omega, \dots\}$ is a total set in $H$. Then, the spectral theorem states that there exists a unique probability measure $\mu$ on $\mathbb{R}$ such that the linear mapping $I$ given through

$$I\Omega = 1, \quad IA^n\Omega = x^n, \quad n \in \mathbb{N},$$

extends by continuity to a unitary operator

$$I : H \to L^2(\mathbb{R}, \mu).$$

Under $I$, the operator $A$ goes over into the operator of multiplication by $x$, given by

$$D\{x\cdot\} = \{f \in L^2(\mathbb{R}, \mu) : xf(x) \in L^2(\mathbb{R}, \mu)\}$$

and $(x \cdot f)(x) = xf(x)$, $f \in D(x\cdot)$. Thus, $IAI^{-1} = x\cdot$. In fact, the measure $\mu$ is given by

$$\mu(\alpha) = (E(\alpha)\Omega, \Omega)_H, \quad \alpha \in \mathcal{B}(\mathbb{R}),$$

where $E(\cdot)$ is the resolution of the identity of $A$. The measure $\mu$ is called the spectral measure of $A$ (at $\Omega$).

The following theorem generalizes the above result to the case of a family of commuting self-adjoint operators indexed by elements of a nuclear space.

**Theorem 2.15. (Projection spectral theorem for a family of commuting, self-adjoint operators)** *Assume that we have two Gel'fand triples $\Phi' \supset H \supset \Phi$ and $\Psi' \supset \mathcal{F} \supset \Psi$, where $H$ and $\mathcal{F}$ are seperable Hilbert spaces. Also assume that we have a family $(A(\varphi))_{\varphi\in\Phi}$ of Hermitian operators in $\mathcal{F}$ such that*

1. *$D(A(\varphi)) = \Psi$, $\varphi \in \Phi$;*
2. *$A(\varphi)\Psi \subset \Psi$ for each $\varphi \in \Phi$, and furthermore $A(\varphi) : \Psi \to \Psi$ is continuous;*
3. *$A(\varphi_1)A(\varphi_2)f = A(\varphi_2)A(\varphi_1)f$, $f \in \Psi$, (i.e., the $A(\varphi)$'s algebraically commute on $\Psi$);*
4. *for all $f, g \in \Psi$, the mapping*

$$\Phi \ni \varphi \mapsto (A(\varphi)f, g)_{\mathcal{F}} \in \mathbb{R}$$

*is continuous;*

5. *There exists a vector* $\Omega$ *in* $\mathcal{F}$ *which is cyclic for* $(A(\varphi))_{\varphi\in\Phi}$*, i.e., the set*

$$\{\Omega,\ A(\varphi_1)\cdots A(\varphi_k)\Omega \mid \varphi_1,\dots,\varphi_k \in \Phi,\ k \in \mathbb{N}\}$$

   *is total in* $\mathcal{F}$*;*

6. *for any* $f \in \Psi$ *and* $\varphi \in \Phi$*, the vector* $f$ *is analytic for the operator* $A(\varphi)$*.*

*Then, each operator* $A(\varphi)$*,* $\varphi \in \Phi$*, is essentially self-adjoint and we denote its closure by* $(\tilde{A}(\varphi), D(\tilde{A}(\varphi)))$*. These operators commute in the sense of their resilutions of the identity. Furthermore, there exists a unique probability measure* $\mu$ *on* $(\Phi', \mathcal{C}(\Phi'))$ *such that the linear operator* $I : \mathcal{F} \to L^2(\Phi', \mu)$ *given through* $I\Omega = 1$ *and*

$$\begin{aligned} I(\tilde{A}(\varphi_1)\cdots\tilde{A}(\varphi_n)\Omega) &= I(A(\varphi_1)\cdots A(\varphi_n)\Omega) \\ &= \langle\varphi_1,\omega\rangle\cdots\langle\varphi_n,\omega\rangle \in L^2(\Phi',\mu) \end{aligned}$$

*is unitary. Under the action of* $I$*, each operator* $(\tilde{A}(\varphi), D(\tilde{A}(\varphi)))$*,* $\varphi \in \Phi$ *goes over into the operator of multiplication by* $\langle\omega,\varphi\rangle$ *in* $L^2(\Phi',\mu)$*, given by*

$$D(\langle\varphi,\omega\rangle\cdot) = \{F \in L^2(\Phi',\mu) : \langle\varphi,\omega\rangle F(\omega) \in L^2(\Phi',\mu)\}$$

*and for each* $F \in D(\langle\varphi,\omega\rangle\cdot)$*,*

$$(\langle\varphi,\omega\rangle\cdot F)(\omega) = \langle\varphi,\omega\rangle F(\omega).$$

## 2.7 Tensor product

We will now construct a tensor product of $n$ Hilbert spaces, $n \in \mathbb{N}$, $n \geq 2$. For simplicity of notation, we will only consider the case of tensor product of the identical Hilbert spaces (the general case of different Hilbert spaces may be treated by complete analogy).

So, let $H$ be a real separable Hilbert space and let $(e_j)_{j=0}^{\infty}$ be an orthonormal basis in $H$. Let $n \in \mathbb{N}$, $n \geq 2$. We construct formal products

$$e_\alpha = e_{\alpha_1} \otimes e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_n},$$

where $\alpha \in \mathbb{Z}_+^n$. We define the separable Hilbert space

$$H^{\otimes n} = \underbrace{H \otimes H \otimes \cdots \otimes H}_{n\text{-times}}$$

as the real Hilbert space with orthonormal basis $(e_\alpha)_{\alpha \in \mathbb{Z}_+^n}$. Thus, vectors from $H^{\otimes n}$ have the form

$$f = \sum_{\alpha \in \mathbb{Z}_+^n} f_\alpha e_\alpha, \quad f_\alpha \in \mathbb{R},$$
$$\|f\|^2{}_{H^{\otimes n}} = \sum_{\alpha \in \mathbb{Z}_+^n} f_\alpha^2,$$
$$(f, g)_{H^{\otimes n}} = \sum_{\alpha \in \mathbb{Z}_+^n} f_\alpha g_\alpha.$$

Let $f^{(k)} = \sum_{j=0}^{\infty} f_j^{(k)} e_j$, $k = 1, \dots, n$, be some vectors from $H$. Then, we define the vector $f^{(1)} \otimes f^{(2)} \otimes \cdots \otimes f^{(n)}$ as the element of $H^{\otimes n}$ given by

$$f^{(1)} \otimes \cdots \otimes f^{(n)} = \sum_{\alpha \in \mathbb{Z}_+^n} f_{\alpha_1}^{(1)} \cdots f_{\alpha_n}^{(n)} e_\alpha.$$

As easily seen,

$$\|f^{(1)} \otimes \cdots \otimes f^{(n)}\|_{H^{\otimes n}} = \|f^{(1)}\|_H \cdots \|f^{(n)}\|_H.$$

The above definition of tensor product depends on the choice of an orthonormal basis in $H$. However, a change of orthonormal basis leads to a tensor product being isomorphic to the initial one. In particular, for any

$f^{(1)}, f^{(2)}, \dots, f^{(n)} \in H$, the construction of the vector $f^{(1)} \otimes f^{(2)} \otimes \dots \otimes f^{(n)}$ does not depend on the choice of orthonormal basis (under this isomorphism).

A typical example of tensor product is the $n$-th tensor power of an $L^2$-space $L^2(R, \mathcal{R}, \mu)$, which is nothing else but $L^2(R^n, \mathcal{R}^n, \mu^{\otimes n})$. Even more generally, for $L^2$-spaces $H_i = L^2(R_i, \mathcal{R}_i, \mu_i)$, the tensor product

$$H_1 \otimes H_2 \otimes \dots \otimes H_n = L^2(R_1 \times R_2 \times \dots \times R_n, \mathcal{R}_1 \times \mathcal{R}_2 \times \dots \times \mathcal{R}_n, \mu_1 \otimes \mu_2 \otimes \dots \otimes \mu_n).$$

Let us discuss a tensor product of linear, generally speaking unbounded operators. For simplicity of notation, we will construct a tensor product of two operators acting in the same Hilbert space $H$. So, let $(A, D(A))$ and $(B, D(B))$ be linear operators in a real, separable Hilbert space $H$. Our aim is to construct a linear operator $A \otimes B$ in $H \otimes H$. As domain of $A \otimes B$, $D(A \otimes B)$, we take the linear span of vectors of the form $f \otimes g$, where $f \in D(A)$ and $g \in D(B)$. Then, we get

$$AB\ f \otimes g = (Af) \otimes (Bg),$$

and extend this definition by linearity to the whole set $D(A \otimes B)$.

The linear unbounded operator $(A \otimes B, D(A \otimes B))$ is called the tensor product of $(A, D(A))$ and $(B, D(B)$.

Let us consider the special case, where the operators $(A, D(A)$ and $(B, D(B)$ are Hermitian and $A$ and $B$ are essentially self-adjoint on $D(A)$ and $D(B)$, respectively. Then, one can prove that the tensor product $(A \otimes B, D(A \otimes B))$ is a Hermitian operator which is essentially self-adjoint.

Let us now assume that we have a Gel'fand triple

$$\Phi = \operatorname*{proj\,lim}_{\tau \in T} H_\tau \subset H_0 \subset \operatorname*{ind\,lim}_{\tau \in T} H_{-\tau} = \Phi'.$$

Let $n \in \mathbb{N}$, $n \geq 2$. For each $\tau \in T$, we can evidently construct the $n$-th tensor power of $H_\tau$, i.e., $H_\tau^{\otimes n}$. As easily seen, the Hilbert space $H_\tau^{\otimes n}$ is topologically imbedded into $H_0^{\otimes n}$. One can prove that the dual space of $H_\tau^{\otimes n}$ with respect to the zero space $H_0^{\otimes n}$ is $H_{-\tau}^{\otimes n}$, so that we get the following rigging of $H_0^{\otimes n}$:

$$H_\tau^{\otimes n} \subset H_0^{\otimes n} \subset H_{-\tau}^{\otimes n}.$$

Furthermore, a straightforward calculation shows that, if the imbedding $H_\tau \subset H_0$ is quasi-nuclear, then so is the imbedding $H_\tau^{\otimes n} \subset H_0^{\otimes n}$. Finally, one can show that the intersection of all $H_\tau^{\otimes n}$, $\tau \in T$, is dense in each $H_\tau^{\otimes n}$. Therefore, $\bigcap_{\tau\in T} H_\tau^{\otimes n}$ may be considered as a nuclear space, which is usually denoted by $\Phi^{\otimes n}$. Furthermore, the dual of $\Phi^{\otimes n}$ with respect to centre space $H_0^{\otimes n}$ is denoted by $\Phi'^{\otimes n}$, and is equal to $\operatorname{ind\,lim}_{\tau\in T} H_{-\tau}^{\otimes n}$. Thus, for each $n \geq 2$, we get the Gel'fand triple

$$\Phi^{\otimes n} = \operatorname*{proj\,lim}_{\tau\in T} H_\tau^{\otimes n} \subset H_0^{\otimes n} \subset \operatorname*{ind\,lim}_{\tau\in T} H_{-\tau}^{\otimes n} = \Phi'^{\otimes n}. \tag{2.28}$$

## 2.8 Symmetric tensor product

We define on $H_0^{\otimes n}$ the operator $\mathrm{Sym}_n$ by

$$\mathrm{Sym}_n\, f_1 \otimes \cdots \otimes f_n := \frac{1}{n!} \sum_{\sigma\in S_n} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)},$$

where $S_n$ is the set of all permutations of $\{1, \dots, n\}$. As easily seen, $\mathrm{Sym}_n$ extends to an orthogonal projection in $H_0^{\otimes n}$.

Denote $H_0^{\odot n} := \mathrm{Sym}_n\, H_0^{\otimes n}$, where $\mathrm{Sym}_n\, H_0^{\otimes n}$ is the subspace of $H_0^{\otimes n}$ onto which $\mathrm{Sym}_n$ projects. Also, for $f_1, \dots, f_n \in H_0$, we denote

$$f_1 \odot \cdots \odot f_n = \mathrm{Sym}_n(f_1 \otimes \cdots \otimes f_n).$$

Note that, for each $\sigma \in S_n$,

$$f_1 \odot \cdots \odot f_n = f_{\sigma(1)} \odot \cdots \odot f_{\sigma(n)}.$$

The space $H_0^{\odot n}$ is called the $n$-th symmetric tensor power of $H_0$, and for $f_1, \dots, f_n \in H_0$, $f_1 \odot \cdots \odot f_n$ is called symmetric tensor product of $f_1, \dots, f_n$. Clearly, for each $f \in H_0$, $f^{\otimes n} = f^{\odot n}$.

In the case where $H_0 = L^2(R, \mu)$, so that $H_0^{\otimes n} = L^2(R^n, \mu^{\otimes n})$, the symmetric tensor power $H_0^{\odot n}$ is the subspace of $L^2(R^n, \mu^{\otimes n})$ which consists of all functions $f \in L^2(R^n, \mu^{\otimes n})$ which remain invariant under the permutation of their variables, i.e., for each $\sigma \in S_n$,

$$f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \quad \mu^{\otimes n}\text{-a.e.}$$

In the case of Gel'fand triple (2.28), we have that $\bigcap_{\tau \in T} H_\tau^{\odot n}$ is dense in $H_0^{\odot n}$, and if $H_{\tau'} \subset H_\tau$, then $H_{\tau'}^{\odot n} \subset H_\tau^{\odot n}$ topologically, and if the former inclusion was quasi-nuclear, then so is the latter inclusion. Thus, $\operatorname{proj\,lim}_{\tau \in T} H_\tau^{\odot n}$ is a nuclear space, which is denote by $\Phi^{\odot n}$. Next, one can easily shows that for each $\tau \in T$ the dual space of $H_\tau^{\odot n}$ with respect to zero space $H_0^{\odot n}$ is $H_{-\tau}^{\odot n}$, so that we get the Gel'fand triple

$$\Phi^{\odot n} = \operatorname*{proj\,lim}_{\tau \in T} H_\tau^{\odot n} \subset H_0^{\odot n} \subset \operatorname*{ind\,lim}_{\tau \in T} H_{-\tau}^{\odot n} = \Phi'^{\odot n}.$$

## 2.9 Symmetric Fock space

The Fock space is the Hilbert space made of the direct sum of symmetric tensor powers of a single-particle Hilbert space.

Below, we will need the following notation: If $H$ is a Hilbert space and $a > 0$, then $aH$ is the Hilbert space which coincides with $H$ as a set and the inner product in $aH$ is given by

$$(f, g)_{aH} = a(f, g)_H.$$

Let $H$ be a separable, real Hilbert space. We define the symmetric Fock space over $H$ as

$$\mathcal{F}(H) := \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}(H),$$

where $\bigoplus_{n=0}^{\infty}$ denotes orthogonal sum of Hilbert spaces, for each $n \in \mathbb{N}$

$$\mathcal{F}^{(n)}(H) := H^{\odot n} n!, \tag{2.29}$$

and $\mathcal{F}_0(H) := \mathbb{R}$. That is, $\mathcal{F}(H)$ consists of all sequences $f = (f^{(0)}, f^{(1)}, \dots)$ where $f^{(0)} \in \mathbb{R}$, $f^{(1)} \in H$, $f^{(2)} \in H^{\odot 2}$, $f^{(3)} \in H^{\odot 3}, \dots$,

$$\|f\|^2_{\mathcal{F}(H)} = |f_0|^2 + \sum_{n=1}^{\infty} \|f^{(n)}\|^2_{\mathcal{F}^{(n)}(H)} = |f_0|^2 + \sum_{n=1}^{\infty} \|f^{(n)}\|^2_{H^{\odot n}} n! < \infty,$$

and for any $f \in \mathcal{F}(H)$ as above and $g = (g^{(0)}, g^{(1)}, \dots) \in \mathcal{F}(H)$

$$(f,g)_{\mathcal{F}(H)} = f^{(0)} g^{(0)} + \sum_{n=1}^{\infty} (f^{(n)}, g^{(n)})_{H^{\odot n}} n!\,.$$

The subspaces $\mathcal{F}^{(n)}(H)$ are often called the $n$-particle subspaces, and the vector $\Omega = (1, 0, 0, \cdots) \in \mathcal{F}^{(0)}(H)$ is called the vacuum.

In view of the definition of the Fock space, we will treat any $H^{\odot n}$ as a subspace of $\mathcal{F}(H)$, so that any vector $f^{(n)} \in H^{\odot n}$ will be identified with the vector

$$(0, \dots, 0, f^{(n)}, 0, 0, \dots)$$

in $\mathcal{F}(H)$.

We will now construct a special orthonormal basis in $\mathcal{F}(H)$. Below, we will denote by $\mathbb{Z}^{\infty}_{+,0}$ the set of all infinite sequences of the form $(\alpha_0, \alpha_1, \dots, \alpha_n, 0, 0, \dots)$, where $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{Z}_+$, $n \in \mathbb{Z}_+$. For $\alpha \in \mathbb{Z}^{\infty}_{+,0}$, we denote $|\alpha| := \alpha_0 + \alpha_1 + \cdots$. If $(e_n)_{n=0}^{\infty}$ is an orthonormal basis of $H$, then we can construct an orthonormal basis of $\mathcal{F}(H)$ as follows

$$e_\alpha = \Big(\frac{1}{\alpha_0! \alpha_1! \alpha_2! \cdots}\Big)^{\frac{1}{2}} e_0^{\otimes \alpha_0} \odot e_1^{\otimes \alpha_1} \odot e_2^{\otimes \alpha_2} \odot \cdots, \quad \alpha \in \mathbb{Z}^{\infty}_{+,0}, \tag{2.30}$$

where if $\alpha = (0, 0, \dots)$, then $e_\alpha = (1, 0, 0, \dots) = \Omega$. Note that $e_\alpha \in \mathcal{F}^{(|\alpha|)}(H)$. The basis $(e_\alpha)_{\alpha \in \mathbb{Z}^\infty_{+,0}}$ is called the basis of occupation numbers.

## 2.10 Rigging of a Fock space

Let $H$ be a real, separable Hilbert space. For any sequence $q = (q_n)_{n=0}^\infty$, $q_n \geq 1$, we define the weighted Fock space $\mathcal{F}(H, q)$ as follows:

$$\mathcal{F}(H, q) = \bigoplus_{n=0}^\infty \mathcal{F}_n(H) q_n.$$

(Recall the notation introduced in the beginning of Section 2.9). In particular, if $q_n = 1$ for all $n \geq 0$, $\mathcal{F}(H, q) = \mathcal{F}(H)$.

Let $H_+ \subset H_0$ quasi-nuclearly. Fix any sequence $q$ as above. As we discussed in Section 2.7, for each $n \geq 2$, the inclusion $H_+^{\odot n} \subset H_0^{\odot n}$ is also quasi-nuclear. Therefore, for $q = (q_n)_{n=0}^\infty$ as above, one can find another sequence $q' = (q'_n)_{n=0}^\infty$, $q'_n \geq q_n$, such that

$$\mathcal{F}(H_+, q') \subset \mathcal{F}(H_0, q)$$

quasi-nuclearly. Indeed, denoted by $\|O_n\|_{HS}$ the Hilbert-Schmidt norm of the inclusion operator $O_n : H_+^{\odot n} \to H_0^{\odot n}$. We know that $\|O_n\|_{HS} < \infty$. Fix any $q' = (q'_n)_{n=0}^\infty$, $q'_n \geq q_n$. Then, clearly, the imbedding operator

$$O : \mathcal{F}(H_+, q) \to \mathcal{F}(H_0, q)$$

is continuous. Then the Hilbert-Schmidt norm of $O$ is equal to

$$\|O\|_{HS}^2 = \sum_{n=0}^\infty \|O_n\|_{HS}^2 \frac{q_n}{q'_n}$$

which is finite for sufficiently quickly growing $(q'_n)_{n=0}^\infty$, e.g. take

$$q'_n = (\|O_n\|_{HS}^2 q_n 2^n)^{-1}.$$

Let us take any Gel'fand triple

$$\Phi = \operatorname*{proj\,lim}_{\tau \in T} H_\tau \subset H_0 \subset \operatorname*{ind\,lim}_{\tau \in T} H_{-\tau} = \Phi'.$$

It follows from the above that

$$\operatorname*{proj\,lim}_{\tau \in T,\, q=(q_n)_{n=0}^\infty,\, q_n \geq 1} \mathcal{F}(H_\tau, p) \tag{2.31}$$

is a nuclear space. Indeed, fix any $\tau$ and $p$ as above. Choose first $\tau' \in T$ such that $H'_\tau \subset H_\tau$ quasi-nuclearly and then choose $q' = (q'_n)_{n=0}^\infty$, $q'_n \geq q_n$, so that $\mathcal{F}(H_{\tau'}, q') \subset \mathcal{F}(H_\tau, q)$ quasi-nuclearly. In fact, the space (2.31) consists of all finite sequences $(f^{(0)}, f^{(1)}, \dots, f^{(n)}, 0, 0, \dots)$ such that $f^{(i)} \in \Phi^{\odot i}$, $i = 0, 1, \dots, n$, $n \in \mathbb{N}$. We denote this space by $\mathcal{F}_{\mathrm{fin}}(\Phi)$. The convergence in this space means uniform finiteness and coordinate-wise convergence in each $\Phi^{\odot n}$.

Thus, we get the Gel'fand triple

$$\mathcal{F}_{\mathrm{fin}}(\Phi) \subset \mathcal{F}(H_0) \subset \mathcal{F}^*_{\mathrm{fin}}(\Phi),$$

where $\mathcal{F}^*_{\mathrm{fin}}(\Phi)$ is the dual space of $\mathcal{F}_{\mathrm{fin}}(\Phi)$ with respect to the zero space $\mathcal{F}(H_0)$. This space consists of all sequences $F = (F^{(0)}, F^{(1)}, \dots)$, where $F^{(n)} \in \Phi'^{\odot n}$, and the dual pairing between $F$ and $f = (f^{(0)}, f^{(1)}, \dots, f^{(n)}, 0, 0, \dots)$ is given by

$$(F, f)_{\mathcal{F}(H_0)} = \sum_{i=0}^{n} (F^{(i)}, f^{(i)})_{H_0^{\odot n}} n!.$$

## 2.11 Creation, annhilation and neutral operators

Let us introduce some linear operators in the Fock space which will be heavily used in our research.

Let $H$ be a real, separable Hilbert space. Denote by $\mathcal{F}_{\mathsf{fin}}(H)$ the subspace of the Fock space $\mathcal{F}(H)$ consisting of all vectors of the form

$$f = (f^{(0)}, f^{(1)}, \dots, f^{(n)}, 0, 0, \dots), \quad f^{(i)} \in \mathcal{F}^{(i)}(H).$$

We can endow $\mathcal{F}_{\mathsf{fin}}(H)$ with a topology such that convergence in $\mathcal{F}_{\mathsf{fin}}(H)$ means uniform finiteness and coordinate-wise convergence in each $\mathcal{F}^{(i)}(H)$ (this topology is similar to that of $\mathbb{R}_0^\infty$).

Let $f \in H$. We define a creation operator $a^+(f)$ as a linear continuous operator on $\mathcal{F}_{\mathsf{fin}}(H)$ given through

$$a^+(f)g^{(n)} = f \odot g^{(n)}, \quad g^{(n)} \in \mathcal{F}^{(n)}(H), \quad n \in \mathbb{Z}_+.$$

Note that, since

$$\mathrm{Sym}_{n+1}(\mathbf{1} \otimes \mathrm{Sym}_n) = \mathrm{Sym}_{n+1},$$

we have, for each $u^{(n)} \in H^{\otimes n}$,

$$a^+(f)\ \mathrm{Sym}_n\, u^{(n)} = \mathrm{Sym}_{n+1}(f \otimes u^{(n)}). \tag{2.32}$$

Also we can write

$$(g^{(0)}, g^{(1)}, \dots, g^{(n)}, 0, 0, \dots) \overset{a^+(f)}{\to} (0, g^{(0)}f, f \odot g^{(1)}, \dots, f \odot g^{(n)}, 0, \dots).$$

Next, we define an annihilation operator $a^-(f)$ as a linear continuous operator on $\mathcal{F}_{\mathsf{fin}}(H)$ given through

$$\begin{aligned} &a^-(f)g^{\odot n} = n(f,g)_H g^{\odot(n-1)}, \quad n \in \mathbb{N}, \\ &a^-(f)\Omega = 0. \end{aligned} \tag{2.33}$$

Assume that $H = L^2(R, \mu)$, so that $H^{\odot n} = L^2_{\mathsf{sym}}(R^n, \mu^{\otimes n})$. Then, (2.33) implies that for each $g^{(n)} \in H^{\odot n}$,

$$(a^-(f)g^{(n)})(x_1, x_2, \dots, x_{n-1}) = n \int_R f(x) g^{(n)}(x, x_1, x_2, \dots, x_{n-1}) \mu(dx).$$

Note also that

$$(g^{(0)}, g^{(1)}, \dots, g^{(n)}, 0, 0, \dots) \overset{a^-(f)}{\to} (a^-(f)g^{(1)}, a^-(f)g^{(2)}, \dots, a^-(f)g^{(n)}, 0, 0, \dots).$$

Assume again that $H = L^2(R, \mu)$, and let $f$ be a bounded, measurable function on $R$. We define the neutral (also called preservation) operator $a^0(f)$ as a linear continuous operator on $\mathcal{F}_{\mathrm{fin}}(H)$ given through

$$\begin{aligned} a^0(f)\Omega &= 0, \\ a^0(f)g^{\odot n} &= n(fg) \odot g^{\otimes(n-1)} \in \mathcal{F}^{(n)}(H), \end{aligned} \tag{2.34}$$

where $fg$ denotes the point-wise product of function $f$ and $g$. (Note that since $f$ is bounded $fg \in L^2(R, \mu)$.)

*Remark* 2.16. In what follows, we will also use a neutral operator $a^0(f)$ for functions $f$ which are not necessarily bounded. In that case, the domain of $a^0(f)$ must be reduced in order to allow the vector $fg$ in (2.34) to be an element of $L^2(R, \mu)$. For example, if $f \in L^2(R, \mu)$, the function $g \in L^2(R, \mu)$ could be bounded.

Direct calculations show that $a^-(f)$ is the restriction to $\mathcal{F}_{\mathrm{fin}}(H)$ of the adjoint operator of $a^+(f)$ in $\mathcal{F}(H)$:

$$(a^+(f)F, G)_{\mathcal{F}(H)} = (F, a^-(f)G)_{\mathcal{F}(H)}, \quad F, G \in \mathcal{F}_{\mathrm{fin}}(H).$$

On the other hand, the neutral operator $a^0(f)$ is symmetric in $\mathcal{F}(H)$,

$$(a^0(f)F, G)_{\mathcal{F}(H)} = (F, a^0(f)G)_{\mathcal{F}(H)}, \quad F, G \in \mathcal{F}_{\mathrm{fin}}(H).$$

*Remark* 2.17. Note that formulas (2.33) and (2.34) imply that, for $f, g_1, \dots, g_n \in H$

$$a^-(f)g_1 \odot g_2 \odot \cdots \odot g_n = (f, g_1)_H g_2 \odot \cdots \odot g_n + (f, g_2)_H g_1 \odot g_3 \odot \cdots \odot g_n$$

$$+\cdots+(f,g_n)_H g_1 \odot \cdots \odot g_{n-1}, \tag{2.35}$$

and

$$\begin{aligned} a^0(f) g_1 \odot g_2 \odot \cdots \odot g_n = &(f g_1) \odot g_2 \odot \cdots \odot g_n + g_1 \odot (f g_2) \odot g_3 \odot \cdots \odot g_n \\ &+ \cdots + g_1 \odot g_2 \odot \cdots \odot g_{n-1} \odot (f g_n). \end{aligned} \tag{2.36}$$

# Chapter 3

# Generalized stochastic processes with independent values through the projection spectral theorem

We will now discuss how a generalized stochastic process with independent values (to be defined below) can be constructed by using the projection spectral theorem (Theorem 2.15).

Assume that for each $x \in \mathbb{R}^d$, $\sigma(x, ds)$ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which contains an infinite number of points in its support. We also assume that for each $\Delta \in \mathcal{B}(\mathbb{R})$,

$$\mathbb{R}^d \ni x \mapsto \sigma(x, \Delta) \tag{3.1}$$

is a measurable mapping. (Note that, if $d = 1$, $\sigma(x, ds)$ is just a Markov kernel on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.) Hence, we can define a $\sigma$-finite measure $dx\,\sigma(x, ds)$ on $(\mathbb{R}^d \times \mathbb{R}, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}))$.

Let $\mathcal{B}_0(\mathbb{R}^d)$ denote the collection of all sets $\Lambda \in \mathcal{B}(\mathbb{R}^d)$ which are bounded. We will additionally assume that, for each $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$, there exists for $C_\Lambda > 0$ such that

$$\int_{\mathbb{R}} |s|^n \sigma(x, ds) \leq C_\Lambda^n n! \quad n \in \mathbb{N}, \tag{3.2}$$

for all $x \in \Lambda$. In particular, for each fixed $x \in \mathbb{R}^d$, the measure $\sigma(x, ds)$ satisfies conditions (i), or equivalently condition (ii), of Theorem 2.6.

We fix the Hilbert space

$$H_0 = L^2(\mathbb{R}^d \times \mathbb{R}, dx\, \sigma(x, ds)). \tag{3.3}$$

Recall the nuclear space $\mathcal{D} = C_0^\infty(\mathbb{R}^d)$ from Example 2.12. Recall the nuclear space $\mathcal{P}$ from Example 2.13. We construct the nuclear space

$$\mathscr{S} = \mathcal{D} \otimes \mathcal{P}. \tag{3.4}$$

This space consists of all functions of the form

$$f(x, s) = \sum_{k=0}^{n} s^k a_k(x),$$

where $n \in \mathbb{N}$ and $a_0(x), a_1(x), \cdots, a_n(x) \in \mathcal{D}$.

