The role of convection on spreading speeds and linear determinacy for reaction-diffusion-convection systems.

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The Role of Convection on Spreading Speeds and Linear Determinacy for Reaction-Diffusion-Convection Systems

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Abstract

This thesis is concerned with spreading speeds and linear determinacy for both discrete-time recursion models $u_{n+1} = Q[u_n]$ and reaction-diffusion-convection systems (PDE) under a co-operative assumption. In this thesis we are interested in the role of convection terms in propagation and linear determinacy. Such reaction-diffusion-convection systems have monotone travelling-wave solutions of the form $w(x - ct)$ that describe the propagation of species as a wave with a fixed speed c, connecting two equilibria, a stable equilibrium β and an unstable equilibrium 0 of the reaction term. The concept of spreading speeds was introduced by Aronson and Weinberger in [2] as a description of asymptotic speeds of spread, and in fact they showed that this spreading speed can be characterized as a minimal travelling wave speed. We discuss a characterization theory of spreading speeds of the PDE system in terms of critical travelling wave speeds. We present sufficient conditions involving both the reaction and convection terms of the PDE system for spreading speeds to equal values obtained from the linearization of the travelling-wave problem of the PDE system about the unstable equilibrium 0. These conditions guarantee the linear determinacy for the discrete-time recursion models and the PDE systems. As a result of the asymmetry in propagation that is caused by the convection terms in the PDE system, and a corresponding lack of reflection invariance in the abstract system $u_{n+1} = Q[u_n]$, we present separate conditions for non-increasing and non-decreasing initial data, called *right* and *left* conditions respectively, and we consider *right* and *left* spreading speeds. Weinberger, Lewis and Li in [42] allowed there to be more equilibria other than 0 and β, in which case different components may spread at different speeds. This implies the need for both slowest and fastest spreading speeds, called *right* and *left* slowest (fastest) spreading speeds corresponding, to non-increasing and non-decreasing initial data respectively. We also give sufficient conditions on the reaction and convection terms such that right (left) slowest spreading speed equals right (left) fastest spreading speed for the PDE system, which implies that the system has a *right* (*left*) single spreading speed. Examples are included that illustrate the key propositions and theorems, for instance, the existence of reaction and (non-trivial) convection terms for which the right and left linear determinacy conditions are simultaneously satisfied, as well as a system that is right (left) linearly determinate in absence of convection terms, but it is not left linearly determinate in the presence of a convection term.

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Table 1: General notation

Symbol	Definition
$u \geq v \in \mathbb{R}^k$	For all $i, 1 \leq i \leq k$, $u_i \geq v_i$, the ith component of u is larger than or equal the ith component of v Note; if u, v are functions from $\mathbb{R} \to \mathbb{R}^k$, then for all x and i, $u_i(x) \geq v_i(x)$
$u > v \in \mathbb{R}^k$	For all $i, 1 \leq i \leq k$, $u_i > v_i$, the ith component of u is strictly larger than the ith component of v Note; if u, v are functions from $\mathbb{R} \to \mathbb{R}^k$, then for all x and i, $u_i(x) > v_i(x)$
$[a, b]$	The set of $x \in \mathbb{R}^k$ such that $a \leq x \leq b$
$\mathbb{R}^k_+$	The set of the positive cone in $\mathbb{R}^k$ of non-negative vectors
$[\alpha, \beta]$	The set of all vectors v such that each ith component of v_i, $\alpha \leq v_i \leq \beta$
$\sup A, A \in \mathbb{R}^k$	$(\sup A)_i := \sup A_i$, $1 \leq i \leq k$
$\mathbb{R}^{k \times k}$	The set of real $k \times k$ matrices
$diag\ z_i$	$diag\ (z_1, ..., z_k)$
$P^{k \times k}$	The set of real $k \times k$ matrices with strictly positive off-diagonal elements
$f^{'}(\alpha)$	The Jacobian matrix of $f : \mathbb{R}^k \to \mathbb{R}^k$ at α
$\mathfrak{F}_{pf}(B)$	Perron-Frobenius eigenvalue of the matrix B
$L^\infty(I, \mathbb{R}^k)$	$\{u : u \to \mathbb{R}^k : u$ is Lebesgue measurable, $\|u\|_{L^\infty(I,\mathbb{R}^k)} < \infty\}$ where $\|u\|_{L^\infty(I,\mathbb{R}^k)} = ess \sup_{\mathbb{R}} |u|$, and $I = \mathbb{R}$ or $I = [-\infty, \alpha]$
$C^p(I, \mathbb{R}^k)$	The space of functions $f : \mathbb{R} \to \mathbb{R}^k$ such that f and its derivatives for $p \geq 1$ are continuous on I where $I = \mathbb{R}^k$ or $I = [0, 1]$
$BUC(\mathbb{R}, \mathbb{R}^k)$	The space of functions $f : \mathbb{R} \to \mathbb{R}^k$ such that f is bounded and uniformly continuous on $\mathbb{R}$
$BUC^p(\mathbb{R}, \mathbb{R}^k)$	The space of functions $f : \mathbb{R} \to \mathbb{R}^k$ such that f and its derivatives for $p \geq 1$ are bounded and uniformly continuous on $\mathbb{R}$
$B_{BUC^1}(0, R)$	$\{u \in BUC^1 : \|u\|_{1,\infty} < R\}$
$\|u\|_\infty$	$\sup_{x \in I} |u(x)|$, where $u : I \to \mathbb{R}^k$
$\|u\|_{1,\infty}$	$\|u\|_\infty + \|u^{'}\|_\infty$

Table 2: Meaning of various speeds

Symbol	Meaning
c_0	Minimal travelling wave speed of non-increasing travelling waves for reaction-diffusion-convection system
$\bar{c}$	(Right) linear value speed for reaction-diffusion-convection equation
$\hat{c}$	Left slowest spreading speed for reaction-diffusion-convection system
$\hat{\hat{c}}$	Left linear value speed for reaction-diffusion-convection system
$\hat{c}_f$	Left fastest spreading speed for reaction-diffusion-convection system
$\hat{c}^+$	Maximum of left linear values for reaction-diffusion-convection system
$\mathring{c}$	Right slowest spreading speed for reaction-diffusion-convection system
$\mathring{\mathring{c}}$	Right linear value speed for reaction-diffusion-convection system
$\mathring{c}_f$	Right fastest spreading speed for reaction-diffusion-convection system
$\mathring{c}^+$	Maximum of right linear values for reaction-diffusion-convection system
$\hat{c}_{lin}$	Linear value corresponding to stable monotone eigenvalue
$\mathring{c}_{lin}$	Linear value corresponding to unstable monotone eigenvalue

Chapter 1

Introduction

The reaction-diffusion equation $u_t = du_{xx} + f(u),\ x \in \mathbb{R}$, is well-known as a simple model of one-dimensional phenomena in, for instance, population growth, chemical reaction, flame propagation, *etc*, where $d > 0$ is a diffusion coefficient and f is a reaction term. For the classical Fisher case [18], $f(u) = ru(1-u)$, Kolmogorov-Petrowsky and Piscounov [24] showed that there exist non-increasing travelling waves, joining the equilibria 1 and 0, for all speeds $c \geq 2\sqrt{dr}$. A travelling wave solution has the form $u(x,t) = w(x-ct)$ and describes the propagation of a species as a wave with a fixed shape and a fixed speed c. It was also shown in [24] that there are no such non-increasing waves of speed slower than the speed c. The concept of asymptotic speeds of spread, known as spreading speeds, was first introduced by Aronson and Weinberger [2] for reaction-diffusion equations. The spreading speed is defined using initial condition $u_0(x)$ of the reaction-diffusion equation that is identically zero at one end, but a wider class of initial conditions will also spread at this spreading speed. Aronson and Weinberger showed also that the spreading speed can be characterized as a minimal wave speed of a class of travelling wave solution (see also [42], [26] and [29]). If the initial condition u_0 converges more slowly, the solution $u(x,t)$ may converge to a travelling wave of speed that is not the minimal wave speed.

In many applications, however, there is convective motion in addition to diffusion and reaction, which can have a major impact on the behaviour of solutions. An example of such convection terms arises in a simple one-dimensional model of the motion of chemotactic cells, based on a model of Keller and Segel [23], that is presented in Benguria, Depassier and Mendez [6], where ρ denotes the density of bacteria chemotactic to a single chemical element of concentration s, the density evolves according to $\rho_t = [D\rho_x - \rho\chi s_x]_x + f(\rho)$, D is a diffusion constant and χ is the chemotactic sensitivity. For travelling front solutions, $s = s(x-ct), \rho = \rho(x-ct)$, we have $s_t = -cs_x$, $s_x = K\rho/c$, and the problem then reduces to a single differential equation for ρ, namely

$$\rho_t = D\rho_{xx} - \frac{\chi K}{c}(\rho)_x^2 + f(\rho).$$

Motivated by such models, we study systems of partial differential equations (PDE) in the presence of convection terms under a co-operative assumption. The theory of spreading speed and travelling waves is presented in [29], [42], [26], [43], and [40] in the context of *co-operative operators* of a discrete time recursion $u_{n+1} = Q[u_n]$. Such recursions are presented as an abstract tool that can be applied to study the spreading speed of reaction-diffusion-convection systems. We present theory about the recursion $u_{n+1} = Q[u_n]$, based on [29], and others, and extend this theory to remove a reflection invariance assumption to allow application to PDE systems involving convection terms and to consider both non-increasing and non-decreasing initial conditions. Note that due to the presence of convection terms, there is a lack of symmetry in propagation in the PDE system and a corresponding lack of reflection invariance in the recursion $u_{n+1} = Q[u_n]$ that imposes us to present separate conditions for non-increasing and non-decreasing initial data to ensure that the spreading speeds equal values obtained from the linearization of the travelling wave problem of the PDE system about the unstable equilibrium 0. These conditions involve both the reaction and convection terms of the PDE system and are denoted by *right (left)* conditions respectively. It is shown in Lui [29] that all components spread at the same speed when there are only two equilibria 0 and β of the reaction term, whereas Weinberger,Lewis and Li in [42] also considered more equilibria but they allowed that these components spread at different speeds. This implies the need for another spreading speed to take account of the fact that different components may spread at different speeds. As a result and corresponding to non-increasing and non-decreasing initial data we have *right (left)* slowest spreading speed and *right (left)* fastest spreading speed. Further, we prove results about spreading speeds for the abstract operator Q which gives us information about discrete recursion and later we apply it to continuous time systems such as reaction-diffusion-convection systems by taking Q to be the time-t map of the reaction-diffusion-convection systems.

A travelling wave of the form $w(x - nc)$ and $w(x - ct)$ is a special solution of recursion $u_{n+1} = Q[u_n]$ and of reaction-diffusion-convection systems, whereas the spreading speed characterises the evolution of the solution with specific kind of initial condition, for instance, when the initial condition is identically zero at one end. Li, Weinberger and Lewis in [26] showed that the spreading speed of reaction-diffusion systems with special linear convection term of the form Eu_x where E is a constant diagonal matrix, can be characterized as the slowest speed of a class of travelling waves. We extend this characterization theory of spreading speed in terms of critical travelling wave speeds to reaction-diffusion systems with non-linear convection term of the form $h_i'(u_i)u_{i,x} = diag\ \left(h_1'(u_1)u_{1,x}, ..., h_k'(u_k)u_{k,x}\right)$. Moreover, we give characterization properties of spreading speeds for the continuous time system, as well as a characterization of slowest spreading speed in terms of travelling waves for the PDE system. Note that in Chapter 2, we discuss non-increasing travelling wave solutions, the minimal speed of which equals a slowest spreading speed for the reaction-diffusion-convection equation as will be established in Chapter 5. We remark that in Chapter 2, we consider non-increasing waves, the analogous results hold for non-decreasing waves, and some results for such waves will be discussed in the context of the PDE system in Chapter 5.

Before presenting the theory of the operator Q, in Chapter 2, we begin by considering travelling-wave solutions of a reaction-diffusion-convection equation of the form

$$u_t + h'(u)u_x = u_{xx} + f(u), \qquad x \in \mathbb{R}, \tag{1.1}$$

with a monostable reaction term $f(u)$ in which 0 is an unstable equilibrium, there is a stable equilibrium $\beta > 0$, and there are no equilibria of f between 0 and β. The definition of *stable* equilibrium α of $f : \mathbb{R}^k \to \mathbb{R}^k$ is that all solutions of $u_t = f'(\alpha)u$ tend to 0 as $t \to \infty$, whereas α is *unstable* if for some initial condition $u(0)$, the solution does not tend to 0. Hence in particular, this definition can be applied in the scalar case when $f : \mathbb{R} \to \mathbb{R}$ and $f'(\alpha)$ is a number instead of a matrix. The monostable reaction function $f(u)$ thus satisfies $f(0) = f(\beta) = 0$ and $f'(\beta) < 0, f'(0) > 0$. Note that for (1.1) with $h'(u) \equiv 0$, there is, of course, reflection symmetry, which means that if $u(x,t)$ is a solution of (1.1), and we define $\hat{u}(x,t) := u(-x,t)$, then $\hat{u}$ also satisfies equation (1.1). Thus corresponding to a non-increasing travelling-wave solution $w(x-ct)$, there is a non-decreasing travelling wave $\hat{w}(x+ct)$ with $\hat{w}(+\infty) = \beta, \hat{w}(-\infty) = 0$, $0 \leq \hat{w} \leq \beta$ and $w(\xi) = \hat{w}(-\xi)$, so that $w(x-ct) = \hat{w}(x-(-c)t)$. On the other hand, it is clear that the presence of the term $h'(u)u_x$ will break this symmetry between non-increasing and non-decreasing waves, in the sense that $\hat{u}(x,t)$ will be the solution of a different problem, in which $h'(u)u_x$ is replaced in (1.1) by $-h'(u)u_x$. Convection terms will clearly affect the values of propagation speeds in comparison in the case when $h'(u) \equiv 0$.

When $h'(u) \equiv 0$, Hadeler and Rothe [20] showed that there exist non-increasing travelling waves $u(x,t) = w(x-ct)$ of (1.1) with $w(-\infty) = \beta, w(+\infty) = 0$, $0 \leq w \leq \beta$ for (1.1) of all speeds $c \geq c_0$, and gave a formula for the minimal travelling wave speed c_0. In this chapter, we begin by presenting a generalization of this formula to the reaction-diffusion-convection equation (1.1). This minimal speed c_0 is bounded below by a critical parameter $\bar{c} \in \mathbb{R}$ determined by the linearization of the travelling wave equation for (1.1). This is because [38, Lemma 2.4, p.136] (see also Theorem 3.7 in [16]) implies that the existence of a real negative eigenvalue for the linearization of the travelling wave equation for (1.1) of speed c is a necessary condition for the existence of the travelling wave with the same speed c, which then implies that $c_0 \geq \bar{c}$. We refer to such a critical value as the *linear value*, that is obtained from the linearization of the travelling-wave equation for (1.1) about the unstable equilibrium 0 when we have a non-increasing travelling-wave solution which converges to 0 at $+\infty$. We also present a sufficient condition to guarantee that the minimal wave speed c_0 equals the linear value $\bar{c} = h'(0) + 2\sqrt{f'(0)}$, extending [20, Corollary 9] to now involve both the functions f and h. This condition is

$$h'(u) + \frac{f(u)}{u\sqrt{f'(0)}} \leq h'(0) + \sqrt{f'(0)} \qquad \text{for all } u \in (0,1), \tag{1.2}$$

which generalises the classical Hadeler-Rothe condition,

$$f(u) \leq uf^{'}(0) \qquad \text{for all } \; u \in (0,1), \tag{1.3}$$

that applies when $h^{'}(u) \equiv 0$. The fact that condition (1.2) is not also a necessary condition for $c_0 = \bar{c}$ is illustrated in Example 2.3. Note that Benguria, Depassier and Mendez [7] give an alternative sufficient condition to ensure that $c_0 = \bar{c}$ which again involves both functions f and h and is based on an alternative variational expression from which the minimal travelling wave speed can be estimated. For more references on work in the same direction, see [19], [31] and [28]. We mention also that Weinberger [41] recently followed and extended the approach of Hadeler and Rothe [20] to introduce a new condition in the case $h^{'}(u) \equiv 0$ that involves replacing u in the right hand side of (1.3) by a suitable choice of function $K(u)$, and briefly discussed such generalised conditions in the presence of $h^{'}(u)$, but we do not pursue this approach further here.

For (1.1), if we have $c_0 = \bar{c}$, then we say that the problem is *right linearly determinate.* Correspondingly, we say that (1.1) is *left linearly determinate* when the maximal travelling wave speed for non-decreasing travelling waves $u(x,t) = w(x - ct)$ equals the speed obtained from the linearization of the travelling wave equation (1.1) about the unstable state, this time with the leading edge tending to the equilibrium 0 at $-\infty$ instead of $+\infty$, in which case $\hat{\bar{c}} = h^{'}(0) - 2\sqrt{f^{'}(0)}$. Linear determinacy for propagation into an unstable state means that the spread rate in the fully nonlinear model equals the spread rate in the corresponding travelling-wave problem linearized about the unstable state, which is the speed associated with the leading edge of the wave. Note that in the absence of a convection term, the right linear value is the negative left linear value. It is useful to determine conditions that ensure (right and/or left) linear determinacy both because it is easier to calculate a minimal wave speed if it equals the corresponding linear value which is determined by an algebraic problem, and because the minimal wave speed, being equal to a spreading speed, is important for applications to, for example, predicting the speed of spread of biological invasions. For further background and results on linear determinacy, that focusses mainly on problems without convection, see, for instance, [20], [42], and also [9], [27], [28].

On the other hand, we also present Proposition 2.3 that gives a sufficient condition for (1.1) that ensures that there is a strict inequality between the right minimal travelling speed c_0 and the right linear value $\bar{c}$. This condition is

$$-h(0) + \sqrt{(h(\beta))^2 + 2\int_0^\beta f(u)du} > 2\sqrt{f'(0)} + h'(0), \tag{1.4}$$

which reduces to the condition of Berestycki and Nirenberg in [8, Remark 10.2] when $h \equiv 0$,

$$\sqrt{2\int_0^\beta f(u)du} > 2\sqrt{f'(0)}. \tag{1.5}$$

We give an example to illustrate that it is possible for a given function f to satisfy the sufficient condition (1.3) to have $c_0 = \bar{c}$, but with the addition of a function h, condition (1.4) is sufficient to guarantee that $c_0 > \bar{c}$. Moreover an example can be constricted an equation which is right but not left linearly determinate is presented in [1, Example 2.7].

Weinberger-Lewis and Li in [42] (see also [29], [39], [26], [43], and [40]), showed that the theory of spreading speeds and monostable travelling waves could be established for discrete-time recursions of the form

$$u_{n+1} = Q[u_n] \qquad n \in \mathbb{N}, \tag{1.6}$$

This operator is order-preserving, which means that if we have $u \leq v$ (in the sense of Table 1), then $Q[u] \leq Q[v]$, $Q[0] = 0$, and $Q[\beta] = \beta$, which says that $0, \beta$ are equilibria. Moreover, translation and reflection invariance properties are also satisfied for this operator,

$$Q[T_y[v]] = T_y[Q[v]] \quad \text{where} \quad T_y[v](x) := v(x-y) \qquad \text{for all } x, y \in \mathbb{R}. \tag{1.7}$$

$$Q[R[v]] = R[Q[v]] \quad \text{where} \quad R[v](x) := v(-x) \qquad \text{for all } x \in \mathbb{R}. \tag{1.8}$$

We will present this background theory in Chapter 3, extending the previous theory for operator Q in (1.6) to remove the reflection invariance property (1.8), and consider left and right spreading speeds for the recursion. Later in the thesis, in Chapter 5, we use this operator Q as a tool, taking Q to be the time-t map and apply it to reaction-diffusion-convection systems. Note that we remove the reflection-invariance assumption (1.8) for Q since we want to consider applications to partial differential equation systems that have convection terms, and if Q comes from a PDE system, then the system can only contain derivatives of even order if Q satisfies (1.8). The presence of convection terms with one derivative u_x breaks the symmetry between x and $-x$, because for $\hat{u}(x,t) = u(-x,t)$, we have $\frac{\partial}{\partial x^p}\hat{u}(x,t) = (-1)^p \frac{\partial}{\partial(-x)^p} u(-x,t)$, and when Q is the time t map of recursion $u_{n+1} = Q[u_n]$ in (1.6), (1.8) does not hold in the presence of convection term $h'(u)u_x$.

For the operator Q, we generalize the definition of slowest spreading speed corresponding to non-increasing initial data that is presented in [29], [42], and [26] to one that treats a recursion that can later be applied to a PDE system with convection term (1.6), and we refer to this slowest spreading speed as *the right slowest spreading speed*, $\mathring{c}$. Corresponding to this right slowest spreading speed, there is *the left slowest spreading speed*, $\hat{c}$ for (1.6) with non-decreasing initial data u_0. In [26, Theorem 2.1], the right slowest spreading speed $\mathring{c}$ is characterized by

$$\lim_{n\to\infty}\left[\sup_{x\geq n(\mathring{c}+\epsilon)} \{u_n\}_i(x)\right] = 0, \ \lim_{n\to\infty}\left[\sup_{x\leq n(\mathring{c}-\epsilon)} \{\beta - u_n(x)\}\right] = 0, \tag{1.9}$$

which says that there exists an index i such that the ith component spreads at a speed no higher than $\mathring{c}$, and no component spreads at a lower speed. We modify [26, Theorem 2.1] to show that the left slowest spreading speed $\hat{c}$ can be characterized by

$$\lim_{n\to\infty}\left[\sup_{x\leq n(\hat{c}-\epsilon)} \{u_n\}_j(x)\right] = 0, \ \lim_{n\to\infty}\left[\sup_{x\geq n(\hat{c}+\epsilon)} \{\beta - u_n(x)\}\right] = 0, \tag{1.10}$$

which says that there exists an index j such that the jth component spreads at a speed no less than $\hat{c}$, and no component spreads at a higher speed. Note that for this modification we do not need that the operator Q satisfies the reflection property (1.8).

We remark that an alternative way to obtain a left slowest spreading speed for the operator Q with non-decreasing initial data from the result that we have for non-increasing initial data, is by defining a new operator $\tilde{Q}$ by

$$\tilde{Q}[v](x) := Q[R[v]](-x) \qquad \text{for all } x, \tag{1.11}$$

where R defined in (1.8) and we denote the right slowest spreading speed of $\tilde{Q}$ by $\tilde{c}$. It is clear that, in general, $\tilde{c} \neq \mathring{c}$ but if the reflection invariance (1.8) holds for the operator Q in (1.6), then $Q = \tilde{Q}$ and hence $\tilde{c} = \mathring{c}$. On the other hand, we present a lemma to show that the left slowest spreading speed $\hat{c}$ for operator Q in (1.6) equals the value $-\tilde{c}$ obtained from $\tilde{Q}$.

It is proved in [29, Theorem 3.1, 3.2] that when there are no extra equilibria other than 0 and β in ψ_β, where $\psi_\beta = \left\{u \in BUC(\mathbb{R}, \mathbb{R}^k) : \ 0 \leq u(x) \leq \beta \ \text{ for all } x \in \mathbb{R}\right\}$, there exists a single spreading speed which means that all components spread at the same speed and such a property (1.9) ((1.10)) that was hold for the ith component, it will hold for all the components. This single speed in [29] is noted by c^*. On the other hand, Weinberger, Lewis and Li [42], discussed the case when there are extra equilibria in ψ_β other than 0 and β, motivated by the fact that models of multiple species interaction, such as in population genetics and in population ecology [40], often have such extra equilibria. Our Hypotheses 3.1 allows there to be more than just the equilibria 0 and β in ψ_β. Under these conditions, as noted in [42], not all components of u_n necessarily spread at the same speed, and as we mentioned before, it is natural to introduce a second speed, called the *right (left) fastest spreading speed*, $\mathring{c}_f(\hat{c}_f)$, in addition to the right (left) slowest spreading speed $\mathring{c}(\hat{c})$. In biological terms, it clearly sometimes happens that different species spread at different rates, which means that in general, there should be a right (left) slowest spreading speed $\mathring{c}(\hat{c})$ and a right (left) fastest spreading speed $\mathring{c}_f(\hat{c}_f)$. A characterization theorem for the left fastest spreading speed $\hat{c}_f$ can be shown via a modification of [26, Theorem 2.2] for non-decreasing initial data for the operator Q in (1.6), that is

$$\limsup_{n\to\infty} \left[\inf_{x \geq n(\hat{c}_f + \epsilon)} \{u_n\}_j (x)\right] > 0, \ \lim_{n\to\infty} \left[\sup_{x \leq n(\hat{c}_f - \epsilon)} u_n(x)\right] = 0, \tag{1.12}$$

which says that there exists an index j such that the jth component spreads at a speed no higher than $\hat{c}_f$, and no component spreads at a lower speed.

Note that it might be more correct to use the ' velocity' rather than 'speed', but that it is common to use the word 'speed' for travelling waves when c is both positive and negative, and we keep to this convention here. In [42], it is shown that the right fastest spreading speed is larger than or equal the right slowest spreading speed, in our notation, $\mathring{c}_f \geq \mathring{c}$, whereas when we have non-decreasing initial data, we show that $\hat{c}_f \leq \hat{c}$. The fact that we use the word 'fastest' despite having $\hat{c}_f \leq \hat{c}$ is because for both kinds of initial data, non-increasing and non-decreasing, the characterization properties of spreading speed, such as (1.9) and (1.10) for the slowest spreading speed and the similar two properties for the fastest spreading speed, always involve the quantity $x - nc$, and so they give speeds 'to the right' if c is positive and 'to the left' if c is negative. It is thus natural to have $\hat{c}_f \leq \hat{c}$, because for instance, in the case when the reflection invariance property (1.8) is satisfied, for non-decreasing initial data, both spreading speeds $\hat{c}_f$ and $\hat{c}$ will be negative, so we have $|\hat{c}_f| \geq |\hat{c}|$. The fact that $\hat{c}_f \leq \hat{c}$ means that the solution is going faster to the left.

If the right fastest spreading speed equals the right slowest spreading speed, we say that the recursion (1.6) has a *single right spreading speed.* Corresponding to this single right speed, we present a result that gives a sufficient condition to guarantee that the recursion (1.6) has *single left spreading speed.* Note that in the case when the recursion (1.6) has right (left) single spreading speed, then this means that all components of u_n spread at the same speed. Hence the characterization properties of, for instance, (1.10), (1.12) whenever there is a single spreading speed, the limits that in general only hold for component i must in fact hold for all components.

A linear operator M is the linearization of Q at 0 if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $||u||_\infty \leq \delta$ implies that $||Q[u] - M[u]||_\infty \leq \epsilon ||u||_\infty$, and such as noted in [42] that

$\lim_{\rho\to 0}\|(1/\rho)Q[\rho u] - M[u]\|_\infty = 0$. This linear operator has the representation,

$$(M[u](x))_i = \sum_{j=1}^{k}\int_{-\infty}^{\infty} u_j(x-y)m_{ij}(y,dy), \tag{1.13}$$

where m_{ij} is a bounded non-negative measure. We introduce the matrix B_μ that is defined in [42] by

$$B_\mu = \left(\int_{-\infty}^{\infty} e^{\mu y} m_{ij}(y,dy)\right), \tag{1.14}$$

and it can be characterized using M as follows: for every positive μ,

$$B_\mu \alpha = M[\alpha e^{-\mu x}]|_{x=0}.$$

We assume that the entries of B_μ are finite for all μ. Lui assumed that there is only one block in the matrix B_μ and so it is an irreducible matrix, whereas Weinberger, Lewis and Li in [42] reordered the coordinates of B_μ if necessary to put it into a block lower triangular form, so it is in Frobenius form, in which all the diagonal blocks are irreducible. See Theorem 3.1; we also assume that B_μ is in Frobenius form.

Similarly to the scalar case, in Chapter 4, we present sufficient conditions for right (left) linear determinacy for the recursion $u_{n+1} = Q[u_n]$ in (1.6). A recursion (1.6) is said to be *left (right) linearly determinate* if the left (right) single speed equals a speed that is obtained from the recursion (1.6) when the operator Q is replaced by its linearization M at the unstable equilibrium 0, which we then call the *left (right) linear value.*

As an application of this operator Q, in Chapter 5, we consider a co-operative system of partial differential equations of the form

$$u_{i,t} + h_i'(u_i)u_{i,x} = d_i u_{i,xx} + f_i(u), \qquad i = 1,2,...,k, \tag{1.15}$$

$$u(0,x) = u_0(x) \qquad \text{for all } x \in \mathbb{R},$$

where $d_i > 0$, the reaction terms $f_1, f_2, ..., f_k$ are independent of x and t and satisfy the co-operative assumption $\frac{\partial f_i}{\partial u_j}(u) \geq 0$, $i \neq j$, the convection functions $h_i^{'}(u_i)$ have the "diagonal" form of convection terms as diag $(h_1^{'}(u_1), h_2^{'}(u_2), .., h_k^{'}(u_k))$, $u = (u_1, u_2, ..., u_k) \in \mathbb{R}^k$, and the initial condition $u_0 \in BUC^1(\mathbb{R}, \mathbb{R}^k)$. Note that the system (1.15) is order-preserving by the Comparison Theorem 5.1, and Example 5.1 illustrates that if the co-operative assumption on f is not satisfied, then such a comparison result could fail. Note that in Chapter 3, we consider the recursion $u_{n+1} = Q[u_n]$ defined in (1.6) with initial condition $u_0 \in BUC(\mathbb{R}, \mathbb{R}^k)$, but in Chapter 5, which deals with the application to the PDE systems, we will use this recursion as an abstract tool for the spreading speed of a reaction-diffusion system with convection terms and we hence restrict the initial condition to be $u_0 \in BUC^1(\mathbb{R}, \mathbb{R}^k)$ (the space of functions $p : \mathbb{R} \to \mathbb{R}^k$ such that p and $p^{'}$ are bounded and uniformly continuous on $\mathbb{R}$) because of the presence of the convection terms.

In order to apply the spreading speed and travelling wave theory based on the recursion $u_{n+1} = Q[u_n]$ in (1.6), to the PDE system defined in (1.15) under conditions that ensure (1.15) satisfies the Comparison Theorem 5.1, we define an operator Q_t. If $u(x,t)$ is a solution of (1.15) and t is any positive number, then the sequence of functions $u_n(x) := u(x, nt)$ is shown to satisfy the recursion (1.6) with an operator $Q_t[u_0]$ that is defined by

$$Q_t[u_0](x) := u(x,t), \tag{1.16}$$

This Q_t is called the *time t map* of (1.15), where u_0 is the initial data of the partial differential equation system in (1.15) at time $t > 0$. As noted in [26], Q_t satisfies the semigroup properties

(1) $Q_{t_1}[Q_{t_2}[v]] = Q_{t_1+t_2}[v]$, for all positive t_1 and t_2,

(2) $\lim_{t\to 0} Q_t[v] = v$,

where (2) is satisfied in the sense that $||Q_t[v] - v||_\infty \to 0$ as $t \to 0$. We prove that $Q_t[v]$ defined in (1.16) satisfies the hypotheses required for the operator Q in (1.6) with slightly modified versions of Hypotheses 3.1 q_4, q_5, q_7, which we call $q_4^{'}, q_5^{'}$, and $q_7^{'}$. The modifications involve requiring that the initial condition u_0 belongs to the set $\psi_\beta \cap B_{BUC^1}(0, R)$ for some fixed $R > 0$, which we impose because of the presence of the convection terms.

We want to be sure that the derivatives $u_{i,x}$ are uniformly bounded all the way down to $t = 0$, and hence choose an initial condition that not only lies between 0 and β, but also lies in a fixed bounded set in BUC^1. The operator Q_t in (1.16) links the discrete-time system (1.6) and continuous-time system (1.15). Moreover, we show that the spreading speed of the time 1 map of the PDE system (1.15) gives a spreading speed for solutions of the system (1.15) itself where the initial condition is non-decreasing, in the sense that the characterization properties in Theorem 5.5 are satisfied.

Lui [29] gave sufficient conditions for spreading speeds to equal linear values that are obtained from the recursion $u_{n+1} = Q[u_n]$ when the operator Q defined in (1.6) is replaced by its linearization M at the unstable equilibrium 0 in the special case of a system with only two equilibria 0, $\beta, \beta > 0$ and $f'(0)$ an irreducible matrix. These results were generalized by Weinberger, Lewis and Li [42] to systems where the Frobenius form of the matrix B_μ may have multiple diagonal blocks and there may be more equilibria other than 0 and β in $[0, \beta]$ provided any additional equilibrium ν has $\nu_i = 0$ for at least one $i \in 1, 2, ..., k$, and they gave a sufficient condition for linear determinacy for the reaction-diffusion systems. This condition is

$$f_i(\rho\zeta(\bar{\mu})) \leq \rho(f'(0)\zeta(\bar{\mu}))_i \qquad \text{for all } \rho > 0, \tag{1.17}$$

where $\zeta(\bar{\mu})$ is a strictly positive eigenvector of the coefficient matrix C_μ, defined in (5.33) below, that is obtained from the linearization of the travelling-wave problem for the system (1.15) about the unstable equilibrium 0 at $+\infty$, and $\bar{\mu}$ is the value of $\mu > 0$ at which the infimum in definition (5.37) is attained. Note that in the scalar case, (1.17) reduces to the well-known Hadeler-Rothe condition (1.3). On the other hand, our sufficient condition for right linear determinacy for the reaction-diffusion-convection systems (1.15), which generalises (1.2) in the scalar case, is that for all positive $\rho \in \mathbb{R}$,

$$f_i(\rho\zeta(\bar{\mu})) \leq \rho\bar{\mu}\left[h_i'(0) - h_i'(\rho\zeta_i(\bar{\mu}))\right]\zeta_i(\bar{\mu}) + \rho(f'(0)\zeta(\bar{\mu}))_i \qquad 1 \leq i \leq k, \tag{1.18}$$

we refer to this condition as the *right combined condition* since it involves a combination of the functions f and h. Clearly (1.18) extends [42, (4.9)] and reduces to it when $h'(u) \equiv 0$. There is a corresponding condition for non-decreasing travelling-front solutions of (1.15), called the *left combined condition*, that ensures the system (1.15) is left linearly determinate, namely that for all positive $\rho \in \mathbb{R}$,

$$f_i(\rho\hat{\zeta}(\hat{\bar{\mu}})) \leq \rho\hat{\bar{\mu}}\left[h_i'(\rho\zeta_i(\hat{\bar{\mu}})) - h_i'(0)\right]\hat{\zeta}_i(\hat{\bar{\mu}}) + \rho(f'(0)\hat{\zeta}(\hat{\bar{\mu}}))_i \qquad 1 \leq i \leq k, \tag{1.19}$$

where $\hat{\zeta}(\bar{\mu})$ is a strictly positive eigenvector of the coefficient matrix $\hat{C}_\mu$, defined in (5.39) below, which is obtained from the linearization of the travelling wave problem for (1.15) about 0 at $-\infty$, and $\hat{\bar{\mu}}$ is the value of $\hat{\mu} > 0$ at which the infimum in definition (5.40) is attained; see Theorems 5.7, 5.10. Note that, in the absence of the function $h'(u)$, the eigenvectors $\zeta(\bar{\mu})$ and $\hat{\zeta}(\bar{\mu})$ are clearly equal. Moreover, in the scalar case, when the eigenvectors both just equal one, we still have two conditions because the function $h'(u)$ is still present and there is asymmetry between non-increasing and non-decreasing initial data.

The rest of our work is organized as follows. In Chapter 6, we compare between two different concepts of linear value. The linear value $\mathring{c}_{lin}$ ($\hat{c}_{lin}$) is defined as the minimum (maximum) of the values of c for which there exists a stable (unstable) monotone eigenvalue λ for the matrix $M(\lambda, c)$ which is presented in [38] and [12], and is defined by

$$M(\lambda, c) := \lambda^2 A + \lambda(cI - D) + B, \tag{1.20}$$

where A is the positive-diagonal matrix of diffusion coefficients $diag\ (d_1, .., d_k)$, D is the diagonal matrix of convection terms $diag\ \left(h_1'(0), ..., h_k'(0)\right)$, and $B = f'(0)$. On the other hand, the linear value $\mathring{\bar{c}}$ ($\hat{\bar{c}}$) is obtained from the linearization operator M at 0 of the time one map Q, which is defined in (5.37), ((5.40)) respectively by

$$\mathring{\bar{c}} := \inf_{\mu>0}\left\{\frac{\gamma_1(\mu)}{\mu}\right\}, \quad \hat{\bar{c}} := \inf_{\hat{\mu}>0}\left\{\frac{\hat{\gamma}_1(\hat{\mu})}{\hat{\mu}}\right\},$$

where $\gamma_1(\mu)$ ($\hat{\gamma}_1(\hat{\mu})$) is the principal eigenvalue of the matrix C_μ $\left(\hat{C}_{\hat{\mu}}\right)$ defined in (5.33) ((5.39)). Note that the right linear value $\mathring{\bar{c}}$ is the same as $\bar{c}$ in [42]. We focus in Chapter 6 mainly on the case when the Frobenius form of $f'(0)$ contains only one block and for simplicity, we suppose that $f'(0) \in P^{k\times k}$, the set of matrices with strictly positive off-diagonal entries, and in Lemma 6.3, we prove that $\mathring{c}_{lin}$ ($\hat{c}_{lin}$) equals $\mathring{c}$ ($\hat{c}$) respectively when f has this form. Note that in the absence of a convection term, the reason that $\mathring{c} = -\hat{c}$ is that $(\lambda^2 A + \lambda c I + B)\, q = 0$ if and only if $((-\lambda)^2 A + (-\lambda)(-c) I + B)\, q = 0$, where q is the eigenvector for the matrix $M(\lambda, c)$ without the convection term D.

In addition, in Section 6.2, we discuss the case when $f'(0)$ contains more than one irreducible block. The proof of [38, Lemma 2.4, p.136] shows that for such f, a necessary condition for the existence of a travelling wave converging to 0 at ∞ ($-\infty$) is the existence of a stable (unstable) monotone eigenvalue corresponding to a non-negative eigenvector X, but not necessarily a strictly positive eigenvector X. Hence when there is more than one block in $f'(0)$, we discuss two possible cases for the eigenvector, a non-negative eigenvector or a strictly positive eigenvector. In Lemma 6.7 we generalize parts (1), (2), (4) and (5) of Lemma 6.2, only in the case when we keep the requirement that $X > 0$, whereas when $X \geq 0, X \neq 0$, under certain additional conditions, we generalize the first part of Lemma 6.2 only. Moreover, in Proposition 6.1, we have a partial generalization of part (3) in Lemma 6.2 and we present an example, Example 6.1, that fully analyzes eigenvalues and eigenvectors in the case when $M(\lambda, c) \in \mathbb{R}^{2\times 2}$ in order to show the reason for the partial generalization of part (3) of Lemma 6.2. This example illustrates that for sufficiently large c, we do indeed have at least one stable monotone eigenvalue with a strictly positive eigenvector (part (2) in Lemma 6.7), but also shows that if we have a stable monotone eigenvalue λ with a strictly positive eigenvector for some value of c, then as c increases, this particular stable monotone eigenvalue λ will necessarily persist under small perturbation. Further, we give an example in the case when $M(\lambda, c) \in \mathbb{R}^{3\times 3}$, Example 6.2, to show that it is possible that for some values of c, there exists a stable monotone eigenvalue with a strictly positive eigenvector, but for larger values of c, there does not exist such a stable monotone eigenvalue with a strictly positive eigenvector. This example illustrates that the generalization of part (3) in Lemma 6.2 must be partial, and thus we can not fully generalize Lemma 6.2 in the case of $M(\lambda, c)$ has multiple blocks.

Finally, Chapter 7 deals with examples for the PDE systems. Firstly we begin with a PDE equation where $f : \mathbb{R} \to \mathbb{R}$, or a PDE system where $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $f'(0) \in \mathbb{R}^{2\times 2}$ has two irreducible blocks. For certain functions f, examples illustrate that we can find functions h so that both the right and left combined conditions (1.18) and (1.19) are satisfied, which implies that a single equation and a system of two equations are each both right and left linearly determinate. We also present examples of a system of two equations under some conditions on the parameters and convection terms that guarantee the system has a right (left) single speed, meaning that the slowest spreading speed equals the fastest spreading speed. Secondly, we present examples of a system of two equations in the case when $f'(0)$ is one irreducible block. We give an example of a system that is left linearly determinate both in the presence and the absence of convection terms. On the other hand, we give an example that shows that under a different condition on the convection term, the system will not be left linearly determinate. Note that when there is one irreducible block in $f'(0)$, it is not easy to calculate explicitly the linear value in spite of the fact it comes from the algebraic problem, but we succeed to estimate it. One of the tools used in deriving our examples is the comparison between one of the equations of the original system and a Fisher-type equation with convection term, of the form

$$u_t = du_{xx} - h'(u)u_x + u(\omega - u), \tag{1.21}$$

where $\omega > 0$. Such Fisher type equations are obtained either from the first or/and the second equation of the original system.

In the Appendix, we prove the Continuous Dependence Theorem 5.4. This result is needed to allow us to show that a variation of Hypothesis q_5 in Hypotheses 3.1 is satisfied by the operator $Q_t[v]$ that is defined in (1.16) in the case when we restrict the initial condition u_0 of the recursion $u_{n+1} = Q[u_n]$ to belong to $BUC^1(\mathbb{R}, \mathbb{R}^k)$. This modified hypothesis says that for a given sequence $\{v_n\}_{n\in\mathbb{N}} \subset \psi_\beta \cap B_{BUC^1}(0, R)$ and $v \in \psi_\beta \cap B_{BUC^1}(0, R)$ such that $\{v_n\}$ converges to v uniformly on every bounded set, $Q_t[v_n]$ converges to $Q_t[v]$ uniformly on every bounded set.

Chapter 2

The minimal travelling wave speed and linear determinacy for a reaction-diffusion-convection equation

This chapter deals with a reaction-diffusion-convection equation (2.2) with a monostable reaction term $f(u)$ in which 0 is an unstable equilibrium, $\beta > 0$ is a stable equilibrium, and there are no equilibria of f between 0 and β. Hadeler and Rothe [20] gave a formula for the minimal travelling wave speed c_0 of the reaction-diffusion-convection equation (2.2) when $h^{'}(u) \equiv 0$, and we present a generalization of this formula c_0 to the reaction-diffusion-convection equation (2.2). This minimal speed c_0 is bounded below by a critical parameter $\bar{c} \in \mathbb{R}$, which we refer to as the *linear value* that is obtained from the linearization of the travelling-wave equation for (2.2) about the unstable equilibria 0 when we have a non-increasing travelling-wave solution which converges to 0 at $+\infty$. Further, a sufficient condition is presented to guarantee that the minimal wave speed c_0 equals the linear value $\bar{c}$ which is also a generalization of the classical Hadeler-Rothe condition [20, Corollary 9]

$$f(u) \leq uf^{'}(0) \qquad \text{for all } \ u \in (0,1), \tag{2.1}$$

that applies when $h'(u) \equiv 0$, to now involve both the functions f and h. On the other hand, a sufficient condition for a strict inequality between c_0 and $\bar{c}$ for the equation (2.2) is given in Proposition 2.3 below. In this chapter, we restrict our attention to non-increasing travelling waves; analogues results hold for non-decreasing-travelling waves. Note that it will be shown later, in subsection 5.3.2, that the minimal speed of such non-increasing waves corresponds to the right slowest spreading speed of the reaction-diffusion-convection equation (system).

2.1 A formula of the minimal travelling wave speed

Consider a reaction-diffusion-convection equation

$$u_t + h'(u)u_x = u_{xx} + f(u) \qquad x \in \mathbb{R},\ t \in (0,\infty), \tag{2.2}$$

where $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$. The functions h and f satisfy the following hypotheses:
E_1: $f \in C^1[0,1]$ and $h \in C^2[0,1]$.
E_2: $f(0) = f(1) = 0, \quad f(u) > 0$ for $u \in (0,1)$.
E_3: $f'(0) > 0, \quad f'(1) < 0$.
A travelling wave is a solution u of (2.2) such that $u(x,t) = w(x - ct)$, where w is here taken to be a non-increasing function such that

$$w(-\infty) = 1, \quad w(\infty) = 0 \qquad\qquad 0 \leq w \leq 1, \tag{2.3}$$

and the speed $c \in \mathbb{R}$ is a constant. Clearly w and c satisfy the ordinary differential equation

$$-w'' = cw' - h'(w)w' + f(w). \tag{2.4}$$

The following preliminary lemma shows that $w : \mathbb{R} \to \mathbb{R}$ satisfying (2.3) and (2.4) must have $w'(+\infty) = w'(-\infty) = 0$. To prove this, we use Landau's inequality [22, Theorem 5.3.1.] which states that if w, w' and w'' are uniformly bounded on an unbounded interval $I \subset \mathbb{R}$, then

$$\|w'\|_\infty^2 \leq 4\|w\|_\infty \|w''\|_\infty,$$

where $\|y\|_\infty = \|y\|_{L^\infty(I,\mathbb{R}^k)} = \sup\{|y(x)| : x \in I\}$ when either $I = \mathbb{R}$ or $I = [\alpha, \infty), (-\infty, \alpha]$ for some α. Note that this inequality holds only when I is the whole line or a half line but not when I is a bounded interval. A counter example for this is that, for $I = [a, b]$ such that $a < b$ and $a, b \in \mathbb{R}$, if we have $u(x) = x$ then $u^{'}(x) = 1$, whereas $u^{''}(x) = 0$. On the whole line $\mathbb{R}$, however there is no $M > 0$ such that $|x| \leq M$ for all $x \in \mathbb{R}$, so Landau's inequality does not apply for this choice of u.

Lemma 2.1. *If w satisfies the equations* (2.3) *and* (2.4), *then*

$$w^{'}(+\infty) = w^{'}(-\infty) = 0.$$

Proof. We first show that $|w^{'}(\xi)|$ is uniformly bounded on $\mathbb{R}$. If $\xi \leq 0$, then

$$w^{'}(0) - w^{'}(\xi) = \int_\xi^0 w^{''}(s)ds \leq -\int_\xi^0 \left[(c - h^{'}(w(s)))\right] w^{'}(s)ds,$$

since $f(w(s)) \geq 0$, so $\int_\xi^0 f(w(s)) \geq 0$, and hence

$$w^{'}(0) - w^{'}(\xi) \leq -h(w(\xi)) + cw(\xi) + h(w(0)) - cw(0). \tag{2.5}$$

Since the right-hand side of (2.5) is bounded independently of ξ and $w^{'}(\xi) \leq 0$, it follows that $|w^{'}(\xi)|$ is uniformly bounded on $(-\infty, 0)$.

Now suppose, for contradiction, that there is a sequence $\xi_n \to \infty$ with $|w^{'}(\xi_n)| \to \infty$ and $|w^{'}(\xi_n)| = \sup_{\xi \leq \xi_n} |w^{'}(\xi)|$. Define $y_n(\xi) = \dfrac{w_n(\xi + \xi_n)}{w_n^{'}(\xi_n)}$. Then y_n satisfies

$$y_n^{''} + \left(c - h^{'}(w_n(\xi + \xi_n))\right) y_n^{'} + \frac{f\left(w_n(\xi + \xi_n)\right)}{w_n^{'}(\xi_n)} = 0, \tag{2.6}$$

and $\sup_{\xi \leq 0} |y_n(\xi)| \to 0$ as $n \to \infty$, $1 = |y_n^{'}(0)| = \sup_{\xi \leq 0} |y_n^{'}(\xi)|$, so by (2.6) we get that there exists $C > 0$ such that

$$\sup_{\xi \leq 0} |y_n^{''}(\xi)| \leq C \qquad \text{for all} \quad n \in \mathbb{N}.$$

But Landau's inequality implies that

$$\left(\sup_{\xi\le 0}|y_n^{'}(\xi)|\right)^2 \le 4\left(\sup_{\xi\le 0}|y_n^{''}(\xi)|\right)\left(\sup_{\xi\le 0}|y_n(\xi)|\right),$$

and hence $1 \le 4C\left(\sup_{\xi\le 0}|y_n(\xi)|\right) \to 0$ as $n \to \infty$, which is a contradiction. So $|w^{'}(\xi)|$ is uniformly bounded on $\mathbb{R}$. It then follows from equation (2.4), that $|w^{''}(\xi)|$ is also uniformly bounded on $\mathbb{R}$.

Now we re-apply Landau's inequality in the case when $I = (-\infty, -n]$. Then

$$\|w^{'}\|^2_{L^\infty((-\infty,-n])} \le 4\|1-w\|_{L^\infty((-\infty,-n])}\|w^{''}\|_{L^\infty((-\infty,-n])},$$

which, since $\|w^{''}\|_{L^\infty((-\infty,-n])} \le M$ and $4\|1-w\|_{L^\infty((-\infty,-n])} \to 0$ as $n \to \infty$ yields that $\|w^{'}\|_{L^\infty((-\infty,-n])} \to 0$ as $n \to \infty$, and hence $w^{'}(-\infty) = 0$.

Moreover, in the case of $I = [n, \infty)$, yields that $\|w\| = \sup_{x\ge n}|w(x)| \to 0$ as $n \to \infty$ so we also get that $\|w^{'}\|^2_{L^\infty([n,\infty))} \to 0$ as $n \to \infty$ which means that $w^{'}(+\infty) = 0$. Thus the lemma is proved. $\square$

It is shown in Hadeler and Rothe [20] that when $h \equiv 0$, a travelling wave satisfying (2.3) exists for each $c \ge c_0$, with

$$c_0 = \inf_{\rho\in\Lambda}\sup_{0<w<1}\left\{\rho^{'}(w) + \frac{f(w)}{\rho(w)}\right\}$$

where the set of functions Λ is defined by

$$\Lambda := \Big\{\rho : [0,1] \to [0,\infty) : \rho \text{ is continuously differentiable, } \rho(0) = 0, \rho^{'}(0) > 0, \\ \text{and } \rho(w) > 0 \text{ for } w \in (0,1)\Big\}. \tag{2.7}$$

The following proposition characterizes the minimal speed c_0, and its proof is similar to that in [20, Theorem 8]. Gilding and Kersner [19, Theorem 8.2] also discuss this extension of [20] and it additionally follows from the special case of [13, Lemmas 2.1, 2.2] when there is only one equation using the fact that one can write $w^{'} = -\rho(w)$ for a function $\rho \in \Lambda$ whenever w is a solution of (2.2) and satisfies (2.3).

Proposition 2.1. *There exists a decreasing travelling-wave solution of* (2.2) *that satisfies properties* (2.3) *for all speeds* $c \geq c_0$, *where* c_0 *is characterized by*

$$c_0 = \inf_{\rho \in \Lambda} \sup_{0<w<1} \left\{ \rho^{'}(w) + h^{'}(w) + \frac{f(w)}{\rho(w)} \right\},$$

and Λ *is defined in* (2.7).

