Braided Hopf algebras and non-trivially associated tensor categories.

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BRAIDED HOPF ALGEBRAS AND NON-TRIVIALLY ASSOCIATED TENSOR CATEGORIES

Thesis submitted to the University of Wales Swansea
by

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To

My Parents

Declaration

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

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Statement 1

This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by citations giving explicit references. A bibliography is appended.

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ABSTRACT

The rigid non-trivially associated tensor category $\mathcal{C}$ is constructed from left coset representatives M of a subgroup G of a finite group X. There is also a braided category $\mathcal{D}$ made from $\mathcal{C}$ by a double construction. In this thesis we consider some basic useful facts about $\mathcal{D}$, including the fact that it is a modular category (modulo a matrix being invertible). Also we give a definition of the character of an object in this category as an element of a braided Hopf algebra in the category. The definition is shown to be adjoint invariant and multiplicative. A detailed example is given. Next we show an equivalence of categories between the non-trivially associated double $\mathcal{D}$ and the trivially associated category of representations of the double of the group $D(X)$. Moreover, we show that the braiding for $\mathcal{D}$ extends to a partially defined braiding on $\mathcal{C}$, and also we look at an algebra $A \in \mathcal{C}$, using this partial braiding. Finally, ideas for further research are included.

NOTATION.
In this thesis references are indicated by square brackets [ ] and equations are numbered in round brackets ( ), where (a.b) denotes equation b in chapter a.
This thesis has been typeset using LaTeX, except the appendix has been done using the Mathematica program, and some figures using the Visio program.

Contents

1 Introduction

Group factorizations are very common in mathematics. Among their uses is the bicrossproduct construction which is one of the primary sources of non-commutative and non-cocommutative Hopf algebras.

It is well known that for every factorization $X = GM$ of a group into two subgroups G and M, a Hopf algebra $H = kM \triangleright\!\!\blacktriangleleft k(G)$ can be constructed, where kM is the group Hopf algebra of M and $k(G)$ is the Hopf algebra of functions on G. Here kM has a basis given by the elements of M, with multiplication given by the group product in M (in general not commutative), and comultiplication given by $\Delta s = s \otimes s$ (which is cocommutative). The functions $k(G)$ have basis given by δ_u for $u \in G$ (assuming that G is finite). The product is just multiplication of functions, which is commutative, and the coproduct is

$$\Delta\delta_u = \sum_{v,w\in G:vw=u} \delta_v \otimes \delta_w ,$$

which is in general not cocommutative. In the symbol $kM \triangleright\!\!\blacktriangleleft k(G)$, the $\triangleright$ part means that kM acts on $k(G)$, and the $\blacktriangleleft$ part means that $k(G)$ coacts on kM. The bicrossproduct Hopf algebras are generally non-commutative and non-cocommutative. These bicrossproduct Hopf algebras arose in the work by S. Majid in an algebraic approach to quantum-gravity [16]. Also they have been noted in the work by M. Takeuchi in connection with extension theory [25].

A number of Hopf algebra constructions related to knot and three manifold invariants have been developed. Among them is the Quantum Double construction of V. G. Drinfeld which associates to a general Hopf algebra H a quasitriangular Hopf algebra $D(H)$, and induces a braiding on the category of its representations [7]. In [5] E. J. Beggs, J. Gould and S. Majid have computed the quantum double and braiding for the bicrossproduct Hopf algebras associated to the factorization $X = GM$ which led to an interesting generalization of crossed modules to bicrossed bimodules.

The coproduct on a Hopf algebra means that a tensor product can be defined for representations. The idea of tensor product is formalized in the definition of a tensor or monoidal category. If the Hopf algebra has a quasitriangular structure, there is a map of representations from $V \otimes W$ to $W \otimes V$, making the category of representations into a braided tensor category. These categories have a description in terms of diagrams of crossing lines, giving a direct calculation of the associated knot invariants.

In [4] E. Beggs has introduced a construction of a non-trivially associated tensor category $\mathcal{C}$ from data which is a choice of left coset representatives M for a subgroup G of a finite group X. This introduces a binary operation $\cdot$ and a G-valued 'cocycle' τ on M. There is also a double construction where X is viewed as a subgroup of a larger group. This gives rise to a braided category $\mathcal{D}$, which is the category of representations of an algebra D, which is itself in the category, and it is this category that we concentrate on in this thesis.

It is our aim in this thesis to find more results using this coset construction introduced in [4]. For example, we show that the non-trivially associated Hopf algebra D has representations which have characters in the same way that the representations of a finite group have characters, and also that the category of its representations has a modular structure in the same way that the category of representations of the double of a group has a modular structure.

This thesis will make continual use of formulae and ideas from [4] which is itself based on the papers [5, 6], but is mostly self contained in terms of notation and definitions. The book [19] has been used as a standard reference for tensor categories and braided Hopf algebras, and [29, 3] as references for modular categories.

An outline of the thesis is the following:

In **chapter 1** we include some important definitions and results which are related to our work. We begin by giving basic algebraic definitions, then the definitions of algebras and coalgebras are given. Next we give the definitions of bialgebras and Hopf algebras as algebraic systems in which the structures of algebras and coalgebras are interrelated by certain laws. The definitions of monoidal categories, rigid categories and braided categories are given with examples in the second section. The coset construction for braided categories is explained depending on [4] in the third section.

In the last section we include some standard definitions, theorems and results for the representations and characters of finite groups. We will consider Hopf algebra analogues of some of these results later.

In **chapter 2** we consider some basic facts about the braided category $\mathcal{D}$ which are useful but do not fit in any subsequent chapter. We begin with an example of the braided category $\mathcal{D}$ which will be used later. Then the category $\mathcal{D}$ is shown to be a ribbon category. The question of whether the braided Hopf algebra D is braided commutative or cocommutative in the category $\mathcal{D}$ is considered. Finally we look at the construction of integrals in the category $\mathcal{D}$.

In general the category $\mathcal{C}$ is not braided, but it contains the braided category $\mathcal{D}$ by forgetting the G-grading and the M-action. In **chapter 3** we show that the braiding for $\mathcal{D}$ extends to a partially defined braiding on $\mathcal{C}$. Moreover, we look at the algebra $A \in \mathcal{C}$

again, using this partial braiding and find a strange one sided braided counit. This may have some relevance to the work by J. Green, D. Nichols and E. Taft on one sided Hopf structures [8]. The chapter continues by showing that A is isomorphic to A^* as objects in $\mathcal{C}$, and calculate the coproduct on A^*.

There remain the problems of a star structure on the objects in $\mathcal{C}$ and the existence of an antipode on A. While no definitive conclusion was reached on these matters, it is shown in the last section that more morphisms can be added to $\mathcal{C}$ to make a richer structure. More work in this direction may possibly shed light on the problems mentioned.

We begin **chapter 4** by describing the indecomposable objects in $\mathcal{C}$, in a similar manner to that used in [6]. A detailed example is given using the group D_6. Then we show how to find the dual objects in the category, and again illustrate this with the example. Next we explicitly evaluate in $\mathcal{D}$ the standard diagram for trace in a ribbon category [19]. Then we define the character of an object in $\mathcal{D}$ as an element of the dual of the braided Hopf algebra D. This element is shown to be right adjoint invariant. Additionally, we show that the character is multiplicative for the tensor product of objects. A formula is found for the character in $\mathcal{D}$ in terms of characters of group representations. Finally we use integrals to construct abstract projection operators to show that general objects in $\mathcal{D}$ can be split into a sum of simple objects.

In **chapter 5** we show that the category $\mathcal{D}$ of the representations of the non-trivially associated algebra D has a modular structure, in the same way that the category of representations of the double of a group has a modular structure. We begin the chapter by giving the definition of modular category, and some other important definitions and results. The ribbon maps are calculated for the indecomposable objects in our example category of section 4.2. The last ingredient needed for a modular category is the trace of the double braiding, and this is calculated in $\mathcal{D}$ in terms of group characters. Then the matrices S, T and C implementing the modular representation are calculated explicitly for our example.

In **chapter 6** we show an equivalence of categories between the double $\mathcal{D}$ of the non-trivially associated tensor category, constructed from left coset representatives of a subgroup of a finite group X, and the category of representations of the Drinfeld double of the group, $D(X)$.

In **chapter 7**, the last chapter, we consider ideas for further research. This includes some detailed calculations on the matrix group SU_2, where there is no single continuous choice of coset representatives for the subgroup of diagonal matrices.

In the appendix we include the main part of the Mathematica files that show the modularity of the category $\mathcal{D}$ for the example discussed in chapter 5.

Chapter 4 (except section 4.6), chapter 5 and chapter 6 have already been sent for publication as a paper by myself and my supervisor E. J. Beggs [2].

Throughout the thesis we assume that all groups mentioned, unless otherwise stated, are finite, and that all vector spaces are finite dimensional. We take the base field k to be the complex numbers $\mathbb{C}$.

Chapter 1

Preliminaries

1.1 Hopf Algebra

Hopf algebras or quantum groups are an exciting generalization of group algebras, and also of function algebras on groups. "They have many remarkable properties and they come with a wealth of examples and applications in pure mathematics and mathematical physics"[20]. Moreover, quantum groups are clearly indicative of a more general non-commutative geometry.

Hopf algebras first appeared in the work of H. Hopf in connection with the cohomology of groups, and also in the work of G. I. Kac in the study of group duals. Hopf algebra also came up in the representation theory of Lie groups and algebraic groups. The text books [24], [1] and [13] are important references for Hopf algebras.

There are also finite-dimensional Hopf algebras such as bicrossproduct Hopf algebras associated to the factorization of finite groups (see [6], [5] and [4]).

In this section we give the definition of Hopf algebras. We begin by giving basic algebraic definitions, then the definitions of algebras and coalgebras are given. Finally, we give the definitions of bialgebras and Hopf algebras as algebraic systems in which the structures of algebras and coalgebras are interrelated by certain laws.

Recall that **a group** $(G, \cdot)$ is a set G with an associative multiplication $\cdot$, a unit element e such that $e \cdot g = g = g \cdot e$ for all $g \in G$, and for which every element g has an inverse g^{-1} such that $gg^{-1} = g^{-1}g = e$. If $g_1 g_2 = g_2 g_1$ for all $g_1, g_2 \in G$, then G is called an abelian group.

For the groups G and H, a **homomorphism** $\phi : G \to H$ is a map satisfying

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2) \quad \text{for all} \quad g_1, g_2 \in G.$$

A group G can **act** on a structure from the left or from the right. We say that G acts on a set M from the right, if for every element $u \in G$ there is a map from M to M, say $s \to (s \triangleleft u)$, such that

$$(s \triangleleft v) \triangleleft u = s \triangleleft vu, \quad \text{for all } u, v \in G, \text{ and } s \in M\,.$$

In the same way, G can act on M from the left [19].

Finally, the **transposition map**, or the **twist map**, $\tau : V \otimes W \to W \otimes V$ is defined by

$$\tau(v \otimes w) = w \otimes v, \quad \text{for } v \in V \text{ and } w \in W.$$

Definition 1.1.1 *[19] An* **algebra***, or an associative k-algebra with unit, is a k-vector space A together with two k-linear maps, multiplication* $\cdot : A \otimes A \to A$ *and unit* $\eta : k \to A$ *such that the following diagrams are commutative:*

a) associativity

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\cdot \otimes id} & A \otimes A \\ {\scriptstyle id \otimes \cdot}\downarrow & & \downarrow{\scriptstyle \cdot} \\ A \otimes A & \xrightarrow{\cdot} & A \end{array}$$

b) unit

$$\begin{array}{ccccc} k \otimes A & \xrightarrow{\eta \otimes id} & A \otimes A & \xleftarrow{id \otimes \eta} & A \otimes k \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cdot} & & \downarrow{\scriptstyle \cong} \\ A & \xrightarrow{id} & A & \xleftarrow{id} & A \end{array}$$

The two outside vertical maps in b) are given by scalar multiplication. Also in b) the usual identity element in A is given by setting $1_A = \eta(1_k)$.