Let $f_n \to f$ as $n \to \infty$ in $\mathscr{S}$. Then, as easily seen, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} f_n(x, s) &= \sum_{k=0}^{n} s^k a_k^{(n)}(x), \quad n \in \mathbb{N}, \\ f(x, s) &= \sum_{k=0}^{n} s^k a_k(x), \end{aligned} \tag{3.5}$$

where $a_k^{(n)}(x), a_k(x) \in \mathcal{D}$ and

$$a_k^{(n)}(x) \to a_k(x) \text{ in } \mathcal{D} \text{ as } n \to \infty. \tag{3.6}$$

**Lemma 3.1.** *The space $\mathscr{S}$ is topologically, i.e., densely and continuously, embedded into*

$$H_0 = L^2(\mathbb{R}^d \times \mathbb{R}, dx\, \sigma(x, ds)).$$

*Proof.* Let us show that $\mathscr{S}$ is a dense subspace of $H_0$. Equivalently, we have to prove that the orthogonal compliment to $\mathscr{S}$ in $H_0$ is the zero space, i.e., $\mathscr{S}^\perp = \{0\}$. Let $g \in \mathscr{S}^\perp$, i.e., $g \in H_0$ is such that

$$(g, f)_{H_0} = 0 \quad \forall f \in \mathscr{S}.$$

Hence for any $a \in \mathcal{D}$ and $k \geq 0$

$$\int_{\mathbb{R}^d} dx \int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)\, a(x)\, s^k = 0.$$

Fix any compact set $\Lambda$ in $\mathbb{R}^d$ and let $a \in \mathcal{D}$ be such that the support of $a$ is a subset of $\Lambda$. Then,

$$\int_{\mathbb{R}^d} dx\, a(x) \left( \int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)\ s^k \right) = 0,$$

hence

$$\int_{\Lambda} dx\, a(x) \left( \int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)\, s^k \right) = 0. \tag{3.7}$$

We state that the function

$$\Lambda \ni x \mapsto \int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)\, s^k$$

belongs to $L^2(\Lambda,\, dx)$. Indeed, by Cauchy's inequality and (3.2),

$$\begin{aligned}
&\int_{\Lambda} dx \left( \int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)\, s^k \right)^2 \\
&\quad \leq \int_{\Lambda} dx \int_{\mathbb{R}} \sigma(x, ds_1)\, g(x, s_1)^2 \int_{\mathbb{R}} \sigma(x, ds_2)\, s_2^{2k} \\
&\quad \leq C_\Lambda^{2k}\, (2k)! \int_{\Lambda} dx \int_{\mathbb{R}} \sigma(x, ds_1)\, g(x, s_1)^2 < \infty.
\end{aligned}$$

Since the set of all functions $a \in \mathcal{D}$ with support in $\Lambda$ is dense in $L^2(\Lambda, dx)$, we therefore conclude from (3.7) that, for a.a $x \in \Lambda$,

$$\int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)\, s^k = 0, \quad \forall k \geq 0. \tag{3.8}$$

Since $g \in H_0$, we get that, for a.a. $x \in \mathbb{R}^d$, $g(x, \cdot) \in L^2(\mathbb{R}, \sigma(x, ds))$. By (3.2), the set of all monomials $s^k$ is a total set in $L^2(\mathbb{R}, \sigma(x, ds))$. Hence, by (3.8), for $dx$-a.a. $x \in \Lambda$ $g(x, s) = 0$ for $\sigma(x, ds)$-a.a. $s \in \mathbb{R}$. Since $\Lambda$ was arbitrary, we get for $dx$-a.a. $x \in \mathbb{R}^d$ $g(x, s) = 0$ for $\sigma(x, ds)$-a.a. $s \in \mathbb{R}$. Hence, for any $\Theta \in \mathcal{B}(\mathbb{R}^d)$ and $\Delta \in \mathcal{B}(\mathbb{R})$, we have

$$\int_{\Theta \times \Delta} dx\, \sigma(x, ds)\, |g(x, s)| = \int_{\Theta} dx \left( \int_{\Delta} \sigma(x, ds)\, |g(x, s)| \right) = 0,$$

since for a.a. $x \in \mathbb{R}^d$, $\int_\Delta \sigma(x, ds)\, |g(x, s)| = 0$. Consider the measure

$$dx\, \sigma(x, ds)\, |g(x, s)|$$

on $\mathbb{R}^d \times \mathbb{R}$. This measure is equal to zero on all sets of the form $\Theta \times \Delta$. Hence, it must be the zero measure on $(\mathbb{R}^d \times \mathbb{R}, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}))$, since the collection of the sets of the form $\Theta \times \Delta$ generates the $\sigma$-algebra $\mathcal{B}(\mathbb{R}^d \times \mathbb{R})$ and it is $\bigcap$-stable (see e.g. [5, Ch. I, Theorem 5.4]). Thus, for each $\Psi \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R})$, $\int_\Psi dx\, \sigma(x, ds)\, |g(x, s)| = 0$, which implies that $|g(x, s)| = 0$ for $dx\, \sigma(x, ds)$-a.a. $(x, s) \in \mathbb{R}^d \times \mathbb{R}$, i.e., $g = 0$ as an element of $H_0$.

Let us now prove the continuity of the embedding of $\mathscr{S}$ into $H_0$. Let $f_n \to f$ as $n \to \infty$ in $\mathscr{S}$. Recall formulas (3.5) and (3.6). Using the dominated convergence theorem, we hence conclude that $f_n \to f$ in $H_0$. $\square$

Thus, we get a Gel'fand triple

$$\mathscr{S} \subset H_0 \subset \mathscr{S}',$$

where $\mathscr{S}'$ is the dual space of $\mathscr{S}$ with respect to the zero space $H_0$.

Thus, by Section 2.10, we get the Gel'fand triple

$$\mathcal{F}_{\mathsf{fin}}(\mathscr{S}) \subset \mathcal{F}(H_0) \subset \mathcal{F}^*_{\mathsf{fin}}(\mathscr{S}).$$

Our aim is to construct a special probability measure on $\mathcal{D}'$ using the projection spectral theorem for a family of commuting self-adjoint operators (Theorem 2.15).

So, we set $\Psi := \mathcal{F}_{\mathsf{fin}}(\mathscr{S})$, $\mathcal{F} := \mathcal{F}(H_0)$, $\Psi^* := \mathcal{F}^*_{\mathsf{fin}}(\mathscr{S})$, $\Phi = \mathcal{D}$, $H = L^2(\mathbb{R}^d, dx)$, $\Phi' = \mathcal{D}'$.

For each $\varphi \in \mathcal{D}$, we define

$$A(\varphi) := a^+(\varphi \otimes 1) + a^-(\varphi \otimes 1) + a^0(\varphi(x)s), \tag{3.9}$$

where $(\varphi \otimes 1)(x, s) := \varphi(x)$. We set $D(A(\varphi)) = \Psi$, and as easily seen $A(\varphi)\Psi \subset \Psi$.

**Theorem 3.2.** *The operators $(A(\varphi))_{\varphi\in\Phi}$ and the Gel'fand triples $\Phi' \supset H \supset \Phi$ and $\Psi' \supset \mathcal{F} \supset \Psi$ satisfy the conditions of Theorem 2.15, so that the statement of this theorem holds for these operators and Gel'fand triples.*

*Proof.* We check the conditions of Theorem 2.15.

1. This condition is clearly satisfied.

2. We already know that $A(\varphi)\Psi \subset \Psi$, $\varphi \in \Phi$. For each $\varphi \in \mathcal{D}$, the linear operator

$$\mathcal{D} \ni f \mapsto \varphi f \in \mathcal{D}$$

is continuous (see Example 2.12 for the description of convergence in $\mathcal{D}$) and the linear operator

$$\mathcal{P} \ni f \mapsto sf(s) \in \mathcal{P}$$

is also continuous (see Example 2.13 for the description of convergence in $\mathcal{P}$). This implies that for each $n \in \mathbb{N}$ the linear operator

$$a^0(\varphi(x)s) : (\mathcal{D} \otimes \mathcal{P})^{\odot n} \to (\mathcal{D} \otimes \mathcal{P})^{\odot n}$$

is continuous. Therefore,

$$a^0(\varphi \otimes s) : \Psi \to \Psi$$

is continuous.

Next the continuity of

$$a^+(\varphi \otimes 1) : \Psi \to \Psi \quad \text{and} \quad a^-(\varphi \otimes 1) : \Psi \to \Psi$$

easily follows from their definition. Thus, the operator

$$A(\varphi) : \Psi \mapsto \Psi$$

is continuous.

3. For any linear operators $A$, $B$, we denote by $[A, B]$ the commutator of $A$, $B$: $[A, B] := AB - BA$.

Let $\mathcal{H} = L^2(R, \nu)$ be an $L^2$-space and let $\mathcal{F}(\mathcal{H})$ be the corresponding Fock space. Then, for any $f_1, f_2 \in \mathcal{H}$,

$$[a^+(f_1),\ a^+(f_2)] = 0. \tag{3.10}$$

Indeed, for each $g^{\odot n} \in \mathcal{F}(\mathcal{H})$,

$$\begin{aligned} a^+(f_1)a^+(f_2)g^{\odot n} &= f_1 \odot f_2 \odot g^{\odot n} \\ &= f_2 \odot f_1 \odot g^{\odot n} \\ &= a^+(f_2)a^+(f_1)g^{\odot n}. \end{aligned}$$

Taking the adjoint operators in (3.10), we get

$$[a^-(f_1),\ a^-(f_2)] = 0. \tag{3.11}$$

Next

$$[a^-(f_1),\ a^+(f_2)] = (f_1, f_2)_{\mathcal{H}}\ 1. \tag{3.12}$$

Indeed by (2.35),

$$\begin{aligned} a^-(f_1)a^+(f_2)g^{\otimes n} &= a^-(f_1)\ f_2 \odot g^{\otimes n} \\ &= (f_1, f_2)_{\mathcal{H}} g^{\otimes n} + n(f_1, g)_{\mathcal{H}}\ f_2 \odot g^{\otimes(n-1)}, \end{aligned}$$

and

$$\begin{aligned} a^+(f_2)a^-(f_1)g^{\otimes n} &= a^+(f_2)(f_1, g)_{\mathcal{H}}\ n\ g^{\otimes(n-1)} \\ &= n(f_1, g)_{\mathcal{H}}\ f_2 \odot g^{\otimes(n-1)}, \end{aligned}$$

so that

$$\big(a^-(f_1)a^+(f_2) - a^+(f_2)a^-(f_1)\big)g^{\otimes n} = (f_1, f_2)_{\mathcal{H}} g^{\otimes n},$$

which proves (3.12).

Next

$$[a^0(f_1), a^0(f_2)] = 0.$$

Indeed, by (2.36),

$$\begin{aligned} a^0(f_1)a^0(f_2)g^{\otimes n} &= a^0(f_1)\ n\ (f_2 g) \odot g^{(n-1)}, \\ &= n(f_1 f_2 g) \odot g^{(n-1)} + n(n-1)(f_1 g) \odot (f_2 g) \odot g^{(n-2)}, \\ &= a^0(f_2)a^0(f_1)\ g^{\otimes n}. \end{aligned}$$

Next

$$[a^0(f_1), a^+(f_2)] = a^+(f_1 f_2). \tag{3.13}$$

Indeed by (2.36)

$$\begin{aligned} a^0(f_1)a^+(f_2)g^{\otimes n} &= a^0(f_1) f_2 \odot g^{\otimes n} \\ &= (f_1 f_2) \odot g^{\otimes n} + n f_2 \odot (f_1 g) \odot g^{\otimes(n-1)} \\ &= a^+(f_1 f_2)g^{\otimes n} + a^+(f_2)a^0(f_1)g^{\otimes n}. \end{aligned}$$

Note that (3.13) means that

$$a^0(f_1)a^+(f_2) - a^+(f_2)a^0(f_1) = a^+(f_1 f_2),$$

and taking the adjoint of these operators we get

$$a^-(f_1)a^0(f_2) - a^0(f_1)a^-(f_2) = a^-(f_1 f_2).$$

Thus

$$[a^-(f_1),\ a^+(f_2)] = a^-(f_1 f_2). \tag{3.14}$$

By (3.10)–(3.14), taking a function $\lambda$, we get;

$$\begin{aligned}
&[a^+(f_1) + a^-(f_1) + a^0(\lambda f_1), a^+(f_2) + a^-(f_2) + a^0(\lambda f_2)] \\
&= [a^+(f_1), a^+(f_2)] + [a^+(f_1), a^-(f_2)] + [a^+(f_1), a^0(\lambda f_2)] \\
&+ [a^-(f_1), a^+(f_2)] + [a^-(f_1), a^-(f_2)] + [a^-(f_1), a^0(\lambda f_2)] \\
&+ [a^0(\lambda f_1), a^+(f_2)] + [a^0(\lambda f_1), a^-(f_2)] + [a^0(\lambda f_1), a^0(\lambda f_2)] \\
&= -(f_1, f_2)_{\mathcal{H}} 1 - a^+(\lambda f_1 f_2) + (f_1, f_2)_{\mathcal{H}} 1 \\
&+ a^-(\lambda f_1 f_2) + a^+(\lambda f_1 f_2) - a^-(\lambda f_1 f_2) \\
&= 0.
\end{aligned} \tag{3.15}$$

Applying formula (3.15) to our case with function $\lambda(x, s) = s$, we get that

$$[A(\varphi_1), A(\varphi_2)] = 0.$$

4. To prove condition 4, we need to check that for any $f, g \in \Psi$, the mappings

$$\mathcal{D} \ni \varphi \mapsto (a^+(\varphi \otimes 1)f, g)_{\mathcal{F}}, \tag{3.16}$$

$$\mathcal{D} \ni \varphi \mapsto (a^-(\varphi \otimes 1)f, g)_{\mathcal{F}}, \tag{3.17}$$

$$\mathcal{D} \ni \varphi \mapsto (a^0(\varphi(x)s)f, g)_{\mathcal{F}} \tag{3.18}$$

are continuous. Indeed, for any $f^{(n)} \in (\mathcal{D} \otimes \mathcal{P})^{\odot n}$, $g^{(n+1)} \in (\mathcal{D} \otimes \mathcal{P})^{\odot (n+1)}$,

$$\begin{aligned}
&\big(a^+(\varphi \otimes 1) f^{(n)}, g^{(n+1)}\big)_{H_0^{\odot(n+1)}} = \big((\varphi \otimes 1) \odot f^{(n)}, g^{(n+1)}\big)_{H_0^{\odot(n+1)}} \\
&= \big((\varphi \otimes 1) \otimes f^{(n)}, g^{(n+1)}\big)_{H_0^{\otimes(n+1)}} \\
&= \int_{(\mathbb{R}^d \times \mathbb{R})^{(n+1)}} \varphi(x_1) f^{(n)}(x_2, s_2, \dots, x_{n+1}, s_{n+1}) g^{(n+1)}(x_1, s_1, \dots, x_{n+1}, s_{n+1}) \\
&\times dx_1 \cdots dx_{n+1}\, \sigma(x_1, ds_1) \cdots \sigma(x_{n+1}, ds_{n+1}),
\end{aligned}$$

which continuously depends on $\varphi \in \mathcal{D}$, by the dominated convergence theorem. This proves continuity of (3.16). Next, we note that the mapping (3.17) can be written as

$$\big(f, a^+(\varphi \otimes 1) g\big)_{\mathcal{F}} = \big(a^+(\varphi \otimes 1) g, f\big)_{\mathcal{F}},$$

which is continuous in $\varphi$ by the proved above.

Finally, for any $f^{(n)}, g^{(n)} \in (\mathcal{D} \otimes \mathcal{P})^{\odot n}$ by (2.34),

$$\begin{aligned}
&\big(a^0(\varphi(x)s) f^{(n)}, g^{(n)}\big)_{H_0^{\odot n}} \\
&= \int_{(\mathbb{R}^d \times \mathbb{R})^n} \Big(\sum_{i=1}^n \varphi(x_i) s_i\Big) f^{(n)}(x_1, s_1, \dots, x_n, s_n)\, g^{(n)}(x_1, s_1, \dots, x_n, s_n) \\
&\times dx_1 \cdots dx_n\, \sigma(x_1, ds_1) \cdots \sigma(x_n, ds_n),
\end{aligned}$$

and again by the dominated convergence theorem, (3.18) is continuous in $\varphi \in \mathcal{D}$.

5. We will now prove that $\Omega \in \mathcal{F}$ is cyclic for $(A(\varphi))_{\varphi \in \mathcal{D}}$. We note that this fact is not trivial sense the set $\{\varphi \otimes 1, \varphi \in \mathcal{D}\}$ is not total in $H_0$. For each $\Delta \in \mathcal{B}_0(\mathbb{R}^d)$, we denoted by $\chi_\Delta$ the indicator function of $\Delta$. For each $\Delta \in \mathcal{B}_0(\mathbb{R}^d)$, we define

$$A(\Delta) = a^+(\chi_\Delta \otimes 1) + a^-(\chi_\Delta \otimes 1) + a^0(\chi_\Delta(x)s).$$

As domain for $A(\Delta)$, we will choose all finite sequences

$$(f^{(0)}, f^{(1)}, \dots, f^{(n)}, 0, 0, \dots), \quad n \in \mathbb{N},$$

where for each $i = 0, 1, 2, \ldots, n$, $f^{(i)}(x_1, s_1 \ldots, x_i, s_i)$ is a finite sum of functions of the form $f_1^{(i)}(x_1, \ldots, x_i) f_2^{(i)}(s_1, \ldots, s_i)$, where $f_1^{(i)}$ is a measurable, bounded, symmetric functions on $(\mathbb{R}^d)^i$ with bounded support and $f_2^{(i)}$ is a symmetric polynomial on $\mathbb{R}^i$.

For each $\Delta \in \mathcal{B}_0(\mathbb{R}^d))$, we can always find a sequence $(\varphi_n)_{n=1}^{\infty}$, $\varphi_n \in \mathcal{D}$, such that $\varphi_n \to \chi_\Delta(x)$ a.e. and the sequence $(\varphi_n)_{n=1}^{\infty}$ is uniformly finite (i.e., $\bigcup_{n \in \mathbb{N}} \operatorname{supp} \varphi_n$ is bounded) and uniformly bounded.

From here it follows that it suffices to prove that $\Omega$ is cyclic for the family $(A(\Delta))_{\Delta \in \mathcal{B}_0(\mathbb{R}^d)}$ in $\mathcal{F}(H_0)$. Denote by $F$ the closed linear span of the vectors

$$\{\Omega, A(\Delta_1) \cdots A(\Delta_n)\Omega,\ \Delta_1, \ldots, \Delta_n \in \mathcal{B}_0(\mathbb{R}^d),\ n \in \mathbb{N}\}.$$

Thus, we have to prove that $F = \mathcal{F}(H_0)$.

**Lemma 3.3.** *For any sets $\Delta_1, \ldots, \Delta_i \in \mathcal{B}_0(\mathbb{R}^d)$ which are mutually disjoint, and any $l_1, \ldots, l_i \in \mathbb{N}$ such that $l_1 + \cdots + l_i = n$, $n \in \mathbb{N}$.*

$$(\chi_{\Delta_1}(x_1) s_1^{l_1 - 1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i) s_i^{l_i - 1}) \in F.$$

*Proof.* We will prove by induction in $n$. We have

$$A(\Delta)\Omega = \chi_\Delta \otimes 1,$$

which is the statement for $n = 1$. Let us assume that the statement holds up to $n$, and let us prove it for $n + 1$. So, let $l_1, \ldots, l_i \in \mathbb{N}$, $l_1 + \cdots + l_i = n + 1$, and let $\Delta_1, \ldots, \Delta_i$ be mutually disjoint. We have to consider two cases.

Case 1: $l_1 = 1$. Since $\Delta_1 \cap \Delta_j = \varnothing$ for each $j = 1, 3, \ldots, i$,

$$\begin{aligned}
&(\chi_{\Delta_1}(x_1) 1) \odot (\chi_{\Delta_2}(x_2) s_2^{l_2 - 1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i) s_i^{l_i - 1}) \\
&= A(\Delta_1)(\chi_{\Delta_2}(x_2) s_2^{l_2 - 1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i) s_i^{l_i - 1})
\end{aligned}$$

and by the assumption

$$(\chi_{\Delta_2}(x_2) s_2^{l_2 - 1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i) s_i^{l_i - 1}) \in F,$$

since $l_2 + l_3 + \cdots + l_i = n$.

Case 2: $l_1 > 1$. We get

$$
\begin{aligned}
&A(\Delta_1)(\chi_{\Delta_1}(x_1)s_1^{l_1-2}) \odot (\chi_{\Delta_2}(x_2)s_2^{l_2-1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i)s_i^{l_i-1}) \\
&= (\chi_{\Delta_1}(x_1)1) \odot (\chi_{\Delta_1}(x_1)s_1^{l_1-2}) \odot (\chi_{\Delta_2}(x_2)s_2^{l_2-1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i)s_i^{l_i-1}) \\
&+ \left(\int_{\Delta_1} dx \int_{\mathbb{R}} \sigma(x, ds_1)s_1^{l_1-2}\right) \times (\chi_{\Delta_2}(x_2)s_2^{l_2-1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i)s_i^{l_i-1}) \\
&+ (\chi_{\Delta_1}(x_1)s_1^{l_1-1}) \odot (\chi_{\Delta_2}(x_2)s_2^{l_2-1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i)s_i^{l_i-1}).
\end{aligned} \tag{3.19}
$$

The left hand side of the equation (3.19) belongs to $F$, since

$$(l_1 - 2) + l_2 + \cdots + l_i = n,$$

the vector

$$\int_{\Delta_1} dx \int_{\mathbb{R}} \sigma(x, ds_1)s_1^{l_1-2} \times (\chi_{\Delta_2}(x_2)s_2^{l_2-1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i)s_i^{l_i-1})$$

belongs to $F$, since $l_2 + \cdots + l_i \le n - 1$, and

$$(\chi_{\Delta_1}(x_1)1) \odot (\chi_{\Delta_1}(x_1)s_1^{l_1-2}) \odot (\chi_{\Delta_2}(x_2)s_2^{l_2-1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i)s_i^{l_i-1})$$

belongs to $F$, since

$$1 + (l_1 - 1) + l_2 + \cdots + l_i = n + 1$$

and we use Case 1. □

**Lemma 3.4.** *$\mathcal{F}(H_0)$ coincides with the closed linear span of the functions of the form*

$$(\chi_{\Delta_1}(x_1)s_1^{l_1-1}) \odot \cdots \odot (\chi_{\Delta_i}(x_i)s_i^{l_i-1}),$$

*where sets $\Delta_1, \ldots, \Delta_i \in \mathcal{B}_0(\mathbb{R}^d)$ are mutually disjoint, and $l_1, \ldots, l_i \in \mathbb{N}$*

*Proof.* Let us first prove that the closed linear span of the set

$$\{\chi_\Delta(x)s^n \mid \Delta \in \mathcal{B}_0(\mathbb{R}^d),\ n \in \mathbb{Z}_+\}$$

is dense in $L^2(\mathbb{R}^d \times \mathbb{R}, dx\,\sigma(x, ds))$. Indeed, let $g \in L^2(\mathbb{R}^d \times \mathbb{R}, dx\,\sigma(x, ds))$ and let $g$ be orthogonal to all elements of this set, i.e,,

$$\int_\Delta dx \int_\mathbb{R} \sigma(x, ds)s^n g(x, s) = 0 \quad \text{for all } \Delta \in \mathcal{B}_0(x),\ n \in \mathbb{Z}_+.$$

By the Cauchy inequality,

$$\int_\Delta dx \int_\mathbb{R} \sigma(x, ds)|s|^n |g(x, s)|$$
$$\le \left(\int_\Delta dx \int_\mathbb{R} \sigma(x, ds)|s|^{2n}\right)^{1/2} \left(\int_\Delta dx \int_\mathbb{R} \sigma(x, ds) g(x, s)^2\right)^{1/2} < \infty$$

by assumption (3.2). Hence, the function

$$|s|^n |g(x, s)| \in L^1(\Delta \times \mathbb{R}, dx\,\sigma(x, ds)),$$

so

$$\int_\mathbb{R} \sigma(x, ds)s^n g(x, s) \in L^1(\Delta, dx).$$

For any $\Delta' \in \mathcal{B}_0(\mathbb{R}^d)$, $\Delta' \subset \Delta$,

$$\int_{\Delta'} \int_\mathbb{R} \sigma(x, ds)s^n g(x, s) = 0,$$

hence for $dx$-a.a. $x \in \mathbb{R}^d$

$$\int_\mathbb{R} \sigma(x, ds)s^n g(x, s) = 0. \tag{3.20}$$

On the other hand, for $dx$-a.a. $x \in \mathbb{R}^d$

$$g(x, s) \in L^2(\mathbb{R}, \sigma(x, ds))$$

and by (3.2), the set of all monomials $s^n$, $n \in \mathbb{Z}_+$, is dense in $L^2(\mathbb{R}, \sigma(x, ds))$. Hence $g(x, ds) = 0$ for $\sigma(x, ds)$-a.a. $s \in \mathbb{R}$. Thus, $g(x, ds) = 0$ for a.a.

$dx\,\sigma(x, ds)$-a.a. $(x, s) \in \mathbb{R}^d \times \mathbb{R}$, i.e., $g = 0$ as an element of $L^2(\mathbb{R}^d \times \mathbb{R}, dx\,\sigma(x, ds))$.

Since, the Lebesgue measure $dx$ is non-atomic, we can analogously show that for each $i \in \mathbb{N}$ the closed linear span of all functions of the form

$$(\chi_{\Delta_1}(x_1)s_1^{l_1-1}) \cdots (\chi_{\Delta_i}(x_i)s_i^{l_i-1}),$$

with $\Delta_1, \dots, \Delta_i \in \mathcal{B}_0(\mathbb{R}^d)$, mutually disjoint, and $l_1, \dots, l_i \in \mathbb{N}$ coincides with $L^2((\mathbb{R}^d \times \mathbb{R})^i, dx_1\,\sigma(x_1, ds_1) \cdots dx_i\,\sigma(x_i, ds_i))$. From here, by applying the symmetrization projection onto $L^2(\mathbb{R}^d \times \mathbb{R}, dx\sigma(x, ds))^{\odot i}$ we conclude the statement. □

By Lemma 3.3 and Lemma 3.4, we get $F = \mathcal{F}(H_0)$.

6. Finally, let us prove that each vector from $\mathcal{F}_{\mathrm{fin}}(\mathscr{S})$ is analytic for each operator $A(\varphi)$, $\varphi \in \mathcal{D}$.

We start with the following lemma.

**Lemma 3.5.** *Fix any $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$. Then, for $n, m \in \mathbb{N}$ and any $x_1, \dots, x_n \in \Lambda$,*

$$\begin{aligned}&\int_{\mathbb{R}^n} (|s_1| + |s_2| + \cdots + |s_n|)^m \sigma(x_1, ds_1) \cdots \sigma(x_n, ds_n)\\ &\le C_\Lambda^m \frac{(n+m)!}{n!}.\end{aligned}$$

*Proof.* For any $x_1, x_2, \dots, x_n$, we have, by (3.2) and using an easy combinatoric formula,

$$\begin{aligned}&\int_{\mathbb{R}^n} (|s_1| + |s_2| + \cdots + |s_n|)^m \sigma(x_1, ds_1) \cdots \sigma(x_n, ds_n)\\ &= \sum_{l_1, \dots, l_n \in \mathbb{Z}_+,\, l_1 + l_2 + \cdots + l_n = m} \frac{m!}{l_1!\, l_2! \cdots l_n!} \int_{\mathbb{R}^n} |s_1|^{l_1} |s_2|^{l_2} \cdots |s_n|^{l_n}\\ &\quad \sigma(x_1, ds_1) \cdots \sigma(x_n, ds_n)\\ &= \sum_{l_1, \dots, l_n \in \mathbb{Z}_+,\, l_1 + l_2 + \cdots + l_n = m} \frac{m!}{l_1!\, l_2! \cdots l_n!} \int_{\mathbb{R}} |s_1|^{l_1} \sigma(x_1, ds_1) \int_{\mathbb{R}} |s_2|^{l_2}\end{aligned}$$

$$
\begin{aligned}
&\sigma(x_2, ds_2) \cdots \int_{\mathbb{R}} |s_n|^{l_n} \sigma(x_n, ds_n) \\
&\leq C_\Lambda^m \sum_{l_1,\dots,l_n \in \mathbb{Z}_+,\, l_1+l_2+\cdots+l_n=m} \frac{m!}{l_1!\, l_2! \cdots l_n!} l_1!\, l_2! \cdots l_n! \\
&= C_\Lambda^m\, m! \sum_{l_1,\dots,l_n \in \mathbb{Z}_+, l_1+l_2+\cdots+l_n=m} 1 \\
&\leq C_\Lambda^m m! \frac{(n+m)!}{m!\, n!} \\
&= C_\Lambda^m \frac{(n+m)!}{n!}.
\end{aligned}
$$

□

*Remark* 3.6. In fact, we will use the following weaker estimate:

$$\int_{\mathbb{R}^n} (|s_1| + |s_2| + \cdots + |s_n|)^m \sigma(x_1, ds_1) \cdots \sigma(x_n, ds_n) \leq C_\Lambda^m (m+n)! \quad (3.21)$$

for all $x_1, x_2, \dots, x_n \in \Lambda$.

It suffices to prove that each vector of the form

$$f^{(m)}(x_1, s_1, x_2, s_2 \dots, x_m, s_m) = g^{(m)}(x_1, x_2, \dots, x_m) s_1^{l_1} \odot \cdots \odot s_m^{l_m},$$

where $g^{(m)} \in \mathcal{D}^{\odot m}$, $l_1, \dots, l_m \geq 0$, $m \in \mathbb{N}$, is analytic for each $A(\varphi)$, $\varphi \in \mathcal{D}$. We will denote $n := l_1 + l_2 + \cdots + l_m$.