Proof. If such a wave solution $u(x,t) = w(x-ct)$ exists for the reaction-diffusion-convection equation (2.2) , then w solves the ordinary differential equation (2.2) with boundary conditions (2.3). Applying the substitution $w \mapsto 1-w$, we have

$$w^{'} = v =: M(w,v), \tag{2.8}$$

$$v^{'} = -cv + h^{'}(1-w)v + f(1-w) =: N(w,v), \tag{2.9}$$

where $M, N : \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable functions. Suppose a function $\theta = (\theta_1, \theta_2)$ is such that $\theta_1, \theta_2 : \mathbb{R} \to \mathbb{R}$ satisfy the following properties

1. $0 < \theta_1(t) < 1, \;\; 0 < \theta_2(t), \;\; \theta_1^{'} = \theta_2$
2. $\theta_1 \to 1, \theta_2 \to 0$ for $t \to +\infty, \quad \theta_1 \to 0, \theta_2 \to \overline{\theta} \geq 0$ for $t \to -\infty$
3. $|\theta_1^{'}| + |\theta_2^{'}| \neq 0, \; |\theta_1^{'}| + |\theta_2^{'}| \to 0$ for $|t| \to \infty, \quad -\infty < \lim_{t\to\infty} \dfrac{\theta_2^{'}}{\theta_1^{'}} < 0.$

and

$$M(\theta_1,\theta_2)\theta_2^{'} - N(\theta_1,\theta_2)\theta_1^{'} \geq 0 \tag{2.10}$$

$$M(0,v) \geq 0 \quad \text{for } 0 < v < \overline{\theta}, \;\; N(w,0) \geq 0 \quad \text{for } 0 < w < 1.$$

With M and N given by (2.8), (2.9), condition (2.10) becomes

$$\theta_2\theta_2^{'} - [-c\theta_2 + h^{'}(1-\theta_1)\theta_2 + f(1-\theta_1)]\theta_1^{'} \geq 0. \tag{2.11}$$

Since $\theta_1^{'} = \theta_2 > 0$ and θ_1 is a strictly increasing function, we can represent the function

$\theta = (\theta_1, \theta_2)$ as $\theta_2 =: \tilde{\rho}(\theta_1)$, then from Properties 2 and 3 respectively, $\tilde{\rho}$ satisfies

$$\tilde{\rho}(1) = 0, \ \tilde{\rho}^{'}(1) < 0. \tag{2.12}$$

It follows from the definition of $\tilde{\rho}$, that $\theta_2^{'} = \tilde{\rho}^{'}(\theta_1)\theta_1^{'} = \tilde{\rho}^{'}(\theta_1)\theta_2 = \tilde{\rho}^{'}(\theta_1)\tilde{\rho}(\theta_1)$. Then condition (2.11) becomes

$$\tilde{\rho}(\theta_1)\tilde{\rho}(\theta_1)\tilde{\rho}^{'}(\theta_1) + c\tilde{\rho}(\theta_1)\tilde{\rho}(\theta_1) - h^{'}(1-\theta_1)\tilde{\rho}(\theta_1)\tilde{\rho}(\theta_1) - f(1-\theta_1)\tilde{\rho}(\theta_1) \geq 0.$$

Since $\theta_2 = \tilde{\rho}(\theta_1) > 0$, this implies

$$c\tilde{\rho}(\theta_1) \geq -\tilde{\rho}(\theta_1)\tilde{\rho}^{'}(\theta_1) + h^{'}(1-\theta_1)\tilde{\rho}(\theta_1) + f(1-\theta_1) \qquad \text{for all } t \in \mathbb{R}$$

which is equivalent to,

$$\begin{aligned} c\tilde{\rho}(w) &\geq -\tilde{\rho}(w)\tilde{\rho}^{'}(w) + h^{'}(1-w)\tilde{\rho}(w) + f(1-w) \\ c &\geq -\tilde{\rho}^{'}(w) + h^{'}(1-w) + \frac{f(1-w)}{\tilde{\rho}(w)} \qquad w \in (0,1). \end{aligned}$$

Since $\tilde{\rho}$ satisfies (2.12) and $\tilde{\rho}(w) > 0$ for $w \in (0,1)$, then the function $\rho(w) := \tilde{\rho}(1-w)$ belongs to the set Λ defined in (2.7). Then (2.10) is satisfied if

$$c \geq \rho^{'}(w) + h^{'}(w) + \frac{f(w)}{\rho(w)} \qquad w \in (0,1),$$

and hence by [20, Corollary 6] we get

$$c_0 = \inf_{\rho\in\Lambda} \sup_{0<w<1} \left\{ \rho^{'}(w) + h^{'}(w) + \frac{f(w)}{\rho(w)} \right\}. \tag{2.13}$$

□

2.2 Sufficient conditions for linear determinacy

The variational formula (2.13) clearly yields upper bounds for c_0 using specific choices of test functions $\rho \in \Lambda$. In particular, if we define

$$L = \sup_{0<w<1} \left\{ \frac{f(w)}{w} \right\}, \quad J = \sup_{0<w<1} \left\{ h^{'}(w) \right\},$$

then as in the proof of [20, corollary 9], and noted also by Malaguti and Marcelli [31], we can obtain estimates for the numerical value of c_0 by taking a particular family of functions $\rho_k \in \Lambda$, where Λ is defined in (2.7) and ρ_k is defined by $\rho_k(w) := kw$, $k > 0$. The minimum of the expression

$$\sup_{0<w<1} \left\{ \rho_k^{'}(w) + h^{'}(w) + \frac{f(w)}{\rho_k(w)} \right\}, \tag{2.14}$$

with respect to k, yields that $1 - (\frac{1}{k^2})L = 0$ if and only if

$$\frac{k^2 - L}{k^2} = 0, \quad \text{which implies that} \quad k = \sqrt{L}. \tag{2.15}$$

Thus the minimum of (2.14) over $k > 0$ is attained at $k = \sqrt{L}$, and (2.14) is thus bounded above by $\sqrt{L} + \dfrac{L}{\sqrt{L}} + J = 2\sqrt{L} + J$. By Proposition 2.1 we thus obtain that

$$c_0 \leq 2\sqrt{L} + J. \tag{2.16}$$

Moreover, the value of the minimal speed c_0 is bounded below by a critical parameter $\bar{c} \in \mathbb{R}$ known as the *linear value*, see [16, Theorem 3.7][1]. This linear value can be defined by the property that the non-linear travelling-wave problem (2.4) can be written as a first-order system of 2 equations,

$$\begin{pmatrix} v^{'} \\ w^{'} \end{pmatrix} = \begin{pmatrix} -cv + h^{'}(w)v - f(w) \\ v \end{pmatrix} = \begin{pmatrix} N(v,w) \\ M(v,w) \end{pmatrix}$$

[1]Note that, in fact $\bar{c} = \mathring{\bar{c}}$, where $\mathring{\bar{c}}$ is the right linear value introduced in (4.10) in Chapter 4, which is a lower bound for the right slowest spreading speed $\mathring{c}$, but for clarity, we keep the simple notation $\bar{c}$ in Chapter 2.

whose linearization about the unstable equilibrium 0 can be written as

$$\begin{pmatrix} v' \\ w' \end{pmatrix} = \begin{pmatrix} -(c-h'(0)) & -f'(0) \\ I & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}, \tag{2.17}$$

and the definition of $\bar{c}$ is that the matrix $\begin{pmatrix} -(c-h'(0)) & -f'(0) \\ I & 0 \end{pmatrix}$ has a real negative eigenvalue λ if and only if $c \geq \bar{c}$. Such an eigenvalue λ satisfies the quadratic equation $\lambda^2 + \lambda(c - h'(0)) + f'(0) = 0$ which is obtained from

$$\begin{vmatrix} -c + h'(0) - \lambda & -f'(0) \\ 1 & -\lambda \end{vmatrix} = 0.$$

Now such λ exist if $c - h'(0) \geq 2\sqrt{f'(0)}$, and hence since $\bar{c}$ is the smallest speed for which such an eigenvalue exists, we have

$$\bar{c} = 2\sqrt{f'(0)} + h'(0) \leq c_0, \tag{2.18}$$

and thus

$$\bar{c} \leq c_0 \leq 2\sqrt{L} + J.$$

This estimate clearly yields a set of sufficient conditions, which generalize the sufficient condition (2.1) that is presented in [20, Corollary 9] to the case of (2.2) with $h \neq 0$, that guarantee that the linear value $\bar{c}$ equals the minimal wave speed c_0, namely if

$$2\sqrt{L} + J \leq 2\sqrt{f'(0)} + h'(0), \tag{2.19}$$

then $c_0 = \bar{c}$. In particular, $\bar{c} = c_0$ if

$$\sup_{0<w<1} \frac{f(w)}{w} = f'(0) \qquad \text{and} \qquad \sup_{0<w<1} h'(w) = h'(0). \tag{2.20}$$

The following proposition gives an alternative sufficient condition (2.21) that ensures $\bar{c} = c_0$. Note that if (2.20) holds then (2.21) is satisfied and we will show in Example 2.2 that (2.21) can hold even when (2.19) is violated.

Proposition 2.2. *A sufficient condition to guarantee that the linear value* $\bar{c} = 2\sqrt{f'(0)} + h'(0)$ *for problem* (2.2) *is equal to the minimal travelling wave speed* c_0 *is that*

$$h'(w) + \frac{f(w)}{\sqrt{f'(0)}w} \leq h'(0) + \sqrt{f'(0)}, \qquad \textit{for all}\, w \in (0,1). \tag{2.21}$$

Proof. Define a function $y : [0,1] \to \mathbb{R}$ by

$$y(w) = \begin{cases} \sqrt{f'(0)} + h'(w) + \dfrac{f(w)}{\sqrt{f'(0)}w} & \text{for all} \quad w \in (0,1] \\ 2\sqrt{f'(0)} + h'(0) & \text{if} \qquad w = 0, \end{cases}$$

and note that (2.21) implies that $y(w) \leq y(0)$ for all $w \in (0,1)$.

By Proposition 2.1, and $\rho \in \Lambda$ defined by $\rho(w) := \sqrt{f'(0)}w$ in (2.13), and (2.21) together imply that

$$c_0 \leq \sup_{0<w<1} \{y(w)\} \leq y(0), \tag{2.22}$$

and hence $c_0 \leq 2\sqrt{f'(0)} + h'(0) = \bar{c}$, so by (2.16), we get $c_0 = \bar{c}$. □

In particular, (2.21) holds if the condition (2.23) in the following lemma is satisfied, since condition (2.23) ensures that the function $u \longmapsto h'(w) + \dfrac{f(w)}{\sqrt{f'(0)}w}$ is non-increasing on $(0,1)$.

Lemma 2.2. *A sufficient condition to guarantee that the linear value* $\bar{c} = 2\sqrt{f'(0)} + h'(0)$ *equals the minimal speed* c_0 *is that*

$$h''(w) + \frac{1}{\sqrt{f'(0)}} \left\{ \frac{f'(w)}{w} - \frac{f(w)}{w^2} \right\} \leq 0 \qquad \textit{for all}\, w \in (0,1). \tag{2.23}$$

Proof. Define $y(w) := h'(w) + \dfrac{f(w)}{\sqrt{f'(0)}w}$ for $w \in (0,1)$, therefore

$$y'(w) = h''(w) + \left\{ \frac{f'(w)}{\sqrt{f'(0)}w} - \frac{f(w)}{\sqrt{f'(0)}w^2} \right\}.$$

Since (2.23) implies that $h''(w) + \left\{ \dfrac{f'(w)}{\sqrt{f'(0)}w} - \dfrac{f(w)}{\sqrt{f'(0)}w^2} \right\} \leq 0$, $y'(w) \leq 0$ for all $w \in$

$(0,1)$, and hence we obtain the condition (2.21), that is

$$h^{'}(w)+\frac{f(w)}{\sqrt{f^{'}(0)}w}\leq h^{'}(0)+\sqrt{f^{'}(0)}.$$

$\square$

An alternative condition that ensures $c_0=\bar{c}$ is given by Benguria, Depassier and Mendez in [7], namely

$$\frac{f^{''}(w)}{\sqrt{f^{'}(0)}}+h^{''}(w)>0 \qquad \text{for all } w\in(0,1). \tag{2.24}$$

This condition is derived using a different variational characterization of c_0 for their equation $w_t+\mu\phi(w)w_x=w_{xx}+f(w)$, $\mu>0$, where the reaction term f satisfies Hypotheses E_1-E_3 and with a non-increasing wave $w(x-ct)$ joining the stable equilibrium $w=1$ to the unstable equilibrium $w=0$. This characterization is that

$$c_0=\sup_{g\in S}\xi(g),$$

with

$$\xi(g)=\frac{\int_0^1\left\{2\sqrt{f(w)g(w)[-g^{'}(w)]}+\mu\phi(w)g(w)\right\}dw}{\int_0^1 g(w)dw},$$

where S is the set of all positive decreasing functions $g(w)$ for which this integral exists and $g(1)=0$. Note that our convection term $h'(w)$ is replaced in [7] by $\mu\phi(w)$, where ϕ is a C^1-function such that, for simplicity, it is assumed that $\phi(0)=0$, but this restriction on $\phi(0)$ clearly only affects the numerical value of $\bar{c}$, not the condition (2.24), and can be removed.

2.2.1 Illustrative examples

The following two examples compare our condition (2.21) with (2.24), (2.19) and (2.20), and in particular, illustrate that functions f and h can be found which satisfy (2.24) but not (2.21) and, vice versa, that there exist functions which satisfy (2.21) but not (2.24).

Example 2.1. Choose $f(w)=w(1-w)$ and $h(w)=(\frac{\delta}{2})w^2$, $\delta\in\mathbb{R}$. Then f satisfies

Hypotheses $E_1 - E_3$, and for this function f, condition (2.23) says that

$$\begin{aligned} h''(w) &\leq -\frac{1}{\sqrt{f'(0)}}\left\{\frac{f'(w)}{w} - \frac{f(w)}{w^2}\right\} \\ &= -1\left\{\frac{1-2w}{w} - \frac{1-w}{w}\right\} \\ &= 1 \qquad \text{for all } \ w \in (0,1), \end{aligned}$$

which holds if and only if $\delta \leq 1$. On the other hand, condition (2.24) requires that $h''(w) < -\frac{f''(w)}{\sqrt{f'(0)}}$, which is satisfied if $\delta = h''(u) < 2$ for all $w \in (0,1)$. Hence if $\delta \in (1,2)$, then (2.24) satisfied but (2.23) is not, and moreover, it is easy to check that our weaker condition (2.21) is also not satisfied for such δ. □

The next example shows that functions f and h can be found which satisfy the condition (2.21) but not (2.24).

Example 2.2. Choose $f(w) = w(1-w)(\epsilon+w)$, where $\epsilon > 0$. Then f satisfies Hypotheses $E_1 - E_3$, and for this function f, condition (2.23) says that

$$\begin{aligned} h''(w) \ &\leq -\frac{1}{\sqrt{\epsilon}}\left\{\frac{\epsilon + 2w - 2\epsilon w - 3w^2}{w} - \frac{w(1-w)(\epsilon+w)}{w^2}\right\} \\ &= -\frac{1}{\sqrt{\epsilon}}\left\{\frac{w - \epsilon w - 2w^2}{w}\right\} \\ &= \frac{1}{\sqrt{\epsilon}}\{\epsilon - 1 + 2w\} \quad \text{for all } w \in (0,1), \end{aligned}$$

whereas condition (2.24) is satisfied if

$$h''(w) < -\frac{f''(w)}{\sqrt{f'(0)}} = \frac{1}{\sqrt{\epsilon}}\{2\epsilon - 2 + 6w\} \quad \text{for all } w \in (0,1).$$

Thus to have that condition (2.23) is satisfied but (2.24) is not, we need that for some $w \in (0,1)$,

$$2\epsilon - 2 + 6w < \epsilon - 1 + 2w \quad \text{if and only if} \quad \frac{1-\epsilon}{4} > w. \tag{2.25}$$

So in particular, if we choose $\epsilon = \frac{1}{2}$, then (2.25) holds for $w \in (0, \frac{1}{8})$, and (2.23) is then satisfied provided

$$h''(w) \leq \frac{1}{\sqrt{\epsilon}}\{\epsilon - 1 + 2w\} = \sqrt{2}(2w - \frac{1}{2}) \qquad \text{for all } w \in (0,1), \tag{2.26}$$

and (2.24) is not satisfied provided

$$\sqrt{2}(6w-1) = \frac{1}{\sqrt{\epsilon}}\{2\epsilon - 2 + 6w\} < h''(w) \qquad \text{for some } w \in (0,1). \tag{2.27}$$

For instance, (2.26) and (2.27) are clearly both satisfied by taking $h''(w) = \sqrt{2}(2w - \frac{1}{2})$, and thus

$$h(w) = \frac{\sqrt{2}}{3}w^3 - \frac{1}{2\sqrt{2}}w^2 + Aw + B \qquad A, B \in \mathbb{R}.$$

We conclude that the chosen functions f and h satisfy condition (2.21) but not condition (2.24). Note that for this function $h(u)$, neither the Benguria-Depassier-Mendez nor the Malaguti-Marcelli alternative conditions (2.24) and (2.19) are satisfied. Moreover, if $\epsilon < 1$, the function f does not satisfy the classical Hadeler-Rothe condition (2.1), while condition (2.23) does hold for a pair of functions f and h if h satisfies (2.26). □

The next example illustrates that there exist functions f and h such that the sufficient condition for linearly determinate (2.21) is not satisfied, but the minimal travelling speed is linear determinacy.

Example 2.3. Consider the equation

$$u_t = u_{xx} - \beta u u_x + \gamma u(1-u)(1+2\gamma u), \tag{2.28}$$

where the functions f and h satisfy Hypotheses $E_1 - E_3$, $\beta = 2(\sqrt{\gamma} - \gamma)$, and $0 < \gamma < 1$. Since the linear value of (2.28) is $\bar{c} = 2\sqrt{\gamma}$, the left-hand inequality in (2.22) becomes

$$2\sqrt{\gamma} \leq c_0 \leq \sup_{0<u<1} y(u),$$

where

$$y(u) := \sqrt{f'(0)} + h'(u) + \frac{f(u)}{u\sqrt{f'(0)}} = 2\sqrt{\gamma} + u(\beta + 2\gamma\sqrt{\gamma} - \gamma) - 2\gamma u^2.$$

For a function $w_{explicit}(\xi) := \left(1 + \exp(\sqrt{\gamma}\xi)\right)^{-1}$, which is a solution of (2.28), the travelling wave speed is $c_{explicit} = 2\sqrt{\gamma}$. We observe that because $\gamma \in (0,1)$, $\sup\limits_{0<u<1} y(u) > 2\sqrt{\gamma}$, which implies that

$$\bar{c} = 2\sqrt{\gamma} \leq c_0 \leq \sup_{0<u<1} y(u) \quad \text{and} \quad \sup_{0<u<1} y(u) > 2\sqrt{\gamma}.$$

Thus the minimal travelling wave speed c_0 is linearly determinate. □

2.3 A sufficient condition for not linear determinacy

In this section we present a proposition that gives a sufficient condition for strict inequality between c_0 and $\bar{c}$. This result is a modification of a result of Berestycki and Nirenberg [8, Remark 10.2] to include the extra term $h^{'}(w)w^{'}$. It shows that when the functions f and h satisfy Hypotheses $E_1 - E_3$ and condition (2.21) is not satisfied, then the minimal speed and the linear value do not necessarily coincide.

Proposition 2.3. *Consider the equation*

$$w^{''} + cw^{'} - h^{'}(w)w^{'} + f(w) = 0 \tag{2.29}$$

$$w(-\infty) = 1, \; w(+\infty) = 0 \qquad \textit{for all } \; w \in (0,1)$$

where the functions f and h satisfy Hypotheses $E_1 - E_3$ and w is a non-increasing travelling wave solution. Suppose in addition that the function h satisfies

$$h(w) \geq 0 \quad \textit{for all } \; w \in (0,1) \quad \textit{and} \quad h(1) \geq h(0). \tag{2.30}$$

Then a sufficient condition to have $c_0 > \bar{c}$ is that

$$-h(0) + \sqrt{(h(1))^2 + 2\int_0^1 f(w)dw} \; > \; 2\sqrt{f'(0)} + h^{'}(0). \tag{2.31}$$

Proof. To show that $c > 0$ for functions f and h that satisfy $E_1 - E_3$ and (2.30), we integrate equation (2.29) to get

$$-c\int_{-\infty}^{+\infty} w^{'}d\xi = \int_{-\infty}^{+\infty} w^{''}d\xi - \int_{-\infty}^{+\infty} h^{'}(w)w^{'}d\xi + \int_{-\infty}^{+\infty} f(w)d\xi,$$

which implies that

$$-c[w(+\infty) - w(-\infty)] = w^{'}(+\infty) - w^{'}(-\infty) - \int_{w(-\infty)}^{w(+\infty)} h^{'}(w)dw + \int_{-\infty}^{+\infty} f(w)d\xi.$$

Then by Lemma 2.1, we have $w^{'}(+\infty) = w^{'}(-\infty) = 0$, and hence

$$\begin{aligned} c = -\int_{1}^{0} h^{'}(w)dw + \int_{-\infty}^{+\infty} f(w)d\xi &= \int_{0}^{1} h^{'}(w)dw + \int_{-\infty}^{+\infty} f(w)d\xi \\ &= h(1) - h(0) + \int_{-\infty}^{+\infty} f(w)d\xi. \end{aligned}$$

So by (2.30) and Hypothesis E_2, it follows that $c > 0$.

Now we want to obtain an estimate of the term $\int_{-\infty}^{+\infty} c(w^{'}(\xi))^2 d\xi$ by firstly, multiplying equation (2.29) by $w^{'}$ and integrating the obtained equation, and secondly, multiplying (2.29) by $1 - w$ and integrating the obtained equation. After that we will compare between the two estimates of $\int_{-\infty}^{+\infty} c(w^{'}(\xi))^2 d\xi$ that we have obtained.

Multiplying equation (2.29) by $w^{'}$ and integrating over $\mathbb{R}$ yields

$$\int_{-\infty}^{+\infty} w^{''}w^{'}d\xi + \int_{-\infty}^{+\infty} c(w^{'}(\xi))^2 d\xi - \int_{-\infty}^{+\infty} h^{'}(w)(w^{'})^2 d\xi + \int_{-\infty}^{+\infty} f(w)w^{'}d\xi = 0. \quad (2.32)$$

By evaluating some terms in equation (2.32) separately we have the following:

1. The first term in equation (2.32) with applying Lemma 2.1 yields

$$\int_{-\infty}^{+\infty} w^{''}w^{'}d\xi = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{d}{d\xi}(w^{'})^2 d\xi = \frac{1}{2}\left[(w^{'}(+\infty))^2 - (w^{'}(-\infty))^2\right] = 0.$$

2. The third term in (2.32) is

$$-\int_{-\infty}^{+\infty} h^{'}(w)(w^{'}(\xi))^2 d\xi = -\int_{-\infty}^{+\infty} h^{'}(w)w^{'}w^{'}d\xi$$
$$= -\Big[h(w).w^{'}\Big]_{\xi=-\infty}^{\xi=+\infty} + \int_{-\infty}^{+\infty} w^{''}h(w)d\xi = \int_{-\infty}^{+\infty} w^{''}h(w)d\xi.$$

Since $w^{''} = -cw^{'} + h^{'}(w)w^{'} - f(w)$,

$$\begin{aligned}
&\int_{-\infty}^{+\infty} w^{''}h(w)d\xi \\
&\quad = -c\int_{-\infty}^{+\infty} h(w)w^{'}d\xi + \int_{-\infty}^{+\infty} h^{'}(w)w^{'}h(w)d\xi - \int_{-\infty}^{+\infty} f(w)h(w)d\xi \\
&\quad = -c\int_{1}^{0} h(w)dw + \int_{-\infty}^{+\infty} \frac{d}{d\xi}\left(\frac{(h(w))^2}{2}\right)d\xi - \int_{-\infty}^{+\infty} f(w)h(w)d\xi \\
&\quad = c\int_{0}^{1} h(w)dw + \frac{1}{2}\left[(h(0))^2 - (h(1))^2\right] - \int_{-\infty}^{+\infty} f(w)h(w)d\xi.
\end{aligned}$$

Thus we have

$$-\int_{-\infty}^{+\infty} h^{'}(w)(w^{'}(\xi))^2 d\xi = c\int_{0}^{1} h(w)dw + \frac{1}{2}\left[h(0)^2 - h(1)^2\right] - \int_{-\infty}^{+\infty} f(w)h(w)d\xi.$$

3. The fourth term in (2.32) gives us

$$\int_{-\infty}^{+\infty} f(w)w^{'}d\xi = \int_{w(-\infty)}^{w(+\infty)} f(w)dw = \int_{1}^{0} f(w)dw = -\int_{0}^{1} f(w)dw.$$

Thus the three integrations together give,

$$\begin{aligned}
c\int_{-\infty}^{+\infty}(w^{'}(\xi))^2 d\xi + c\int_{0}^{1} h(w)dw + \frac{h(0)^2 - h(1)^2}{2}& \\
- \int_{-\infty}^{+\infty} f(w)h(w)d\xi - \int_{0}^{1} f(w)dw = 0.& \qquad (2.33)
\end{aligned}$$

Then by (2.30) and Hypothesis E_2, we obtain that $\int_{-\infty}^{+\infty} f(w)h(w)d\xi > 0$. Note that we need to estimate this term because we do not know the explicit form of the function w and as a result of this we cannot evaluate this term. We estimate it since the integration is with respect to $d\xi$ not dw. This give

$$c\int_{-\infty}^{+\infty}(w^{'}(\xi))^2d\xi + c\int_0^1 h(w)dw \geq \frac{h(1)^2-h(0)^2}{2} + \int_0^1 f(w)dw = 0. \tag{2.34}$$

We now multiply equation (2.29) by $1-w$ to get

$$w^{''}(1-w) + cw^{'}(1-w) - h^{'}(w)w^{'}(1-w) + f(w)(1-w) = 0, \tag{2.35}$$

and then integrate each term separately over $\mathbb{R}$.

1. The first term in (2.35) gives us

$$\begin{aligned}\int_{-\infty}^{+\infty}(1-w)w^{''}d\xi &= \Big[w^{'}(\xi)(1-w(\xi))\Big]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} w^{'}(\xi)(-w^{'}(\xi))d\xi \\ &= \int_{-\infty}^{+\infty}(w^{'}(\xi))^2d\xi.\end{aligned}$$

2. The second term in (2.35) is

$$c\int_{-\infty}^{+\infty} w^{'}(1-w)d\xi = c\int_{w(-\infty)}^{w(+\infty)}(1-w)dw = c\int_1^0(1-w)dw = -c\left[w-\frac{w^2}{2}\right]_0^1 = -\frac{c}{2}.$$

3. The third term in (2.35) is

$$\begin{aligned}-\int_{-\infty}^{+\infty} h^{'}(w)w^{'}(1-w)d\xi &= -\int_{-\infty}^{+\infty} h^{'}(w)w^{'}d\xi + \int_{-\infty}^{+\infty} h^{'}(w)w^{'}wd\xi \\ &= -\int_1^0 h^{'}(w)dw + \int_1^0 h^{'}(w)wdw \\ &= [h(w)]_0^1 + \left\{[-h(w)w]_0^1 + \int_0^1 h(w)dw\right\} \\ &= [h(w)]_0^1 - [h(w)w]_0^1 + \int_0^1 h(w)dw \\ &= -h(0) + \int_0^1 h(w)dw.\end{aligned}$$

4. Since the function f satisfies Hypothesis E_2, integration of the last term gives us the

following estimate

$$\int_{-\infty}^{+\infty} f(w)(1-w)d\xi > 0. \tag{2.36}$$

Thus

$$\int_{-\infty}^{+\infty} (w'(\xi))^2 d\xi + \int_0^1 h(w)dw = \frac{c}{2} + h(0) - \int_{-\infty}^{+\infty} f(w)(1-w)d\xi. \tag{2.37}$$

So by (2.36), equation (2.37) gives

$$\int_{-\infty}^{+\infty} (w'(\xi))^2 d\xi + \int_0^1 h(w)dw \leq \frac{c}{2} + h(0),$$

and since $c > 0$, this implies

$$c\int_{-\infty}^{+\infty} (w'(\xi))^2 d\xi + c\int_0^1 h(w)dw \leq \frac{c^2}{2} + ch(0). \tag{2.38}$$

Then comparing (2.38) with (2.34) yields that

$$\frac{c^2}{2} + ch(0) - \frac{h(1)^2 - h(0)^2}{2} - \int_0^1 f(w)dw \geq 0. \tag{2.39}$$

In order to understand what inequality (2.39) tells us about c, we study the roots of

$$s^2 + 2sh(0) - (h(1)^2 - h(0)^2) - 2\int_0^1 f(w)dw = 0, \tag{2.40}$$

which is a quadratic equation in s, that has solutions

$$s = -h(0) \mp \sqrt{h(1)^2 + 2\int_0^1 f(w)dw}.$$

Using Hypothesis E_2, we obtain that (2.40) has a positive root which is

$$s_+ = -h(0) + \sqrt{h(1)^2 + 2\int_0^1 f(w)dw},$$

Thus since $c > 0$, (2.39) implies that $c \geq s_+$, and by definition of the minimum value of

c_0, it follows that

$$c_0 \geq -h(0) + \sqrt{h(1)^2 + 2\int_0^1 f(w)dw}.$$

Since the linear value for equation (2.29) is $\bar{c} = 2\sqrt{f'(0)} + h'(0)$, a sufficient condition to have $c_0 > \bar{c}$, is thus that

$$-h(0) + \sqrt{(h(1))^2 + 2\int_0^1 f(w)dw} \;>\; 2\sqrt{f'(0)} + h'(0). \tag{2.41}$$

$\square$

Note that condition (2.41) reduces to the condition of Berestycki and Nirenberg in [8] when $h \equiv 0$ namely, that

$$\sqrt{2\int_0^1 f(w)dw} \;>\; 2\sqrt{f'(0)} \quad \text{if and only if} \quad \int_0^1 f(w)dw \;>\; 2f'(0). \tag{2.42}$$

The following example illustrates the fact that it is possible for a given function f to satisfy the sufficient condition (2.42) for $c_0 = \bar{c}$ when $h \equiv 0$, but one can find a function h so (2.41) will be sufficient to ensure that $c_0 > \bar{c}$ when we add the term $h'(w)w'$.

Example 2.4. Choose $f(u) = u(1-u)$ and $h(u) = \delta u^2$, $\delta \geq 0$, so (2.41) says that $\sqrt{\dfrac{3\delta^2+1}{3}} > 2$, which is satisfied for $\delta > \sqrt{\frac{11}{3}}$. Hence (2.41) holds for such δ. But when $h \equiv 0$, (2.42) clearly fails since $\sqrt{\dfrac{1}{3}} \;>\; 2$, is false, and in fact, this function f satisfies the condition

$$f(u) \leq f'(0)u \qquad \text{for all } u \in (0,1), \tag{2.43}$$

which ensures that $c_0 = \bar{c}$ if $h \equiv 0$. Note that (2.43) implies that

$$\int_0^1 f(u)du \leq f'(0)\int_0^1 udu = f'(0)\left[\frac{u^2}{2}\right]_0^1 = \frac{f'(0)}{2},$$

and hence

$$\int_0^1 f(u)du \leq \frac{f'(0)}{2}. \tag{2.44}$$

Clearly no function f can satisfy both (2.44) and (2.42).

Chapter 3

A discrete-time recursion system

This chapter is mainly background material, that describes and slightly extends the work in [29], [42], [26], [39], [26], [43], and [40]. We consider a discrete-time recursion system of the form

$$u_{n+1} = Q[u_n] \qquad n \in \mathbb{N}, \tag{3.1}$$

with an initial condition $u_0 \in BUC(\mathbb{R}, \mathbb{R}^k)$ (the space of functions $p : \mathbb{R} \to \mathbb{R}^k$ such that p is bounded and uniformly continuous on $\mathbb{R}$), and where $u_n(\cdot) = ((u_n)_1(\cdot), ..., (u_n)_k(\cdot))$ is a vector-valued function such that $u_n \in BUC(\mathbb{R}, \mathbb{R}^k)$ which represents the population densities at time n of k interacting species or age classes. The operator $Q : BUC(\mathbb{R}, \mathbb{R}^k) \to BUC(\mathbb{R}, \mathbb{R}^k)$ in (3.1) is assumed to be *an order-preserving operator*, which means that if we have $u \leq v \in BUC(\mathbb{R}, \mathbb{R}^k)$, in the sense that the vectors $u(x), v(x) \in \mathbb{R}^k$ satisfy $u(x) \leq v(x)$ for each $x \in \mathbb{R}$, then this implies that $Q[u](x) \leq Q[v](x)$ for each $x \in \mathbb{R}$.

The linear operator M is the linearization of Q at 0 if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\|u\|_\infty \leq \delta$ implies that $\|Q[u] - M[u]\|_\infty \leq \epsilon \|u\|_\infty$ where $M[u] = \lim\limits_{\rho \searrow 0}[(1/\rho)Q[\rho u]]$. As described in [42], this linear operator has the representation,

$$(M[u](x))_i = \sum_{j=1}^{k} \int_{-\infty}^{\infty} u_j(x-y) m_{ij}(y, dy), \tag{3.2}$$

where m_{ij} is a bounded non-negative measure which allows us to introduce the matrix

B_μ by

$$B_\mu := \left(\int_{-\infty}^{\infty} e^{\mu y} m_{ij}(y, dy) \right), \tag{3.3}$$

which can characterized by

$$B_\mu \alpha = M[\alpha e^{-\mu x}]|_{x=0} \qquad \text{for every constant} \quad \alpha \in \mathbb{R}^k. \tag{3.4}$$

We assume that the entries of B_μ are finite for all μ. Since the m_{ij} are non-negative, the entries of the matrix B_μ are non-negative, and an entry of B_μ is 0 if and only if m_{ij} is identically zero, which means that either all the B_μ are *irreducible* or they are all *reducible.* The matrix B_μ is said to be *reducible* if it can be put into lower block triangular form by re-ordering the coordinates, whereas if this can not be done, the matrix is said to be *irreducible.* If a reducible matrix is in lower block triangular form and all the diagonal blocks are irreducible, the matrix is said to be in *Frobenius form.* The statement of *Perron - Frobenius* theorem is the following.

Theorem 3.1. *(Perron-Frobenius theorem[35]) Any non-zero irreducible matrix with non-negative entries has a unique positive eigenvalue, called the principal eigenvalue, which has a corresponding strictly positive principal eigenvector. In addition, the absolute values of all the other eigenvalues are less than or equal the principal eigenvalue.*

A useful corollary about an irreducible matrix with non-negative off-diagonal entries is the following.

Corollary 3.1. *Given any irreducible matrix with off-diagonal entries non-negative, there exists a unique real eigenvalue, called the principal eigenvalue with a corresponding strictly positive principal eigenvector. In addition, the real parts of all other eigenvalues are strictly less than the principal eigenvalue.*

Proof. Let M be an irreducible matrix with non-negative off-diagonal entries. Then there exists $\alpha > 0$ such that $M + \alpha I$ is a non-zero irreducible matrix with non-negative entries. By Theorem 3.1, there exists a positive eigenvalue λ with positive eigenvector q such that $(M + \alpha I)q = \lambda q$, and the absolute values of all other eigenvalues of $M + \alpha I$ are less than or equal λ. Then $Mq = (\lambda - \alpha)q$, so M has a real eigenvalue $\lambda - \alpha$ with positive eigenvector q. Moreover, if μ is an eigenvalue of $M + \alpha I$ other than λ, then $|\mu| \leq \lambda$, so if ν is an eigenvalue of the matrix M other than $\lambda - \alpha$, then $|\nu + \alpha| \leq \lambda$ and hence $Re(\nu) < \lambda - \alpha$. $\square$

3.1 Hypotheses of discrete-time recursion system

We start with some notation. Here 0 denotes the constant vector in which all components are 0. If $\beta > 0$ such that $\beta \in \mathbb{R}^k$, we define the set of functions,

$$\psi_\beta = \left\{ u \in BUC(\mathbb{R}, \mathbb{R}^k) : \, 0 \leq u(x) \leq \beta \ \text{ for all } x \in \mathbb{R} \right\}. \tag{3.5}$$

If $Q[w] = w$, the function $w(\cdot)$ is said to be an *equilibrium* of Q, so that if $u_l = w$ in the recursion (3.1) for some $l \in \mathbb{N}$, then $u_n = w$ for all $n \geq l$. Note that the operator Q satisfies the translation invariance property (1.7), but it is not assumed to satisfy a reflection invariance property (1.8) as was assumed in [42].
The reason for removing the reflection invariance assumption is due to the fact that we will consider applications to partial differential equation systems that have convection terms in Chapter 5, and the presence of these convection terms, which involve first-order derivatives, means that we do not have a symmetry between x and $-x$. In fact when Q is the time t map of the recursion (3.1), then property (1.8) does not hold in the presence of a convection term.

We make the following assumptions about the operator Q in the recursion (3.1):

q_1. $Q[0] = 0$, there is a constant $\beta > 0$ such that $Q[\beta] = \beta$, and there is no constant vector $v \in \mathbb{R}^k, v \neq \beta$ such that $Q[v] = v$ where $0 < v \leq \beta$.

q_2. The operator Q is order-preserving on non-negative functions, in the sense that if $u \geq v \geq 0$ are any two functions in ψ_β, then $Q[u] \geq Q[v] \geq 0$.

q_3. Q satisfies the translation-invariance property (1.7).

q_4. If $(v_n)_{n \in \mathbb{N}}$ such that $v_n \in \psi_\beta$ converges to v uniformly on each bounded subset of $\mathbb{R}$, then $Q[v_n]$ converges to $Q[v]$ uniformly on each bounded subset of $\mathbb{R}$.

q_5. Given a sequence $(v_n)_{n\in\mathbb{N}}$ such that $v_n \in \psi_\beta$, there exists a subsequence v_{n_l} such that $Q[v_{n_l}]$ converges uniformly on every bounded subset of $\mathbb{R}$.

q_6. a) The matrix B_μ in (3.3) has finite entries for all μ and is in *Frobenius* form, with $\lambda_\sigma(\mu)$ the *principal eigenvalue* of the σth diagonal block from the top of B_μ.

b) $1 < \lambda_1(0)$, and $\lambda_1(0) > \lambda_\sigma(0)$ for every $\sigma > 1$.

c) The matrix B_0 has at least one nonzero entry to the left of each of its diagonal blocks other than the uppermost one.

q_7. There exists a family of bounded linear order-preserving operators on $\mathbb{R}^k$-valued functions $M^{(\kappa)}$ which satisfies the following properties:

(i) For every sufficiently large $\kappa \geq 0$ and $v : \mathbb{R} \to \mathbb{R}^k$, there is a constant vector $\omega > 0$ such that

$$Q[v] \geq M^{(\kappa)}[v] \quad \text{when} \quad 0 \leq v \leq \omega.$$

(ii) For every positive μ the matrices $B_\mu^{(\kappa)}$ that can be characterized by

$$B_\mu^{(\kappa)}\alpha := M^{(\kappa)}[\exp(-\mu x)\alpha]|_{x=0}$$

converge to B_μ as $\kappa \to \infty$.

3.2 Slowest spreading speed for discrete-time recursion system (3.1)

In this section we firstly present some results that will be useful for the definition of the right (left) slowest spreading speed. The following proposition shows that if the initial condition u_0 for the recursion (3.1) lies between 0 and β, so $Q[u_0]$ also does.

Proposition 3.1. *An operator Q that satisfies Hypotheses $q_1 - q_7$ maps ψ_β into itself.*

Proof. Let $u_0 \in \psi_\beta$, since Q is an order-preserving operator, 0 and β are equilibria and the initial condition u_0 lies between 0 and β, thus $0 \leq Q[u_0] \leq \beta$, from which it follows that $Q[u_0] \in \psi_\beta$. $\square$

Note that it follows from Hypothesis q_6 $(a), (c)$ we can say there is an eigenvector $\zeta(0) > 0$ of B_0 corresponding to the principal eigenvalue $\lambda_1(0)$, $B_0\zeta(0) = \lambda_1(0)\zeta(0)$. We prove the strict positivity of the eigenvector later, in the simple case when we have only two blocks of $f'(0)$ in Chapter 5, and in the general case when we have multiple blocks in Chapter 6. The following lemma shows that for any positive constant vector u_0, the constants u_n defined in (3.1) converge to β. In biological terms, this means that β is a globally stable *coexistence equilibrium.* A coexistence equilibrium is one in which all of the components are strictly positive. A globally stable equilibrium means that, for some set of initial conditions are not necessarily that close to the equilibrium, the solution u_n tends to this equilibrium as $n \to \infty$.

Lemma 3.1. *Suppose that $u_0 \in [0, \beta], u_0 > 0$ is any constant vector, then $\lim_{n\to\infty} u_n = \beta$ where $(u_n)_{n\in\mathrm{N}}$ is the sequence of constant vectors obtained from the recursion* (3.1).

Proof. From (3.4) we have $B_0\zeta(0) = M[\zeta(0)]$, recall that $M[u] = \lim_{\rho\searrow 0}[(1/\rho)Q[\rho u]]$. Then

$$\frac{1}{\rho}Q[\rho\zeta(0)] \longrightarrow M[\zeta(0)] = \lambda_1(0)\zeta(0) \quad \text{as } \rho \to 0. \tag{3.6}$$

Then (3.6) and the fact that $\lambda_1(0) > 1$ together imply that for $\rho > 0$ sufficiently small, we have

$$\frac{1}{\rho}Q[\rho\zeta(0)] > \zeta(0). \tag{3.7}$$

Now let $\alpha := \rho\zeta(0)$. Then $\alpha > 0$, and (3.7) says that $Q[\alpha] > \alpha$, so if we define a constant vector $\alpha_0 = \alpha$ and then α_n by

$$\alpha_{n+1} = Q[\alpha_n], \tag{3.8}$$

we have $\alpha_1 > \alpha_0$, and since Q is an order-preserving operator, it follows by induction that $\alpha_{n+1} \geq \alpha_n$ for all $n \in \mathrm{N}$. We also have that $0 < \alpha < \beta$, and since Q is order-preserving, it follows that $0 = Q[0] < \alpha_n < Q[\beta] = \beta$ for all $n \in \mathrm{N}$ and hence we get $\alpha_n \leq \beta$ for all n. So α_n is a non-decreasing sequence that is bounded above by β. Thus $\gamma = \lim_{n\to\infty} \alpha_n$ exists, and by (3.8) we have $\gamma = Q[\gamma]$. Since $0 < \alpha \leq \alpha_n \leq \gamma \leq \beta$, it follows by Hypothesis q_1 that $\gamma = \beta$.

Then given any constant vector $u_0 \in [0, \beta]$, $u_0 > 0$, there exists $\rho > 0$ small enough that both (3.7) holds and $\rho\zeta(0) < u_0$. Then since $Q[\rho\zeta(0)] < Q[u_0]$, it follows that the sequence defined by $u_{n+1} = Q[u_n]$ must also converge to β as $n \to \infty$ because $\alpha_n \leq u_n \leq \beta$, and the lemma is proved. $\square$

The following lemma is a crucial tool needed to define the left slowest spreading speed for the recursion (3.1) with a non-decreasing initial function $u_0 = \phi(\cdot)$. The definition of this slowest spreading speed in (3.12) below is a modification of the definition presented in [26, (2.4)] with a non-decreasing initial function $u_0 = \phi$ instead of a non-increasing initial function.

Lemma 3.2 (*Comparison Lemma*). *Let $Q : BUC(\mathbb{R}, \mathbb{R}^k) \to BUC(\mathbb{R}, \mathbb{R}^k)$ satisfy Hypothesis q_2. If the sequences u_n and v_n satisfy the inequalities $u_{n+1} \leq Q[u_n]$ and $v_{n+1} \geq Q[v_n]$ for all n respectively, and if $u_0 \leq v_0$, then $u_n \leq v_n$ for all n.*

Proof. Suppose that $v_n \geq u_n$. Since we have

$$v_{n+1} \geq Q[v_n], \text{ and } u_{n+1} \leq Q[u_n], \tag{3.9}$$

and by Hypothesis q_2, we obtain

$$v_{n+1} \geq Q[v_n] \geq Q[u_n] \geq u_{n+1}.$$

Since we also know that $u_0 \leq v_0$, it follows by induction that $u_n \leq v_n$ for all $n \in \mathbb{N}$. $\square$

Now choose a continuous vector-valued function $\phi \in BUC(\mathbb{R}, \mathbb{R}^k)$ with the following properties:

e_1. $\phi(x)$ is non-decreasing in x,

e_2. $\phi(x) = 0 \quad$ for all $x \leq 0$,

e_3. $0 < \phi(+\infty) < \beta$.

Letting $a_0(c; s) := \phi(s)$, we define the sequence $a_n(c; s)$ by the recursion

$$a_{n+1}(c; s) = \max\left\{\phi(s), Q[a_n(c; \cdot)](s + c)\right\}, \tag{3.10}$$

where the operator Q satisfies Hypotheses q_1-q_7. By definition, $a_1(c;s) \geq \phi(s) = a_0(c;s)$. Suppose that $a_{k+1}(c;s) \geq a_k(c;s)$ is true for some $k \geq 0$. Then

$$\begin{aligned} a_{k+2}(c;s) &= \max\{\phi(s), Q[a_{k+1}(c;\cdot)](s+c)\} \\ &\geq \max\{\phi(s), Q[a_k(c;\cdot)](s+c)\} \\ &= a_{k+1}(c;s). \end{aligned}$$

So, since $a_0 \leq a_1$, by induction we have $a_n(c;s) \leq a_{n+1}(c;s)$ for all n. Moreover, since $a_0(c;s) = \phi(s) \leq \beta$, and if we assume that $a_n(c;s) \leq \beta$ for all s is true, then

$$a_{n+1}(c;s) = \max\{\phi(s), Q[a_n(c;\cdot)](s+c)\} \leq \beta$$

from which it follows by induction that $a_n \leq a_{n+1} \leq \beta$ for all n. Since $a_n(c;s)$ is a non-decreasing function in s for all n, the translation $s \mapsto s+c$ applied to a non-decreasing function is non-decreasing in c, from which it follows that $a_n(c;s)$ is also non-decreasing in c for all n. The fact that the vectors $a(c;\mp\infty)$ are equilibria of Q follows using arguments similar to these in [29, Lemma 2.6], and Hypothesis q_1 then implies that $a(c;+\infty) = \beta$.

In order to show that the function $a(c;s)$ does not depend on s for sufficiently positive c, we first prove the following lemma which shows that the function $a(c;s) = \beta$ for all s if (3.11) is satisfied, and then we use this lemma to obtain that $a(c;s) \equiv \beta$, which is equivalent to showing that $a(c;-\infty) = \beta$ since we already know that $a(c;+\infty) = \beta$. Note that this following result is a modification of [39, Lemma 5.3] when a is now a non-decreasing function of s and c instead of a non-increasing function.

Lemma 3.3. $a(c;-\infty) = \beta$ *if and only if there exists* $n \in \mathbb{N}$ *such that*

$$a_n(c;0) > \phi(\infty). \tag{3.11}$$

Proof. Suppose that $a(c;-\infty) = \beta$, then we can say that $a(c;s) = \beta$ for all s, since we already know that $a(c;\infty) = \beta$, and in particular, $a(c;0) = \beta$. By Property e_3 and the fact that $a_n(c;0)$ converges to $a(c;0)$ when $n \to \infty$, there exists an $n \in \mathbb{N}$ such that (3.11) holds. Suppose, on the other hand, that there exists n_0 such that

$$a_{n_0}(c;0) > \phi(\infty).$$

Since we also know that a_{n_0}, ϕ are non-decreasing functions, and that $\phi = 0$ for all $s \leq 0$ and $\phi \leq \phi(+\infty)$, it follows from the continuity of a_n and ϕ that there exists $\delta > 0$ such that

$$a_{n_0}(c; s-\delta) > \phi(s) = a_0(c;s) \quad \text{for all } s \in \mathbb{R}.$$

Now suppose that $a_{n_0+k}(c; s-\delta) \geq a_k(c;s)$ is true for some $k \geq 0$. Then $a_{n_0+k+1}(c; s-\delta) = \max\{a_{n_0}(c; s-\delta), Q[a_{n_0+k}(c;\cdot)](s-\delta+c)\}$ and since $Q[a_{n_0+k}(c;\cdot)](s-\delta+c) = Q[a_{n_0+k}(c;\cdot-\delta)](s+c) \geq Q[a_k(c;\cdot)](s+c)$, thus

$$a_{n_0+k+1}(c; s-\delta) \geq \max\{\phi(s), Q[a_k(c;\cdot)](s+c)\} = a_{k+1}(c;s).$$

So, by induction we have shown that $a_{n_0+k}(c; s-\delta) \geq a_k(c;s)$ for all $k \in \mathbb{N} \cup \{0\}$. Letting $k \to \infty$, which then shows that $a(c; s-\delta) \geq a(c;s)$ for all s. But since $\delta > 0$, and a is non-decreasing in s, we also know that $a(c; s-\delta) \leq a(c;s)$. Thus

$$a(c; s-\delta) = a(c;s) \quad \text{for all } s,$$

and therefore a is a constant, because the only non-decreasing periodic functions are constant. It follows that the function a does not depend on s. Since $a(c;\infty) = \beta$, $a(c;s) = \beta$ for all s. □

To define a slowest spreading speed, we will next show that $a(c;s) \equiv \beta$, or equivalently that $a(c;-\infty) = \beta$ for a sufficiently positive c. It follows from [39, Lemma 5.2] that for n sufficiently large, $a_n(0,\infty) = \alpha_n$ where $\alpha_0 \equiv \phi(\infty)$ and $\alpha_{n+1} = Q[\alpha_n]$.
Since $0 < \alpha_0 = \phi(\infty) < \beta$, we know that $\alpha_n \to \beta$ as $n \to \infty$ and so for some $\tilde{n}$ sufficiently large, we have $a_{\tilde{n}}(0,\infty) > \phi(\infty)$. Hence for $t > 0$ sufficiently large and from the monotonicity of ϕ, we have $a_{\tilde{n}}(0,t) > \phi(\infty) > \phi(t)$, and thus

$$a_{\tilde{n}}(0,t) = \max\{\phi(t), Q[a_{\tilde{n}-1}(0,\cdot+t)](0)\} = Q[a_{\tilde{n}-1}(0,\cdot+t)](0),$$

whereas

$$a_{\tilde{n}}(t,0) = \max\{\phi(0), Q[a_{\tilde{n}-1}(t,\cdot)](0+t)\} = \max\{0, Q[a_{\tilde{n}-1}(t,\cdot)](t)\},$$

by Hypothesis q_3, we have $a_{\tilde{n}}(t,0) = Q[a_{\tilde{n}-1}(t,\cdot+t)](0)$ and since $a_n(c;s)$ is non-decreasing in c, thus

$$Q[a_{\tilde{n}-1}(t,\cdot+t)](0) \geq Q[a_{\tilde{n}-1}(0,\cdot+t)](0) = a_{\tilde{n}}(0,t) > \phi(\infty).$$

By Lemma 3.3 we thus have $a(t,-\infty) = \beta$, which implies that $a(c;-\infty) = \beta$ for all $c \leq t$. Note The function $a(c;\cdot)$ clearly depends on the choice of the initial function ϕ, but the vector $a(c;-\infty)$ is independent of the initial function ϕ. To prove this, we define a new function $\mathring{\phi}$ with the same Properties $e_1 - e_3$ for ϕ, then we obtain a different sequence $\mathring{a}_n(c;\cdot)$ and a different limit function $\mathring{a}(c;\cdot)$. Again, by Hypothesis q_1, $\lim\limits_{n\to\infty} \mathring{a}_n(c;\infty) = \beta > \mathring{\phi}(\infty)$. Then there exists $N \in \mathbb{N}$ and a translation τ such that $a_N(c;x-\tau) \geq \mathring{\phi}(x) = \mathring{a}_0(c;x)$ for all $x \in \mathbb{R}$. By the definition of a in (3.10), and using the Comparison Lemma 3.2 with the operator $\mathcal{Q}$ defined by $\mathcal{Q}[v(\cdot)](s) := Q[v(\cdot)](s+c)$, we obtain $a(c;x-\tau) \geq \mathring{a}(c;x)$ for all $x \in \mathbb{R}$. In particular, $a(c;-\infty) \geq \mathring{a}(c;-\infty)$. By exchanging the role of ϕ and $\mathring{\phi}$, we also obtain that $\mathring{a}(c;-\infty) \leq a(c;-\infty)$ and hence $a(c;-\infty) = \mathring{a}(c;-\infty)$, which means that the vector $a(c;-\infty)$ is independent of the initial function ϕ.
So the left slowest spreading speed $\hat{c} \geq -\infty$ of (3.1) can be defined as

$$\hat{c} := \inf\{c : a(c;-\infty) = \beta\}. \tag{3.12}$$

3.2.1 Characterization properties of the slowest spreading speed

The following theorem is a characterization of the left slowest spreading speed $\hat{c}$ in (3.12) where the initial condition ϕ is a non-decreasing function. This theorem is a modification of Theorem 2.1 in [26] for the characterization the left slowest spreading speed c^* in [26, 2.4] where the initial condition is a non-increasing function.