For two algebras A and B, the tensor product $A \otimes B$ is an algebra with multiplication

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2, \qquad \text{for all } a_1, a_2 \in A \text{ and } b_1, b_2 \in B,$$

and has vector space given by the tensor product of the vector spaces A and B.

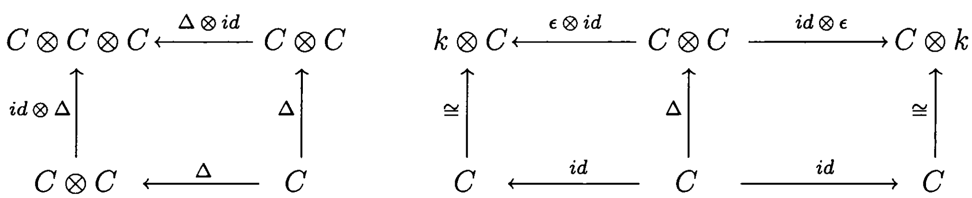
The algebra A is commutative if and only if $\cdot \circ \tau = \cdot$, where τ is the twist map. The linear map f is said to be an algebra map if it respects the algebra structure, i.e.

$$f(ab) = f(a)f(b) \qquad \text{and} \qquad f(1) = 1.$$

Now we dualize the notion of algebra to get the following definition.

Definition 1.1.2 *[19] A* **coalgebra***, or a coassociative k-coalgebra with counit, is a k-vector space C together with two k-linear maps, comultiplication* $\Delta : C \to C \otimes C$ *and* *counit* $\epsilon : C \to k$ *such that the following diagrams are commutative:*

a) coassociativity b) counit



The two outside vertical maps in b) are given by $c \mapsto 1 \otimes c$ *and* $c \mapsto c \otimes 1$, *for any* $c \in C$. *Note that the commutative diagrams here are obtained from the commutiave diagrams in the definition of an algebra by reversing arrows.*

In symbols, coassociativity means $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$, and that ϵ is a counit means $(\epsilon \otimes \mathrm{id}) \circ \Delta(\mathrm{c}) = \mathrm{c} = (\mathrm{id} \otimes \epsilon) \circ \Delta(\mathrm{c})$ for all $\mathrm{c} \in \mathrm{C}$.

The coalgebra C is said to be cocommutative if $\tau \circ \Delta = \Delta$, where τ is the twist map.

For two coalgebras C and D, the tensor product $C \otimes D$ is also a coalgebra with vector space given by the tensor product of the vector spaces C and D, and comultiplication given by $\Delta(c \otimes d) = c_1 \otimes d_1 \otimes c_2 \otimes d_2$ where, $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$ and $\Delta(d) = \sum d_{(1)} \otimes d_{(2)}$, for all $c \in C$ and $d \in D$.

The map f is said to be a coalgebra map if it respects the coalgebra structure, i.e.

$$(f \otimes f) \circ \Delta = \Delta \circ f, \qquad \text{and} \qquad \epsilon \circ f = \epsilon.$$

Notation. [24, 22] Let C be any coalgebra with comultiplication $\Delta : C \to C \otimes C$. We give a sigma notation for Δ as follows:

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)} \qquad \text{for any} \quad c \in C.$$

The subscripts (1) and (2) are symbolic and do not indicate particular elements of C. When Δ must be applied more than once the power of the notation becomes apparent. For example, the coassociativity diagram gives that $\sum c_{(1)} \otimes c_{(2)_{(1)}} \otimes c_{(2)_{(2)}} = \sum c_{(1)_{(1)}} \otimes c_{(1)_{(2)}} \otimes c_{(2)}$, which can simply written as $\Delta_2(c) = \sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$. In general we write

$$\Delta_{n-1}(c) = \sum c_{(1)} \otimes \ldots . \otimes c_{(n)} \,.$$

$\Delta_{n-1}(c)$ is the element obtained by applying the coassociativity $(n-1)$ times. According to this notation the counit diagram says that

$$\sum \epsilon(c_{(1)})c_{(2)} = \sum \epsilon(c_{(2)})c_{(1)} = c \qquad \text{for all} \quad c \in C.$$

Definition 1.1.3 *[22] A k-vector space H is a* **bialgebra** *if $(H, \cdot, \eta)$ is an algebra, (H, Δ, ϵ) is a coalgebra and either of the following equivalent conditions holds:*

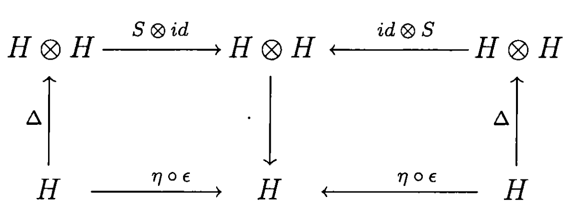
1) Δ and ϵ are algebra maps.

2) $\cdot$ and η are coalgebra maps.

Definition 1.1.4 *[19] A* **Hopf algebra** *H is a bialgebra equipped with a linear map $S : H \longrightarrow H$, called the antipode, satisfying*

$$\cdot(S \otimes \mathrm{id}) \circ \Delta = \cdot(\mathrm{id} \otimes \mathrm{S}) \circ \Delta = \eta \circ \epsilon \,,$$

which can be illustrated by the following commutative diagram:



Note that in sigma notation, S satisfies

$$\sum (Sh_{(1)})\, h_{(2)} = \sum h_{(1)}\, (Sh_{(2)}) = \epsilon(h)\, 1_h \qquad \text{for all} \quad h \in H.$$

The antipode map in Hopf algebra plays a similar role to that of the inverse map which sends each element to its inverse in a group, although it is not required that $S^2 = \text{id}$. Moreover, we do not assume that S as a linear map has an inverse S^{-1} although this is always so if H is a finite dimensional Hopf algebra.

Proposition 1.1.5 *[19] The antipode map S of a Hopf algebra H is unique and also is:*

1- an antialgebra map, i.e. $S(h_{(1)}\, h_{(1)}) = S(h_{(2)})\, S(h_{(1)})$ *and* $S(1) = 1$,

2- an anticoalgebra map, i.e. $(S \otimes S) \circ \Delta(h) = \tau \circ \Delta \circ S(h)$ *and* $\epsilon S(h) = \epsilon(h)$, *for all* $h \in H$, *where τ is the twist map.*

Proof. For the complete proof of this proposition see [19].

We shall now see that for every finite Hopf algebra H there is a **dual Hopf algebra** $H^* = Hom_k(H, k)$. H and H^* determine a non-degenerate bilinear form $\langle\, ,\, \rangle : H^* \otimes H \to k$ via $\langle \phi, h \rangle = \phi(h)$. By non-degenerate we mean that $\langle \phi, h \rangle = 0$ for all $h \in H$ implies that $\phi = 0$, and $\langle \phi, h \rangle = 0$ for all $\phi \in H^*$ implies that $h = 0$. If $\phi : V \to W$ is k-linear, then $\phi^* : W^* \to V^*$ is given by $\phi^*(f)(v) = f\big(\phi(v)\big)$ for all $f \in W^*$ and $\mathrm{v} \in \mathrm{V}$. The explicit formulae that determine the Hopf algebra structure on H^* from that on H are as follows:

$$\begin{gathered} \langle \phi\psi, h \rangle = \langle \phi \otimes \psi, \Delta h \rangle\,, \quad \langle 1, h \rangle = \epsilon(h)\,, \quad \langle \phi, 1 \rangle = \epsilon(\phi)\,, \\ \langle \phi, hg \rangle = \langle \Delta\phi, h \otimes g \rangle\,, \quad \langle S(\phi), h \rangle = \langle \phi, S(h) \rangle\,, \end{gathered} \tag{1.1}$$

for all $h\,, g \in H$ and $\phi\,, \psi \in H^*$. Two Hopf algebras A and B are said to be dual if there is a non-degenerate bilinear form $\langle\, ,\, \rangle : A \otimes B \to k$, satisfying the rules mentioned in (1.1).

Example 1.1.6 *[20] Let G be any finite group and let $H = kG$ be its group algebra (the vector space with basis $\sum \lambda_g \cdot g$ where $g \in G$ and $\lambda_g \in k$, which just means that the elements of kG is a linear combination of the elements of G). Then H is bialgebra via the following: The product in G, $1 = e$, $\Delta g = g \otimes g$ and $\epsilon(g) = 1$ for all $g \in G$ where e is the identity element of G. Moreover, H is a Hopf algebra by defining $S(g) = g^{-1}$ for each $g \in G$. In fact, G does not need to be finite for kG to be a Hopf algebra, but we will be interested in the finite case.*

1.2 Braided Categories

The idea of Hopf algebras in braided categories goes back to Milnor and Moore [21]. The notion of braided category plays an important role in quantum group theory. S. Majid (see [19]) studies Hopf algebras in braided categories under the name "braided groups" with an algebraic motivation from biproduct construction as well as many motivations from physics [27]. In this section, as well as throughout the thesis, with respect to the braided categories and Hopf algebras in braided categories, we heavily rely on the work by S. Majid in [19].

Definition 1.2.1 *[19] A category $\mathcal{C}$ is a collection (class) of objects V, W, Z, U, etc, denoted by $obj(\mathcal{C})$, and a set $Mor_{\mathcal{C}}(V,W)$ of morphisms for each pair (V,W) of objects of $\mathcal{C}$ (if $\mathcal{C}$ is clear we write $Mor_{\mathcal{C}}(V,W) = Mor(V,W)$). The sets $Mor(V,W)$ and $Mor(Z,U)$ are disjoint unless $(V,W) = (Z,U)$. There should also be a composition operation $\circ$ such that for all V, W, Z, $U \in \mathcal{C}$ and $\phi \in Mor(V,W)$, $\varphi \in Mor(W,Z)$ and $\psi \in Mor(Z,U)$:*

1) There should be an element $\varphi \circ \phi$ in $Mor(V,Z)$, *2) the associativity property of* $\circ$ *holds, i.e.* $(\phi \circ \varphi) \circ \psi = \phi \circ (\varphi \circ \psi)$,

3) every set $Mor(W, W)$ *should contain an identity element* $\mathrm{id_W}$ *such that* $\varphi \circ \mathrm{id_W} = \varphi$ *and* $\mathrm{id_W} \circ \phi = \phi$.

A morphism $\phi \in Mor(V, W)$ *is called an* **isomorphism** *if there exists a morphism* $\phi^{-1} \in Mor(W, V)$ *such that* $\phi \circ \phi^{-1} \in Mor(W, W)$ *and* $\phi^{-1} \circ \phi \in Mor(V, V)$ *are identity morphisms.*

Definition 1.2.2 *A category* $\mathcal{D}$ *is called a* **subcategory** *of the category* $\mathcal{C}$ *if the objects of* $\mathcal{D}$ *form a subclass of the objects of* $\mathcal{C}$*, i.e.* $obj(\mathcal{D}) \subset obj(\mathcal{C})$*, and for any two objects* V, W *of* $\mathcal{D}$, $Mor_{\mathcal{D}}(V, W) \subset Mor_{\mathcal{C}}(V, W)$*, in particular* $1_V \in Mor_{\mathcal{D}}(V, V)$ *is the same as* $1_V \in Mor_{\mathcal{C}}(V, V)$*. If* $Mor_{\mathcal{D}}(V, W) = Mor_{\mathcal{C}}(V, W)$ *for all* V, W *in* $\mathcal{D}$*, then* $\mathcal{D}$ *is called a* **full subcategory** *of* $\mathcal{C}$.