Below we will denote by $C$ different positive constants whose explicit values are not essential for us. So, we fix $\varphi \in \mathcal{D}$ and we have to prove that there exists $C$ such that

$$\|A^n(\varphi) f^{(m)}\|_{\mathcal{F}(H_0)} \leq C^n n!, \quad n \in \mathbb{N}. \quad (3.22)$$

Since

$$
\begin{aligned}
\|A^n(\varphi) f^{(m)}\|^2_{\mathcal{F}(H_0)} &= (A^n(\varphi) f^{(m)}, A^n(\varphi) f^{(m)})_{\mathcal{F}(H_0)} \\
&= (A^{2n}(\varphi) f^{(m)}, f^{(m)})_{\mathcal{F}(H_0)},
\end{aligned}
$$

(3.22) is equivalent to

$$(A^{2n}(\varphi)f^{(m)}, f^{(m)})_{\mathcal{F}(H_0)} \leq C^n(n!)^2. \tag{3.23}$$

Choose $\Delta \in \mathcal{B}_0(\mathbb{R}^d)$ such that

$$|\varphi| \leq C\chi_\Delta, \quad |g^{(m)}| \leq C\chi_\Delta^{\otimes m}.$$

Then (3.23) would follow from

$$\left(B^{2n}(\Delta)\psi^{(m)}, \psi^{(m)}\right)_{\mathcal{F}} \leq C^n(n!)^2, \tag{3.24}$$

where

$$B(\Delta) = a^+(\chi_\Delta(x)1(s)) + a^-(\chi_\Delta(x)1(s)) + a^0(\chi_\Delta(x)|s|)$$

and

$$\psi^{(m)}(x_1, s_1, x_2, s_2, \ldots, x_m, s_m) = \chi_\Delta^{\otimes m}(x_1, x_2, \ldots, x_m)|s_1|^{l_1} \odot \cdots \odot |s_m|^{l_m}.$$

We see that

$$B^{2n}(\Delta) = \left(a^+(\chi_\Delta(x)1(s)) + a^-(\chi_\Delta(x)1(s)) + a^0(\chi_\Delta(x)|s|)\right)^{2n},$$

which is a sum of $3^{2n}$ terms where every term is a product of $2n$ operators each of which is one of the operators $a^+(\chi_\Delta(x)1(s))$, $a^-(\chi_\Delta(x)1(s))$, $a^0(\chi_\Delta(x)|s|)$. Since we have to estimate $\left(B^{2n}(\Delta)\psi^{(m)}, \psi^{(m)}\right)$, we are interested only in those terms which have the same number of creation and annihilation operators. Denote this number by $k$. Thus, we consider a term in which we have a product (in arbitrary order) of $k$ creation operators $a^+(\chi_\Delta(x)1(s))$, $k$ annihilation operators $a^-(\chi_\Delta(x)1(s))$ and $(2n-2k) = 2(n-k)$ neutral operators $a^0(\chi_\Delta(x)|s|)$. Denote such a term by $D$. Without loss of generality, we assume that

$$\operatorname{vol}(\Delta) := \int_\Delta dx$$

is $\geq 1$ (otherwise we need to extent the set $\Delta$).

By using (3.21), for any $q, r \in \mathbb{N}$, $f \in L^2(\mathbb{R}^q, \sigma(x_1, ds_1) \otimes \cdots \otimes \sigma(x_q, ds_q))$, $x_1, \ldots, x_q \in \Delta$, by the Cauchy inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^n} (|s_1| + |s_2| + \cdots + |s_q|)^r f(s_1, s_2, \ldots, s_q) \sigma(x_1, ds_1) \sigma(x_2, ds_2) \cdots \sigma(x_q, ds_q) \\
&\quad \leq \left( \int_{\mathbb{R}^n} (|s_1| + |s_2| + \cdots + |s_q|)^{2r} \sigma(x_1, ds_1) \sigma(x_2, ds_2) \cdots \sigma(x_q, ds_q) \right)^{\frac{1}{2}} \\
&\qquad \times \|f\|_{L^2(\mathbb{R}^q, \sigma(x_1, ds_1) \otimes \cdots \otimes \sigma(x_q, ds_q))} \\
&\quad \leq C^{2r+q} ((2r+q)!)^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^q, \sigma(x_1, ds_1) \otimes \cdots \otimes \sigma(x_q, ds_q))} \\
&\quad \leq C^{2r+q} ((2(r+q))!)^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^q, \sigma(x_1, ds_1) \otimes \cdots \otimes \sigma(x_q, ds_q))} \\
&\quad \leq C^{r+q} (r+q)! \|f\|_{L^2(\mathbb{R}^q, \sigma(x_1, ds_1) \otimes \cdots \otimes \sigma(x_q, ds_q))},
\end{aligned} \tag{3.25}$$

where we used the inequality:

$$\begin{aligned}
(2l)! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdots (2l-1) \cdot (2l) \\
&\leq 2 \cdot 2 \cdot 4 \cdot 4 \cdots (2l)(2l) \\
&= (2 \cdot 4 \cdot 6 \cdots (2l))^2 \\
&= (2^l 1 \cdot 2 \cdot 3 \cdots l)^2 \\
&= 2^{2l} (l!)^2, \quad l \in \mathbb{N}.
\end{aligned}$$

Therefore,

$$(D\psi^{(m)}, \psi^{(m)})_{\mathcal{F}} \leq C^n m! (m+1)(m+2) \cdots (m+k)(m+k+2(n-k))! \tag{3.26}$$

where the factor $m!$ comes from the fact that $\psi^{(m)}$ belongs to $\mathcal{F}^{(m)} L^2(\mathbb{R} \times \mathbb{R}, dx\, \sigma(x, ds))$, the factor $(m+1)(m+2) \cdots (m+k)$ comes from the fact that we have $k$ annihilation operators, and the factor $(m+k+2(n-k))!$ comes from the estimate (3.25) and the fact that we have $2(n-k)$ neutral operators.

Hence, by (3.26)

$$\begin{aligned}(D\psi^{(m)},\psi^{(m)})_{\mathcal{F}} &\le C^n(m+k)!\,((m+k)+2(n-k))!\\ &\le C^n(2(m+k)+2(n-k))!\\ &= C^n(2m+2n)!\\ &\le C^n(2n)!\\ &\le C^n(n!)^2,\end{aligned}$$

where we used that, for a fixed $m$,

$$(2n+2m)! \le C^n(2n).$$

From here the estimate (3.24) follows. □

Let us find the Fourier transform of the spectral measure $\mu$ of the family $(A(\varphi))_{\varphi\in\mathcal{D}}$ in $\mathcal{F}(H_0)$ from Theorem 3.2. $\sigma(x,ds)$.

**Theorem 3.7.** *The spectral measure $\mu$ of the family of operators $(A(\varphi))_{\varphi\in\mathcal{D}}$ in $\mathcal{F}(H_0)$, which exists due to Theorem 3.2, has the following Fourier transform:*

$$\begin{aligned}\int_{\mathcal{D}'} e^{i\langle\varphi,\omega\rangle}\mu(d\omega) = \exp\Big[ &-\frac{1}{2}\int_{\mathbb{R}^d} dx\,\sigma(x,\{0\})\varphi(x)^2\\ &+\int_{\mathbb{R}^d} dx\int_{\mathbb{R}^*}\sigma(x,ds)\frac{1}{s^2}[e^{i\varphi(x)s}-i\varphi(x)s-1]\Big],\end{aligned} \tag{3.27}$$

*where* $\mathbb{R}^* := \mathbb{R}\setminus\{0\}$.

*Proof.* We will divide the proof into several steps.

**Step 1.** Formula (3.27) is evidently equivalent to the following formula:

$$\begin{aligned}\int_{\mathcal{D}'} e^{it\langle\varphi,\omega\rangle}\mu(d\omega) = \exp\Big[ &-\frac{1}{2}\int_{\mathbb{R}^d} dx\,\sigma(x,\{0\})\varphi(x)^2t^2\\ &+\int_{\mathbb{R}^d} dx\int_{\mathbb{R}^*}\sigma(x,ds)\frac{1}{s^2}[e^{i\varphi(x)st}-i\varphi(x)st-1]\Big],\end{aligned} \tag{3.28}$$

where $\varphi \in \mathcal{D}$ and $t \in \mathbb{R}$. Fix any $\varphi \in \mathcal{D}$. Consider the following measurable mapping:

$$\mathbb{R}^d \times \mathbb{R}^* \ni (x, s) \mapsto \varphi(x)s \in \mathbb{R}. \tag{3.29}$$

Let $\xi(dz)$ be the image of the measure $dx\, \sigma(x, ds)\frac{1}{s^2}$ under the mapping (3.29). Let also

$$a := \int_{\mathbb{R}^d} dx\, \sigma(x, \{0\})\varphi(x)^2.$$

Then, the right hand side of the formula (3.28) can be written in the form

$$\exp\left[\frac{1}{2}a(it)^2 + \int_{\mathbb{R}} \xi(dz)[e^{itz} - itz - 1]\right].$$

Since the function $e^{itz} - itz - 1$ vanishes when $z = 0$, we continue

$$= \exp\left[\frac{1}{2}a(it)^2 + \int_{\mathbb{R}^*} \xi(dz)[e^{itz} - itz - 1]\right]. \tag{3.30}$$

Let us check that $\xi(dz)$ is a Lévy measure, i.e., it satisfies

$$\int_{\mathbb{R}^*} (z^2 \wedge 1)\xi(dz) < \infty.$$

So, let us first show that

$$\int_{[-1,1]\setminus\{0\}} z^2\xi(dz) < \infty.$$

Indeed

$$\begin{aligned}\int_{[-1,1]\setminus\{0\}} z^2\xi(dz) &\le \int_{\mathbb{R}} z^2\xi(dz)\\ &= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x, ds)\frac{1}{s^2}(s\varphi(x))^2\\ &= \int_{\mathbb{R}^d} dx\, \varphi(x)^2 \int_{\mathbb{R}^*} \sigma(x, ds) < \infty.\end{aligned}$$

Next,

$$\begin{aligned}
\int_{\mathbb{R}} \chi_{\{|z|\geq 1\}}(z)\,\xi(dz) &= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x,ds)\frac{1}{s^2}\chi_{\{|s\varphi(x)|\geq 1\}}(s,x) \\
&= \int_{\text{supp }\varphi} dx \int_{\mathbb{R}^*} \sigma(x,ds)\frac{1}{s^2}\chi_{\{|s\varphi(x)|\geq 1\}}(s,x) \\
&\leq \int_{\text{supp }\varphi} dx \int_{\mathbb{R}^*} \sigma(x,ds)\frac{1}{s^2}\chi_{\{|s|\geq \frac{1}{\sup_{y\in\mathbb{R}^d}|\varphi(y)|}\}}(s,x) \\
&\leq \int_{\text{supp }\varphi} dx \left(\sup_{y\in\mathbb{R}^d}|\varphi(y)|\right)^2 \int_{\mathbb{R}^*} \sigma(x,ds) < +\infty.
\end{aligned}$$

Therefore, the expression in (3.30) is the Fourier transform of an infinitely divisible random variable (see e.g. [4]). In particular, the right hand side of formula (3.28), considered as a function of $t$, is the Fourier transform of a random variable. We state the Lapace transform of an infinite divisible random variable can be extended to a function of complex variable which is analytic in a neighborhood of zero.

Indeed, we first note that the right hand side of (3.28) can be written as

$$\begin{aligned}
&\exp\Bigg[\frac{(it)^2}{2}\int_{\mathbb{R}^d} dx\,\sigma(x,\{0\})\varphi(x)^2 \\
&\quad + \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x,ds)\sum_{n=2}^{\infty}\frac{\varphi(x)^n s^{n-2}(it)^n}{n!}\Bigg]. \qquad (3.31)
\end{aligned}$$

By (3.2),

$$\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x,ds)\sum_{n=2}^{\infty}\frac{|\varphi(x)|^n|s|^{n-2}|t|^n}{n!} < \infty$$

for $|t| < C_{\text{supp}(\varphi)}$. Hence, (3.31) can be extended to an analytic function

$$\begin{aligned}
&\{z\in\mathbb{C} \mid |z| < C_{\text{supp}(\varphi)}\} \ni z \mapsto \exp\Bigg[\frac{z^2}{2}\int_{\mathbb{R}^d} dx\,\sigma(x,\{0\})\varphi(x)^2 \\
&\quad + \sum_{n=2}^{\infty}\frac{z^n}{n!}\int_{\mathbb{R}^d}\varphi(x)^n\,dx\int_{\mathbb{R}^*}\sigma(x,ds)s^{n-2}\Bigg].
\end{aligned}$$

On the other hand, by (3.22),

$$\int_{\mathcal{D}'} |\langle\varphi,\omega\rangle|^n\,\mu(d\omega) \leq C^n n!.$$

Hence, by Theorem 2.6 and Remark 2.7, if we show that equality (3.28) holds for all $t$ from a neighborhood of zero in $\mathbb{R}$, then it will follow that equality (3.28) holds for all $t \in \mathbb{R}$, and so (3.27) holds.

**Step 2.** For each $x \in \mathbb{R}^d$, we define a measure $\nu(x, ds)$ on $\mathbb{R}^*$ by

$$\nu(x, ds) := \frac{1}{s^2}\sigma(x, ds).$$

We also define a measure $\varkappa(x, ds)$ on $\mathbb{R}$ by

$$\varkappa(x, ds) = \sigma(x, \{0\})\delta_0(ds) + \nu(x, ds). \tag{3.32}$$

Here $\delta_0(ds)$ is the Dirac measure at 0. Note that the measure $\sigma(x, \{0\})\delta_0(ds)$ is concentrated at 0, while the measure $\nu(x, ds)$ is concentrated on $\mathbb{R}^*$. Define a Hilbert space

$$\mathcal{H}_0 := L^2(\mathbb{R}^d \times \mathbb{R}, dx\, \varkappa(x, ds)). \tag{3.33}$$

We construct a unitary isomorphism

$$\mathcal{U} : H_0 \to \mathcal{H}_0 \tag{3.34}$$

by

$$(\mathcal{U}f)(x, s) = \begin{cases} f(x, 0) & \text{if } s = 0, \\ f(x, s)s & \text{if } s \neq 0. \end{cases} \tag{3.35}$$

We naturally extend this isomorphism to a unitary operator

$$\mathcal{U} : \mathcal{F}(H_0) \to \mathcal{F}(\mathcal{H}_0). \tag{3.36}$$

We will use the same notation for operators in $\mathcal{F}(H_0)$ and their images under $\mathcal{U}$, i.e. operators in $\mathcal{F}(\mathcal{H}_0)$. An easy calculation shows that an operator $A(\varphi)$ in $\mathcal{F}(\mathcal{H}_0)$ has the form

$$A(\varphi) = a^+(\varphi(x)s)+a^-(\varphi(x)s)+a^0(\varphi(x)s)+a^+(\varphi(x)\chi_{\{0\}}(s))+a^-(\varphi(x)\chi_{\{0\}}(s)). \tag{3.37}$$

The initial domain of this operator (before closure) is $\mathcal{U}\mathcal{F}_{\mathrm{fin}}(\mathscr{S})$.

**Step 3.** Fix any $\Lambda_1,\dots,\Lambda_n\in\mathcal{B}_0(\mathbb{R}^d)$, disjoint. Denote $\Lambda=\bigcup_{i=1}^n\Lambda_i$. Fix any $\Delta_1,\dots,\Delta_m$— disjoint, bounded, measurable subsets of $\mathbb{R}^*$, and set $\Delta_0:=\{0\}$. (We consider a metric on $\mathbb{R}^*$ such that the distance from any point in $\mathbb{R}^*$ to 0 (in the limiting sense) is $+\infty$).

Consider the functions

$$e_{ij}=\chi_{\Lambda_i\times\Delta_j},\quad i=1,\dots,n,\ j=0,1,\dots,m \tag{3.38}$$

in $\mathcal{H}_0$. These functions are evidently orthogonal. Let $R$ be the subspace of $\mathcal{H}_0$ which is the closed linear span of the functions (3.38). Thus, $(e_{ij})_{i=1,\dots,n,\,j=0,1,\dots,m}$ form an orthogonal basis in $R$. Consider the symmetric Fock space over $R$, i.e., $\mathcal{F}(R)$. An orthogonal basis of this space is formed by the vectors

$$e_{10}^{\otimes\alpha_{10}}\odot e_{11}^{\otimes\alpha_{11}}\odot\dots\odot e_{nm}^{\otimes\alpha_{nm}}=:e_{(\alpha_{10},\dots,\alpha_{nm})} \tag{3.39}$$

where $\alpha_{ij}\in\mathbb{Z}_+$. Denote by $G(R)$ the linear span of the vectors (3.39). Consider operators

$$a_{ij}:=\begin{cases}a^+(e_{ij})+a^-(e_{ij})+a^0(e_{ij}), & \text{if } j\neq 0,\\ a^+(e_{i0})+a^-(e_{i0}), & \text{if } j=0\end{cases} \tag{3.40}$$

in $\mathcal{F}(R)$ with domain $G(R)$. We have

$$\begin{aligned}
a^+(e_{ij})e_{(\alpha_{10},\dots,\alpha_{nm})}&=e_{(\alpha_{10},\dots,\alpha_{ij}+1,\dots,\alpha_{nm})},\\
a^0(e_{ij})e_{(\alpha_{10},\dots,\alpha_{nm})}&=\alpha_{ij}e_{(\alpha_{10},\dots,\alpha_{nm})}\quad \text{if } j\neq 0,\\
a^-(e_{ij})e_{(\alpha_{10},\dots,\alpha_{nm})}&=\alpha_{ij}\int_{\Lambda_i}dx\int_{\Delta_j}\varkappa(x,ds)\,e_{(\alpha_{10},\dots,\alpha_{ij}-1,\dots,\alpha_{nm})}.
\end{aligned} \tag{3.41}$$

Denote by $\mathcal{F}_{ij}$ the closed subspace of $\mathcal{F}(R)$ in which vectors $(e_{ij}^{\otimes\alpha_{ij}})_{\alpha_{ij}\in\mathbb{Z}_+}$ form an orthogonal basis. Consider the tensor product of Hilbert spaces

$$\mathcal{F}_{10}\otimes\mathcal{F}_{11}\otimes\dots\otimes\mathcal{F}_{nm}.$$

By the definition of symmetric tensor product, we may construct a unitary isomorphism

$$S : \mathcal{F}_{10} \otimes \mathcal{F}_{11} \otimes \cdots \otimes \mathcal{F}_{nm} \to \mathcal{F}(R)$$

by setting

$$S e_{10}^{\otimes \alpha_{10}} \otimes e_{11}^{\otimes \alpha_{11}} \otimes \cdots \otimes e_{nm}^{\otimes \alpha_{nm}} = e_{10}^{\otimes \alpha_{10}} \odot e_{11}^{\otimes \alpha_{11}} \odot \cdots \odot e_{nm}^{\otimes \alpha_{nm}}.$$

Then it follows from (3.40) and (3.41) that each operator

$$S^{-1} a_{ij} S$$

has the form

$$\mathbf{1} \otimes \cdots \otimes A_{ij} \otimes \cdots \otimes \mathbf{1},$$

where the operator $A_{ij}$ (staying at the $ij$-th place) is the operator $a_{ij}$ acting in $\mathcal{F}_{ij}$ with domain which is the linear span of the vectors $(e_{ij}^{\otimes \alpha_{ij}})_{\alpha_{ij} \in \mathbb{Z}_+}$. As easily seen from our previous considerations, each operator $A_{ij}$ is essentially self-adjoint in $\mathcal{F}_{ij}$.

By [7, Chapter 3], we can construct the Fourier transform of the finite family of operators $(A_{ij})_{i=1,\dots,n,\, j=0,1,\dots,m}$ in the Hilbert space $\mathcal{F}_{10} \otimes \mathcal{F}_{11} \otimes \cdots \otimes \mathcal{F}_{nm}$. Its spectral measure, denoted by $\gamma$, is the product measure

$$\gamma = \gamma_{10} \otimes \gamma_{11} \otimes \cdots \otimes \gamma_{nm}$$

on $\mathbb{R}^{n(m+1)}$, where $\gamma_{ij}$ is the spectral measure of the operator $A_{ij}$ in $\mathcal{F}_{ij}$ at the vacuum state $e_{ij}^{\otimes 0}$. By formula (3.41), we have

$$A_{ij} e_{ij}^{\otimes \alpha_{ij}} = e_{ij}^{\otimes (\alpha_{ij}+1)} + \alpha_{ij} e_{ij}^{\otimes \alpha_{ij}} + \alpha_{ij} \int_{\Lambda_i} dx \int_{\Delta_j} \nu(x, ds)\, e_{ij}^{\otimes (\alpha_{ij}-1)}$$

if $j \neq 0$, and

$$A_{i0} e_{i0}^{\otimes \alpha_{i0}} = e_{i0}^{\otimes (\alpha_{i0}+1)} + \alpha_{i0} \int_{\Lambda_i} dx\, \sigma(x, \{0\}) e_{i0}^{\otimes (\alpha_{i0}-1)}.$$

From here we immediately find that $\gamma_{ij}$ is the centered Poisson measure with parameter $\int_{\Lambda_i} dx \int_{\Delta_j} \nu(x, ds)$ if $j \neq 0$, and the Gaussian measure with mean 0 and variance $\int_{\Lambda_i} dx\, \sigma(x, \{0\})$ if $j = 0$ (see [15, Chap. 1, Sec. 4]). Hence, we have, for $r_{10}, \dots r_{nm} \in \mathbb{R}$,

$$\begin{aligned}
&\int_{\mathbb{R}^{n(m+1)}} e^{i(y_{10}r_{10}+\cdots+y_{nm}r_{nm})} d\gamma_{10}(y_{10}) \cdots d\gamma_{nm}(y_{nm}) \\
&= \exp\Bigg[ -\frac{1}{2}\bigg( \int_{\Lambda_1} dx\, \sigma(x,\{0\})\, r_{10}^2 + \cdots + \int_{\Lambda_n} dx\, \sigma(x,\{0\})\, r_{n0}^2 \bigg) \\
&\quad + \int_{\Lambda_1} dx \int_{\Delta_1} \nu(x, ds)(e^{ir_{11}} - ir_{11} - 1) \\
&\quad + \cdots + \int_{\Lambda_n} dx \int_{\Delta_m} \nu(x, ds)(e^{ir_{nm}} - ir_{nm} - 1) \Bigg] \\
&= \exp\Bigg[ -\frac{1}{2} \int_{\mathbb{R}^d} dx\, \sigma(x,\{0\}) (\chi_{\Lambda_1}(x) r_{10} + \cdots + \chi_{\Lambda_n}(x) r_{n0})^2 \\
&\quad + \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \nu(x, ds)(e^{if(x,s)} - if(x,s) - 1) \Bigg],
\end{aligned} \tag{3.42}$$

where

$$f(x,s) = \chi_{\Lambda_{11}}(x)\chi_{\Delta_1}(s) r_{11} + \cdots + \chi_{\Lambda_{nm}}(x)\chi_{\Delta_m}(s) r_{nm}.$$

**Step 4.** We define, for a function $g(x,s)$, an operator

$$B(g) := a^+(g) + a^-(g) + a^0(g(x,s)\chi_{\mathbb{R}^*}(s))$$

in $\mathcal{F}(\mathcal{H}_0)$ (on a proper domain). We now set

$$\begin{aligned}
g(x,s) :&= \chi_{\Lambda_1}(x)\chi_{\{0\}}(s) r_{10} + \cdots + \chi_{\Lambda_n}(x)\chi_{\{0\}}(s) r_{n0} \\
&\quad + \chi_{\Lambda_1}(x)\chi_{\Delta_1}(s) r_{11} + \cdots + \chi_{\Lambda_n}(x)\chi_{\Delta_m}(s) r_{nm}.
\end{aligned} \tag{3.43}$$

By Step 3 and estimate (3.24), for any $z \in \mathbb{C}$ with $|z|$ sufficiently small, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{z^n (B(g)^n \Omega, \Omega)_{\mathcal{F}(\mathcal{H}_0)}}{n!} &= \exp\Bigg[ \frac{1}{2} \int_{\mathbb{R}^d} dx\, \sigma(x,\{0\}) z^2 g(x,0)^2 \\
&\quad + \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \nu(x, ds)(e^{z\, g(x,s)} - z\, g(x,s) - 1) \Bigg].
\end{aligned}$$

**Step 5.** Let us fix sets $\Lambda_1, \dots, \Lambda_n$ as above, let $r_1, \dots, r_n \in \mathbb{R}$. Set

$$\psi(x) := \chi_{\Lambda_1}(x) r_1 + \cdots + \chi_{\Lambda_n}(x) r_n. \tag{3.44}$$

We now approximate the function

$$f(x, s) := \psi(x)\chi_{\{0\}}(s) + \psi(x)s$$

point-wise by function as in (3.43). Then, at least informally, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{z^n (B(f)^n \Omega, \Omega)_{\mathcal{F}(\mathcal{H}_0)}}{n!} &= \exp\Bigg[\frac{z^2}{2} \int_{\mathbb{R}^d} \sigma(x, \{0\}) \psi(x)^2 dx \\
&\quad + \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \nu(x, ds)(e^{z\psi(x)s} - z\,\psi(x)s - 1)\Bigg] \\
&= \exp\Bigg[\frac{z^2}{2} \int_{\mathbb{R}^d} \sigma(x, \{0\}) \psi(x)^2 dx \\
&\quad + \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \frac{\sigma(x, ds)}{s^2}(e^{z\psi(x)s} - z\,\psi(x)z - 1)\Bigg].
\end{aligned} \tag{3.45}$$

Let us justify this limit. We can assume that the functions $g_k(x, s)$ of the form (3.43) by which we approximate the function $f(x, s)$ satisfy

$$|g_k(x, s)| \le C\chi_\Lambda(x)(|s| + \chi_{\{0\}}(s)), \tag{3.46}$$

for all $k \in \mathbb{N}$, where $C > 0$.

We have, for $x \in \mathbb{R}^d$, $s \in \mathbb{R}^*$, and $z \in \mathbb{C}$,

$$\frac{1}{s^2} \sum_{n=2}^{\infty} \frac{|z|^n |g_k(x, s)|^n}{n!} \le \chi_\Lambda(x) \sum_{n=2}^{\infty} \frac{|z|^n C^n |s|^{n-2}}{n!}. \tag{3.47}$$

Hence, by (3.2), (3.47), and the dominated convergence theorem,

$$\begin{aligned}
&\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x, ds) s^{-2} (e^{z\, g_k(x,s)} - z\, g_k(x, s) - 1) \\
&\qquad \to \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x, ds) s^{-2} (e^{z\,\psi(x)s} - z\,\psi(x)s - 1),
\end{aligned}$$

as $k \to \infty$, for $z$ from a neighborhood of zero in $\mathbb{C}$. Also by (3.46) and dominated convergence theorem

$$\int_{\mathbb{R}^d} dx\, \sigma(x, \{0\}) g_k(x, 0)^2 \to \int_{\mathbb{R}^d} dx\, \sigma(x, \{0\}) \psi(x)^2.$$

Hence, for $z$ from a neighborhood of zero in $\mathbb{C}$,

$$\begin{aligned}
&\exp\left[\frac{z^2}{2} \int_{\mathbb{R}^d} dx\, \sigma(x, \{0\}) g_k(x, 0)^2 \right.\\
&\quad \left. + \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x, ds)\, s^{-2} (e^{z g_k(x,s)} - z\, g_k(x, s) - 1)\right]\\
&\to \exp\left[\frac{z^2}{2} \int_{\mathbb{R}^d} dx\, \sigma(x, \{0\}) \psi(x)^2 \right.\\
&\quad \left. + \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x, ds)\, s^{-2} (e^{z\psi(x)s} - z\, \psi(x)s - 1)\right].
\end{aligned}$$

Next, let us show that

$$\sum_{n=0}^{\infty} \frac{z^n (B(g_k)^n \Omega, \Omega)_{\mathcal{F}(\mathcal{H}_0)}}{n!} \to \sum_{n=0}^{\infty} \frac{z^n (A(\psi)^n \Omega, \Omega)_{\mathcal{F}(\mathcal{H}_0)}}{n!} \tag{3.48}$$

as $k \to \infty$ for $z$ from a neighborhood of zero in $\mathbb{C}$. We first note, by the dominated convergence theorem and (3.46), that for a fixed $n \in \mathbb{N}$,

$$(B(g_k)^n \Omega, \Omega)_{\mathcal{F}(\mathcal{H}_0)} \to (A(\psi)^n \Omega, \Omega)_{\mathcal{F}(\mathcal{H}_0)}$$

as $k \to \infty$. Furthermore, as follows from (3.24), there exists a constant $C > 0$ such that

$$|(B(g_k)^n \Omega, \Omega)_{\mathcal{F}(\mathcal{H}_0)}| \le C^n\, n! \tag{3.49}$$

for all $k \in \mathbb{N}$. Therefore, (3.48) holds by the dominated convergence theorem.

Hence, we conclude that, for some $\epsilon > 0$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{z^n (A(\psi)^n \Omega, \Omega)_{\mathcal{F}(\mathcal{H}_0)}}{n!} &= \exp\left[\frac{z^2}{2} \int_{\mathbb{R}^d} dx\, \sigma(x, \{0\}) \psi(x, 0)^2 \right.\\
&\quad \left. + \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x, ds)\, s^{-2} (e^{z\psi(x)s} - z\, \psi(x)s - 1)\right]
\end{aligned} \tag{3.50}$$

for all $z \in \mathbb{C}$, $|z| < \epsilon$. As easily seen from the above considerations, $\epsilon$ depends only on $\Lambda = \bigcup_{i=1}^{n} \Lambda_i$ and on $\sup_{x\in\Lambda} |\psi(x)| = \max_{i=1,\dots,n} |r_i|$.