Theorem 3.2. *Suppose that the initial function u_0 satisfies $u_0(x) = 0$ for all sufficiently negative x, and that there are positive constants $0 < \rho \leq \sigma < 1$ such that $0 \leq u_0 \leq \sigma\beta$ for all x and $u_0 \geq \rho\beta$ for all sufficiently positive x. Then there exists an index j such* *that for any positive ϵ, the solution u_n of the recursion* (3.1) *has the properties*

$$\lim_{n\to\infty}\left[\sup_{x\leq n(\hat{c}-\epsilon)}\{u_n\}_j(x)\right]=0, \tag{3.13}$$

and

$$\lim_{n\to\infty}\left[\sup_{x\geq n(\hat{c}+\epsilon)}\{\beta-u_n(x)\}\right]=0. \tag{3.14}$$

That is, the jth component spreads at a speed no less than $\hat{c}$, and no component spreads at a higher speed.

Proof. Since $u_0=0$ for $x\leq 0$ and $u_0\leq\sigma\beta$, we can choose a function $\phi(x)$ with properties e_1-e_3 and a number δ such that $u_0(x)\leq\phi(x-\delta)$.

Recall that $a_{n+1}(c;s)\geq Q[a_n](s+c)$, so if we define the function v_0 such that $v_0:=a_0(c;x-\delta)=\phi(x-\delta)$, and v_n such that $v_n=a_n(c;x-\delta-nc)$, then

$$v_{n+1}=a_{n+1}(c;x-\delta-(n+1)c)\geq Q[a_n](x-\delta-nc)=Q[v_n].$$

So, we have $u_{n+1}=Q[u_n]$, $v_{n+1}\geq Q[v_n]$ and $u_0\leq\phi(x-\delta)=v_0$, which means that we can apply the Comparison Lemma 3.2 to get that $u_n\leq v_n$, and hence $u_n(x)\leq a_n(c;x-\delta-nc)$ for all $x\in\mathbb{R}$. Then $\sup\limits_{x\leq n(\hat{c}-\epsilon)}[u_n(x)]\leq\sup\limits_{x\leq n(\hat{c}-\epsilon)}[a_n(c;x-\delta-nc)]$, so since a_n is non-decreasing of x and by letting $\hat{c}=c-\dfrac{\epsilon}{2}$, we have

$$\sup_{x\leq n(c-\frac{3}{2}\epsilon)}[u_n(x)]\leq a_n(c;nc-\frac{3}{2}n\epsilon-\delta-nc)=a_n(c;-\frac{3}{2}n\epsilon-\delta),$$

and since a_n is non-decreasing sequence in n, we thus obtain $\sup\limits_{x\leq n(c-\frac{3}{2}\epsilon)}[u_n(x)]\leq a(c;-\frac{3}{2}n\epsilon-\delta)$, from which it follows that

$$\lim_{n\to\infty}[\sup_{x\leq n(c-\frac{3}{2}\epsilon)}u_n(x)]\leq\lim_{n\to\infty}a(c;-\frac{3}{2}n\epsilon-\delta)=a(c,-\infty).$$

Now $a(c;-\infty)$ is an equilibrium, and since $c=\hat{c}-\dfrac{\epsilon}{2}<\hat{c}$, $a(c;-\infty)$ is an equilibrium other than β. So by Hypothesis q_1, $a(c;-\infty)$ has at least one zero component, say the

j-th component is zero, and therefore

$$\lim_{n\to\infty}[\sup_{x\leq n(\hat{c}-\epsilon)}\{u_n\}_j(x)] = 0.$$

To derive the second property (3.14), assume firstly that Q has the additional properties

(1) If α is a constant vector that satisfies $0 \leq \alpha < \beta$, then $Q[\alpha] < \beta$.

(2) If $u(x)$ vanishes for $x \leq \eta$, then there is a number γ such that

$$Q[u](x) = 0 \quad \text{for } x \leq \eta + \gamma.$$

Choose a function $\hat{\phi}$ that satisfies Properties $e_1 - e_3$ with $\hat{\phi}(x) \leq u_0(x)$ and let $c > \hat{c}$. By the definition of the left slowest spreading speed $\hat{c}$ in (3.12), $a_n(c;0)$ increases to β as $n \to \infty$. Thus there exists an index N such that $a_N(c;0) > \phi(\infty)$. Since both a_N and $\hat{\phi}$ are non-decreasing in x and $\hat{\phi}$ vanishes for $x \leq 0$, it follows that $a_N(c;x) > \phi(x)$ for all $x \in \mathbb{R}$.

Let b_n be the solution of the recursion $b_{n+1}(\cdot) = Q[b_n](\cdot)$ with the initial condition $b_0(x) = \hat{\phi}(x)$ for all $x \in \mathbb{R}$. Since $u_0 \geq \hat{\phi}(x)$, by the Comparison Lemma 3.2, we have $u_n(x) \geq b_n(x)$ for all n and $x \in \mathbb{R}$, and Lemma 3.1 implies that $b_n(\infty)$ converges to β as $n \to \infty$. Moreover, $a_0 = \hat{\phi}$, there exists $\alpha < \beta$ such that $\hat{\phi} \leq \alpha$ by Property e_3, and Property (1) says that $Q[\alpha] < \beta$. So by the Comparison Lemma 3.2, $Q[\alpha] \geq Q[\hat{\phi}] = Q[a_0]$, and the definition of a in (3.10) implies that

$$a_1(c;s) = \max\left\{\hat{\phi}(s), Q[a_0(c;\cdot)](s+c)\right\} \leq \max\{\alpha, Q[\alpha]\} < \beta.$$

Suppose that $a_k(c;s) < \beta$ is true for some $k \geq 0$. Then it follows by Hypotheses q_2 and q_1 that $Q[a_k(c;\cdot)](s) < Q[\beta] = \beta$ for all s. Then

$$a_{k+1}(c;s) = \max\left\{\hat{\phi}(s), Q[a_k](s+c)\right\} < \beta,$$

so $a_N(c;\infty) < \beta$, since we already know that $a_1(c;\cdot) < \beta$. Thus there exists $M \geq N$ such that $b_M(\infty) > a_N(\infty)$. Property (2) then gives $a_N(c;x) = 0$ for $x \leq N(\gamma + c)$ for some $\gamma \in \mathbb{R}$, and hence there is a number τ such that $b_M(\cdot) \geq a_N(c; \cdot + \tau)$. Since a_n is non-decreasing in n, so $a_n(c; x) > a_N(c; x) > \hat{\phi}(x) = a_0$ for $n \geq N$ and $x \in \mathbb{R}$. Then

$$a_{n+1}(c; s) = \max\left\{\hat{\phi}(s), Q[a_n(c; \cdot)](s + c)\right\} = Q[a_n(c; \cdot)](s + c) =: \mathcal{Q}[a_n(c; \cdot)](s),$$

and by following the same procedure for b_0 we get that

$$b_{n+1}(c; s) = Q[b_n(c; \cdot)](s + c) = \mathcal{Q}[b_n(c; \cdot)](s).$$

Therefore both the functions $b_n(\cdot + nc)$ and $a_n(c; \cdot + \tau)$ satisfy the same recursion (3.1) for $n \geq N$. By applying the Comparison Lemma 3.2 with the operator $\mathcal{Q}$ we get $b_n(\cdot + (n - M)c) \geq a_{n+N-M}(c; \cdot + \tau)$ when $n \geq M$. Therefore

$$u_n(x + nc) \geq b_n(x + nc) \geq a_{n+N-M}(c; x + \tau),$$

it follows that $\beta - u_n(x + nc) \leq \beta - a_{n+N-M}(c; x + \tau)$ which is equivalent to $\beta - u_n(y) \leq \beta - a_{n+N-M}(c; y - nc + \tau)$.

Since $a_n(c; s)$ is non-decreasing in s and by letting $c = \hat{c} + 2\epsilon$ we obtain that

$$\begin{aligned}
\sup_{y \geq n(c-\epsilon)} \{\beta - u_n(y)\} &\leq \sup_{y \geq n(c-\epsilon)} \{\beta - a_{n+N-M}(c; y - nc + \tau)\} \\
&= \beta - a_{n+N-M}(c; nc - n\epsilon - nc + \tau) \\
&= \beta - a_{n+N-M}(c; -n\epsilon + \tau) \\
&= \beta - a_{n+N-M}(\hat{c} + 2\epsilon; -n\epsilon + \tau).
\end{aligned}$$

From the definition of the recursion in (3.10) we have $\beta - a_{n+N-M}(\hat{c} + 2\epsilon; -n\epsilon + \tau) = \beta - a_{n+N-M}(\hat{c} + \epsilon; \tau)$ and as $n \to \infty$ we get $\lim_{n \to \infty} \beta - a_{n+N-M}(\hat{c} + \epsilon; \tau) = \beta - a(\hat{c} + \epsilon; \tau)$. Since we have that $a(c, +\infty) = \beta$ and the definition of the left slowest spreading speed $\hat{c}$ in (3.12) says that for all $\hat{c} + \epsilon > \hat{c}$ we have $a(c, -\infty) = \beta$, then this means that $a(\hat{c} + \epsilon; \tau) = \beta$ since $a(\hat{c} + \epsilon; \cdot)$ is monotone and $a(\hat{c} + \epsilon; \infty) = a(\hat{c} + \epsilon; -\infty) = \beta$, so

$a(\hat{c}+\epsilon;\cdot) \equiv \beta$. So the right-hand side converges to zero, which yields

$$\lim_{n\to\infty}\left[\sup_{x\geq n(\hat{c}+\epsilon)}\{\beta - u_n(x)\}\right] = 0.$$

Secondly, we show that (3.14) holds even in the case when we do not have these additional two properties (1) and (2) for the operator Q, by defining a new operator $\bar{Q}$ such as that considered in [26], [40], by

$$\bar{Q}[v](y) := \min\left\{Q\left[\zeta\left(\frac{|y-\cdot|}{\alpha_1}\right)v(\cdot)\right](y),\ (1-\delta)Q\left[\zeta\left(\frac{|y-\cdot|}{\alpha_1}\right)v(\cdot)\right](y)+\delta v(y)\right\}, \tag{3.15}$$

where α_1 and δ are two positive parameters and we define the cut-off function $\zeta(s)$ to be a smooth scalar function with the following properties

- $\zeta(s)$ is non-negative and non-increasing for $s \geq 0$;
- $\zeta(s) = 0$ for $s \geq 1$;
- $\zeta(s) = 1$ for $0 \leq s \leq \frac{1}{2}$.

First we need to show that the two properties (1) and (2) hold for $\bar{Q}$. Let v be a constant vector such that $0 \leq v < \beta$. Then $0 \leq \zeta\left(\frac{|y-\cdot|}{\alpha}\right)v(\cdot) \leq \beta$, which implies $0 \leq Q\left[\zeta\left(\frac{|y-\cdot|}{\alpha}\right)v(\cdot)\right] \leq Q[\beta] = \beta$. By definition of $\bar{Q}$ in (3.15) it follows that $\bar{Q} < \beta$, and Property (1) holds.

In order to prove Property (2), we need to show that if v vanishes for $x \leq \eta$, then there exists α_1 such that $\bar{Q}[v](y) \equiv 0$ for $y \leq \eta - \alpha_1$. Choose $v(x) = 0$ for $x \leq \eta$. Then

$$\zeta\left(\frac{|y-x|}{\alpha_1}\right)v(x) = 0 \quad \text{if} \quad \begin{cases} x \leq \eta \\ \dfrac{|y-x|}{\alpha_1} \geq 1 \quad \text{if and only if} \quad |y-x| \geq \alpha_1 \end{cases}$$

For $y \leq \eta - \alpha_1$ we have two cases, (a) if $x \geq \eta$, then $|x-y| \geq \alpha_1$ which implies that $\zeta\left(\frac{|y-x|}{\alpha_1}\right)v(x) = 0$ and (b) if $x \leq \eta$, then $\zeta\left(\frac{|y-x|}{\alpha_1}\right)v(x) = 0$. So from (a) and (b) we conclude that if $y \leq \eta - \alpha_1$, then $\zeta\left(\frac{|y-x|}{\alpha_1}\right)v(x) = 0$ for all $x \in \mathbb{R}$.

Then if we define $\tilde{v}_y(\cdot) := \zeta\left(\frac{|y-\cdot|}{\alpha_1}\right)v(\cdot)$, then $\tilde{v}_y(\cdot) \equiv 0$ if $y \leq \eta - \alpha_1$, and thus $Q[\tilde{v}_y(\cdot)] \equiv 0$ and then $Q[\tilde{v}_y(\cdot)](y) = 0$ for such y. Thus $\bar{Q}[v](y) = 0$ if $y \leq \eta - \alpha_1$, and so Property (2) also holds for $\bar{Q}$.

Therefore the operator $\bar{Q}$ satisfies the two Properties (1) and (2), and since the operator Q is order-preserving, it follows that $\bar{Q}$ is order-preserving also. We can follow the same idea in the proof of [40, Lemma 5.1] (see [26, Theorem 2.1]) in order to prove that when $\alpha_1 \to \infty$ and $\delta \to 0$, the operator $\bar{Q}[v]$ converges to $Q[v]$. Firstly we need to show that as $\alpha_1 \to \infty$ the operator $Q\left[\zeta\left(\frac{|y-\cdot|}{\alpha_1}\right)v(\cdot)\right]$ converges uniformly on each bounded set to the operator $Q[v]$. Since $\zeta\left(\frac{|y-x|}{\alpha_1}\right) = 1$ if and only if $\frac{|y-x|}{\alpha_1} \leq \frac{1}{2}$, it follows that $\zeta\left(\frac{|y-x|}{\alpha_1}\right)v(x) = v(x)$ whenever $|y-x| \leq \alpha_1/2$, and hence as $\alpha_1 \to \infty$, $\zeta\left(\frac{|y-\cdot|}{\alpha}\right)v(\cdot)$ converges uniformly on each bounded set to v and by Hypothesis q_4, we get that $Q[\zeta\left(\frac{|y-\cdot|}{\alpha_1}\right)v(\cdot)]$ converges uniformly on each bounded set to $Q[v]$. Therefore as $\alpha_1 \to \infty$ and for fixed δ we have shown that $\bar{Q}[v](y)$ converges uniformly on each bounded set to

$$\hat{Q}[v](y) = \min\left\{Q[v](y), (1-\delta)Q[v](y) + \delta v(y)\right\}. \tag{3.16}$$

Secondly we want to show that when δ goes to 0, we have $\hat{Q}[v] \to Q[v]$. By the definition of $\hat{Q}[v]$ in (3.16) we have either that the minimum is equal to $Q[v](y)$, in which case there is nothing to do, or the minimum is equal to $(1-\delta)Q[v](y) + \delta v(y)$, in which case $(1-\delta)Q[v](y) + \delta v(y) \leq Q[v](y)$, so $\delta\left[v(y) - Q[v](y)\right] \leq 0$, and thus $Q[v](y) \geq v(y)$ since $\delta > 0$. So the difference between $Q[v](y)$ and $\hat{Q}[v](y)$ gives

$$-\hat{Q}[v](y) + Q[v](y) = \delta\left[Q[v](y) - v(y)\right],$$

and since we know that $Q[v](y)$ and $v(y)$ are bounded between 0 and β, this difference tends to 0 uniformly on $\mathbb{R}$ as δ tends to 0. Then by applying the above argument with $\bar{Q}$ instead of Q we obtain as in ([40],[26]) that (3.14) is valid, and hence the theorem is established. □

Note that for the recursion (3.1) with non-decreasing initial data ϕ and operator Q, we have defined the left slowest spreading speed $\hat{c}$, which we call *the left slowest spreading speed.* On the other hand, for non-increasing initial data with the same operator Q, we have *the right slowest spreading speed*, that we denote by $\mathring{c}$, the existence of which follows straightforwardly the proof of the existence of the slowest spreading speed in [26, (2.4)] under the same assumptions on the operator Q. Note that Q is not assumed to satisfy the reflection property (1.8). The following theorem justifies the use of the terminology 'slowest spreading speed', because it shows that all components spread at least this speed, and at at least one component spreads at exactly this speed. This result characterizes the right slowest spreading speed $\mathring{c}$, and it is exactly the same as Theorem 2.1 in [26] with the notation $\mathring{c}$ instead of c^*.

Theorem 3.3. *Suppose that the initial function u_0 satisfies $u_0(x) = 0$ for all sufficiently large x, and that there are positive constants $0 < \rho \leq \sigma < 1$ such that $0 \leq u_0 \leq \sigma\beta$ for all x and $u_0 \geq \rho\beta$ for all sufficiently negative x. Then there exists an index i such that for any positive ϵ, the solution u_n of the recursion* (3.1) *has the properties*

$$\lim_{n\to\infty}\left[\sup_{x\geq n(\mathring{c}+\epsilon)} \{u_n\}_i(x)\right] = 0, \tag{3.17}$$

and

$$\lim_{n\to\infty}\left[\sup_{x\leq n(\mathring{c}-\epsilon)} \{\beta - u_n(x)\}\right] = 0. \tag{3.18}$$

That is, the ith component spreads at a speed no higher than $\mathring{c}$, and no component spreads at a lower speed.

Next, as an alternative approach for getting the left slowest spreading speed $\hat{c}$ for non-decreasing initial data ϕ with operator Q, we can consider non-increasing initial data $\tilde{\phi}$ defined by $\tilde{\phi}(x) = \phi(-x)$, with a new operator $\tilde{Q}$ which is defined by

$$\tilde{Q}[v](x) := Q[R[v]](-x) \quad \text{for all } x, \tag{3.19}$$

where $R[v](x) := v(-x)$ for all x. By applying [26, Theorem 2.1] with $\tilde{\phi}$ and $\tilde{Q}$, we can define a right slowest spreading speed $\tilde{c}$ by

$$\tilde{c} = \sup \{c : \tilde{a}(c, \infty) = \beta\}, \tag{3.20}$$

such that $\tilde{a}_n$ is the non-increasing sequence in n and defined by the recursion

$$\tilde{a}_{n+1}(c; s) = \max \left\{ \tilde{\phi}(s), \tilde{Q}[\tilde{a}_n(c; \cdot)](s + c) \right\}$$

where the initial condition is $\tilde{a}_0(c; s) = \tilde{\phi}(s)$ and $\tilde{\phi}$ satisfies the following properties

e_1'. $\tilde{\phi}(x)$ is non-increasing in x,

e_2'. $\tilde{\phi}(x) = 0 \quad$ for all $x \geq 0$

e_3'. $0 < \tilde{\phi}(-\infty) < \beta$.

The following lemma is an important tool and shows that $\tilde{Q}$ satisfies Hypotheses $q_1 - q_7$ provided that Q satisfies these hypotheses. This lemma will allow us to characterize the slowest spreading speed $\tilde{c}$ using [26, Theorem 2.1].

Lemma 3.4. *If the operator $Q[v]$ in the recursion* (3.1) *satisfies Hypotheses $q_1 - q_7$, then the operator $\tilde{Q}[v]$ does also.*

Proof. (1) We need to prove that the operator $\tilde{Q}$ is order-preserving, so we to show that if $v \geq u$, then $\tilde{Q}[v] \geq \tilde{Q}[u]$. By the definition (3.19), we have $\tilde{Q}[v] = Q[R(v)](-x)$, $\tilde{Q}[u] = Q[R(u)](-x)$, and if $v(x) \geq u(x)$ for all x, then $v(-x) \geq u(-x)$ and hence $R(v)(x) \geq R(u)(x)$ for all x. Since Q is an order-preserving operator by Hypothesis q_2, then $Q[R(v)](x) \geq Q[R(u)](x)$, so $Q[R(v)](-x) \geq Q[R(u)](-x)$, for all x. Therefore $\tilde{Q}[v] \geq \tilde{Q}[u]$, as required.

(2) We need to show that $\tilde{Q}[0] = 0$ and $\tilde{Q}[\beta] = \beta$. Since $Q[0](x) = 0$, so $Q[0](-x) = 0$ for all x. Then $Q[R(0)](x) = 0$ implies that $Q[R(0)](-x) = 0$ for all x. Thus $\tilde{Q}[0] = 0$ for all x. By following the same steps we can prove that $\tilde{Q}[\beta] = \beta$ for all x.

(3) We need to show that $\tilde{Q}[T_y[v]](x)$ is translation invariant operator. We have

$$\begin{aligned}\tilde{Q}[T_y[v]](x) &= Q[R[T_y[v]]](-x) = Q[T_{-y}[R(v)]](-x)\\ &= T_{-y}[Q[R(v)]](-x) = Q[R(v)](-x+y) = Q[R(v)](-(x-y))\\ &= \tilde{Q}[v](x-y) = T_y\tilde{Q}[v](x).\end{aligned}$$

Therefore $\tilde{Q}$ is a translation invariant operator.

(4) We need to show that $\tilde{Q}[v_n]$ converges uniformly on every bounded set to $\tilde{Q}[v]$ when v_n converges uniformly on every bounded sets to v as $n \to \infty$. Let $M > 0$. Then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N, x \in [-M, M]$, we have $|\tilde{Q}[v_n](x) - \tilde{Q}[v](x)| < \epsilon$ if and only if $|Q[R[v_n]](-x) - Q[R[v]](-x)| < \epsilon$ which implies that

$$|Q[R[v_n]](x) - Q[R[v]](x)| < \epsilon \qquad \text{for all } x \in [-M, M].$$

Now we need to prove that $R[v_n]$ converges uniformly on bounded sets to $R[v]$. Suppose that the sequence $v_n \in \psi_\beta$ is such that for each $\delta > 0$ and $M > 0$, there exists N such that for $n \geq N$ implies that $|v_n(x) - v(x)| < \delta$ for all $x \in [-M, M]$, which is equivalent to $|R[v_n](x) - R[v](x)| < \delta$ for all $x \in [-M, M]$. Therefore the sequence $R[v_n]$ converges uniformly on bounded sets to $R[v]$, and hence $Q[R[v_n]]$ converges uniformly on bounded sets to $Q[R[v]]$. Thus for all $\epsilon > 0$ and $M > 0$, there exists $\hat{N}$ such that $n \geq \hat{N}$ which implies that $|Q[R[v_n]](x) - Q[R[v]](x)| < \epsilon$ for all $x \in [-M, M]$ which yields $|Q[R[v_n]](-x) - Q[R[v]](-x)| < \epsilon$ for all $x \in [-M, M]$, which is equivalent to $|\tilde{Q}[v_n](x) - \tilde{Q}[v](x)| < \epsilon$ for all $x \in [-M, M]$. So the property is proved.

(5) We need to prove that given a sequence $(v_n)_{n\in\mathbb{N}}$, $v_n \in \psi_\beta$, there exists a subsequence $(v_{n_l})_{l\in\mathbb{N}}$ such that $\tilde{Q}[v_{n_l}]$ converges uniformly on each bounded set. Take a sequence $(v_n), n \in \mathbb{N}$ such that $(v_n) \in \psi_\beta$. Then $R(v_n)_{n\in\mathbb{N}}$ is a sequence in ψ_β. Thus there exists a subsequence $R(v_{n_l})_{n\in\mathbb{N}}$ such that $Q[R[v_{n_l}]]$ converges uniformly on each bounded set $[-M, M]$. So there exists a function $L \in \psi_\beta$ such that $L : \mathbb{R} \to \mathbb{R}^k$

such that for all $\delta > 0$ and $M > 0$, there exists N such that for $n \geq N$ implies that

$$|Q[R[v_{nl}]](x) - L(x)| < \delta \text{ if and only if } |Q[R[v_{nl}]](-x) - L(-x)| < \delta$$

for all $x \in [-M, M]$, which is equivalent to $|\tilde{Q}[v_{nl}](x) - L(x)| < \delta$ for all $x \in [-M, M]$.

(6) We need to characterize the matrix $\tilde{B}_{\tilde{\mu}}$ to prove Hypothesis $q_6((a),(b),(c))$ for the operator $\tilde{Q}$. Define the linear operator $\tilde{M}$ to be the linearization of the operator $\tilde{Q}$ at 0. Then to characterize $\tilde{M}$ in terms of M note that if for any $\epsilon > 0$ there is $\delta > 0$ such that if $\|u\|_\infty \leq \delta$, then $\|\tilde{Q}[u] - \tilde{M}[u]\|_\infty \leq \epsilon \|u\|_\infty$, then we have

$$\|\tilde{Q}[u] - \tilde{M}[u]\|_\infty = \sup_{x \in \mathbb{R}} |\tilde{Q}[u](x) - \tilde{M}[u](x)| = \sup_{x \in \mathbb{R}} |Q[R[u]](-x) - \tilde{M}[u](x)|.$$

Since $\|R[u]\|_\infty = \sup_{x \in \mathbb{R}} |R(u(x))| = \sup_{x \in \mathbb{R}} |u(-x)| = \sup_{-x \in \mathbb{R}} |u(-x)| = \|u\|_\infty$, we get

$$\begin{aligned}
\sup_{x \in \mathbb{R}} |Q[R[u]](-x) - \tilde{M}[u](x)| &= \sup_{-x \in \mathbb{R}} |Q[R[u]](-x) - \tilde{M}[u](x)| \\
&= \sup_{t \in \mathbb{R}} |Q[R[u]](t) - \tilde{M}[u](-t)| \\
&= \sup_{t \in \mathbb{R}} |Q[v](t) - \tilde{M}[R[v]](-t)|,
\end{aligned}$$

where $v = Ru$. Now $\|u\|_\infty = \|v\|_\infty < \delta$, and $u(x) = v(-x) = (Rv)(x)$, so $u = Rv$, and hence $\sup_{t \in \mathbb{R}} |Q[v](t) - \tilde{M}[R[v]](-t)| = \|Q[v] - R\hat{M}[R[v]]\|_\infty$.
Thus for all $\epsilon > 0$, there exists $\delta > 0$ such that if $\|v\|_\infty < \delta$, then

$$\|Q[v] - R\tilde{M}[R[v]]\|_\infty \leq \epsilon \|v\|_\infty,$$

and because of the uniqueness property for the Fréchet derivative for the linear operator, we conclude that $\tilde{M}[u] = RMR[u]$, and hence $\tilde{M} = RMR$. It follows that $\tilde{M}$ has the same representation for the linear operator M in (3.2) which allows us to introduce the matrix $\tilde{B}_{\tilde{\mu}}$ corresponding to the matrix B_μ in (3.4) by

$$\tilde{B}_{\tilde{\mu}}\alpha = \tilde{M}[\alpha e^{-\tilde{\mu}x}]|_{x=0} = RM[R[\alpha e^{-\tilde{\mu}x}]]|_{x=0} = RM[\alpha e^{\tilde{\mu}x}]|_{x=0} = M[\alpha e^{\tilde{\mu}x}]|_{x=0}. \tag{3.21}$$

Thus since B_μ is in Frobenius form by Hypothesis $q_6(a)$, then the matrix $\tilde{B}_{\tilde{\mu}}$ is in Frobenius form, and the Perron-Frobenius Theorem 3.1 says that for each block there is a principal eigenvalue $\tilde{\lambda}(\tilde{\mu})$ with a corresponding strictly positive principal eigenvector $\tilde{\zeta}(\tilde{\mu})$. It is clear that Hypothesis q_6 (c) is satisfied for $\tilde{B}_0$, and $B_0 = \tilde{B}_0$, from which it follows that Hypothesis q_6 $(a),(b),(c)$ hold for $\tilde{B}_{\tilde{\mu}}$ whenever it holds for B_μ.

(7) We need to prove that a family of bounded linear order-preserving operators on $\mathbb{R}^k$-valued functions, $\tilde{M}^{(\kappa)}$ satisfies the properties (i) and (ii) in Hypothesis q_7 for $\tilde{Q}$ and $\tilde{B}_{\tilde{\mu}}^{(\kappa)}$. Hypothesis q_7 (i) says that for every sufficiently large $\kappa > 0$, there is a constant vector $w > 0$ such that $Q[u](x) \geq M^{(\kappa)}[u](x)$ when $0 \leq u \leq w$. So from the definition of $\tilde{Q}$ in (3.19) and the information of $\tilde{M}$ we get that

$$Q[R[u]](-x) \geq M^{(\kappa)}[R[u]](-x) \quad \text{if and only if} \quad \tilde{Q}[u](x) \geq \tilde{M}^{(\kappa)}[u](x),$$

when $0 \leq u \leq w$, and Property (i) of q_7 holds for $\tilde{Q}$. Moreover, equation (3.21) implies Property (ii) of q_7, and thus Hypothesis $q_7(i),(ii)$ is satisfied for $\tilde{Q}$. This finishes the proof of the lemma. $\square$

We remark that in the notation of [26, (2.4)], c^* is the slowest spreading speed with operator Q and non-increasing initial data, which corresponds here to the right slowest-spreading speed $\tilde{c}$ with operator $\tilde{Q}$. Note also that, if the reflection property (1.8) is satisfied, then the earlier operator Q equals the operator $\tilde{Q}$, because for all x we have $Q[R[v]](x) = R[Q[v]](x) \;\Leftrightarrow\; Q[R[v]](x) = Q[v](-x)$ which implies that $Q[R[v]](-x) = Q[v](x) \;\Leftrightarrow\; \tilde{Q}[v](x) = Q[v](x)$.

The following theorem characterizes the right slowest spreading speed $\tilde{c}$. This result is the same as Theorem 3.3 with the notation $\tilde{c}$ and $\tilde{Q}$ instead of $\mathring{c}$ and Q. In general, $\tilde{c} \neq \mathring{c}$, but if the reflection invariance property (1.8) holds, then $\tilde{c} = \mathring{c}$ since $\tilde{Q} = Q$.

Theorem 3.4. *Suppose that the initial function v_0 satisfies $v_0(x) = 0$ for all sufficiently large x, and that there are positive constants $0 < \rho \leq \sigma < 1$ such that $0 \leq v_0 \leq \sigma\beta$ for all x and $v_0 \geq \rho\beta$ for all sufficiently negative x. Then there exists an index i such that for any positive ϵ, the solution v_n of the recursion $v_{n+1} = \tilde{Q}[v_n]$ has the properties*

$$\lim_{n\to\infty} \left[\sup_{x \geq n(\tilde{c}+\epsilon)} \{v_n\}_i(x) \right] = 0, \tag{3.22}$$

and

$$\lim_{n\to\infty} \left[\sup_{x \leq n(\tilde{c}-\epsilon)} \{\beta - v_n(x)\} \right] = 0. \tag{3.23}$$

That is, the ith component spreads at a speed no higher than $\tilde{c}$, and no component spreads at a lower speed.

Proof. This follows from [26, Theorem 2.1] with Q replaced by $\tilde{Q}$. □

Clearly there is a relationship between the left slowest spreading speed $\hat{c}$ which is defined in (3.12), and the right slowest spreading speed $\tilde{c}$ that is defined in (3.20). The following lemma explains this relationship.

Lemma 3.5. *The left slowest spreading speed $\hat{c}$ equals the value $-\tilde{c}$, where $\tilde{c}$ is the right slowest spreading speed.*

Proof. From the characterization properties (3.22), (3.23) for the left slowest spreading speed $\tilde{c}$ for v_n, we will first extract information for the sequence u_n that is obtained from non-decreasing initial data with the operator Q to obtain equivalent characterization properties for u_n.

Define $u_n(-x) := v_n(x)$ for all n, where v_n is the sequence as in Theorem 3.4. Then we have $v_0(x) = u_0(-x)$ and $u_{n+1}(-x) = v_{n+1}(x)$. So $u_{n+1}(-x) = Q[u_n](-x)$, and by the definition of $\tilde{Q}$ in (3.19) we have $\tilde{Q}[v_n](x) = Q[u_n](-x)$. By Lemma 3.4, the operator $\tilde{Q}$ satisfies $q_1 - q_7$. Since the initial function $v_0(x)$ satisfies the conditions of Theorem 3.4, we have

$$\lim_{n\to\infty} \left[\sup_{x \geq n(\tilde{c}+\epsilon)} \{v_n\}_i(x) \right] = 0.$$

Since we know that $\{v_n\}_i(x) = \{u_n\}_i(-x)$, we obtain

$$\sup_{x\geq n(\tilde{c}+\epsilon)} \{v_n\}_i(x) = \sup_{x\geq n(\tilde{c}+\epsilon)} \{u_n\}_i(-x) = \sup_{-x\leq -n(\tilde{c}+\epsilon)} \{u_n\}_i(-x) = \sup_{x\leq -n(\tilde{c}+\epsilon)} \{u_n\}_i(x).$$

Therefore we have

$$\lim_{n\to\infty} \left[\sup_{x\leq n(-\tilde{c}-\epsilon)} \{u_n\}_i(x) \right] = 0 \tag{3.24}$$

Statement (3.23) in Theorem 3.4 says that $\lim_{n\to\infty} \left[\sup_{x\leq n(\tilde{c}-\epsilon)} \{\beta - v_n(x)\} \right] = 0$. Then

$$\sup_{x\leq n(\tilde{c}-\epsilon)} \{\beta - v_n(x)\} = \sup_{x\leq n(\tilde{c}-\epsilon)} \{\beta - u_n(-x)\} = \sup_{-x\geq -n(\tilde{c}-\epsilon)} \{\beta - u_n(-x)\}$$

$$= \sup_{x\geq -n(\tilde{c}-\epsilon)} \{\beta - u_n(x)\}.$$

Thus we get

$$\lim_{n\to\infty} \left[\sup_{x\geq n(-\tilde{c}+\epsilon)} \{\beta - u_n(x)\} \right] = 0. \tag{3.25}$$

Now we complete the proof by comparing (3.13) with (3.25), and (3.14) with (3.24). First we compare between the statements (3.13) in Theorem 3.2 and (3.25). From equation (3.13) we know $\{u_n\}_j(n(\hat{c}-\epsilon)) \leq \sup_{x\leq n(\hat{c}-\epsilon)} \{u_n\}_j(x)$, so

$$\lim_{n\to\infty} \left[u_{n_j}(n(\hat{c}-\epsilon)) \right] = 0. \tag{3.26}$$

On the other hand, from equation (3.25) we have
$\{\beta - u_n(n(-\tilde{c}+\epsilon))\} \leq \sup_{x\geq (n(-\tilde{c}+\epsilon))} \{\beta - u_n(x)\}$, so $\lim_{n\to\infty} \{\beta - u_n(n(-\tilde{c}+\epsilon))\} = 0$.
Suppose that $\hat{c} > -\tilde{c}$. Then $-\tilde{c}+\epsilon = \hat{c}-\epsilon$, so (3.25) gives

$$\lim_{n\to\infty} \left[\sup_{x\geq n(\hat{c}-\epsilon)} \{\beta - u_n(x)\} \right] = 0, \tag{3.27}$$

from which it follows that $\lim_{n\to\infty} \{\beta - u_n(n(\hat{c}-\epsilon))\} = 0$, and hence

$$\lim_{n\to\infty} \{u_n\}_j(n(\hat{c}-\epsilon)) = \beta_j \neq 0 \quad \text{for each index } j \in \{1, ..., k\}.$$

This is a contradiction with (3.26), so $\hat{c} \leq -\tilde{c}$.

It can be shown similarly that $-\tilde{c} \leq \hat{c}$ by comparing (3.14) with (3.24). Thus we have $-\tilde{c} = \hat{c}$ and the lemma is proved. □

3.3 Fastest spreading speed for discrete-time recursion system (3.1)

It is shown in Lui [29, Theorem 3.1, 3.2] that there exists a single spreading speed when there are no extra equilibria other than 0 and β in ψ_β, in the sense that (3.13) in Theorem 3.2 in fact holds for all components of u_n, not only the jth component, which, together with (3.14), shows that all components spread at the same speed. Lui also assumed that there is only one diagonal block in B_μ. In biological terms, it clearly sometimes happens that different species spread at different rates. If the assumption of there being no extra equilibrium is dropped, then there may be an equilibrium ν other that 0 and β in ψ_β. This possibility of extra equilibria was first discussed in Weinberger, Lewis and Li [42], motivated by the fact that models of species interaction often have such extra equilibria. Our Hypothesis q_1 allows there to be more than just the equilibria 0 and β in ψ_β. Under these conditions, as already noted in [42], not all components of u_n necessarily spread at the same speed, and it is natural to introduce a second speed, called the left fastest spreading speed $\hat{c}_f$. Thus in general, there should be right (left) slowest spreading speed and right (left) fastest spreading speed. A *single right (left) spreading speed* means that the right (left) slowest-spreading speed of the recursion (3.1) equals the right (left) fastest spreading speed.

Corresponding to the left slowest spreading speed $\hat{c}$, we have the left fastest-spreading speed $\hat{c}_f$ and similarly for the right slowest spreading speed $\mathring{c}$, we will introduce the right fastest-spreading speed, $\mathring{c}_f$.

Now in order to define the left fastest spreading speed $\hat{c}_f$, we use a similar argument to that used previously with a_n by choosing a function ϕ that satisfies Properties $e_1 - e_3$

and letting $b_n(x)$ be the solution of the recursion

$$b_{n+1} = Q[b_n]$$

with $b_0(x) = \phi(x)$. Define a function

$$B(c; x) := \lim_{n\to\infty} \sup b_n(x + nc). \tag{3.28}$$

Since $b_n(x + nc)$ is a non-decreasing function in x and c for each n, hence $B(c; x)$ is a non-decreasing function of x and c also. The following lemma is an important tool for the definition of the left fastest spreading speed. The purpose of this lemma is to show that $B(c; -\infty) = \beta$ for sufficiently positive c, from which it follows that the set $\{c : B(c; -\infty) \neq 0\}$ is not empty. Note that the following result is a modification of [39, Lemma 5.3] which involves the sequence a_n, whereas here we adapt the argument of [39] to treat the sequence b_n, and b_n is now a non-decreasing function of x instead of a_n which is a non-increasing function of x and s. Note that we prove the following lemma with more details than the proof of the corresponding lemma [39, Lemma 5.3], and we use a similar argument to that used in the proof of Lemma 3.3.

Lemma 3.6. *$B(c; -\infty) = \beta$ if and only if there is an $n \in \mathbb{N}$ such that*

$$b_n(0 + nc) = b_n(nc) > \phi(\infty).$$

Proof. If $B(c; -\infty) = \beta$, then $B(c; s) = \beta$ for all s. In particular, for $s = 0$, $B(c; 0) = \beta > \phi(\infty)$. Thus by the definition of B in (3.28) we have $b_n(nc) > \phi(\infty)$ for some n sufficiently large.

On the other hand, suppose that $b_n(nc) > \phi(\infty)$ holds for $n = n_0$. Since b_{n_0} and ϕ are non-decreasing functions, $\phi = 0$ for all $s \leq 0$, and $\phi \leq \phi(\infty)$, it follows from the continuity of b_{n_0} and ϕ that there exists $\delta > 0$ such that

$$b_{n_0}(n_0 c + s - \delta) \geq \phi(s) = b_0(s) \qquad \text{for all } s \in \mathbb{R}.$$

Suppose that $b_{n_0+k}((n_0+k)c+s-\delta) \geq b_k(kc+s)$ is true for some $k \geq 0$. Then

$$\begin{aligned}
b_{n_0+k+1}((n_0+k+1)c+s-\delta) &= Q[b_{n_0+k}(\cdot)]((n_0+k+1)c+s-\delta) \\
&= Q[b_{n_0+k}((n_0+k)c+\cdot-\delta)](s+c) \geq Q[b_k(kc+\cdot)](s+c) \\
&= Q[b_k(\cdot)]((k+1)c+s) = b_{k+1}((k+1)c+s).
\end{aligned}$$

So, by induction we have shown that

$$b_{n_0+k+1}((n_0+k+1)c+s-\delta) \geq b_{k+1}((k+1)c+s) \quad \text{for all } k \geq 0.$$

For fixed s, letting $k \to \infty$ through a subsequence on each side we find

$$B(c; s-\delta) \geq B(c; s) \quad \text{for all } s.$$

But the function B is non-decreasing in s and $\delta > 0$, so $B(c; s-\delta) \leq B(c; s)$ for all s. It follows that $B(c; s-\delta) = B(c; s)$ for all s. Therefore $B(c; s)$ is constant because the only non-decreasing periodic functions are constant. Since $B(c; \infty) = \beta$, then $B(c; s) = \beta$ for all s. In particular,

$$B(c; -\infty) = \beta.$$

□

The fact that $B(c; \mp\infty)$ are equilibria of Q follows by using similar arguments to these in [29, Lemma 2.6], and Hypothesis q_1 then implies that $B(c; +\infty) = \beta$.

To show that $B(c; -\infty) = \beta$ for a sufficiently positive c, define $P_n(c; x) := b_n(x+nc)$, so we have

$$P_n(c; x) = b_n(x+nc) = Q[b_{n-1}(\cdot)](x+nc) = Q[b_{n-1}(\cdot+(n-1)c)](x+c) = Q[P_{n-1}(c; \cdot)](x+c).$$

It follows from [39, Lemma 5.2] that for n sufficiently large, $P_n(0, \infty) = \alpha_n$ where $\alpha_0 \equiv \phi(\infty)$ and $\alpha_{n+1} = Q[\alpha_n]$.

Since $0 < \alpha_0 = \phi(\infty) < \beta$, we know that $\alpha_n \to \beta$ as $n \to \infty$ and so for some $\tilde{n}$ sufficiently large, we have $P_{\tilde{n}}(0, \infty) > \phi(\infty)$. Hence for $t > 0$ sufficiently large and from the monotonicity of ϕ we have $P_{\tilde{n}}(0,t) > \phi(\infty) > \phi(t)$, thus $P_{\tilde{n}}(0,t) = Q[P_{\tilde{n}-1}(0,\cdot{+}t)](0)$. On the other hand

$$P_{\tilde{n}}(t,0) = Q[P_{\tilde{n}-1}(t,\cdot)](0+t) = Q[P_{\tilde{n}-1}(t,\cdot)](t),$$

by Hypothesis q_3, we have $P_{\tilde{n}}(t,0) = Q[P_{\tilde{n}-1}(t,\cdot{+}t)](0)$ and since $P_n(c;\cdot)$ is non-decreasing in c, thus

$$Q[P_{\tilde{n}-1}(t,\cdot+t)](0) \geq Q[P_{\tilde{n}-1}(0,\cdot+t)](0) = P_{\tilde{n}}(0,t) > \phi(\infty).$$

By Lemma 3.6 we thus have $B(t,-\infty) = \beta$, which implies that $B(c;-\infty) = \beta$ for all $c \leq t$.

To prove that the function $B(c;-\infty)$ is independent of the initial function ϕ, we use the same argument that was used for the function a by defining a new function ϕ^* and having a different sequence $P_n^*(c;\cdot)$, to obtain that $P^*(c;-\infty) = P(c;-\infty)$, which means that the vector $P(c;-\infty)$ does not depend on the initial function $\phi(x)$.
It follows that $B(c;-\infty)$ is independent of ϕ also and thus we define the left fastest spreading speed $\hat{c}_f$ by

$$\hat{c}_f := \inf\{c : B(c;-\infty) \neq 0\}. \tag{3.29}$$

Note that we define the left slowest spreading speed $\hat{c}$ in (3.12) to be the infimum of the set where for each c, $a(c;-\infty) = \beta$ such that $a_n(c;s)$ is the sequence that is defined in (3.10) with an initial condition that satisfies Properties $e_1 - e_3$, whereas we define the left fastest spreading speed in (3.29) to be the infimum of the set where for each c, $B(c;-\infty) \neq 0$ such that $B(c;x)$ is a function that is defined in (3.28). The reason for the definition of a_{n+1} in (3.10) to be the maximum of two objects is to ensure that we have a non-decreasing function in n, in the sense that $a_{n+1} \geq a_n$, which means that $\lim\limits_{n\to\infty} a_n$ exists.

We also present an alternative 'fastest' spreading speed $\dot{c}_f$ in (4.1), defined as the infimum of the set where for each c, $a(c;-\infty) \neq 0$ such that $a_n(c;s)$ is the sequence with an initial condition that satisfies Properties $e_1 - e_3$. Then $\dot{c}_f$ is a modification of the quantity that was introduced in [42, (2.9)] with non-decreasing initial data instead of non-increasing initial data. In fact the reason for presenting (3.29) instead of the alternative formula $\dot{c}_f$

is due to the fact that it was noted in [43] that $\dot{c}_f$ is not, in fact, a good definition for fastest spreading speed, because one of the characterization properties (3.30) and (3.31) does not necessarily hold with the definition (4.1) of the fastest spreading speed $\dot{c}_f$ using a_n, as presented in [42, (2.9)]. However, in [43, Theorem 4.1] an extra hypothesis is added under which for the formula $\dot{c}_f$ of the fastest spreading speed in [42], both characterization properties (3.30) and (3.31) hold. Note that $\dot{c}_f$ is also a convenient tool for the proof of Theorem 4.1, that characterizes spreading speeds in terms of travelling-wave speeds.

3.3.1 Characterization properties of the fastest spreading speed

The following theorem is to characterize the left fastest spreading speed $\hat{c}_f$ where the initial condition ϕ is a non-decreasing function. This result is a modification of [26, Theorem 2.2] where the initial condition is a non-increasing function. Only a brief justification was given in [26], and we prove it with full details here.

Theorem 3.5. *Suppose that the initial function u_0 satisfies $u_0(x) = 0$ for all sufficiently negative x, and that there are positive constants $0 < \rho \le \sigma < 1$ such that $0 \le u_0 \le \sigma\beta$ for all x and $u_0 \ge \rho\beta$ for all sufficiently positive x. Then there exists an index l such that for each positive ϵ, the solution u_n of the recursion* (3.1) *has the properties*

$$\limsup_{n\to\infty} \left[\inf_{x \ge n(\hat{c}_f+\epsilon)} \{u_n\}_l(x) \right] > 0, \tag{3.30}$$

and

$$\lim_{n\to\infty} \left[\sup_{x \le n(\hat{c}_f-\epsilon)} u_n(x) \right] = 0. \tag{3.31}$$

That is, the lth component spreads at a speed no higher than $\hat{c}_f$, and no component spreads at a lower speed.

Proof. Choose a function ϕ which has Properties $e_1 - e_3$ and satisfies $\phi \le u_0$. By the Comparison Lemma 3.2 we have $u_n(x) \ge b_n(x)$. Since b_n is a non-decreasing function in x, we have

$$\inf_{x\ge n(\hat{c}_f+\epsilon)} u_n(x) \ge \inf_{x\ge n(\hat{c}_f+\epsilon)} b_n(x) = b_n(n(\hat{c}_f+\epsilon)),$$

which implies that

$$\limsup_{n\to\infty}\left[\inf_{x\geq n(\hat{c}_f+\epsilon)} u_n(x)\right] \geq \limsup_{n\to\infty} b_n(n(\hat{c}_f+\epsilon)) = B(\hat{c}_f+\epsilon;0).$$

Since $\hat{c}_f < \hat{c}_f + \epsilon$ and by the definition of $\hat{c}_f$ in (3.29), it follows that

$$\limsup_{n\to\infty}\left[\inf_{x\geq n(\hat{c}_f+\epsilon)} u_n(x)\right] \geq B(\hat{c}_f+\epsilon;-\infty) \neq 0,$$

and thus there exists an index l such that $B_l(\hat{c}_f+\epsilon;-\infty) > 0$ and

$$\limsup_{n\to\infty}\left[\inf_{x\geq n(\hat{c}_f+\epsilon)} \{u_n\}_l(x)\right] > 0.$$

In order to prove the second statement (3.31), we choose a function ϕ that satisfies Properties $e_1 - e_3$ with an additional property that $\phi(x-\eta) \geq u_0(x)$ for some $\eta > 0$. The Comparison Lemma 3.2 implies that, $u_n(x) \leq b_n(x-\eta)$, and since b_n is non-decreasing function, we have

$$\sup_{x\leq n(\hat{c}_f-\epsilon)} u_n(x) \leq b_n\left(n(\hat{c}_f - \frac{1}{2}\epsilon) - \frac{1}{2}n\epsilon - \eta\right) \leq b_n\left(n(\hat{c}_f-\frac{1}{2}\epsilon)+\tau\right),$$

for $\tau = -\frac{1}{2}n\epsilon - \eta$ and a sufficiently large n. Thus we get that

$$\limsup_{n\to\infty}\left[\sup_{x\leq n(\hat{c}_f-\epsilon)} u_n(x)\right] \leq \limsup_{n\to\infty} b_n\left(n(\hat{c}_f-\frac{1}{2}\epsilon)+\tau\right),$$

for each $s \leq -\eta$, and since b_n is non-decreasing function in n, we have

$$\begin{aligned}\limsup_{n\to\infty} b_n\left(n(\hat{c}_f-\frac{1}{2}\epsilon) - \frac{n}{2}\epsilon - \eta\right) &\leq \limsup_{n\to\infty} b_n\left(n(\hat{c}_f-\frac{1}{2}\epsilon)+s\right)\\ &= B\left((\hat{c}_f-\frac{1}{2}\epsilon);s\right),\end{aligned}$$

so

$$\limsup_{n\to\infty} b_n\left(n(\hat{c}_f-\frac{1}{2}\epsilon) - \frac{n}{2}\epsilon - \eta\right) \leq \lim_{s\to-\infty} B\left(\hat{c}_f - \frac{1}{2}\epsilon;s\right) = B\left(\hat{c}_f-\frac{1}{2}\epsilon;-\infty\right).$$

Since we know that $\hat{c}_f - \frac{1}{2}\epsilon < \hat{c}_f$, then for each c, $\hat{c}_f - \frac{1}{2}\epsilon < \inf\{c : B(c; -\infty) \neq 0\}$, so $\hat{c}_f - \frac{1}{2}\epsilon \notin \{c : B(c; -\infty) \neq 0\}$ which implies $B(c; -\infty) = 0$. Therefore we obtain

$$\lim_{n\to\infty} \sup \left[\sup_{x \leq n(\hat{c}_f - \epsilon)} u_n(x) \right] = 0.$$

Then the theorem is proved. □

Note that in the proof of Theorem 3.5 we use a similar argument that used in the proof of Theorem 3.2 concerning the characterization properties for the left slowest spreading speed $\hat{c}$, but without assuming the additional properties (1), (2) for Q or defining a new operator $\bar{Q}$ that defined in (3.15).