Definition 1.2.3 *[19] Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two categories. A (covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ *is a 'map' between the two categories which respects their structure. Thus for every* $V \in \mathcal{C}$*, we specify an object* $F(V) \in \mathcal{D}$*, and for every morphism* $\phi : V \to W$*, we specify a morphism* $F(\phi) : F(V) \to F(W)$ *such that* $F(\phi \circ \psi) = F(\phi) \circ F(\psi)$ *for any two morphisms* ϕ, ψ *for which* $\phi \circ \psi$ *is defined in* $\mathcal{C}$*, and* $F(\mathrm{id_V}) = \mathrm{id_{F(V)}}$.

Definition 1.2.4 *[19] Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two categories. A contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ *is a 'map' between the two categories which for every* $V \in \mathcal{C}$ *specifies an object* $F(V) \in \mathcal{D}$*, and for every morphism* $\phi : V \to W$ *specifies a morphism* $F(\phi) : F(W) \to F(V)$ *such that* $F(\phi \circ \psi) = F(\psi) \circ F(\phi)$ *for any two morphisms* ϕ, ψ *for which* $\phi \circ \psi$ *is defined in* $\mathcal{C}$*, and* $F(\mathrm{id_V}) = \mathrm{id_{F(V)}}$.

Example 1.2.5 *a) Set, the category of all sets with $Mor(A, B)$ the set of all maps from A to B for any two sets A, B. The composition $f \circ g$ for $f \in Mor(A, B)$, $g \in Mor(B, C)$ for any objects A, B, C in $obj(Set)$ is the usual composition of maps and $1_A \in Mor(A, A)$ is the identity map on A.*

b) Let $\mathcal{C} = Group$ be the category of all groups. $Obj(Group)$ is the class of all groups. If G and H are two groups, then $Mor(G, H)$ is the set of all group homomorphisms from G to H. The composition of two group homomorphisms is also a group homomorphism and for any three group homomorphisms the associativity property is satisfied. The identity map $\mathrm{id_G \in Mor(G, G)}$ *is also a group homomorphism. The category Group is a subcategory of the category Set but it is not a full subcategory.*

Example 1.2.6 *For a unital algebra A, let $\mathcal{C} = \mathcal{M}_A$ be the category of right A-modules. The objects of $\mathcal{M}_A$, $obj(\mathcal{M}_A)$, is the class of all vector spaces on which A acts. The morphisms are the linear maps that commute with the action of A. An example of a functor is the* **forgetful functor** *$F : \mathcal{M}_A \to Vec$ that assigns to each representation its underlying vector space (i.e. throws away the action of A).*

Definition 1.2.7 *[19] A* **monoidal category** *is $(\mathcal{C}, \otimes, \underline{1}, \Phi, l, r)$, where $\mathcal{C}$ is a category and $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor which is associative in the sense that there is a natural equivalence $\Phi : (\ \otimes\ )\otimes \to \otimes(\ \otimes\ )$ which just means that there are given functorial isomorphisms*

$$\Phi_{V,W,Z} : (V \otimes W) \otimes Z \cong V \otimes (W \otimes Z)\,, \quad \text{for all} \quad V, W, Z \in \mathcal{C}\,,$$

obeying the pentagon condition given in the diagram below . A unit object $\underline{1}$ is also required and natural equivalences between the functors $(\ ) \otimes \underline{1}$, $\underline{1} \otimes (\ )$ and the identity *functor $\mathcal{C} \to \mathcal{C}$, i.e. there should be given functorial isomorphisms $l_V : V \cong V \otimes \underline{1}$ and $r_V : V \cong \underline{1} \otimes V$, obeying the triangle condition given in the diagram below.*

$$
\begin{array}{ccc}
(V \otimes \underline{1}) \otimes W & \xrightarrow{\Phi} & V \otimes (\underline{1} \otimes W) \\
{\scriptstyle l \otimes \mathrm{id}} \nwarrow & & \nearrow {\scriptstyle \mathrm{id} \otimes r} \\
& V \otimes W &
\end{array}
\qquad
\begin{array}{ccc}
& (V \otimes W) \otimes (Z \otimes U) & \\
{\scriptstyle \Phi} \nearrow & & \searrow {\scriptstyle \Phi} \\
((V \otimes W) \otimes Z) \otimes U & & V \otimes (W \otimes (Z \otimes U)) \\
{\scriptstyle \Phi \otimes \mathrm{id}} \downarrow & & \uparrow {\scriptstyle \mathrm{id} \otimes \Phi} \\
(V \otimes (W \otimes Z)) \otimes U & \xrightarrow[\Phi]{} & V \otimes ((W \otimes Z) \otimes U)
\end{array}
$$

a) The triangle condition b) The pentagon condition

Example 1.2.8 *The category Set of sets is a monoidal category with $\otimes = \times$, the direct product of sets. The category Vec of vector spaces is also monoidal category with $\otimes$ the usual tensor product. In both cases, the isomorphisms are the obvious ones. The unit objects are the singleton set for the category Set and the field k for the category Vec.*

Definition 1.2.9 *[19] A* **braided monoidal** *(or quasitensor)* **category** $(\mathcal{C}, \otimes, \Psi)$ *is a monoidal category which is commutative in the sense that there is a natural equivalence between the two functors $\otimes, \otimes^{op} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, which just means that there are given functorial isomorphisms*

$$\Psi_{V,W} : V \otimes W \to W \otimes V, \quad \text{for all} \quad V, W \in \mathcal{C},$$

obeying the hexagon conditions in the following diagram:

$$
\begin{array}{ccc}
& V \otimes (W \otimes Z) & \\
{\scriptstyle \mathrm{id} \otimes \Psi} \swarrow & & \searrow {\scriptstyle \Phi^{-1}} \\
V \otimes (Z \otimes W) & & (V \otimes W) \otimes Z \\
{\scriptstyle \Phi^{-1}} \downarrow & & \downarrow {\scriptstyle \Psi} \\
(V \otimes Z) \otimes W & & Z \otimes (V \otimes W) \\
{\scriptstyle \Psi \otimes \mathrm{id}} \searrow & & \swarrow {\scriptstyle \Phi^{-1}} \\
& (Z \otimes V) \otimes W &
\end{array}
\qquad
\begin{array}{ccc}
& (V \otimes W) \otimes Z & \\
{\scriptstyle \Phi} \swarrow & & \searrow {\scriptstyle \Psi \otimes \mathrm{id}} \\
V \otimes (W \otimes Z) & & (W \otimes V) \otimes Z \\
{\scriptstyle \Psi} \downarrow & & \downarrow {\scriptstyle \Phi} \\
(W \otimes Z) \otimes V & & W \otimes (V \otimes Z) \\
{\scriptstyle \Phi} \searrow & & \swarrow {\scriptstyle \mathrm{id} \otimes \Psi} \\
& W \otimes (Z \otimes V) &
\end{array}
$$

If we suppress Φ, the hexagon conditions can be given by the following formulas:

$$\Psi_{V\otimes W,Z} = \Psi_{V,Z} \circ \Psi_{W,Z}\,, \quad \Psi_{V,W\otimes Z} = \Psi_{V,Z} \circ \Psi_{V,W}\,, \quad \text{for all} \quad V,W,Z \in \mathcal{C}\,.$$

Definition 1.2.10 *[19] An object V in a monoidal category $\mathcal{C}$ has a left dual or is rigid if there is an object V^* and morphisms* $\mathrm{ev_V} : \mathrm{V^* \otimes V} \to \underline{1}$, $\mathrm{coev_V} : \underline{1} \to \mathrm{V \otimes V^*}$ *such that*

$$V \cong \underline{1} \otimes V \overset{\mathrm{coev}\otimes\mathrm{id}}{\longrightarrow} (V \otimes V^*) \otimes V \overset{\Phi}{\longrightarrow} V \otimes (V^* \otimes V) \overset{\mathrm{id}\otimes\mathrm{ev}}{\longrightarrow} V \otimes \underline{1} \cong V\,,$$

$$V^* \cong V^* \otimes \underline{1} \overset{\mathrm{id}\otimes\mathrm{coev}}{\longrightarrow} V^* \otimes (V \otimes V^*) \overset{\Phi^{-1}}{\longrightarrow} (V^* \otimes V) \otimes V^* \overset{\mathrm{ev}\otimes\mathrm{id}}{\longrightarrow} \underline{1} \otimes V^* \cong V^*\,,$$

compose to $\mathrm{id_V}$ *and* $\mathrm{id_{V^*}}$*, respectively. Also if V and W are rigid in $\mathcal{C}$ and $\phi : V \to W$ is a morphism in the category, then* $\phi^* = (\mathrm{ev_V} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \phi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{coev_W}) : \mathrm{W^*} \to \mathrm{V^*}$, *is called the dual or the adjoint morphism of ϕ.*

Definition 1.2.11 *[19] If every object in the monoidal category $\mathcal{C}$ has a dual, then we say that $\mathcal{C}$ is a* **rigid** *monoidal category.*

A morphism $T : V \to W$, a tensor product $F : V \otimes W \to Y$, the braid $\Psi_{V,W} : V \otimes W \to W \otimes V$ and the maps $\mathrm{ev_V} : \mathrm{V^* \otimes V} \to \underline{1}$ and $\mathrm{coev_V} : \underline{1} \to \mathrm{V \otimes V^*}$ in tensor categories are represented in terms of diagrams as the following in order:

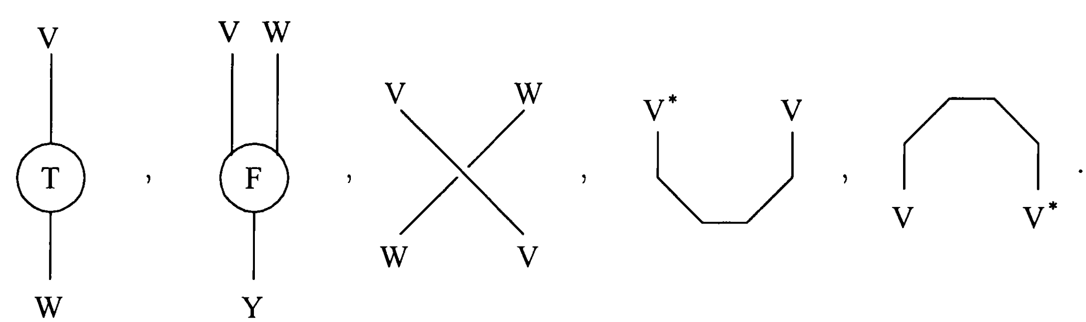


figure 1.1

The definition of dual and the adjoint morphism can be given in terms of diagrams as the following, in order, read from top to bottom:

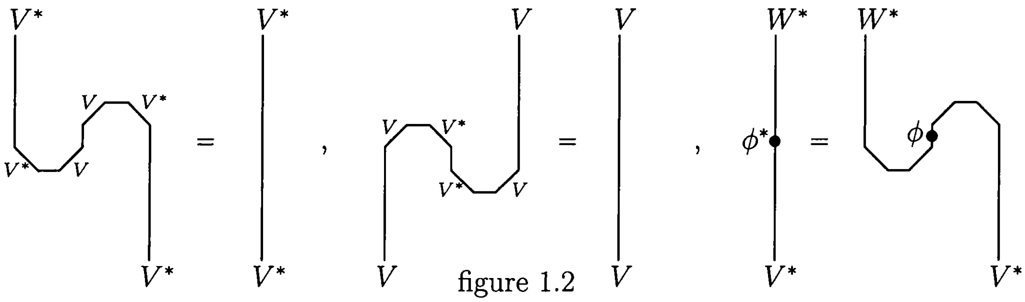


figure 1.2

Some axioms of a Hopf algebra can be illustrated for the unit, counit, associativity, coassociativity and the antipode, in order, by the following diagrams:

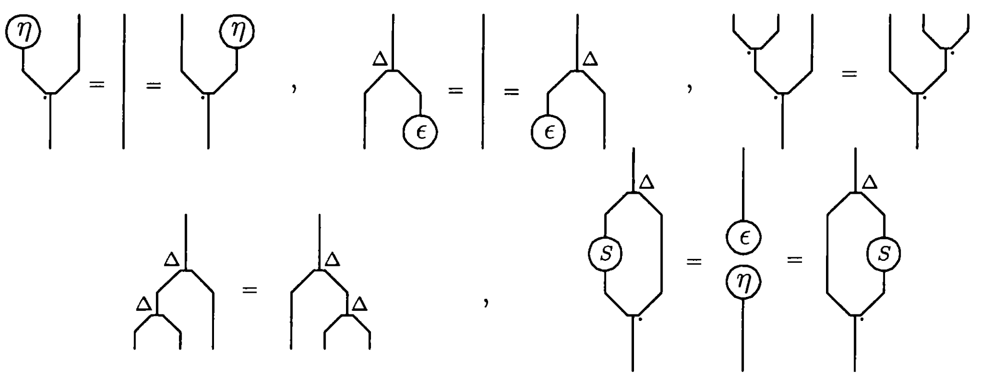


figure 1.3

In the following diagram we give the homomorphism property for a braided coproduct and the braided antihomomorphism property of S in order, where H is any braided Hopf algebra:

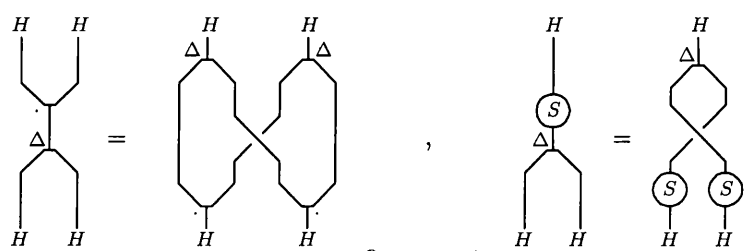


figure 1.4

Proposition 1.2.12 *[20] If H is an algebra in a rigid tensor category, then its dual H^* is a coalgebra in the category using the following definitions:*

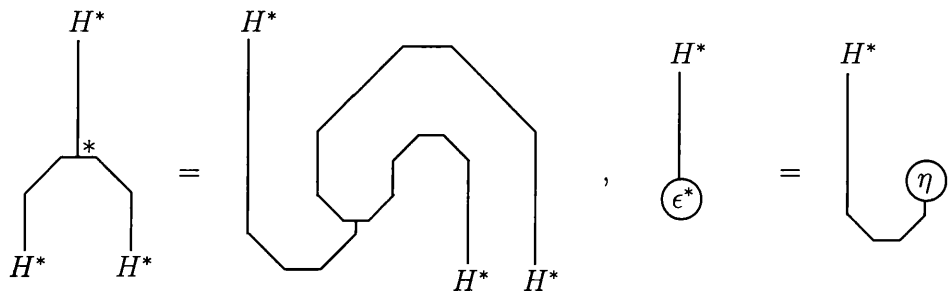
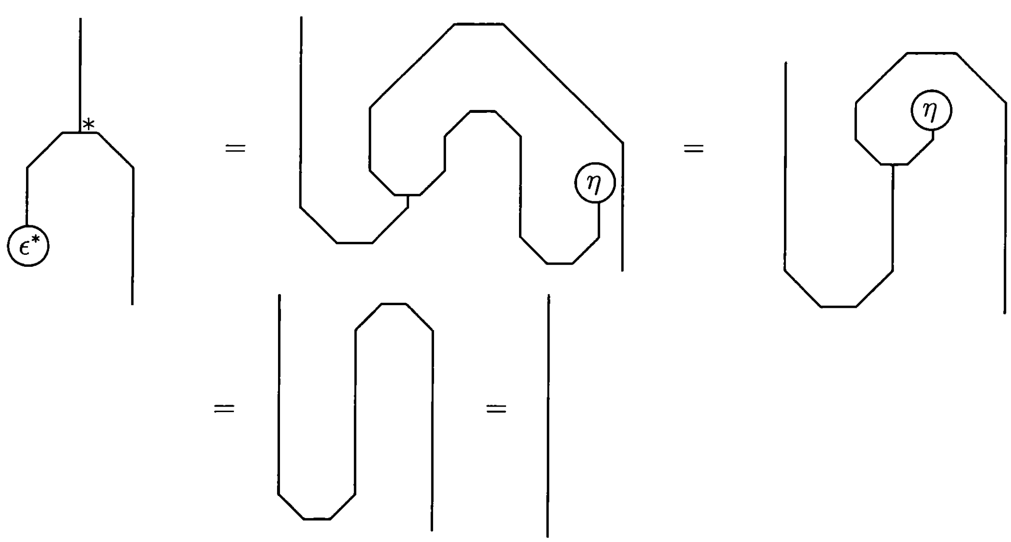
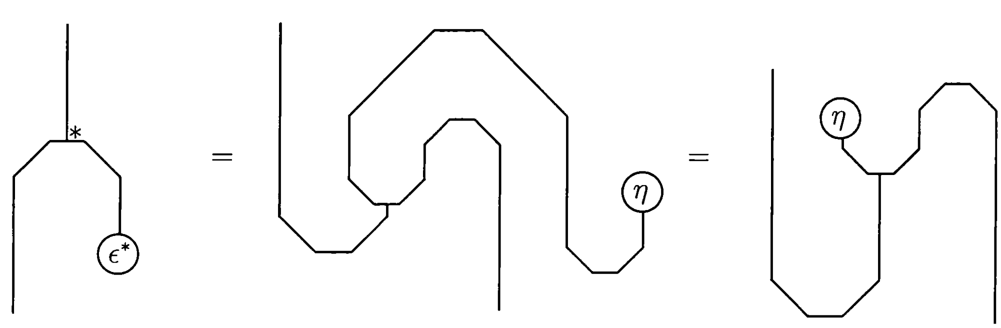
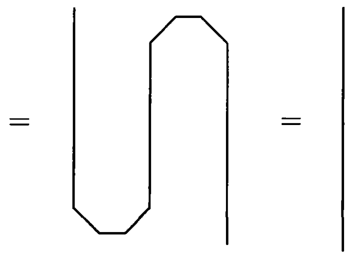


figure 1.4: *a-comultiplication* *b- counit*

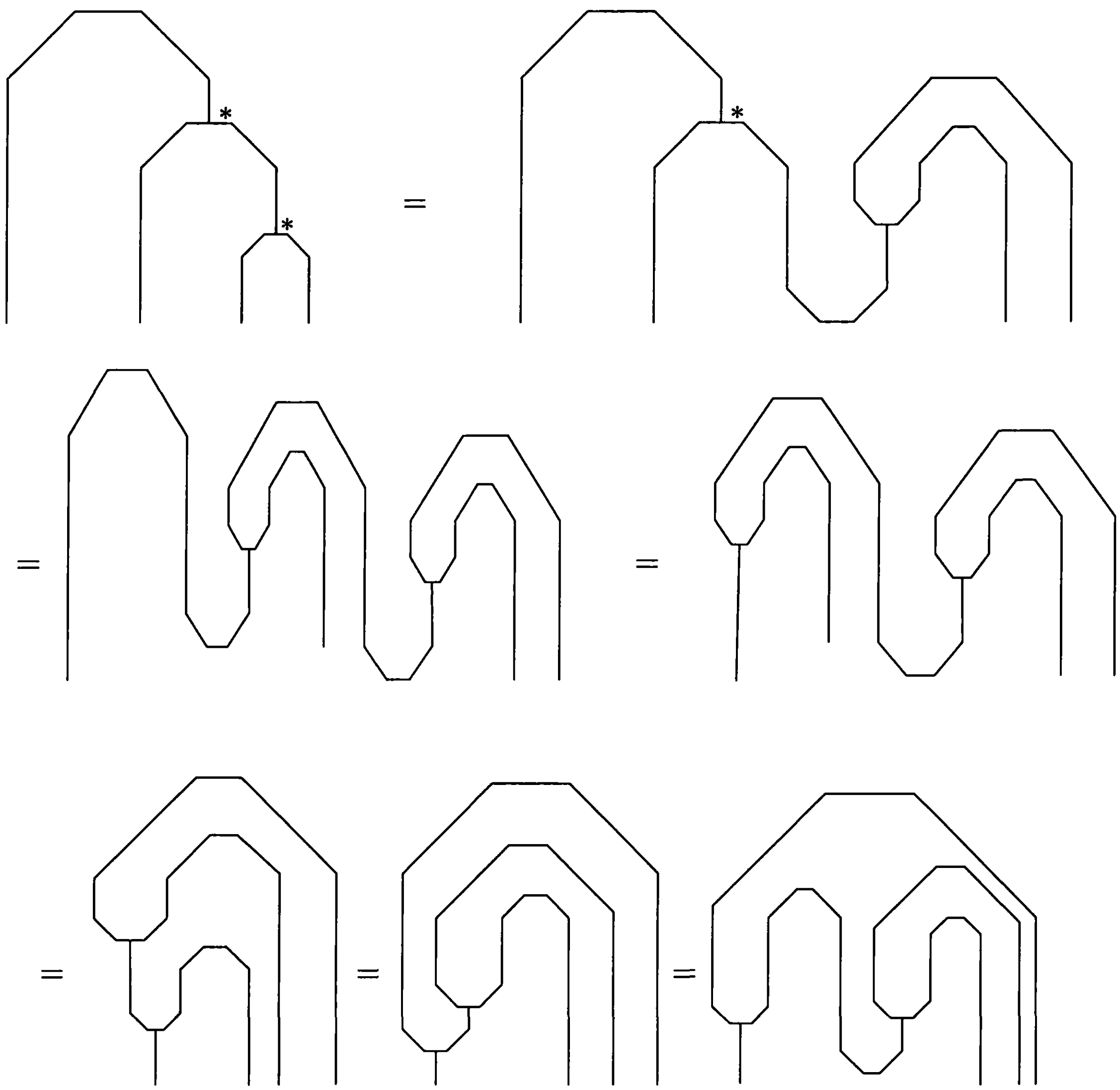
Proof. First we check the counit property for ϵ^* as following