**Step 6.** We fix any $\varphi \in \mathcal{D}$. Let $\Lambda$ be the support of the function $\varphi$. We will now approximate $\varphi$ by functions $\psi_k$ as in (3.44). For each $k$, we will denote corresponding $\Lambda$-sets by $\Lambda_1^{(k)}, \Lambda_2^{(k)}, \dots, \Lambda_{n_k}^{(k)}$, so that $\bigcup_{i=1}^{n_k} \Lambda_{n_i} = \Lambda$. We will also assume that $\sup_{x\in\Lambda} |\psi_k(x)| \le C$ for all $k \in \mathbb{N}$. By the dominated convergence theorem, we get

$$\begin{aligned}
&\exp\left[\frac{z^2}{2}\int_{\mathbb{R}^d} dx\,\sigma(x,\{0\})\psi_k(x)^2\right.\\
&\quad\left.+\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x,ds)\, s^{-2}(e^{z\,\psi_k(x)s} - z\,\psi_k(x)s - 1)\right]\\
&\to \exp\left[\frac{z^2}{2}\int_{\mathbb{R}^d} dx\,\sigma(x,\{0\})\,\varphi(x)^2\right.\\
&\quad\left.+\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x,ds)\, s^{-2}(e^{z\,\varphi(x)s} - z\,\varphi(x)s - 1)\right]
\end{aligned}$$

for $z \in \mathbb{C}$ from a neighborhood of zero. So, to prove the theorem, it remains to show that, for $z \in \mathbb{C}$ from a neighborhood of zero,

$$\sum_{n=0}^{\infty} \frac{z^n (A(\psi_k)^n\Omega, \Omega)_{\mathcal{F}(\mathcal{H}_0)}}{n!} \to \sum_{n=0}^{\infty} \frac{z^n (A(\varphi)^n\Omega, \Omega)_{\mathcal{F}(\mathcal{H}_0)}}{n!}. \tag{3.51}$$

Similarly to (3.49), we have, for all $k \in \mathbb{R}$,

$$|(A(\psi_k)^n\Omega, \Omega)_{\mathcal{F}(\mathcal{H}_0)}| \le C^n\, n!$$

Hence, by the dominated convergence theorem, formula (3.51) would follow if we show that, for each $n \in \mathbb{N}$

$$(A(\psi_k)^n\Omega, \Omega)_{\mathcal{F}(\mathcal{H}_0)} \to (A(\varphi)^n\Omega, \Omega)_{\mathcal{F}(\mathcal{H}_0)} \quad \text{as } k \to \infty, \tag{3.52}$$

which again follows by the dominated convergence theorem. $\square$

We will now summarize the main results of this chapter.

**Corollary 3.8.** *Assume that condition* (3.2) *is satisfied. Let a Hilbert space* $H_0$ *be given by* (3.3)*, and let a nuclear space* $\mathscr{S}$ *is given by* (3.4)*. For each* $\varphi \in \mathcal{D}$*, let* $A(\varphi)$ *be the Hermitian operator on* $\mathcal{F}(H_0)$ *defined by* (3.9)*, with domain* $\mathcal{F}_{\mathrm{fin}}(\mathscr{S})$*. Then these operators are essentially selfadjoint on* $\mathcal{F}_{\mathrm{fin}}(\mathscr{S})$ *and their closures are denoted by* $(\tilde{A}(\varphi), D(\tilde{A}(\varphi)))$*. The latter selfadjoint operators commute in the sense of their resolutions of the identity. Furthermore, there exists a unique probability measure* $\mu$ *on* $(\mathcal{D}', \mathcal{C}(\mathcal{D}'))$ *such that the linear operator* $I : \mathcal{F}(H_0) \to L^2(\mathcal{D}', \mu)$ *given through* $I\Omega = 1$ *and*

$$\begin{aligned} I(\tilde{A}(\varphi_1) \cdots \tilde{A}(\varphi_n)\Omega) &= I(A(\varphi_1) \cdots A(\varphi_n)\Omega) \\ &= \langle \varphi_1, \omega\rangle \cdots \langle \varphi_n, \omega\rangle \in L^2(\mathcal{D}', \mu) \end{aligned}$$

*is unitary. Under the action of* $I$*, each operator* $(\tilde{A}(\varphi), D(\tilde{A}(\varphi)))$*,* $\varphi \in \mathcal{D}$ *goes over into the operator of multiplication by* $\langle \omega, \varphi\rangle$ *in* $L^2(\mathcal{D}', \mu)$*. The Fourier transform of the measure* $\mu$ *is given by* (3.27)*.*

Denote by $B_0(\mathbb{R}^d)$ the linear space of all measurable bounded, real-valued functions on $\mathbb{R}^d$. For each $f \in B_0(\mathbb{R}^d)$, we may define a random variable $\langle f, \omega\rangle$ as an $L^2(\mathcal{D}', \mu)$-limit of functions $\langle \varphi_n, \omega\rangle$ with $\varphi_n \in \mathcal{D}$, $n \in \mathbb{N}$, such that $\varphi_n \to f$ in $L^2(\mathbb{R}^d, dx)$. The Fourier transform $\int_{\mathcal{D}'} e^{i\langle f, \omega\rangle}\, \mu(d\omega)$ is clearly given by the right hand side of (3.27) in which $\varphi$ is replaced by $f$.

Let $\Lambda_1, \dots, \Lambda_n \in \mathcal{B}_0(\mathbb{R}^d)$, mutually disjoint. Then for any $t_1, \dots, t_n \in \mathbb{R}$, by (3.27),

$$\int_{\mathcal{D}'} \exp\left[i(t_1\langle \chi_{\Lambda_1}, \omega\rangle + \cdots + t_n\langle \chi_{\Lambda_n}, \omega\rangle)\right] \mu(d\omega) = \prod_{i=1}^{n} \int_{\mathcal{D}'} \exp[it_1\langle \chi_{\Lambda_i}, \omega\rangle]\, \mu(d\omega).$$

So, the random variables $\langle \chi_{\Lambda_1}, \omega\rangle, \dots, \langle \chi_{\Lambda_n}, \omega\rangle$ are independent. Thus, the probability measure $\mu$ is a *generalized stochastic process with independent values*, see [19].

# Chapter 4

# Nualart–Schoutens decomposition

The aim of this chapter is to generalize the Naulart–Schoutens chaotic decomposition of the $L^2$-space of a Lévy process ( [36], see also [31, 40]) to the case of a rather general generalized stochastic process with independent values.

We start this chapter with a discussion of an orthogonal decomposition of a general Fock space. This decomposition generalizes, in some sense, the well-known basis of occupation numbers in the Fock space, see Section 2.9.

Let $H$ be a real seperable Hilbert space. Let $(H_k)_{k=0}^{\infty}$ be a sequence of closed subspaces of $H$ such that

$$H = \bigoplus_{k=0}^{\infty} H_k.$$

Let $n \geq 2$. Then clearly

$$\begin{aligned} H^{\otimes n} &= \left( \bigoplus_{k_1=0}^{\infty} H_{k_1} \right) \otimes \left( \bigoplus_{k_2=0}^{\infty} H_{k_2} \right) \otimes \cdots \otimes \left( \bigoplus_{k_n=0}^{\infty} H_{k_n} \right) \\ &= \bigoplus_{(k_1,k_2,\dots,k_n)\in\mathbb{Z}_+^n} H_{k_1} \otimes H_{k_2} \otimes \cdots \otimes H_{k_n}. \end{aligned} \tag{4.1}$$

Denote by $\mathrm{Sym}_n$ the orthogonal projection of $H^{\otimes n}$ onto $H^{\odot n}$. Recall that, for any $f_1, f_2, \dots, f_n \in H$

$$n!\,\mathrm{Sym}_n\ f_1 \otimes \cdots \otimes f_n = \sum_{\sigma \in S_n} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}. \tag{4.2}$$

For each $(k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n$, let $H_{k_1} \odot H_{k_2} \odot \cdots \odot H_{k_n}$ denote the Hilbert space $\mathrm{Sym}_n(H_{k_1} \otimes H_{k_2} \otimes \cdots \otimes H_{k_n})$, i.e., the space of all $\mathrm{Sym}_n$-projections of elements of $H_{k_1} \otimes H_{k_2} \otimes \cdots \otimes H_{k_n}$.

Assume that $(k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n$ , $(l_1, l_2, \dots, l_n) \in \mathbb{Z}_+^n$ are such that there exists a permutation $\sigma \in S_n$ such that

$$(k_1, k_2, \dots, k_n) = (l_{\sigma(1)}, l_{\sigma(2)}, \dots, l_{\sigma(n)}). \tag{4.3}$$

Then

$$H_{k_1} \odot H_{k_2} \odot \cdots \odot H_{k_n} = H_{l_1} \odot H_{l_2} \odot \cdots \odot H_{l_n}. \tag{4.4}$$

Indeed, take any $f_1 \in H_{l_1}, f_2 \in H_{l_2}, \dots, f_n \in H_{l_n}$. Then

$$f_1 \odot f_2 \odot \cdots \odot f_n = f_{\sigma(1)} \odot f_{\sigma(2)} \odot \cdots \odot f_{\sigma(n)}. \tag{4.5}$$

We have $f_{\sigma(i)} \in H_{l_{\sigma(i)}} = H_{k_i}$. Therefore, the vector in (4.5) belongs to $H_{k_1} \odot H_{k_2} \odot \cdots \odot H_{k_n}$. Since the set of all vectors of the form $f_1 \odot f_2 \odot \cdots \odot f_n$ with $f_i \in H_{l_i}$ is total in $H_{l_1} \odot H_{l_2} \odot \cdots \odot H_{l_n}$, we therefore conclude that

$$H_{l_1} \odot H_{l_2} \odot \cdots \odot H_{l_n} \subset H_{k_1} \odot H_{k_2} \odot \cdots \odot H_{k_n}$$

By inverting the argument, we obtain the inverse conclusion, and so formula (4.4) holds.

If no permutation $\sigma \in S_n$ exists which satisfies (4.3), then

$$H_{k_1} \odot H_{k_2} \odot \cdots \odot H_{k_n} \perp H_{l_1} \odot H_{l_2} \odot \cdots \odot H_{l_n}. \tag{4.6}$$

Indeed, take any $f_i \in H_{k_i}$, $g_i \in H_{l_i}$, $i = 1, 2, \dots, n$. Then

$$\begin{aligned}
&\Big(f_1 \odot f_2 \odot \cdots \odot f_n, g_1 \odot g_2 \odot \cdots \odot g_n\Big)_{H^{\odot n}} \\
&\quad = \Big(\mathrm{Sym}_n\left(f_1 \otimes f_2 \otimes \cdots \otimes f_n\right), g_1 \otimes g_2 \otimes \cdots \otimes g_n\Big)_{H^{\otimes n}} \\
&\quad = \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^{n} (f_{\sigma(i)}, g_i)_H \\
&\quad = \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^{n} (f_i, g_{\sigma(i)})_H = 0.
\end{aligned}$$

Since the vectors of the form $f_1 \odot f_2 \odot \cdots \odot f_n$ with $f_i \in H_{k_i}$ and $g_1 \odot g_2 \odot \cdots \odot g_n$ with $g_i \in H_{l_i}$ form a total set in $H_{k_1} \odot H_{k_2} \odot \cdots \odot H_{k_n}$ and $H_{l_1} \odot H_{l_2} \odot \cdots \odot H_{l_n}$, respectively, we get (4.6).

By (4.1), the closed linear span of the spaces $H_{k_1} \odot H_{k_2} \odot \cdots \odot H_{k_n}$ with $(k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n$ coincides with $H^{\odot n}$. Hence, by (4.4) and (4.6), we get the orthogonal decomposition

$$H^{\odot n} = \bigoplus_{\alpha \in \mathbb{Z}_{+,0}^{\infty},\, |\alpha| = n} H_0^{\odot \alpha_0} \odot H_1^{\odot \alpha_1} \odot H_2^{\odot \alpha_2} \odot \cdots . \tag{4.7}$$

Hence, by (4.7) and the definition of $\mathcal{F}(H)$, we get the following

**Lemma 4.1.** *We have the orthogonal decomposition of the symmetric Fock space $\mathcal{F}(H)$:*

$$\mathcal{F}(H) = \bigoplus_{\alpha \in \mathbb{Z}_{+,0}^{\infty}} H_0^{\odot \alpha_0} \odot H_1^{\odot \alpha_1} \odot H_2^{\odot \alpha_2} \cdots (|\alpha|!). \tag{4.8}$$

Next, we have:

**Lemma 4.2.** *Let $\alpha \in \mathbb{Z}_{+,0}^{\infty}$, $|\alpha| \geq 2$. Then*

$$\begin{aligned}
&\mathrm{Sym}_n : \left(H_0^{\odot \alpha_0} \otimes H_1^{\odot \alpha_1} \otimes H_2^{\odot \alpha_2} \otimes \cdots\right) \alpha_0! \alpha_1! \alpha_2! \cdots \\
&\qquad \to \left(H_0^{\odot \alpha_0} \odot H_1^{\odot \alpha_1} \odot H_2^{\odot \alpha_2} \odot \cdots\right) |\alpha|!
\end{aligned} \tag{4.9}$$

*is a unitary operator.*

*Proof.* We start the proof with the following well-known observation (see e.g. Proposition 2.4 in [14], where this statement is shown in a much more general setting). Let $k, l \geq 1$, $n := k + l$. Then we have:

$$\mathrm{Sym}_n = \mathrm{Sym}_n(\mathrm{Sym}_k \otimes \mathrm{Sym}_l).$$

Hence, for any $\alpha \in \mathbb{Z}^{\infty}_{+,0}$, $|\alpha| = n$, we get

$$\mathrm{Sym}_n = \mathrm{Sym}_n(\mathrm{Sym}_{\alpha_0} \otimes \mathrm{Sym}_{\alpha_1} \otimes \mathrm{Sym}_{\alpha_2} \otimes \cdots).$$

Therefore, we have the following equality of subspaces of $H^{\otimes n}$ :

$$\begin{aligned}
&H_0^{\odot\alpha_0} \odot H_1^{\odot\alpha_1} \odot H_2^{\odot\alpha_2} \odot \cdots \\
&\quad = \mathrm{Sym}_n \left(H_0^{\otimes\alpha_0} \otimes H_1^{\otimes\alpha_1} \otimes H_2^{\otimes\alpha_2} \otimes \cdots\right) \\
&\quad = \mathrm{Sym}_n \left( \mathrm{Sym}_{\alpha_0} \otimes \mathrm{Sym}_{\alpha_1} \otimes \mathrm{Sym}_{\alpha_2} \otimes \cdots \right)\left(H_0^{\otimes\alpha_0} \otimes H_1^{\otimes\alpha_1} \otimes H_2^{\otimes\alpha_2} \otimes \cdots\right) \\
&\quad = \mathrm{Sym}_n \left(H_0^{\odot\alpha_0} \otimes H_1^{\odot\alpha_1} \otimes H_2^{\odot\alpha_2} \otimes \cdots\right).
\end{aligned}$$

This shows that the image of the operator $\mathrm{Sym}_n$ in (4.9) is the whole space $H_0^{\odot\alpha_0} \odot H_1^{\odot\alpha_1} \odot H_2^{\odot\alpha_2} \odot \cdots n!$. Hence, we only need to prove that this operator is an isometry.

Fix any $f_i, g_i \in H_i$ with $i \in \mathbb{Z}_+$ and any $\alpha \in \mathbb{Z}^{\infty}_{+,0}$. Then, by (4.2)

$$\begin{aligned}
&\Big( \mathrm{Sym}_n \left(f_0^{\otimes\alpha_0} \otimes f_1^{\otimes\alpha_1} \otimes f_2^{\otimes\alpha_2} \otimes \cdots\right), \mathrm{Sym}_n \left(g_0^{\otimes\alpha_0} \otimes g_1^{\otimes\alpha_1} \otimes g_2^{\otimes\alpha_2} \otimes \cdots\right)\Big)_{H^{\odot n}} n! \\
&\quad = \Big( \mathrm{Sym}_n \left(f_0^{\otimes\alpha_0} \otimes f_1^{\otimes\alpha_1} \otimes f_2^{\otimes\alpha_2} \otimes \cdots\right), g_0^{\otimes\alpha_0} \otimes g_1^{\otimes\alpha_1} \otimes g_2^{\otimes\alpha_2} \otimes \cdots\Big)_{H^{\otimes n}} n! \\
&\quad = \sum_{\sigma_0 \in S_{\alpha_0}} \Big(f_0, g_0\Big)_{H_0}^{\alpha_0} \cdot \sum_{\sigma_1 \in S_{\alpha_1}} \Big(f_1, g_1\Big)_{H_1}^{\alpha_1} \cdots \\
&\quad = \Big(f_0^{\otimes\alpha_0}, g_0^{\otimes\alpha_0}\Big)_{H_0^{\odot\alpha_0}} \alpha_0! \Big(f_1^{\otimes\alpha_1}, g_1^{\otimes\alpha_1}\Big)_{H_1^{\odot\alpha_1}} \alpha_1! \cdots \\
&\quad = \Big(f_0^{\otimes\alpha_0} \otimes f_1^{\otimes\alpha_1} \otimes \cdots, g_0^{\otimes\alpha_0} \otimes g_1^{\otimes\alpha_1} \otimes \cdots\Big)_{H_0^{\odot\alpha_0} \otimes H_1^{\odot\alpha_1} \otimes \cdots} \alpha_0! \alpha_1! \cdots .
\end{aligned} \tag{4.10}$$

Since the set of all vectors of the form $f_i^{\otimes\alpha_i}$ with $f_i \in H_i$ is a total subset of $H_i^{\odot\alpha_i}$, we conclude from (4.10) that the operator in (4.9) is indeed an isometry. $\square$

By Lemmas 4.1 and 4.2, we get

**Lemma 4.3.** *The symmetrization operator*

$$\mathrm{Sym} : \bigoplus_{\alpha \in Z_{+,0}^{\infty}} \left(H_0^{\odot\alpha_0} \otimes H_1^{\odot\alpha_1} \otimes H_2^{\odot\alpha_2} \otimes \cdots\right)\alpha_0!\alpha_1!\alpha_2!\cdots \to \mathcal{F}(H)$$

*is unitary.*

*Remark* 4.4. Let us assume that each Hilbert space $H_k$ is one-dimensional and in each $H_k$ we fix a vector $e_k \in H_k$ such that $||e_k|| = 1$. Thus, $(e_k)_{k=0}^{\infty}$ is an orthonormal basis of $H$. By Lemma 4.3, the set of the vectors

$$\left((\alpha_0!\alpha_1!\alpha_2!\cdots)^{-\frac{1}{2}} e_0^{\otimes\alpha_0} \odot e_1^{\otimes\alpha_1} \odot e_2^{\otimes\alpha_2} \odot \cdots\right)_{\alpha \in \mathbb{Z}_{+,0}^{\infty}}$$

is an orthonormal basis of $\mathcal{F}(H)$, which is a basis of occupation numbers.

Now, we want to apply the general result about the orthogonal decomposition of the Fock space to the case of $\mathcal{F}(H)$, where $H = L^2(\mathbb{R}^d \times \mathbb{R}, dx\,\sigma(x, ds))$ is as in Chapter 3. (We have dropped the lower index 0 in $H_0$).

We denote by $(q^{(n)}(x,s))_{n\geq 0}$ the sequence of monic polynomials which are orthogonal with respect to the measure $\sigma(x, ds)$. By Section 2.2, we have the following recursive formula:

$$\begin{aligned} sq^{(n)}(x,s) &= q^{(n+1)}(x,s) + b_n(x)q^{(n)}(x,s) + a_n(x)q^{(n-1)}(x,s), \quad n \geq 1, \\ sq^{(0)}(x,s) &= q^{(1)}(x,s) + b_0(x). \end{aligned} \tag{4.11}$$

From now on, we will assume that the following condition is satisfied:

(A) For each $n \in \mathbb{N}$, the function $a_n(x)$ from (4.11) is locally bounded on $\mathbb{R}^d$, i.e., for each $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$, $\sup_{x\in\Lambda} a_n(x) < \infty$.

Denote by $\mathfrak{L}$ the linear space of all functions on $\mathbb{R}^d \times \mathbb{R}$ which have the form

$$f(x,s) = \sum_{k=0}^{n} \varphi_k(x) q^{(k)}(x,s), \tag{4.12}$$

where $n \in \mathbb{N}$, $\varphi \in \mathcal{D}$, $k = 0, 1, \dots, n$, and each $q^{(k)}(x,\cdot)$ is the $k$-th order monic orthogonal polynomial on $\mathbb{R}$ with respect to the measure $\sigma(x, ds)$, $x \in \mathbb{R}^d$. Analogously to Lemma 3.1, we get the following

**Lemma 4.5.** *The space $\mathfrak{L}$ is densely embedded into*

$$H = L^2(\mathbb{R}^d \times \mathbb{R}, dx\, \sigma(x, ds)).$$

*Proof.* Let $f(x,s) = \varphi(x) q^{(k)}(x,s)$, where $\varphi \in \mathcal{D}$. Let us show that $f \in H$. Denote $\Lambda := \operatorname{supp}(a)$. We have

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}} \sigma(x, ds) f(x,s)^2 \le C \int_\Lambda dx \int_{\mathbb{R}} \sigma(x, ds)\, q^{(k)}(x,s)^2. \tag{4.13}$$

If $k = 0$, then $q^{(0)}(x,s) = 1$, and the right hand side of (4.13) is evidently finite. By the theory of orthogonal polynomials (see e.g. [15] or [6])

$$\int_{\mathbb{R}} \sigma(x, ds_2)\, q^{(k)}(x,s)^2 = a_1(x) a_2(x) \cdots a_k(x), \quad k \ge 1. \tag{4.14}$$

Hence we continue (4.13)

$$\le C \int_\Lambda dx \int_{\mathbb{R}} \sigma(x, ds) a_1(x) a_2(x) \cdots a_k(x) < \infty$$

by (A). Thus, $\mathfrak{L} \subset H$.

We now have to show that $\mathfrak{L}$ is a dense subset of $H$. Let $g \in H$ be such that

$$(g, f)_H = 0 \quad \forall f \in \mathfrak{L}.$$

Hence for any $\varphi \in \mathcal{D}$ and $k \geq 0$

$$\int_{\mathbb{R}^d} dx \int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)\, \varphi(x)\, q^{(k)}(x, s) = 0.$$

Fix any compact set $\Lambda$ in $\mathbb{R}^d$ and let $\varphi \in \mathcal{D}$ be such that the support of $\varphi$ is a subset of $\Lambda$. Then,

$$\int_{\mathbb{R}^d} dx\, \varphi(x) \left( \int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)\ q^{(k)}(x, s) \right) = 0.$$

Hence

$$\int_{\Lambda} dx\, \varphi(x) \left( \int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)\, q^{(k)}(x, s) \right) = 0. \tag{4.15}$$

We state that the function

$$\Lambda \ni x \mapsto \int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)\, q^{(k)}(x, s)$$

belongs to $L^2(\Lambda,\, dx)$. Indeed, if $k = 0$, then $q^{(0)}(x, s) = 1$ and this statement evidently follows from Cauchy's inequality. Assume that $k \geq 1$. Then by Cauchy's inequality, (4.13), and condition (A),

$$\begin{aligned}
&\int_{\Lambda} dx \left( \int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)\, q^{(k)}(x, s) \right)^2 \\
&\quad \leq \int_{\Lambda} dx \int_{\mathbb{R}} \sigma(x, ds_1)\, g(x, s_1)^2 \int_{\mathbb{R}} \sigma(x, ds_2)\, q^{(k)}(x, s_2)^2 \\
&= \int_{\Lambda} dx \int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)^2\, a_1(x) a_2(x) \cdots a_k(x) \\
&\leq \left( \prod_{i=1}^{k} \sup_{x \in \Lambda} a_i(x) \right) \int_{\Lambda} dx \int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)^2 < \infty.
\end{aligned}$$

Since the set of all functions $\varphi \in \mathcal{D}$ with support in $\Lambda$ is dense in $L^2(\Lambda, dx)$, we therefore conclude from (4.15) that, for $dx$-a.a $x \in \Lambda$,

$$\int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)\, q^{(k)}(x, s) = 0, \quad \forall k \geq 0. \tag{4.16}$$

Since $g \in H$, we get that, for $dx$-a.a. $x \in \mathbb{R}^d$, $g(x, \cdot) \in L^2(\mathbb{R}, \sigma(x, ds))$. Since $\{q^{(k)}(x, \cdot)\}_{k=0}^{\infty}$ form an orthogonal basis in $L^2(\mathbb{R}, \sigma(x, ds))$, we conclude from

(4.16) that for $dx$-a.a. $x \in \mathbb{R}^d$ $g(x,s) = 0$ for $\sigma(x,ds)$-a.a. $s \in \mathbb{R}$. From here, analogously to the proof of Lemma 3.1, we get that $g = 0$ as an element of $H$. Hence $\mathfrak{L}$ is indeed dense in $H$. $\square$

For each $n \in \mathbb{Z}_+$, we define

$$\mathfrak{L}_n := \{g_n(x,s) = f_n(x)\, q^{(n)}(x,s) \mid f_n \in \mathcal{D}\}.$$

We have $\mathfrak{L}_n \subset \mathfrak{L}$, and the linear span of the $\mathfrak{L}_n$ spaces coincides with $\mathfrak{L}$. For any $g_n(x,s) = f_n(x)\, q^{(n)}(x,s) \in \mathfrak{L}_n$ and $g_m(x,s) = f_m(x)\, q^{(m)}(x,s) \in \mathfrak{L}_m$, $n, m \in \mathbb{Z}_+$, we have

$$\begin{aligned}(g_n, g_m)_H &= \int_{\mathbb{R}^d \times \mathbb{R}} g_n(x,s)\, g_m(x,s) dx\, \sigma(x,ds) \\ &= \int_{\mathbb{R}^d} f_n(x)\, f_m(x) \Big( \int_{\mathbb{R}} q^{(n)}(x,s)\, q^{(m)}(x,s)\, \sigma(x,ds) \Big) dx.\end{aligned} \tag{4.17}$$

Hence, if $n \neq m$, then

$$(g_n, g_m)_H = 0,$$

which implies that the linear spaces $\{\mathfrak{L}_n\}_{n=0}^{\infty}$ are mutually orthogonal in $H$. Denote by $H_n$ the closure of $\mathfrak{L}_n$ in $H$. Then by Lemma 4.5,

$$H = \bigoplus_{n=0}^{\infty} H_n.$$

By (4.17), setting $n = m$, we get

$$\begin{aligned}\|g_n\|_{H_n}^2 &= \int_{\mathbb{R}^d} f_n^2(x) \Big( \int_{\mathbb{R}} q^{(n)}(x,s)^2\, \sigma(x,ds) \Big) dx \\ &= \int_{\mathbb{R}^d} f_n^2(x) \rho_n(dx),\end{aligned} \tag{4.18}$$

where

$$\rho_n(dx) = \Big( \int_{\mathbb{R}} q^{(n)}(x,s)^2\, \sigma(x,ds) \Big) dx$$

is a measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Consider a linear operator

$$\mathcal{D} \ni f_n \mapsto (J_n f_n)(x,s) := f_n(x) q^{(n)}(x,s) \in \mathfrak{L}_n.$$

The image of $J_n$ is clearly the whole $\mathfrak{L}_n$. Now, $\mathfrak{L}_n$ is dense in $H_n$, while $\mathcal{D}$ is evidently dense in $L^2(\mathbb{R}^d, \rho_n(dx))$. By (4.18), for each $f_n \in \mathcal{D}$,

$$\|J_n f_n\|_{H_n} = \|f_n\|_{L^2(\mathbb{R}^d,\, \rho_n(dx))}.$$

Therefore, we can extend the operator $J_n$ by continuity to a unitary operator

$$J_n : L^2(\mathbb{R}^d, \rho_n(dx)) \to H_n. \tag{4.19}$$

In particular,

$$H_n = \left\{ f_n(x)\, q^{(n)}(x, s) \mid f_n \in L^2(\mathbb{R}^d, \rho_n(dx)) \right\}.$$

Therefore, for each $k \geq 2$

$$\begin{aligned} H_n^{\otimes k} = \Big\{ & f_n^{(k)}(x_1, \dots, x_k)\, q^{(n)}(x_1, s_1) \cdots q^{(n)}(x_k, s_k) \mid \\ & f_n^{(k)} \in L^2(\mathbb{R}^d, \rho_n(dx))^{\otimes k} = L^2\big((\mathbb{R}^d)^k, \rho_n(dx_1) \cdots \rho_n(dx_k)\big) \Big\}. \end{aligned}$$

Since the operator $J_n$ in (4.19) is unitary, we get that the operator

$$J_n^{\otimes k} : L^2(\mathbb{R}^d, \rho_n(dx))^{\otimes k} \to H_n^{\otimes k}$$

is also unitary. The restriction of $J_n^{\otimes k}$ to $L^2(\mathbb{R}^d, \rho_n(dx))^{\odot k}$ is a unitary operator

$$J_n^{\otimes k} : L^2(\mathbb{R}^d, \rho_n(dx))^{\odot k} \to H_n^{\odot k}. \tag{4.20}$$

Indeed, take any $f_n \in L^2(\mathbb{R}^d, \rho_n(dx))$. Then $f_n^{\otimes k} \in L^2(\mathbb{R}^d, \rho_n(dx))^{\odot k}$ and the set of all such vectors is total in $L^2(\mathbb{R}^d, \rho_n(dx))^{\odot k}$. Now, by the definition of $J_n^{\otimes k}$, we get

$$J_n^{\otimes k} f_n^{\otimes k} = (J_n\, f_n)^{\otimes k} \in H_n^{\odot k},$$

and furthermore the set of all vectors of the form $(J_n\, f_n)^{\otimes k}$ is total in $H_n^{\odot k}$. Hence, the statement follows.