Similarly to what we did earlier, we can define the right fastest spreading speed $\mathring{c}_f$, now using non-increasing initial data ϕ that satisfies Properties $e_1^{'} - e_3^{'}$ and the corresponding function B defined in (3.28) by

$$\mathring{c}_f := \sup\{c : B(c; \infty) \neq 0\}. \tag{3.32}$$

The following theorem characterizes the right fastest spreading speed $\mathring{c}_f$. This result is a straightforward consequence of Theorem 2.2 in [26] with the notation c_f^* replaced by $\mathring{c}_f$.

Theorem 3.6. *Suppose that the initial function u_0 satisfies $u_0(x) = 0$ for all sufficiently large x, and that there are positive constants $0 < \rho \leq \sigma < 1$ such that $0 \leq u_0 \leq \sigma\beta$ for all x and $u_0 \geq \rho\beta$ for all sufficiently negative x. Then there exists an index j such that for each positive ϵ, the solution u_n of the recursion* (3.1) *has the properties*

$$\lim_{n\to\infty} \sup \left[\inf_{x \leq n(\mathring{c}_f - \epsilon)} \{u_n\}_j (x) \right] > 0, \tag{3.33}$$

and

$$\lim_{n\to\infty} \left[\sup_{x \geq n(\mathring{c}_f + \epsilon)} u_n(x) \right] = 0. \tag{3.34}$$

That is, the jth component spreads at a speed no less that $\mathring{c}_f$, and no component spreads at a higher speed.

The following proposition shows that $\hat{c}_f \leq \hat{c}$. Note, as we mentioned before, that the following convention, we use the word 'speed' when it might be technically more accurate to use the word 'velocity' because it is possible to have either $c > 0$ or $c < 0$. Moreover, in the case of using non-decreasing initial data, we have $\hat{c}_f \leq \hat{c}$, which ensures that $\hat{c}_f$ is going faster to the left than $\hat{c}$, whereas, in the case of using non-increasing initial data, we have $\mathring{c}_f \geq \mathring{c}$, as shown in [42], which ensures $\mathring{c}_f$ is going faster to the right than $\mathring{c}$.

Proposition 3.2. *For non-decreasing initial data of the recursion* (3.1)*, the left fastest spreading speed $\hat{c}_f$ is less than or equal the left slowest spreading speed $\hat{c}$.*

Proof. The idea of the proof depends on comparing (3.14) in Theorem 3.2 and (3.31) in Theorem 3.5 and using a contradiction argument. Thus we compare

$$\lim_{n\to\infty}\left[\sup_{x\geq n(\hat{c}+\epsilon)}\{\beta - u_n(x)\}\right] = 0, \tag{3.35}$$

$$\lim_{n\to\infty}\left[\sup_{x\leq n(\hat{c}_f-\epsilon)} u_n(x)\right] = 0. \tag{3.36}$$

In (3.35) we have $\{\beta - u_n(n(\hat{c}+\epsilon))\} \leq \sup_{x\geq n(\hat{c}+\epsilon)}\{\beta - u_n(x)\}$. Thus as $n \to \infty$ we get

$$\lim_{n\to\infty}\{\beta - u_n(n(\hat{c}+\epsilon))\} = 0. \tag{3.37}$$

Suppose $\hat{c}_f > \hat{c}$ and let $\epsilon = \dfrac{\hat{c}_f - \hat{c}}{2}$. Then $\hat{c}_f - \epsilon = \hat{c} + \epsilon$, so (3.37) becomes

$$\lim_{n\to\infty}\{\beta - u_n(n(\hat{c}_f - \epsilon))\} = 0. \tag{3.38}$$

On the other hand, we know that $u_n(n(\hat{c}_f - \epsilon)) \leq \sup_{x\leq n(\hat{c}_f-\epsilon)} u_n(x)$, which implies from equation (3.36) that as $n \to \infty$, $\lim_{n\to\infty} u_n(n(\hat{c}_f - \epsilon)) = 0$, and this contradicts (3.38). Therefore $\hat{c}_f \leq \hat{c}$. □

Chapter 4

Characterization of slowest spreading speeds using travelling waves and linear determinacy for discrete-time systems

In this chapter we begin by presenting a theorem that shows that the left slowest spreading speed $\hat{c}$ defined in (3.12) can be characterized as the left slowest speed of a class of travelling waves c. We will then present sufficient conditions that ensure the recursion (3.1) is right (left) linearly determinate.

4.1 Characterization of slowest spreading speed $\hat{c}$ as slowest speed of a family of travelling waves

A travelling wave is a solution of the recursion $u_{n+1} = Q[u_n]$, $n \in \mathbb{N}$, which has the form

$$u_n(x) = w(x - nc) \qquad x \in \mathbb{R},$$

for some $c \in \mathbb{R}$. The value c is called the speed of the wave. The following lemma is an important tool for the next theorem, since it explains the relationship between the fastest spreading speed $\hat{c}_f$ that is defined in (3.29), and the alternative fastest speed $\dot{c}_f$ defined by

$$\dot{c}_f = \inf\{c; a(c, -\infty) \neq 0\}. \tag{4.1}$$

This quantity $\dot{c}_f$ is a modification of the fastest spreading speed that was introduced in [42, (2.9)] with non-decreasing initial data satisfying Properties $e_1 - e_3$ instead of non-increasing initial data.

Lemma 4.1. *Suppose that the operator Q satisfies Hypotheses 3.1. Then*

$$\dot{c}_f \leq \hat{c}_f.$$

Proof. The idea of the proof is to exploit the relationship between the definition of $\dot{c}_f$ in (4.1) and the definition of $\hat{c}_f$ in (3.29). Consider $v_n = b_n(\cdot + nc)$ with $v_0 = b_0 = a_0$. Then

$$v_{n+1}(s) = b_{n+1}(\cdot+(n+1)c)(s) = Q[b_n(\cdot+(n+1)c)](s) = Q[b_n(\cdot+nc)](s+c) = Q[v_n](s+c),$$

whereas from the definition of the sequence a_n in (3.10), we have

$$a_{n+1}(s) = \max\{\phi(s), Q[a_n(c,\cdot)](s+c)\} \geq Q[a_n(c,\cdot)](s+c).$$

Define an order-preserving operator $\mathfrak{Q}$ by $\mathfrak{Q}[w](s) := Q[w](s+c)$, so we have $v_{n+1}(s) = \mathfrak{Q}[v_n](s)$ and $a_{n+1}(s) \geq \mathfrak{Q}[a_n](s)$. The Comparison Lemma 3.2 with operator $\mathfrak{Q}$ then implies that $v_n \leq a_n$ for all n, and hence $b_n(\cdot + nc) \leq a_n(c,\cdot) \leq a(c,\cdot)$. Then by the definition of $B(c;x)$ in (3.28), namely $\lim_{n\to\infty} \sup b_n(x+nc) =: B(c;x)$, and since we know that $\lim_{n\to\infty} a_n(c,x) = a(c;x)$, it follows that $B(c,x) \leq a(c,x)$ for all x, from which it follows that if we have $B(c,-\infty) \neq 0$, then $a(c,-\infty) \neq 0$. Therefore, as sets, we have $\{c; B(c,-\infty) \neq 0\} \subset \{c; a(c,-\infty) \neq 0\}$, which yields that $\inf\{c; a(c,-\infty) \neq 0\} \leq \inf\{c; B(c,-\infty) \neq 0\}$, which is equivalent to saying that

$$\dot{c}_f \leq \hat{c}_f. \tag{4.2}$$

by the definitions of $\dot{c}_f$ in (4.1) and the definition of $\hat{c}_f$ in (3.29). $\square$

The following theorem relates the left slowest spreading speed to non-decreasing travelling waves and gives a condition that is sufficient to guarantee that the recursion has a single left spreading speed. This result is similar to [26, Theorem 3.1] but we adapt the proof to present the case when the profile w of the travelling wave solution $w(x-nc)$ is non-decreasing instead of the non-increasing.

It is useful for the next theorem to recall the definition of a lower semicontinuous real-valued function $f : I \to \mathbb{R}$, (see, for example, [21, p.89]), that is, for every real number α and x_0 such that $\alpha < f(x_0)$, there is a neighbourhood U of x_0 such that $\alpha < f(x)$ for all $x \in U$.

Theorem 4.1. *Suppose that the operator Q satisfies Hypotheses $q_1 - q_7$, and let $\hat{c}, \hat{c}_f$ be the left slowest and left fastest spreading speeds respectively. Then*

(i) If $c \leq \hat{c}$, there is a non-decreasing travelling wave solution $w(x-nc)$ of speed c with $w(\infty) = \beta$ and $w(-\infty)$ an equilibrium of Q other than β.

(ii) If there is a travelling wave $w(x-nc)$ with $w(\infty) = \beta$ such that

$$\liminf_{x \to -\infty} w_i(x) = 0 \qquad \textit{for at least one component } i, \tag{4.3}$$

then $c \leq \hat{c}$.

(iii) If (4.3) *holds for all components of w, then $c \leq \hat{c}_f$.*

(iv) If there are no constant equilibria of Q other than 0 and β in ψ_β, then $\hat{c} = \hat{c}_f$, which says that the recursion (3.1) *has a single left spreading speed.*

Proof. (i) Choose a fixed vector-valued initial function $\phi(s)$ with Properties $e_1 - e_3$. We can define a sequence $a_n(c,l,s)$ for each $l \geq 0$ by the recursion

$$a_{n+1}(c,l,s) = \max\{l\phi(s), Q[a_n(c,l,x)](s+c)\} \tag{4.4}$$

where $a_0(c,l,s) = l\phi(s)$. Since $a_n(c,l,s)$ is a non-decreasing function in n as well as in s and c, it follows that as $n \to \infty$, $\lim_{n\to\infty} a_n(c,l,s) = a(c,l,s)$, which is non-decreasing in c and s. This means that $a(c,l,s)$ is the limit of a non-decreasing family of continuous functions in n, so it is a lower semicontinuous function of c. Now when $c < \hat{c}$, it follows from (3.12) and [29, Lemma 2.6] that $a(c, l; -\infty)$ is a constant equilibrium other than β, and since $a(c, l; -\infty)$ is a lower semicontinuous function of c, that $a(\hat{c}, l; -\infty)$ is also a constant equilibrium other than β. Since $a(c, l, -\infty)$ does not depend on the initial function as shown before (p.47) when we define the slowest spreading speed $\hat{c}$ in (3.12), then $a(c, l, -\infty)$ is, in fact, independent of l.

(ii) By Hypothesis q_5, there is a sequence n_i such that $Q[a_{n_i}(c, l; \cdot + c)](y)$ converges uniformly for y on bounded sets. Since a_n is a non-decreasing sequence in n and Q is an order-preserving operator, the whole sequence $Q[a_n(c, l; x + c)](\cdot)$ converges uniformly on bounded sets. It shown in the proof of [26, Theorem 3.1] that we can take the limits $(n \to \infty)$ in (4.4) to get

$$a(c, l; s) = \max \{l\phi(s), Q[a(c, l; \cdot)](s + c)\}. \tag{4.5}$$

Since β is the only equilibrium in the interior of ψ_β, by Hypothesis q_1, we can choose $\eta > 0$ so small that there is no constant equilibrium other than β in the set $\{u \in \psi_\beta : |\beta - u| \ \leq \eta\}$. Since $0 < \eta < |\beta - \eta|$, there exists $\epsilon > 0$ such that $0 < \epsilon < \eta < |\beta - \eta| - \epsilon < |\beta - \eta|$,and since the continuous function $|\beta - a(c, l; s)|$ decreases from $|\beta - \nu| > \eta$ to 0, there exists M such that for $s \leq -M$, we have $|\beta - a(c, l; s)| \geq |\beta - \nu| - \epsilon$, and for $s \geq M$, $|\beta - a(c, l; s)| \leq \epsilon$.
The intermediate value theorem on $[-M, M]$ then says that there exists $L(l) \in [-M, M]$ such that $|\beta - a(c, l; L(l))| = \eta$.
Now by (4.5) and Hypothesis q_5, there is a sequence $l_i \to 0$ such that $a(c, l_i; \cdot + L(l_i))$ converges uniformly on bounded sets to a function $w(\cdot)$. Thus we can take limits in (4.5) by replacing l by l_i and s by $y + L(l_i) - (n+1)c$ and Hypothesis q_3 to get that

$$w(y - (n+1)c) = Q[w(\cdot - nc)](y), \qquad y \in \mathbb{R}. \tag{4.6}$$

Therefore, $u_n(x) = w(x - nc)$ is a travelling wave solution of the recursion (3.1), with $|\beta - a(c, l; L(l))| = |\beta - w(0)| = \eta$. Again following the same approach as in the proof of Theorem 3.1 in [26], we find that $\liminf_{x \to -\infty} w_i(x) = 0$.

(iii) Suppose that there is a wave $w(\cdot - nc)$ with $w(\infty) = \beta$. Choose a function $\phi(\cdot)$ with Properties $e_1 - e_3$ such that $\phi(x) \le w(x)$ for all x. If we define a sequence a_n such in (3.10) with $a_0(c;x) = \phi(x)$ and since $\phi(x) \le w(x)$, then $a_0(c;x) \le w(x)$. Suppose that $a_\kappa(c;\cdot) \le w(\cdot)$ is true for some $\kappa \ge 0$. Then $Q[a_\kappa(c;\cdot)](s) \le Q[w](s)$, since

$$Q[w(\cdot - nc)](s) = Q[w(\cdot)](s - nc) = w(s - (n+1)c) \quad \text{for all } s,$$

it follows that $Q[w(\cdot)](s) = w(s - c)$, which implies that $Q[a_\kappa(c;\cdot)](s + c) \le w(s - c + c) = w(s)$. Thus $a_{\kappa+1}(c;s) = \max\{\phi(s), Q[a_\kappa(c;\cdot)](s + c)\} \le w(s)$ for all s. It follows by induction that $a_\kappa(c;\cdot) \le w(\cdot)$ for all κ, which yields $a(c;x) \le w(x)$ for all x, and when $x \to -\infty$ we have $a(c, -\infty) \le \liminf_{x \to -\infty} w(x)$. Then from assumption (4.3) it follows that $a_i(c;-\infty) = 0$ for some i, which implies that $a(c;-\infty) \ne \beta$. The definition of $\hat{c}$ in (3.12) says that $\hat{c} = \inf\{c : a(c;-\infty) = \beta\}$, so the fact that $a(c;-\infty) \ne \beta$ implies that we must have $c \le \hat{c}$. However, if (4.3) holds for all components, we must have $c \le \dot{c}_f$, and by Lemma 4.1, we then obtain that

$$c \le \dot{c}_f \le \hat{c}_f. \tag{4.7}$$

(iv) If we have a travelling wave w with velocity c that satisfies (4.3) for all i, then we get $c \le \hat{c}_f$ by (4.7). On the other hand, we know from parts (i)-(iii) that for all $c \le \hat{c}$, there exists a travelling wave w with velocity c that satisfies (4.3) for some i, and then since the only equilibrium in ψ_β other than β is 0, we must have $w(-\infty) = 0$. In particular, there exists a travelling wave w with velocity $\hat{c}$ such that (4.3) holds for all i. Then we obtain that

$$\hat{c} \le \hat{c}_f. \tag{4.8}$$

But we already know from Proposition 3.2 that

$$\hat{c}_f \le \hat{c}. \tag{4.9}$$

Thus from (4.8) and (4.9) together yield that $\hat{c} = \hat{c}_f$. Thus the recursion (3.1) has single left spreading speed and the theorem is established. $\square$

4.2 Sufficient conditions for single speed and linear determinacy

The recursion (3.1) is said to be *linearly determinate* if the right (left) slowest spreading speed equals the right (left) fastest spreading speed, so there is a right (left) single spreading speed, and this right (left) single speed agrees with the speed that obtained from the recursion (3.1) when the operator Q replaced by its linearization M at 0.

We refer to a speed that is obtained from the linearization as a *linear value speed.* Since we have non-increasing and non-decreasing initial data, so we consider two kinds of linear determinacy, namely *the right linear determinacy* and *the left linear determinacy* corresponding to the two initial data respectively. It is presented in [42] that there are conditions on the recursion (3.1) where Q satisfies Hypotheses $q_1 - q_7$ with non-increasing initial data u_0, that ensure that the recursion has *right linear determinacy.* We can extract from this information on right linear determinacy sufficient conditions to ensure *left linear determinacy* for non-decreasing initial data with the same operator Q in (3.1).

A first tool to find sufficient conditions for right linear determinacy will be given in the following lemma. This result is exactly [42, lemma 3.1] where we refer the notation $\mathring{c}_f$ to be the right fastest spreading speed, $\mathring{c}$ to be the right slowest spreading speed and $\mathring{\bar{c}}$ is the right linear value for the recursion (3.1), which is defined in [42, (2.19)] as

$$\mathring{\bar{c}} := \inf_{\mu>0} \left\{\mu^{-1} \ln \lambda_1(\mu)\right\}, \tag{4.10}$$

where $\lambda_1(\mu)$ is the principal eigenvalue of the first diagonal block in B_μ defined in (3.3), this matrix B_μ being assumed by Hypothesis q_6 (a) to be in Frobenius form. This linear value $\mathring{\bar{c}}$ is denoted in [42] by *the slower speed.* There is also a *faster speed,* $\mathring{c}^+$, that is defined in [42, (2.20)] as

$$\mathring{c}^+ := \max_{\sigma} \left[\inf_{\mu>0} \{\mu^{-1} \ln \lambda_\sigma(\mu)\} \right], \tag{4.11}$$

where $\lambda_\sigma(\mu)$ is the principal eigenvalue of the σth diagonal block in B_μ, and we define $\bar{c}_\sigma$ by $\bar{c}_\sigma := \inf_{\mu>0} \{\mu^{-1} \ln \lambda_\sigma(\mu)\}$.

We prove the following lemma, which corresponds to [42, Lemma 2.2 (2.11)], giving more detail here. The proof depends on characterizing the spreading speed using the projection operator P_σ. We define P_σ by saying that $P_\sigma[v]$ has the same components as v in the directions corresponding to the σth diagonal block of the matrix B_0, whereas in the other directions the components are zero. Note that Hypothesis q_7 used to show that $c_\sigma \geq \bar{c}_\sigma$.

Lemma 4.2. *Suppose that the operator Q satisfies Hypotheses $q_1 - q_7$ and let $\mathring{c}, \mathring{c}_f$ be the right slowest and right fastest spreading speeds respectively. Then*

$$\mathring{c} \geq \mathring{\bar{c}} \quad \textit{and} \quad \mathring{c}_f \geq \mathring{c}^+. \tag{4.12}$$

Proof. Let l_σ be the dimension of the σth diagonal block of B_0. For any l_σ vector, $\omega(\cdot) \in \mathbb{R}^{l_\sigma}$, we define the vector-valued function $\bar{\omega} \in \mathbb{R}^k$ by saying that the components of $\bar{\omega}$ are those of the function ω, and its other components are zero. Then we define the auxiliary operator by

$Q_\sigma[\omega] :=$ the l_σ vector whose entries are those coordinates of $Q[\bar{\omega}]$ which correspond to the σth block.

It follows from [29, Theorem 3.5] that Q_σ has the right single speed c_σ and that Hypothesis q_7 implies that $c_\sigma \geq \bar{c}_\sigma$ for each σ, then by [42, Lemma 2.2 (2.11)] we have

$$0 = \lim_{n\to\infty} \left[\sup_{x \geq n(\mathring{c}+\epsilon)} P_1[u_n](x) \right] = \lim_{n\to\infty} \left[\sup_{x \geq n(\mathring{c}+\epsilon)} P_1 Q[u_{n-1}](x) \right],$$

where u_n satisfies (3.1). Since $u_{n-1} \geq 0$ and by the definition of P_1 we have $u_{n-1} \geq P_1[u_{n-1}]$. Then

$$P_1 Q[u_{n-1}](x) \geq P_1 Q[P_1[u_{n-1}]](x),$$

which implies that

$$0 = \lim_{n\to\infty}\left[\sup_{x\geq n(\mathring{c}+\epsilon)} P_1Q[u_{n-1}](x)\right] \geq \lim_{n\to\infty}\left[\sup_{x\geq n(\mathring{c}+\epsilon)} P_1Q[P_1[u_{n-1}](x)]\right] \geq 0.$$

By the definition of Q_σ, this implies that $0 = \lim_{n\to\infty}\left[\sup_{x\geq n(\mathring{c}+\epsilon)} Q_1[P_1[u_{n-1}]](x)\right]$. Hence $\mathring{c} \geq c_1 \leq \bar{c}_1 = \mathring{\bar{c}}$ where $c_1, \bar{c}_1$ are the speeds for the operator Q_1. Thus we have $\mathring{c} \geq \mathring{\bar{c}}$.

Now to show that $\mathring{c}_f \geq \mathring{c}^+$, from [42, Lemma 2.2 (2.10)] we have

$$0 = \lim_{n\to\infty}\left[\sup_{x\geq n(\mathring{c}_f+\epsilon)} u_n(x)\right] = \lim_{n\to\infty}\left[\sup_{x\geq n(\mathring{c}_f+\epsilon)} Q[u_{n-1}](x)\right] \geq \lim_{n\to\infty}\left[\sup_{x\geq n(\mathring{c}_f+\epsilon)} Q[P_\sigma[u_{n-1}]](x)\right] \geq 0.$$

Since $u_{n-1} \geq 0$, it follows that $u_{n-1} \geq P_\sigma[u_{n-1}]$. This yields

$$0 = \lim_{n\to\infty}\left[\sup_{x\geq n(\mathring{c}_f+\epsilon)} Q[P_\sigma[u_{n-1}]](x)\right].$$

By definition of Q_σ, for all $\epsilon > 0$, we can say that

$$0 = \lim_{n\to\infty}\left[\sup_{x\geq n(\mathring{c}_f+\epsilon)} Q_\sigma[P_\sigma[u_{n-1}]](x)\right].$$

Therefore

$$\mathring{c}_f \geq c_\sigma = \bar{c}_\sigma = \inf_{\mu>0}\mu^{-1}\ln\lambda_\sigma(\mu) \qquad \text{for each } \sigma,$$

and hence

$$\mathring{c}_f \geq \max_\sigma\left\{\inf_{\mu>0}\mu^{-1}\ln\lambda_\sigma(\mu)\right\} = \mathring{c}^+,$$

and the lemma is proved. □

The following theorem is [42, Theorem 3.1] and gives a simple condition under which the recursion (3.1) has right linear determinacy and we omit the proof (see [42], Remarks after Theorem 3.1 and Theorem 4.2). Note that to obtain this result we do not need the reflection invariance for the operator Q that was assumed in [42].

Theorem 4.2. *([42, Theorem 3.1]) Suppose that the operator Q satisfies Hypotheses $q_1 - q_7$ and the infimum in (4.10) is attained at $\bar{\mu} \in (0, \infty]$. Assume that either*

1. $\bar{\mu}$ is finite,

$$\lambda_1(\bar{\mu}) > \lambda_\sigma(\bar{\mu}) \quad \textit{for all } \sigma > 1 \quad \textit{and} \quad Q[\min\left\{e^{-\bar{\mu}x}\zeta(\bar{\mu}), \beta\right\}] \leq e^{-\bar{\mu}(x-\bar{c})}\zeta(\bar{\mu}),$$

or

2. There exists a sequence $\nu \to \bar{\mu}$ such that

$$\lambda_1(\mu_\nu) > \lambda_\sigma(\mu_\nu) \quad \textit{for all } \sigma > 1 \quad \textit{and} \quad Q[\min\{e^{\mu_\nu x}\zeta(\mu_\nu), \beta_\nu\}] \leq e^{-\mu_\nu(x-\bar{c}_\nu)}\zeta(\mu_\nu)].$$

Then

$$\mathring{c}_f = \mathring{c} = \mathring{\bar{c}} = \mathring{c}^+,$$

which means that (3.1) *has a single right speed and is right linearly determinate.*

Note that the condition $\lambda_1(\bar{\mu}) > \lambda_\sigma(\bar{\mu})$ for all $\sigma > 1$ is only used to prove that $\mathring{\bar{c}} = \mathring{c}^+$.

We re-apply Theorem 4.2 for right linear determinacy to obtain a result about sufficient conditions for left linear determinacy for the recursion (3.1) by using the operator $\tilde{Q}$ that is previously defined in (3.19). Let $\tilde{\lambda}_1(\tilde{\mu})$ denote the principal eigenvalue of the first diagonal block in $\tilde{B}_{\tilde{\mu}}$ defined in (3.21). Then we can define the right linear value for the recursion $u_{n+1} = \tilde{Q}[u_n]$ by

$$\tilde{\bar{c}} := \inf_{\tilde{\mu}>0}\left\{\tilde{\mu}^{-1}\ln\tilde{\lambda}_1(\tilde{\mu})\right\}, \tag{4.13}$$

and we define the speed $\tilde{c}^+$ by

$$\tilde{c}^+ := \max_\sigma\left[\inf_{\tilde{\mu}>0}\left\{\tilde{\mu}^{-1}\ln\tilde{\lambda}_\sigma(\tilde{\mu})\right\}\right], \tag{4.14}$$

where $\tilde{\lambda}_\sigma(\tilde{\mu})$ is the principal eigenvalue of the σth diagonal block in $\tilde{B}_{\tilde{\mu}}$. Then we will get a result that gives a condition under which the recursion $u_{n+1} = \tilde{Q}[u_n]$ has a single right speed and the recursion is right linearly determinate. Moreover, since we know $\tilde{c} \geq \tilde{\bar{c}}$ and Lemma 3.5 implies that, $\tilde{c} = -\hat{c}$, so we obtain that $-\tilde{c} \leq -\tilde{\bar{c}}$. Then we define the left linear value for Q and its linearization M with non-decreasing initial data by

$$\hat{\bar{c}} := -\inf_{\tilde{\mu}>0} \left\{ \tilde{\mu}^{-1} \ln \tilde{\lambda}_1(\tilde{\mu}) \right\}, \tag{4.15}$$

and it follows that

$$\hat{c}^+ := -\max_{\sigma} \left[\inf_{\tilde{\mu}>0} \left\{ \tilde{\mu}^{-1} \ln \tilde{\lambda}_\sigma(\tilde{\mu}) \right\} \right]. \tag{4.16}$$

Therefore, in the case when the initial condition for the recursion (3.1) is non-decreasing with the same operator Q, we get from Lemma 4.2 that $\hat{c} \leq \hat{\bar{c}}$ and $\hat{c}_f \leq \hat{c}^+$. From the definitions of $\tilde{Q}$ in (3.19), the assumption $\tilde{Q}[\min\left\{e^{-\tilde{\mu}x}\tilde{\zeta}(\bar{\tilde{\mu}}), \tilde{\beta}\right\}] \leq e^{-\tilde{\mu}(x-\hat{\bar{c}})}\tilde{\zeta}(\bar{\tilde{\mu}})$ is equivalent to assume that $Q[\min\left\{e^{\tilde{\mu}x}\tilde{\zeta}(\bar{\tilde{\mu}}), \tilde{\beta}\right\}] \leq e^{\tilde{\mu}(x-\hat{\bar{c}})}\tilde{\zeta}(\bar{\tilde{\mu}})$, and hence Theorem 4.3 below gives sufficient conditions for the left linear determinacy for the recursion (3.1).

Theorem 4.3. *Suppose that the operator Q satisfies Hypotheses $q_1 - q_7$ and the infimum in (4.15) is attained at $\bar{\tilde{\mu}} \in (0, \infty]$. Assume that either*

1. *$\bar{\tilde{\mu}}$ is finite,*

$$\tilde{\lambda}_1(\bar{\tilde{\mu}}) > \tilde{\lambda}_\sigma(\bar{\tilde{\mu}}) \quad \textit{for all}\, \sigma > 1 \quad \textit{and} \quad Q[\min\left\{e^{\tilde{\mu}x}\tilde{\zeta}(\bar{\tilde{\mu}}), \tilde{\beta}\right\}] \leq e^{\tilde{\mu}(x-\hat{\bar{c}})}\tilde{\zeta}(\bar{\tilde{\mu}}).$$

or

2. *There exists a sequence $\nu \to \bar{\tilde{\mu}}$ such that*

$$\tilde{\lambda}_1(\tilde{\mu}_\nu) > \tilde{\lambda}_\sigma(\tilde{\mu}_\nu) \quad \textit{for all}\, \sigma > 1 \quad \textit{and} \quad Q[\min\left\{e^{\tilde{\mu}_\nu x}\tilde{\zeta}(\bar{\tilde{\mu}}_\nu), \tilde{\beta}_\nu\right\}] \leq e^{\tilde{\mu}_\nu(x-\hat{\bar{c}}_\nu)}\tilde{\zeta}(\bar{\tilde{\mu}}_\nu).$$

Then

$$\hat{c}_f = \hat{c} = \hat{\bar{c}} = \hat{c}^+,$$

which means that the recursion (3.1) *has a single left speed and is left linearly determinate.*

Theorems 4.2, 4.3 will be used to establish results for the PDE systems in Chapter 5, and illustrative examples will be given in Chapter 7.

Chapter 5

Applications to reaction-diffusion-convection systems

In this chapter we discuss a continuous-time model that can be studied with the help of the recursion (3.1). We consider a system of partial differential equations (PDE), namely a co-operative system of reaction-diffusion-convection equations of the form

$$u_{i,t} + h_i^{'}(u_i)u_{i,x} = d_i u_{i,xx} + f_i(u) \qquad i = 1, 2, ..., k, \tag{5.1}$$

$$u(0, x) = u_0(x) \qquad \text{for all } x \in \mathbb{R},$$

where $d_i > 0$, the reaction terms $f_1, f_2, ..., f_k$ are independent of x and t and satisfy the co-operative assumption $\dfrac{\partial f_i}{\partial u_j}(u) \geq 0$, $i \neq j$, the convection functions $h_i^{'}(u_i)$ give the "diagonal" form of convection term diag $(h_1^{'}(u_1), h_2^{'}(u_2), ..., h_k^{'}(u_k))$, $u = (u_1, u_2, ..., u_k) \in \mathbb{R}^k$, and the initial condition $u_0 \in BUC^1(\mathbb{R}, \mathbb{R}^k)$ (the space of functions $p : \mathbb{R} \to \mathbb{R}^k$ such that p and $p^{'}$ are bounded and uniformly continuous on $\mathbb{R}$).

For $T > 0$, denote $\Gamma_T = \{u : \mathbb{R} \times [0, T] \to \mathbb{R}^k : u$ is bounded, continuous, u_t, u_x, u_{xx} exist and are continuous on $\mathbb{R} \times (0, T]\}$, and for $(x, t) \in \mathbb{R} \times (0, T]$ and $u \in \Gamma_T$, define

$$N(u)(x, t) := -u_t(x, t) + Au_{xx}(x, t) - h'(u)u_x(x, t) + f(u)(x, t),$$

where the reaction term $f : \mathbb{R}^k \to \mathbb{R}^k$, $A = \text{diag}(d_1, d_2, ..., d_k)$, and the convection term

$h'(u) := \mathrm{diag}(h_1'(u_1), h_2'(u_2), ..., h_k'(u_k))$. We say that the function $\bar{u} \in \Gamma_T$ is a *supersolution* of (5.1) if $N(\bar{u})(x,t) \leq 0$ for all $(x,t) \in \mathbb{R} \times (0,T]$, and the function $\underline{u} \in \Gamma_T$ is a *subsolution* of (5.1) if $N(\underline{u})(x,t) \geq 0$ for all $(x,t) \in \mathbb{R} \times (0,T]$.

The following theorem is an useful tool for system (5.1). Note that, of course, a reaction-diffusion-convection system does not, in general, possess a comparison principle. But the diagonal structure of $h'(u)$ and the co-operative assumption on f together ensure that such a principle does hold here.

Theorem 5.1. *(Comparison principle) Let the function $f \in C^1(\mathbb{R}^k, \mathbb{R}^k)$ satisfy $\dfrac{\partial f_i}{\partial u_j}(u) \geq 0$, $i \neq j$, and $\underline{u}, \bar{u} \in \Gamma_T$ be such that $\underline{u}, \bar{u}$ are continuous on $\mathbb{R} \times (0,T]$, $\underline{u}_x, \bar{u}_x$ are bounded and uniformly continuous on $\mathbb{R}$, and $N(\bar{u})(x,t) \leq 0$, and $N(\underline{u})(x,t) \geq 0$ for $(x,t) \in \mathbb{R} \times (0,T)$. Suppose that $\bar{u}(x,0) \geq \underline{u}(x,0)$ for all $x \in \mathbb{R}$. Then $\bar{u}(x,t) \geq \underline{u}(x,t)$ for all $(x,t) \in \mathbb{R} \times (0,T]$.*

Proof. Suppose that $\underline{u}$ is a subsolution and $\bar{u}$ is a supersolution for (5.1) for all $(x,t) \in \mathbb{R} \times (0,T]$, and we have

$$\underline{u}(x,0) \leq \bar{u}(x,0), \tag{5.2}$$

for all $x \in \mathbb{R}$. Then $\underline{u}_i$, $\bar{u}_i$ satisfy

$$\underline{u}_{i,t} \leq d_i \underline{u}_{i,xx} - h_i'(\underline{u}_i)\underline{u}_{i,x} + f_i(\underline{u}), \tag{5.3}$$

$$\bar{u}_{i,t} \geq d_i \bar{u}_{i,xx} - h_i'(\bar{u}_i)\bar{u}_{i,x} + f_i(\bar{u}), \tag{5.4}$$

for each $i \in \{1, 2, ..., k\}$, and we can re-write equation (5.4) as

$$-\bar{u}_{i,t} \leq -d_i \bar{u}_{i,xx} + h_i'(\bar{u}_i)\bar{u}_{i,x} - f_i(\bar{u}),$$

which together with equation (5.3) gives that

$$(\underline{u} - \bar{u})_{i,t} \leq d_i(\underline{u} - \bar{u})_{i,xx} - h_i'(\underline{u}_i)\underline{u}_{i,x} + h_i'(\bar{u}_i)\bar{u}_{i,x} + f_i(\underline{u}) - f_i(\bar{u}).$$

Define $w_i := (\underline{u}_i - \bar{u}_i)$. Then

$$w_{i,t} \leq d_i w_{i,xx} - h_i^{'}(\underline{u}_i)\underline{u}_{i,x} + h_i^{'}(\bar{u}_i)\bar{u}_{i,x} + f_i(\underline{u}) - f_i(\bar{u}). \tag{5.5}$$

In order to relate $f_i(\underline{u})$ and $f_i(\bar{u})$, we can define a function $\varphi_i(\theta)$ to be $\varphi_i(\theta) := f_i(\theta\underline{u} + (1-\theta)\bar{u})$, so that

$$\varphi_i^{'}(\theta) = \frac{\partial f_i}{\partial u}\left[\theta(\underline{u}) + (1-\theta)(\bar{u})\right](\underline{u}_i - \bar{u}_i) = \left(\frac{\partial f_i}{\partial u}\left[\theta(\underline{u}) + (1-\theta)(\bar{u})\right]\right) w_i.$$

Therefore

$$f_i(\underline{u}) - f_i(\bar{u}) = \varphi_i(1) - \varphi_i(0) = \left(\int_0^1 df_i[\theta(\underline{u}) + (1-\theta)(\bar{u})]\, d\theta\right) w_i =: \mathfrak{B}_i(x,t) w_i.$$

We can re-write the term $h_i^{'}(\bar{u}_i)\bar{u}_{i,x} - h_i^{'}(\underline{u}_i)\underline{u}_{i,x}$ as

$$h_i^{'}(\bar{u}_i)\bar{u}_{i,x} - h_i^{'}(\underline{u}_i)\underline{u}_{i,x} = \left[h_i^{'}(\bar{u}_i) - h_i^{'}(\underline{u}_i)\right]\bar{u}_{i,x} - h_i^{'}(\underline{u}_i)\left[\underline{u}_{i,x} - \bar{u}_{i,x}\right],$$

and by using the same procedure we have

$$\begin{aligned} h_i^{'}(\bar{u}_i) - h_i^{'}(\underline{u}_i) &= \int_0^1 h_i^{''}(\theta\bar{u}_i + (1-\theta)\underline{u}_i)(\bar{u}_i - \underline{u}_i)d\theta \\ &= \left(\int_0^1 h_i^{''}(\theta\bar{u}_i + (1-\theta)\underline{u}_i)\, d\theta\right) w_i =: \mathfrak{M}_i(x,t) w_i. \end{aligned}$$

Let $h_i^{'}(\underline{u}_i) =: \alpha_i(x,t)$ and $\mathfrak{N}_i(x,t) := \mathfrak{M}_i(x,t)\, \bar{u}_{i,x}$. Then (5.5) implies

$$w_{i,t} \leq d_i w_{i,xx} + \mathfrak{N}_i(x,t) w_i - \alpha_i(x,t) w_{i,x} + \mathfrak{B}_i(x,t) w_i. \tag{5.6}$$

If we let $\mathfrak{C}_i(x,t) = (\mathfrak{B}_i(x,t) + \mathfrak{N}_i(x,t))$, then (5.6) can be written as

$$w_{i,t} \leq d_i w_{i,xx} - \alpha_i(x,t) w_{i,x} + \mathfrak{C}_i(x,t) w_i,$$

for $i = 1, 2, ..., k$. Applying [38, Theorem 5.3] with the initial condition (5.2) gives us $w_i = \underline{u}_i - \bar{u}_i \leq 0$ which implies that $\underline{u}_i(x,t) \leq \bar{u}_i(x,t)$ for all $(x,t) \in \mathbb{R} \times (0,T]$. Then the result is proved. □

The following example illustrates that the Comparison Theorem 5.1 does not necessarily hold if the co-operative assumption is not satisfied. Note that a related example, that is presented in [13], illustrates that without the diagonality assumption on the convection term, which means that when u_x appears in the v-equation or v_x appears in the u-equation, a comparison result could fail.

Example 5.1. Consider the system below for $x \in (0,1),\ t \in (0,T)$,

$$\begin{aligned} u_t &= u_{xx} - h_1^{'}(u)u_x - v, \\ v_t &= v_{xx} - h_2^{'}(v)v_x. \end{aligned} \tag{5.7}$$

We seek candidate sub $\underline{u}, \underline{v}$ and super $\bar{u}, \bar{v}$ solutions, for $(x,t) \in (0,1) \times (0,T)$, for the system (5.7), in the form

$$\bar{u}(x,t) = 1 - t(x - x^2),\ \underline{u}(x,t) = \frac{1}{2},\ \bar{v}(x,t) = \varepsilon, \text{ and } \ \underline{v}(x,t) = -\frac{1}{2}x. \tag{5.8}$$

Then $\bar{u}$ satisfies

$$\begin{aligned} \bar{u}_t - \bar{u}_{xx} + h_1^{'}(\bar{u})\bar{u}_x + \bar{v} &= -(x - x^2) - 2t + h_1^{'}(\bar{u})(-t + 2tx) + \varepsilon \\ &\geq -\frac{1}{4} - 2t + h_1^{'}(\bar{u})t(2x - 1) + \varepsilon \geq 0, \end{aligned} \tag{5.9}$$

if $\varepsilon \geq \frac{1}{4} + 2t - h_1^{'}(\bar{u})t(2x-1) = \frac{1}{4} + t[2 - h_1^{'}(\bar{u})(2x-1)]$. Since for $x \in (0,1)$, the supremum value of $|2x - 1| = 1$, it follows that $2 - h_1^{'}(\bar{u})(2x-1) \leq 2 + |h_1^{'}(\bar{u})|$, and thus $t[2 - h_1^{'}(\bar{u})(2x-1)] \leq t[2 + |h_1^{'}(\bar{u})|]$, so $\bar{u}_t - \bar{u}_{xx} + h_1^{'}(\bar{u})\bar{u}_x + \bar{v} \geq 0$ provided $h_1(\bar{u})$ and ε are chosen such that $\varepsilon > \dfrac{1}{4} + t[2 + |h_1^{'}(\bar{u})|]$.
Now for $\underline{u}$, we clearly have

$$\underline{u}_t - \underline{u}_{xx} + h_1^{'}(\underline{u})\underline{u}_x + \underline{v} = -\frac{1}{2}x \ \leq 0 \quad \text{for all } x \in (0,1), t \in (0,T).$$

By following the same procedure for $\underline{v}$ and $\bar{v}$, we obtain

$$\bar{v}_t - \bar{v}_{xx} + h_2^{'}(\bar{v})\bar{v}_x = 0\ (\geq 0) \qquad \text{for all } (x,t),$$

and

$$\underline{v}_t - \underline{v}_{xx} + h_2'(\underline{v})\underline{v}_x = -\frac{1}{2}h_2'(\underline{v}) \ \leq 0 \quad \text{provided that } \ h_2'(\underline{v}) \geq 0.$$

If the Comparison Theorem 5.1 held, we would have $\bar{u}(x,t) \geq \underline{u}(x,t)$ and $\bar{v}(x,t) \geq \underline{v}(x,t)$ for $(x,t) \in (0,1) \times (0,T)$ whenever $(\underline{u}, \underline{v})$ and $(\bar{u}, \bar{v})$ are sub/super solutions on $(0,1) \times (0,T)$ respectively, so in particular

$$\bar{u}(x,t) - \underline{u}(x,t) \geq 0, \tag{5.10}$$

which says that $t(x-x^2) \leq \frac{1}{2}$. However, for $x = \frac{1}{2}$, (5.10) does not hold for $t > 2$, whereas for $t \in (0,3]$ we have

$$\frac{1}{4} + t[2 - h_1'(\bar{u})(2x-1)] \leq \frac{1}{4} + 3[2 + |h_1'(\bar{u})|],$$

for all $x \in (0,1)$. So for $\varepsilon = 8$, and assuming that h_1 is such that $|h_1'(\bar{u})| \leq 11/12$, then (5.9) holds, and hence $(\bar{u}, \bar{v})$ is a supersolution of the system (5.7) for all $(x,t) \in (0,1) \times (0,3]$, and provided we also have $h_2'(\underline{v}) \geq 0$, $(\underline{u}, \underline{v})$ is a subsolution of (5.7) on $(0,1) \times (0,3]$. Thus the Comparison Theorem 5.1 does not hold for system (5.7).

5.1 Hypotheses of the reaction-diffusion-convection systems

We assume in the following that the functions $f : \mathbb{R}^k \to \mathbb{R}^k$ and $h : \mathbb{R}^k \to \mathbb{R}^k$ in system (5.1) satisfy the hypotheses:

s_1: $\dfrac{\partial f_i}{\partial u_j} \geq 0$ for $i \neq j$;

s_2: $f(0) = 0$ there exists $\beta > 0$ such that $f(\beta) = 0$, and there is no $\nu > 0$ other than β such that $f(\nu) = 0$ and $0 < \nu \leq \beta$;

s_3: Neither f nor h depends explicitly on either x or t, and $d_i > 0$ is constant for $i = 1, 2, .., k \in \mathbb{R}$;

s_4: h has the diagonal form of convection terms diag $(h_i(u_i))$ for $i = 1, 2, ..., k \in \mathbb{R}$;

s_5: The functions f and h are continuously differentiable at α for each $0 \leq \alpha \leq \beta$;

s_6: The Jacobian matrix $f'(0)$ is in Frobenius form (see, Theorem 3.1) and such that the principal eigenvalue $\gamma_1(0)$ of the upper left diagonal block of $f'(0)$ is positive and is strictly larger than the principal eigenvalue of the other diagonal blocks, and there is at least one nonzero entry to the left of the each diagonal block other than the first one.

Remark 5.1. (i) Property s_1 says that the system 5.1 is co-operative, which ensures that it is order-preserving by the Comparison Theorem 5.1.

(ii) In the general case of Property s_6, for a system of k equations that satisfy Hypotheses $s_1 - s_5$, when there might be more than two diagonal blocks in $f'(0)$, [36, Theorem 2.1], which we quote in Chapter 6 as Theorem 6.3, ensures that the eigenvector of $f'(0)$ corresponding to the principal eigenvalue of the first block is strictly positive (we will discuss in Chapter 6, the strict positivity of the eigenvector of $f'(0)$ corresponding to the principal eigenvalue of its first block).

(iii) In most of our examples in Chapter 7, we consider a system of two equations. Thus $f'(0)$ has the form, $f'(0) = \begin{pmatrix} \alpha & \delta \\ \varrho & \sigma \end{pmatrix}$ where $\delta, \varrho \geq 0$. In the case when $f'(0) = \begin{pmatrix} \alpha & 0 \\ \varrho & \sigma \end{pmatrix}$ where $\alpha, \varrho > 0$ and $\alpha > \sigma$, it is easy to show, by an elementary calculation, that the eigenvector of $f'(0)$ corresponding to the principal eigenvalue α of the first block is strictly positive. Indeed, suppose the principal eigenvector of $f'(0)$ corresponding to the principal eigenvalue α is $z = (x, y)^T$. Then $\varrho x + \sigma y = \alpha y \Leftrightarrow \varrho x = (\alpha - \sigma)y$. Since $\alpha > \sigma$ and $\varrho > 0$ it follows that x and y have the same sign, and thus can be chosen so that $z = (x, y)^T$ is strictly positive.

5.2 Important results for the PDE system (5.1)

In this section we present results that are important tools for the PDE system (5.1) and will be used to show that the operator Q_t defined in (5.18) satisfies Hypotheses $q_1 - q_5$. The following proposition shows the existence of a unique solution of (5.1) and continuous dependence in $BUC^1(\mathbb{R}, \mathbb{R}^k)$ on the initial data u_0 for time t such that $0 \leq t < \tau(u_0)$. See [15, Proposition A.3] with $c = 0$ and $-h'(u)u_x + f(u)$ in place of $f(u, u_x)$.

Proposition 5.1. *Suppose that f and h satisfy Hypotheses $s_1 - s_6$ and the initial condition $u_0 \in BUC^1(\mathbb{R}, \mathbb{R}^k)$. Then there exists a maximal $\tau(u_0) \in (0, \infty]$ such that there exists a function $U^{u_0} \in C^1\left((0, \tau(u_0)), BUC^1(\mathbb{R}, \mathbb{R}^k)\right)$ such that u^{u_0} defined by $u^{u_0}(x,t) = U^{u_0}(t)(x)$ for each $x \in \mathbb{R}, t \in [0, \tau(u_0))$ satisfies (5.1) and its initial data. Moreover, there is a unique function $U^{u_0} : [0, \tau(u_0)) \to BUC^1(\mathbb{R}, \mathbb{R}^k)$ with these properties. In addition, given $0 < T < \tau(u_0)$, there exist $r, K > 0$, depending on u_0 and T, such that if $\tilde{u_0} \in BUC^1(\mathbb{R}, \mathbb{R}^k)$ is such that $\|u_0 - \tilde{u_0}\|_{1,\infty} < r$, then $\tau(\tilde{u_0}) \geq T$ and*

$$\|u^{u_0}(\cdot, t) - u^{\tilde{u_0}}(\cdot, t)\|_{1,\infty} \leq K\|u_0 - \tilde{u_0}\|_{1,\infty} \qquad \textit{for each } 0 \leq t \leq T. \tag{5.11}$$

The next proposition gives a condition under which a unique solution of (5.1) exists for all time t. See [15, Proposition A.4].

Proposition 5.2. *Suppose that that f and h satisfy Hypotheses $s_1 - s_6$. Let $u_0 \in BUC^1(\mathbb{R}, \mathbb{R}^k)$ be such that*

$$\sup_{0 \leq \tilde{t} < \tau(u_0)} \|u^{u_0}(\cdot, \tilde{t})\|_\infty = K < \infty, \tag{5.12}$$

where u^{u_0} and $\tau(u_0)$ are as in Proposition 5.1. Then $\tau(u_0) = \infty$.

The following theorem shows the existence of a unique solution of (5.1) for all time t provided the initial condition lies between the equilibria 0 and β, in which case the solution still lies between these values by the Comparison Theorem 5.1, which allows Proposition 5.2 to be applied. This result is a modification of [15, Theorem A.7].

Theorem 5.2. *Suppose that f and h satisfy Hypotheses $s_1 - s_6$. Then if $u_0 \in BUC^1(\mathbb{R}, \mathbb{R}^k)$ is such that*

$$0 \leq u_0 \leq \beta \qquad \textit{for all } x \in \mathbb{R}, \tag{5.13}$$

then $\tau(u_0) = \infty$, and

$$0 \leq u^{u_0}(x,t) \leq \beta \qquad \textit{for all } x \in \mathbb{R}, t \geq 0. \tag{5.14}$$

Proof. Let $u_0 \in BUC^1(\mathbb{R}, \mathbb{R}^k)$ satisfy (5.13). Since 0 and β are equilibria, if we have initial data $\underline{u}_0 \equiv 0$, the solution of (5.1) is $\underline{u}(x,t) \equiv 0$ and similarly, if we have an initial data $\bar{u}_0 \equiv \beta$, the solution of (5.1) is $\bar{u}(x,t) \equiv \beta$. Then the Comparison Theorem 5.1 implies that, if $0 \leq u_0 \leq \beta$, then

$$\underline{u}(x,t) \leq u(x,t) \leq \bar{u}(x,t), \tag{5.15}$$

where $u(x,t)$ is the solution of (5.1) with $u(x,0) = u_0(x)$. Thus $0 \leq u(x,t) \leq \beta$ for $x \in \mathbb{R}, 0 \leq t < \tau(u_0)$. Hence condition (5.12) in Proposition 5.2 is satisfied and the results follows by applying the Comparison Theorem 5.1. □

Note that in the following we will always assume that u_0 satisfies (5.13).

The following theorem states that for given initial data $u_0 \in \psi_\beta \cap B_{BUC^1}(0,R)$, where $B_{BUC^1}(0,R) = \{u \in BUC^1 : \|u\|_{1,\infty} < R\}$, there is a uniform bound for $\|u_x(\cdot,t)\|_\infty$. We will always assume in the following that the initial condition u_0 belong to a set $\psi_\beta \cap B_{BUC^1}(0,R)$ for some fixed $R > 0$, because we have convection terms and the initial condition u_0 lies between 0 and β, and we want to be sure that the derivatives u_{ix} are uniformly bounded all the way down to $t = 0$. This result follows from [25, Theorem 3.1, p.437], and we omit the proof.