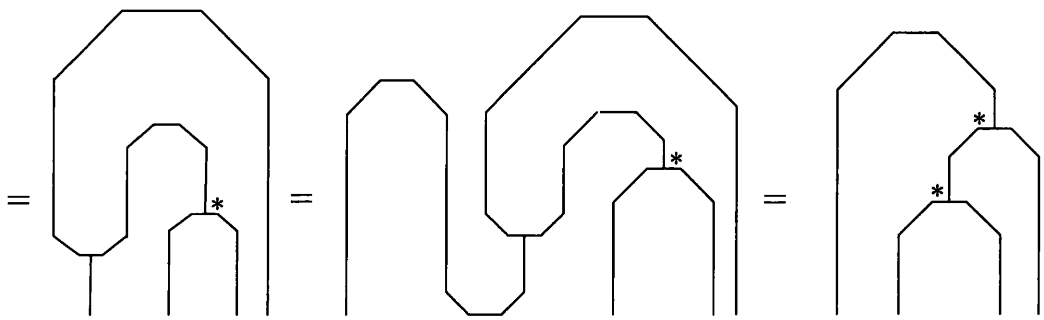






Now we check the coassociativity of the comultiplication





Proposition 1.2.13 *[20] If H is a coalgebra in a rigid tensor category, then its dual H^* is an algebra in the category using the following definitions:*

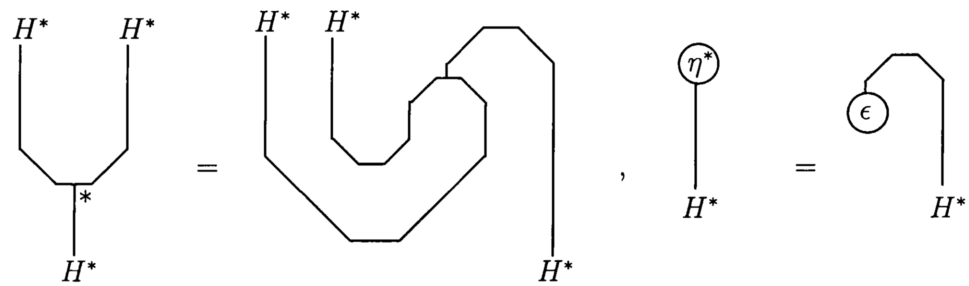
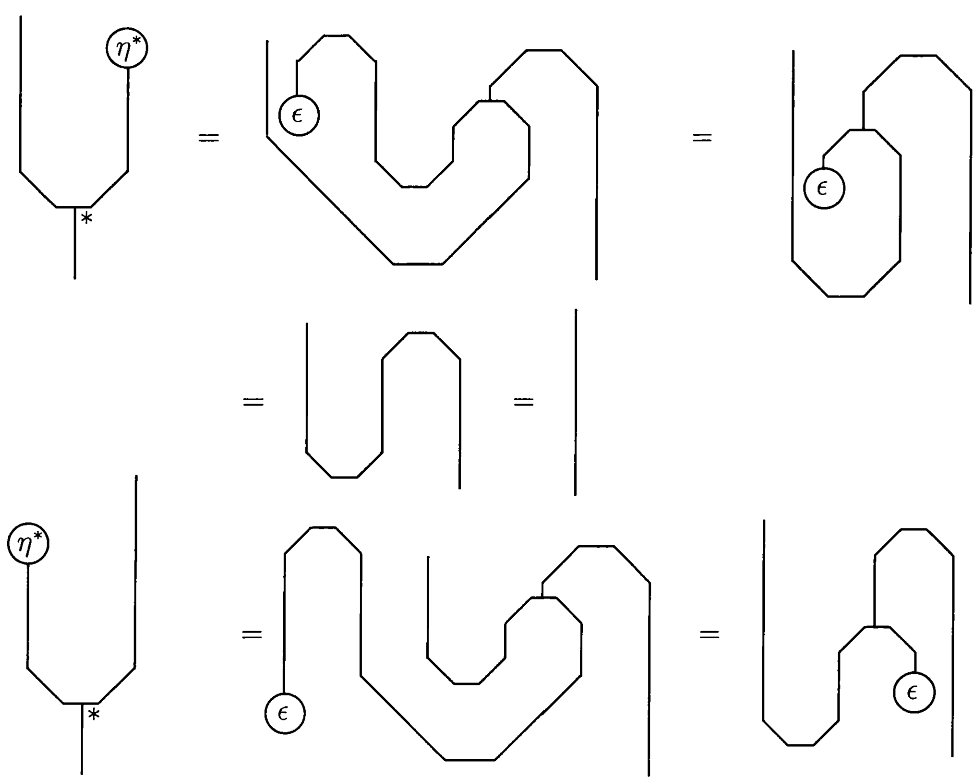
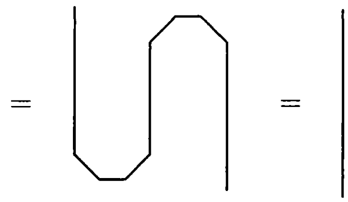


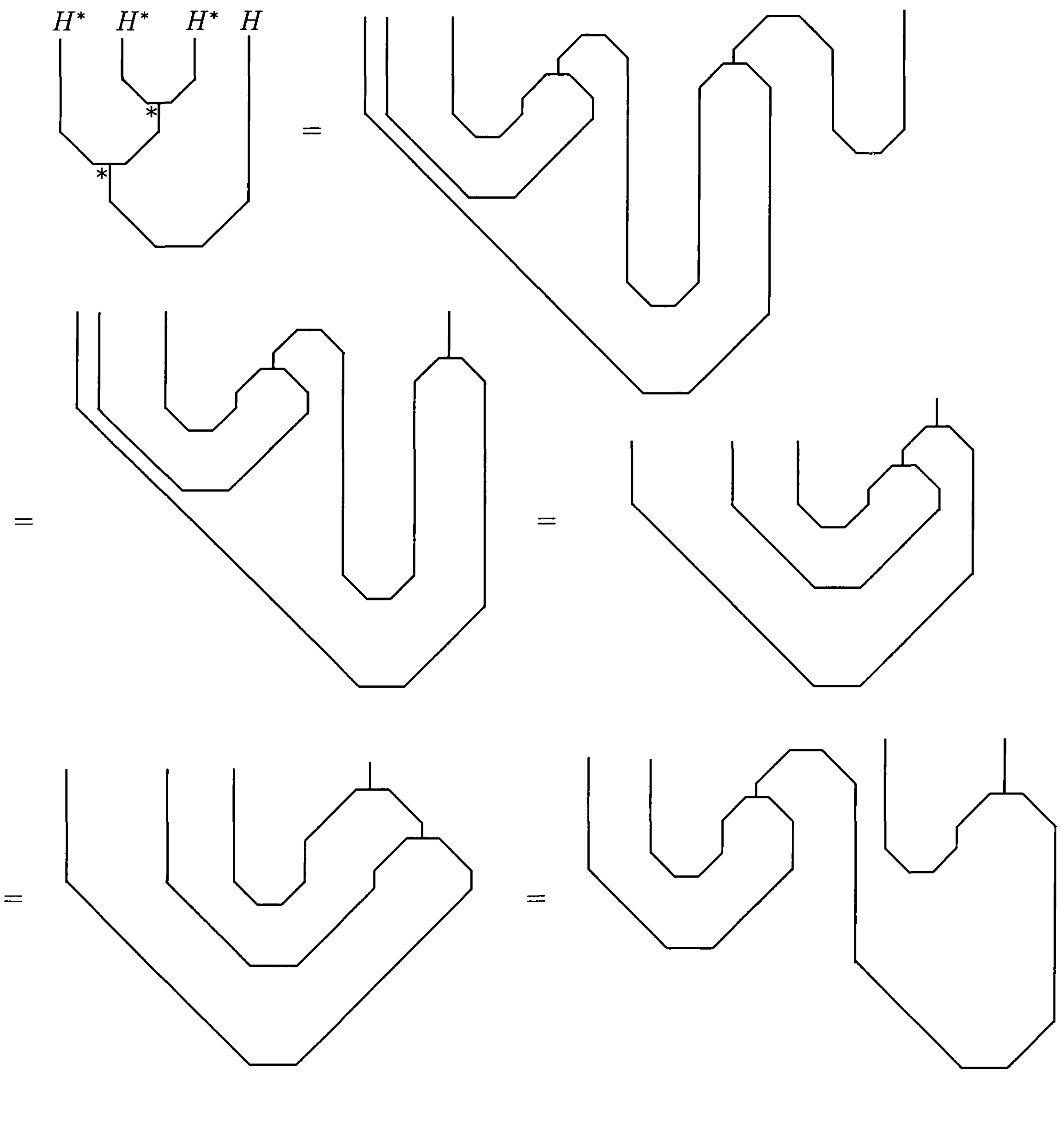
figure 1.5: *a-multiplication* *b- unit*

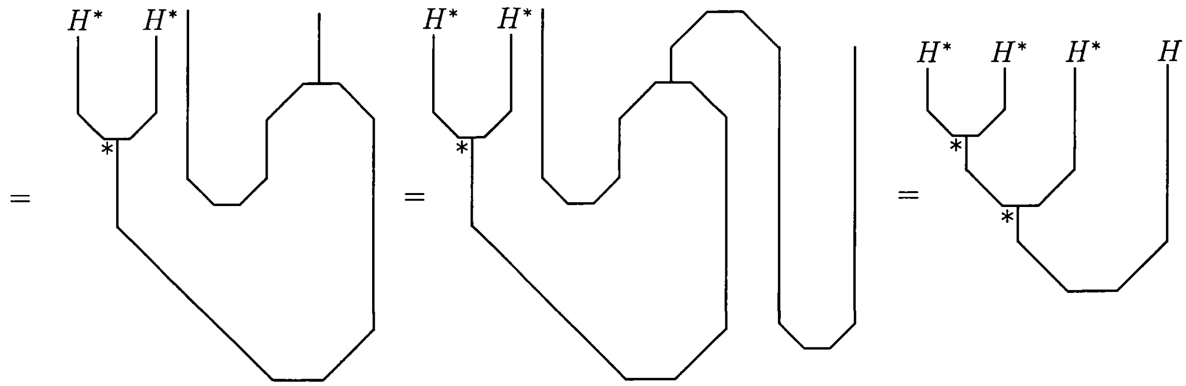
Proof. First we check the unit property for η^*





Now we check the associativity property for the multiplication





Proposition 1.2.14 *[20] If H is a braided Hopf algebra in a rigid braided category, then we can make H^* into a braided Hopf algebra by the following definitions:*