For any $f_n^{(k)} \in L^2(\mathbb{R}^d, \rho_n(dx))^{\otimes k}$,

$$\big(J_n^{\otimes k} f_n^{(k)}\big)(x_1, s_1, \dots, x_k, s_k) = f_n^{(k)}(x_1, \dots, x_k) q^{(n)}(x_1, s_1) \cdots q^{(n)}(x_k, s_k).$$

Hence, the unitary operator (4.20) acts as follows

$$\begin{aligned} &L^2(\mathbb{R}^d, \rho_n(dx))^{\odot k} \ni f_n^{(k)}(x_1, \dots, x_k) \\ &\quad \mapsto \big(J_n^{\otimes k} f_n^{(k)}\big)(x_1, s_1, \dots, x_k, s_k) = f_n^{(k)}(x_1, \dots, x_k) q^{(n)}(x_1, s_1) \cdots q^{(n)}(x_k, s_k). \end{aligned}$$

Thus, each function $g_n^{(k)} \in H_n^{\odot k}$ has a representation

$$g_n^{(k)}(x_1, s_1, \dots, x_k, s_k) = f_n^{(k)}(x_1, \dots, x_k) q^{(n)}(x_1, s_1) \cdots q^{(n)}(x_k, s_k),$$

where $f_n^{(k)} \in L^2(\mathbb{R}^d, \rho_n(dx))^{\odot k}$ and $\|g_n^{(k)}\|_{H_n^{\odot k}} = \|f_n^{(k)}\|_{L^2(\mathbb{R}^d, \rho_n(dx))^{\odot k}}$.

For each $\alpha \in \mathbb{Z}_{+,0}^{\infty}$, we consider the Hilbert space

$$L^2_\alpha\big((\mathbb{R}^d)^{|\alpha|}\big) := L^2(\mathbb{R}^d, \rho_0(dx))^{\odot \alpha_0} \otimes L^2(\mathbb{R}^d, \rho_1(dx))^{\odot \alpha_1} \otimes \cdots . \tag{4.21}$$

We now define a unitary operator

$$J_\alpha : L^2_\alpha\big((\mathbb{R}^d)^{|\alpha|}\big) \to H_0^{\odot \alpha_0} \otimes H_1^{\odot \alpha_1} \otimes \cdots ,$$

where

$$J_\alpha = J_0^{\otimes \alpha_0} \otimes J_1^{\otimes \alpha_1} \otimes \cdots .$$

We evidently have, for each $f_\alpha \in L^2_\alpha\big((\mathbb{R}^d)^{|\alpha|}\big)$,

$$\begin{aligned} &\big(J_\alpha f_\alpha\big)(x_1, s_1, x_2, s_2, \dots, x_{|\alpha|}, s_{|\alpha|}) \\ &\quad = f_\alpha(x_1, x_2, \dots, x_{|\alpha|}) q^{(0)}(x_1, s_1) \cdots q^{(0)}(x_{\alpha_0}, s_{\alpha_0}) \\ &\quad \times q^{(1)}(x_{\alpha_0+1}, s_{\alpha_0+1}) \cdots q^{(1)}(x_{\alpha_0+\alpha_1}, s_{\alpha_0+\alpha_1}) \cdots . \end{aligned}$$

For each $\alpha \in \mathbb{Z}_{+,0}^{\infty}$, we define a Hilbert space

$$\mathcal{G}_\alpha := L^2_\alpha\big((\mathbb{R}^d)^{|\alpha|}\big) \alpha_0! \alpha_1! \cdots .$$

The $J_\alpha$ is evidently a unitary operator

$$J_\alpha : \mathcal{G}_\alpha \to (H_0^{\odot\alpha_0} \otimes H_1^{\odot\alpha_1} \otimes \cdots)\alpha_0!\alpha_1! \cdots .$$

Denote

$$\mathcal{G} := \bigoplus_{\alpha \in \mathbb{Z}^\infty_{+,0}} \mathcal{G}_\alpha.$$

Hence, we can construct a unitary operator

$$J : \mathcal{G} \to \bigoplus_{\alpha \in \mathbb{Z}^\infty_{+,0}} (H_0^{\odot\alpha_0} \otimes H_1^{\odot\alpha_1} \otimes \cdots)\alpha_0!\alpha_1! \cdots$$

by setting

$$J := \bigoplus_{\alpha \in \mathbb{Z}^\infty_{+,0}} J_\alpha.$$

By Lemma 4.3, we get a unitary operator

$$\mathcal{R} : \mathcal{G} \to \mathcal{F}(H),$$

by setting

$$\mathcal{R} := \operatorname{Sym} J.$$

Thus, by Theorem 3.2, we get

**Theorem 4.6.** *Let condition (A) be satisfied. We have a unitary isomorphism*

$$\mathcal{K} : \mathcal{G} \to L^2(\mathcal{D}', \mu)$$

*given by* $\mathcal{K} := I\mathcal{R}$*, where the unitary operator* $I : \mathcal{F}(H) \to L^2(\mathcal{D}', \mu)$ *is from Theorem 3.2.*

We will now give an interpretation of the unitary isomorphism $\mathcal{K}$ in terms of multiple stochastic integrals. Since the results below will not be used anywhere else in this dissertation, we will present only a sketch of the proof, omitting some technical details.

Let us recall that the operators $A(\varphi)$ in $\mathcal{F}(H)$ are given by

$$A(\varphi) := a^+(\varphi(x)1(s)) + a^0(\varphi(x)\, s) + a^-(\varphi(x)1(s)).$$

Now, for each $k \in \mathbb{N}$, we define operators

$$A^{(k)}(\varphi) := a^+(\varphi(x)\, s^{k-1}) + a^0(\varphi(x)\, s^k) + a^-(\varphi(x)\, s^{k-1}).$$

In particular, $A^{(1)}(\varphi) = A(\varphi)$. We state that, *after the closure, the operator $A^{(k)}(\varphi)$ go over, under the unitary isomorphism $I$, into the operator of multiplication by $I(\varphi(x)\, s^{k-1})$.*

Let us explain this result in the case where $\varphi(x) = \chi_\Delta(x)$, where $\Delta = (a_1, b_1) \times \cdots \times (a_d, b_d)$ (recall $d$ is the dimension of the underlying space). For each $n \in \mathbb{N}$, let us consider a partition

$$\Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_n = \Delta$$

of the set $\Delta$ into mutually disjoint sets $\Delta_1, \Delta_2, \ldots, \Delta_n$ such that

$$\int_{\Delta_i} dx = \frac{1}{n} \int_\Delta dx\,, \quad i = 1, 2, \ldots, n,$$

for example

$$\Delta_i = \Big(a_1 + \frac{b_1 - a_1}{n}(i-1),\, a_1 + \frac{b_1 - a_1}{n} i\Big) \times (a_2, b_2) \times \cdots \times (a_d, b_d).$$

Let us first consider the case where $k = 2$. We state that, in the Fock space $\mathcal{F}(L^2(\mathbb{R}^d \times \mathbb{R}, dx\, \sigma(x, ds)))$,

$$\chi_\Delta(x)\, s = \lim_{n \to \infty} \Big[\sum_{i=1}^n A^{(1)}(\chi_{\Delta_i})^2 \Omega - \int_\Delta dx\, \Omega\Big]. \tag{4.22}$$

We denote by $\mathbf{P}_n$ the orthogonal projection of the symmetric Fock space onto its $n$-particle subspace. Then

$$\mathbf{P}_2\Big(\sum_{i=1}^{n} A^{(1)}(\chi_{\Delta_i})^2\Omega\Big) = \sum_{i=1}^{n}(\chi_{\Delta_i}(x)s)^{\otimes 2},$$

and

$$\begin{aligned}
&\|\sum_{i=1}^{n}(\chi_{\Delta_i}(x)1(s))^{\otimes 2}\|^2_{L^2(\mathbb{R}^d\times\mathbb{R},dx\,\sigma(x,ds))^{\otimes 2}}\\
&\quad=\sum_{i=1}^{n}\|(\chi_{\Delta_i}(x)1(s))^{\otimes 2}\|^2_{L^2(\mathbb{R}^d\times\mathbb{R},dx\,\sigma(x,ds))^{\otimes 2}}\\
&\quad=\sum_{i=1}^{n}\Big(\int_{\Delta_i}dx\int_{\mathbb{R}}1(s)\,\sigma(x,ds)\Big)^2\\
&\quad=\sum_{i=1}^{n}\Big(\int_{\Delta_i}dx\Big)^2\\
&\quad=\Big(\int_{\Delta}dx\frac{1}{n}\Big)^2 n\\
&\quad=\Big(\int_{\Delta}dx\Big)^2\frac{1}{n}\to 0\quad\text{as } n\to\infty.
\end{aligned}$$

Next

$$\begin{aligned}
\mathbf{P}_1\Big(\sum_{i=1}^{n} A^{(1)}(\chi_{\Delta_i})^2\Omega\Big) &= \sum_{i=1}^{n} a^0(\chi_{\Delta_i}(x)s)a^+(\chi_{\Delta_i}(x)\,1(s))\\
&= \sum_{i=1}^{n}\chi_{\Delta_i}(x)s = \chi_{\Delta}(x)s,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{P}_0\Big(\sum_{i=1}^{n} A^{(1)}(\chi_{\Delta_i})^2\Omega\Big) &= \sum_{i=1}^{n} a^-(\chi_{\Delta_i}(x)\,1(s))a^+(\chi_{\Delta_i}(x)\,1(s))\Omega\\
&= \sum_{i=1}^{n}\int_{\Delta_i}dx\int_{\mathbb{R}}\sigma(x,ds)\,1(s)\Omega\\
&= \int_{\Delta}dx\,\Omega,
\end{aligned}$$

which proves (4.22).

Next, we state that, for each element from a proper domain,

$$A^{(2)}(\chi_\Delta) = \lim_{n\to\infty} \Big[ \sum_{i=1}^{n} A^{(1)}(\chi_{\Delta_i})^2 - \int_\Delta dx\, \mathbf{1} \Big], \tag{4.23}$$

where the limit is understood in the strong sense in the Fock space $\mathcal{F}(H)$. Clearly, formula (4.22) is a special case of (4.23) when we apply this equality to the vacuum vector $\Omega$. It is sufficient to check (4.23) on any vector $f^{(k)} \in L^2(\mathbb{R}^d \times \mathbb{R}, dx\, \sigma(x, ds))^{\odot k}$.

Analogously to the above, we get:

$$\begin{aligned}
&\sum_{i=1}^{n} a^+(\chi_{\Delta_i}(x)\, 1(s)) a^+(\chi_{\Delta_i}(x)\, 1(s)) f^{(k)} \\
&\quad = \Big( \sum_{i=1}^{n} (\chi_{\Delta_i}(x)\, 1(s))^{\otimes 2} \Big) \odot f^{(k)} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Next, let us show that

$$\sum_{i=1}^{n} a^+(\chi_{\Delta_i}(x) 1(s)) a^0(\chi_{\Delta_i}(x) s) f^{(k)} \to 0 \quad \text{as } n \to \infty.$$

It is easy to see that it suffices to check this statement for $k = 1$ and $f^{(1)}(x, s) = g(x) s^l$. Then

$$\begin{aligned}
&\| \sum_{i=1}^{n} a^+(\chi_{\Delta_i}(x)\, 1(s)) a^0(\chi_{\Delta_i}(x) s) g(x) s^l \|^2_{L^2(\mathbb{R}^d \times \mathbb{R}, dx\, \sigma(x,ds))^{\odot 2}} \\
&\quad = \| \sum_{i=1}^{n} a^+(\chi_{\Delta_i}(x)\, 1(s)) (\chi_{\Delta_i}(x)\, g(x)\, s^{l+1}) \|^2_{L^2(\mathbb{R}^d \times \mathbb{R}, dx\, \sigma(x,ds))^{\odot 2}} \\
&\quad = \| \sum_{i=1}^{n} (\chi_{\Delta_i}(x)\, 1(s)) \odot (\chi_{\Delta_i}(x)\, g(x)\, s^{l+1}) \|^2_{L^2(\mathbb{R}^d \times \mathbb{R}, dx\, \sigma(x,ds))^{\odot 2}} \\
&\quad = \| \operatorname{Sym}_2 \Big( \sum_{i=1}^{n} (\chi_{\Delta_i}(x)\, 1(s)) \otimes (\chi_{\Delta_i}(x)\, g(x)\, s^{l+1}) \Big) \|^2_{L^2(\mathbb{R}^d \times \mathbb{R}, dx\, \sigma(x,ds))^{\odot 2}} \\
&\quad \le \| \sum_{i=1}^{n} (\chi_{\Delta_i}(x)\, 1(s)) \otimes (\chi_{\Delta_i}(x)\, g(x)\, s^{l+1}) \|^2_{L^2(\mathbb{R}^d \times \mathbb{R}, dx\, \sigma(x,ds))^{\otimes 2}} \\
&\quad = \sum_{i=1}^{n} \| (\chi_{\Delta_i}(x)\, 1(s)) \|^2_{L^2(\mathbb{R}^d \times \mathbb{R}, dx\, \sigma(x,ds))} \cdot \| (\chi_{\Delta_i}(x)\, g(x)\, s^{l+1}) \|^2_{L^2(\mathbb{R}^d \times \mathbb{R}, dx\, \sigma(x,ds))}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \int_{\Delta_i} dx \int_{\mathbb{R}} \sigma(x, ds)\ 1(s) \cdot \int_{\Delta_i} dx' g(x')^2 \int_{\mathbb{R}} (s')^{2(l+1)}\, \sigma(x', ds') \\
&\le C \sum_{i=1}^{n} \int_{\Delta_i} dx \cdot \int_{\Delta_i} dx' g(x')^2 \\
&= \frac{C}{n} \int_{\Delta} dx \cdot \sum_{i=1}^{n} \int_{\Delta_i} dx' g(x')^2 \\
&= \frac{1}{n} C \int_{\Delta} dx \cdot \int_{\Delta} dx' g(x')^2 \to 0 \quad \text{as}\, n \to \infty. \qquad (4.24)
\end{aligned}
$$

Next, we state that

$$
\sum_{i=1}^{n} a^0(\chi_{\Delta_i}(x)s) a^+(\chi_{\Delta_i}(x)\, 1(s)) f^{(k)} \to a^+(\chi_{\Delta}(x)s) f^{(k)}.
$$

Indeed,

$$
\begin{aligned}
&\Big(a^+(\chi_{\Delta_i}(x)\, 1(s)) f^{(k)}\Big)(x_1, s_1, \dots, x_{k+1}, s_{k+1}) \\
&\qquad = \mathrm{Sym}_{k+1}\Big(\chi_{\Delta_i}(x_1)\, 1(s_1) f^{(k)}(x_2, s_2, \dots, x_{k+1}, s_{k+1})\Big).
\end{aligned}
$$

Hence, analogously to the above,

$$
\begin{aligned}
&\sum_{i=1}^{n} \Big(a^0(\chi_{\Delta_i}(x)s) a^+(\chi_{\Delta_i}(x)\, 1(s)) f^{(k)}\Big)(x_1, s_1, \dots, x_{k+1}, s_{k+1}) \\
&\quad = \sum_{i=1}^{n} \mathrm{Sym}_{k+1}\Big(\sum_{j=1}^{k+1} \chi_{\Delta_i}(x_j) s_j \chi_{\Delta_i}(x_1)\, 1(s_1) f^{(k)}(x_2, s_2, \dots, x_{k+1}, s_{k+1})\Big) \\
&\quad = \sum_{i=1}^{n} \mathrm{Sym}_{k+1}\Big(\chi_{\Delta_i}(x_1) s_1 f^{(k)}(x_2, s_2, \dots, x_{k+1}, s_{k+1})\Big) \\
&\quad + \sum_{i=1}^{n} \mathrm{Sym}_{k+1}\Big(\sum_{j=2}^{k+1} \chi_{\Delta_i}(x_j) s_j \chi_{\Delta_i}(x_1)\, 1(s_1) f^{(k)}(x_2, s_2, \dots, x_{k+1}, s_{k+1})\Big) \\
&\quad \to \sum_{i=1}^{n} \mathrm{Sym}_{k+1}\Big(\chi_{\Delta_i}(x_1) s_1 f^{(k)}(x_2, s_2, \dots, x_{k+1}, s_{k+1})\Big) \\
&\quad = \big(a^+(\chi_{\Delta}(x)s) f^{(k)}\big)(x_1, s_1, \dots, x_{k+1}, s_{k+1}) \quad \text{as}\, n \to \infty.
\end{aligned}
$$

Similarly to the above, we then have

$$
\sum_{i=1}^{n} a^-(\chi_{\Delta_i}(x)\, 1(s)) a^-(\chi_{\Delta_i}(x)\, 1(s)) f^{(k)} \to 0,
$$

$$\sum_{i=1}^{n} a^0(\chi_{\Delta_i}(x)s)a^-(\chi_{\Delta_i}(x)\,1(s))f^{(k)} \to 0,$$
$$\sum_{i=1}^{n} a^-(\chi_{\Delta_i}(x)\,1(s))a^0(\chi_{\Delta_i}(x)s)f^{(k)} \to a^-(\chi_{\Delta}(x)s)f^{(k)} \quad \text{as } n \to \infty.$$

Next,

$$\begin{aligned}
&\sum_{i=1}^{n} a^-(\chi_{\Delta_i}(x)\,1(s))a^+(\chi_{\Delta_i}(x)\,1(s))f^{(k)} \\
&= \sum_{i=1}^{n} a^-(\chi_{\Delta_i}(x)\,1(s))\,\mathrm{Sym}_{k+1}\Big[\chi_{\Delta_i}(x_1)\,1(s_1)f^{(k)}(x_2,s_2,\dots,x_{k+1},s_{k+1})\Big] \\
&= \sum_{i=1}^{n}\Big[\int_{\Delta_i} dx \int_{\mathbb{R}} \sigma(x,ds)\,1(s)f^{(k)}(x_1,s_1,\dots,x_k,s_k) \\
&+ k\,\mathrm{Sym}_k\Big(\chi_{\Delta_i}(x_1)1(s_1)\int_{\Delta_i} dy \int_{\mathbb{R}} \sigma(y,du)\,1(u)f^{(k)}(y,u,x_2,s_2,\dots,x_k,s_k)\Big)\Big] \\
&= \Big(\int_{\Delta} dx\Big) f^{(k)}(x_1,s_1,\dots,x_k,s_k) \\
&+ \mathrm{Sym}_k\Big(k\sum_{i=1}^{n}\chi_{\Delta_i}(x_1)1(s_1)\int_{\Delta_i} dy \int_{\mathbb{R}} \sigma(y,du)\,f^{(k)}(y,u,x_2,s_2,\dots,x_k,s_k)\Big) \\
&\to \Big(\int_{\Delta} dx\Big) f^{(k)}(x_1,s_1,\dots,x_k,s_k)
\end{aligned}$$

analogously to (4.24).

Similarly,

$$\sum_{i=1}^{n} a^+(\chi_{\Delta_i}(x)\,1(s))a^-(\chi_{\Delta_i}(x)\,1(s))f^{(k)} \to 0 \quad \text{as } n \to \infty$$

Finally, we should treat the term $\sum_{i=1}^{n} a^0(\chi_{\Delta_i}(x)s)a^0(\chi_{\Delta_i}(x)s)f^{(k)}$. We can write

$$\begin{aligned}
&\sum_{i=1}^{n} a^0(\chi_{\Delta_i}(x)s)a^0(\chi_{\Delta_i}(x)s)f^{(k)} \\
&\quad = \mathrm{Sym}_k\Big(\sum_{i=1}^{n}\sum_{l=1}^{k}\chi_{\Delta_i}(x_l)s_l\sum_{m=1}^{k}\chi_{\Delta_i}(x_m)s_m f^{(k)}(x_1,s_1,\dots,x_k,s_k)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Sym}_k \Big( \sum_{i=1}^{n} \sum_{l=1}^{k} \chi_{\Delta_i}(x_l) s_l^2 f^{(k)}(x_1, s_1, \dots, x_k, s_k) \Big) \\
&+ \mathrm{Sym}_k \Big( \sum_{i=1}^{n} \sum_{\substack{l,m=1,\dots,k \\ l\neq m}} \chi_{\Delta_i}(x_l) s_l \chi_{\Delta_i}(x_m) s_m \, f^{(k)}(x_1, s_1, \dots, x_k, s_k) \Big) \\
&= \mathrm{Sym}_k \Big( \sum_{l=1}^{k} \chi_{\Delta}(x_l) s_l^2 f^{(k)}(x_1, s_1, \dots, x_k, s_k) \Big) \\
&+ \mathrm{Sym}_k \Big( \sum_{i=1}^{n} \sum_{\substack{l,m=1,\dots,k \\ l\neq m}} \chi_{\Delta_i}(x_l) s_l \chi_{\Delta_i}(x_m) s_m \, f^{(k)}(x_1, s_1, \dots, x_k, s_k) \Big) \\
&= a^0(\chi_\Delta(x) s^2) f^{(k)} \\
&+ \mathrm{Sym}_k \Big( \sum_{i=1}^{n} \sum_{\substack{l,m=1,\dots,k \\ l\neq m}} \chi_{\Delta_i}(x_l) s_l \chi_{\Delta_i}(x_m) s_m \, f^{(k)}(x_1, s_1, \dots, x_k, s_k) \Big) \\
&\to a^0(\chi_\Delta(x) s^2) f^{(k)} \quad \text{as } n \to 0.
\end{aligned}
$$

Thus, equality (4.23) is proven.

We know that, under the isomorphism $I$, each operator $A^{(1)}(\chi_{\Delta_i})$ goes over into an operator of multiplication. Therefore, for each $n$, under the isomorphism $I$, the operator

$$\sum_{i=1}^{n} A^{(1)}(\chi_{\Delta_i})^2 - \int_\Delta dx\, \mathbf{1}$$

also goes over into an operator of multiplication. But by (4.23), the operator $A^{(2)}(\chi_\Delta)$ is the limit of the operators $\sum_{i=1}^{n} A^{(1)}(\chi_{\Delta_i})^2 - \int_\Delta dx\, \mathbf{1}$ as $n \to \infty$. Hence the limiting operator $A^{(2)}(\chi_\Delta)$ should also be an operator of multiplication. Since $I\Omega = 1(\omega)$, by (4.23) the image of $A^{(2)}(\chi_\Delta)$ is the operator of multiplication by $I(\chi_\Delta(x)\, s)$. From here, we conclude the general statement that $A^{(2)}(\varphi)$ goes over into the operator of multiplication by $I(\varphi(x)\, s)$. Thus the statement is proven for $k = 2$.

For a general $k$, analogously to (4.22) and (4.23), we get

$$\chi_\Delta(x)\, s^k = \lim_{n\to\infty} \Big[\sum_{i=1}^n A^{(k)}(\chi_{\Delta_i})A^{(1)}(\chi_{\Delta_i})\Omega - \int_\Delta dx \int_{\mathbb{R}} \sigma(x, ds)\, s^{k-1}\Omega\Big]$$

$$A^{(k+1)}(\chi_\Delta) = \lim_{n\to\infty} \Big[\sum_{i=1}^n A^{(k)}(\chi_{\Delta_i})A^{(1)}(\chi_{\Delta_i}) - \int_\Delta dx \int_{\mathbb{R}} \sigma(x, ds)\, s^{k-1}\Omega\Big].$$

From here, by induction, we conclude that the operator $A^{(k+1)}(\varphi)$ goes over into the operator of multiplication by $I(\varphi(x)\, s^k)$.

For each $k = 0, 1, 2, \dots$ and $\varphi \in \mathcal{D}$ we define

$$Y^{(k)}(\varphi) := I(\varphi(x)\, s^k).$$

As we have just shown, each operator $A^{(k+1)}(\varphi)$ goes over, under $I$, into the operator of multiplication by $Y^{(k)}(\varphi)$.

Suppose, for a moment, that the measures $\sigma(x, ds)$ do not depend on $x \in \mathbb{R}^d$. For a fixed $\varphi \in \mathcal{D}$, let us orthogonalize in $L^2(\mathcal{D}', \mu)$ the functions $(Y^{(k)}(\varphi))_{k=0}^\infty$. This is of course equivalent to the orthogonalization of the monomials $(s^k)_{k=0}^\infty$ in $L^2(\mathbb{R}, \sigma)$. Denote by $(q^{(k)})_{k=0}^\infty$ the system of monic orthogonal polynomials with respect to the measure $\sigma$. Thus, the random variables $(Z^{(k)}(\varphi))_{k=0}^\infty$, where

$$Z^{(k)}(\varphi) := I(\varphi(x)\, q^{(k)}(s)),$$

appear as a result of the orthogonalization of $(Y^{(k)}(\varphi))_{k=0}^\infty$. Since $q^{(0)}(s) = 1$, we have

$$Z^{(0)}(\varphi) = Y^{(0)}(\varphi) = \langle \varphi, w\rangle.$$

For each $k \geq 1$, we have a representation of $q^{(k)}(s)$ as follows:

$$q^{(k)}(s) = \sum_{i=0}^k b_i^{(k)}\, s^i.$$

Thus,

$$
\begin{aligned}
Z^{(k)}(\varphi) &= I(\varphi(x)\, q^{(k)}(s)) \\
&= \sum_{i=0}^{k} b_i^{(k)} I(\varphi(x)\, s^i) \\
&= \sum_{i=0}^{k} b_i^{(k)} Y^{(i)}(\varphi).
\end{aligned}
$$

Hence, the image under $I^{-1}$ of the operator of multiplication by $Z^{(k)}(\varphi)$ is the operator

$$
\begin{aligned}
R^{(k)}(\varphi) :&= \sum_{i=0}^{k} b_i^{(k)}(a^+(\varphi(x)s^i) + a^-(\varphi(x)s^i) + a^0(\varphi(x)s^{i+1})) \\
&= a^+\Big(\varphi(x)\sum_{i=0}^{k} b_i^{(k)} s^i\Big) + a^-\Big(\varphi(x)\sum_{i=0}^{k} b_i^{(k)} s^i\Big) + a^0\Big(\varphi(x)\sum_{i=0}^{k} b_i^{(k)} s^{i+1}\Big) \\
&= a^+(\varphi(x)q^{(k)}(s)) + a^-(\varphi(x)q^{(k)}(s)) + a^0(\varphi(x)s\, q^{(k)}(s)).
\end{aligned}
$$

Let us now consider the general case, i.e., the measure $\sigma(x, ds)$ does depend on $x \in \mathbb{R}^d$. We are using the monic polynomial $(q^{(k)}(x,\cdot))_{k=0}^{\infty}$ which are orthogonal with respect to the measure $\sigma(x, ds)$. We have

$$
q^{(k)}(x,s) = \sum_{i=0}^{k} b_i^{(k)}(x)\, s^i.
$$

We now define

$$
\begin{aligned}
Z^{(k)}(\varphi) :&= I(\varphi(x)\, q^{(k)}(x,s)) \\
&= \sum_{i=0}^{k} I(\varphi(x) b_i^{(k)}(x)\, s^i) \\
&= \sum_{i=0}^{k} Y^{(i)}(\varphi b_i^{(k)}).
\end{aligned}
$$

Hence, the image under $I^{-1}$ of the operator of multiplication by $Z^{(k)}(\varphi)$ is the operator

$$
R^{(k)}(\varphi) := \sum_{i=0}^{k} \big(a^+(\varphi(x)b_i^{(k)}(x)s^i) + a^-(\varphi(x)b_i^{(k)}(x)s^i) + a^0(\varphi(x)b_i^{(k)}(x)s^{i+1})\big)
$$

$$
\begin{aligned}
&= a^+\Big(\varphi(x)\sum_{i=0}^{k} b_i^{(k)}(x)\, s^i\Big) + a^-\Big(\varphi(x)\sum_{i=0}^{k} b_i^{(k)}(x)\, s^i\Big)\\
&\quad + a^0\Big(\varphi(x)\sum_{i=0}^{k} b_i^{(k)}(x)\, s^{i+1}\Big)\\
&= a^+(\varphi(x)q^{(k)}(x,s)) + a^-(\varphi(x)q^{(k)}(x,s)) + a^0(\varphi(x)s\, q^{(k)}(x,s)).
\end{aligned}
$$

We will now introduce a multiple Wiener–Itô integral with respect to $Z^{(k)}$'s. So, we fix any $\alpha \in \mathbb{Z}_{+,0}^{\infty}$, $|\alpha| = n$, $n \in \mathbb{N}$. Take any $\Delta_1, \dots, \Delta_n \in \mathcal{B}_0(\mathbb{R}^d)$, mutually disjoint. Then we define

$$
\begin{aligned}
&\int_{\Delta_1\times\Delta_2\times\cdots\times\Delta_n} dZ^{(0)}(x_1)\cdots dZ^{(0)}(x_{\alpha_0})dZ^{(1)}(x_{\alpha_0+1})\cdots dZ^{(1)}(x_{\alpha_0+\alpha_1})\\
&\quad \times dZ^{(2)}(x_{\alpha_0+\alpha_1+1})\cdots\\
&= \int_{(\mathbb{R}^d)^n} \chi_{\Delta_1}(x_1)\chi_{\Delta_2}(x_2)\cdots\chi_{\Delta_n}(x_n)dZ^{(0)}(x_1)\cdots dZ^{(0)}(x_{\alpha_0})\\
&\quad \times dZ^{(1)}(x_{\alpha_0+1})\cdots dZ^{(1)}(x_{\alpha_0+\alpha_1})dZ^{(2)}(x_{\alpha_0+\alpha_1+1})\cdots\\
&:= Z^{(0)}(\Delta_1)\cdots Z^{(0)}(\Delta_{\alpha_0})Z^{(1)}(\Delta_{\alpha_0+1})\cdots Z^{(1)}(\Delta_{\alpha_0+\alpha_1})Z^{(2)}(\Delta_{\alpha_0+\alpha_1+1})\cdots .
\end{aligned}
$$