Theorem 5.3. *If u is a solution of (5.1) with initial condition $u_0 \in \psi_\beta \cap B_{BUC^1}(0,R)$, then for given $R > 0$, there exists $M > 0$ such that*

$$\|u_x(\cdot,t)\|_\infty \leq M \qquad \text{for all}\, t \geq 0.$$

The following theorem shows the continuous dependence in a weighted norm for different solutions $u, \tilde{u}$ of the PDE system (5.1) corresponding to different initial conditions $u_0, \tilde{u}_0$ (see [30, p.263-265]). This controls differences between $u(\cdot,t)$ and $\tilde{u}(\cdot,t)$ in the supremum norm on bounded sets, which is needed to fit the PDE system into framework of Q. In particular, this theorem will be useful to prove Hypothesis q_4' in Lemma 5.2 below. We give the proof of Theorem 5.4 in the Appendix.

Theorem 5.4. *(Continuous Dependence Theorem) Given $R > 0$, $T > 0$, there exists $C > 0$ such that if $u, \tilde{u}$ are solutions of* (5.1) *with initial data $u_0, \tilde{u}_0 \in \psi_\beta \cap B_{BUC^1}(0, R)$, then for each $0 \le t \le T$,*

$$\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{\infty,\eta} + t^{1/2}\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{1,\infty,\eta} \le C\|u_0(\cdot,t) - \tilde{u}_0(\cdot,t)\|_{\infty,\eta}, \tag{5.16}$$

where

$$\eta(x) := \frac{1}{1+x^2}, \quad \|z\|_{\infty,\eta} := \sup_{x\in\mathbb{R}} |\eta(x)z(x)|, \quad \|z\|_{1,\infty,\eta} := \|z\|_{\infty,\eta} + \|z_x\|_{\infty,\eta}. \tag{5.17}$$

We also include the following lemma which shows that if the initial condition u_0 of the system (5.1) is non-increasing, then the solution of this system remains non-increasing.

Lemma 5.1. *If the initial condition $u_0 \in BUC^1(\mathbb{R}^k, \mathbb{R}^k)$ of the system* (5.1) *is non-increasing, then the corresponding solution $u(x,t)$ of* (5.1) *remains non-increasing in x for all t.*

Proof. The proof depends on the Comparison Theorem 5.1. Suppose that u_0 is the initial condition of (5.1) for the solution $u(x,t)$, and consider the initial condition $u_0(\cdot + \Delta x)$ where $\Delta x > 0$, so that, by translation invariance, the corresponding solution of (5.1) is $u(\cdot + \Delta x, t)$. Since u_0 is non-increasing and $\Delta x > 0$, then $u_0(\cdot + \Delta x) \le u_0(\cdot)$. The Comparison Theorem 5.1 implies that $u(\cdot + \Delta x, t) \le u(\cdot, t)$, and hence $u(\cdot + \Delta x, t) - u(\cdot, t) \le 0$. Since $\Delta x > 0$ was arbitrary, it follows that $u_x(\cdot, t) \le 0$ since $u_x(\cdot,t) = \lim\limits_{\Delta x\to 0} \dfrac{u(\cdot + \Delta x, t) - u(\cdot, t)}{\Delta x}$. □

5.2.1 Results for the abstract tool Q_t for the PDE system (5.1)

In the absence of the convection terms in (5.1), it is shown in [42] that such a PDE system can be related to (3.1) by taking Q to be its time-t map, that is,

$$Q_t[u_0](x) := u(x,t), \tag{5.18}$$

where $u(x,t)$ is the solution of the problem (5.1) at time $t > 0$ and the sequence of functions $u_n(x) := u(x, nt)$ satisfies the recursion (3.1) with Q replaced by Q_t. We note that Q_t, as noted in [26], satisfies the following semigroup properties

(g_1) $Q_{t_1}[Q_{t_2}[v]] = Q_{t_1+t_2}[v]$ for all positive t_1 and t_2

(g_2) $\lim_{t\to 0} Q_t[v] = v$,

in the sense that $\|Q_t[v] - v\|_\infty \to 0$ as $t \to 0$.

The following lemma shows that the operator Q_t defined in (5.18) satisfies Hypotheses $q_1 - q_3$, modified Hypotheses q_4', q_5', Hypothesis q_6, and modified Hypothesis q_7'. We will modify versions of Hypotheses q_4 and q_5, denoted by q_4', q_5', by assuming that the initial condition u_0 belongs to the set $\psi_\beta \cap B_{BUC^1}(0, R)$ for some fixed $R > 0$. This is because of the presence of the convection terms, as a result of which it is useful that the derivatives $u_{i,x}$ are uniformly bounded all the way down to $t = 0$. Later we will also prove a modified version of Hypothesis q_7, denoted by q_7', in Lemma 5.4, whereas in subsection 5.3.1 we discuss Hypothesis q_6 for Q_t. This result connects the discrete recursion (3.1) and the continuous-time system (5.1). The modified hypotheses q_4' and q_5' are:

q_4'. For a given sequence $\{v_n\}_{n\in\mathbb{N}} \subset \psi_\beta \cap B_{BUC^1}(0, R)$ and $v \in \psi_\beta \cap B_{BUC^1}(0, R)$ such that $\{v_n\}$ converges to v uniformly on every bounded set, then $Q_t[v_n]$ converges to $Q_t[v]$ uniformly on every bounded set.

q_5'. For a given sequence $\{v_n\}_{n\in\mathbb{N}} \subset \psi_\beta \cap B_{BUC^1}(0, R)$, there exists a subsequence $\{v_{n_l}\}_{n_l\in\mathbb{N}}$ such that $Q_t[v_{n_l}]$ converges uniformly on each bounded set.

Lemma 5.2. *The operator $Q_t[v]$ that is defined in (5.18) satisfies Hypotheses $q_1 - q_3$ and q_4', q_5'.*

Proof. (q_1) We need to prove that the operator Q_t is order-preserving, which means that if $v \geq u$, then $Q_t[v] \geq Q_t[u]$.

Since we have from the definition of Q_t that $Q_t[v] = v(x,t)$ and $Q_t[u] = u(x,t)$ for all $x, t \in \mathbb{R}$, then by the Comparison Theorem 5.1 we get $Q_t[v] \geq Q_t[u]$ for all $x, t \in \mathbb{R}$.

(q_2) Since $f(0) = f(\beta) = 0$, then $Q_t[0] = 0$ and $Q_t[\beta] = \beta$.

(q_3) We need to prove that the translation invariance property (1.7) holds for the operator Q_t. For the initial data u_0 there is a corresponding solution $u(x,t)$ of (5.1) and Proposition 5.1 says that this solution is unique. Suppose that $u_0(x) := v_0(x-y)$. From the definition of Q_t in (5.18) and the translation invariance in (1.7) we have that $Q_t[T_y[v]](x) = v(x-y,t)$. If we consider $v_0(x)$ as initial data, the function $v(x-y,t)$ is a solution of (5.18) which by Proposition 5.1 is the unique solution $v(x-y,t)$ of (5.1) with initial condition $v_0(x)$. Since $v(x-y,t) = u_0(x)$, then the solution that we have is a translation of $u(x,t)$ which means that we have got the solution $T_y[Q_t[v]](x)$ and it follows that $Q_t[T_y[v]](x) = T_y[Q_t[v]](x)$.

(q_4') Inequality (5.16) in Theorem 5.4 says that, for $C > 0$

$$\|Q_t[v_n](\cdot) - Q_t[v](\cdot)\|_{\infty,\eta} + t^{1/2}\|Q_t[v_n](\cdot) - Q_t[v](\cdot)\|_{1,\infty,\eta} \leq C\|v_n(\cdot) - v(\cdot)\|_{\infty,\eta},$$

and in particular,

$$\|Q_t[v_n](\cdot) - Q_t[v](\cdot)\|_{\infty,\eta} \leq C\|v_n(\cdot) - v(\cdot)\|_{\infty,\eta}. \tag{5.19}$$

Consider $[-L, L]$. Then we want to show that $Q_t[v_n] \to Q_t[v]$ uniformly on $[-L, L]$, which means that for a given $\epsilon > 0$, there exists N such that $n \geq N$ implies that

$$\sup_{x\in[-L,L]} |Q_t[v_n](x) - Q_t[v](x)| < \epsilon. \tag{5.20}$$

Since $\|Q_t[v_n](\cdot) - Q_t[v](\cdot)\|_{\infty,\eta} = \sup\limits_{x\in\mathbb{R}} \eta(x)|Q_t[v_n](x) - Q_t[v](x)|$, then

$$\begin{aligned}\sup_{x\in[-L,L]} |Q_t[v_n](x) - Q_t[v](x)| &= \sup_{x\in[-L,L]} (1+x^2)\frac{1}{1+x^2}|Q_t[v_n](x) - Q_t[v](x)| \\ &\leq (1+L^2)\sup_{x\in[-L,L]}\frac{1}{1+x^2}|Q_t[v_n](x) - Q_t[v](x)|.\end{aligned} \tag{5.21}$$

For $\delta > 0$, (5.19) and (5.21) give

$$\sup_{x\in[-L,L]} |Q_t[v_n](x) - Q_t[v](x)| \leq (1+L^2)\delta \quad \text{if} \quad \|v_n - v\|_{\infty,\eta} \leq \delta/C.$$

Therefore

$$\sup_{x\in[-L,L]} |Q_t[v_n](x) - Q_t[v](x)| \le \epsilon \quad \text{if} \quad \|v_n - v\|_{\infty,\eta} \le \epsilon/C(1+L^2).$$

Now since we know that for a given $B > 0$, $\sup\limits_{x\in[-B,B]} |v_n(x) - v(x)| \to 0$ as $n \to \infty$, and from Proposition 5.1 we have that if $|x| \ge B$, then $|\eta(x)v(x)| = \dfrac{1}{1+x^2}|v(x)| \le \dfrac{M}{1+x^2} \le \dfrac{M}{1+B^2}$ where M is a constant. We know that

$$\sup_{x\in\mathbb{R}}|\eta(x)\,(v_n(x) - v(x))|$$
$$= \max\left\{ \sup_{|x|\le B} \eta(x)|v_n(x) - v(x)|,\ \sup_{|x|\ge B} |\eta(x)(v_n(x) - v(x))| \right\},$$

where

$$\sup_{|x|\ge B} |\frac{1}{1+x^2}\,(v_n(x) - v(x))| \le \sup_{|x|\ge B} \frac{1}{1+x^2}\,(|v_n(x)| + |v(x)|) \le \sup_{|x|\ge B} \frac{2M}{1+x^2} \le \frac{2M}{1+B^2}.$$

Now choose N_0 sufficiently large such that for $n \ge N_0$,

$$\sup_{x\in[-B,B]} |\eta(x)\,(v_n(x) - v(x))| \le \frac{\epsilon}{C(1+L^2)}.$$

Then $\dfrac{2M}{1+B^2} \le \dfrac{\epsilon}{C(1+L^2)}$ if $B \ge \sqrt{\dfrac{2MC(1+L^2)}{\epsilon}}$. For fixed such B, and since we know that $|\eta(x)| \le 1$, we have that

$$\sup_{|x|\le B} \eta(x)|v_n(x) - v(x)| \le \sup_{|x|\le B} |v_n(x) - v(x)|,$$

and since $\sup\limits_{x\in[-B,B]} |v_n(x) - v(x)| \to 0$ as $n \to \infty$, there exists $\tilde{N}_0$ such that $\sup\limits_{x\in[-B,B]} |v_n(x) - v(x)| \le \dfrac{\epsilon}{C(1+L^2)}$ when $n \ge \tilde{N}_0$. Thus

$$\max\left\{ \sup_{|x|\le B} \eta(x)|v_n(x) - v(x)|,\ \sup_{|x|\ge B} |\eta(x)(v_n(x) - v(x))| \right\} \le \frac{\epsilon}{C(1+L^2)},$$

and hence $Q_t[v_n] \to Q_t[v]$ uniformly on $[-L, L]$.

(q_5') Define $z_n := Q_t[v_n]$. Then we want to show that $\{z_n\}_{n\in\mathbb{N}}$ is uniformly bounded and uniformly equicontinuous on each bounded set in $\mathbb{R}$.

Fix $t > 0$. Since 0 is an equilibrium, (5.11) in Proposition 5.1 implies that $\|(z_n)_x\|_\infty \leq M$, and $\|z_n\|_\infty \leq M$, for some constant M independent of $n \in \mathbb{N}$. So $\sup_{x\in\mathbb{R},n\in\mathbb{N}} |z_n(x)| \leq M$, and hence $\|Q_t[v_n]\|_\infty \leq C$ for a constant $C > 0$, for all $n \in \mathbb{N}$. So $Q_t[v_n]$ is uniformly bounded.

Moreover, since $Q_t[v_n] = u_n(\cdot, t)$ where $u_n(\cdot, 0) = v_n$. By the Mean-Value Theorem, there exists $c \in (x, y)$ such that

$$\|u_n(x,t) - u_n(y,t)\|_\infty \leq \|(u_n)_x(c,t)(x-y)\|_\infty \leq \|(u_n)_x(c,t)\|_\infty |x-y| \leq M|x-y|.$$

Therefore for $\epsilon > 0$, $|u_n(x,t) - u_n(y,t)| < \epsilon$ when $|x - y| < \dfrac{\epsilon}{M}$ for $x, y \in \mathbb{R}, t \geq 0$, which means that $Q_t[v_n]$ is uniformly equicontinuous on $[-L, L]$. Thus we have that $Q_t[v_n]$ is uniformly bounded and uniformly equicontinuous on $\mathbb{R}$. Then, for a given bounded set $[-L, L]$, $Q_t[v_n]$ is uniformly bounded and uniformly equicontinuous on $[-L, L]$, so Arzela-Ascoli's Theorem (see for instance [34, Theorem 2.5, p.49]) implies that there exists a subsequence $\{v_{n_l}\}_{n_l\in\mathbb{N}}$ such that $Q_t[v_{n_l}]$ converges uniformly on $[-L, L]$. By a diagonal subsequence argument, there exists a sub-subsequence $\{v_{n_{l_k}}\}_{n_{l_k}\in\mathbb{N}}$ such that $Q_t[v_{n_{l_k}}]$ converges uniformly on each bounded set. □

The following important theorem shows that the left slowest spreading speed of the time 1 map of the PDE system gives a spreading speed for solutions of the system (5.1) itself where the initial data is non-decreasing in the sense that (5.22) and (5.23) are satisfied. This result gives us important information about the continuous-time problem (5.1) using the discrete recursion (3.1). This theorem is a modification of [42, Theorem 4.1], with non-decreasing initial data instead of non-increasing and with the introduction of the convection term $h'(u)u_x$. Note that here we use Theorem 5.2 and Theorem 5.3 to ensure that $h'(u)u_x$ and $f(u)$ both are uniformly bounded.

Theorem 5.5. *Suppose that the function f satisfies Hypotheses $E_1 - E_3$ and let Q_t be the time t map in* (5.18). *If the left spreading speed corresponding to non-decreasing initial* *data $\hat{c}$ in (3.12) is defined to be $\hat{c}_1$, then the left spreading speed for Q_t map, $\hat{c}_t = t\hat{c}$, and for any initial function $u_0(x) \in \psi_\beta \cap BUC^1$, the solution u of (5.1) has the properties that for each $\epsilon > 0$*

$$\lim_{t\to\infty}\left[\max_{x\le t(\hat{c}-\epsilon)} u_j(x,t)\right] = 0 \qquad \textit{for some index } j, \tag{5.22}$$

$$\lim_{t\to\infty}\left[\max_{x\ge t(\hat{c}+\epsilon)} \{\beta - u(x,t)\}\right] = 0. \tag{5.23}$$

Proof. Since ψ_β is closed and bounded and we know by the Comparison Theorem 5.1 that $0 \le u(x,t) \le \beta$ for all (x,t), then u is bounded. Since f and $h'(\cdot)$ are continuous then $|f(u)| \le M$ and $|h'(u)| \le \tilde{M}$ for all $u \in \psi_\beta$. Since $u_0 \in BUC^1$, so Theorem 5.3 implies that $|u_x(x,t)|$ is bounded for all $x \in \mathbb{R}$, $t \ge 0$. It follows that $h'(u)u_x$ is uniformly bounded for $(x,t) \in [0,1] \times [0,T]$. Thus there exists $\rho > 0$ such that

$$|f(u) + h'(u)u_x| \le \rho \qquad \text{for } u \in \psi_\beta. \tag{5.24}$$

Let $\zeta(0)$ be a positive principal eigenvector of B_0. For any $\epsilon, \delta \ge 0$, there exists a large integer l such that

$$\rho/l \le (\delta/4)\zeta(0). \tag{5.25}$$

Now we are using the left spreading speed $\hat{c}$ for Q to be applied to the time 1 map Q_1, so Property 3.13 which holds for the index j, can be apply to the time 1 map Q_1 and the time $1/l$ map $Q_{1/l}$ of the system (5.18) gives $\hat{c}_{1/l} = \frac{\hat{c}_1}{l} := \frac{\hat{c}}{l}$. Property 3.13 for $Q_{1/l}$ with ϵ replaced by $\epsilon/2$ shows that there exists a number N_δ such that

$$u_j(y, n/l) \le (\delta/2)\zeta_j(0) \qquad \text{when } y \le n(\hat{c}/l - \frac{\epsilon}{2}) \quad \text{and } n \ge N_\delta. \tag{5.26}$$

Since we have (5.24), so we have that $u_{j,t} - d_j u_{j,xx} \le \rho$. We now compare this equation with the heat equation $v_{j,t} - d_j v_{j,xx} = \rho$. Since u_j is a subsolution of the heat equation with the same non-decreasing initial condition $(u_0)_j$, then the Comparison Theorem 5.1 gives $u_j \le v_j$. The standard formula of solution of the heat equation is the following, see

[5, Theorem 9.1,p.249].

$$v_j(x,t) = \int_{\mathbb{R}} \Gamma_j(x-y, t-n/l) u_{0_j}(y) dy + \rho \int_{n/l}^{t} \int_{\mathbb{R}} \Gamma_j(x-y, t-n/l) dy dl,$$

where $\Gamma_j(x,t) = \dfrac{1}{\sqrt{4\pi d_j t}} e^{-x^2/4d_j t}$. Suppose that for the index j, and a constant $R \in \mathbb{R}$, $u_{0_j}(y) \le \delta/2\zeta_j(0)$ if $|y| \le R$, whereas $u_{0_j}(y) \le \beta_j$ if $|y| > R$. Since we know that

$$\int_{\mathbb{R}} \Gamma_j(x-y,t) dy = \int_{\mathbb{R}} \Gamma_j(x-y, t-n/l) dy = 1,$$

thus for $(x,t) \in (0,1) \times (0,T)$, we have

$$\begin{aligned}
&u_j(x,t) \le \int_{\mathbb{R}} \Gamma_j(x-y,(t-n/l)) u_{0_j}(y) dy + \rho\,[t-n/l] \\
&\le \int_{|y|\le R} \Gamma_j(x-y,(t-n/l))(\delta/2)\zeta_j(0) dy + \int_{|y|>R} \Gamma_j(x-y,(t-n/l))\beta dy + \rho\,[t-n/l] \\
&\le (\delta/2)\zeta_j(0) \int_{|y|\le R} \Gamma_j(x-y,(t-n/l)) dy + \beta_j \int_{|y|>R} \Gamma_j(x-y,(t-n/l)) dy + \rho\,[t-n/l] \\
&\le (\delta/2)\zeta_j(0) + \beta_j \int_{|y|>R} \Gamma_J(x-y,(t-n/l)) dy + \rho\,[t-n/l]\,.
\end{aligned}$$

Now we want to evaluate the term $\int_{|y|>R} \Gamma_j(x-y,(t-n/l))dy$ by substituting the form of Γ_j to estimate $u_j(0,t)$ and then apply the shift to estimate $u_j(x,t)$ as follows,

$$\begin{aligned}
&\int_{|y|>R} \frac{1}{4\pi d_j(t-n/l)} \exp\left\{-(x-y)^2/4d_j(t-n/l)\right\} dy|_{x=0} \\
&\qquad\qquad = \int_{|y|>R} \frac{1}{4\pi d_j(t-n/l)} \exp\left\{-(y)^2/4d_j(t-n/l)\right\} dy,
\end{aligned}$$

that gives

$$\begin{aligned}
&\int_{-\infty}^{-R} \frac{1}{4\pi d_j(t-n/l)} \exp\left\{-(y)^2/4d_j(t-n/l)\right\} dy \\
&\qquad\qquad + \int_{R}^{\infty} \frac{1}{4\pi d_j(t-n/l)} \exp\left\{-(y)^2/4d_j(t-n/l)\right\} dy \\
&\qquad\qquad = 2\int_{R}^{\infty} \frac{1}{4\pi d_j(t-n/l)} \exp\left\{-(y)^2/4d_j(t-n/l)\right\} dy. \quad (5.27)
\end{aligned}$$

Since $\int_0^\infty e^{-ay^2}dy = \frac{1}{2}\sqrt{\frac{\pi}{a}}$, so if we let $b = y - R$, $a = \frac{1}{4d_j(t-n/l)}$, then since $(b+R)^2 = b^2 + 2bR + R^2 \geq b^2 + R^2$, we obtain that

$$\int_0^\infty e^{-a(b+R)^2}db \leq \int_0^\infty e^{-a(b^2+R^2)}db = e^{-aR^2}\int_0^\infty e^{-ab^2}db = e^{-aR^2}\cdot\frac{1}{2}\sqrt{\frac{\pi}{a}}.$$

Thus (5.27) becomes

$$\frac{1}{\sqrt{\pi d_j(t-n/l)}}\int_0^\infty \exp\left\{-\frac{1}{4d_j(t-n/l)}(b+R)^2\right\}db$$

$$\leq \frac{1}{\sqrt{\pi d_j(t-n/l)}}\left[\frac{1}{2}\exp\left(\frac{-R^2}{4d_j(t-n/l)}\right)\sqrt{\frac{\pi}{4d_j(t-n/l)}}\right]$$
$$= \left(\frac{1}{4d_j(t-n/l)}\right)\exp\left(\frac{-R^2}{4d_j(t-n/l)}\right).$$

This means that if $u_j(y,n/l) \leq (\delta/2)\zeta_j(0)$ for $y \leq R$ and $u_j(y,n/l) \leq \beta_j$ for all y, then for $0 \leq t - n/l \leq 1/l$,

$$u_j(0,t) \leq \rho\,[t-n/l] + (\delta/2)\zeta_j(0) + \beta_j\left[\frac{1}{4}\left(d_j(t-n/l)\right)^{-1}\exp\left(\frac{-R^2}{4d_j(t-n/l)}\right)\right]. \tag{5.28}$$

Then if we choose $R = R_\delta$ sufficiently large, we get that $\beta_j \exp\left(\frac{-R^2}{4d_j(t-n/l)}\right)$ is bounded by $(\delta/4)\zeta_j(0)$ when $0 \leq t - n/l \leq 1/l$. Thus (5.28) implies that

$$u_j(0,t) \leq \rho\,[t-n/l] + (\delta/2)\zeta_j(0) + (\delta/4)\zeta_j(0).$$

By (5.26) we get that for $x \leq t(\hat{c} - \frac{\epsilon}{2}) - R_\delta$,

$$u_j(x,t) \leq (\delta/4)\zeta_j(0) + (\delta/2)\zeta_j(0) + (\delta/4)\zeta_j(0) = \delta\zeta_j(0),$$

which means that $u_j(x,t)$ is bounded by $\delta\zeta_j(0)$ when

$$x \leq t(\hat{c} - \frac{\epsilon}{2}) - R_\delta, \tag{5.29}$$

$0 \leq t - n/l \leq 1/l$, and $n \geq N_\delta$. So (5.29) is implied by the inequality $x \leq t(\hat{c} - \epsilon)$, since

$x \leq t(\hat{c} - \epsilon)$, and if $t \geq \max\{N_\delta/l, 2R_\delta/\epsilon\}$, then it implies that $x \leq t(\hat{c} - \epsilon/2) - R_\delta$ if $t \geq 2R_\delta/\epsilon$. Since δ is arbitrary, for the index j we have

$$\lim_{t\to\infty} \left[\max_{x \leq t(\hat{c}-\epsilon)} u_j(x,t) \right] \leq \delta \zeta_j(0).$$

By applying the same procedure to the function $\beta - u(x,t)$, the theorem is proved. Note that in this theorem and in contrast to [42], we are allowing that $\hat{c} \neq \hat{c}_f$, since we apply the characterization property (3.13) for the left spreading speed that holds only for the index j but not for all the components of $\max_{x \leq t(\hat{c}-\epsilon)} u_j(x,t)$. □

The following theorem establishes the existence of travelling waves for the continuous-time recursion, introduced on [26, p.91], satisfying

$$u(x, t_1 + t_2) = Q_{t_2}[u(\cdot, t_1)](x). \tag{5.30}$$

This theorem extends Theorem 4.1 from the discrete-time recursion (3.1) to the continuous-time recursion (5.30). It is important for the following theorem to note that, in terms of Q_t, Theorem 5.2 implies that $Q_t[\psi_\beta \cap BUC^1] \subset \psi_\beta$. This result is a modification of [26, Theorem 4.1] for non-decreasing travelling wave solutions instead of non-increasing.

Theorem 5.6. *Suppose that Q_t is a family of operators defined on the set ψ_β that satisfy the semigroup properties (g_1) and (g_2) and such that Lemma 5.2 holds for Q_t for each $t > 0$. Let the left slowest spreading speed of the recursion (3.1) be $\hat{c}$ with Q replaced by Q_1. Then*

(i) If $c \leq \hat{c}$, there is a non-decreasing travelling wave solution $Q_t[w](x) = w(x - ct)$ of (5.30) of speed c with $w(\infty) = \beta$ and $w(-\infty)$ an equilibrium other than β.

(ii) If there is a travelling wave $w(x - ct)$ with $w(\infty) = \beta$ such that

$$\liminf_{x\to-\infty} w_i(x) = 0 \qquad \text{for at least one component } i, \tag{5.31}$$

then $c \leq \hat{c}$.

(iii) If (5.31) holds for all components of w, then $c \leq \hat{c}_f$, where $\hat{c}_f$ is defined in (3.29).

(iv) If there is $t_0 > 0$ such that the recursion (3.1) *with Q replaced by Q_{t_0} has no constant equilibria other than* 0 *and β in ψ_β defined in* (3.5)*, then the left spreading speed for the time t map Q_t, $\hat{c}[Q_t] = \hat{c}_f[Q_t] = t\hat{c}_f[Q_1]$ for all $t > 0$ which means that the recursion has a single left spreading speed.*

Proof. For the proof of the first three statements $(i) - (iii)$, we can follow the proof of the corresponding three statements in [26, Theorem 4.1].
(iv) We want to prove that $\hat{c}_f[Q_t] = \hat{c}[Q_t]$. We proved in Theorem 5.5 that $\hat{c}[Q_t] = t\hat{c}[Q_1]$, and by using the same argument for $\hat{c}_f[Q_t]$, we get that $\hat{c}_f[Q_t] = t\hat{c}_f[Q_1]$. Thus for t_0, we have $\hat{c}_f[Q_{t_0}] = t_0\hat{c}_f[Q_1]$ and $\hat{c}[Q_{t_0}] = t_0\hat{c}[Q_1]$. From Theorem 4.1 (iv) we have $t_0\hat{c}_f[Q_1] = t_0\hat{c}[Q_1]$, which implies that $\hat{c}_f[Q_1] = \hat{c}[Q_1]$. Since t is arbitrary, then we obtain that $\hat{c}_f[Q_t] = \hat{c}[Q_t]$ and the theorem is proved. □

Note that in subsection 5.3.2 we will present results for the PDE system (5.1) showing that the left slowest spreading speed $\hat{c}$ can be characterized in terms of a class of travelling waves, and give a condition that guarantees that (5.1) has single left speed.

5.3 Single speed and linear determinacy for the PDE system (5.1)

We can use Theorems 4.2, 4.3 from Chapter 4, that contain conditions for single right (left) spreading speed and right (left) linear determinacy for the discrete recursion (3.1), to deduce results for the continuous-time system (5.1). Again, we need to consider both right linear determinacy, corresponding to non-increasing initial data, and left linear determinacy, corresponding to non-decreasing initial data.

5.3.1 Linearization operator M for the PDE system (5.1)

The linearization operator M at 0 of the time 1 map Q_1 is the time 1 map of the linearized system (5.1) at 0, which is

$$u_{i,t} + h_i'(0)u_{i,x} = d_i u_{i,xx} + (f'(0)u)_i, \quad i = 1, 2, ...k. \tag{5.32}$$

In order to derive an explicit characterization of M, we seek a solution of (5.32) of the form $u(x,t) = e^{-\mu x}\eta(t)$, where $\mu \in \mathbb{R}$. For each i, we have

$$\eta_t e^{-\mu x} = \left(\mu^2 \operatorname{diag} d_i + \mu \operatorname{diag} h_i^{'}(0) + f^{'}(0)\right) e^{-\mu x}\eta(t),$$

thus

$$C_\mu = \mu^2 \operatorname{diag} d_i + \mu \operatorname{diag} h_i^{'}(0) + f^{'}(0). \tag{5.33}$$

By Property s_1, the off-diagonal entries of C_μ are nonnegative.

The vector-valued function η is a solution of the system of ordinary differential equation with constant coefficients C_μ that satisfies

$$\eta_t = C_\mu \eta \quad \text{with } \eta(0) = \alpha \in \mathbb{R}^k. \tag{5.34}$$

Since we have $\eta(t) = \exp(C_\mu t)\eta(0)$, see [4, p.169], then $u(x,t) = e^{-\mu x}\exp(C_\mu t)\eta(0)$, which implies that the time 1 map M is

$$M_1[e^{-\mu x}\alpha] = e^{-\mu x}\exp(C_\mu t)\alpha. \tag{5.35}$$

Then the using characterization of the matrix B_μ in (3.4) and (5.35) with $t = 1, x = 0$, provided that $\eta(0) = \alpha$, we find that

$$B_\mu \alpha = \exp\left[C_\mu\right]\alpha, \tag{5.36}$$

and hence $B_\mu = \exp[C_\mu]$. Since C_μ defined in (5.33), is in Frobenius form, so [37, p.86, Theorem 8.1, p.257] with an induction argument imply that $\exp[C_\mu] = B_\mu$ is in Frobenius form. Moreover, by [10, Theorem 2.52, p.168], we have $\lambda_\sigma(\mu) = e^{\gamma_\sigma(\mu)}$ where γ_σ denotes the principal eigenvalue of the σth block of the matrix C_μ defined in (5.33). Hence we deduce that Hypothesis q_6 holds for Q_t.

Corresponding to the definition of linear value in (4.10), we can define the linear value for the first block of the matrix C_μ, called the *right linear value*, by

$$\mathring{\bar{c}} := \inf_{\mu>0} \left\{ \frac{\gamma_1(\mu)}{\mu} \right\}. \tag{5.37}$$

Note that we use the notation $\bar{c}$ here, as in Chapter 2, because in the case when $f'(0)$ is actually irreducible (so has only one irreducible block in the Frobenius form), it is easy to see that this definition of $\bar{c}$ coincides with the natural extension to the system (5.1) of the definition of $\bar{c}$ in the scalar case discussed in Chapter 2, namely that the travelling-wave problem linearized about the unstable equilibrium 0 has a real negative eigenvalue corresponding to a strictly positive eigenvector if and only if $c \geq \bar{c}$. In fact, (5.37) yields an alternative characterization of $\bar{c}$ in the scalar case as well. When there are two or more blocks in the Frobenius form of $f'(0)$, which we will discuss in Section 6.2, we cannot conclude that these two definitions of $\bar{c}$ necessarily coincide.

In order to consider linear determinacy for *non-decreasing* initial data as well as *non-increasing* initial data, note next that if we define $\hat{u}(x,t) = u(-x,t)$ where u is a solution of system (5.1), then $\hat{u}$ is a solution of the system

$$\hat{u}_{i,t} - \hat{h}_i^{'}(\hat{u}_i)\hat{u}_{i,x} = d_i \hat{u}_{i,xx} + f_i(u_i) \tag{5.38}$$

for which the related coefficient matrix is

$$\hat{C}_{\hat{\mu}} = \hat{\mu}^2 \text{ diag } d_i - \hat{\mu} \text{ diag } \hat{h}_i^{'}(0) + f^{'}(0). \tag{5.39}$$

Clearly system (5.39) is obtained from system (5.1) simply by replacing h by $\hat{h} := -h$, where f and $\hat{h}$ satisfy Hypotheses $s_1 - s_6$ if and only if these hypotheses hold for f and h. So results for non-decreasing initial data of (5.1) can be deduced immediately from results on non-increasing initial data of (5.38).

For $\bar{\mu}$ which is defined to be the value of $\mu > 0$ at which the infimum in the definition of $\mathring{\bar{c}}$ in (5.37) is attained, the following lemma shows that in the case when we have a system of two equations, as in most of our examples, there is a sufficient condition for $\bar{\mu}$ to equal

$\hat{\bar{\mu}}$, the value at which the infimum in the definition of the linear value corresponding to $\hat{C}_{\hat{\mu}}$, that we called *left linear value*

$$\hat{\bar{c}} := \inf_{\hat{\mu}>0}\left\{\frac{\hat{\gamma}_1(\hat{\mu})}{\hat{\mu}}\right\}, \tag{5.40}$$

is attained, where $\hat{\gamma}_1$ is the principal eigenvalue of the first block of $\hat{C}_{\hat{\mu}}$. Note that for such $\bar{\mu}$, (5.37) and (5.40) become

$$\mathring{\bar{c}} = \frac{\gamma_1(\bar{\mu})}{\bar{\mu}}, \quad \hat{\bar{c}} = \frac{\hat{\gamma}_1(\hat{\bar{\mu}})}{\hat{\bar{\mu}}}. \tag{5.41}$$

Here and in the following, we denote by $\zeta(\bar{\mu})$ an eigenvector of C_μ corresponding to the eigenvalue $\gamma_1(\bar{\mu})$ and by $\hat{\zeta}(\hat{\bar{\mu}})$ an eigenvector of $\hat{C}_{\hat{\mu}}$ corresponding to the eigenvalue $\hat{\gamma}_1(\bar{\mu})$.

Lemma 5.3. *In the case that* (5.1) *is a system of two equations, if f and h satisfy Hypotheses $s_1 - s_6$, and* $f'(0) = \begin{pmatrix} \alpha & 0 \\ \varrho & \sigma \end{pmatrix}$ *where $\alpha, \varrho > 0$ and $\alpha > \sigma$, then (i) $\bar{\mu} = \hat{\bar{\mu}}$, and (ii) the eigenvector $\zeta(\bar{\mu})$ of C_μ corresponding to $\gamma_1(\bar{\mu})$ can be chosen equal to the eigenvector $\hat{\zeta}(\hat{\bar{\mu}})$ of $\hat{C}_{\hat{\mu}}$ corresponding to $\hat{\gamma}_1(\bar{\mu})$ if and only if $h_1'(0) = h_2'(0)$.*

Proof. Part (i) is immediate from the definitions of $\mathring{\bar{c}}$ and $\hat{\bar{c}}$. For (ii), let $h'(0) = \text{diag}(a, b)$ and $\zeta(\bar{\mu}) = \begin{pmatrix} 1 & \alpha_2 \end{pmatrix}^T$. Then equation (5.33) yields

$$C_\mu = \begin{pmatrix} d_1\mu^2 + a\mu + \alpha & 0 \\ \varrho & d_2\mu^2 + \mu b + \sigma \end{pmatrix}. \tag{5.42}$$

Let $E(\mu) := \dfrac{\gamma_1(\mu)}{\mu} = \dfrac{d_1\mu^2 + a\mu + \alpha}{\mu}$. Then $E'(\mu) = d_1 - \dfrac{\alpha}{\mu^2} = 0$ if and only if $\mu = \sqrt{\alpha/d_1}$. So $\bar{\mu} = \sqrt{\alpha/d_1}$, and therefore

$$C_{\bar{\mu}} = \begin{pmatrix} 2\alpha + a\sqrt{\alpha/d_1} & 0 \\ \varrho & d_2\dfrac{\alpha}{d_1} + b\sqrt{\alpha/d_1} + \sigma \end{pmatrix}. \tag{5.43}$$

In order to find the principal eigenvector $\zeta(\bar{\mu})$ corresponding to the principal eigenvalue

$\gamma_1(\bar{\mu})$, we have

$$\begin{pmatrix} 2\alpha + a\sqrt{\alpha/d_1} & 0 \\ \varrho & d_2\dfrac{\alpha}{d_1} + b\sqrt{\alpha/d_1} + \sigma \end{pmatrix} \begin{pmatrix} 1 \\ \alpha_2 \end{pmatrix} = \left(2\alpha + a\sqrt{\alpha/d_1}\right) \begin{pmatrix} 1 \\ \alpha_2 \end{pmatrix}, \quad (5.44)$$

which yields that

$$\alpha_2 = -\varrho / \left(\sqrt{\alpha/d_1}(b-a) + \alpha(\frac{d_2}{d_1} - 2) + \sigma\right).$$

On the other hand, since part (i) holds, then equation (5.39) yields

$$\hat{C}_\mu = \begin{pmatrix} d_1\mu^2 - a\mu + \alpha & 0 \\ \varrho & d_2\mu^2 - \mu b + \sigma \end{pmatrix}, \quad (5.45)$$

which implies that

$$\hat{C}_{\bar{\mu}} = \begin{pmatrix} 2\alpha - a\sqrt{\alpha/d_1} & 0 \\ \varrho & d_2\dfrac{\alpha}{dd_1} - b\sqrt{\alpha/d_1} + \sigma \end{pmatrix}.$$

The second component of the principal eigenvector $\hat{\zeta}(\bar{\mu}) = \begin{pmatrix} 1 & \hat{\alpha}_2 \end{pmatrix}^T$ corresponding to the principal eigenvalue $\hat{\gamma}_1(\bar{\mu})$ is

$$\hat{\alpha}_2 = -\varrho / \left(\sqrt{\alpha/d_1}(a-b) + \alpha(\frac{d_2}{d_1} - 2) + \sigma\right).$$

It is then clear that $\zeta(\bar{\mu}) = \hat{\zeta}(\bar{\mu})$ if and only if $a = b$. $\square$

For a system with only two equilibria 0, β with $\beta > 0$ and $f'(0)$ an irreducible matrix, Lui [29] gave sufficient conditions for spreading speeds to equal linear values, and these results were generalized by [42] to systems where the Frobenius form may have multiple diagonal blocks and there may be more equilibria other than 0 and β in $[0, \beta]$ provided any additional equilibrium ν has $\nu_i = 0$ for at least one $i \in 1, 2, ..., k$. Note that in Chapter 2, we discuss the case of a single equation, and have only two equilibria 0 and β, whereas in this chapter, we have a system, which, by Hypothesis s_2, may have equilibria in addition to 0 and β if they have at least one component equal to zero.

The following lemma proves that a modification of Hypothesis q_7, denoted by $q_7^{'}$, is satisfied by the time 1 map of the system (5.1). We modify Hypothesis q_7 to the case when the initial condition $v \in BUC^1$ of (5.1) is small enough not only in $\|\cdot\|_\infty$ but also in $\|\cdot\|_{1,\infty}$. The reason for this modification is to control $h^{'}(u)u_x$ in the proof of Lemma 5.4. Since the role of $q_7^{'}$ is to ensure that $c_\sigma \geq \bar{c}_\sigma$, it is sufficient to consider initial conditions that are small in $\|\cdot\|_{1,\infty}$ in order to estimate the spreading speed c_σ. Lemma 5.4 is a modification of [42, Lemma 4.1] to include the convection terms $h_i^{'}(u_i)u_{i,x}$. It is needed to ensure that the right spreading speed of the recursion (3.1) is bigger than or equal to the right linear value. The modified hypothesis $q_7^{'}$ is :

$q_7^{'}$. A family of bounded linear order-preserving operators on $\mathbb{R}^k$-valued functions $M^{(\kappa)}$ satisfies the following properties:

(i) For every large $\kappa \geq 0$ and $v : \mathbb{R} \to \mathbb{R}^k$, there is a constant vector $w > 0$ and $\delta > 0$ such that $Q[v] \geq M^\kappa[v]$ and $\|v\|_{1,\infty} < \delta$.

(ii) For every positive μ, the matrices $B_\mu^{(\kappa)}$ that can be characterized by $B_\mu^{(\kappa)}\alpha :=$ $M^\kappa[\exp(-\mu x)\alpha]|_{x=0}$ converge to B_μ as $\kappa \to \infty$.

Lemma 5.4. *If the functions f and h satisfy Hypotheses $s_1 - s_6$, then there exists a family of bounded linear order-preserving operators on $\mathbb{R}^k$-valued functions $M^{(\kappa)}$ which satisfies Hypothesis $q_7^{'}$.*

Proof. Choose $\rho \geq 0$ such that the diagonal elements of the matrix $f^{'}(0) + \rho I$ are strictly positive. Hypothesis s_1 then ensures that all the entries of this matrix are non-negative. For any $\kappa > 1$ and $\mu > 0$ we define $M_1^{(\kappa)}[v]$ to be the time one map of the linear system

$$w_{i,t} = \text{ diag } d_i w_{i,xx} - \text{diag } h_i^{'}(0)w_{i,x} + (1-\kappa^{-1})f^{'}(0)w_i - \kappa^{-1}w_i(\rho+1), \tag{5.46}$$

$$w(x,0) = v(x).$$

That is, $M^{(\kappa)}[v](x) := w(x,1)$. The idea of the proof is to show that for a sufficiently small initial condition v, we get $Q[v] \geq M^\kappa[v]$. As a tool, first consider the case when

v is instead given by $v = e^{-\mu x}\alpha$ with $\mu > 0$, v is non-increasing. The solution w of the system (5.1) is $w(x,t) = e^{-\mu x}\eta(t)$, where η satisfies that

$$\eta' = \left[\mu^2 \text{ diag } d_i + \mu \text{ diag } h_i'(0) + (1-\kappa^{-1})f'(0) - \kappa^{-1}(\rho+1)I\right]\eta(t).$$

For the initial condition $v = e^{-\mu x}\alpha$, and with $B_\mu^{(\kappa)}$ that is defined in Hypothesis q_7 (ii), we have that

$$B_\mu^{(\kappa)}\alpha = M^{(\kappa)}[e^{-\mu x}\alpha] = e^{-\mu x}\ e^{\mu^2 \text{ diag } d_i + \mu \text{ diag } h_i'(0) + (1-\kappa^{-1})f'(0) - \kappa^{-1}(\rho+1)I}\alpha,$$

and at $x=0$, $B_\mu^{(\kappa)} = e^{\mu^2 \text{ diag } d_i + \mu \text{ diag } h_i'(0) + (1-\kappa^{-1})f'(0) - k^{-1}I(\rho+1)}$. When $\kappa^{-1} \to 0$, the matrix $B_\mu^{(\kappa)}$ converges to the matrix B_μ as $\kappa \to \infty$. Thus we have proved Property (ii) of Hypothesis q_7'.

Now in order to establish Property i of q_7', we define for each i the projection

$$\{\Pi_i[\alpha]\}_j = \left[\begin{array}{lll} \alpha_j & if & \{f'(0)+\rho I\}_{ij} > 0 \\ 0 & if & \{f'(0)+\rho I\}_{ij} = 0. \end{array}\right.$$

Note that since $\rho \geq 0$, $\{\Pi_i[\alpha]\}_i = \alpha_i$, and that $\Pi_i[\alpha] \leq \alpha$ when $\alpha \geq 0$. Hypothesis s_1 ensures that

$$f_i(\alpha) \geq f_i(\Pi_i[\alpha]). \tag{5.47}$$

Moreover,

$$\Pi_i[\alpha]\cdot\nabla f_i(0) = \sum_{j=1}^{n}\{\Pi_i[\alpha]\}_j \cdot \frac{\partial f_i}{\partial u_j}(0), \tag{5.48}$$

and since $\{\Pi_i[\alpha]\}_j = \alpha_j$ if $\{f'(0)+\rho I\}_{ij} > 0$, then (5.48) becomes

$$\Pi_i[\alpha]\cdot\nabla f_i(0) = \sum_{j=1}^{n}\alpha_j \cdot \frac{\partial f_i}{\partial u_j}(0) = \left(f'(0)\alpha\right)_i \qquad \text{for all } \alpha. \tag{5.49}$$

Suppose that σ is a positive lower bound for the strictly positive terms $\{f'(0)+\rho I\}_{ij} > 0$, that is $0 < \sigma \leq \{f'(0)+\rho I\}_{ij}$, which implies that $1 \leq \sigma^{-1}\{f'(0)+\rho I\}_{ij}$. Since

$|\Pi_i[\alpha]| = \left(\sum_{j=1}^n \{\Pi_i[\alpha]\}_j^2\right)^{1/2} \leq \sum_{j=1}^n \{\Pi_i[\alpha]\}_j$, and

$$\sum_{j=1}^n \{\Pi_i[\alpha]\}_j \leq \sigma^{-1}\left\{\left(f'(0)+\rho I\right)\alpha\right\}_i = \sigma^{-1}\sum_{j=1}^n \left(f'(0)+\rho I\right)_{ij}\alpha_j,$$

so for all $\alpha \geq 0$ we have

$$|\Pi_i[\alpha]| \leq \sum_{j=1}^n \{\Pi_i[\alpha]\}_j \leq \sigma^{-1}\sum_{j=1}^n \left(f'(0)+\rho I\right)_{ij}\alpha_j = \sigma^{-1}\left[\Pi_i[\alpha]\cdot\nabla f'(0)+\rho\alpha_i\right], \tag{5.50}$$

If we let $\zeta(0) > 0$ be the eigenvector of B_0 with $\|\delta(0)\|_1 = 1$, there is $\delta_\kappa \geq 0$ such that if $0 \leq \alpha \leq \delta_\kappa\zeta(0)$, then for all i, the differentiability of f_i at 0 shows that for given $\kappa \geq 1$ and for $\epsilon = \sigma/\kappa$, there exists $\delta \geq 0$ such that if $|\alpha| \leq \delta$, then $|\nabla f_i(0)\cdot\Pi_i[\alpha] - f_i(\Pi_i[\alpha])| \leq (\sigma/\kappa)|\Pi_i[\alpha]|$. By substituting (5.47), (5.49) and (5.50) into this inequality, we get that

$$(f'(0)\alpha)_i - (1/\kappa)\left[(f'(0)\alpha)_i + \rho\alpha_i\right] \leq f(\alpha), \tag{5.51}$$

when $0 \leq \alpha \leq \delta_\kappa\zeta(0)$. Now we observe that the solution of the system (5.46) with initial condition $v = \delta_\kappa e^{-\gamma_1(0)}\zeta(0)$ is $\delta_\kappa e^{-\gamma_1(0)}e^{[(1-\kappa^{-1})\gamma_1(0)-\kappa^{-1}\rho]t}\zeta(0)$ and for $\gamma_1(0) > 0$, κ large, we have $(1-\kappa^{-1})\gamma_1(0) - \kappa^{-1}\rho > 0$. Therefore for $0 \leq t \leq 1$ we have

$$\begin{aligned}\delta_\kappa e^{-\gamma_1(0)}e^{[(1-\kappa^{-1})\gamma_1(0)-\kappa^{-1}\rho]t}\zeta(0) &\leq \delta_\kappa e^{-\gamma_1(0)}e^{[(1-\kappa^{-1})\gamma_1(0)-\kappa^{-1}\rho]}\zeta(0)\\ &= \delta_\kappa e^{-\kappa^{-1}(\gamma_1(0)+\rho)}\zeta(0) \leq \delta_\kappa\zeta(0).\end{aligned}$$

Thus if we take the initial data $\|v\|_{1,\infty}$ small enough such that $0 \leq v \leq \delta_\kappa e^{-\gamma_1(0)}\zeta(0)$, then the corresponding solution w_i of (5.46) satisfies $0 \leq w_i \leq \delta_\kappa\zeta(0)$ for $0 \leq t \leq 1$. Then we show that w_i is a subsolution of the system (5.1), as follows. Since we know that $w_{i,t} = \text{ diag } d_i w_{i,xx} - \text{diag } h_i'(0)w_{i,x} + (1-\kappa^{-1})f'(0)w_i - \kappa^{-1}w_i(\rho+1)$, which implies that $w_{i,t} \leq \text{ diag } d_i w_{i,xx} - \text{diag } h_i'(w_i)w_{i,x} + f_i(w)$ holds if and only if

$$-\text{ diag } h_i'(0)w_{i,x} + (1-\kappa^{-1})\left(f'(0)w\right)_i - \kappa^{-1}w_i(\rho+1) \leq -\text{diag } h_i'(w_i)w_{i,x} + f_i(w). \tag{5.52}$$

Since $0 \leq w_i \leq \delta_\kappa\zeta(0)$ and (5.51) holds for $0 \leq \alpha \leq \delta_\kappa\zeta(0)$, so taking $\alpha = w_i$ in (5.51), (5.52) becomes

$$\left(\text{diag } h_i'(w_i) - \text{diag } h_i'(0)\right) w_{i,x} + (1-\kappa^{-1})\left(f'(0)w\right)_i - \kappa^{-1}(\rho w)_i - f_i(w) - \kappa^{-1}w_i \le 0.$$

Now the Mean Value Theorem implies that $h_i'(w_i) - h_i'(0) = h_i''(\xi)w_i$, therefore $|h_i'(w_i) - h_i'(0)| \le M|w_i|$, where M is a constant such that $|h_i''(\xi)| \le M$ for $\xi \in [0, \beta_i]$. Moreover, by Proposition 5.1 with $\tilde{u}_0 \equiv 0$ and $T = 2$, there exists K such that (5.11) implies that

$$\|w_i(\cdot, t)\|_{1,\infty} \le K\|v\|_{1,\infty} \qquad \text{for each } 0 \le t \le 2,$$

from which it follows that $\|w_i\|_{1,\infty} \le K\|v\|_{1,\infty} \le \dfrac{\kappa^{-1}}{M}$ if $\|v\|_{1,\infty} \le \dfrac{\kappa^{-1}}{KM}$, and hence for all $t \in [0,1]$, $\|w_{i,x}\|_\infty \le \dfrac{\kappa^{-1}}{M}$. Thus we can choose δ_κ smaller if necessary to ensure that if $\|v\|_{1,\infty} \le \delta_\kappa$, then $|\left(h_i'(0) - h_i'(w_i)\right) w_{i,x}| \le M|w_i| \cdot \kappa^{-1}/M = \kappa^{-1}w_i$ for all $t \in [0,1]$. Therefore

$$\begin{aligned}
&\left(\text{diag } h_i'(w) - \text{diag } h_i'(0)\right) w_{i,x} + (1-\kappa^{-1})\left(f'(0)w\right)_i - \kappa^{-1}(\rho w)_i - f_i(w) - \kappa^{-1}w_i \\
&\le |\text{diag } h_i'(w_i) - \text{diag } h_i'(0)||w_{i,x}| - \kappa^{-1}w_i + (1-\kappa^{-1})\left(f'(0)w\right)_i - \kappa^{-1}(\rho w)_i - f_i(w) \\
&\le \kappa^{-1}w_i - \kappa^{-1}w_i + (1-\kappa^{-1})\left(f'(0)w\right)_i - \kappa^{-1}(\rho w)_i - f_i(w) \\
&= (1-\kappa^{-1})\left(f'(0)w\right)_i - \kappa^{-1}(\rho w)_i - f_i(w) \le 0. \qquad (5.53)
\end{aligned}$$

Then (5.53) shows that w_i is a subsolution for the non-linear system (5.1). By applying the Comparison Theorem 5.1 with the initial condition v and noting that $w(x,1) = M^{(\kappa)}[v](x)$, it follows that $Q[v] \ge M^{(\kappa)}[v]$, and the lemma is proved. □

5.3.2 Travelling waves and spreading speeds for the PDE system (5.1)

The following theorem shows that the left slowest spreading speed $\hat{c}$ can be characterized as the maximum speed of a class of travelling waves. This result gives a condition to guarantee that the PDE system (5.1) has single left speed. This result is a modification of [26, Theorem 4.2] to the case of non-decreasing initial data.