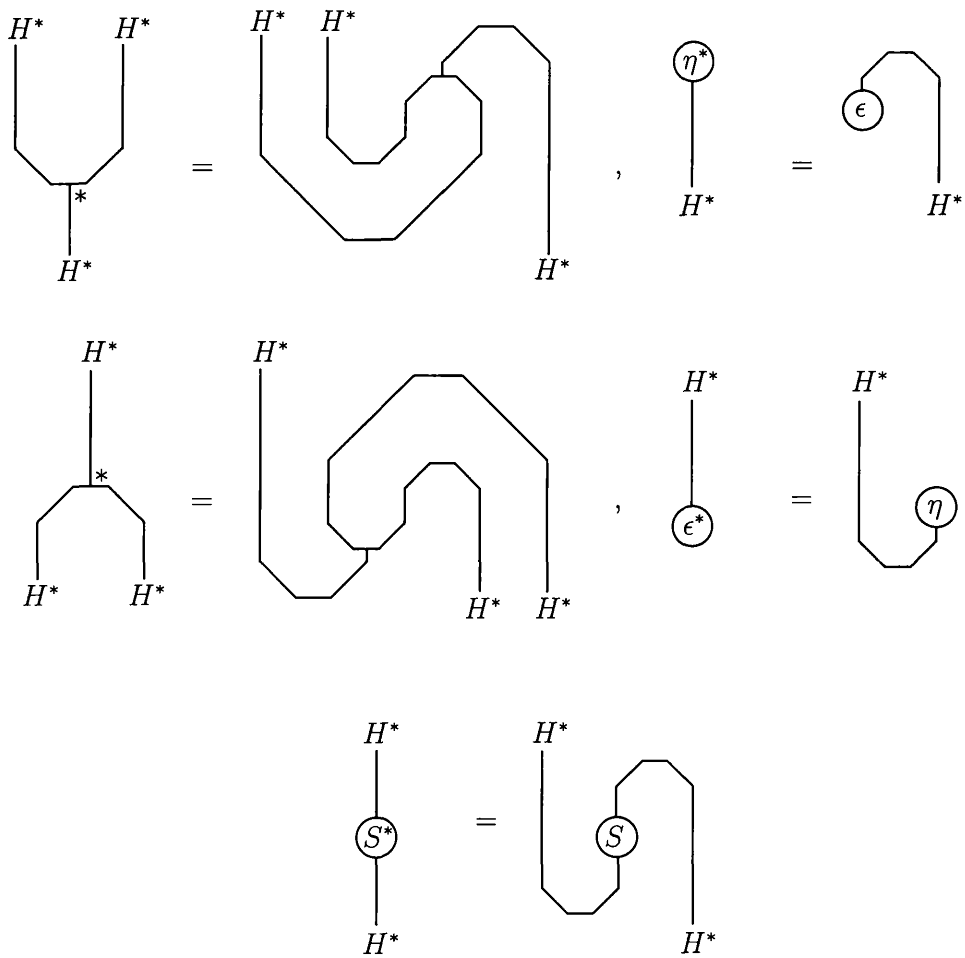
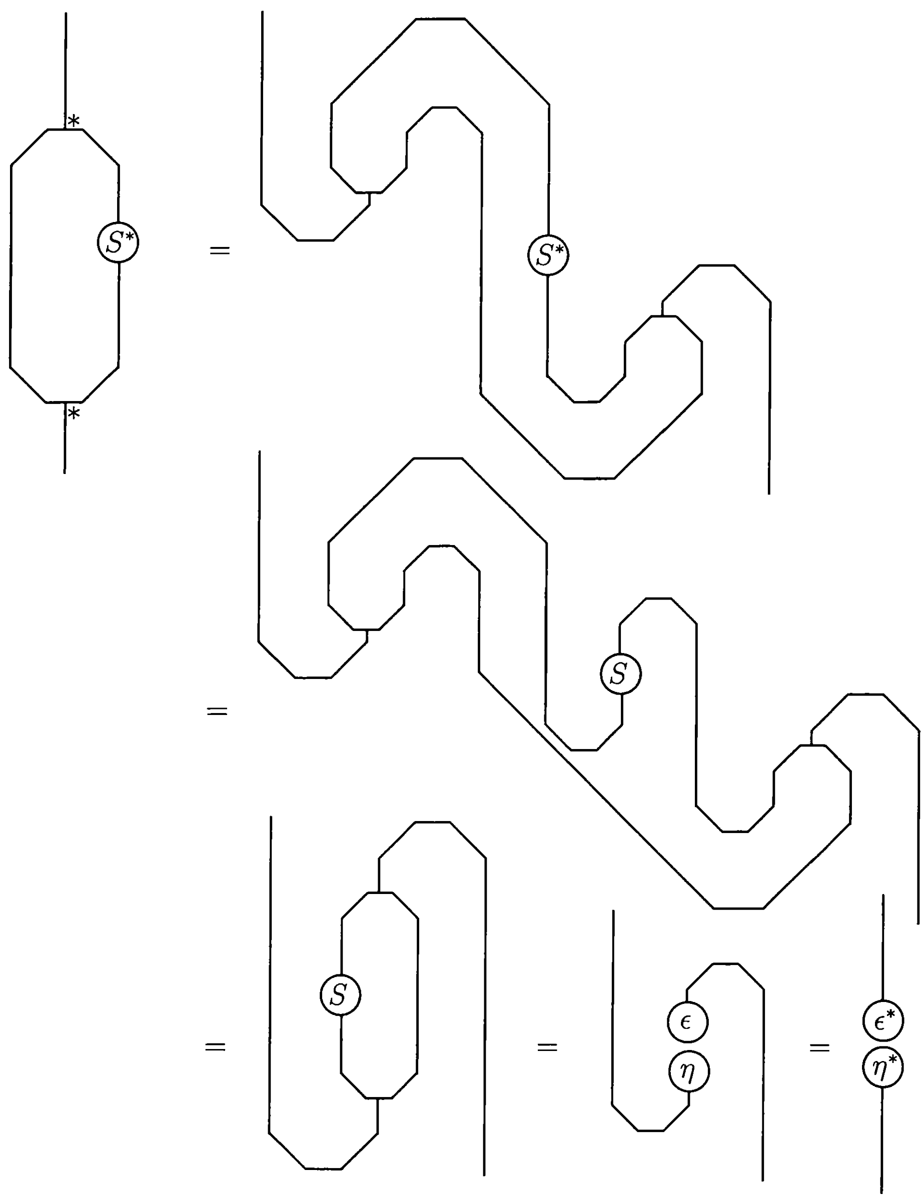
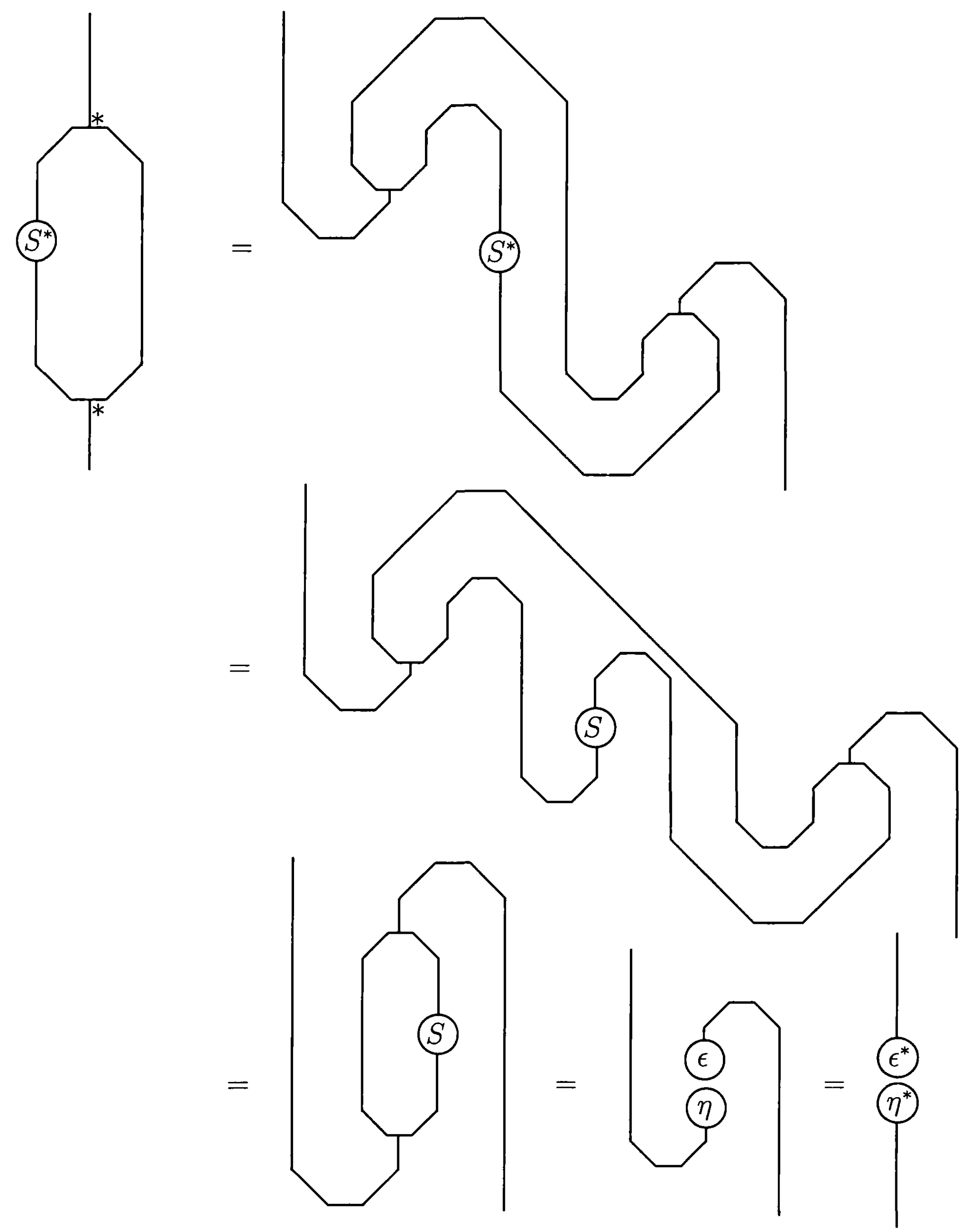


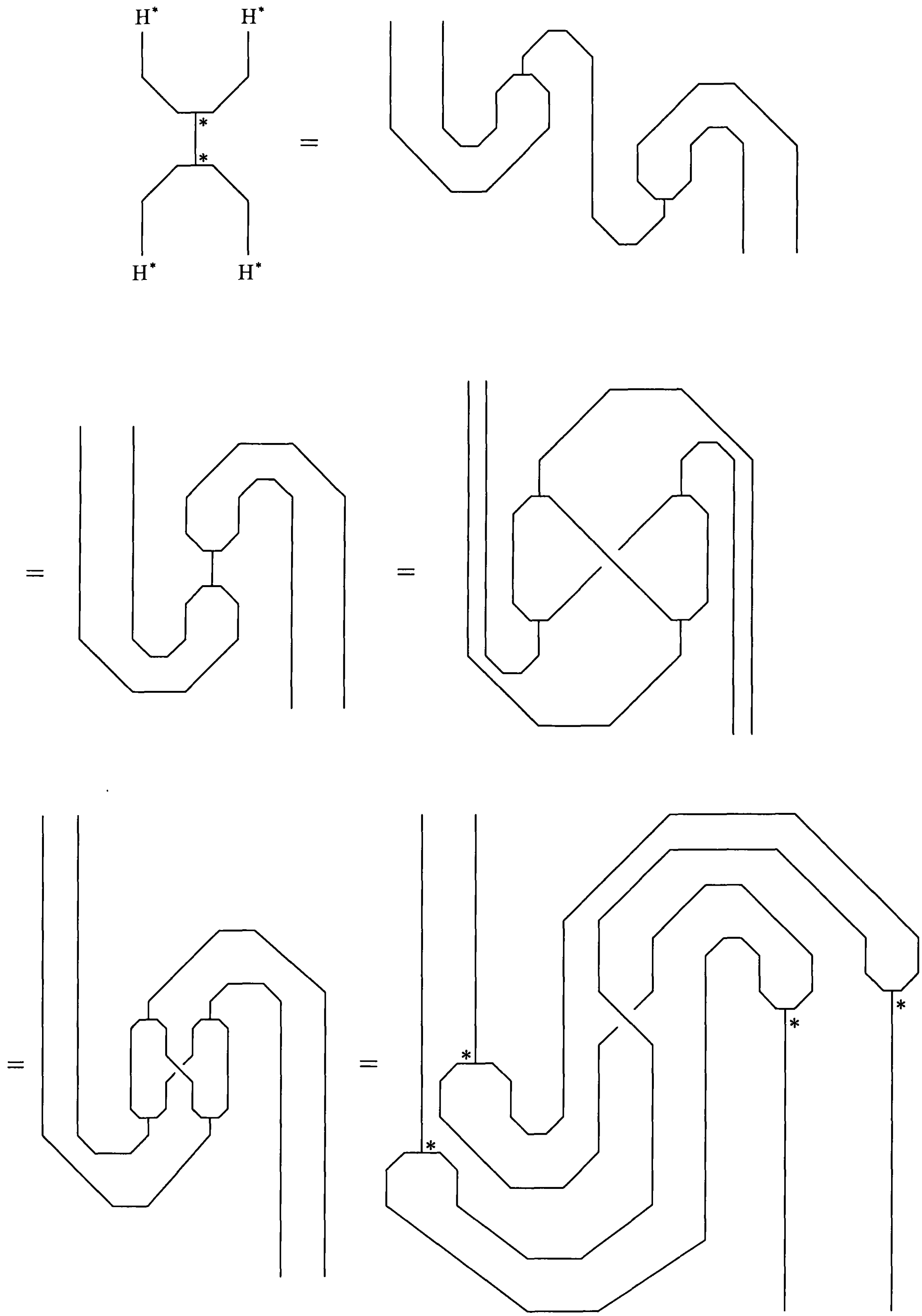
figure 1.6: Definitions of multiplication, unit, comultiplication, counit, and antipode on H^*