Here

$$
Z^{(k)}(\Delta) := Z^{(k)}(\chi_\Delta).
$$

Using that the sets $\Delta_1, \dots, \Delta_n$ are mutually disjoint,

$$
\begin{aligned}
&I^{-1}(Z^{(0)}(\Delta_1)\cdots Z^{(0)}(\Delta_{\alpha_0})Z^{(1)}(\Delta_{\alpha_0+1})\cdots Z^{(1)}(\Delta_{\alpha_0+\alpha_1})Z^{(2)}(\Delta_{\alpha_0+\alpha_1+1})\cdots)\\
&= R^{(0)}(\chi_{\Delta_1})\cdots R^{(0)}(\chi_{\Delta_{\alpha_0}})R^{(1)}(\chi_{\Delta_{\alpha_0+1}})\cdots R^{(1)}(\chi_{\Delta_{\alpha_0+\alpha_1}})R^{(2)}(\chi_{\Delta_{\alpha_0+\alpha_1+1}})\cdots\\
&= a^+(\chi_{\Delta_1}q^{(0)})\cdots a^+(\chi_{\Delta_{\alpha_0}}q^{(0)})a^+(\chi_{\Delta_{\alpha_0+1}}q^{(1)})\cdots a^+(\chi_{\Delta_{\alpha_0+\alpha_1}}q^{(1)})\\
&\quad \times a^+(\chi_{\Delta_{\alpha_0+\alpha_1+1}}q^{(2)})\cdots\Omega\\
&= (\chi_{\Delta_1}q^{(0)})\odot\cdots\odot(\chi_{\Delta_{\alpha_0}}q^{(0)})\odot(\chi_{\Delta_{\alpha_0+1}}q^{(1)})\odot\cdots\odot(\chi_{\Delta_{\alpha_0+\alpha_1}}q^{(1)})\\
&\quad \odot(\chi_{\Delta_{\alpha_0+\alpha_1+1}}q^{(2)})\odot\cdots\\
&= \mathrm{Sym}_n\Big(\Big[(\chi_{\Delta_1}q^{(0)})\odot\cdots\odot(\chi_{\Delta_{\alpha_0}}q^{(0)})\Big]\otimes\Big[(\chi_{\Delta_{\alpha_0+1}}q^{(1)})\odot\cdots
\end{aligned}
$$

$$
\begin{aligned}
&\odot (\chi_{\Delta_{\alpha_0+\alpha_1}} q^{(1)})\Big] \otimes \cdots \Big) \\
&= \mathrm{Sym}_n \bigg( \Big[ (\chi_{\Delta_1} \odot \cdots \odot \chi_{\Delta_{\alpha_0}})(x_1, \ldots, x_{\alpha_0}) q^{(0)}(x_1, s_1) \cdots q^{(0)}(x_{\alpha_0}, s_{\alpha_0}) \Big] \\
&\otimes \Big[ (\chi_{\Delta_{\alpha_0+1}} \odot \cdots \odot \chi_{\Delta_{\alpha_0+\alpha_1}})(x_{\alpha_0+1}, \ldots, x_{\alpha_0+\alpha_1})\, q^{(1)}(x_{\alpha_0+1}, s_{\alpha_0+1}) \\
&\cdots q^{(1)}(x_{\alpha_0+\alpha_1}, s_{\alpha_0+\alpha_1}) \Big] \otimes \cdots \bigg) \\
&= \mathcal{R}\Big( (\chi_{\Delta_1} \odot \cdots \odot \chi_{\Delta_{\alpha_0}}) \otimes (\chi_{\Delta_{\alpha_0+1}} \odot \cdots \odot \chi_{\Delta_{\alpha_0+\alpha_1}}) \otimes \cdots \Big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&Z^{(0)}(\Delta_1) \cdots Z^{(0)}(\Delta_{\alpha_0}) Z^{(1)}(\Delta_{\alpha_0+1}) \cdots Z^{(1)}(\Delta_{\alpha_0+\alpha_1}) Z^{(2)}(\Delta_{\alpha_0+\alpha_1+1}) \cdots \\
&= \mathcal{K}((\chi_{\Delta_1} \odot \cdots \odot \chi_{\Delta_{\alpha_0}}) \otimes (\chi_{\Delta_{\alpha_0+1}} \odot \cdots \odot \chi_{\Delta_{\alpha_0+\alpha_1}}) \otimes \cdots).
\end{aligned}
$$

The set of all vectors of the form

$$
((\chi_{\Delta_1} \odot \cdots \odot \chi_{\Delta_{\alpha_0}}) \otimes (\chi_{\Delta_{\alpha_0+1}} \odot \cdots \odot \chi_{\Delta_{\alpha_0+\alpha_1}}) \otimes \cdots)
$$

is total in $\mathcal{G}_\alpha$. Therefore, by linearity and continuity, we can extend the definition of the multiple Wiener–Itô integral to the whole space $\mathcal{G}_\alpha$. Thus, we get, for each $f_\alpha \in \mathcal{G}_\alpha$,

$$
\begin{aligned}
&\int_{(\mathbb{R}^d)^{|\alpha|}} f_\alpha(x_1, \ldots, x_{|\alpha|}) dZ^{(0)}(x_1) \cdots dZ^{(0)}(x_{\alpha_0}) dZ^{(1)}(x_{\alpha_0+1}) \cdots dZ^{(1)}(x_{\alpha_0+\alpha_1}) \\
&\times dZ^{(2)}(x_{\alpha_0+\alpha_1+1}) \cdots = \mathcal{K} f_\alpha.
\end{aligned}
$$

Thus, we have the following theorem.

**Theorem 4.7.** *The unitary isomorphism*

$$
\mathcal{K} : \mathcal{G} \to L^2(\mathcal{D}', \mu)
$$

*from Theorem 4.6 is given by*

$$
\mathcal{G} = \bigoplus_{\alpha \in \mathbb{Z}^\infty_{+,0}} \mathcal{G}_\alpha \ni (f_\alpha)_{\alpha \in \mathbb{Z}^\infty_{+,0}} = f \mapsto \mathcal{K} f
$$

$$= \sum_{\alpha\in\mathbb{Z}_{+,0}^{\infty}} \int_{(\mathbb{R}^d)^{|\alpha|}} f_\alpha(x_1,\dots,x_{|\alpha|}) dZ^{(0)}(x_1)\cdots dZ^{(0)}(x_{\alpha_0})$$
$$\times\, dZ^{(1)}(x_{\alpha_0+1})\cdots dZ^{(1)}(x_{\alpha_0+\alpha_1}) dZ^{(2)}(x_{\alpha_0+\alpha_1+1})\cdots .$$

*Remark* 4.8. Let us recall that in Step 2 of the proof of Theorem 3.7, we have constructed an equivalent representation (3.37) (see also (3.32)–(3.34)) of the operator $A(\varphi)$ in the symmetric Fock space $\mathcal{F}(L^2(\mathbb{R}^d\times\mathbb{R}, dx\, \varkappa(x,ds)))$. The part

$$\mathcal{G}(\varphi) := a^+(\varphi(x)\chi_{\{0\}}(s)) + a^-(\varphi(x)\chi_{\{0\}}(s))$$

of this operator describes the Gaussian part of the process, while the part

$$\mathscr{J}(\varphi) = a^+(\varphi(x)s) + a^-(\varphi(x)s) + a^0(\varphi(x)s)$$

describes the jump part.

It is easy to see that, under the unitary isomorphism (3.36), for $k\geq 2$, the operator $A^{(k)}(\varphi)$ goes over into the operator

$$\mathscr{J}^{(k)}(\varphi) = a^+(\varphi(x)\, s^k) + a^-(\varphi(x)\, s^k) + a^0(\varphi(x)\, s^k).$$

Recall that the operator $\mathscr{J}^{(1)}(\varphi) := \mathscr{J}(\varphi)$ describe the jump part of the Lévy process. Thus, for $k\geq 2$, the operators $(\mathscr{J}^{(k)}(\varphi))_{\varphi\in\mathcal{D}}$ describe the same jump process as $\mathscr{J}(\varphi)$, but with jumps having value $s^k$, rather than $s$.

To be more precise, consider for simplicity the case where $\sigma(x,ds)=\sigma(ds)$ does not depend on $x$ and $\nu(ds)=\frac{1}{s^2}\sigma(ds)$ is a measure on $\mathbb{R}^*$. Assume also that $\int_{\mathbb{R}^*} s\,\nu(ds) = \int_{\mathbb{R}^*}\frac{1}{s}\sigma(ds) < +\infty$. Then, the application of the projection spectral theorem to the family $(\mathscr{J}(\varphi))_{\varphi\in\mathcal{D}}$ leads to a probability measure $\eta(dw)$ on $\mathcal{D}'$ having the Fourier transform

$$\int_{\mathcal{D}'} e^{i\langle\varphi,\omega\rangle}\eta(d\omega) = \exp\Big[\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*}\nu(ds)(e^{i\varphi(x)s} - i\varphi(x)s - 1)\Big],\quad \varphi\in\mathcal{D}.$$

In fact, the measure $\nu$ is concentrated on the subset $\mathcal{K} \subset \mathcal{D}'$ given by

$$\mathcal{K} := \left\{ \sum_{(x,\, s) \in \gamma} \delta_x\, s \mid \gamma \in \Gamma \right\},$$

where

$$\Gamma := \Big\{ \gamma \subset \mathbb{R}^d \times \mathbb{R}^* \mid \text{if } (x_1, s_1), (x_2, s_2) \in \Gamma,\ (x_1, s_1) \neq (x_2, s_2) \text{ then } x_1 \neq x_2$$
$$\text{and for each bounded } \Lambda \subset \mathbb{R}^d \text{ and } \epsilon > 0\text{:}$$
$$|\gamma \cap (\Lambda \times \{|s| > \epsilon\})| < \infty, \qquad \sum_{(x,\, s) \in \gamma \cap (\Lambda \times \mathbb{R}^*)} |s| < \infty \Big\}.$$

Here, for a set $A$, $|A|$ denotes the the number of points of the set $A$, and $\delta_x$ denotes the Dirac measure at $x$.

If we now apply the projection spectral theorem to the family $(\mathscr{J}^{(k)}(\varphi))_{\varphi \in \mathcal{D}}$, $k \geq 2$, then this will lead us to the probability measure $\eta^{(k)}(dw)$ on $\mathcal{D}'$ having the Fourier transform

$$\int_{\mathcal{D}'} e^{i\langle \varphi, \omega \rangle} \eta^{(k)}(d\omega) = \exp\Big[ \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \nu(ds) (e^{i\varphi(x) s^k} - i\varphi(x) s^k - 1) \Big], \quad \varphi \in \mathcal{D}.$$

Each measure $\eta^{(k)}$ is concentrated on the set

$$\mathcal{K}^{(k)} := \left\{ \sum_{(x,\, s) \in \gamma} \delta_x\, s \mid \gamma \in \Gamma^{(k)} \right\},$$

where

$$\Gamma^{(k)} := \Big\{ \gamma \subset \mathbb{R}^d \times \mathbb{R}^* \mid \text{if } (x_1, s_1), (x_2, s_2) \in \Gamma,\ (x_1, s_1) \neq (x_2, s_2) \text{ then } x_1 \neq x_2$$
$$\text{and for each bounded } \Lambda \subset \mathbb{R}^d \text{ and } \epsilon > 0\text{:}$$
$$|\gamma \cap (\Lambda \times \{|s| > \epsilon\})| < \infty, \qquad \sum_{(x,\, s) \in \gamma \cap (\Lambda \times \mathbb{R}^*)} |s|^k < \infty \Big\},$$

and $\eta^{(k)}$ is the push-forward (image) of the measure $\nu$ under the transformation

$$\sum_{(x,\, s) \in \gamma} \delta_x\, s \mapsto \sum_{(x,\, s) \in \gamma} \delta_x\, s^k.$$

# Chapter 5

# Decomposition in orthogonal polynomials

Let us recall that, in Chapters 3 and 4 we have constructed the following unitary operators:

$$\mathcal{F}(H) \xrightarrow{I} L^2(\mathcal{D}', \mu),$$

$$\mathcal{G} = \bigoplus_{\alpha \in \mathbb{Z}^\infty_{+,0}} \mathcal{G}_\alpha \xrightarrow{\mathcal{R}} \mathcal{F}(H),$$

$$\mathcal{G} = \bigoplus_{\alpha \in \mathbb{Z}^\infty_{+,0}} \mathcal{G}_\alpha \xrightarrow{\mathcal{K} = I\mathcal{R}} L^2(\mathcal{D}', \mu).$$

For any $f_1, f_2, \dots, f_n \in \mathcal{D}$, we call the function

$$\mathcal{D}' \ni \omega \mapsto \langle f_1, \omega \rangle \cdots \langle f_n, \omega \rangle = \langle f_1 \odot \cdots \odot f_n, \omega^{\otimes n} \rangle \tag{5.1}$$

an algebraic monomial of $n$-th order on $\mathcal{D}'$. Let $\mathcal{P}_{\mathrm{alg}}(\mathcal{D}')$ denote the linear spaces of all algebraic polynomials on $\mathcal{D}'$, thus each element of $\mathcal{P}_{\mathrm{alg}}(\mathcal{D}')$ is a finite sum of function of the form (5.1) and constants.

We note that, by Theorem 3.2, $\mathcal{P}_{\mathrm{alg}}(\mathcal{D}')$ is a dense subset of $L^2(\mathcal{D}', \mu)$. Denote by $\mathcal{P}^{(n)}_{\mathrm{alg}}(\mathcal{D}')$ the linear space of all algebraic polynomials of order $\leq n$.

Denote $\overline{\mathcal{P}_{\mathrm{alg}}^{(n)}(\mathcal{D}')}$ the closure of $\mathcal{P}_{\mathrm{alg}}^{(n)}(\mathcal{D}')$ in $L^2(\mathcal{D}', \mu)$. Elements of $\overline{\mathcal{P}_{\mathrm{alg}}^{(n)}(\mathcal{D}')}$ are usually called measurable polynomials of order $\leq n$. Denote

$$\mathbb{P}^{(n)}(\mathcal{D}') = \overline{\mathcal{P}_{\mathrm{alg}}^{(n)}(\mathcal{D}')} \ominus \overline{\mathcal{P}_{\mathrm{alg}}^{(n-1)}(\mathcal{D}')}$$

where $\ominus$ denotes orthogonal difference in $L^2(\mathcal{D}', \mu)$. Elements of $\mathbb{P}^{(n)}(\mathcal{D}')$ will be called measurable orthogonal polynomials of order $n$. By construction, for each $n$,

$$\overline{\mathcal{P}_{\mathrm{alg}}^{(n)}(\mathcal{D}')} = \bigoplus_{k=0}^{n} \mathbb{P}^{(k)}(\mathcal{D}'). \tag{5.2}$$

Now consider the space $\bigoplus_{n=0}^{\infty} \mathbb{P}^{(n)}(\mathcal{D}')$. By (5.2),

$$\mathcal{P}_{\mathrm{alg}}(\mathcal{D}') \subset \bigoplus_{n=0}^{\infty} \mathbb{P}^{(n)}(\mathcal{D}'). \tag{5.3}$$

We know that $\mathcal{P}_{\mathrm{alg}}(\mathcal{D}')$ is dense in $L^2(\mathcal{D}', \mu)$. Hence, its closure coincides with $L^2(\mathcal{D}', \mu)$. Therefore, by (5.3), the closure of $\bigoplus_{n=0}^{\infty} \mathbb{P}^{(n)}(\mathcal{D}')$ also coincides with $L^2(\mathcal{D}', \mu)$. But $\bigoplus_{n=0}^{\infty} \mathbb{P}^{(n)}(\mathcal{D}')$ is closed, as orthogonal sum of closed subspaces. Therefore, we get the following trivial proposition, see e.g. [42]

**Proposition 5.1.** *We have*

$$L^2(\mathcal{D}', \mu) = \bigoplus_{n=0}^{\infty} \mathbb{P}^{(n)}(\mathcal{D}').$$

We now want to explicitly describe the space $\mathcal{K}^{-1}\mathbb{P}^{(n)}(\mathcal{D}')$ as a subspace of $\mathcal{G}$.

From now on, we will assume that the following condition is satisfied. This conditions is evidently stronger than condition (A).

(B) The functions $a_n(x)$ and $b_n(x)$ from (4.11) are locally bounded on $\mathbb{R}^d$.

**Theorem 5.2.** *For each $n \geq 0$, we have*

$$\mathcal{K}^{-1}\mathbb{P}^{(n)}(\mathcal{D}') = \bigoplus_{\substack{\alpha\in\mathbb{Z}_{+,0}^{\infty}\\ 1\alpha_0+2\alpha_1+3\alpha_2+\cdots=n}} \mathcal{G}_\alpha$$

*Proof.* We will use the notations from Chapter 4. For $\alpha \in \mathbb{Z}_{+,0}^{\infty}$, we denote by $G_\alpha$ the subspace of $\mathcal{F}(H)$ given by

$$G_\alpha := \mathcal{R}\,\mathcal{G}_\alpha = \mathcal{R}_\alpha\,\mathcal{G}_\alpha = (H_0^{\odot\alpha_0} \odot H_1^{\odot\alpha_1} \odot \cdots)|\alpha|!\,.$$

We need to prove that

$$I^{-1}\mathbb{P}^{(n)}(\mathcal{D}') = \bigoplus_{\substack{\alpha\in\mathbb{Z}_{+,0}^{\infty}\\ 1\alpha_0+2\alpha_1+3\alpha_2+\cdots=n}} G_\alpha =: G^{(n)}. \tag{5.4}$$

We will first obtain a description of the space $\bigoplus_{k=0}^{n} G^{(k)} =: M^{(n)}$.

As follows from the proof of Theorem 4.6, each element of the space $G_\alpha$ has a representation

$$\begin{aligned}\operatorname{Sym}_{|\alpha|}\big(&f(x_1,x_2,\dots,x_{|\alpha|})\,q^{(0)}(x_1,s_1)\cdots q^{(0)}(x_{\alpha_0},s_{\alpha_0})\\ &\times q^{(1)}(x_{\alpha_0+1},s_{\alpha_0+1})\cdots q^{(1)}(x_{\alpha_0+\alpha+1},s_{\alpha_0+\alpha_1})\cdots\big),\end{aligned} \tag{5.5}$$

where

$$f \in L^2(\mathbb{R}^d,\rho_0(dx))^{\otimes\alpha_0} \otimes L^2(\mathbb{R}^d,\rho_1(dx))^{\otimes\alpha_1} \otimes \cdots \tag{5.6}$$

(Note that, due to symmetrizator $\operatorname{Sym}_{|\alpha|}$, we may take a function $f$ as in (5.6), rather than from the space $L^2_\alpha((\mathbb{R}^d)^{|\alpha|})$, see (4.21).) Recall that

$$\begin{aligned}\rho_n(dx) &= \Big(\int_{\mathbb{R}} q^{(n)}(x,s)^2\sigma(x,ds)\Big)dx\\ &= a_1(x)a_2(x)\cdots a_n(x)\,dx,\end{aligned}$$

for $n \geq 1$, and $\rho_0(dx) = 1$. By assumption (A) the functions $a_i(x)$ are locally bounded. Therefore, each measurable, bounded function on $(\mathbb{R}^d)^{|\alpha|}$ with compact support (i.e, $f \in B_0(\mathbb{R}^{d|\alpha|})$) belongs to the space $L^2(\mathbb{R}^d, \rho_0(dx))^{\otimes\alpha_0}\otimes L^2(\mathbb{R}^d, \rho_1(dx))^{\otimes\alpha_1}\otimes\cdots$. Furthermore, the set of functions as in (5.5) with $f \in B_0(\mathbb{R}^{d|\alpha|})$ is dense in $G_\alpha$. Hence a total set in $M^{(n)}$ is obtained by taking all functions of the form (5.5) with $f \in B_0(\mathbb{R}^{d|\alpha|})$, where $1\alpha_0+2\alpha_1+3\alpha_2+\cdots \le n$ and the vacuum vector $\Omega$.

**Lemma 5.3.** *For each $\alpha \in \mathbb{Z}_{+,0}^\infty$ with $1\alpha_0 + 2\alpha_1 + 3\alpha_2 + \cdots \le n$, and for each $f \in B_0(\mathbb{R}^{d|\alpha|})$, consider the function*

$$\begin{aligned}\mathrm{Sym}_{|\alpha|}\big(f(x_1, x_2, \dots, x_{|\alpha|})\, s_1^0 \cdots s_{\alpha_0}^0 s_{\alpha_0+1}^1 \cdots s_{\alpha_0+\alpha_1}^1 \\ \times s_{\alpha_0+\alpha_1+1}^2 \cdots s_{\alpha_0+\alpha_1+\alpha_2}^2 \cdots\big).\end{aligned} \tag{5.7}$$

*Then the set of all such functions is a total set in $M^{(n)}$.*

*Proof.* To simplify the proof a little bit, we will assume that, for each $x \in \mathbb{R}^d$, the measure $\sigma(x, ds)$ has infinite support. (If this is not the case, the proof below requires an easy modification.)

Recall that $q^{(0)}(x, s) = 1$ and

$$q^{(n+1)}(x,s) = sq^{(n)}(x,s) - b_n(x)q^{(n)}(x,s) - a_n(x)q^{(n-1)}(x,s).$$

By condition (B), the functions $a_n(x)$ and $b_n(x)$ are locally bounded. We therefore have that

$$q^{(n)}(x,s) = \sum_{l=0}^{n} C_l^{(n)}(x)\, s^l, \tag{5.8}$$

where each $C_l^{(n)}(x)$ is a measurable, locally bounded functions on $\mathbb{R}^d$. By substituting (5.8) into (5.5), we see that each function of the form (5.6) with $f \in B_0(\mathbb{R}^{d|\alpha|})$ can be represented as a finite sum of functions as in (5.7) with $1\alpha_0 + 2\alpha_1 + 3\alpha_2 + \cdots \le n$.

Next, we note that, since $q^{(n)}(x,s)$ is a monic polynomial in $s$, we in fact have

$$q^{(n)}(x,s) = s^n + \sum_{l=0}^{n-1} C_l^{(n)}(x)\, s^l.$$

Therefore,

$$s^n = q^{(n)}(x,s) - \sum_{l=0}^{n-1} C_l^{(n)}(x)\, s^l.$$

From here, we can conclude by induction that

$$s^n = q^{(n)}(x,s) + \sum_{l=0}^{n-1} d_l^{(n)}(x)\, q^{(n)}(x,s), \tag{5.9}$$

where $d_l^{(n)}(x)$ are measurable, locally bounded functions on $\mathbb{R}^d$. By substituting (5.9) into (5.7), we see that each function of the form (5.7) with $f \in B_0(\mathbb{R}^d)^{|\alpha|}$ can be represented as a finite sum of functions as in (5.5) with $1\alpha_0 + 2\alpha_1 + 3\alpha_2 + \cdots \leq n$. □

For any $f_1, f_2. \dots, f_n \in \mathcal{D}$, let us consider the monomial

$$\langle f_1 \odot f_2 \odot \cdots \odot f_n, \omega^{\otimes n}\rangle.$$

Then

$$I^{-1}\langle f_1 \odot f_2 \odot \cdots \odot f_n, \omega^{\otimes n}\rangle = A(f_1)\cdots A(f_n)\,\Omega.$$

**Lemma 5.4.** *For any $f_1, f_2, \dots, f_n \in \mathcal{D}$, $A(f_1)\cdots A(f_n)\,\Omega$ can be represented as a finite sum of functions as in* (5.7) *with $1\alpha_0 + 2\alpha_1 + 3\alpha_2 + \cdots \leq n$.*

*Proof.* We will prove this statement by induction. For $n = 1$,

$$A(f_1)\Omega = f_1(x)\, s^0,$$

so the statement holds. Let us assume that the statement holds for $1, 2, \dots, n$, and we want to prove it for $n+1$. It suffices to show that, for each element as in (5.7) with $1\alpha_0 + 2\alpha_1 + 3\alpha_2 + \cdots \leq n$, the image of this elements under the action of the operator $A(\varphi)$, $\varphi \in \mathcal{D}$, can be represented by a finite sum of elements as in (5.7) with $1\alpha_0 + 2\alpha_1 + 3\alpha_2 + \cdots \le (n+1)$. So we fix any $\alpha \in \mathbb{Z}_{+,0}^\infty$ with $1\alpha_0 + 2\alpha_1 + 3\alpha_2 + \cdots \le n$ and $f \in B_0(\mathbb{R}^{d|\alpha|})$. Then

$$
\begin{aligned}
&A(\varphi)\ \mathrm{Sym}_{|\alpha|}\left(f(x_1, x_2, \ldots, x_{|\alpha|})\, s_1^0 \cdots s_{\alpha_0}^0\, s_{\alpha_0+1}^1 \cdots s_{\alpha_0+\alpha_1}^1 \cdots\right)\\
&= (a^+(\varphi(x)\,1(s)) + a^0(\varphi(x)\,s) + a^-(\varphi(x)\,1(s)))\\
&\quad \times \mathrm{Sym}_{|\alpha|}\left(f(x_1, x_2, \ldots, x_{|\alpha|})\, s_1^0 \cdots s_{\alpha_0}^0\, s_{\alpha_0+1}^1 \cdots s_{\alpha_0+\alpha_1}^1 \cdots\right)\\
&= \mathrm{Sym}_{|\alpha|+1}\left(\varphi(x_1) f(x_2, x_3, \ldots, x_{|\alpha|+1})\, s_1^0\, s_2^0 \cdots s_{\alpha_0+1}^0\, s_{\alpha_0+2}^1 \cdots s_{\alpha_0+\alpha_1+1}^1 \cdots\right)\\
&\quad + \sum_{l=1}^{|\alpha|} \mathrm{Sym}_{|\alpha|}\left(\varphi(x_l)\, s_l\, f(x_1, x_2, \ldots, x_{|\alpha|})\, s_1^0 \cdots s_{\alpha_0}^0\, s_{\alpha_0+1}^1 \cdots s_{\alpha_0+\alpha_1}^1 \cdots\right)\\
&\quad + \sum_{l=1}^{|\alpha|} \int_{\mathbb{R}^d \times \mathbb{R}} dy\, \sigma(y, du) \varphi(y)\, 1(u)\\
&\quad \times \mathrm{Sym}_{|\alpha|-1}\Big(f(x_1, x_2, \ldots, x_{l-1}, y, x_{l+1}, \ldots, x_{|\alpha|-1})\\
&\quad \times s_1^0\, s_2^0 \cdots s_{\alpha_0}^0 s_{\alpha_0+1}^1 \cdots s_{\alpha_0+\alpha_1}^1 \cdots |_{s_l = u}\Big).
\end{aligned}
\tag{5.10}
$$

For the first term in this sum, we have

$$
1(\alpha_0 + 1) + 2\alpha_1 + 3\alpha_2 + \cdots = n + 1. \tag{5.11}
$$

For elements in the sum corresponding to the neutral operator, we note that

$$
\begin{aligned}
&1\alpha_0 + 2\alpha_1 + \cdots + (j+1)(\alpha_j - 1) + (j+2)(\alpha_{j+1} + 1) + \cdots\\
&\quad = 1\alpha_0 + 2\alpha_1 + \cdots + (j+1)\alpha_j - (j+1) + (j+2)\alpha_{j+1} + (j+2) + \cdots\\
&\quad = n - (j+1) + (j+2) = n + 1.
\end{aligned}
\tag{5.12}
$$

Finally, for elements in the sum corresponding to the annihilation operator we evidently have that the elements have the corresponding value $\le n - 1$.

Hence, by (5.10), (5.11) and (5.12), we conclude the statement. □

Thus by Lemma 5.4, for any $f_1, f_2, \ldots, f_k \in \mathcal{D}$, $k \le n$,

$$
A(f_1)\, A(f_2) \cdots A(f_k)\, \Omega \subset M^{(n)}.
$$

To finish the proof of the Theorem, we only need to show that every function as in (5.7) with $1\alpha_0 + 2\alpha_1 + 3\alpha_2 + \cdots = n$ can be approximated in $\mathcal{F}(H)$ by linear combinations of vectors of the form $A(\varphi_1)\cdots A(\varphi_n)\Omega$, where $\varphi_1, \dots, \varphi_n \in \mathcal{D}$. But this directly follows from the proof of Theorem 3.2, see in particular, the proof of Lemma 3.3 $\square$

For any $f_1, \dots, f_n \in \mathcal{D}$, we evidently have

$$\langle f_1, \omega\rangle \cdots \langle f_n, \omega\rangle = \langle f_1 \odot \cdots \odot f_n, \omega^{\otimes n}\rangle \in \mathcal{P}^{(n)}_{\mathrm{alg}}(\mathcal{D}') \subset \overline{\mathcal{P}^{(n)}_{\mathrm{alg}}(\mathcal{D}')}.$$

We denote by $P^{(n)}(f_1 \odot \cdots \odot f_n, \omega)$ the element of $L^2(\mathcal{D}', \mu)$, which is obtained as the orthogonal projection of $\langle f_1 \odot \cdots \odot f_n, \omega^{\otimes n}\rangle$ onto $\mathbb{P}^{(n)}(\mathcal{D}')$. Our next aim is to obtain the explicit form of the vector $I^{-1}P^{(n)}(f_1 \odot \cdots \odot f_n, \omega) \in \mathcal{F}(H)$.