Theorem 5.7. *If the system* (5.1) *satisfies Hypotheses* $s_1 - s_6$*, then for every* $c \leq \hat{c}$*, the system* (5.1) *has a non-decreasing travelling wave solution* $w(x-ct)$ *of speed* c *with* $w(\infty) = \beta$ *and* $w(-\infty)$ *a zero of* f *other than* β*. If there is a travelling wave solution* $w(x-ct)$ *with* $w(\infty) = \beta$ *such that for at least one component* i*,*

$$\liminf_{x \to -\infty} w_i(x) = 0,$$

then $c \leq \hat{c}$*. Moreover, if this property holds for all components of* w*, then* $c \leq \hat{c}_f$*.*
If there are no constant equilibria other that 0 *and* β *in* ψ_β*, then* $\hat{c} = \hat{c}_f$*, which means that the system* (5.1) *has single left spreading speed.*

Proof. It is shown in Lemma 5.2 and Lemma 5.4 that Q_t satisfies Hypotheses $q_1 - q_3$, $q_4^{'} - q_5^{'}$, q_6, $q_7^{'}$, and since Q_t defined as the time t map of the PDE system (5.1), so we can apply Theorem 5.6 for Q_t which corresponds to [26, Theorem 4.1] to get the results and then the theorem is proved. □

For non-increasing initial data of system (5.1), the analogue of Theorem 5.7 for characterizing right spreading speed as the minimum speed of a class of travelling waves and gives a sufficient condition for (5.1) to have single right speed is the following.

Theorem 5.8. *If the system* (5.1) *satisfies Hypotheses* $s_1 - s_6$*, then for every* $c \geq \mathring{c}$*, this system has a non-increasing travelling wave solution* $w(x-ct)$ *of speed* c *with* $w(-\infty) = \beta$ *and* $w(\infty)$ *a zero of* f *other than* β*. If there is a travelling wave solution* $w(x-ct)$ *with* $w(-\infty) = \beta$ *such that for at least one component* i*,*

$$\liminf_{x \to \infty} w_i(x) = 0,$$

then $c \geq \mathring{c}$*. Moreover, if this property holds for all components of* w*, then* $c \geq \mathring{c}_f$*.*
If there are no constant equilibria other than 0 *and* β *in* ψ_β *defined in* (3.5)*, then* $\mathring{c} = \mathring{c}_f$*, which means that the system* (5.1) *has single right speed.*

5.3.3 Sufficient conditions for linear determinacy for (5.1)

A simple combined condition, involving both f and h, that ensures that (5.1) is right linear determinate, will be given in Theorem 5.9. The following lemma shows that the solution $u(x,t)$ of (5.1) with continuous and piecewise C^1 initial condition u_0, exists under a certain condition. Lemma 5.5 is an important tool for Theorem 5.9.

Lemma 5.5. *If the initial condition u_0 is continuous and piecewise C^1 and satisfies that $u_0 \in [0,\beta]$, then the solution $u(x,t)$ of the PDE system (5.1) exists and is such that there exists $M > 0$ such that $|u_x(x,t)| \leq M$ for all $x \in \mathbb{R}$, $t > 0$, and $u(x,t) \in [0,\beta]$ for all $x \in \mathbb{R}, t > 0$.*

Proof. Since u_0 is continuous and piecewise C^1, there exist $\xi^1, \xi^2, ..., \xi^m$ such that u_0 is C^1 on $\mathbb{R} \setminus \{\xi^1, \xi^2, ..., \xi^m\}$, and continuous on $\mathbb{R}$, so there exists a sequence $u_n \in B_{BUC^1}(0,R) \cap \psi_\beta$, for some $R > 0$, such that $\|u_n - u_0\|_\infty \to 0$ as $n \to \infty$. Then for each n, there exists a solution v^n of the PDE system (5.1) with $v^n(x,0) = u_n(x)$. Moreover, there exists $M \geq 0$, independent of n, such that $|v^n_x(x,t)| \leq M$ for all $x \in \mathbb{R}, t, n > 0$, and $v^n(x,t) \in [0,\beta]$ for all $x \in \mathbb{R}, t, n > 0$. Then there exists a subsequence v^u_ν and limit $u \in C(\mathbb{R} \times [0,\infty))$ such that $v^u_\nu \to u$ in $C([-M,M],[0,T])$, $v^u_\nu \to u$ in $C^{2+\alpha,1+\alpha}([-M,M],[\delta,T])$ for each $\delta, T > 0$, some $\alpha > 0$, and $|u_x(x,t)| \leq M$ for all $x \in \mathbb{R}, t > 0$, and u is a solution of the PDE system (5.1) on $\mathbb{R} \times (0,\infty)$, that satisfies that $u \in [0,\beta]$. $\square$

The following result uses a modification of the ideas in [42, Theorem 4.2]. This result corresponds to Theorem 4.3 for the discrete-time recursion (3.1). It gives simple condition to guarantee that the system (5.1) with non-increasing initial data, has the right linearly determinacy.

Theorem 5.9. *Suppose that the functions f and h in system (5.1) satisfy Hypotheses $s_1 - s_6$. Assume that either*

(i) *$\bar{\mu}$ is finite,*

$$\gamma_1(\bar{\mu}) > \gamma_\sigma(\bar{\mu}), \qquad \textit{for all}\, \sigma > 1, \tag{5.54}$$

and

$$f_i(\rho\zeta(\bar{\mu})) \leq \rho\bar{\mu}\left[h_i'(0) - h_i'(\rho\zeta_i(\bar{\mu}))\right]\zeta_i(\bar{\mu}) + \rho(f'(0)\zeta(\bar{\mu}))_i \qquad \textit{for all}\, \rho > 0, \tag{5.55}$$

or

(ii) For some sequence $\nu \to \bar{\mu}$, for each ν the two inequalities in part (i) hold with $\bar{\mu}$ replaced by μ_ν.

Then

$$\mathring{c}_f = \mathring{c} = \mathring{\bar{c}} = \mathring{c}^+,$$

where $\bar{\mu}$ defined in p.98, and these speeds defined in (3.32), (4.10) *and* (4.11) *respectively. Thus the system is right linearly determinate.*

Proof. The key to the proof is to show that $S(x,t) := \min\left\{e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta(\bar{\mu}), \beta\right\}$ is a super-solution of (5.1). First note that $S(x,0) = \min\{e^{-\bar{\mu}x}\zeta(\bar{\mu}), \beta\}$, Then we will show that for each i,

$$S_{i,t} \geq d_i S_{i,xx} - h_i^{'}(S_i)S_{i,x} + f_i(S), \tag{5.56}$$

at each (x,t) at which S_i is smooth. There are four possibilities of $S(x,t)$, namely
(1) when $S(x,t) = e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta(\bar{\mu})$ in a neighbourhood around the point (x_0,t_0), then $S_{i,t} = \mathring{\bar{c}}\bar{\mu}e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta_i(\bar{\mu})$ for each i, and thus

$$\begin{aligned}
&d_i S_{i,xx} - h_i^{'}(S_i)S_{i,x} + f_i(S)\\
&\qquad = d_i(\bar{\mu})^2 \cdot e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta_i(\bar{\mu}) + \bar{\mu}h_i^{'}(e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta_i(\bar{\mu}))(e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta_i(\bar{\mu})) + f_i(e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta(\bar{\mu})).
\end{aligned}$$

Then (5.55) can be re-written as

$$f_i(\rho\zeta(\bar{\mu})) + \rho\bar{\mu}h_i^{'}(\rho\zeta_i)\zeta_i(\bar{\mu}) \;\leq\; \rho\bar{\mu}h_i^{'}(0)\zeta_i(\bar{\mu}) + \rho(f^{'}(0)\zeta(\bar{\mu}))_i \qquad \rho > 0,$$

so taking $\rho = e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}$, (5.55) implies that

$$\begin{aligned}
d_i(\bar{\mu})^2 \cdot e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta_i(\bar{\mu}) &+ \bar{\mu}h_i^{'}(e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta_i(\bar{\mu}))e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta_i(\bar{\mu}) + f_i(e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta(\bar{\mu}))\\
&\leq d_i(\bar{\mu})^2 e^{-\bar{\mu}(x-\mathring{\bar{c}}t)} + \bar{\mu}h_i^{'}(0) \cdot e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta_i(\bar{\mu}) + e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}(f^{'}(0)\zeta(\bar{\mu}))_i\\
&= e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\left\{\left[d\bar{\mu}^2 + \bar{\mu}h^{'}(0) + f^{'}(0)\right]\zeta(\bar{\mu})\right\}_i\\
&= e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\gamma_1(\bar{\mu})\zeta_i(\bar{\mu})\\
&= S_{i,t},
\end{aligned}$$

where $\mathring{\bar{c}}$ is defined in (5.41), and hence (5.56) holds.
(2) Since $f(\beta) = 0$ and β is a solution for (5.1), then when $S(x,t) = \beta$ in a neighbourhood around the point (x_0, t_0), (5.56) holds.

Now we consider the cases when S_i is given by either $e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta_i(\bar{\mu})$ or β_i in a neighbourhood around the point (x_0, t_0), but S is neither $e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta(\bar{\mu})$ nor β. Thus we have
(3) $S_i = e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta_i(\bar{\mu})$, and $S_j = \beta_j$ for some j. From the definition of $S(x,t)$, (5.56) holds for such i if and only if

$$\bar{\mu}\mathring{\bar{c}}e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta_i(\bar{\mu}) \geq d_i(\bar{\mu})^2 e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta_i(\bar{\mu}) - h_i^{'}(S^i)(-\bar{\mu})e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta_i(\bar{\mu}) + f_i(S). \tag{5.57}$$

Since we know that $S \leq e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta(\bar{\mu})$ and $S_i = e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta_i(\bar{\mu})$, by Hypothesis s_1, we have that $f_i(S) \leq f_i(e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta(\bar{\mu}))$, so if we know that

$$\bar{\mu}\mathring{\bar{c}}e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta_i(\bar{\mu}) \geq d_i(\bar{\mu})^2 e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta(\bar{\mu}) - h_i^{'}(S_i)(-\bar{\mu})e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta_i(\bar{\mu}) + f_i(e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta(\bar{\mu})),$$

then (5.57) holds, which is the same inequality that we have in case (1), and hence (5.56) holds.
(4) $S_i = \beta_i$, and $S_j = e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta_j(\bar{\mu})$ for some j. (5.56) is satisfied if and only if $0 \geq f_i(S)$ and by the definition of S we know that $S \leq \beta$, whereas $S_i = \beta_i$. Therefore $f_i(S) \leq f_i(\beta) = 0$, and hence (5.56) holds.
Thus (5.56) holds whenever S_i is smooth.

Now define $z(x,t) := S(x,t) - u(x,t)$, where $u(x,t)$ is the solution of the PDE system (5.1) with initial condition $u_0(x) = S(x,0)$. Then $z(x,0) = 0$, and at (x,t) where S_i is smooth, we have

$$\begin{aligned} z_{i,t} &\geq d_i z_{i,xx} - \left(h_i^{'}(S_i) - h_i^{'}(u_i)\right) u_{i,x} - (S_{i,x} - u_{i,x})h_i^{'}(S_i) + f_i(S) - f_i(u) \\ &= d_i z_{i,xx} + H_i(x,t) z_{i,x} + (F(x,t)z)_i\,, \end{aligned} \tag{5.58}$$

where $H = \text{diag}\ (H_1, ..., H_k)$ is diagonal and bounded, and $F \in \mathbb{R}^{k\times k}$ is bounded with non-negative off-diagonal elements.

Since (5.58) only holds at points where S_i is smooth, we need to modify the proofs of [38, p.245-247, Lemma 5.1, Lemma 5.2, Theorem 5.3] to show that $z(x,t) \geq 0$ for all $(x,t) \in \mathbb{R} \times (0,T)$ for each fixed T. It suffices to extend [38, Lemma 5.2] to cover the case where we want to show that $z(x,t) \geq 0$ everywhere, but we have (5.58) only at the points for which z_i is smooth.

Choose x_1, x_2 so that all points $(x,t) \in \mathbb{R} \times (0,T)$ at which z is not smooth are contained in $[x_1, x_2] \times (0,T)$. Then [38, Lemma 5.1], which concerns the points outside this bounded interval, will hold as in [38].

We need to show that, for $\epsilon > 0$, $p = (1, 1, ..., 1)$, if

$$(z + \epsilon p)_{i,t} > d_i \, (z + \epsilon p)_{i,xx} + H_i(x,t) \, (z + \epsilon p)_{i,x} + (F(x,t) \, (z + \epsilon p))_i \, , \tag{5.59}$$

when z_i is smooth, and $z(x_k, t) > 0$ $(k = 1, 2)$ for $t \in [0,T]$, then $z + \epsilon p > 0$ in $\Lambda = [x_1, x_2] \times [0,T]$. We will use a contradiction argument. Suppose that there exists a point (x_0, t_0) such that $z(x,t) + \epsilon p \geq 0$ for $0 \leq t \leq t_0$, $x_1 \leq x \leq x_2$, and for some component i, $(z(x_0,t_0) + \epsilon p)_i = 0$. Then $(x_0, t_0) \in (x_1, x_2) \times (0,T]$. Now for such a component i, and at (x_0, t_0), S_i is either given by $e^{-\bar{\mu}(x - \mathring{c}t)} \zeta_i(\bar{\mu})$ or β_i. Since we know, by Lemma 5.5, that $u(x,t) \in [0, \beta]$, it follows that $0 \leq u_i \leq \beta_i$. If $S_i = \beta_i$, then $z_i = S_i - u_i = \beta_i - u_i \geq 0$, which implies that $(z + \epsilon p)_i > 0$, and hence we cannot have $(z + \epsilon p)_i = 0$ at a point (x_0, t_0) at which $S_i(x_0, t_0) = \beta_i$. Then at (x_0, t_0), S_i is equal to $e^{-\bar{\mu}(x - \mathring{c}t)} \zeta_i(\bar{\mu})$ but not equal to β_i, then there must be a neighbourhood around (x_0, t_0) where S_i is also given by $e^{-\bar{\mu}(x - \mathring{c}t)} \zeta_i(\bar{\mu})$, and hence S_i is smooth on a neighbourhood of (x_0, t_0), so the proof of [38, Lemma 5.2] applies to show that $z + \epsilon p > 0$ on $[x_1, x_2] \times [0,T]$, because from (5.59) and at (x_0, t_0), we have $(z + \epsilon p)_{i,t} \leq 0, (z + \epsilon p)_{i,xx} \geq 0$ and $(z + \epsilon p)_{i,x} = 0$. Moreover, since F has non-negative off-diagonal elements and $(z + \epsilon p)_i \, (x_0, t_0) = 0$, then

$$(F(x,t) \, (z + \epsilon p))_i = \sum_{j \neq i} F_{ij}(x,t) \, (z + \epsilon p)_j \geq 0.$$

So the right-hand side of (5.59) is non-negative, whereas the left-hand side is non-positive, which is a contradiction.

Then since H is diagonal and bounded, F is bounded with non-negative off-diagonal elements, [38, Theorem 5.3] implies immediately that $z(x,t) \geq 0$ for all $(x,t) \in \mathbb{R} \times (0,T)$. Thus we have $u(x,t) \leq S(x,t)$, and in particular, $u(x,1) \leq S(x,1)$, which implies that $Q[\min\{e^{-\bar{\mu}x}\zeta(\bar{\mu}), \beta\}] \leq e^{-\bar{\mu}(x-\mathring{\bar{c}}t)}\zeta(\bar{\mu})$, and the theorem is proved. $\square$

Note that in the following we refer to (5.55) as a *right combined condition* corresponding to non-increasing initial data, and there is also a *left combined condition*, (5.61), corresponding to non-decreasing initial data.

The next theorem gives a sufficient condition to ensure that system (5.1) is left linearly determinate. To prove this, we can use the same argument that used in proof of Theorem 5.9 but corresponding to non-decreasing initial data.

Theorem 5.10. *Suppose that the functions f and h satisfy Hypotheses $s_1 - s_6$. Assume that either*

(i) *$\hat{\bar{\mu}}$ is finite,*

$$\hat{\gamma}_1(\hat{\bar{\mu}}) > \hat{\gamma}_\sigma(\hat{\bar{\mu}}), \quad \textit{for all } \sigma > 1, \tag{5.60}$$

and

$$f_i(\rho\hat{\zeta}(\hat{\bar{\mu}})) \leq \rho\hat{\bar{\mu}}\left[h_i'(\rho\hat{\zeta}_i(\hat{\bar{\mu}})) - h_i'(0)\right]\hat{\zeta}_i(\hat{\bar{\mu}}) + \rho(f'(0)\hat{\zeta}(\bar{\mu}))_i \qquad \textit{for all } \rho > 0, \tag{5.61}$$

or

(ii) *For some sequence $\nu \to \bar{\mu}$, for each ν the two inequalities in part (i) hold with $\bar{\mu}$ replaced by μ_ν.*

Then

$$\hat{c}_f = \hat{c} = \hat{\bar{c}} = \hat{c}^+,$$

where $\hat{\bar{\mu}}$ defined in p.99, and these speeds defined in (3.29), (3.12), (4.15), *and* (4.16) *respectively. Thus the system has single left speed and is left linearly determinate.*

The following lemma gives a sufficient condition to guarantee that (5.54) and (5.60) hold for the matrices C_μ and $\hat{C}_\mu$ respectively in the particular case that (5.1) consists of two equations which means that $f'(0)$ has two blocks.

Lemma 5.6. *If* (5.1) *consists of two equations, for given matrices* $f'(0)$ *and* $h'(0)$ *as in Lemma 5.3, a sufficient condition to have that* (5.54) *and* (5.60) *are satisfied for the matrices* C_μ *and* $\hat{C}_\mu$ *respectively is that*

$$\frac{\sigma - \alpha(2-d_2)}{\sqrt{\alpha}} < a - b < \frac{\alpha(2-d_2) - \sigma}{\sqrt{\alpha}}. \qquad (5.62)$$

Furthermore, if (5.62) *holds, then the eigenvectors* $\zeta(\bar{\mu}), \hat{\zeta}(\hat{\mu})$ *corresponding to* $\gamma_1(\bar{\mu}), \hat{\gamma}_1(\hat{\mu})$ *are strictly positive.*

Proof. Note that Lemma 5.3 yields that $\bar{\mu} = \hat{\mu}$, and straightforward calculation shows that for $d_1 = 1$, $\bar{\mu} = \hat{\bar{\mu}} = \sqrt{\alpha}$. Then we have $\gamma_1(\bar{\mu}) = 2\alpha + a\sqrt{\alpha}$ and $\gamma_2(\bar{\mu}) = \alpha d_2 + b\sqrt{\alpha} + \sigma$. It follows that $\gamma_1(\bar{\mu}) > \gamma_2(\bar{\mu})$ if $2\alpha + a\sqrt{\alpha} > \alpha d_2 + b\sqrt{\alpha} - \sigma$, which implies that $\alpha(2-d_2) + \sqrt{\alpha}(a-b) + \sigma > 0$. So $\gamma_1(\bar{\mu}) > \gamma_2(\bar{\mu})$ if and only if $a - b > (\sigma - \alpha(2-d_2))/\sqrt{\alpha}$. On the other hand, $\hat{\gamma}_1(\bar{\mu}) = 2\alpha - a\sqrt{\alpha}$ and $\hat{\gamma}_2(\bar{\mu}) = \alpha(d_2) - b\sqrt{\alpha} + \sigma$. It follows that $\hat{\gamma}_1(\bar{\mu}) > \hat{\gamma}_2(\bar{\mu})$ if $2\alpha - a\sqrt{\alpha} - \alpha d_2 + b\sqrt{\alpha} - \sigma > 0$, which implies that $-\alpha(2-d_2) + \sqrt{\alpha}(a-b) + \sigma < 0$. Thus $\hat{\gamma}_1(\bar{\mu}) > \hat{\gamma}_2(\bar{\mu})$ if and only if $a - b < (\alpha(2-d_2) - \sigma)/\sqrt{\alpha}$. Then straightforward calculation shows that (5.62) ensures the eigenvectors $\zeta(\bar{\mu}), \hat{\zeta}(\hat{\mu})$ are strictly positive, since $\varrho > 0$ and (5.54), (5.60) hold (note that the analogous observation for eigenvectors of $f'(0) = C_0$ already mentioned in Remark 5.1 (ii)). □

Remark 5.2. Since (5.55) in Theorem 5.9 applies to a system of k equations, so in particular, when there is only one equation in (5.1), $\bar{\mu} = \sqrt{f'(0)}$, $\zeta(\bar{\mu}) = 1$, and then condition (5.55) is equivalent to (2.21) in Proposition 2.2, because setting $\hat{\rho} = \rho\zeta(\bar{\mu}) = u$ in (2.21) gives

$$h'(\hat{\rho}) + \frac{f(\hat{\rho})}{\hat{\rho}\bar{\mu}} \leq h'(0) + \frac{f'(0)}{\bar{\mu}}, \qquad \text{for all } \hat{\rho} \in (0,1),$$

which is precisely the scalar analogue of (5.55).

The next proposition gives a necessary condition for the existence of a function h satisfying both the 'right combined condition', (5.55),

$$f_i(\rho\zeta(\bar{\mu})) \leq \rho\bar{\mu}\left[h_i'(0) - h_i'(\rho\zeta_i(\bar{\mu}))\right]\zeta_i(\bar{\mu}) + \rho(f'(0)\zeta(\bar{\mu}))_i \qquad \text{for all } \rho > 0, \qquad (5.63)$$

and the 'left combined condition', (5.61),

$$f_i(\rho\hat{\zeta}(\hat{\mu})) \leq \rho\hat{\mu}\left[h_i'(\rho\hat{\zeta}_i(\hat{\mu})) - h_i'(0)\right]\hat{\zeta}_i(\hat{\mu}) + \rho(f'(0)\zeta(\bar{\mu}))_i \qquad \text{for all } \rho > 0. \tag{5.64}$$

Proposition 5.3. *Suppose that the functions f and h are such that $\bar{\mu} = \hat{\mu}$, $\zeta(\bar{\mu}) = \hat{\zeta}(\bar{\mu})$, and $\zeta(\bar{\mu}), \hat{\zeta}(\bar{\mu})$ are strictly positive. Then a necessary condition for both the right combined condition* (5.63) *and the left combined condition* (5.64) *to be satisfied is that the function f satisfies*

$$f_i(\rho\zeta(\bar{\mu})) \leq \rho(f'(0)\zeta(\bar{\mu}))_i \qquad \textit{for all } \rho > 0. \tag{5.65}$$

Proof. Introduce the notation

$$\Lambda_i(\rho) := \rho(f'(0)\zeta(\bar{\mu}))_i - f_i(\rho\zeta(\bar{\mu})) \quad \rho > 0, \tag{5.66}$$

and note that both (5.63) and (5.64) are satisfied if and only if

$$-\Lambda_i(\rho) \;\leq\; \rho\bar{\mu}\left[h_i'(0) - h_i'(\rho\zeta(\bar{\mu}))\right]\zeta_i(\bar{\mu}) \;\leq\; \Lambda_i(\rho) \quad \text{for all } \rho > 0. \tag{5.67}$$

The result is then immediate from the fact that (5.67) can only hold if $\Lambda_i(\rho) \geq 0$ for all $\rho > 0$, which is equivalent to (5.65). □

Remark 5.3. Lemma 5.3 shows that when (5.1) is a system of two equations, $\bar{\mu} = \hat{\mu}$ if the function f is as in Remark 5.1 (iii), and $\zeta(\bar{\mu}) = \hat{\zeta}(\bar{\mu})$ if we also have that $h_1'(0) = h_2'(0)$. Lemma 5.6 gives conditions that ensure $\zeta(\bar{\mu}), \hat{\zeta}(\bar{\mu})$ are strictly positive. Note that in the scalar case, it is obvious we have $\zeta(\bar{\mu}) = \hat{\zeta}(\bar{\mu}) = 1$, and $\bar{\mu} = \hat{\mu}$ because the term $h'(0)$ does not play a role in the value of $\bar{\mu} = \hat{\mu}$, and from (2.15), we have $\bar{\mu} = \hat{\mu} = \sqrt{f'(0)}$. Thus this means (5.65) in the scalar case is clearly equivalent to the classical condition (2.1) when $h \equiv 0$.

Chapter 6

Correspondence between different concepts of linear values of $f'(0)$

In this chapter we discuss the correspondence between two different concepts of linear value in the case when the Frobenius form of $f'(0)$ contains only one block. For simplicity, we will suppose that $f'(0) \in P^{k \times k}$, the set of real $k \times k$ matrices with strictly positive off-diagonal elements. In addition to the right (left) linear value $\mathring{\bar{c}}$ $(\hat{\bar{c}})$ that was introduced in Chapter 5 (5.37) ((5.40)), we will define an alternative right (left) linear value speed, $\mathring{c}_{lin}$ $(\hat{c}_{lin})$, determined by the values of c such for which there exists a monotone eigenvalue, in a sense defined in Definition 6.1, of the linearization of the travelling wave problem for this c about the unstable equilibrium 0, (see (6.2)).
We can write the system (5.1) as

$$u_{i,t} + h_i'(u_i)u_{i,x} = d_i u_{i,xx} + f_i(u) \qquad \text{for } i = 1, 2, ..., k.$$

By substituting the travelling wave $u(x,t) = w(x - ct)$, we get

$$-cw_i' + h_i'(w_i)w_i' = d_i w_i'' + f_i(w) \qquad \text{for } i = 1, 2, ..., k,$$

of which the linearization about $w = 0$ is

$$-cw' + h_i'(0)w_i' = d_i w_i'' + f_i'(0)w \qquad \text{for } i = 1, 2, ..., k.$$

Let us define a matrix D as

$$D := h^{'}(0) = \text{diag } \left(h_1^{'}(0)\ h_2^{'}(0)\ ...\ h_k^{'}(0)\right), \tag{6.1}$$

and recall that $A = \text{diag } (d_1, d_2, ...d_k)$. Then the linearization of travelling wave problem becomes

$$Aw^{''} + (cI - D)w^{'} + f^{'}(0)w = 0. \tag{6.2}$$

Let $v = w^{'}$. Then (6.2) becomes $Av^{'} + (cI - D)v + f^{'}(0)w = 0$,so $v^{'} = -A^{-1}(cI - D)v - A^{-1}f^{'}(0)w$, and hence (6.2) can be re-written as

$$\begin{bmatrix} v^{'} \\ w^{'} \end{bmatrix} = \begin{bmatrix} -A^{-1}(cI - D) & -A^{-1}f^{'}(0) \\ I & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}.$$

The eigenvalue λ of the matrix $\begin{bmatrix} -A^{-1}(cI - D) & -A^{-1}f^{'}(0) \\ I & 0 \end{bmatrix}$ corresponding to the eigenvector $(y, z)^T$ satisfies

$$\begin{bmatrix} -A^{-1}(cI - D) & -A^{-1}f^{'}(0) \\ I & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \lambda \begin{bmatrix} y \\ z \end{bmatrix},$$

which holds if and only if $-A^{-1}(cI - D)y - A^{-1}f^{'}(0)z = \lambda y$ and $y = \lambda z$. Then $-A^{-1}(cI - D)\lambda z - A^{-1}f^{'}(0)z = \lambda^2 z$, and hence

$$\left(\lambda^2 A + \lambda(cI - D) + f^{'}(0)\right) z = 0.$$

If $z > 0$, then it follows by the Perron-Frobenius Theorem 3.1 that

$$\mathfrak{F}_{pf}\left(\lambda^2 A + \lambda(cI - D) + f^{'}(0)\right) = 0,$$

where $\mathfrak{F}_{pf}$ is defined in Table 1. Note that if we seek a solution of (6.2) of the form $w(\xi) = \exp(\lambda\xi)q$, $q > 0$, then $\left(\lambda^2 A + \lambda(cI - D) + f^{'}(0)\right) q = 0$.

Define the matrix $M(\lambda, c)$ that depends on λ and c by

$$M(\lambda, c) := \lambda^2 A + \lambda(cI - D) + f'(0). \tag{6.3}$$

Now it is useful to introduce a definition of stability and instability of an eigenvalue λ of the travelling wave problem (6.2) linearized about the equilibrium point 0.

Definition 6.1. Given $c \in \mathbb{R}$, we say that an eigenvalue λ is a *stable (unstable)* monotone eigenvalue of the travelling wave problem (6.2) linearized about 0, with corresponding eigenvector X, if:

1. $M(\lambda, c)X = 0$,
2. λ is real and strictly negative (positive), and
3. the eigenvector X of $M(\lambda, c)$ has strictly positive components.

Note that λ is a stable monotone eigenvalue of (6.2) if and only if $\mathfrak{F}_{pf}(M(\lambda, c)) = 0$. In the case when the Frobenius form of $f'(0)$ contains only one block, an alternative right (left) linear value speed can be defined as follows.

Definition 6.2. $\mathring{c}_{lin}$ $(\hat{c}_{lin})$ is defined as the infimum (supremum) of the values of c for which there exists a stable (unstable) monotone eigenvalue λ of (6.2) for this value of c.

We will later establish directly from Definition 6.2, in Theorem 6.2 and Lemma 6.2, the existence of $\mathring{c}_{lin}$ $(\hat{c}_{lin}) \in \mathbb{R}$. Note that [38, Lemma 2.4, p.136] (see also, [16, Theorem 3.7]) shows that the existence of a stable (unstable) monotone eigenvalue is a necessary condition for the existence of a travelling wave that converges to 0 (β) at $+\infty$ $(-\infty)$. In addition, because it is a necessary condition and we have seen earlier in Theorem 5.6, that the minimal non-increasing travelling wave speed exists, so it immediately tells us that the minimum speed $c_0 \geq \mathring{c}_{lin}$.
Clearly, the definition of $\mathring{c}_{lin}$ $(\hat{c}_{lin})$ is different from the definition of the right (left) linear value $\mathring{c}$ $(\hat{c})$ that is defined earlier in (5.37) ((5.40)) respectively. However, the definitions being different does not automatically mean that the values are different. Here we focus on the case when we have only one block for the Frobenius matrix $f'(0)$, which means

that we also will have one block for the Frobenius matrix $M(\lambda, c)$ that is defined in (6.3), and suppose $f'(0) \in P^{k\times k}$ for simplicity. From Hypothesis s_6, it follows that the Frobenius eigenvalue $\mathfrak{F}_{pf}(f'(0)) > 0$. We will then prove that $\mathring{c}_{lin}$ $(\hat{c}_{lin})$ equals $\mathring{c}$ $(\hat{c})$ in Lemma 6.3.

6.1 Eigenvalues and eigenvectors if $M(\lambda, c)$ has a single irreducible block

The following lemma gives a condition on a matrix $N \in P^{k\times k}$ that ensures that $\mathfrak{F}_{pf}(N) > 0$, which is useful for Lemma 6.2. This result is the same as [12, Corollary 1.6].

Lemma 6.1. *If the matrix $N \in P^{k\times k}$ and there exists $u \in \mathbb{R}^k \setminus \{0\}$, $-u \notin \mathbb{R}^k_+$ such that $Nu > 0$, then $\mathfrak{F}_{pf}(N) > 0$.*

Proof. Since $Nu > 0$, there exists $\beta > 0$ such that $Nu > \beta u$. There exists $\alpha > 0$ such that for the matrix $N+\alpha I$ all entries are strictly positive (> 0) and $(N+\alpha I)u > (\alpha+\beta)u$. By [32, Theorem 2.3] we have $\mathfrak{F}_{pf}(N+\alpha I) \geq \alpha + \beta$, which means that

$$\mathfrak{F}_{pf}(N) \geq \beta > 0.$$

□

We also quote a related result, which is a variant [36, Theorem 1.6].

Theorem 6.1. *Let $N \in P^{k\times k}$ and suppose that there exists $u \in \mathbb{R}^k_+$ and $u \notin \{0\}$ such that if $Nu < 0$, then $\mathfrak{F}_{pf}(N) < 0$.*

The following lemma shows whether 0 is the Perron-Frobenius eigenvalues of $M(\lambda, c)$ in (6.3) or not. This result is an adaptation of [12, Lemma 3.4] to treat the case when we have a diagonal matrix D in (6.1).

Lemma 6.2. *Let $f'(0) \in P^{k\times k}$. Then*

1. *When c is sufficiently negative, there are no stable monotone eigenvalues λ of the linearized travelling-wave problem (6.2).*

2. *When c is sufficiently positive, there exists a stable monotone eigenvalue λ of the linearized travelling-wave problem* (6.2).

3. *If there exists a stable monotone eigenvalue when $c = c_a$, then for all values c with $c \geq c_a$, a stable monotone eigenvalue exists.*

4. *When c is sufficiently positive, there are no unstable monotone eigenvalues of the linearized travelling-wave problem* (6.2).

5. *When c is sufficiently negative, there exists an unstable monotone eigenvalue of the linearized travelling-wave problem* (6.2).

6. *If there exists an unstable monotone eigenvalue when $c = c_b$, then for all $c \leq c_b$, an unstable monotone eigenvalue exists.*

Proof. 1) Suppose $q > 0$ is a Perron-Frobenius eigenvector of $f^{'}(0)$. Then

$$\left(\lambda^2 A + f^{'}(0)\right) q = \lambda^2 A q + \mathfrak{F}_{pf}(f^{'}(0))q > 0.$$

For c sufficiently negative, $(cI - D)$ is a diagonal matrix with strictly negative diagonal entries, in which case whenever $\lambda < 0$, then $\lambda(cI - D)$ is a diagonal matrix with strictly positive diagonal entries. Thus we have $M(\lambda, c)q = \left(A\lambda^2 + \lambda(cI - D) + f^{'}(0)\right) q > 0$. By Lemma 6.1, we get that $\mathfrak{F}_{pf}\left(M(\lambda, c)\right) > 0$ for such $\lambda < 0$. $cI - D$ is a diagonal matrix with strictly negative diagonal-entries for such c. So $\mathfrak{F}_{pf}\left(M(\lambda, c)\right) \neq 0$, and it follows that there is no stable eigenvalue for such c.

2) Take $\lambda = -1$. Then

$$M(-1, c) = A - (cI - D) + f^{'}(0) = A + D + f^{'}(0) - cI.$$

Since $\mathfrak{F}_{pf}(M(-1, c)) = \mathfrak{F}_{pf}(A + D + f^{'}(0)) - c$, then $\mathfrak{F}_{pf}(M(-1, c)) < 0$ if c is sufficiently large and positive. So since $\mathfrak{F}_{pf}(M(0, c)) = \mathfrak{F}_{pf}(f^{'}(0)) > 0$, the continuous dependence of $M(\lambda, c)$ on λ implies that there exists γ such that $-1 < \gamma < 0$ where $\mathfrak{F}_{pf}(M(\gamma, c)) = 0$, which means that γ is a stable monotone eigenvalue.

3) Assume that there is a stable monotone eigenvalue $\lambda < 0$ at $c = c_a$ with corresponding eigenvector $X > 0$. Then $\left(\lambda^2 A + \lambda(c_a I - D) + f'(0)\right) X = 0$. But for $\delta > 0$ and $c = c_a + \delta$, we have

$$\begin{aligned}\left(A\lambda^2 + \lambda(cI - D) + f'(0)\right) X &= \left(A\lambda^2 + \lambda(c_a I - D) + f'(0)\right) X + \lambda\delta X \\ &= \lambda\delta X \quad < 0,\end{aligned}$$

since $X > 0, \lambda < 0$. Then Theorem 6.1 implies that

$$\mathfrak{F}_{pf}\left(A\lambda^2 + \lambda(cI - D) + f'(0)\right) < 0.$$

Since we know $\mathfrak{F}_{pf}(M(0, c)) > 0$, then there exists a stable monotone eigenvalue $\gamma \in (\lambda, 0)$ such that $\mathfrak{F}_{pf}(M(\gamma, c)) = 0$, by the continuous dependence of $M(\lambda, c)$ on λ and c.

To prove parts (4), (5) and (6), we can follow the same procedure as in the proof of parts (1), (2) and (3) respectively, but for unstable monotone eigenvalues instead of stable eigenvalues, and the lemma is proved. □

6.1.1 An alternative linear value speed $\mathring{c}_{lin}$ $(\hat{c}_{lin})$ for (6.2)

For the existence of $\mathring{c}_{lin}$ $(\hat{c}_{lin}) \in \mathbb{R}$, it is useful to define a set V by

$$V := \left\{c \in \mathbb{R} : \left(A\lambda^2 + \lambda(cI - D) + f'(0)\right) y = 0 \text{ for some } \lambda < 0 \text{ and } y \in \mathbb{R}^k, y > 0\right\}.$$

Note that $c \in V$ if and only if there exists a stable monotone eigenvalue for this c, and that V is non-empty and bounded below by Lemma 6.2 (1), (2). So $\mathring{c}_{lin} = \inf V \in \mathbb{R}$ exists. Likewise, for the existence of $\hat{c}_{lin}$, we define a set $\tilde{V}$ by

$$\tilde{V} := \left\{c \in \mathbb{R} : \left(A\lambda^2 + \lambda(cI - D) + f'(0)\right) y = 0 \text{ for some } \lambda > 0 \text{ and } y \in \mathbb{R}^k, y > 0\right\}.$$

Note again that $c \in \tilde{V}$ if and only if there exists an unstable monotone eigenvalue for this c, and that $\tilde{V} \in \mathbb{R}$ is non-empty and bounded above by Lemma 6.2 (4), (5). So $\hat{c}_{lin} = -\inf \tilde{V}$ exists.

The following lemma shows that the alternative right (left) linear value $\mathring{c}_{lin}$ ($\hat{c}_{lin}$) equals the right (left) linear value $\mathring{\bar{c}}$ ($\hat{\bar{c}}$) that is defined in (5.37) ((5.40)) respectively.

Lemma 6.3. *The right linear value $\mathring{\bar{c}}$ defined in* (5.37) *equals the alternative right linear value $\mathring{c}_{lin}$.*

Proof. We can re-write the matrix C_μ in (5.33) in terms of matrix D in (6.1) as $C_\mu = A\mu^2 + \mu D + f^{'}(0)$ and compare this with (6.3), to obtain that $C_\mu = M(-\mu, c) + \mu c I$. It follows that $\mathfrak{F}_{pf}(C_\mu) = \mathfrak{F}_{pf}(M(-\mu, c)) + \mu c$, and hence $\mu^{-1}\mathfrak{F}_{pf}(C_\mu) = \mu^{-1}\mathfrak{F}_{pf}(M(-\mu, c)) + c$, for all $\mu > 0$ and c.
Suppose that there exists a stable monotone eigenvalue $\lambda = -\mu_1$, $\mu_1 > 0$, then $\mathfrak{F}_{pf}(M(-\mu_1, c_1)) = 0$, which implies that

$$\mu_1^{-1}\mathfrak{F}_{pf}(C_{\mu_1}) = \mu_1^{-1}\mathfrak{F}_{pf}(M(-\mu_1, c_1)) + c_1 = c_1,$$

and since $\mathring{\bar{c}} = \inf\limits_{\mu>0} \mu^{-1}\mathfrak{F}_{pf}(C_\mu)$, so $\mathring{\bar{c}} \leq c_1$. Let $\mathring{c}_{lin}$ be the minimal value of c which a stable monotone eigenvalue exists, so we have $\mathring{\bar{c}} \leq \mathring{c}_{lin}$. We need to show that $\mathring{\bar{c}} \geq \mathring{c}_{lin}$. Since $\mathring{\bar{c}} = \inf\limits_{\mu>0}\mu^{-1}\mathfrak{F}_{pf}(C_\mu)$, then for $\bar{\mu}$ that is defined to be the value of μ at which the infimum in the definition of $\mathring{\bar{c}}$ in (5.37) is attained, we have

$$\mathring{\bar{c}} = \inf_{\mu>0}\mu^{-1}\mathfrak{F}_{pf}(C_\mu) = \bar{\mu}^{-1}\mathfrak{F}_{pf}(C_{\bar{\mu}}) = \bar{\mu}^{-1}\mathfrak{F}_{pf}(M(-\bar{\mu}, \mathring{\bar{c}})) + \mathring{\bar{c}}.$$

Therefore $\mathfrak{F}_{pf}(M(-\bar{\mu}, \mathring{\bar{c}})) = 0$, thus $-\bar{\mu}$ is a stable monotone eigenvalue at $\mathring{\bar{c}}$. Hence $\mathring{\bar{c}} \geq \mathring{c}_{lin}$. It follows that $\mathring{\bar{c}} = \mathring{c}_{lin}$. The lemma is established. $\square$

Note that we can follow the same procedure to prove that $\hat{\bar{c}} = \hat{c}_{lin}$.
The following theorem is a generalization of [12, Theorem 3.5] to treat both right $\mathring{\bar{c}}$ and left $\hat{\bar{c}}$ linear values.

Theorem 6.2. *Let $f^{'}(0) \in P^{k\times k}$. Then*

(i) For each $c \geq \mathring{\bar{c}}$, there is a stable monotone eigenvalue λ of the linearized travelling wave problem (6.2), *whereas for $c < \mathring{\bar{c}}$, there is no such λ.*

(ii) For each $c \le \hat{c}$, there is an unstable monotone eigenvalue λ of the linearized travelling wave problem (6.2)*, whereas for $c > \hat{c}$, there is no such λ.*

Proof. (i) Recall that $\mathring{c} = \mathring{c}_{lin} = \inf V$. For any $c \in V$ and for $\delta > 0$, Lemma 6.2 (3) says that we have $c + \delta \in V$, which implies that $(\mathring{c}, \infty) \subset V$. Now we use a contradiction argument to prove that $\mathring{c} \in V$. Take a sequence $c_l \to \mathring{c}$ with eigenvalues $\lambda_l < 0$ with corresponding eigenvectors $z^l > 0$ such that

$$\left(A\lambda_l^2 + \lambda_l(c_l I - D) + f'(0)\right) z^l = 0,$$

where z^l is a positive Perron-Frobenius eigenvector with $\|z^l\|_\infty = 1$. Then for a subsequence we can let $z^l \to z$ such that $\|z\|_\infty = 1$. Since for each l, $\lambda_l \neq 0$, then as $l \to \infty$, $\left(A\lambda_l - D + \lambda_l^{-1} f'(0)\right) z^l \to -\mathring{c} z$ and thus $\{\lambda_l\}_{l \in \mathbb{N}}$ is bounded sequence in $\mathbb{R}$, because if it were not, then $\lambda_l \to -\infty$, and since $Az^l \to Az$, A is a positive diagonal matrix and $z \geq 0$ but $z \neq 0$ because $\|z\|_\infty = 1$, then when $\lambda_l \to -\infty$, we have $\lambda_l^{-1}\left(A\lambda_l - D + \lambda_l^{-1} f'(0)\right) z^l \to 0$ which implies that $Az^l \to 0$ which contradicts $Az^l \to Az > 0$. Hence there is a convergent subsequence, say $\lambda_l \to \lambda \leq 0$. Taking the limit as $l \to \infty$ gives

$$\left(A\lambda^2 + \lambda(\mathring{c} I - D) + f'(0)\right) z = 0,$$

Now $z \geq 0$, but since the off-diagonal elements of the matrix $f'(0)$ are strictly positive, and z is a Perron-Frobenius eigenvector, so we have $z > 0$. It follows that if $\lambda = 0$, then $f'(0)z = 0$ for $z > 0$, which contradicts the fact that $\mathfrak{F}_{pf}\left(f'(0)\right) > 0$. Thus $\mathring{c} \in V$, which is equivalent to saying that the set V contains all $c \geq \mathring{c}$.

(ii) We can follow the same procedure for unstable eigenvalues to prove that $\hat{c} \in \tilde{V}$. Thus the theorem is proved. □

The following lemma shows that the Perron-Frobenius eigenvalue of the matrix $M(\lambda, c)$ in (6.3) is a convex function of λ. This result is a modification of [12, Lemma 3.7] to treat the case when we have a diagonal matrix D in (6.1). Cohen in [11] proves that, $\mathfrak{F}_{pf}$ is a convex function of a diagonal matrix D, in the sense that, given diagonal matrices

D_1 and D_2 and $M \in P^{n\times n}$, then for $0 < \alpha < 1$,

$$\mathfrak{F}_{pf}\left(\alpha D_1 + (1-\alpha)D_2 + M\right) \leq \alpha\mathfrak{F}_{pf}\left(D_1 + M\right) + (1-\alpha)\,\mathfrak{F}_{pf}\left(D_2 + M\right). \tag{6.4}$$

Lemma 6.4. *If $f'(0) \in P^{n\times n}$, A is positive diagonal matrix, D is a diagonal matrix, and $c \in \mathbb{R}$, then the Perron-Frobenius eigenvalue of $M(\lambda, c)$ in (6.3) is a strictly convex function of λ.*

Proof. Let $\lambda_1, \lambda_2 \in \mathbb{R}$ be such that $\lambda_1 \neq \lambda_2$, and note that for $0 < t < 1$ we have $(t\lambda_1 + (1-t)\lambda_2)^2 < t\lambda_1^2 + (1-t)\lambda_2^2$. Since A is positive diagonal matrix, then

$$\begin{aligned}
&\mathfrak{F}_{pf}\left(A(t\lambda_1 + (1-t)\lambda_2)^2 + (t\lambda_1 + (1-t)\lambda_2)(cI - D) + f'(0)\right) \\
&\qquad < \mathfrak{F}_{pf}\left(A(t\lambda_1^2 + (1-t)\lambda_2^2) + (t\lambda_1 + (1-t)\lambda_2)(cI - D) + f'(0)\right) \\
&\qquad = \mathfrak{F}_{pf}\left(t(A\lambda_1^2 + \lambda_1(cI - D)) + (1-t)(A\lambda_2^2 + \lambda_2(cI - D)) + f'(0)\right) \\
&\qquad \leq t\mathfrak{F}_{pf}\left(A\lambda_1^2 + \lambda_1(cI - D) + f'(0)\right) + (1-t)\left((A\lambda_2^2 + \lambda_2(cI - D)) + f'(0)\right),
\end{aligned}$$

since for $k = 1, 2$, $A\lambda_k^2 + \lambda_k(cI - D)$ are diagonal matrices, so (6.4) gives the last inequality. □

The following lemma shows that we cannot have the case that both stable and unstable eigenvalues exist for a given value of c.

Lemma 6.5. *There is no value of c for which both a stable and an unstable monotone eigenvalue exist.*

Proof. Suppose, for contradiction, that there exists a stable monotone eigenvalue $\lambda_1 < 0$ and an unstable monotone eigenvalue $\lambda_2 > 0$. Then $\mathfrak{F}_{pf}(M(\lambda_1, c)) = 0 = \mathfrak{F}_{pf}(M(\lambda_2, c))$, and by Lemma 6.4 it follows that $\mathfrak{F}_{pf}(M(0, c)) < 0$, which contradicts the fact that $\mathfrak{F}_{pf}(M(0, c)) = \mathfrak{F}_{pf}\left(f'(0)\right) > 0$. □

The following lemma shows the relationship between the right linear value $\mathring{\bar{c}}$ defined in (5.37) and the left linear value $\hat{\bar{c}}$ defined in (5.40), using the relationship between the alternative right linear value $\mathring{c}_{lin}$ and the alternative left linear value $\hat{c}_{lin}$. Note that Lemma (6.3) ensures that $\mathring{\bar{c}} = \mathring{c}_{lin}, \hat{\bar{c}} = \hat{c}_{lin}$.

Lemma 6.6. *The linear value $\mathring{c}_{lin}$ for which there exists a stable monotone eigenvalue of* (6.2) *is strictly larger than the linear value $\hat{c}_{lin}$ for which there exists an unstable monotone eigenvalue of* (6.2).

Proof. It follows from Theorem 6.2 that there exists a stable monotone eigenvalue for $c \in [\mathring{\bar{c}}, \infty)$ and an unstable monotone eigenvalue for $c \in (-\infty, \hat{\bar{c}}]$. Hence by Lemma 6.5, which says that we cannot have both a stable and an unstable monotone eigenvalue for any value of c, we conclude that the linear value for which there exists a stable eigenvalue $\mathring{c}_{lin} > \hat{c}_{lin}$, and the lemma is proved. □

Note that Lemma 6.3 implies that $\mathring{\bar{c}} > \hat{\bar{c}}$, and we can use the analogue of Lemma 4.2 in Chapter 4, to obtain $\hat{c} \leq \hat{\bar{c}}$. Thus Lemma 4.2 together with Lemma 6.6 implies that $\mathring{c} \geq \mathring{\bar{c}} > \hat{\bar{c}} \geq \hat{c}$, and hence $\mathring{c} > \hat{c}$. That is, the right slowest spreading speed is strictly larger than the left slowest-spreading speed.

6.2 Eigenvalues and eigenvectors if $M(\lambda, c)$ has multiple irreducible blocks

Suppose now that $f'(0)$, which is in Frobenius form by Hypothesis s_6, contains more than one irreducible block. In this case, [38, Lemma 2.4, p.136] (see also, [16, Theorem 3.7]) shows that if there exists a travelling wave, then there exists a stable (unstable) monotone eigenvalue $\lambda \leq 0$ ($\lambda \geq 0$) such that $M(\lambda, c)X = 0$ for $X \geq 0, X \neq 0$. The proof of this lemma suggests that in the case when $f'(0)$ has more than one block, it is natural to allow the eigenvector X of $M(\lambda, c)$ be $X \geq 0$, $X \neq 0$ instead of $X > 0$. This is because, when there are multiple blocks in $M(\lambda, c)$, the argument that the existence of a travelling wave implies the existence of a stable (unstable) monotone eigenvalue yields that it is necessary to have a non-negative eigenvector but not necessarily a strictly positive eigenvector.

In such a case, when the Frobenius form of $f'(0)$ has more than one block, one can thus define the stable (unstable) monotone eigenvalue similarly to Definition 6.1 (2), but it is now natural to ask that the eigenvector is non-negative and non-zero but not necessarily strictly positive.

We can thus consider two possibilities for the eigenvector X, either $X \geq 0, X \neq 0$, or we keep that $X > 0$ as in the case when $f^{'}(0)$ is one irreducible block. So we will discuss these two cases. We can generalize the parts (1), (2), (4) and (5) of Lemma 6.2 just in the case when we keep that $X > 0$, whereas when $X \geq 0, X \neq 0$, under certain condition on the Perron-Frobenius eigenvalue of the first block of $f^{'}(0)$, we can generalize the first part of Lemma 6.2 only. In addition, we give a partial generalization of part (3) of Lemma 6.2, that is proved later in Proposition 6.1.