Proof. We have already proved in the previous two propositions that unit, counit, associativity and coassociativity properties are held. So we only need to prove the property for S^* and the compatibility condition. We now check the property for S^* as the following:





Lastly we check the compatibility condition between the multiplication and the comultiplication as the following


H*
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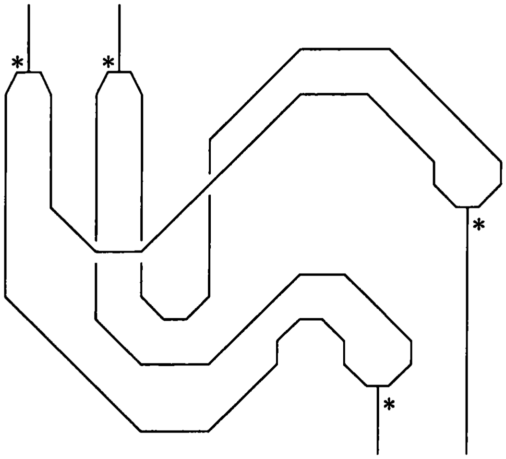
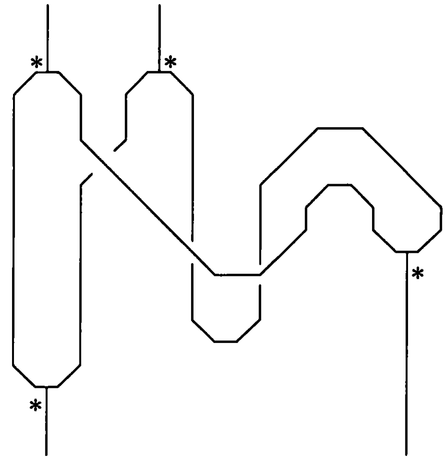
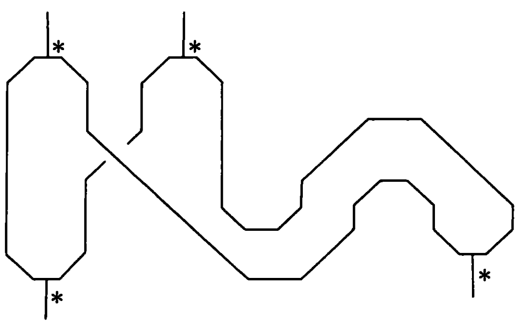
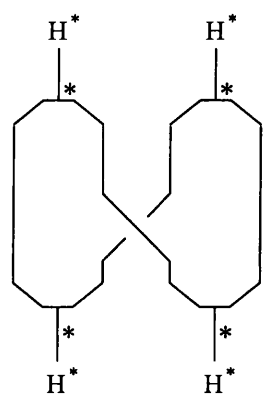
=  = 

=  = 

1.3 The coset construction for braided categories

Although all the results in this section are in [4], they are included for the sake of completeness and as they are the base for many parts of the thesis. The definitions and the main results only are included and those who are interested in the details and the proofs can see the original paper [4] by E. Beggs. In this section, frequently the same phrasing is used as in [4].

Definition 1.3.1 *[4] For a group X and a subgroup G, we call $M \subset X$ a set of left* *coset representatives if for every $x \in X$ there is a unique $s \in M$ so that $x \in Gs$. The decomposition $x = us$ for $u \in G$ and $s \in M$ is called the unique factorization of x.*

In what follows, $M \subset X$ is assumed to be a set of left coset representatives for the subgroup $G \subset X$. In addition, the identity in X will be denoted by e.

Definition 1.3.2 *[4] For $s, t \in M$ we define $\tau(s,t) \in G$ and $s \cdot t \in M$ by the unique factorization $st = \tau(s,t)(s \cdot t)$ in X. The functions $\triangleright : M \times G \to G$ and $\triangleleft : M \times G \to M$ are also defined by the unique factorization $su = (s \triangleright u)(s \triangleleft u)$ for $s, s \triangleleft u \in M$ and $u, s \triangleright u \in G$.*

It was shown that the binary operation $(M, \cdot)$ has a unique left identity $e_m \in M$ and also has the right division property (i.e. there is a unique solution $p \in M$ to the equation $p \cdot s = t$ for all $s, t \in M$). If $e \in M$ then $e_m = e$ is also a right identity [4].

The result of the next proposition will be used at many places in the thesis:

Proposition 1.3.3 *[4] For $t, s, p \in M$ and $u, v \in G$, the following identities between $(M, \cdot)$ and τ hold:*

$$s \triangleright (t \triangleright u) = \tau(s,t)\big((s \cdot t) \triangleright u\big)\tau\big(s \triangleleft (t \triangleright u), t \triangleleft u\big)^{-1} \quad \text{and} \quad (s \cdot t) \triangleleft u = \big(s \triangleleft (t \triangleright u)\big) \cdot (t \triangleleft u),$$

$$s \triangleright uv = (s \triangleright u)\big((s \triangleleft u) \triangleright v\big) \quad \text{and} \quad s \triangleleft uv = (s \triangleleft u) \triangleleft v,$$

$$\tau(p,s)\tau(p \cdot s, t) = \big(p \triangleright \tau(s,t)\big)\tau\big(p \triangleleft \tau(s,t), s \cdot t\big) \quad \text{and} \quad \big(p \triangleleft \tau(s,t)\big) \cdot (s \cdot t) = (p \cdot s) \cdot t.$$

In what follows, unless otherwise stated, we assume that $e \in M$ to make things easier.It was proved in [4] that for all $t \in M$ and $v \in G$, the following identities hold:

$$e \triangleleft v = e, \; e \triangleright v = v, \; t \triangleright e = e, \; t \triangleleft e = t.$$

Example 1.3.4 *[4] Let X be the dihedral group* $D_6 = \langle a, b : a^6 = b^2 = e, ab = ba^5 \rangle$, *whose elements we list as* $\{e, a, a^2, a^3, a^4, a^5, b, ba, ba^2, ba^3, ba^4, ba^5\}$, *and let G be the non-abelian normal subgroup of order 6 generated by* a^2 *and* b, *i.e.* $G = \{e, a^2, a^4, b, ba^2, ba^4\}$. *We choose* $M = \{e, a\}$. *The* τ *function is given by* $\tau(a, a) = a^2$, *and all other combinations giving e. The operation* $\triangleleft$ *is trivial, and* $\triangleright$ *is given by a acting on G as the permutation* (b, ba^4, ba^2), *i.e.* $a \triangleright b = ba^4$ *etc. Note that though* $(M, \cdot)$ *is a group,* $\triangleright$ *is not a group action.*

The tensor category $\mathcal{C}$ was defined in [4] as the following: Take a category $\mathcal{C}$ of finite dimensional vector spaces over a field k, whose objects are right representations of the group G and have M-gradings, i.e. an object V can be written as $\bigoplus_{s \in M} V_s$. ξ is said to be a homogeneous element of V if $\xi \in V_s$ for some $s \in M$, with grade $\langle \xi \rangle = s$. In this thesis we assume in our formulae that we have chosen homogeneous elements of the relevant objects, as the general elements are just linear combinations of the homogeneous elements. The action for the representation is written as $\bar{\triangleleft} : V \times G \to V$. In addition it is supposed that the action and the grading satisfy the compatibility condition, i.e. $\langle \xi \bar{\triangleleft} u \rangle = \langle \xi \rangle \triangleleft u$. The morphisms in the category $\mathcal{C}$ were defined to be linear maps which preserve both the grading and the action, i.e. for a morphism $\vartheta : V \to W$ we have $\langle \vartheta(\xi) \rangle = \langle \xi \rangle$ and $\vartheta(\xi) \bar{\triangleleft} u = \vartheta(\xi \bar{\triangleleft} u)$ for all $\xi \in V$ and $u \in G$ [4].

Proposition 1.3.5 *[4]* $\mathcal{C}$ *can be made into a tensor category by taking* $V \otimes W$ *to be the usual vector space tensor product, with actions and gradings given by*

$$\langle \xi \otimes \eta \rangle = \langle \xi \rangle \cdot \langle \eta \rangle \quad \text{and} \quad (\xi \otimes \eta) \bar{\triangleleft} u = \xi \bar{\triangleleft} (\langle \eta \rangle \triangleright u) \otimes \eta \bar{\triangleleft} u \,.$$

For morphisms $\theta : V \to \widetilde{V}$ *and* $\vartheta : W \to \widetilde{W}$, *the morphism* $\theta \otimes \vartheta : V \otimes W \to \widetilde{V} \otimes \widetilde{W}$ *is* *defined by* $(\theta\otimes\vartheta)(\xi\otimes\eta)=\theta(\xi)\otimes\vartheta(\eta)$ *for* $\xi\in V$ *and* $\eta\in W$.

For the tensor operation, the identity is just the vector space k with trivial G-action and grade $e\in M$. For any object V the morphisms $l_V:V\to V\otimes k$ and $r_V:V\to k\otimes V$ are given by $l_V(\xi)=\xi\otimes 1$ and $r_V(\xi)=1\otimes\xi$, where 1 is the multiplicative identity in k [4].