To this end, let us recall the definition of a second quantization operator. Let $(A, D(A))$ be an (unbounded) linear operator in the Hilbert space $H$. We want to define an unbounded linear operator $(d\Gamma(A), D(d\Gamma(A)))$ in $\mathcal{F}(H)$. As the domain $D(d\Gamma(A))$ of this operator we will choose the linear span of the vacuum vector $\Omega$ and vectors of the form $f_1 \odot \cdots \odot f_n$, where $f_1, \dots, f_n \in D(A)$, $n \in \mathbb{N}$. The action of $d\Gamma(A)$ is then defined by

$$\begin{aligned} &d\Gamma(A)\,\Omega = 0, \\ &d\Gamma(A)\, f_1 \odot \cdots \odot f_n \\ &\quad = \sum_{i=1}^{n} f_1 \odot \cdots \odot f_{i-1} \odot (Af_i) \odot f_{i+1} \odot \cdots \odot f_n, \quad n \in \mathbb{N}. \end{aligned} \tag{5.13}$$

For example, the neutral operator $a^0(\varphi(x)s)$ is an (extension of) the differential second quantization of the operator of multiplication by $\varphi(x)s$ in $L^2(\mathbb{R}^d \times \mathbb{R}, dx\,\sigma(x, ds))$:

$$a^0(\varphi(x)\,s) = d\Gamma(M_{\varphi(x)\,s}). \tag{5.14}$$

Analogously to the differential second quantization operator $d\Gamma(A)$ in $\mathcal{F}(H)$, we can define an operator $N(A)$ in the full Fock space $\mathcal{F}_{\text{full}}(H) = \bigoplus_{n=0}^{\infty} H^{\otimes n} n!$ by the formula

$$N(A)\Omega = 0,$$

$$N(A)\, f_1 \otimes \cdots \otimes f_n = \sum_{i=1}^{n} f_1 \otimes \cdots \otimes f_{i-1} \otimes (Af_i) \otimes f_{i+1} \otimes \cdots \otimes f_n, \quad n \in \mathbb{N},$$

where $f_1, \ldots, f_n \in \mathcal{D}(A)$. Then, for each $f^{(n)} \in H^{\otimes n}$ which belongs to the domain of $N(A)$, we have

$$d\Gamma(A)\ \mathrm{Sym}_n\, f^{(n)} = \mathrm{Sym}_n\, N(A)\, f^{(n)}. \tag{5.15}$$

For each $\varphi \in \mathcal{D}$, we define operators $J^+(\varphi)$, $J^0(\varphi)$ and $J^-(\varphi)$ in $L^2(\mathbb{R}^d \times \mathbb{R}, dx\, \sigma(x, ds))$ by

$$\begin{aligned} J^+(\varphi)\big(f(x)\, q^{(n)}(x,s)\big) &= \varphi(x)\, f(x)\, q^{(n+1)}(x,s), \\ J^0(\varphi)\big(f(x)\, q^{(n)}(x,s)\big) &= \varphi(x)\, f(x)\, b_n(x)\, q^{(n)}(x,s), \\ J^-(\varphi)\big(f(x)\, q^{(n)}(x,s)\big) &= \varphi(x)\, f(x)\, a_n(x)\, q^{(n-1)}(x,s), \end{aligned}$$

where $f \in \mathcal{D}$, $n \geq 0$. Thus, by (4.11) the operator $M_{\varphi(x)s}$ of multiplication by $\varphi(x)s$ has representation

$$M_{\varphi(x)s} = J^+(\varphi) + J^0(\varphi) + J^-(\varphi).$$

Hence, by (5.14), we get

$$a^0(\varphi(x)s) = d\Gamma\big(J^+(\varphi)\big) + d\Gamma(J^0(\varphi)) + d\Gamma(J^-(\varphi)\big).$$

Therefore,

$$A(\varphi) = a^+(\varphi(x)\, 1(s)) + a^-(\varphi(x)\, 1(s)) + d\Gamma(J^+(\varphi)) + d\Gamma(J^0(\varphi)) + d\Gamma(J^-(\varphi)).$$

We denote

$$
\begin{aligned}
&A^+(\varphi) := a^+(\varphi(x)\,1(s)) + d\Gamma(J^+(\varphi)),\\
&A^0(\varphi) := d\Gamma(J^0(\varphi)),\\
&A^-(\varphi) := a^-(\varphi(x)\,1(s)) + d\Gamma(J^-(\varphi)),
\end{aligned}
$$

so that

$$A(\varphi) = A^+(\varphi) + A^0(\varphi) + A^-(\varphi).$$

**Theorem 5.5.** *For any $f_1, \dots, f_n \in \mathcal{D}$,*

$$I^{-1}P^{(n)}(f_1 \odot \cdots \odot f_n, \omega) = A^+(f_1)\cdots A^+(f_n)\Omega.$$

*Proof.* We note that

$$I^{-1}\langle f_1 \odot \cdots \odot f_n, \omega^{\otimes n}\rangle = A(f_1)\cdots A(f_n)\Omega.$$

Thus, by (5.4), we are interested in the projection of the vector $A(f_1)\cdots A(f_n)\Omega$ onto $G^{(n)}$. Hence, to prove the theorem it suffices to show that, for each $\varphi \in \mathcal{D}$, $A^+(\varphi)$ maps $G^{(n)}$ into $G^{(n+1)}$, $A^0(\varphi)$ maps $G^{(n)}$ into $G^{(n)}$, and $A^-(\varphi)$ maps $G^{(n)}$ into $G^{(n-1)}$.

By (5.5), (5.6) and analogously (5.11) and (5.12), we conclude that $A^+(\varphi)$ maps $G^{(n)}$ into $G^{(n+1)}$. Clearly, we also get that $A^0(\varphi)$ maps $G^{(n)}$ into $G^{(n)}$. Next, we see that the operator $a^-(\varphi(x)\,1(s))$ maps a function as in (5.5) into a sum of functions of 'order' $n-1$, by annihilating one polynomial $q^{(0)}(x_i, s_i)$ and zeros, which are obtained by annihilating polynomials $q^{(l)}(x,s)$ of order $l \geq 1$, by orthogonality of these polynomials to $q^{(0)}(x,s) = 1$. Thus, $a^-(\varphi(x)\,1(s))$ maps $G^{(n)}$ into $G^{(n-1)}$. Also easily seen, $d\Gamma(J^-(\varphi))$ maps $G^{(n)}$ into $G^{(n-1)}$. Thus, $A^-(\varphi)$ maps $G^{(n)}$ into $G^{(n-1)}$.

□

Recall that, for each $n \in \mathbb{N}$, we denote by $B_0((\mathbb{R}^d)^n)$ the linear space of all bounded, measurable, real-valued functions on $(\mathbb{R}^d)^n$ with compact support. We introduce a topology on $B_0((\mathbb{R}^d)^n)$ which yields the following notion of convergence: $f_n \to f$ as $n \to \infty$ means that there exists a compact set $\Delta \subset \mathbb{R}^d$ such that $\operatorname{supp}(f_n) \subset \Delta$ for all $n \in \mathbb{N}$ and $\sup_{x\in\mathbb{R}^d} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.

We denote by $\mathcal{P}(n)$ the set of all (unordered) partitions of the set $\{1, \dots, n\}$. For each partition $\theta = \{\theta_1, \dots, \theta_l\} \in \mathcal{P}(n)$, we set $|\theta| := l$. For each $\theta \in \mathcal{P}(n)$, we denote by $(\mathbb{R}^d)^{(n)}_\theta$ the subset of $(\mathbb{R}^d)^n$ which consists of all $(x_1, \dots, x_n) \in (\mathbb{R}^d)^n$ such that, for all $1 \le i < j \le n$, $x_i = x_j$ if and only if $i$ and $j$ belong to the same element of the partition $\theta$. Note that the sets $(\mathbb{R}^d)^{(n)}_\theta$ with $\theta \in \mathcal{P}(n)$ form a partition of $(\mathbb{R}^n)^n$.

For example, for $n = 2$, $\mathcal{P}(2)$ has 2 elements: $\theta = \{\{1\}, \{2\}\}$ and $\eta = \{\{1, 2\}\}$. Then,

$$
\begin{aligned}
(\mathbb{R}^d)^{(2)}_\theta &= \{(x_1, x_2) \in (\mathbb{R}^d)^2 \mid x_1 \neq x_2\}, \\
(\mathbb{R}^d)^{(2)}_\eta &= \{(x_1, x_2) \in (\mathbb{R}^d)^2 \mid x_1 = x_2\}.
\end{aligned}
$$

Of course,

$$(\mathbb{R}^d)^2 = (\mathbb{R}^d)^{(2)}_\theta \sqcup (\mathbb{R}^d)^{(2)}_\eta.$$

For $n = 3$, $\mathcal{P}(3)$ has 5 elements: $\alpha = \{\{1, 2, 3\}\}$, $\beta = \{\{1, 2\}, \{3\}\}$, $\gamma = \{\{1, 3\}, \{2\}\}$, $\theta = \{\{1\}, \{2, 3\}\}$ and $\eta = \{\{1\}, \{2\}, \{3\}\}$, so that

$$
\begin{aligned}
(\mathbb{R}^d)^{(3)}_\alpha &= \{(x_1, x_2, x_3) \in (\mathbb{R}^d)^2 \mid x_1 = x_2 = x_3\}, \\
(\mathbb{R}^d)^{(3)}_\beta &= \{(x_1, x_2, x_3) \in (\mathbb{R}^d)^2 \mid x_1 = x_2 \neq x_3\}, \\
(\mathbb{R}^d)^{(3)}_\gamma &= \{(x_1, x_2, x_3) \in (\mathbb{R}^d)^2 \mid x_1 = x_3 \neq x_2\}, \\
(\mathbb{R}^d)^{(3)}_\theta &= \{(x_1, x_2, x_3) \in (\mathbb{R}^d)^2 \mid x_1 \neq x_2 = x_3\}, \\
(\mathbb{R}^d)^{(3)}_\eta &= \{(x_1, x_2, x_3) \in (\mathbb{R}^d)^2 \mid x_1,\ x_2,\ x_3 \text{ different}\}.
\end{aligned}
$$

For each $f^{(n)} \in B_0((\mathbb{R}^d)^n)$ and $\theta = \{\theta_1, \dots, \theta_l\} \in \mathcal{P}(n)$ with

$$\max\theta_1 < \max\theta_2 < \cdots < \max\theta_l, \tag{5.16}$$

we define

$$\begin{aligned}(\mathcal{I}_\theta f^{(n)})&(x_1, s_1, \dots, x_l, s_l) :\\ &= f_\theta^{(n)}(x_1, \dots, x_l) q^{(|\theta_1|-1)}(x_1, s_1) q^{(|\theta_2|-1)}(x_2, s_2) \cdots q^{(|\theta_l|-1)}(x_l, s_l),\end{aligned} \tag{5.17}$$

where $|\theta_i|$ denotes the number of elements in the set $\theta_i$, and the function $f_\theta^{(n)}(x_1, \dots, x_l)$ is obtained from the function $f^{(n)}(y_1, \dots, y_n)$ by replacing, for all $i \in \theta_1$, $y_i$ with $x_1$, for all $i \in \theta_2$, $y_i$ with $x_2$, and so on.

For example, for the partition $\theta = \{\{2\}, \{1,3\}\}$, we have $\theta_1 = \{2\}$, $\theta_2 = \{1,3\}$ since $\max\theta_1 = 2 < \max\theta_2 = 3$, and $f_\theta^{(3)}(x_1, x_2) = f_\theta^{(3)}(x_2, x_1, x_2)$.

**Theorem 5.6.** *For any $h_1, \dots, h_n \in \mathcal{D}$, $n \in \mathbb{N}$, and setting $f^{(n)}(x_1, \dots, x_n) = h_1(x_1) \cdots h_n(x_n)$, we have*

$$I^{-1} P^{(n)}(h_1 \odot \cdots \odot h_n, \omega) = \sum_{\theta \in \mathcal{P}(n)} \mathrm{Sym}_{|\theta|}(\mathcal{I}_\theta f^{(n)}).$$

*Proof.* By Theorem 5.5, (2.32) and (5.15)

$$\begin{aligned}I^{-1} P^{(n)}&(h_1 \odot \cdots \odot h_n, \omega) = A^+(h_1) \cdots A^+(h_n)\Omega\\ &= (a^+(h_1(x)1(s)) + d\Gamma(J^+(h_1))) \cdots (a^+(h_n(x)1(s)) + d\Gamma(J^+(h_n)))\Omega\\ &= \mathrm{Sym}_n(R^+(h_1(x)1(s)) + N(J^+(h_1))) \cdots (R^+(h_n(x)1(s)) + N(J^+(h_n)))\Omega,\end{aligned} \tag{5.18}$$

where for $g \in H$, $R^+(g)$ is the free creation operator in the full Fock space $\mathcal{F}_{\mathrm{full}}(H)$:

$$R^+(g) f^{(n)} = g \otimes f^{(n)}, \quad f^{(n)} \in H^{\otimes n}.$$

Thus, by formula (5.18), to prove the theorem, we need to show that

$$(R^+(h_1(x)1(s)) + N(J^+(h_1))) \cdots (R^+(h_n(x)1(s)) + N(J^+(h_n)))\Omega$$

$$= \sum_{\theta \in \mathcal{P}(n)} \mathcal{I}_\theta(h_1 \otimes \cdots \otimes h_n).$$

We will prove this statement by induction in $n$. For $n = 1$, we evidently have

$$(R^+(h_1(x)1(s)) + N(J^+(h_1)))\Omega = h_1(x)1 = \mathcal{I}_{\{\{1\}\}} h_1,$$

so the statement holds. Let us assume that, for any $h_2, \ldots, h_{n+1} \in \mathcal{D}$

$$\begin{aligned}
&(R^+(h_2(x)1(s)) + N(J^+(h_2))) \cdots (R^+(h_{n+1}(x)1(s)) + N(J^+(h_{n+1})))\Omega \\
&\quad = \sum_{\theta \in \mathcal{P}(2,\ldots,n+1)} \mathcal{I}_\theta(h_2 \otimes \cdots \otimes h_{n+1}),
\end{aligned}$$

where $\mathcal{P}(2, \ldots, n+1)$ denotes the set of all unordered partitions of $\{2, \ldots, n+1\}$ and $\mathcal{I}_\theta(h_2 \otimes \cdots \otimes h_{n+1})$ is defined by analogy with (5.17). Hence, for any $h_1 \in \mathcal{D}$,

$$\begin{aligned}
&(R^+(h_1(x)1(s)) + N(J^+(h_1))) \cdots (R^+(h_{n+1}(x)1(s)) + N(J^+(h_{n+1})))\Omega \\
&\quad = (R^+(h_1(x)1) + N(J^+(h_1))) \sum_{\theta \in \mathcal{P}(2,\ldots,n+1)} \mathcal{I}_\theta(h_2 \otimes \cdots \otimes h_{n+1}),
\end{aligned}$$

Fix any $\theta = \{\theta_1, \ldots, \theta_k\} \in \mathcal{P}(2, \ldots, n+1)$. Then

$$\begin{aligned}
&R^+(h_1(x)1(s))\mathcal{I}_\theta(h_2 \otimes \cdots \otimes h_{n+1}) \\
&\quad = (h_1(x)1(s)) \otimes \mathcal{I}_\theta(h_2 \otimes \cdots \otimes h_{n+1}) \\
&\quad = \mathcal{I}_{\theta^+}(h_1 \otimes h_2 \otimes \cdots \otimes h_{n+1}),
\end{aligned}$$

where $\theta^+ \in \mathcal{P}(n+1)$ is given by

$$\theta^+ = \{\{1\}, \theta_1, \ldots, \theta_k\}.$$

Next,

$$N(J^+(h_1))\mathcal{I}_\theta(h_2 \otimes \cdots \otimes h_{n+1}) = \sum_{j=1}^{k} \mathcal{I}_{\theta_j^0}(h_1 \otimes h_2 \otimes \cdots \otimes h_{n+1}),$$

where $\theta_j^0 \in \mathcal{P}(n+1)$ is given by

$$\theta_j^0 = \{\theta_1, \dots, \theta_{j-1}, \theta_j \cup \{1\}, \theta_{j+1}, \dots, \theta_k\}.$$

From here, it follows that

$$(R^+(h_1(x)1) + N(J^+(h_1))) \cdots (R^+(h_n(x)1) + N(J^+(h_n)))\Omega$$
$$= \sum_{\xi \in \mathcal{P}(n+1)} \mathcal{I}_\xi(h_1 \otimes \cdots \otimes h_{n+1}).$$

□

In order to calculate the scalar product of orthogonal polynomials $P^{(n)}(f^{(n)}, \omega)$ and $P^{(n)}(g^{(n)}, \omega)$, we proceed as follows.

Let us fix a sequence $c = (c_k)_{k=1}^\infty$ such that each $c_k$ is a measurable function from $\mathbb{R}^d$ to $[0, +\infty)$ and $c_1(x) \equiv 1$ for all $x \in \mathbb{R}^d$. We will now construct an extended symmetric Fock space $\mathbf{F}_c(H)$.

Let us fix $n \in \mathbb{N}$ and a partition $\theta = \{\theta_1, \dots, \theta_l\} \in \mathcal{P}(n)$ satisfying (5.16) We define a measure $\zeta_{c,\theta}^{(n)}$ on $(\mathbb{R}^d)_\theta^{(n)}$ as the push- forward of the measure

$$\left(c_{|\theta_1|}(x_1) \cdots c_{|\theta_l|}(x_l)\right) n! \left(|\theta_1|! \cdots |\theta_l|!\right)^{-1} dx_1 \cdots dx_l \tag{5.19}$$

on $(\mathbb{R}^d)^{(l)}$ under the mapping

$$(\mathbb{R}^d)^{(l)} \ni y = (y_1, \dots, y_l) \mapsto (R_\theta^1 y, \dots, R_\theta^n y) \in (\mathbb{R}^d)_\theta^{(n)}, \tag{5.20}$$

where

$$R_\theta^i y := y_j \quad \text{for } i \in \theta_j$$

and

$$(\mathbb{R}^d)^{(l)} = \{(y_1, \dots, y_l) \in (\mathbb{R}^d)^l \mid y_i \neq y_j \text{ if } i \neq j\}. \tag{5.21}$$

For example, if $n=3$, $\theta=\{\theta_1,\theta_2\}$, with $\theta_1=\{2\}$, $\theta_2=\{1,3\}$, then the mapping (5.20) is

$$(\mathbb{R}^d)^{(2)}\ni y=(y_1,y_2)\mapsto(y_2,y_1,y_2)\in(\mathbb{R}^d)^{(3)}_\theta.$$

Recalling that the sets $(\mathbb{R}^d)^{(n)}_\theta$ with $\theta\in\mathcal{P}(n)$ form a partition of $(\mathbb{R}^d)^n$, we define a measure $\zeta_c^{(n)}$ on $(\mathbb{R}^d)^n$ such that the restriction of $\zeta_c^{(n)}$ to each $(\mathbb{R}^d)^{(n)}_\theta$ is equal to $\zeta_{c,\theta}^{(n)}$. For example, for $n=2$,

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^2} f^{(2)}(x_1,x_2)\,\zeta_c^{(2)}(dx_1\times dx_2)\\
&=\int_{\{x_1\neq x_2\}} f^{(2)}(x_1,x_2)dx_1dx_2\cdot 2+\int_{\mathbb{R}^d} f^{(2)}(x,x)c_2(x)dx\\
&=2\int_{(\mathbb{R}^d)^2} f^{(2)}(x_1,x_2)dx_1dx_2+\int_{\mathbb{R}^d} f^{(2)}(x,x)c_2(x)dx.
\end{aligned}$$

Let us fix a permutation $\pi\in S_n$ and a partition $\theta=\{\theta_1,\dots,\theta_l\}\in\mathcal{P}(n)$ satisfying (5.16). The permutation $\pi$ maps the partition $\theta$ into a new partition

$$\{\pi\theta_1,\dots,\pi\theta_l\}\in\mathcal{P}(n).$$

We call this new partition $\beta=\{\beta_1,\dots,\beta_l\}$, where the elements of the partition $\beta$ are enumerated in such a way that

$$\max\beta_1<\max\beta_2<\cdots<\max\beta_l.$$

Thus, the permutation $\pi\in S_n$ identifies a permutation $\hat\pi\in S_l$ (dependent on $\theta$) such that

$$\pi\theta_i=\beta_{\hat\pi(i)},\quad i=1,\dots,l.$$

For example, let $n=3$, $\theta=\{\theta_1,\theta_2\}$ with $\theta_1=\{2\}$, $\theta_2=\{1,3\}$. Let $\pi\in S_3$ be given by

$$\begin{pmatrix}1\,2\,3\\2\,3\,1\end{pmatrix}.$$

Then $\pi\theta_1 = \{3\}$, $\pi\theta_2 = \{1,2\}$, $\beta_1 = \{\beta_1, \beta_2\}$ with $\beta_1 = \{1,2\}$, $\beta_2 = \{3\}$ and the permutation $\hat{\pi} \in S_2$ is given by

$$\hat{\pi} = \begin{pmatrix} 1\,2 \\ 2\,1 \end{pmatrix},$$

since

$$\pi\theta_1 = \beta_2 = \beta_{\hat{\pi}(1)},$$
$$\pi\theta_2 = \beta_1 = \beta_{\hat{\pi}(2)}.$$

For each function $f^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}$, we define its symmetrization by

$$\big(\operatorname{Sym}_n f^{(n)}\big)(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\pi \in S_n} f^{(n)}(x_{\sigma(1)}, \dots, x_{\sigma(n)}), \quad (x_1, \dots, x_n) \in (\mathbb{R}^d)^n.$$

For any functions $f_1, \dots, f_n : \mathbb{R}^d \to \mathbb{R}$, we denote

$$f_1 \odot \cdots \odot f_n := \operatorname{Sym}_n(f_1 \otimes \cdots \otimes f_n).$$

**Lemma 5.7.** *For each $n \in \mathbb{N}$, $\operatorname{Sym}_n$ is an orthogonal projection in the Hilbert space $L^2\big((\mathbb{R}^d)^n, \zeta_c^{(n)}\big)$.*

*Proof.* We evidently have that $\operatorname{Sym}_n^2 = \operatorname{Sym}_n$. By construction, the measure $\zeta_c^{(n)}$ remains invariant under the transformation

$$(\mathbb{R}^d)^n \ni (x_1, \dots, x_n) \mapsto (x_{\pi(1)}, \dots, x_{\pi(n)}) \in (\mathbb{R}^d)^n$$

for each $\pi \in S_n$. Therefore, the operator $\operatorname{Sym}_n$ is bounded and self-adjoint in $L^2\big((\mathbb{R}^d)^n, d\zeta_c^{(n)}\big)$. Hence, it is an orthogonal projection. $\square$

For each $n \in \mathbb{N}$, we denote by $\mathbf{F}_{c,n}^{\mathrm{sym}}(H)$ the subspace of $L^2\big((\mathbb{R}^d)^n, \zeta_c^{(n)}\big)$ that is the image of the orthogonal projection $\operatorname{Sym}_n$. Clearly, $\mathbf{F}_{c,n}^{\mathrm{sym}}(H)$ consists of all ($\zeta_c^{(n)}$-versions of ) symmetric functions from $L^2\big((\mathbb{R}^d)^n, \zeta_c^{(n)}\big)$. We define an extended Fock space

$$\mathbf{F}_c^{\mathrm{sym}}(H) := \bigoplus_{n=0}^{\infty} \mathbf{F}_{c,n}^{\mathrm{sym}}(H).$$

We note that, for $c = (1, 0, 0, \ldots)$, we get $\mathbf{F}_c^{\mathrm{sym}}(H) = \mathcal{F}(H)$. Otherwise, i.e, if for some $n \geq 2$, $c_n(x) > 0$ on a set of positive Lebesgue measure, then $\mathcal{F}(H)$ is a proper subset of $\mathbf{F}_c^{\mathrm{sym}}(H)$. Indeed, for each $n \geq 2$ and each measurable subset of $(\mathbb{R}^d)^{(n)}$, $\zeta_c^{(n)}(A) = n! \int_A dx_1 \cdots dx_n$, see (5.19) in the case where $\theta \in \mathcal{P}(n)$, $\theta = \{\theta_1, \ldots, \theta_n\}$, $\theta_i = \{x_i\}$, $i = 1, 2, \ldots, n$. Since $\int_{(\mathbb{R}^d)^n \setminus (\mathbb{R}^d)^{(n)}} dx_1 \cdots dx_n = 0$, we may therefore embed $\mathcal{F}(H)$ into $\mathbf{F}_c^{\mathrm{sym}}(H)$ by identifying each function $f^{(n)} \in H^{\odot n}$ with the function from $\mathbf{F}_c^{\mathrm{sym}}(H)$ which is equal to $f^{(n)}$ on $(\mathbb{R}^d)^{(n)}$, and to zero otherwise. Evidently, the orthogonal compliment to $\mathcal{F}(H)$ in $\mathbf{F}_c^{\mathrm{sym}}(H)$ is a non-zero space.

Next, recall the system $(q^{(k)}(x, s))$ of monic orthogonal polynomials (in the $s$-variable) in $L^2(\mathbb{R}, \sigma(x, ds))$. From now on, we will use the sequence $(c_k)_{k=1}^{\infty}$ defined by

$$c_k(x) = \int_{\mathbb{R}} q^{(k-1)}(x, s)^2 \sigma(x, ds), \quad k \in \mathbb{N}. \tag{5.22}$$

Thus, $c_1(x) \equiv 1$, and, by (4.14), for $k \geq 2$,

$$c_k(x) = a_1(x) a_2(x) \cdots a_{k-1}(x). \tag{5.23}$$

**Lemma 5.8.** *For any $f_1, \ldots, f_n, g_1, \ldots, g_n \in \mathcal{D}$, we have*

$$\begin{aligned}&\left(P^{(n)}(f_1 \odot \cdots \odot f_n, \cdot), P^{(n)}(g_1 \odot \cdots \odot g_n, \cdot)\right)_{L^2(\mathcal{D}'\mu)} \\ &\quad = (f_1 \odot \cdots \odot f_n, g_1 \odot \cdots \odot g_n)_{\mathbf{F}_{c,n}^{\mathrm{sym}}(H)}.\end{aligned}$$

*Proof.* By the polarization identity, it suffices to prove Lemma 5.8 for any $f_1 = g_1, \ldots, f_n = g_n \in \mathcal{D}$. Denote $f^{(n)} = f_1 \otimes \cdots \otimes f_n$. By Theorem 5.6,

$$\begin{aligned}&\left(P^{(n)}(f^{(n)}, \cdot), P^{(n)}(f^{(n)}, \cdot)\right)_{L^2(\mu)} \\ &\quad = \sum_{\theta \in \mathcal{P}(n)} \sum_{\xi \in \mathcal{P}(n)} \left(\mathrm{Sym}_{|\theta|}(\mathcal{I}_\theta f^{(n)}), \mathrm{Sym}_{|\xi|}(\mathcal{I}_\xi f^{(n)})\right)_{\mathcal{F}(H)} \\ &\quad = \sum_{l=1}^{n} \sum_{\substack{\theta, \xi \in \mathcal{P}(n) \\ |\theta| = |\xi| = l}} \left(\mathrm{Sym}_l(\mathcal{I}_\theta f^{(n)}), \mathcal{I}_\xi f^{(n)}\right)_{H^{\otimes l}} l!.\end{aligned} \tag{5.24}$$

On the other hand, by Lemma 5.7

$$
\begin{aligned}
&(\operatorname{Sym}_n f^{(n)}, \operatorname{Sym}_n f^{(n)})_{\mathbf{F}^{\mathrm{sym}}_{c,n}(H)}\\
&\quad = (\operatorname{Sym}_n f^{(n)}, \operatorname{Sym}_n f^{(n)})_{L^2((\mathbb{R}^d)^n, \zeta_c^{(n)})}\\
&\quad = (\operatorname{Sym}_n f^{(n)}, f^{(n)})_{L^2((\mathbb{R}^d)^n, \zeta_c^{(n)})}\\
&\quad = \int_{(\mathbb{R}^d)^n} (\operatorname{Sym}_n f^{(n)}) f^{(n)} d\zeta_c^{(n)}\\
&\quad = \sum_{\xi \in \mathcal{P}(n)} \int_{(\mathbb{R}^d)^{(n)}_\xi} (\operatorname{Sym}_n f^{(n)}) f^{(n)} d\zeta_{c,\xi}^{(n)}.
\end{aligned}
\tag{5.25}
$$

By (5.24) and (5.25), the lemma will follow if we show that, for a fixed $\xi \in \mathcal{P}(n)$ with $|\xi| = l$,

$$
\begin{aligned}
&\sum_{\theta \in \mathcal{P}(n),\, |\theta| = l} \left(\operatorname{Sym}_l(\mathcal{I}_\theta f^{(n)}), \mathcal{I}_\xi f^{(n)})\right)_{H^{\otimes l}} l!\\
&\quad = \int_{(\mathbb{R}^d)^{(n)}_\xi} (\operatorname{Sym}_n f^{(n)}) f^{(n)}\, d\zeta_{c,\xi}^{(n)}.
\end{aligned}
\tag{5.26}
$$

So, we fix a partition $\xi = \{\xi_1, \dots, \xi_l\} \in \mathcal{P}(n)$ and assume that

$$
\max \xi_1 < \max \xi_2 < \cdots < \max \xi_l.
$$

Denote $k_i := |\xi_i|$, $i = 1, \dots, l$. By the definition of $\mathcal{I}_\xi f^{(n)}$ (see (5.17))

$$
\begin{aligned}
&(\mathcal{I}_\xi f^{(n)})(y_1, s_1, \dots, y_l, s_l)\\
&\quad = \Big(\prod_{i_1 \in \xi_1} f_{i_1}\Big)(y_1) \Big(\prod_{i_2 \in \xi_2} f_{i_2}\Big)(y_2) \cdots \Big(\prod_{i_l \in \xi_l} f_{i_l}\Big)(y_l)\\
&\qquad \times q^{(k_1 - 1)}(y_1, s_1) q^{(k_2 - 1)}(y_2, s_2) \cdots q^{(k_l - 1)}(y_l, s_l).
\end{aligned}
\tag{5.27}
$$