The proof of the generalization lemma, Lemma 6.7, depends on considering the first block of the matrix $M(\lambda, c)$, which we refer to as M^1. In addition, F^1 denotes the first block of $f^{'}(0)$, and the parts of matrices A, D corresponding to the first block of $f^{'}(0)$ are denoted by A^1, D^1 respectively, and similarly for the σth block, we define the matrices $A^\sigma, I^\sigma, D^\sigma$, and F^σ for $\sigma > 1$. We first quote the following theorem because it is useful to the proof of Lemma 6.7.

Theorem 6.3. *[36, Theorem 2.1]. Suppose N is a non-negative irreducible matrix, with Perron-Frobenius eigenvalue r. A necessary and sufficient condition for a solution $X, X \geq 0, X \neq 0$, to the equation*

$$(sI - N)\,X = Y,$$

to exist for $Y \geq 0, Y \neq 0$ is that $s > r$. In this case there is only one solution X, which is strictly positive and given by

$$X = (sI - N)^{-1}\,Y.$$

6.2.1 Eigenvalues corresponding to an eigenvector $X > 0$

Suppose we keep the condition that $X > 0$. Then the generalization of parts (1), (2), (4) and (5) of Lemma 6.2 is the following.

Lemma 6.7. *Let $f^{'}(0)$ have more than one block. Then*

1. *When c is sufficiently negative, there are no stable monotone eigenvalues λ of the linearization of the travelling-wave problem* (6.2).

2. When c is sufficiently positive, there exists a stable monotone eigenvalue λ of the linearization travelling-wave problem (6.2).

3. When c is sufficiently positive, there are no unstable monotone eigenvalues of the linearization of the travelling-wave problem (6.2).

4. When c is sufficiently negative, there exists an unstable monotone eigenvalue of the linearization travelling-wave problem (6.2).

Proof. 1) Suppose $X^1 > 0$ is the Perron-Frobenius eigenvector of F^1. Then

$$\left(\lambda^2 A^1 + F^1\right) X^1 = \lambda^2 A^1 X^1 + \mathfrak{F}_{pf}(F^1) X^1 > 0 \qquad \text{for any } \lambda < 0.$$

So by Lemma 6.1, $\mathfrak{F}_{pf}(\lambda^2 A^1 + F^1) > 0$. For c sufficiently negative, $(cI^1 - D^1)$ is a diagonal matrix with diagonal elements strictly negative, whenever $\lambda < 0$ and I^1 is the first part of identity matrix I corresponding to F^1, so $\lambda(cI^1 - D^1)$ is a diagonal matrix with strictly positive diagonal entries. Thus we have $(M^1(\lambda, c)) X^1 = (\lambda^2 A^1 + \lambda(cI^1 - D^1) + F^1) X^1 > 0$. By Lemma 6.1, we thus get that

$$\mathfrak{F}_{pf}\left(M^1(\lambda, c)\right) > 0 \quad \text{for } \lambda < 0, \tag{6.5}$$

where c is chosen such that $cI^1 - D^1$ is a diagonal matrix with strictly negative diagonal-entries. Now since we suppose that the eigenvector X for the Frobenious matrix $M(\lambda, c)$ satisfies $X > 0$, so the part of X corresponding to the first block of $M(\lambda, c)$, which we refer to as X^1 satisfies $X^1 > 0$. Since we know that if there is a stable monotone eigenvalue $\lambda < 0$ with $X^1 > 0$, then $(\lambda^2 A^1 + \lambda(cI^1 - D^1) + F^1) X^1 = 0$, thus $\mathfrak{F}_{pf}\left(M^1(\lambda, c)\right) = 0$, which is a contradiction with (6.5). Therefore it follows that there is no stable monotone eigenvalue.

(2) Choose $\delta > 0$ sufficiently small and let $\lambda = -\delta$. Since we suppose that $X > 0$, so Theorem 6.3 implies that for any λ such that $|\lambda| < \delta$ we have

$$\mathfrak{F}_{pf}\left(\lambda^2 A^1 - \lambda(cI^1 - D^1) + F^1\right) > \mathfrak{F}_{pf}\left(\lambda^2 A^\sigma - \lambda(cI^\sigma - D^\sigma) + F^\sigma\right),$$

for each $\sigma > 1$. Then we have

$$M^1(-\delta, c) = \delta^2 A^1 - \delta(cI^1 - D^1) + F^1 = \delta^2 A^1 + \delta D^1 + F^1 - \delta c I^1.$$

Since $\mathfrak{F}_{pf}(M^1(-\delta, c)) = \mathfrak{F}_{pf}(\delta^2 A^1 + \delta D^1 + F^1) - \delta c I^1$, then $\mathfrak{F}_{pf}(M^1(-\delta, c)) < 0$ if c is sufficiently large and positive.

Since $\mathfrak{F}_{pf}(M^1(0, c)) = \mathfrak{F}_{pf}(F^1) > 0$ and by the continuous dependence of $\mathfrak{F}_{pf}(M(\lambda, c))$ on λ, there exists γ such that $\lambda < \gamma < 0$ where $\mathfrak{F}_{pf}(M^1(\gamma, c)) = 0$. Since we know that $|\lambda| < \delta$, so $|\gamma| < \delta$, and we have

$$\mathfrak{F}_{pf}\left(\gamma^2 A^1 + \gamma(cI^1 - D^1) + F^1\right) > \mathfrak{F}_{pf}\left(\gamma^2 A^\sigma + \gamma(cI^\sigma - D^\sigma) + F^\sigma\right).$$

Then by Theorem 6.3, there is a strictly positive eigenvector X for $M(\gamma, c)$ corresponding to the eigenvalue 0, and thus γ is a stable monotone eigenvalue for $M(\lambda, c)$.

We can follow the previous procedure for unstable monotone eigenvalues instead of stable eigenvalues to prove the parts (3) and (4). The lemma is proved. □

In place of the third part of Lemma 6.2, we have the following proposition. Note that a corresponding result holds for the sixth part of Lemma 6.2.

Proposition 6.1. *If there exists a stable monotone eigenvalue when $c = c_a$, then there exists $\delta_0 > 0$ such that a stable monotone eigenvalue exists corresponding to the eigenvector X_c, for each $c \in [c_a, c_a + \delta_0)$.*

Proof. Assume that there is a stable monotone eigenvalue $\lambda^* < 0$ at $c = c_a$ corresponding to the eigenvector $X_{c_a} > 0$ such that

$$\left(d_i(\lambda^*)^2 + \lambda^*(c_a I - D) + F\right) X_{c_a} = 0, \quad F = f'(0). \tag{6.6}$$

Then since $X_{c_a} > 0$, (6.6) implies that

$$\mathfrak{F}_{pf}\left(M^1(\lambda^*, c_a)\right) > \mathfrak{F}_{pf}\left(M^\sigma(\lambda^*, c_a)\right) \qquad \text{for } \sigma > 1,$$

if and only if $\mathfrak{F}_{pf}\left(M^1(\lambda^*, c)\right) > \mathfrak{F}_{pf}\left(M^\sigma(\lambda^*, c)\right)$ for all c. Then there exists $\epsilon > 0$ such that

$$\mathfrak{F}_{pf}\left(M^1(\lambda,c)\right) > \mathfrak{F}_{pf}\left(M^\sigma(\lambda,c)\right) \qquad \text{when} \quad |\lambda - \lambda^*| < \epsilon. \tag{6.7}$$

Now (6.6) implies that $((\lambda^*)^2 A^1 + \lambda^*(c_a I^1 - D^1) + F^1)\, X^1_{c_a} = 0, X^1_{c_a} > 0$, from which it follows that $\mathfrak{F}_{pf}\left((\lambda^*)^2 A^1 + \lambda^*(c_a I^1 - D^1) + F^1\right) = \mathfrak{F}_{pf}(M^1(\lambda^*, c_a)) = 0$.
Since $\mathfrak{F}_{pf}(M^1(\lambda, c_a))$ is a strictly convex function of λ by Lemma 6.4, then either $\mathfrak{F}_{pf}(M^1(\lambda, c_a)) > 0$ when $\lambda < \lambda^*$, or $\mathfrak{F}_{pf}(M^1(\lambda, c_a)) > 0$ when $\lambda > \lambda^*$, or both.
Suppose that $\mathfrak{F}_{pf}(M^1(\lambda, c_a)) > 0$ when $\lambda < \lambda^*$ (similarly for $\lambda > \lambda^*$). Then choose $\delta_0 > 0$ small enough that $\mathfrak{F}_{pf}(M^1(\lambda^* - \frac{\epsilon}{2}, c_a)) > -\delta_0(\lambda^* - \frac{\epsilon}{2})$, which implies that $\mathfrak{F}_{pf}(M^1(\lambda^* - \frac{\epsilon}{2}, c)) > 0$ whenever $c = c_a + \delta,\ 0 < \delta \leq \delta_0$. But $\mathfrak{F}_{pf}(M^1(\lambda^*, c)) = \delta\lambda^* < 0$, so there exists $\gamma \in \left(\lambda^* - \frac{\epsilon}{2}, \lambda^*\right)$ such that $\mathfrak{F}_{pf}(M^1(\gamma, c)) = 0$. Since $|\gamma - \lambda^*| < \epsilon$, we know that

$$\mathfrak{F}_{pf}(M^1(\gamma, c)) > \mathfrak{F}_{pf}(M^\sigma(\gamma, c)) \qquad \sigma > 1,$$

and hence there is a strictly positive eigenvector $X_c > 0$ of $M(\gamma, c)$, corresponding to the eigenvalue $\mathfrak{F}_{pf}(M^1(\gamma, c))$, by Theorem 6.3. Therefore if there exists a stable monotone eigenvalue with corresponding eigenvector strictly positive at some c_a, then there exists $\delta_0 > 0$ such that there exists a stable monotone eigenvalue with strictly positive eigenvector for each $c \in [c_a, c_a + \delta_0)$. $\square$

Note that this does not, however, show that if a stable eigenvalue with positive eigenvector exists for some c_a, then a perturbation of this eigenvalue exists for all $c > c_a$. In Example 6.1, we will consider a 2×2 matrix to illustrate part (2) of Lemma 6.7, as well as to show that for all values of c sufficiently large, there exists at least one, and sometimes, two stable monotone eigenvalues, depending on the structure of the matrix and the value of c.

6.2.2 Eigenvalues corresponding to an eigenvector $X \geq 0$

In the following proposition we generalize the first part of Lemma 6.2 in the case when the eigenvector $X \geq 0$, $X \neq 0$. In this result we assume that for each σth block, we have $\mathfrak{F}_{pf}(F^\sigma) > 0,\ \sigma > 1$. We already have $\mathfrak{F}_{pf}(F^1) > 0$ from Hypothesis s_2.

Proposition 6.2. *Suppose that $\mathfrak{F}_{pf}(F^\sigma) > 0, \sigma > 1$. Then when c is sufficiently negative, there are no stable monotone eigenvalues λ of the linearization of the travelling-wave problem* (6.2) *corresponding to an eigenvector X satisfying $X \geq 0, X \neq 0$.*

Proof. Suppose $X^1 > 0$ be the Perron-Frobenius eigenvector of F^1, then we have

$$\left(\lambda^2 A^1 + F^1\right) X^1 = \lambda^2 A^1 X^1 + \mathfrak{F}_{pf}(F^1) X^1 > 0 \qquad \text{for any } \lambda < 0.$$

So by Lemma 6.1, $\mathfrak{F}_{pf}(\lambda^2 A^1 + F^1) > 0$. Choose c sufficiently negative to ensure that $(cI^1 - D^1)$ is a diagonal matrix with strictly negative diagonal elements, whenever $\lambda < 0$, so $\lambda(cI^1 - D^1)$ is a diagonal matrix with strictly positive diagonal entries. Thus we have $\left(M^1(\lambda, c)\right) X^1 = \left(\lambda^2 A^1 + \lambda(cI^1 - D^1) + F^1\right) X^1 > 0$. By Lemma 6.1, we get

$$\mathfrak{F}_{pf}\left(M^1(\lambda, c)\right) > 0 \quad \text{for any} \quad \lambda < 0. \tag{6.8}$$

For the other blocks, we can replace X^1 by X^σ where X^σ is the Perron-Frobenius eigenvector of F^σ, and repeat the same argument for X^σ to get that

$$\mathfrak{F}_{pf}\left(M^\sigma(\lambda, c)\right) > 0 \quad \text{for any} \quad \lambda < 0, \sigma > 1, \tag{6.9}$$

when c is sufficiently negative such that for each σ, $(cI^\sigma - D^\sigma)$ is a diagonal matrix with strictly negative diagonal entries.

Now suppose that there exists a stable monotone eigenvalue $\lambda_0 < 0$ corresponding to the eigenvector $X \geq 0, X \neq 0$, but not $X > 0$, such that $M(\lambda_0, c)X = 0$. Note first that if for the σth block and $X^\sigma \geq 0, X^\sigma \neq 0$ we have $M^\sigma(\lambda_0, c)X^\sigma = 0$, so by [36, Theorem 1.6], we get that

$$X^\sigma > 0 \quad \text{and} \quad 0 = \mathfrak{F}_{pf}(M^\sigma(\lambda_0, c)),$$

which contradicts (6.9) for this σ.

Now if $X^1 \neq 0$, so $0 = \mathfrak{F}_{pf}(M^1(\lambda_0, c))$ which contradicts (6.8). However, if $X^1 = 0$, then $M^2(\lambda_0, c)X^2 = 0$ which implies $\mathfrak{F}_{pf}\left(M^2(\lambda_0, c)\right) = 0$ provided that $X^2 > 0, \neq 0$, which is a contradiction with (6.9) when $\sigma = 2$. Again if $X^2 = 0$, then we will get a contradiction with (6.9) when $\sigma = 3$, etc. This means that since $X \geq 0, X \neq 0$, there must be a part of X that is non-zero, which gives us a contradiction with (6.9). Thus we prove the generalized first part of Lemma 6.2. $\square$

Note that when considering the second part of Lemma 6.2, if we do not know that the Perron-Frobenius eigenvalue of $M^1(\lambda, c)$ is strictly larger than the Perron-Frobenius eigenvalues of the other diagonal blocks of $M(\lambda, c)$, we cannot in fact have even a non-negative and non-zero eigenvector X. This is because, if we suppose that

$$M(\lambda, c) = \begin{pmatrix} B^1 & 0 & \ldots\ldots & & 0 \\ S^1 & B^2 & 0 & \ldots. & 0 \\ S^2 & S^3 & B^3 & \ldots & 0 \\ \vdots & \vdots & & \ddots & B^k \end{pmatrix},$$

where $B^1, ..., B^k$ are irreducible diagonal blocks, each of $S^1, ..., S^k$ contains at least one positive entry by (ii) in Remark 5.1, and η^1 is the Perron-Frobenius eigenvalue for B^1 with a strictly positive eigenvector X^1, and such that $\eta^1 > \eta^\sigma$ for $\sigma > 1$. Then

$$\begin{pmatrix} B^1 & 0 & \ldots\ldots & & 0 \\ S^1 & B^2 & 0 & \ldots. & 0 \\ S^2 & S^3 & B^3 & \ldots & 0 \\ \vdots & \vdots & & \ddots & B^k \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \\ X^3 \\ \vdots \\ X^k \end{pmatrix} = \eta^1 \begin{pmatrix} X^1 \\ X^2 \\ X^3 \\ \vdots \\ X^k \end{pmatrix},$$

and $S^1X^1 + B^2X^2 = \eta^1X^2$ which is true if and only if $(\eta^1 - B^2)\, X^2 = S^1X^1$. Since $S^1X^1 \geq 0, \neq 0$, then by Theorem 6.3 , we get $X^2 > 0$. Using Theorem 6.3 again, gives us $X^3 > 0$, since we have $(\eta^1 - B^3)\, X^3 = (S^1X^1 + S^2X^2)\, X^3$, and $S^1X^1 + S^2X^2 \geq 0$. Thus by repeating this argument of Theorem 6.3, k times, we get that the eigenvector $X = \left(X^1, ..., X^k\right)^T$ of $M(\eta^1, c)$ is strictly positive. Moreover, if we have a strictly positive eigenvector X, then the Perron-Frobenius eigenvalue of $M^1(\lambda, c)$ is strictly larger than the Perron-Frobenius eigenvalues of the other diagonal blocks of $M(\lambda, c)$.

The following example illustrates the reason for the need of the extra condition $\mathfrak{F}_{pf}(F^\sigma) > 0, \sigma > 1$ in Proposition 6.2, and shows what happens concerning the existence of stable monotone eigenvalues with strictly positive eigenvector and with non-negative eigenvector when we increase the value of c. We will discuss the possibilities for λ in three cases, and we show that for a large value of c, there exists at least one stable monotone eigenvalue $\lambda < 0$ with strictly positive eigenvector.

Example 6.1. Consider, for simplicity, the 2×2 matrix $M(\lambda, c)$ that contains two irreducible blocks

$$M(\lambda, c) = \begin{pmatrix} \lambda^2 a_1 + \lambda(c - d_1) + b_1 & 0 \\ e & \lambda^2 a_2 + \lambda(c - d_2) + b_2 \end{pmatrix},$$

where $b_1 > b_2$, and $a_1, a_2, b_1, e > 0$. Thus the eigenvalues are $\lambda^2 a_i + \lambda(c - d_i) + b_i, \quad i = 1, 2$, and to have that the eigenvalue of $M(\lambda, c)$ is 0, we need either

(i) $\lambda^2 a_1 + \lambda(c - d_1) + b_1 = 0$ and $\lambda^2 a_2 + \lambda(c - d_2) + b_2 < 0$, in which case the eigenvector is strictly positive, or

(ii) $\lambda^2 a_2 + \lambda(c - d_2) + b_2 = 0$, in which case the eigenvector is $(0, 1)^T$ and $\lambda^2 a_1 + \lambda(c - d_1) + b_1$ can be any value.

Now if $b_2 > 0$, (ii) has a solution for all c sufficiently large, whereas if $b_2 < 0$, (ii) has a solution for all c. This means that if we allow $b_2 < 0$, then allowing a non-negative eigenvector yields that we have the existence of a stable monotone eigenvalue for any c, and thus the condition in Proposition 6.2 is necessary.
If $b_2 = 0$, the solutions of $\lambda^2 a_2 + \lambda(c - d_2) = 0$ are $\lambda = 0$ and $\lambda = \dfrac{d_2 - c}{a_2}$ which is strictly negative (< 0) for c large. Thus (ii) has solutions at least for all $c > c_2$ for some c_2.
To have solutions of (i), we need $\lambda^2 a_1 + \lambda(c - d_1) + b_1 = 0$ and $\lambda^2 a_2 + \lambda(c - d_2) + b_2 < 0$, which is equivalent to

$$\lambda^2 a_1 + \lambda(c - d_1) + b_1 = 0 \quad \text{and} \quad \lambda^2(a_2 - a_1) - \lambda(d_2 - d_1) + b_2 - b_1 < 0. \tag{6.10}$$

The inequality in (6.10) defines a range of λ for which, if $\lambda^2 a_1 + \lambda(c - d_1) + b_1 = 0$, then λ is a stable monotone eigenvalue with strictly positive eigenvector. Since $\lambda^2 a_1 + \lambda(c - d_1) + b_1 =$

0 is a quadratic equation in λ, the solutions are

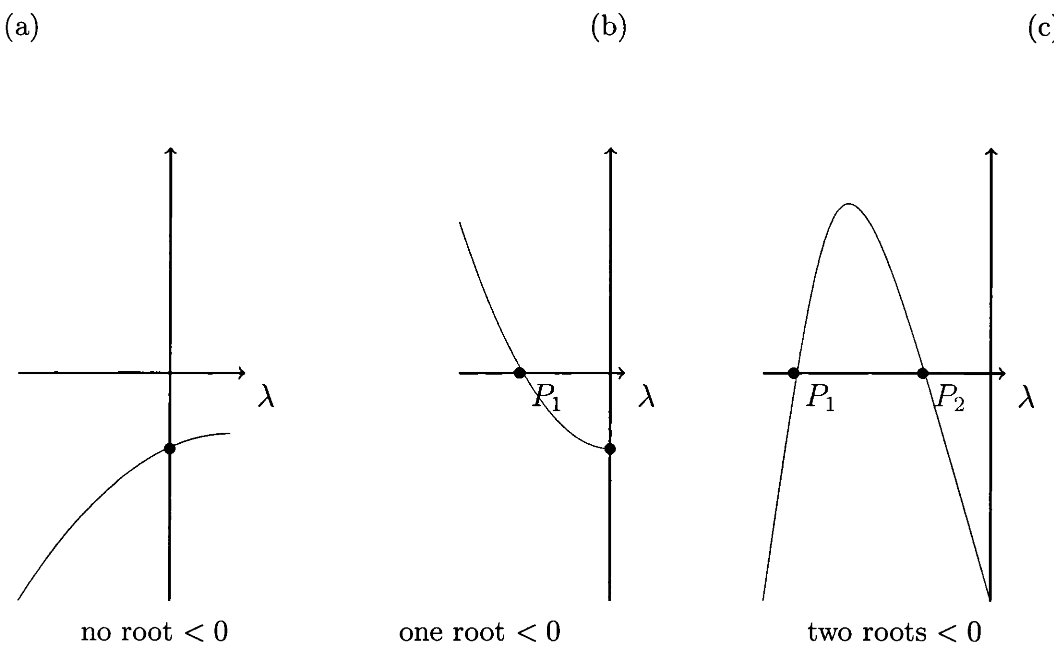
$$\lambda = \frac{-(c-d_1) \mp \sqrt{(c-d_1)^2 - 4a_1b_1}}{2a_1},$$

and hence it has a negative solution λ for all $c \geq d_1 + 2\sqrt{a_1b_1}$. Thus if $c = d_1 + 2\sqrt{a_1b_1}$, there is one stable monotone eigenvalue, λ_{neg} such that $\lambda_{neg} := -\sqrt{\frac{b_1}{a_1}}$, whereas if $c > d_1 + 2\sqrt{a_1b_1}$, there are two negative values λ_1, λ_2 with $\lambda_1 < \lambda_{neg} < \lambda_2$, and $\lambda_1 \to -\infty, \lambda_2 \to 0$ as $c \to \infty$.

There are three possible forms for $\lambda \mapsto \lambda^2(a_2 - a_1) - \lambda(d_2 - d_1) + b_2 - b_1$ depending on the parameters. The roots, P_1, P_2, of $\lambda^2(a_2 - a_1) - \lambda(d_2 - d_1) + b_2 - b_1 = 0$, are given by

$$\frac{(d_2 - d_1) \mp \sqrt{(d_2 - d_1)^2 - 4(a_2 - a_1)(b_2 - b_1)}}{2(a_2 - a_1)},$$

so we will discuss the three cases (a), (b) and (c) as follows.



Firstly, in case (a), the inequality (6.10) is satisfied for any $\lambda < 0$, so whenever $\lambda^2 a_1 + \lambda(c - d_1) + b_1 = 0$, λ is a stable monotone eigenvalue with strictly positive eigenvector. Therefore, there exists a stable monotone eigenvalue with strictly positive eigenvector for all $c \geq d_1 + 2\sqrt{a_1 b_1}$.

Secondly, in case (b), what happens depends on whether $\lambda_{neg} < P_1$ or $\lambda_{neg} > P_1$. If $\lambda_{neg} < P_1$, then for c close to $d_1 + 2\sqrt{a_1 b_1}$, there is no stable monotone eigenvalue, but when c is large enough that $\lambda_2 > P_1$, then λ_2 is a stable monotone eigenvalue.

Finally, in case (c), what happens depends on how λ_{neg} relates to P_1 and P_2. In particular, for $\lambda_{neg} < P_1$, both λ_1, λ_2 are stable monotone eigenvalues for c close to $d_1 + 2\sqrt{a_1 b_1}$, but when λ_2 reaches P_1, it stops being a stable monotone eigenvalue.
Moreover, different things happen when $P_1 < \lambda_{neg} < P_2$, or $P_2 < \lambda_{neg}$. Note that it is clearly not true that if λ is a stable monotone eigenvalue with strictly positive eigenvector for some c, then as c increases to ∞, a perturbation of λ is a stable monotone eigenvalue with strictly positive eigenvector. This is a contrast with Proposition 6.1. The fact that each of the curves in (a), (b) and (c) is negative close to zero, and $\lambda_2 \to 0$ as $c \to \infty$, tells us that there is a stable monotone eigenvalue with strictly positive eigenvector, close to zero when c is large enough. This illustrates the proof in Lemma 6.7 (2).$\square$

The structure of the 2×2 matrix, as we explained above, implies that when c increases, there is at least one stable monotone eigenvalue with strictly positive eigenvector. In the following example, Example 6.2, we consider a 3×3 matrix $M(\lambda, c)$ to illustrate that it is possible that there exists a stable monotone eigenvalue with strictly positive eigenvector for some values of c, but for a larger value of c, there is no such stable monotone eigenvalue corresponding to strictly positive eigenvector.

Example 6.2. Consider the 3×3 matrix $M(\lambda, c)$ such that

$$M(\lambda, c) = \begin{pmatrix} \lambda^2 a_1 + \lambda(c - d_1) + b_1 & 0 & 0 \\ e & \lambda^2 a_2 + \lambda(c - d_2) + b_2 & 0 \\ f & 0 & \lambda^2 a_3 + \lambda(c - d_3) + b_3 \end{pmatrix}.$$

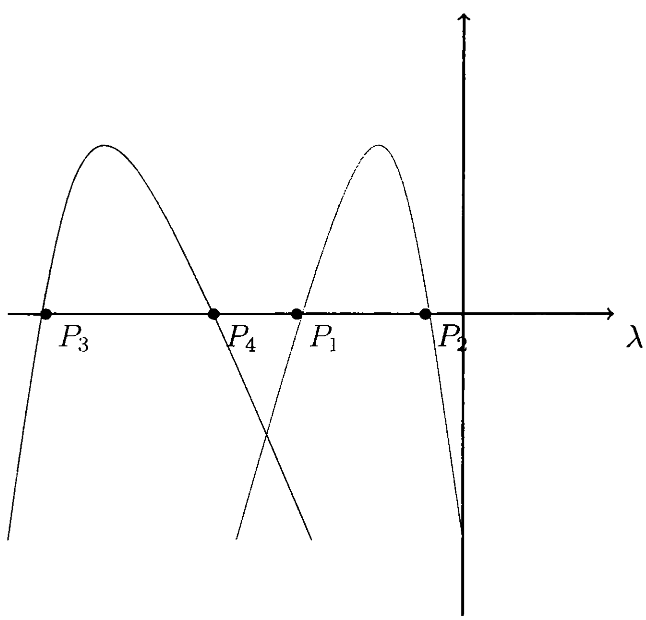
Then, similarly to before, we have a stable monotone eigenvalue with strictly positive eigenvector if

$$\lambda^2 a_1 + \lambda(c - d_1) + b_1 = 0,\ \lambda^2 a_2 + \lambda(c - d_2) + b_2 < 0, \text{ and } \lambda^2 a_3 + \lambda(c - d_3) + b_3 < 0,$$

which is equivalent to

$$\lambda^2 a_1 + \lambda(c - d_1) + b_1 = 0 \text{ and } \lambda^2(a_2 - a_1 - \lambda(d_2 - d_1) + b_2 - b_1 < 0, \text{ and}$$
$$\lambda^2(a_3 - a_1) - \lambda(d_3 - d_1) + b_3 - b_1 < 0.$$

The curves
$\lambda \mapsto \lambda^2(a_2 - a_1 - \lambda(d_2 - d_1) + b_2 - b_1,\ \lambda \mapsto \lambda^2(a_3 - a_1) - \lambda(d_3 - d_1) + b_3 - b_1,$
where $b_2 - b_1, b_3 - b_1 < 0$, depend on the parameters and could have a number of forms. In particular, the root $P_1 - P_4$ could be in the form



In this case, where λ_1, λ_2 and λ_{neg} are as in Example 6.1, if $P_4 < \lambda_{neg} < P_1$, then for c close to $d_1 + 2\sqrt{a_1 b_1}$, there exists a stable monotone eigenvalue with strictly positive eigenvector. But since $\lambda_1 \to -\infty$ and $\lambda_2 \to 0$ as $c \to \infty$, there exist $c^\sharp, c^b > d_1 + 2\sqrt{a_1 b_1}$ such that for $c^\sharp < c < c^b$, we have $P_3 < \lambda_2 < P_4$ and $P_1 < \lambda_1 < P_2$, so neither is a stable monotone eigenvalue with strictly positive eigenvector nor a stable monotone eigenvalue with non-negative eigenvector. $\square$

As a conclusion, note that we cannot directly generalize the third part in Lemma 6.2. As seen in Example 6.1 earlier, in a 2×2 matrix, when c increases, there is at least one stable monotone eigenvalue with strictly positive eigenvector, whereas in Example 6.2, in a 3×3 matrix, because of the structure of the cases for the eigenvalues that we have here, it is possible that there exist such eigenvalues for some c, and for c very large, but for some values of c in between such stable monotone eigenvalue with strictly positive eigenvector do not exist. Example 6.2 thus shows that the generalization of part (3) of Lemma 6.2 in Proposition 6.1, must be partial, and hence we cannot fully generalize Lemma 6.2 in the case of $M(\lambda, c)$ having multiple blocks.

Chapter 7

Examples

In this chapter we present some examples that illustrate key propositions and theorems. For instance, we illustrate Proposition 5.3, that gives a necessary condition for both right and left combined conditions for linear determinacy to be satisfied, Theorem 5.7, Theorem 5.8 about single right and single left spreading speeds, Theorem 5.9, and Theorem 5.10 concerning right and left linear determinacy.

Some examples illustrate that for a chosen function f and under some condition on the convection term h, an equation (scalar case) and a system (containing two equations) are each both right and left linearly determinate. We present examples of a system of two equations under some conditions on the parameters and convection terms that guarantee that a system has a right (left) single speed, which means that the right (left) slowest spreading speed equals the right (left) fastest spreading speed. We give examples to illustrate that a system is right (left) linearly determinate in the presence and absence of convection terms. On the other hand, there is an example showing that under a condition on the convection term, the system will not be left linearly determinate.

7.1 Examples illustrating sufficient conditions for both left and right linear determinacy

The first example considers the scalar case for a given choice of reaction term f, and illustrates conditions on the convection term h under which both the right and left conditions for linear determinacy (5.63) and (5.64) are satisfied.

Example 7.1. Choose $f : \mathbb{R} \to \mathbb{R}$ such that $f(u) = u(1-u)(u+\delta)$, where $\delta > 0$ (see also Example 2.2 in Chapter 2). If $\delta \geq 1$, this function f satisfies properties $E_1 - E_3$ and (2.1). Then with $\bar{\mu} = \sqrt{\delta}, \zeta(\bar{\mu}) = 1$, (5.67) becomes

$$-\Lambda(\rho) \leq \rho\sqrt{\delta}\left[h'(0) - h'(\rho)\right] \leq \Lambda(\rho), \quad \rho > 0, \tag{7.1}$$

where $\Lambda(\rho) = \rho f'(0) - f(\rho) = \rho\delta - (\rho^2 + \rho\delta - \rho^3 - \rho^2\delta) = \rho^3 + (\delta - 1)\rho^2 > 0$. An example of a function h satisfying (7.1) can be constructed by, for instance, taking $\rho\sqrt{\delta}\left[h'(0) - h'(\rho)\right] \equiv \Lambda(\rho)$, in which case $h'(\rho) = h'(0) - \dfrac{\Lambda(\rho)}{\rho\sqrt{\delta}} = h'(0) - \dfrac{\rho^2 + \rho(\delta-1)}{\sqrt{\delta}}$ which implies that $h'(\rho) = A + \dfrac{\rho(1-\rho-\delta)}{\sqrt{\delta}}$, where $A := h'(0)$, and hence a function h that satisfies (7.1) is

$$h(\rho) = \frac{(1-\delta)\rho^2}{2\sqrt{\delta}} - \frac{\rho^3}{3\sqrt{\delta}} + A\rho + B, \qquad A, B \in \mathbb{R}.$$

Thus by Proposition 5.3, the equation

$$u_t + h(\rho) = u_{xx} + u(1-u)(u+\delta), \tag{7.2}$$

where $\delta > 0$, is both right and left linearly determinate. On the other hand, if $0 < \delta < 1$, the function f does not satisfy condition (2.1), and thus by Proposition 5.3, it is impossible to find a function h which satisfies both the right and left combined conditions (5.63) and (5.64). $\square$

Our second example employs a function $f : \mathbb{R}^2 \to \mathbb{R}^2$, also used in [42, Example 4.1], that falls into the second category in Remark 5.1 (iii) and so has two blocks in the Frobenius form of $f'(0)$. This reaction function f is obtained from a competition nonlinearity using a well-known change of variables that converts competition systems to co-operative systems; see [42]. Here we derive conditions on the diffusion coefficient d_2 and $a, b \in \mathbb{R}$ which are sufficient to allow the construction of a function $h = (h_1(u_1), h_2(u_2))$ such that $h'(0) = \mathrm{diag}(a, b)$ and both (5.63) and (5.64) are satisfied for f and h. Two separate cases are treated: first, when $h'(0) = \mathrm{diag}(a, a)$ for some $a \in \mathbb{R}$, in which case the eigenvectors

$\zeta(\bar{\mu}), \hat{\zeta}(\bar{\mu})$ are equal, and second, when $h'(0) = \text{diag}(a,b)$ with $a \neq b$, in which case $\zeta(\bar{\mu}) \neq \hat{\zeta}(\bar{\mu})$. □

Example 7.2. Choose

$$f(u_1,u_2) = \begin{pmatrix} 3u_1 - 4u_1^2 + u_1u_2 \\ 5u_2^2 - u_2 + 8u_1 - 4u_2^3 - 8u_1u_2 \end{pmatrix},$$

so that

$$f'(u_1,u_2) = \begin{pmatrix} 3 - 8u_1 + u_2 & u_1 \\ 8 - 8u_2 & 10u_2 - 1 - 12u_2^2 - 8u_1 \end{pmatrix}, \; f'(0) = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix},$$

and denote $h'(0) = \text{diag}\,(a,b)$, where $a, b \in \mathbb{R}$ with possibly $a \neq b$. There are four solutions of $f(u_1,u_2) = (0,0)$ with $u_1, u_2 \geq 0$, namely the four equilibria $(0,0)$, $(0,\frac{1}{4})$, $(0,1)$ and $(1,1)$. Taking $\beta = (1,1)$, Hypotheses $s_1 - s_6$ are clearly satisfied with the minor modification that hypothesis s_1 holds for all $(u,u_2) \in [(0,0),(1,1)]$ rather than for all (u_1,u_2), which is easily seen to be sufficient for the above theory to apply because all solutions (u_1,u_2) of (5.1) considered here lie between the equilibria $(0,0)$ and $\beta = (1,1)$. Then $\bar{\mu} = \hat{\bar{\mu}} = \sqrt{3}$, and the coefficient matrices $C_{\bar{\mu}}$ and $\hat{C}_{\bar{\mu}}$ respectively are

$$C_{\bar{\mu}} = \begin{pmatrix} d_1\bar{\mu}^2 + a\bar{\mu} + 3 & 0 \\ 8 & d_2\bar{\mu}^2 + b\bar{\mu} - 1 \end{pmatrix} = \begin{pmatrix} 6 + a\sqrt{3} & 0 \\ 8 & 3d_2 + b\sqrt{3} - 1 \end{pmatrix},$$

and

$$\hat{C}_{\bar{\mu}} = \begin{pmatrix} d_1\bar{\mu}^2 - a\bar{\mu} + 3 & 0 \\ 8 & d_2\bar{\mu}^2 - b\bar{\mu} - 1 \end{pmatrix} = \begin{pmatrix} 6 - a\sqrt{3} & 0 \\ 8 & 3d - b\sqrt{3} - 1 \end{pmatrix}.$$

Thus the eigenvectors $\zeta(\bar{\mu})$, $\hat{\zeta}(\bar{\mu})$ for $\gamma_1(\bar{\mu}) = 6 + \sqrt{3}a$, $\hat{\gamma}_1(\bar{\mu}) = 6 - \sqrt{3}a$ are

$$\zeta(\bar{\mu}) = \begin{pmatrix} 1 \\ 8/(7 + \sqrt{3}(a-b) - 3d_2) \end{pmatrix} =: \begin{pmatrix} 1 \\ \alpha_2 \end{pmatrix},$$

$$\hat{\zeta}(\bar{\mu}) = \begin{pmatrix} 1 \\ 8/(7 - \sqrt{3}(a-b) - 3d_2) \end{pmatrix} =: \begin{pmatrix} 1 \\ \alpha_2^* \end{pmatrix}.$$

Provided $d_2 < 7/3$, it follows from Lemma 5.6 (and by inspection) that if a, b satisfy

$$\frac{-(7-3d_2)}{\sqrt{3}} < a - b < \frac{(7-3d_2)}{\sqrt{3}},$$

then the eigenvectors $\zeta(\bar{\mu}), \hat{\zeta}(\bar{\mu})$ are strictly positive and (5.54), (5.60) are satisfied.

Now define $\eta := a - b$ and consider the cases $\eta = 0$ and $\eta \neq 0$. If $\eta = 0$, the eigenvectors $\zeta(\bar{\mu})$, $\hat{\zeta}(\bar{\mu})$ are equal (*cf.* Lemma 5.3), and provided d_2 satisfies the stricter restriction that $d_2 \leq 2/3$, the function f satisfies the necessary condition (5.65) of Proposition 5.3, in which case it is clearly possible to construct functions h_1, h_2 for which both (5.63), (5.64) hold by using a similar method to that in our explicit construction of h in Example 7.1.

For $\eta \neq 0$, to have that both conditions (5.63), (5.64) are satisfied for a given h_1, we require

$$h_1'(\rho) \leq a - \left(\frac{\rho^2(-4+\alpha_2)}{\sqrt{3}\rho}\right) = a - \left(\frac{\rho(-4+\alpha_2)}{\sqrt{3}}\right) \qquad \rho > 0, \tag{7.3}$$

and

$$h_1'(\rho) \geq a + \left(\frac{\rho^2(-4+\alpha_2^*)}{\sqrt{3}\rho}\right) = a + \left(\frac{\rho(-4+\alpha_2^*)}{\sqrt{3}}\right) \qquad \rho > 0. \tag{7.4}$$

writing $t = \rho$ in (7.3) and (7.4) respectively, so we obtain that

$$\frac{t}{\sqrt{3}}(\alpha_2^* - 4) \leq h_1'(t) - a \leq \frac{t}{\sqrt{3}}(-\alpha_2 + 4), \quad t > 0,$$

which can be satisfied if $\alpha_2^* - 4 \leq -\alpha_2 + 4$, which holds if

$$\frac{8}{7-\sqrt{3}(a-b)-3d_2} + \frac{8}{7+\sqrt{3}(a-b)-3d_2} \leq 8,$$

thus $\dfrac{1}{7-\sqrt{3}(a-b)-3d_2} + \dfrac{1}{7+\sqrt{3}(a-b)-3d_2} \leq 1$, which yields

$$\frac{7+\sqrt{3}\eta-3d_2+7-\sqrt{3}\eta-3d_2}{(7-\sqrt{3}\eta-3d_2)(7+\sqrt{3}\eta-3d_2)} \leq 1,$$

and gives

$$\frac{14-6d_2}{49+7\sqrt{3}\eta-21d_2-7\sqrt{3}\eta-3\eta^2+3\sqrt{3}\eta d_2-21d_2-3\sqrt{3}\eta d_2+9d_2^2}\leq 1.$$

That is equivalent to requiring

$$9d_2^2-36d_2+35-3\eta^2\geq 0. \tag{7.5}$$

Since (7.5) holds if $d_2\leq(6-\sqrt{1+3\eta^2})/3$, a function h_1 satisfying the first inequality in each of (5.63), (5.64) can be constructed for such d_2,η. Note that the larger values of d_2 that satisfy (7.5) violate the additional requirement that $d_2<7/3$, and that it is clearly necessary to have $\eta^2<35/3$ to be able to construct h_1 for some $d_2>0$.

Now for the existence of a function h_2 such that both (5.63), (5.64) are satisfied, we need for $\rho>0$ that

$$h_2'(\rho\alpha_2)\leq b-\left(\frac{5(\rho\alpha_2)^2-4(\rho\alpha_2)^3-8\rho^2\alpha_2}{\sqrt{3}\rho\alpha_2}\right)=b-\left(\frac{5(\rho\alpha_2)-4(\rho\alpha_2)^2-8\rho}{\sqrt{3}}\right), \tag{7.6}$$

and

$$h_2'(\rho\alpha_2^*)\geq b+\left(\frac{5(\rho\alpha_2^*)^2-4(\rho\alpha_2^*)^3-8\rho^2\alpha_2^*}{\sqrt{3}\rho\alpha_2^*}\right)=b+\left(\frac{5(\rho\alpha_2^*)-4(\rho\alpha_2^*)^2-8\rho}{\sqrt{3}}\right). \tag{7.7}$$

Writing $t=\rho\alpha_2$ and $t=\rho\alpha_2^*$ in (7.6) and (7.7) respectively, so we obtain that

$$\frac{5t-4t^2-8t/\alpha_2^*}{\sqrt{3}}\leq h_2'(t)-b\leq-\left(\frac{5t-4t^2-8t/\alpha_2}{\sqrt{3}}\right),\quad t>0,$$

which can hold if

$$10t-8t^2-8t\left(\frac{7-\sqrt{3}\eta-3d_2}{8}\right)-8t\left(\frac{7+\sqrt{3}\eta-3d_2}{8}\right)\leq 0,$$

and hence it holds if $-8t^2-4t+6td_2\leq 0$. Thus we need that $-2t(2+4t-3d_2)\leq 0$ for all $t>0$, which holds if $d_2\leq(2+4t)/3$ for all $t>0$. Hence a function h_2 satisfying (5.63), (5.64) can always be constructed, regardless of the value of η, provided $d_2\leq 2/3$.

Therefore we have shown that for this choice of $f : \mathbb{R}^2 \to \mathbb{R}^2$, if $0 \leq |a-b| \leq \sqrt{5}$, a sufficient condition to be able to find a function $h = (h_1, h_2)$ so that both combined conditions (5.63), (5.64) are satisfied and $h'(0) = \text{diag}(a,b)$, is $d_2 \leq 2/3$, whereas if $\sqrt{5} \leq |a-b| < \sqrt{35/3}$, a function $h = (h_1, h_2)$ for which (5.63), (5.64) both hold and $h'(0) = \text{diag}(a,b)$ can be constructed provided

$$d_2 \leq \frac{6 - \sqrt{1 + 3(a-b)^2}}{3}.$$

Thus this implies that for this chose of functions f and h, the system is both right and left linearly determinate by Theorem 5.9 and Theorem 5.10. □

7.2 Examples about the single right (left) spreading speed and right (left) linear determinacy

The following example illustrates [26, Theorem 3.1] and [42, Proposition 2.1] about the single speed of a system that has more than two equilibria in the absence of the convection terms, and the generalized result, Theorem 5.8. We consider the system that was discussed in [42, Theorem 4.4] with the addition of convection terms and with a non-increasing initial condition u_0. This example illustrates that the linear value for a co-operative system with convection terms may equal the linear value for a Fisher-type equation with convection term (1.21). Note that a sufficient condition for the right linear value of Fisher equation with convection (1.21) to be equal the right spreading speed for such equation is $h_1'(u_1) \leq h_1'(0)$; see (2.20).

Example 7.3. Consider the co-operative system with convection terms

$$u_{1,t} = u_{1,xx} - h_1'(u_1)u_{1,x} + r_1 u_1(1 - a_1 - u_1 + a_1 u_2),$$

$$u_{2,t} = d_2 u_{2,xx} - h_2'(u_2)u_{2,x} + r_2(1-u_2)(a_2 u_1 - u_2), \tag{7.8}$$

with a non-increasing initial condition u_0, where $r_i, a_i > 0,\ u_i \geq 0$, for $i = 1, 2$, $r_1(1-a_1) >$

$0, a_2 > 1$ and the convection terms satisfy that

$$h_i'(u_i) \leq h_i'(0) \qquad \text{for } i = 1, 2.$$

Now, since the convection terms do not affect the number of equilibria, we can apply Theorem 5.8 to show that the sufficient condition is satisfied for system (7.8). If will be shown that (7.8) has four equilibria and then we will conclude that the system has right single speed. First, we find the equilibria as follows. We want to find (u_1, u_2) such that $f(u_1, u_2) = 0$ which holds if and only if

$$r_1 u_1(1 - a_1 - u_1 + a_1 u_2) = 0, \tag{7.9}$$

$$r_2(1 - u_2)(a_2 u_1 - u_2) = 0. \tag{7.10}$$

From equation (7.10) we obtain, $u_2 = 1$ or $u_1 = \frac{u_2}{a_2}$. Substituting these values in (7.9) we get $u_1 = 0, u_1 = 1$, and $r_1(\frac{u_2}{a_2})(1 - a_1 - \frac{u_2}{a_2} + a_1 u_2) = 0$. This yields $u_1 = \frac{1 - a_1}{1 - a_1 a_2}$ and so

$$(u_1, u_2) = \left(\frac{1 - a_1}{1 - a_1 a_2}, \frac{a_2(1 - a_1)}{1 - a_1 a_2} \right). \tag{7.11}$$

Thus the four equilibria are $(0,0), (0,1), (1,1), (u_1, u_2)$. Since $a_2 > 1$, our equilibria will be $(0,0)$ and $\beta = (1,1)$, which means we are left with just two equilibria, not more, and thus by Theorem 5.8, the system has single right speed. In other words, $\mathring{c} = \mathring{c}_f$.
Now to evaluate the right linear value for the system (7.8) we have

$$f'(u_1, u_2) = \begin{pmatrix} r_1 - a_1 r_1 - 2r_1 u_1 + a_1 r_1 u_2 & r_1 a_1 u_1 \\ r_2 a_2 (1 - u_2) & -r_2 + 2r_2 u_2 - a_2 r_2 u_1 \end{pmatrix},$$

and hence $f'(0,0) = \begin{pmatrix} r_1 - a_1 r_1 & 0 \\ r_2 a_2 & -r_2 \end{pmatrix}$. Thus the coefficient matrix in equation (5.33) is

$$C_\mu = \begin{pmatrix} \mu^2 + a\mu + r_1(1 - a_1) & 0 \\ r_2 a_2 & d_2 \mu^2 + b\mu - r_2 \end{pmatrix},$$

where $h_1'(0) = a, h_2'(0) = b$. Therefore from the definition of $\mathring{c}$ in (5.37), in which the infimum is attained at $\bar{\mu}$, we have

$$\mathring{c} = \bar{\mu}^{-1}(\mu^2 + a\mu + r_1(1-a_1)) = \bar{\mu} + a + \frac{r_1}{\bar{\mu}} - \frac{r_1 a_1}{\bar{\mu}}.$$

Let $E(\mu) = \mu + a + \frac{r_1}{\mu} - \frac{r_1 a_1}{\mu}$, then $E'(\mu) = 1 - \frac{r_1{}^2}{\bar{\mu}} + \frac{r_1 a_1}{\bar{\mu}^2} = 0$ if and only if $\mu = \sqrt{r_1(1-a_1)}$, and hence $\bar{\mu} = \sqrt{r_1(1-a_1)}$. Therefore the right linear value for (7.8) is

$$\mathring{c} = \sqrt{r_1(1-a_1)} + \frac{r_1(1-a_1)}{\sqrt{r_1(1-a_1)}} + a = 2\sqrt{r_1(1-a_1)} + a.$$

On the other hand, the Fisher equation with convection term is

$$u^\sharp_{1,t} = u^\sharp_{1,xx} - h_1'(u^\sharp_1)u^\sharp_{1,x} + r_1 u^\sharp_1(1 - a_1 - u^\sharp_1), \tag{7.12}$$

and since we have $u_1, u_2 \geq 0$, then

$$\begin{aligned} u_{1,t} &= u_{1,xx} - h_1'(u_1)u_{1,x} + r_1 u_1(1 - a_1 - u_1 + a_1 u_2) \\ &\geq u_{1,xx} - h_1'(u_1)u_{1,x} + r_1 u_1(1 - a_1 - u_1). \end{aligned}$$

Consider (7.12) with initial condition given by the first component $(u_0)_1$ of the initial condition u_0 of (7.8) such that $0 \leq (u_0)_1 \leq 1 - a_1$. We choose the initial condition $(u_0)_1$ in this way because the upper equilibrium of (7.12) is $u^\sharp_1 = 1 - a_1$. Thus the first component converges to $1 - a_1\epsilon < 1$. This is a special case of the condition in Theorem 3.3, so it will spread at a speed no slower than the right slowest spreading speed $\mathring{c}$.

Thus the first component u_1 of the solution u of the system (7.8) is a supersolution of (7.12), and hence by the Comparison Theorem 5.1, is bounded below by the solution $u^\sharp$ of (7.12) with initial condition $(u_0)_1$. The right fastest (slowest) spreading speed for the system $\mathring{c}_f(\mathring{c})$ is thus bounded below ($\geq$) by the right spreading speed for the Fisher equation with convection (7.12), which we refer as $c^\sharp_1$, so

$$\mathring{c} \geq c^\sharp_1.$$

Since $h_1'(u_1) \leq h_1'(0)$, then we have $c_1^\sharp = \bar{c}_1$ where $\bar{c}_1$ is the linear value for (7.12) which defined in (2.18). Thus we have $\mathring{c} \geq \bar{c}_1$.

Now in order to evaluate $\bar{c}_1$, since we have from (7.12) that $f_1(u^\sharp) = r_1 u^*(1 - a_1 - u^*)$, then $f_1'(u^\sharp) = r_1 - a_1 r_1 - 2r_1 u^*$ which yields $f_1{}'(0) = r_1(1 - a_1)$ and from (2.18) we get $\bar{c}_1 = 2\sqrt{r_1 - r_1 a_1} + h_1'(0)$. It is clear that $\mathring{\mathring{c}} = \bar{c}_1$, so as a conclusion, we have

$$\mathring{c}_f = \mathring{c} \geq \mathring{\mathring{c}} = \bar{c}_1 = c_1^\sharp.$$

□

The following example illustrates that it is possible for the right fastest spreading speed to be strictly larger than the right slowest spreading speed for a system that involves convection terms, provided the convection terms satisfy a sufficient condition. This example is a modification of [26, Example 4.1] with the addition of convection terms $h_i'(u_i)u_{i,x}$ for $i = 1, 2$ and with a non-increasing initial condition u_0.

Example 7.4. Consider the cooperative two-species Lotka-Volterra model

$$u_{1,t} = u_{1,xx} - h_1'(u_1)u_{1,x} + r_1 u_1(1 - u_1 + a_1 u_2) \tag{7.13}$$

$$u_{2,t} = d_2 u_{2,xx} - h_2'(u_2)u_{2,x} + r_2 u_2(1 - u_2 + a_2 u_1) \tag{7.14}$$

with a non-increasing initial condition u_0, where all parameters are positive, $a_1 a_2 < 1$, the reaction term f, and convection term h satisfy Hypotheses $s_1 - s_6$, and the additional condition that the convection terms satisfy $h_i'(u_i) \leq h_i'(0)$ for all $u_i \in [0, 1]$, $i = 1, 2$. By following the previous procedure for calculating the equilibria, we find that the system (7.13), (7.14) has four equilibria $(0, 0), (0, 1), (1, 0), (u_1^*, u_2^*)$, with

$$(u_1^*, u_2^*) = (\frac{1 + a_1}{1 - a_1 a_2}, \frac{1 + a_2}{1 - a_1 a_2}).$$

Since there are four equilibria, and by Theorem 5.8, the system does not necessarily have a right single speed. Note first that the right linear value for system (7.13), (7.14) equals the right linear value for a Fisher equation with convection term, $\bar{c}_1$, which is obtained from equation (7.13), namely

$$u^{\sharp}_{1,t} = u^{\sharp}_{1,xx} - h_1'(u^{\sharp}_1)u^{\sharp}_{1,x} + r_1 u^{\sharp}_1(1 - u^{\sharp}_1). \tag{7.15}$$

From (2.18), we get $\bar{c}_1 = 2\sqrt{f_1'(0)} + h_1'(0) = 2\sqrt{r_1} + a$ where $a = h_1'(0)$, whereas the right linear value for the system is $\mathring{\bar{c}} = \bar{\mu} + a + r_1\bar{\mu}^{-1}$, where $\bar{\mu} = \sqrt{r_1}$, which implies that $\mathring{\bar{c}} = 2\sqrt{r_1} + a = \bar{c}_1$.

Since we know that

$$\begin{aligned} u_{1,t} &= u_{1,xx} - h_1'(u_1)u_{1,x} + r_1 u_1(1 - u_1 + a_1 u_2) \\ &\geq u_{1,xx} - h_1'(u_1)u_{1,x} + r_1 u_1(1 - u_1), \end{aligned}$$

and since $u_2, u_1 \geq 0$, then u_1 is a supersolution for (7.15) with the initial condition $u_0(x) = (u_0)_1$, and by the Comparison Theorem 5.1, we have $u_1(x,t) \geq u^{\sharp}(x,t)$ where $u^{\sharp}$ is the solution of (7.15) with initial condition $0 \leq u_0 \leq 1$. We choose an initial condition u_0 such that the first component of the system (7.13) converges to $1 - \epsilon$ for some $\epsilon > 0$, so [26, Theorem 3.1] implies that $(u_0)_1$ will spread at a speed no slower than the right slowest spreading speed $\mathring{c}$. Since $u_2 \geq 0$, and by the Comparison Theorem 5.1 we have that u_1 can not spread more slowly than it would if we replaced u_2 by 0 in (7.13), and we conclude that

$$\mathring{c}_f \geq \bar{c}_1 = \mathring{\bar{c}}, \tag{7.16}$$

and since $h_1'(u_1) \leq h_1'(0)$, so $\bar{c}_1 = c^{\sharp}_1$, where $c^{\sharp}_1$ is the right spreading speed for (7.15). Likewise, since we know that $u_1 \leq u^*_1$, we obtain

$$u_{2,t} = d_2 u_{2,xx} - h_2'(u_2)_{2,x} + r_2 u_2(1 - u_2 + a_2 u_1) \tag{7.17}$$

$$\leq d_2 u_{2,xx} - h_2'(u_2)u_{2,x} + r_2 u_2\left(1 - u_2 + a_2(\frac{1+a_1}{1-a_1a_2})\right), \tag{7.18}$$

so u_2 is a subsolution for the equation

$$u^{\sharp}_{2,t} = d_2 u^{\sharp}_{2,xx} - h_2'(u^{\sharp}_2)u^{\sharp}_{2,x} + r_2 u^{\sharp}_2\left(1 - u^{\sharp}_2 + a_2(\frac{1+a_1}{1-a_1a_2})\right), \tag{7.19}$$

and hence, by the Comparison Theorem 5.1, $u_2 \leq u_2^\sharp$, where $u_2^\sharp$ is the solution of (7.19) with initial condition $(u_0)_2$. We choose an initial condition u_0 for (7.13)-(7.14) such that the initial condition of the second component (7.19), $(u_0)_2$, satisfies $0 \leq (u_0)_2 < \dfrac{1+a_2}{1-a_1a_2}$, and it follows that $(u_0)_2$ will spread at a speed no slower than the right slowest spreading speed $\mathring{c}$. The Comparison Theorem 5.1 shows that u_2 can not spread more rapidly than it would if we replaced u_1 by u_1^* in (7.14). Since $h_2^{'}(u_2) \leq h_2^{'}(0)$, the spreading speed of the equation (7.19) equals the linear value $\bar{c}_2$ for equation (7.19) and $\bar{c}_2 = 2\sqrt{d_2 f_2^{'}(0)} + h_2^{'}(0) = 2\sqrt{d_2 r_2 u_2^*} + b$, where $b = h_2^{'}(0)$. So since $u_2, u_2^\sharp$ are non-increasing in x, the fact that $u_2 \leq u_2^\sharp$ implies that

$$\mathring{c} \leq \bar{c}_2 = 2\sqrt{d_2 r_2 u_2^*} + b. \tag{7.20}$$

Therefore if $2\sqrt{r_1} + a > 2\sqrt{d_2 r_2 u_2^*} + b$, it follows from (7.16) and (7.20) that

$$\mathring{c}_f > \mathring{c}.$$

In particular, this implies that if a is sufficiently larger than b, then we have that the right fastest spreading speed is strictly bigger than the right slowest one. □

The following example considers a system that is discussed in [14]. It has four equilibria, but two of them will be outside the rectangle $[T, S]$ such that $T = (0,0), S = (1,1)$ provided certain conditions are satisfied. Note that in this example, S is a stable equilibrium and T is an unstable one (note that, in contrast, S is unstable and T is stable in [14]).

Example 7.5. First consider the system without convection terms

$$u_{1,t} = u_{1,xx} - \alpha_1 u_1 + \alpha_1 u_2 + r_1(u_1 - u_1^2), \tag{7.21}$$

$$u_{2,t} = u_{2,xx} + \alpha_2 u_1 - \alpha_2 u_2 + r_2(u_2 - u_2^2), \tag{7.22}$$

with non-decreasing initial condition u_0, and where all parameters are positive. Following the same procedure to find the four equilibria as previously, we find that from equation (7.22) we have

$$u_1 = \alpha_2^{-1}[\alpha_2 u_2 - r_2(u_2 - u_2^2)] = u_2 - \alpha_2^{-1} r_2 (u_2 - u_2^2). \tag{7.23}$$

Substituting this value in (7.21) we get

$$(u_2 - u_2^2)\left[\alpha_1\alpha_2^{-1}r_2 + r_1 - \alpha_2^{-1}r_1r_2 + 2\alpha_2^{-1}r_2r_1u_2 - \alpha_2^{-2}r_2^{-2}r_1(u_2 - u_2^2)\right] = 0,$$

which gives us $u_2 = 0, u_2 = 1$, and

$$u_2 = \frac{1}{2}\left[(1 - 2\alpha_2 r_2^{-1}) \mp \sqrt{(2\alpha_2 r_2^{-1} - 1)^2 - 4(\alpha_1\alpha_2 r_2^{-1}r_1^{-1} + \alpha_2^2 r_2^{-2} - \alpha_2 r_2^{-1})}\right].$$

This implies that $u_1 = 0, u_1 = 1$ respectively. Now if $r_1 r_2 \geq 4\alpha_1\alpha_2$ (as presented in [33], and noted in [14]), then $(2\alpha_2 r_2^{-1} - 1)^2 - 4(\alpha_1\alpha_2 r_2^{-1}r_1^{-1} + \alpha_2^2 r_2^{-2} - \alpha_2 r_2^{-1}) > 0$, which means that the values of u_2 are

$$u_2 = \frac{1}{2} - \alpha_2 r_2^{-1} \mp \sqrt{\frac{1}{4} - \frac{\alpha_1\alpha_2}{r_1 r_2}}.$$

Let $s = \dfrac{1}{4} - \dfrac{\alpha_1\alpha_2}{r_1 r_2}$. If we substitute these values of u_2 in (7.23) we obtain $u_1 = \dfrac{1}{2} - \dfrac{\alpha_1}{r_1} \mp \sqrt{s}$. So the four equilibria are $(0,0), (1,1), E_- := (u_1^-, u_2^-), E_+ := (u_1^+, u_2^+)$, where

$$(u_1, u_2) = \left(\frac{1}{2} - \frac{\alpha_1}{r_1} \mp \sqrt{s}, \frac{1}{2} - \frac{\alpha_2}{r_2} \pm \sqrt{s}\right).$$

Now to prove that $E_\mp$ are outside the rectangle $[T, S]$, it is enough to show that one component of (u_1, u_2) is less than zero, such as $u_1 < 0$. Thus we need $\dfrac{1}{2} - \dfrac{\alpha_1}{r_1} - \sqrt{s} < 0$, which occurs if and only if

$$\sqrt{\frac{1}{4} - \frac{\alpha_1 r_1}{\alpha_2 r_2}} > \frac{1}{2} - \frac{\alpha_1}{r_1}. \tag{7.24}$$

The inequality in (7.24) is satisfied in three cases, (i) when $\dfrac{1}{2} - \dfrac{\alpha_1}{r_1} < 0$, (ii) when $\dfrac{1}{2} - \dfrac{\alpha_2}{r_2} < 0$ which implies that $\dfrac{1}{2} - \dfrac{\alpha_2}{r_2} - \sqrt{s} < 0$, and (iii) when $\dfrac{1}{2} - \dfrac{\alpha_1}{r_1} \geq 0$ and $\dfrac{1}{2} - \dfrac{\alpha_2}{r_2} \geq 0$ which implies that $1 \geq \dfrac{\alpha_1}{r_1} + \dfrac{\alpha_2}{r_2}$.
So $E_\mp$ are outside the rectangle and we are left with just two equilibria $(0,0), (1,1)$. Thus the system (7.22) has left single speed by Theorem 5.7, which means that $\hat{c}_f = \hat{c}$.

Now we want to show that the function $f(u_1, u_2)$ satisfies the sufficient condition in Theorem 5.10, which is a straightforward modification of [42, Theorem 4.2] for non-decreasing initial condition and in addition of convection terms, from which it follows that the single left spreading speed $\hat{c}$ for the system equals the left linear value $\hat{\hat{c}}$ for the system. Since condition (5.60) is trivially satisfied here, because we have only one block of $f'(0)$, a sufficient condition for left linear determinacy is

$$f_i(u_1, u_2) \le f_i'(0,0)\,(u_1\; u_2)^T \qquad \text{for all } u_1, u_2 \ge 0, i = 1, 2. \tag{7.25}$$

Note that a generalization of condition (7.25) is the condition in [42, Theorem 4.2], that is

$$f\left(\rho\zeta(\bar{\mu})\right) \le \rho f'(0)\zeta(\bar{\mu}). \tag{7.26}$$

But (7.25) is easier to check since we do not need to evaluate $\bar{\mu}, \zeta(\bar{\mu})$, and it happens to be true here. Condition (7.25) is

$$\begin{pmatrix} -\alpha_1 u_1 + \alpha_1 u_2 + r_1(u_1 - u_1^2) \\ \alpha_2 u_1 - \alpha_2 u_2 + r_2(u_2 - u_2^2) \end{pmatrix} \le \begin{pmatrix} -\alpha_1 + r_1 & \alpha_1 \\ \alpha_2 & -\alpha_2 + r_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

which it holds if and only if $-\alpha_1 u_1 + \alpha_1 u_2 + r_1(u_1 - u_1^2) \le (-\alpha_1 + r_1)u_1 + \alpha_1 u_2$. It is require that $-r_1 u_1^2 \le 0$, which is true since $r_1 \ge 0$, and $\alpha_2 u_1 - \alpha_2 u_2 + r_2(u_2 - u_2^2) \le (\alpha_2 u_1 + (-\alpha_2 + r_2)u_2$, which requires that $-r_2 u_2^2 \le 0$, which is true since $r_2 \ge 0$. Thus we have

$$\hat{c}_f = \hat{c} = \hat{\hat{c}}.$$

□

The following example modifies the system (7.21), (7.22) by the addition of the terms $-h_1'(u_1)u_{1,x}, -h_2'(u_2)u_{2,x}$. Using Theorem 5.10 we will show that the system has single left spreading speed that equals its left linear value, which means that the system is left-linearly determinate.

Example 7.6. Consider the system below

$$u_{1,t} = u_{1,xx} - h_1'(u_1)u_{1,x} - \alpha_1 u_1 + \alpha_1 u_2 + r_1(u_1 - u_1^2) \tag{7.27}$$

$$u_{2,t} = u_{2,xx} - h_2'(u_2)u_{2,x} + \alpha_2 u_1 - \alpha_2 u_2 + r_2(u_2 - u_2^2),$$

with non-decreasing initial data, where all parameters are positive, $r_1 r_2 \geq 4\alpha_1\alpha_2$, and $r_1 > \alpha_1$. The reaction term

$$f = (f_1(u_1,u_2), f_2(u_1,u_2)) = \begin{pmatrix} -\alpha_1 u_1 + \alpha_1 u_2 + r_1(u_1 - u_1^2) \\ \alpha_2 u_1 - \alpha_2 u_2 + r_2(u_2 - u_2^2) \end{pmatrix}, \tag{7.28}$$

and convection term h satisfy Hypotheses $s_1 - s_6$ and the function h_i is supposed additionally to satisfy

$$h_i'(0) \leq h_i'(u_i) \qquad \text{for } u_i \in [0,1],\ i = 1,2.$$

As we showed above, the system (7.21), (7.22) has only two equilibria and since the convection terms do not effect the number of equilibria provided the condition $r_1 r_2 \geq 4\alpha_1\alpha_2$ guarantees that there are only two non-negative equilibria, $(0,0)$ and $(1,1)$. So the system (7.27) has single left spreading speed $\hat{c}_f = \hat{c}$. A sufficient condition for left linear determinacy involving functions f and h is

$$f_i(u_1,u_2) - f_i'(0,0)\,(u_1,u_2)^T \leq \bar{\mu}\left[h_i'(u_1,u_2) - h_i'(0,0)\right](u_1,u_2)^T, \quad u_1,u_2 \geq 0, \tag{7.29}$$

since (7.29) implies (5.61). Note that the condition (5.61) is trivially satisfied here since we have only one block in $f'(0)$.

The condition (7.29) is equivalent to

$$\begin{pmatrix} -\alpha_1 u_1 + \alpha_1 u_2 + r_1(u_1 - u_1^2) \\ \alpha_2 u_1 - \alpha_2 u_2 + r_2(u_2 - u_2^2) \end{pmatrix} - \begin{pmatrix} -\alpha_1 + r_1 & \alpha_1 \\ \alpha_2 & -\alpha_2 + r_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \leq$$

$$\bar{\mu}\left[\begin{pmatrix} h_1'(u_1) & 0 \\ 0 & h_2'(u_2) \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{7.30}$$

which is true if and only if $-r_1 u_1^2 - \bar{\mu} u_1(h_1'(u_1) - a) \leq 0$, and $-r_2 u_2^2 - \bar{\mu} u_2(h_2'(u_2) - b) \leq 0$. Since $h_1'(0) = a \leq h_1'(u_1)$, $h_2'(0) = b \leq h_2'(u_2)$, and since $r_1, r_2, u_1, u_2, \bar{\mu} > 0$, it follows that (7.30) holds provided $h_1'(u_1) - a \geq 0$, and $h_2'(u_2) - b \geq 0$. Then Theorem 5.10 implies that $\hat{c}_f = \hat{c} = \hat{\bar{c}}$ for the system (7.27), which means that it is left linearly determinate.

We will use an argument similar to those used above to estimate the linear value $\hat{\hat{c}}$ of (7.27) provided that $h_1'(0) \leq h_1'(u_1)$. The argument depends on comparing with a Fisher equation with convection term (1.21) that is obtained from the first equation in the system but with $u_2 = 0$ and with the non-decreasing initial condition $0 \leq (u_0)_1 \leq r_1 - \alpha_1$ for the system (7.27) which is the same initial condition as for the Fisher equation

$$u_{1,t}^\sharp = u_{1,xx}^\sharp - h_1'(u_1^\sharp)u_{1,x}^\sharp - \alpha_1 u_1^\sharp + r_1(u_1^\sharp - (u^\sharp)_1^2). \tag{7.31}$$

Since $u_2 \geq 0$, we have

$$\begin{aligned} u_{1,t} &= u_{1,xx} - h_1'(u_1)u_{1,x} - \alpha_1 u_1 + \alpha_1 u_2 + r_1(u_1 - u_1^2) \\ &\geq u_{1,xx} - h_1'(u_1)u_{1,x} - \alpha_1 u_1 + r_1(u_1 - u_1^2). \end{aligned}$$

So u_1 is a supersolution of (7.31), and hence the Comparison Theorem 5.1 implies that $u_1 \geq u_1^\sharp$, where $u_1^\sharp$ is the solution of (7.31). Choose an initial condition $0 \leq (u_0)_1 \leq r_1 - \alpha_1$ such that the initial condition of the first component of (7.31) converges to $r_1 - \alpha_1 < 1$, so $(u_0)_1$ will spread at a speed no faster than the left slowest spreading speed. Hence because the initial condition is non-decreasing, we get

$$\hat{c}_f = \hat{c} = \hat{\hat{c}} \leq \bar{c}_1 = h'(0) - 2\sqrt{f_1'(0)} = a - 2\sqrt{r_1 - \alpha_1}, \tag{7.32}$$

where $\bar{c}_1$ denotes the linear value of (7.31), $f_1'(0)$ denotes the first component of $f : \mathbb{R}^2 \to \mathbb{R}^2$, and because $f'(0)$ does not have any zero off-diagonal entries, if it were possible to find such λ and q, then we would know that $\bar{c}_1 \leq \hat{\hat{c}}$. If, however, it is not possible to find such λ and q, then $\hat{\hat{c}} < \bar{c}_1$. It is not straightforward to calculate the linear value of the system (7.27) in this case, in contrast to previous examples. Thus we will estimate it by taking $c = \bar{c}_1$, and we investigate whether or not it is possible for this value of c that there exists an unstable monotone eigenvalue $\lambda > 0$ and a vector $q > 0$ that satisfy

$$M(\lambda, c)q = \left[\lambda^2 + \lambda c I - \lambda \text{ diag } (a, b) + \begin{pmatrix} r_1 - \alpha_1 & \alpha_1 \\ \alpha_2 & r_2 - \alpha_2 \end{pmatrix}\right] q = 0,$$

which is equivalent to

$$\left[\lambda^2 + \lambda(a - 2\sqrt{r_1 - \alpha_1})I - \lambda \text{ diag } (a, b) + \begin{pmatrix} r_1 - \alpha_1 & \alpha_1 \\ \alpha_2 & r_2 - \alpha_2 \end{pmatrix}\right] q = 0. \tag{7.33}$$

For a positive vector $q = (x, y)^T$, where $x, y > 0$, (7.33) becomes two quadratic equations for λ,

$$\begin{aligned} \left(\lambda^2 - 2\lambda\sqrt{r_1 - \alpha_1} + (r_1 - \alpha_1)\right) x + \alpha_1 y = 0, \\ \left(\lambda^2 - \lambda(2\sqrt{r_1 - \alpha_1} - a + b) + (r_2 - \alpha_2)\right) y + \alpha_2 x = 0. \end{aligned} \tag{7.34}$$

By letting $\dfrac{y}{x} = z$, the two equations become

$$\lambda^2 - 2\lambda\sqrt{r_1 - \alpha_1} + r_1 - \alpha_1 + \alpha_1 z = 0, \tag{7.35}$$

$$\lambda^2 - \lambda(2\sqrt{r_1 - \alpha_1} - a + b) + r_2 - \alpha_2 + \alpha_2 z^{-1} = 0. \tag{7.36}$$

Then from (7.35), we find that

$$\lambda = \frac{2\sqrt{r_1 - \alpha_1} \mp \sqrt{4(r_1 - \alpha_1) - 4(r_1 - \alpha_1 + \alpha_1 z)}}{2} = \frac{2\sqrt{r_1 - \alpha_1} \mp \sqrt{-4\alpha_1 z}}{2},$$

and since $-4\alpha_1 z < 0$, we can not have a real solution such that $\lambda > 0, z > 0$. Therefore we must have the strict inequality, $\hat{\bar{c}} < \bar{c}_1 = a - 2\sqrt{r_1 - \alpha_1}$. □

7.3 Example illustrating a sufficient condition on the convection term for no linear determinacy

The final example considers the previous system (7.27) but when the convection term does not satisfy $h_i'(0) \leq h_i'(u_i), i = 1, 2$. We suppose here that the convection term is such that $h_1'(0) = h_2'(0)$, and will show that under certain additional conditions on the convection term and parameters, this system is not left linearly determinant.

Example 7.7. Consider the co-operative system

$$u_{1,t} = u_{1,xx} - h_1'(u_1)u_{1,x} - \alpha_1 u_1 + \alpha_1 u_2 + r_1(u_1 - u_1^2) \tag{7.37}$$

$$u_{2,t} = u_{2,xx} - h_2'(u_2)u_{2,x} + \alpha_2 u_1 - \alpha_2 u_2 + r_2(u_2 - u_2^2),$$

with non-decreasing initial condition u_0, where all parameters are positive, $r_1 r_2 \geq 4\alpha_1\alpha_2$ and $r_1 > \alpha_1$. The functions f in (7.28), and h satisfy Hypotheses $s_1 - s_6$ and $h_1'(0) = h_2'(0)$. As we showed above, the system (7.37) has single left spreading speed $\hat{c}_f = \hat{c}$.

Since it is not straightforward to calculate the left linear value $\hat{\bar{c}}$ of the system, we again will estimate it using a Perron-Frobenius eigenvalue argument as follows. The Perron-Frobenius eigenvalue $\mathfrak{F}_{pf}(f'(0)) =: \gamma$ satisfies

$$\gamma^2 - \gamma\left[(r_1 - \alpha_1) + (r_2 - \alpha_2)\right] + (r_1 - \alpha_1)(r_2 - \alpha_2) - \alpha_1\alpha_2 = 0,$$

which, letting $R_i := r_i - \alpha_i, i = 1, 2$, says that

$$\gamma^2 - \gamma(R_1 + R_2) + R_1 R_2 - \alpha_1\alpha_2 = 0,$$

which is a quadratic function in γ. Thus

$$\gamma = \frac{(R_1 + R_2) \mp \sqrt{(R_1 + R_2)^2 - 4(R_1 R_2 - \alpha_1\alpha_2)}}{2},$$

and since we are interested in the unstable eigenvalue γ, so

$$\gamma = \frac{(R_1 + R_2) + \sqrt{(R_1 - R_2)^2 + 4\alpha_1\alpha_2}}{2}.$$

For this positive eigenvalue, there exists a positive eigenvector $q = (x, y)^T, x, y > 0$ that satisfies $f'(0)q = \mathfrak{F}_{pf}\left(f'(0)\right) q$. Then for this q, the equation

$$M(\lambda, c)q = \left[\lambda^2 + \lambda c I - \lambda \operatorname{diag}(a, b) + \gamma I\right] q = 0,$$

is equivalent to the two quadratic equations in λ,

$$\left(\lambda^2 + \lambda c - \lambda\, a + \gamma\right) x = 0, \tag{7.38}$$

$$\left(\lambda^2 + \lambda c - \lambda\, b + \gamma\right) y = 0. \tag{7.39}$$

Since $h_1'(0) = a = h_2'(0) = b$, dividing (7.38) and (7.39) by x, y respectively yields a single quadratic equation in λ,

$$\lambda^2 + \lambda(c-a) + \frac{(R_1+R_2) + \sqrt{(R_1-R_2)^2 + 4\alpha_1\alpha_2}}{2} = 0,$$

and hence

$$\lambda = \frac{-(c-a) \mp \sqrt{(c-a)^2 - 4\left(\dfrac{(R_1+R_2) + \sqrt{(R_1-R_2)^2 + 4\alpha_1\alpha_2}}{2}\right)}}{2}.$$

Since we are interested in positive λ, the critical value of c for which such λ exists satisfies $(c-a)^2 = 2\left[(R_1+R_2) + \sqrt{(R_1-R_2)^2 + 4\alpha_1\alpha_2}\right]$. It follows that to obtain such positive λ, we need $c - a = -\left(2\left[(R_1+R_2) + \sqrt{(R_1-R_2)^2 + 4\alpha_1\alpha_2}\right]\right)^{1/2}$. Thus we have shown that for this value of c given by

$$c_\lambda := h_1'(0) - \left(2\left[(R_1+R_2) + \sqrt{(R_1-R_2)^2 + 4\alpha_1\alpha_2}\right]\right)^{1/2}, \tag{7.40}$$

we get an unstable monotone eigenvalue λ corresponding with the positive eigenvector q, and then by the definition of the linear value $\hat{\bar{c}}$ of the system (7.37), we have

$$c_\lambda \leq \hat{\bar{c}}. \tag{7.41}$$

Now recall a Fisher equation with convection terms

$$u^\sharp_{1,t} = u^\sharp_{1,xx} - h_1'(u^\sharp_1)u^\sharp_{1,x} - \alpha_1 u^\sharp_1 + r_1(u^\sharp_1 - (u^\sharp)^2_1), \tag{7.42}$$

where $f_1(u_1) = r_1(u_1 - u_1^2) - \alpha_1 u_1 = u_1\left(r_1 - \alpha_1 - r_1 u_1\right)$, so the equilibria of $f_1(u_1)$ are 0 and $\frac{r_1 - \alpha_1}{r_1} = \frac{R_1}{r_1}$. Then the proof of Proposition 2.3

$$-h_1(0) + \sqrt{\left(h_1\left(\frac{R_1}{r_1}\right)\right)^2 + 2\int_0^{\frac{R_1}{r_1}} f_1(u_1)du_1} = -h_1(0) + \sqrt{\left(h_1\left(\frac{R_1}{r_1}\right)\right)^2 + \frac{R_1^3}{3r_1^2}}$$

gives an upper bound for the left spreading speed $c_1^\sharp$ of scalar equation (7.42), that is

$$c_1^\sharp \leq -h_1(0) + \sqrt{\left(h_1\left(\frac{R_1}{r_1}\right)\right)^2 + \frac{R_1^3}{3r_1^2}}. \tag{7.43}$$

Suppose the function h is chosen such that

$$-h_1(0) + \sqrt{\left(h_1\left(\frac{R_1}{r_1}\right)\right)^2 + \frac{R_1^3}{3r_1^2}} < c_\lambda, \tag{7.44}$$

holds (note that, for instance, (7.44) will hold if $h_1'(0)$ is sufficiently large). Since we know from Example 7.6 that u_1 is a supersolution of (7.31), hence $u_1 \geq u_1^\sharp$, where $u_1^\sharp$ is the solution of (7.31) with initial data $(u_0)_1$. So $(u_0)_1$ will spread at most the left slowest spreading speed $\hat{c}$, thus since $(u_0)_1$ is non-decreasing, we get that the left slowest spreading speed for the system $\hat{c}$ is a lower bound for the left slowest spreading speed $c_1^\sharp$ of the scalar problem (7.42). That is, $\hat{c} \leq c_1^\sharp$, hence using (7.43), we obtain that

$$\hat{c} \leq c_1^\sharp \leq -h_1(0) + \sqrt{\left(h_1(\frac{R_1}{r_1})\right)^2 + \frac{R_1^3}{3r_1^2}},$$

and if (7.44) holds, we get

$$\hat{c} \leq c_1^\sharp \leq -h_1(0) + \sqrt{\left(h_1(\frac{R_1}{r_1})\right)^2 + \frac{R_1^3}{3r_1^2}} < c_\lambda \leq \hat{\bar{c}}. \tag{7.45}$$

Hence (7.45) shows directly that the system is not left linearly determinate since single left speed of the system is strictly less that its left linear value. $\square$

Appendix A

Proof of Continuous Dependence Theorem 5.4

Here we prove the Continuous Dependence Theorem 5.4.

The idea of the proof depends on estimating both $\|u(\cdot,t)-\tilde{u}(\cdot,t)\|_{\infty,\eta}$ and $t^{1/2}\|u_0(\cdot,t)-\tilde{u}_0(\cdot,t)\|_{1,\infty,\eta}$, using the definition of $\eta(x)$ in (5.17) and Peetre's inequality (A.1), in order to show that there exists $C>0$ to satisfy the required inequality for all $t\in[0,T]$.

A useful inequality for this following proof is Peetre's inequality (see [3, p.99]), which implies that, for any $\mu\in\mathbb{R}$, and $x,y\in\mathbb{R}$,

$$\left(\frac{1+x^2}{1+y^2}\right)^{\mu}\leq 2^{|\mu|}\left(1+(x-y)^2\right)^{|\mu|}. \tag{A.1}$$

Proof. (Theorem 5.4) Note first that the solution u of (5.1) with initial data u_0 satisfies

$$u_i(x,t)=\int_{\mathbb{R}}\Gamma_i(x-y,t)(u_0)_i(y)dy+\int_0^t\int_{\mathbb{R}}\Gamma_i(x-y,t-s)\gamma_i\left(u(y,s),u_x(y,s)\right)dyds, \tag{A.2}$$

where

$$\gamma_i(u,u_x)=-h_i(u)u_{i,x}+f_i(u),\quad\text{and}\quad\Gamma_i(x,t)=\frac{1}{\sqrt{4\pi d_i t}}\exp\left(-x^2/4d_it\right).$$

So if $u, \tilde{u}$ denote solutions of (5.1) corresponding to initial data $u_0, \tilde{u}_0$, we have

$$u_i(x,t) - \tilde{u}_i(x,t) = \int_{\mathbb{R}} \Gamma_i(x-y,t)\left((u_0)_i(y) - \tilde{u}_{0i}(y)\right) dy +$$
$$\int_0^t \int_{\mathbb{R}} \Gamma_i(x-y,t-s)\left[\gamma_i\left(u(y,s),u_x(y,s)\right) - \gamma_i\left(\tilde{u}(y,s),\tilde{u}_x(y,s)\right)\right] dy,$$

so that if we define $w := u - \tilde{u}$, $w_0 := u_0 - \tilde{u}_0$, and $\phi := \gamma(u,u_x) - \gamma(\tilde{u},\tilde{u}_x)$, we have

$$w_i(x,t) = \int_{\mathbb{R}} \Gamma_i(x-y,t) w_{0i}(y) dy + \int_0^t \int_{\mathbb{R}} \Gamma_i(x-y,t-s)\phi_i(y,s) dyds, \tag{A.3}$$

and hence

$$\eta(x) w_i(x,t) = \eta(x) \int_{\mathbb{R}} \Gamma_i(x-y,t) w_{0i}(y) dy + \eta(x) \int_0^t \int_{\mathbb{R}} \Gamma_i(x-y,t-s)\phi_i(y,s) dyds.$$

Now to prove the continuous dependence Theorem 5.4, we need to estimate both $\|w(\cdot,t)\|_{\infty,\eta}$ and $t^{1/2}\|w(\cdot,t)\|_{1,\infty,\eta}$. Note first that

$$\begin{aligned} |\eta(x) \int_{\mathbb{R}} \Gamma_i(x-y,t) w_{0i}(y) dy| &= |\int_{\mathbb{R}} \Gamma_i(x-y,t) \frac{\eta(x)}{\eta(y)} . \eta(y) w_{0i}(y) dy| \\ &\leq \sup_{y\in\mathbb{R}} |\eta(y) w_{0i}(y)| \int_{\mathbb{R}} \Gamma_i(x-y,t) . \frac{\eta(x)}{\eta(y)} dy. \end{aligned} \tag{A.4}$$

Hence by Peetre's inequality (A.1),

$$\frac{\eta(x)}{\eta(y)} = \frac{1+y^2}{1+x^2} = \left(\frac{1+y^2}{1+x^2}\right)^{-1} \leq 2\left(1+(x-y)^2\right),$$

and thus

$$\int_{\mathbb{R}} \Gamma_i(x-y,t) \frac{\eta(x)}{\eta(y)} dy \leq 2 \int_{\mathbb{R}} \Gamma_i(x-y,t)\left(1+(x-y)^2\right) dy. \tag{A.5}$$

Now since $\int_{\mathbb{R}} \Gamma_i(x-y,t) dy = 1$, and by letting $x - y = z$, we have

$$\int_{\mathbb{R}} \Gamma_i(x-y,t)(x-y)^2 dy = \int_{\mathbb{R}} \Gamma_i(z,t) z^2 dz = \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi d_i t}} \exp\left(\frac{-z^2}{4d_i t}\right) z^2 dz,$$

if we suppose that $v^2 = \frac{z^2}{4d_it}$, then $dv = \frac{dz}{2\sqrt{d_it}}$, and it follows that

$$\int_{\mathbb{R}} \frac{1}{2\sqrt{\pi d_i t}} \exp\left(\frac{-z^2}{4d_it}\right) z^2 dz = \frac{4d_it}{\sqrt{\pi}} \int_{\mathbb{R}} v^2 \exp(-v^2) dv,$$

gives that

$$\int_{\mathbb{R}} \Gamma_i(x-y,t) \frac{\eta(x)}{\eta(y)} dy \leq C(1+t), \tag{A.6}$$

for some constant C depending only on d_i. Then (A.4) and (A.6) together give that

$$|\eta(x) \int_{\mathbb{R}} \Gamma_i(x-y,t) w_{0i}(y) dy| \leq C(1+t) \sup_{y\in\mathbb{R}} |\eta(y) w_{0i}(y)|. \tag{A.7}$$

Next

$$\begin{aligned}
\frac{\partial}{\partial x} \int_{\mathbb{R}} \Gamma_i(x-y,t) w_{0i}(y) dy &= \int_{\mathbb{R}} \frac{\partial}{\partial x} \Gamma_i(x-y,t) w_{0i}(y) dy \\
&= \int_{\mathbb{R}} \frac{\partial}{\partial x} \frac{1}{2\sqrt{\pi d_i t}} \exp\left(\frac{-(x-y)^2}{4d_it}\right) w_{0i}(y) dy \\
&= \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi d_i t}} \frac{-2(x-y)}{4d_it} \cdot \exp\left(\frac{-(x-y)^2}{4d_it}\right) w_{0i}(y) dy \\
&= -\frac{1}{4\sqrt{\pi}(d_it)^{3/2}} \int_{\mathbb{R}} (x-y) \cdot \exp\left(\frac{-(x-y)^2}{4d_it}\right) w_{0i}(y) dy,
\end{aligned}$$

and

$$\begin{aligned}
&|\eta(x) \int_{\mathbb{R}} \frac{\partial}{\partial x} \Gamma_i(x-y,t) w_{0i}(y) dy| \\
&\quad = \frac{1}{4\sqrt{\pi}(d_it)^{3/2}} |\textstyle\int_{\mathbb{R}} (x-y) \exp\left(\frac{-(x-y)^2}{4d_it}\right) \frac{\eta(x)}{\eta(y)} \eta(y) w_{0i}(y) dy| \\
&\quad \leq \sup_{y\in\mathbb{R}} |\eta(y) w_{0i}(y)| \frac{1}{2\sqrt{\pi}(d_it)^{3/2}} \textstyle\int_{\mathbb{R}} |x-y| \exp\left(\frac{-(x-y)^2}{4d_it}\right) (1+(x-y)^2) dy,
\end{aligned} \tag{A.8}$$

and since

$$\begin{aligned}
\frac{1}{2\sqrt{\pi}(d_it)^{3/2}}\int_{\mathbb{R}}&|x-y|.\exp\left(\frac{-(x-y)^2}{4d_it}\right)(1+(x-y)^2)dy\\
&=\frac{1}{2\sqrt{\pi}(d_it)^{3/2}}\cdot\int_{\mathbb{R}}2\sqrt{d_it}|v|\exp(-v^2)\left(1+4d_itv^2\right)2\sqrt{d_it}dv\\
&=\frac{2}{\sqrt{\pi}(d_it)^{1/2}}\int_{\mathbb{R}}|v|\exp(-v^2)\left(1+4d_itv^2\right)dv\\
&\leq\frac{C(1+t)}{t^{1/2}},
\end{aligned}\tag{A.9}$$

where C is a constant (again, depending only on d_i). Then (A.8) and (A.9) together give that

$$\begin{aligned}
t^{1/2}|\eta(x)\frac{\partial}{\partial x}\int_{\mathbb{R}}\Gamma_i(x-y,t)w_{0i}(y)dy|&\leq t^{1/2}\frac{C(1+t)}{t^{1/2}}\sup_{y\in\mathbb{R}}|\eta(y)w_{0i}(y)|\\
&=C(1+t)\sup_{y\in\mathbb{R}}|\eta(y)w_{0i}(y)|.
\end{aligned}\tag{A.10}$$

Now we need to estimate both

$$\eta(x)\int_0^t\int_{\mathbb{R}}\Gamma_i(x-y,t-s)\phi_i(y,s)dyds,\ \text{and}\ t^{1/2}\eta(x)\frac{\partial}{\partial x}\int_0^t\int_{\mathbb{R}}\Gamma_i(x-y,t-s)\phi_i(y,s)dyds.$$

To do this, note first that

$$\begin{aligned}
\phi_i(y,s)&=\gamma_i(u,u_x)(y,s)-\gamma_i(\tilde{u},\tilde{u}_x)(y,s)\\
&=h_i^{'}(u_i)u_{i,x}-h_i^{'}(\tilde{u}_i)\tilde{u}_{i,x}+f_i(u)-f_i(\tilde{u})\\
&=h_i^{'}(u_i)\left(u_{i,x}-\tilde{u}_{i,x}\right)+\tilde{u}_{i,x}\left(h_i^{'}(u_i)-h_i^{'}(\tilde{u}_i)\right)+f_i(u)-f_i(\tilde{u}),
\end{aligned}$$

Define $\psi_i(\theta):=f_i(\theta u+(1-\theta)\tilde{u})$, and $\zeta_i(\theta):=h_i^{'}(\theta u_i+(1-\theta)\tilde{u}_i)$. Then

$$f_i(u)-f_i(\tilde{u})=\int_0^1 df_i[\theta(u)+(1-\theta)(\tilde{u})](u-\tilde{u})d\theta,$$

and

$$h_i^{'}(\tilde{u}_i)-h_i^{'}(u_i)=\int_0^1 h_i^{''}(\theta\tilde{u}_i+(1-\theta)u_i)(\tilde{u}_i-u_i)d\theta,$$

so we have

$$\phi_i(y,s) = \zeta(y,s)w_{i,x} + (\psi(y,s)w)_i\,,$$

where ζ, ψ are bounded uniformly on $\mathbb{R} \times [0,\infty)$. Hence

$$\eta(y)\phi_i(y,s) = \zeta(y,s)\eta(y)w_{i,x} + (\psi(y,s)\eta(y)w)_i\,.$$

Since we know that $u, \tilde{u} \in \psi_\beta$, which means that $0 \le u, \tilde{u} \le \beta$, and $\|w\|_{1,\infty,\eta} = \max\limits_{1\le i\le k}\|w_i\|_{1,\infty,\eta} = \max\limits_{1\le i\le k}\|w_i\|_{\infty,\eta} + \|w_{i,x}\|_{\infty,\eta}$, it follows that

$$|\zeta(y,s)|\eta(y)|w_{i,x}| \le C\|w(\cdot,s)\|_{1,\infty,\eta}, \text{ and } |(\psi(y,s)\eta(y)w)_i| \le C\|w(\cdot,s)\|_{\infty,\eta},$$

thus

$$\eta(y)|\phi_i(y,s)| \le C\|w(\cdot,s)\|_{1,\infty,\eta} \qquad \text{for some constant } C > 0.$$

Then

$$\begin{aligned}
|\eta(x)\int_0^t\int_{\mathbb{R}}\Gamma_i(x-y,t-s)\phi_i(y,s)dyds| &\le \int_0^t\int_{\mathbb{R}}\Gamma_i(x-y,t-s)\frac{\eta(x)}{\eta(y)}\eta(y)|\phi_i(y,s)|dyds \\
&\le 2C\int_0^t \|w(\cdot,s)\|_{1,\infty,\eta}\left(\int_{\mathbb{R}}\Gamma_i(x-y,t-s)(1+(x-y)^2)dy\right)ds \\
&\le \tilde{C}\int_0^t \|w(\cdot,s)\|_{1,\infty,\eta}\,(1+(t-s))\,ds,
\end{aligned}$$

for some constant $\tilde{C}$. Since $\int_{\mathbb{R}}\Gamma_i(x-y,t-s)\,(1+(x-y)^2)\,dy \le C\,(1+(t-s))$, where C is a constant, we have

$$\begin{aligned}
|\eta(x)\int_0^t\int_{\mathbb{R}}\Gamma_i(x-y,t-s)\phi_i(y,s)dyds| &\le \tilde{C}\int_0^t \|w(\cdot,s)\|_{1,\infty,\eta}\,(1+(t-s))\,ds \\
&= \tilde{C}\int_0^t (1+(t-s))\,s^{-1/2}s^{1/2}\|w(\cdot,s)\|_{1,\infty,\eta}ds \\
&\le \tilde{C}\left(\sup_{0\le s\le t} s^{1/2}\|w(\cdot,s)\|_{1,\infty,\eta}\right)\int_0^t (1+t-s)\,s^{-1/2}ds \\
&\le \tilde{\tilde{C}}(1+t)t^{1/2}\left(\sup_{0\le s\le t} s^{1/2}\|w(\cdot,s)\|_{1,\infty,\eta}\right), \qquad \text{(A.11)}
\end{aligned}$$

for some constant $\tilde{\tilde{C}}$. On the other hand,

$$|\eta(x)\frac{\partial}{\partial x}\int_0^t\int_{\mathbb{R}}\Gamma_i(x-y,t-s)\phi_i(y,s)dyds| = |\eta(x)\int_0^t\int_{\mathbb{R}}\frac{\partial}{\partial x}\Gamma_i(x-y,t-s)\phi_i(y,s)dyds|$$
$$\leq \textstyle\int_0^t\int_{\mathbb{R}}|\frac{\partial}{\partial x}\Gamma_i(x-y,t-s)|\frac{\eta(x)}{\eta(y)}\cdot\eta(y)|\phi_i(y,s)|dyds$$
$$\leq 2C\textstyle\int_0^t\|w(\cdot,s)\|_{1,\infty,\eta}\int_{\mathbb{R}}\left(1+(x-y)^2\right)|\frac{\partial}{\partial x}\Gamma_i(x-y,t-s)|dyds. \tag{A.12}$$

Then since

$$\begin{aligned}\frac{\partial}{\partial x}\Gamma_i(x-y,t-s) &= \frac{\partial}{\partial x}\frac{1}{2\sqrt{\pi d_i(t-s)}}\cdot\exp\left(\frac{-(x-y)^2}{4d_i(t-s)}\right)\\ &= \frac{1}{2\sqrt{\pi d_i(t-s)}}\frac{-2(x-y)}{4d_i(t-s)}\exp\left(\frac{-(x-y)^2}{4d_i(t-s)}\right),\end{aligned}$$

and we have

$$\begin{aligned}&\int_{\mathbb{R}}\left(1+(x-y)^2\right)|\frac{\partial}{\partial x}\Gamma_i(x-y,t-s)|dy\\ &= C\int_{\mathbb{R}}\frac{|x-y|}{(t-s)^{3/2}}\left(1+(x-y)^2\right)\cdot\exp\left(\frac{-(x-y)^2}{4d_i(t-s)}\right)dy\\ &= C\int_{\mathbb{R}}\frac{2\sqrt{d_i(t-s)}|v|}{(t-s)^{3/2}}\left(1+4d_i(t-s)v^2\right)\cdot\exp(-v^2)2\sqrt{d_i(t-s)}dv\\ &= \frac{\tilde{C}}{(t-s)^{1/2}}\int_{\mathbb{R}}|v|\left(1+4d_i(t-s)v^2\right)\cdot\exp(-v^2)2\sqrt{d_i(t-s)}dv\\ &\leq \frac{\tilde{\tilde{C}}}{(t-s)^{1/2}} \qquad \text{for some constants } \tilde{C},\tilde{\tilde{C}}.\end{aligned} \tag{A.13}$$

Then (A.12) and (A.14) together give

$$\begin{aligned}&|\eta(x)\frac{\partial}{\partial x}\int_0^t\int_{\mathbb{R}}\Gamma_i(x-y,t-s)\phi_i(y,s)dyds|\\ &\qquad\leq C(1+t)\int_0^t(t-s)^{-1/2}s^{-1/2}s^{1/2}\|w(\cdot,s)\|_{1,\infty,\eta}ds,\end{aligned} \tag{A.14}$$

and since $\int_0^t(t-s)^{-1/2}s^{-1/2}ds \leq 2\int_0^{t/2}s^{-1/2}\left(\frac{t}{2}\right)^{-1/2}ds = 2\left(\frac{t}{2}\right)^{\frac{-1}{2}}2t^{1/2} = C$, for some constant C, we have

$$|\eta(x)\frac{\partial}{\partial x}\int_0^t\int_{\mathbb{R}}\Gamma_i(x-y,t-s)\phi_i(y,s)dyds| \le \tilde{C}(1+t)\sup_{0\le s\le t} s^{1/2}\|w(\cdot,s)\|_{1,\infty,\eta}, \quad \text{(A.15)}$$

which implies that

$$\begin{aligned} t^{1/2}|\eta(x)\frac{\partial}{\partial x}\int_0^t\int_{\mathbb{R}}\Gamma_i(x-y,t-s)\phi_i(y,s)dyds| \\ \le \tilde{C}\, t^{1/2}(1+t)\left(\sup_{0\le s\le t} s^{1/2}\|w(\cdot,s)\|_{1,\infty,\eta}\right). \quad \text{(A.16)}\end{aligned}$$

Then (A.11) and (A.16) give that

$$\begin{aligned} \max\Big\{|\eta(x)\frac{\partial}{\partial x}\int_0^t\int_{\mathbb{R}}\Gamma_i(x-y,t-s)\phi_i(y,s)dyds|, \\ t^{1/2}|\eta(x)\frac{\partial}{\partial x}\textstyle\int_0^t\int_{\mathbb{R}}\Gamma_i(x-y,t-s)\phi_i(y,s)dyds|\Big\}. \\ \le \tilde{C}t^{1/2}(1+t)\left(\sup_{0\le s\le t} s^{1/2}\|w(\cdot,s)\|_{1,\infty,\eta}\right), \quad \text{(A.17)}\end{aligned}$$

for some constant $\tilde{C}$, whereas (A.7) and (A.10) give that

$$\begin{aligned} \max\Big\{|\eta(x)\int_{\mathbb{R}}\Gamma_i(x-y,t)w_{0i}(y)dy|, t^{1/2}|\eta(x)\frac{\partial}{\partial x}\int_{\mathbb{R}}\Gamma_i(x-y,t)w_{0i}(y)dy|\Big\} \\ \le \tilde{C}(1+t)\left(\sup_{0\le s\le t}|\eta(y)w_{0i}(y)|\right) \\ \le C(1+t)\|w_0\|_{\infty,\eta}, \quad \text{(A.18)}\end{aligned}$$

for some constant C. It then follows from (A.3), (A.17) and (A.18) that

$$\begin{aligned} &\|w(\cdot,t)\|_{\infty,\eta} + t^{1/2}\|w(\cdot,t)\|_{1,\infty,\eta} \\ &= \max_{1\le i\le k}\sup_{x\in\mathbb{R}}|\eta(x)w_i(x,t)| + t^{1/2}\max_{1\le i\le k}\sup_{x\in\mathbb{R}}|\eta(x)\frac{\partial}{\partial x}w_i(x,t)| \\ &\le C(1+t)\|w_0\|_{1,\infty,\eta} + \tilde{C}(1+t)t^{1/2}\sup_{0\le s\le t}s^{1/2}\|w(\cdot,s)\|_{1,\infty,\eta}, \end{aligned}$$

which implies that

$$\sup_{0\le s\le t}\left(\|w(\cdot,s)\|_{\infty,\eta}+s^{1/2}\|w(\cdot,s)\|_{1,\infty,\eta}\right)$$
$$\le C(1+T)\|w_0\|_{\infty,\eta}+\tilde{C}(1+T)t^{1/2}\sup_{0\le s\le t}\left(\|w(\cdot,s)\|_{\infty,\eta}+s^{1/2}\|w(\cdot,s)\|_{1,\infty,\eta}\right),$$

and hence

$$\left(1-\tilde{C}(1+T)t^{1/2}\right)\sup_{0\le s\le t}\left(\|w(\cdot,s)\|_{\infty,\eta}+s^{1/2}\|w(\cdot,s)\|_{1,\infty,\eta}\right)\le C(1+T)\|w_0\|_{\infty,\eta}. \quad (A.19)$$

Then (A.19) yields that there exists $\delta > 0$, depending only on T, R and the constants and functions in (A.2), such that if $t \le \delta$, then $1-\tilde{C}(1+T)t^{1/2} \le 1/2$, and hence

$$\sup_{0\le s\le t}\left(\|w(\cdot,s)\|_{\infty,\eta}+s^{1/2}\|w(\cdot,s)\|_{1,\infty,\eta}\right)\le 2C(1+T)\|w_0\|_{\infty,\eta}. \quad (A.20)$$

Then for $\delta/2 \le s \le 3\delta/2$,

$$\sup_{\delta/2\le s\le 3\delta/2}\left(\|w(\cdot,s)\|_{\infty,\eta}+(s-\frac{\delta}{2})^{1/2}\|w(\cdot,s)\|_{1,\infty,\eta}\right)\le 2C(1+T)\|w(\cdot,\delta/2)\|_{\infty,\eta}$$
$$\le \left(2C(1+T)\right)^2\|w_0\|_{\infty,\eta}, \quad (A.21)$$

and since for $s \in [\delta, \frac{3\delta}{2}]$, $s \le 2(s-\frac{\delta}{2})$, so $s^{1/2} \le 2^{1/2}\left(s-\frac{\delta}{2}\right)^{\frac{1}{2}}$, it follows from (A.21) that

$$\sup_{\delta/2\le s\le 3\delta/2}\left(\|w(\cdot,s)\|_{\infty,\eta}+(s)^{1/2}\|w(\cdot,s)\|_{1,\infty,\eta}\right)$$
$$\le 2^{1/2}\sup_{\delta/2\le s\le 3\delta/2}\left(\|w(\cdot,s)\|_{\infty,\eta}+(s-\frac{\delta}{2})^{1/2}\|w(\cdot,s)\|_{1,\infty,\eta}\right)$$
$$\le 2^{1/2}\left(2C(1+T)\right)^2\|w_0\|_{\infty,\eta}. \quad (A.22)$$

Since $T < N_0\delta$ for some $N_0 \in \mathbb{N}$, it follows similarly, by considering the intervals $[\delta, 2\delta]$, $[\frac{3\delta}{2}, \frac{5\delta}{2}]$, etc, that there exists $C > 0$ such that $\sup_{0\le s\le T}\left(\|w(\cdot,s)\|_{\infty,\eta}+s^{1/2}\|w(\cdot,s)\|_{1,\infty,\eta}\right) \le C\|w_0\|_{\infty,\eta}$, as required, and the theorem is established. □

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