Let $\theta = \{\theta_1, \dots, \theta_l\} \in \mathcal{P}(n)$ and assume that (5.16) is satisfied. Let $r_i := |\theta_i|$, $i = 1, \dots, l$. We may assume that there exists a permutation $\widehat{\pi} \in S_l$ such that

$$
r_i = k_{\widehat{\pi}(i)}, \quad i = 1, \dots, l. \tag{5.28}
$$

Indeed, the corresponding term in the sum on the left hand side of (5.26) vanishes, as we will necessarily have, for each $x \in \mathbb{R}^n$, the scalar product of two different orthogonal polynomials in $L^2(\mathbb{R}, \sigma(x, ds))$. Analogously to (5.27), we have

$$\begin{aligned}
&(\mathcal{I}_\theta f^{(n)})(y_1, s_1, \dots, y_l, s_l) \\
&\quad = \Big(\prod_{j_1 \in \theta_1} f_{j_1}\Big)(y_1)\Big(\prod_{j_2 \in \theta_2} f_{j_2}\Big)(y_2) \cdots \Big(\prod_{j_l \in \theta_l} f_{j_l}\Big)(y_l) \\
&\qquad \times q^{(r_1-1)}(y_1, s_1) q^{(r_2-1)}(y_2, s_2) \cdots q^{(r_l-1)}(y_l, s_l).
\end{aligned} \tag{5.29}$$

Hence

$$\begin{aligned}
&l!\ \mathrm{Sym}_l(\mathcal{I}_\theta f^{(n)})(y_1, s_1, \dots, y_l, s_l) \\
&\quad = \sum_{\varkappa \in S_l} (\mathcal{I}_\theta f^{(n)})(y_{\varkappa^{-1}(1)}, s_{\varkappa^{-1}(1)}, \dots, y_{\varkappa^{-1}(l)}, s_{\varkappa^{-1}(l)}) \\
&\quad = \sum_{\varkappa \in S_l} \Big(\prod_{j_1 \in \theta_{\varkappa(1)}} f_{j_1}\Big)(y_1)\Big(\prod_{j_2 \in \theta_{\varkappa(2)}} f_{j_2}\Big)(y_2) \cdots \Big(\prod_{j_l \in \theta_{\varkappa(l)}} f_{j_l}\Big)(y_l) \\
&\qquad \times q^{(r_{\varkappa(1)}-1)}(y_1, s_1) q^{(r_{\varkappa(2)}-1)}(y_2, s_2) \cdots q^{(r_{\varkappa(l)}-1)}(y_l, s_l)
\end{aligned}$$

Hence, by (5.22)

$$\begin{aligned}
&\big(\mathrm{Sym}_l(\mathcal{I}_\theta f^{(n)}), \mathcal{I}_\xi f^{(n)}\big)_{H^{\otimes l}}\, l! \\
&\quad = \sum_{\widehat{\pi}} \Big(\int_{\mathbb{R}^d} \big(\prod_{j_1 \in \theta_{\widehat{\pi}(1)}} f_{j_1}\big)(y_1)\big(\prod_{i_1 \in \xi_1} f_{i_1}\big)(y_1) c_{k_1}(y_1)\, dy_1\Big) \\
&\qquad \cdots \Big(\int_{\mathbb{R}^d} \big(\prod_{j_l \in \theta_{\widehat{\pi}(l)}} f_{j_l}\big)(y_l)\big(\prod_{i_l \in \xi_l} f_{i_l}\big)(y_l) c_{k_l}(y_l)\, dy_l\Big),
\end{aligned} \tag{5.30}$$

where the summation is over all permutation $\widehat{\pi} \in S_l$ which satisfy (5.28). Let us fix such a permutation $\widehat{\pi}$. Then, there exist $k_1! \cdots k_l! = r_1! \cdots r_l!$ permutations $\pi \in S_n$ which satisfy

$$\pi \xi_i = \theta_{\widehat{\pi}(i)}, \quad i = 1, \dots, l. \tag{5.31}$$

Note that, for each permutation $\pi$ satisfying (5.31) and for $(y_1, \dots, y_n) \in (\mathbb{R}^d)^{(n)}_\xi$,

$$\begin{aligned}(f_1 \otimes \cdots \otimes f_n)(y_{\pi^{-1}(1)}, \dots, y_{\pi^{-1}(n)}) &= \big(f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}\big)(y_1, \dots, y_n) \\ &= \Big(\prod_{j_1 \in \pi\xi_1} f_{j_1}\Big)(y_1) \cdots \Big(\prod_{j_l \in \pi\xi_l} f_{j_l}\Big)(y_l) \\ &= \Big(\prod_{j_1 \in \theta_{\widehat{\pi}(1)}} f_{j_1}\Big)(y_1) \cdots \Big(\prod_{j_l \in \theta_{\widehat{\pi}(l)}} f_{j_l}\Big)(y_l). \end{aligned} \tag{5.32}$$

Let $\xi, \theta \in \mathcal{P}(n)$ be such that condition (5.28) is satisfied by some permutation $\widehat{\pi} \in S_l$. Denote by $S_n[\theta, \xi]$ the set of all permutations $\pi \in S_n$ which satisfy (5.31) with some permutation $\widehat{\pi} \in S_l$. (Note the permutation $\widehat{\pi}$ is then completely identified by $\pi$, $\theta$ and $\xi$ and automatically satisfies (5.28).) Clearly, if $\theta$ and $\theta'$ are from $\mathcal{P}(n)$ with $|\theta| = |\theta'| = l$, both satisfying (5.28), and $\theta \neq \theta'$, then

$$S_n[\theta, \xi] \cap S_n[\theta', \xi] = \varnothing. \tag{5.33}$$

Furthermore,

$$\bigcup_{\substack{\theta \in \mathcal{P}(n),\, |\theta| = l \\ \theta \text{ satisfying } (5.28)}} S_n[\theta, \xi] = S_n. \tag{5.34}$$

Hence, by (5.29), (5.32), (5.33) and (5.34),

$$\begin{aligned}&\big(\mathrm{Sym}_l(\mathcal{I}_\theta f^{(n)}), \mathcal{I}_\xi f^{(n)}\big)_{H^{\otimes l}}\, l! \\ &\quad = \frac{1}{n!} \sum_{\pi \in S_n[\theta, \xi]} \int_{(\mathbb{R}^d)^{(n)}_\xi} f^{(n)}(x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)}) f^{(n)}(x_1, \dots, x_n)\, \zeta^{(n)}_\xi(x_1, \dots, x_n).\end{aligned}$$

Hence

$$\sum_{\theta \in \mathcal{P}(n), |\theta| = l} \big(\mathrm{Sym}_l(\mathcal{I}_\theta f^{(n)}), \mathcal{I}_\xi f^{(n)}\big)_{H^{\otimes l}}\, l!$$

$$
\begin{aligned}
&= \sum_{\substack{\theta\in\mathcal{P}(n),\,|\theta|=l \\ \theta\,\text{satisfying}\,(5.28)}} \big(\operatorname{Sym}_l(\mathcal{I}_\theta f^{(n)}), \mathcal{I}_\xi f^{(n)}\big)_{H^{\otimes l}}\, l! \\
&= \frac{1}{n!} \sum_{\substack{\theta\in\mathcal{P}(n),\,|\theta|=l \\ \theta\,\text{satisfying}\,(5.28)}} \sum_{\pi\in S_n[\theta,\xi]} \int_{(\mathbb{R}^d)^{(n)}_\xi} f^{(n)}(x_{\pi^{-1}(1)},\dots,x_{\pi^{-1}(n)}) \\
&\quad \times f^{(n)}(x_1,\dots,x_n)\,d\zeta^{(n)}_{c,\xi}(x_1,\dots,x_n) \\
&= \frac{1}{n!}\sum_{\pi\in S_n} \int_{(\mathbb{R}^d)^{(n)}_\xi} f^{(n)}(x_{\pi^{-1}(1)},\dots,x_{\pi^{-1}(n)}) f^{(n)}(x_1,\dots,x_n)\,d\zeta^{(n)}_{c,\xi}(x_1,\dots,x_n) \\
&= \int_{(\mathbb{R}^d)^{(n)}_\xi} \frac{1}{n!}\sum_{\pi\in S_n} f^{(n)}(x_{\pi(1)},\dots,x_{\pi(n)}) f^{(n)}(x_1,\dots,x_n)\,d\zeta^{(n)}_{c,\xi}(x_1,\dots,x_n) \\
&= \int_{(\mathbb{R}^d)^{(n)}_\xi} (\operatorname{Sym}_n f^{(n)}) f^{(n)}\,d\zeta^{(n)}_{c,\xi}(x_1,\dots,x_n),
\end{aligned}
$$

so that (5.26) is proven. $\square$

For each $n \in \mathbb{N}$, let $R_n$ denote the linear span of functions of the form

$$f_1 \otimes f_2 \otimes \cdots \otimes f_n,$$

where $f_1, f_2, \dots, f_n \in \mathcal{D}$. This is a dense subset of $L^2((\mathbb{R}^d)^n, \zeta_c^{(n)})$. Clearly $\operatorname{Sym}_n R_n$ is the linear span of functions of the form

$$f_1 \odot f_2 \odot \cdots \odot f_n$$

with $f_1, f_2, \dots, f_n \in \mathcal{D}$, and $\operatorname{Sym}_n R_n$ is dense in the Hilbert space

$$\mathbf{F}^{\mathrm{sym}}_{c,n}(H) = \operatorname{Sym}_n L^2((\mathbb{R}^d)^n, \zeta_c^{(n)}).$$

By Lemma 5.8, for any $f^{(n)}, g^{(n)} \in \operatorname{Sym}_n R_n$

$$\big(P^{(n)}(f^{(n)},.), P^{(n)}(g^{(n)},.)\big)_{L^2(\mathcal{D}',\mu)} = (f^{(n)}, g^{(n)})_{\mathbf{F}^{\mathrm{sym}}_{c,n}(H)}.$$

Therefore, the linear mapping

$$\operatorname{Sym}_n R_n \ni f^{(n)} \mapsto P^{(n)}(f^{(n)},\cdot) \in L^2(\mathcal{D}\text{"}, \mu)$$

can be extended by continuity to an isometric operator

$$U_n : \mathbf{F}_{c,n}^{\mathrm{sym}}(H) \mapsto L^2(\mu).$$

For each $f^{(n)} \in \mathbf{F}_{c,n}^{\mathrm{sym}}(H)$, we denote

$$P^{(n)}(f^{(n)}, \cdot) := U_n f^{(n)}.$$

On the other hand we know that the set $\{P^{(n)}(f^{(n)}, \cdot) \mid f^{(n)} \in R_n\}$ is dense in $\mathbb{P}^{(n)}(\mathcal{D}')$, and $\mathbb{P}^{(n)}(\mathcal{D}')$ is a closed subspace of $L^2(\mu)$. Therefore, the image of $U_n$ is $\mathbb{P}^{(n)}(\mathcal{D}')$, and

$$U_n : \mathbf{F}_{c,n}^{\mathrm{sym}}(H) \mapsto \mathbb{P}^{(n)}(\mathcal{D}')$$

is a unitary operator. By Proposition 5.1,

$$L^2(\mathcal{D}', \mu) = \bigoplus_{n=0}^{\infty} \mathbb{P}^{(n)}(\mathcal{D}'),$$

and by definition

$$\mathbf{F}_{c}^{\mathrm{sym}}(H) = \bigoplus_{n=0}^{\infty} \mathbf{F}_{c,n}^{\mathrm{sym}}(H).$$

Thus, we conclude the following decomposition of $L^2(\mathcal{D}', \mu)$ in orthogonal polynoials.

**Theorem 5.9.** *Let $c = (c_k)_{k=1}^{\infty}$, be defined by (5.22). Then we have a unitary isomorphism*

$$\mathbf{F}_{c}^{\mathrm{sym}}(H) \ni F = (f^{(n)})_{n=0}^{\infty} \mapsto UF := f(0) + \sum_{n=1}^{\infty} P^{(n)}(f^{(n)}, \cdot) \in L^2(\mathcal{D}', \mu).$$

# Chapter 6

# An example: Meixner-type processes with independent values

Let us first recall Meixner's classification of orthogonal polynomials which have generating function of exponential type, see e.g. [15] for further detail.

Assume that functions $f(z)$ and $\Psi(z)$ have a Taylor series representation around zero. Also assume that $f(0) = 1$, $\Psi(0) = 0$, and $\Psi'(0) = 1$. Then, the equation

$$G(x, z) := \exp(x\Psi(z))f(z) = \sum_{n=0}^{\infty} \frac{p^{(n)}(x)}{n!} z^n \tag{6.1}$$

determine a system of monic polynomials $p^{(n)}(x)$, $n \in \mathbb{Z}_+$. Meixner [34] found all classes of such polynomials which are orthogonal with respect to a probability measure $\nu$ on $\mathbb{R}$ and have infinite support. In fact, a given system $(p^{(n)}(x))_{n=0}^{\infty}$ of monic polynomials is orthogonal and has generating function (6.1) if and only if there exists $l \in \mathbb{R}$, $\lambda \in \mathbb{R}$, $k > 0$ and $\eta \geq 0$ such that

$$xp^{(n)}(x) = p^{(n+1)}(x) + (n\lambda + l)p^{(n)}(x) + n(k + \eta(n-1))p^{(n-1)}(x). \tag{6.2}$$

If one only considers the case where the measure of orthogonality, $\nu$, is centered, i.e.,

$$\int_{\mathbb{R}} x \, d\nu(x) = 0,$$

then $l = 0$, so that (6.2) becomes

$$xp^{(n)}(x) = p^{(n+1)}(x) + n\lambda p^{(n)}(x) + n(k + \eta(n-1))p^{(n-1)}(x). \qquad (6.3)$$

For fixed parameters $\lambda$ and $\eta$, we define $\alpha, \beta \in \mathbb{C}$ so that

$$\alpha + \beta = -\lambda, \qquad \alpha\beta = \eta, \qquad (6.4)$$

or equivalently

$$1 + \lambda t + \eta t^2 = (1 - \alpha t)(1 - \beta t). \qquad (6.5)$$

Clearly $\lambda \in \mathbb{R}$ and $\eta \geq 0$ if and only if either $\alpha, \beta \in \mathbb{R}$, $\alpha$ and $\beta$ being of the same sign, or $\mathrm{Im}(\alpha) \neq 0$ and $\alpha$ and $\beta$ are complex conjugate.

We have to distinguish the following five cases:

I. (Gaussian case) Now $\alpha = \beta = 0$ (or equivalently $\lambda = \eta = 0$). The orthogonality measure $\nu$ is the Gaussian measure:

$$d\nu(x) = (2\pi k)^{-1/2} \exp\left(-\frac{x^2}{2k}\right) dx. \qquad (6.6)$$

The Fourier transform of the Gaussian measure $\nu$ is given by

$$\int_{\mathbb{R}} \exp(iux) d\nu(x) = \exp\left(-\frac{1}{2}ku^2\right), \qquad u \in \mathbb{R}. \qquad (6.7)$$

Furthermore,

$$\Psi(z) = z, \qquad f(z) = \exp(-\frac{1}{2}kz^2),$$

so that

$$G(x, z) = \exp\left(xz - \frac{1}{2}kz^2\right).$$

The $(p^{(n)})_{n=0}^{\infty}$ is a system of Hermite polynomials.

II. (Poisson case) Assume $\alpha \neq 0$, $\beta = 0$ (which corresponds to the choice of $\lambda \neq 0$ and $\eta = 0$). Now $\nu$ is a centered Poisson measure:

$$d\nu(x) = \exp\left(-\frac{k}{\alpha^2}\right) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{k}{\alpha^2}\right)^n \delta_{(-\alpha n + \frac{k}{\alpha})}(dx),$$

where $\delta_a$ denotes the Dirac measure with mass at $a$. The Fourier transform of $\nu$ is given by

$$\int_{\mathbb{R}} \exp(iux) d\nu(x) = \exp\left(\frac{k}{\alpha^2}(e^{-i\alpha u} - 1 + i\alpha u)\right).$$

Furthermore, in a neighborhood of zero,

$$\begin{aligned} \Psi(z) &= -\frac{1}{\alpha} \log(1 - \alpha z), \\ f(z) &= \exp\left(k(\frac{1}{\alpha^2} \log(1 - \alpha z) + \frac{z}{\alpha})\right), \end{aligned}$$

so that

$$G(x, z) = \exp\left(-\frac{x}{\alpha} \log(1 - \alpha z) + k\left(\frac{1}{\alpha^2} \log(1 - \alpha z) + \frac{z}{\alpha}\right)\right).$$

The $(p^{(n)})_{n=0}^{\infty}$ is a system of Charlier polynomials.

III. (Gamma case) Assume $\alpha = \beta \neq 0$ (which corresponds to $\lambda = -2\alpha \in \mathbb{R}$, $\eta = \alpha^2 > 0$). $\nu$ is a centered gamma measure:

$$\begin{aligned} d\nu(x) &= \chi_{(-\infty, -k/\alpha)}(x) \left(-x + \frac{k}{\alpha}\right)^{-1+k/\alpha^2} e^{x/\alpha}, \qquad \alpha > 0, \\ d\nu(x) &= \chi_{(-k/\alpha, +\infty)}(x) \left(x + \frac{k}{\alpha}\right)^{-1+k/\alpha^2} e^{x/\alpha}, \qquad \alpha < 0. \end{aligned}$$

The Fourier transform of the gamma measure $\nu$ (in a neighborhood of zero) is given by

$$\int_{\mathbb{R}} \exp(iux) d\nu(x) = \exp\left(k\left(\frac{iu}{\alpha} - \frac{1}{\alpha^2} \log(1 + \alpha iu)\right)\right).$$

Next,

$$\Psi(z) = \frac{z}{1-\alpha z},$$
$$f(z) = \exp\left(-k(\frac{1}{\alpha^2}\log(1-\alpha z) + \frac{z}{(1-\alpha z)\alpha})\right),$$

so that

$$G(x,z) = \exp\left(\frac{xz}{1-\alpha z} - k\left(\frac{1}{\alpha^2}\log(1-\alpha z) + \frac{z}{(1-\alpha z)\alpha}\right)\right).$$

The $(p^{(n)})_{n=0}^{\infty}$ is a system of Laguerre polynomials.

IV. (Pascal case) Now $\alpha \neq \beta \neq 0$, $\alpha, \beta \in \mathbb{R}$ (which corresponds to $\eta > 0$ and $\lambda^2 - 4\eta \geq 0$). Then $\nu$ is a centered Pascal measure (negative binomial distribution):

$$d\nu(x) = \sum_{n=0}^{\infty} \left(-\frac{\beta}{\alpha}\right)^n \left(-\frac{k}{\alpha\beta}\right)_n \delta_{(-\frac{k}{\alpha-(\alpha-\beta)n})}(dx),$$

where $(\varkappa)_0 := 1$, $(\varkappa)_n := \varkappa(\varkappa+1)\cdots(\varkappa+n-1)$, $n \in \mathbb{N}$, $x \in \mathbb{R}$, and we assumed that $|\alpha| > |\beta|$. The Fourier transform of $\nu$ (in a neighborhood of zero) is given by

$$\int_{\mathbb{R}} \exp(iux)d\nu(x) = \exp\left(-\frac{k}{\alpha\beta}\log\frac{\alpha e^{-i\beta u} - \beta e^{-i\alpha u}}{\alpha-\beta}\right).$$

Furthermore

$$\Psi(z) = \frac{1}{\alpha-\beta}\log\left(\frac{1-\beta z}{1-\alpha z}\right),$$
$$f(z) = \exp\left(-\frac{k}{\alpha-\beta}\log\left(\frac{(1-\beta z)^{\frac{1}{\beta}}}{(1-\alpha z)^{\frac{1}{\alpha}}}\right)\right),$$

so that

$$G(x,z) = \exp\left(\frac{x}{\alpha-\beta}\log\left(\frac{1-\beta z}{1-\alpha z}\right) - \frac{k}{\alpha-\beta}\log\left(\frac{(1-\beta z)^{\frac{1}{\beta}}}{(1-\alpha z)^{\frac{1}{\alpha}}}\right)\right).$$

The $(p^{(n)})_{n=0}^{\infty}$ is a system of Meixner polynomials of the first kind.

V. (Meixner case). Now $\mathrm{Im}(\alpha) \neq 0$, $\alpha = \overline{\beta}$ (which corresponds to $\eta > 0$ and $\lambda^2 - 4\eta < 0$). The measure $\nu$ is a centered Meixner measure

$$d\nu(x) = \left(\Gamma\left(\frac{k}{2\alpha\beta}\right)\right)^{-2} \left|\Gamma\left(\frac{k}{2\alpha\beta} + \frac{i}{2}\left(x - \frac{(\alpha+\beta)k}{2\alpha\beta}\right)\right)\right|^2 \\ \times \exp\left[-\left(x - \frac{(\alpha+\beta)k}{2\alpha\beta}\right)\arctan\left(\frac{i(\alpha+\beta)}{|\alpha+\beta|}\right)\right],$$

where we assumed that $\mathrm{Im}(\alpha) > \mathrm{Im}(\beta)$. The formulas for the Fourier transform of $\nu$, the functions $\Psi(z)$, $\Psi^{-1}(z)$, and $G(x,z)$ have the same form as in the case IV, but with complex conjugate $\alpha$, $\beta$. The $(p^{(n)}(x))_{n=0}^{\infty}$ is a system of Meixner polynomials of the second kind, or the Meixner–Pollaczek polynomials.

In fact, all the above formulas for the Fourier transform and the generating function can be written down in a common form if one uses infinite sums involving $\alpha$ and $\beta$, see [38].

For each measure of orthogonality of polynomials from the Meixner class, $\nu$, the Laplace transform of $\nu$,

$$\int_{\mathbb{R}} e^{ux}\nu(dx)$$

is well-defined in a neighborhood of zero in $\mathbb{R}$. The function

$$\mathfrak{C}(u) = \log\left(\int_{\mathbb{R}} e^{ux}\nu(dx)\right),$$

also defined in a neighborhood of zero in $\mathbb{R}$, is called the cumulant transform of $\nu$. We denote by $\mathfrak{C}_{\lambda,\eta}(u)$ the cumulant transform of the measure $\nu$ corresponding to parameters $k=1$, $\lambda$, and $\eta$.

For any $\lambda \in \mathbb{R}$ and $\eta > 0$, let $\sigma_{\lambda,\eta}(ds)$ be the probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which is the measure of orthogonality of monic polynomials $(q^{(n)}(s))_{n\geq 0}$ which satisfy the recurrence relation:

$$sq^{(n)}(s) = q^{(n+1)}(s) + \lambda(n+1)q^{(n)}(s) + \eta n(n+1)q^{(n-1)}(s). \tag{6.8}$$

Note that the probability measure $\sigma_{\lambda,\eta}$ belongs to the Meixner class.

We have [40], see also [34]:

**Proposition 6.1.** *For any $k > 0$, $\lambda \in \mathbb{R}$ and $\eta > 0$, let $\nu_{k,\lambda,\eta}$ be the measure of orthogonality of monic polynomials satisfying* (6.3). *Then the measure $\nu_{k,\lambda,\eta}$ is infinitely divisible and $k\, s^{-2}\sigma_{\lambda,\eta}(ds)$ is its Lévy measure:*

$$\int_{\mathbb{R}} \exp(iux)\, \nu_{k,\lambda,\eta}(dx) = \exp\left[k \int_{\mathbb{R}} \sigma_{\lambda,\eta}(ds) s^{-2}(e^{ius} - 1 - ius)\right].$$

*(Note that, for $s = 0$, the function $s^{-2}(e^{ius} - 1 - ius)$ is assumed to take the value $-\frac{s^2}{2}$.) In particular,*

$$\mathfrak{C}_{\lambda,\eta}(u) = \int_{\mathbb{R}} \sigma_{\lambda,\eta}(ds) s^{-2}(e^{us} - 1 - us). \tag{6.9}$$

From now on, we fix any measurable, locally bounded functions $\lambda : \mathbb{R}^d \to \mathbb{R}$ and $\eta : \mathbb{R}^d \to (0, \infty)$. For each $x \in \mathbb{R}^d$, let $\sigma(x, ds)$ be the probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by

$$\sigma(x, ds) := \sigma_{\lambda(x),\eta(x)}(ds),$$

see Proposition 6.1. By (6.8), the corresponding functions $a_n(x)$ and $b_n(x)$ are given by

$$a_n(x) = \eta(x)n(n+1), \quad b_n(x) = \lambda(x)(n+1),$$

and so condition (B) is satisfied. Thus our results in Chapters 3–5 are applicable to $\sigma(x, ds)$. Note that, by (5.23), the corresponding functions $(c_k(x))_{k=1}^{\infty}$ are given by

$$c_k(x) = \eta(x)^{k-1}(k-1)!\, k!, \quad k \in \mathbb{N}.$$

(Here $0^0 := 1$.) Hence, by (5.19), the measure $\zeta_{c,\theta}^{(n)}$ on $(\mathbb{R}^d)_{\theta}^{(n)}$ is the push-forward of the measure

$$\eta(x_1)^{|\theta_1|-1} \cdots \eta(x_l)^{|\theta_l|-1}(|\theta_1| - 1)!\, |\theta_1|! \cdots (|\theta_l| - 1)!\, |\theta_l|!$$

$$\times\, n!(|\theta_1|!\cdots|\theta_l|!)^{-1}\,dx_1\cdots dx_l$$
$$= \eta(x_1)^{|\theta_1|-1}\cdots\eta(x_l)^{|\theta_l|-1}(|\theta_1|-1)!\cdots(|\theta_l|-1)!\,n!\,dx_1\cdots dx_l.$$

Furthermore, by Theorem 3.7, the Fourier transform of the corresponding measure $\mu$ on $\mathcal{D}'$ is given by

$$\int_{\mathcal{D}'} e^{i\langle\varphi,\omega\rangle}\mu(d\omega) = \exp\left[\int_{\mathbb{R}^d} dx \int_{\mathbb{R}} \sigma(x,ds)\frac{1}{s^2}[e^{i\varphi(x)s} - i\varphi(x)s - 1]\right]. \tag{6.10}$$

Thus, our results extend the corresponding results of [30, 31] and [38].

*Remark* 6.2. Recall that in [30, 31] $\lambda$ and $\eta$ were constants, and in [38] the functions $\lambda$ and $\eta$ were assumed to be smooth, whereas we do not even assume that these functions are continuous.

As follows from the proof of Theorem 3.7, the Laplace transform of the measure $\mu$,

$$\varphi \mapsto \int_{\mathcal{D}'} e^{\langle\varphi,\omega\rangle}\mu(d\omega)$$

is well-defined in a neighborhood of zero in $\mathcal{D}$. Hence, we can define its cumulant transform

$$\mathfrak{C}(\varphi) := \log\left(\int_{\mathcal{D}'} e^{\langle\varphi,\omega\rangle}\mu(d\omega)\right),$$

which is also defined in a neighborhood of zero in $\mathcal{D}$.

**Theorem 6.3.** *We have, for $\varphi$ from a neighborhood of zero in $\mathcal{D}$,*

$$\mathfrak{C}(\varphi) = \int_{\mathbb{R}^d} \mathfrak{C}_{\lambda(x),\eta(x)}(\varphi(x))\,dx. \tag{6.11}$$

*Proof.* We will only sketch the proof of this theorem. By approximation, it suffices to prove the following statement.

Fix any $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$ and any constant $\varepsilon > 0$. Then there exists a constant $C > 0$ for which the following holds. Let $\Lambda, \dots, \Lambda \in \mathcal{B}_0(\mathbb{R}^d)$ be mutually disjoint and $\bigcup_{j=1}^{n} \Lambda_j = \Lambda$. Let functions $\lambda$ and $\eta$ take on constant values on each set $\Lambda_j$, $j = 1, \dots, n$, and let the functions $|\lambda|$ and $\eta$ be bounded by $\varepsilon$ on $\Lambda$. Let a function $\varphi$ be given by

$$\varphi(x) = \sum_{j=1}^{n} r_j \chi_{\Lambda_j}(x), \tag{6.12}$$

where

$$\max_{j=1,\dots,n} |r_j| \le C.$$

Then formula (6.11) holds for this function $\varphi$.

Indeed, denote by $\lambda_j$ and $\eta_j$ the value of $\lambda$ and $\eta$ on $\Lambda_j$. By (6.9), (6.10), and (6.12),

$$\begin{aligned}
\mathfrak{C}(\varphi) &= \sum_{j=1}^{n} \int_{\Lambda_j} dx \int_{\mathbb{R}} \sigma(x, ds) s^{-2}(e^{r_j s} - r_j s - 1) \\
&= \sum_{j=1}^{n} \left( \int_{\Lambda_j} dx \right) \int_{\mathbb{R}} \sigma_{\lambda_j,\eta_j}(ds)(e^{r_j s} - r_j s - 1) \\
&= \sum_{j=1}^{n} \left( \int_{\Lambda_j} dx \right) \mathfrak{C}_{\lambda_j,\eta_j}(r_j) \\
&= \int_{\mathbb{R}^d} \sum_{j=1}^{n} \chi_{\Lambda_j}(x) \mathfrak{C}_{\lambda(x),\eta(x)}(\varphi(x))\, dx \\
&= \int_{\mathbb{R}^d} \mathfrak{C}_{\lambda(x),\eta(x)}(\varphi(x))\, dx,
\end{aligned}$$

where we used that $\mathfrak{C}_{\lambda(x),\eta(x)}(0) = 0$. From here the statement follows.

$\square$

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