Proposition 1.3.6 *[4] There is an associator* $\Phi_{UVW}:(U\otimes V)\otimes W\to U\otimes(V\otimes W)$ *given by*

$$\Phi((\xi\otimes\eta)\otimes\zeta)=\xi\bar{\triangleleft}\tau(\langle\eta\rangle,\langle\zeta\rangle)\otimes(\eta\otimes\zeta)\,.$$

Next the **rigidity** of $\mathcal{C}$ was shown in [4] as the following, supposing that $(M,\cdot)$ has right inverses, i.e. for every $s\in M$ there is an $s^R\in M$ so that $s\cdot s^R=e$: Let $V=\bigoplus_{s\in M}V_s$, where $\xi\in V_s$ corresponds to $\langle\xi\rangle=s$. Now take the dual vector space V^*, and set

$$V^*_{s^L}=\{\alpha\in V^*:\alpha|_{V_t}=0\quad\forall t\neq s\}.$$

Then $V^*=\bigoplus_{s\in M}V^*_{s^L}$, and we define $\langle\alpha\rangle=s^L$ when $\alpha\in V^*_{s^L}$. We define the evaluation map $\mathrm{ev}:V^*\otimes V\to k$ by $\mathrm{ev}(\alpha,\xi)=\alpha(\xi)$. The grading on V^* has been designed so that this map preserves gradings. Considering the action $\bar{\triangleleft}u$, if we apply evaluation to $\alpha\bar{\triangleleft}(\langle\xi\rangle\triangleright u)\otimes\xi\bar{\triangleleft}u$ we should get $\alpha(\xi)\bar{\triangleleft}u=\alpha(\xi)$. To do this we define $\big(\alpha\bar{\triangleleft}(\langle\xi\rangle\triangleright u)\big)(\xi\bar{\triangleleft}u)=\alpha(\xi)$, or if we put $\eta=\xi\bar{\triangleleft}u$ we get

$$\big(\alpha\bar{\triangleleft}\big((\langle\eta\rangle\triangleleft u^{-1})\triangleright u\big)\big)=\alpha(\eta\bar{\triangleleft}u^{-1})=\big(\alpha\bar{\triangleleft}(\langle\eta\rangle\triangleright u^{-1})^{-1}\big)(\eta)\,.$$

If this is rearranged to give $\alpha\triangleleft v$, we get the following formula:

$$(\alpha\bar{\triangleleft}v)(\eta)=\alpha\big(\eta\bar{\triangleleft}\tau(\langle\eta\rangle^L,\langle\eta\rangle)^{-1}(\langle\eta\rangle^L\triangleright v^{-1})\tau(\langle\eta\rangle^L\triangleleft v^{-1},(\langle\eta\rangle^L\triangleleft v^{-1})^R)\big).\qquad(1.2)$$

For the coevaluation map to be defined, a basis $\{\xi\}$ of each V_s was taken and a corresponding dual basis $\{\hat{\xi}\}$ of each V^*_{sL}, i.e. $\hat{\eta}(\xi) = \delta_{\xi,\eta}$. Then these bases were put together for all $s \in M$ to get the following definition:

$$\mathrm{coev}(1) = \sum_{\xi \in basis} \xi \bar{\triangleleft} \tau(\langle\xi\rangle^L, \langle\xi\rangle)^{-1} \otimes \hat{\xi}\,,$$

which was proved to be a morphism in $\mathcal{C}$ [4].

The algebra A in the tensor category $\mathcal{C}$ was constructed in [4] so that the group action and the grading in the definition of $\mathcal{C}$ can be combined as the following: Consider a single object A, a vector space spanned by a basis $\delta_s \otimes u$ for $s \in M$ and $u \in G$. For any object V in $\mathcal{C}$ define a map $\bar{\triangleleft} : V \otimes A \to V$ by

$$\xi \bar{\triangleleft} (\delta_s \otimes u) = \delta_{s,\langle\xi\rangle} \xi \bar{\triangleleft} u\,. \tag{1.3}$$

This map was shown to be a morphism in $\mathcal{C}$ only if $\langle\xi\rangle \cdot \langle\delta_s \otimes u\rangle = \langle\xi \bar{\triangleleft} u\rangle$ i.e. $s \cdot \langle\delta_s \otimes u\rangle = s \triangleleft u$ if $\langle\xi\rangle = s$. If we put $a = \langle\delta_s \otimes u\rangle$, the action of $v \in G$ is given by

$$(\delta_s \otimes u) \bar{\triangleleft} v = \delta_{s \triangleleft (a \triangleright v)} \otimes (a \triangleright v)^{-1} uv\,. \tag{1.4}$$

It was also proved in [4] that the action and the grading on A are consistent as well as that the action $\bar{\triangleleft} : V \otimes A \to V$ is a morphism in the category for any object V in $\mathcal{C}$.

Proposition 1.3.7 *[4] The formula for the product μ for A in $\mathcal{C}$ consistent with the action above, where $a = \langle\delta_s \otimes u\rangle$ and $b = \langle\delta_t \otimes v\rangle$, is given by*

$$(\delta_s \otimes u)(\delta_t \otimes v) = \delta_{t, s \triangleleft u} \delta_{s \triangleleft \tau(a,b)} \otimes \tau(a,b)^{-1} uv\,.$$

Proposition 1.3.8 *[4] Multiplication $\mu : A \otimes A \to A$ is a morphism and associative in $\mathcal{C}$. Also there is an identity I for the multiplication and an algebra map $\epsilon : A \to k$ in the category given by*

$$I=\sum \delta_t\otimes e\,,\quad \epsilon(\delta_s\otimes u)=\delta_{s,e}\,.$$

The identity I has the trivial action on all objects in $\mathcal{C}$. Also the action of $h\in A$ on the object k is just multiplication by $\epsilon(h)$, and $\epsilon(I)=1$.

Next **the braided tensor category** $\mathcal{D}$ was defined in [4] as the following: A category $\mathcal{D}$ is obtained from the category $\mathcal{C}$ by considering additional structures of a function $\bar{\triangleright}: M\otimes V\to V$ and a G-grading $|\xi|\in G$ for ξ in every object V in $\mathcal{D}$. The following connections between the gradings and actions are required:

$$\begin{gathered}|\eta\bar{\triangleleft}u|=(\langle\eta\rangle\triangleright u)^{-1}|\eta|u\ ,\ \ s\cdot\langle\eta\rangle=\langle s\bar{\triangleright}\eta\rangle\cdot(s\triangleleft|\eta|)\ ,\\ \tau(s,\langle\eta\rangle)^{-1}(s\triangleright|\eta|)=\tau(\langle s\bar{\triangleright}\eta\rangle,s\triangleleft|\eta|)^{-1}|s\bar{\triangleright}\eta|\,.\end{gathered}\tag{1.5}$$

The operation $\bar{\triangleright}$ is an action of M, which is defined to mean that $t\bar{\triangleright}: V\to V$ is linear for all objects V in $\mathcal{D}$ and all $t\in M$, and also that

$$p\bar{\triangleright}(t\bar{\triangleright}\kappa)=(p^{'}\cdot t\bar{\triangleright}\kappa)\bar{\triangleleft}\tau\big(p^{'}\triangleleft(t\triangleright|\kappa|),t\triangleleft|\kappa|\big)^{-1},\tag{1.6}$$

for any $\kappa\in V$, where $p^{'}=p\triangleleft\tau(\langle t\bar{\triangleright}\kappa\rangle,t\triangleleft|\kappa|)\tau(t,\langle\kappa\rangle)^{-1}$. A cross relation between the two actions is also required,

$$(s\bar{\triangleright}\eta)\bar{\triangleleft}\big((s\triangleleft|\eta|)\triangleright u\big)=\big(s\triangleleft(\langle\eta\rangle\triangleright u)\big)\bar{\triangleright}(\eta\bar{\triangleleft}u)\,.\tag{1.7}$$

Note that the morphisms in the category $\mathcal{D}$ are linear maps preserving both gradings and both actions. From the conditions above, it was shown that the connections between the gradings and the actions can be given by the following factorizations in X [4]:

$$|s\bar{\triangleright}\eta|^{-1}\langle s\bar{\triangleright}\eta\rangle=(s\triangleleft|\eta|)|\eta|^{-1}\langle\eta\rangle(s\triangleleft|\eta|)^{-1}\ \ ,\ \ |\eta\bar{\triangleleft}u|^{-1}\langle\eta\bar{\triangleleft}u\rangle=u^{-1}|\eta|^{-1}\langle\eta\rangle u\,.\tag{1.8}$$

To make $\mathcal{D}$ into a tensor category, the G-grading and the M-action on the tensor products were given as the following:

$$|\xi \otimes \eta| = \tau(\langle\xi\rangle, \langle\eta\rangle)^{-1}|\xi||\eta|\,,$$
$$\big(s\triangleleft\tau(\langle\xi\rangle, \langle\eta\rangle)\big)\bar{\triangleright}(\xi\otimes\eta) = (s\bar{\triangleright}\xi)\bar{\triangleleft}\tau(s\triangleleft|\xi|, \langle\eta\rangle)\tau\big(\langle(s\triangleleft|\xi|)\bar{\triangleright}\eta\rangle, s\triangleleft|\xi||\eta|\big)^{-1}\otimes(s\triangleleft|\xi|)\bar{\triangleright}\eta\,. \tag{1.9}$$

Proposition 1.3.9 *[4] The gradings on the tensor product $V \otimes W$ of objects V and W in $\mathcal{D}$ are given by the following factorization in X: $|\xi\otimes\eta|^{-1}\langle\xi\otimes\eta\rangle = |\eta|^{-1}|\xi|^{-1}\langle\xi\rangle\langle\eta\rangle$.*

Proof.

$|\xi\otimes\eta|^{-1}\langle\xi\otimes\eta\rangle = |\xi\otimes\eta|^{-1}(\langle\xi\rangle\cdot\langle\eta\rangle) = |\xi\otimes\eta|^{-1}\tau(\langle\xi\rangle,\langle\eta\rangle)^{-1}\langle\xi\rangle\langle\eta\rangle = |\eta|^{-1}|\xi|^{-1}\langle\xi\rangle\langle\eta\rangle$. □

It was shown that these gradings are consistent with the actions as specified in (1.8), and the function $\bar{\triangleright}$ applied to $V\otimes W$ satisfies the condition (1.3) to be an M-action. In addition, the functions $\bar{\triangleright}$ and $\bar{\triangleleft}$ satisfy the cross relation (1.7) on $V\otimes W$ [4].

Theorem 1.3.10 *[4] $\mathcal{D}$ is a braided tensor category when the following structures are given:*

The identity object is k, with trivial gradings and actions.

The associator Φ and the maps l and r are defined as for $\mathcal{C}$.

The braiding $\Psi : V\otimes W \to W\otimes V$ is defined by $\Psi(\xi\otimes\eta) = \langle\xi\rangle\bar{\triangleright}\eta\otimes\xi\bar{\triangleleft}|\eta|$.

A **double construction** was defined as the following:

Definition 1.3.11 *[4] Give the set Y, which is identical to the group X, a binary operation $\circ$ defined by*

$$(us)\circ(vt) = vust = vu\,\tau(s,t)(s\cdot t) \quad \textit{for} \quad u,v\in G \ \textit{and} \quad s,t\in M\,.$$

The functions $\tilde{\triangleleft} : Y \times X \to Y$ *and* $\tilde{\tau} : Y \times Y \to X$ *are defined by* $y \tilde{\triangleleft} x = x^{-1} y x$ *and* $\tilde{\tau}(vt, wp) = \tau(t,p)$. *Also we define the function* $\tilde{\triangleright} : Y \times X \to X$ *by*

$$vt \tilde{\triangleright} wp = v^{-1} wpv' = twpt'^{-1}\,, \text{ where } vt \tilde{\triangleleft} wp = v't'\,,\ v', w \in G \text{ and } t', p \in M\,.$$

It was proved in [4] that the maps $\tilde{\triangleleft}$, $\tilde{\triangleright}$ and $\tilde{\tau}$ satisfy all the conditions listed in (1.3.3) by giving $(Y, \circ)$ the place of $(M, \cdot)$, and giving the group X the place of G. Moreover, it was proved that the element $e_y = f_m{}^{-1} e_m = e$ is a left identity for Y, which is not in general a right identity, and that the operation $(Y, \circ)$ has the right division property. It was also shown that the corresponding left inverse is given by the formula $(vt)^L = v^{-1}t^{-1}$, for $v \in G$ and $t \in M$.

Returning to the case where $e \in M$, a Y valued grading on the objects of $\mathcal{D}$ was introduced by $\|\xi\| = |\xi|^{-1}\langle\xi\rangle$. Using previous results, it was shown that $\|\eta \bar{\triangleleft} u\| = \|\eta\| \tilde{\triangleleft} u$, $\|s \bar{\triangleright} \eta\| = \|\eta\| \tilde{\triangleleft} (s \triangleleft |\eta|)^{-1}$ and $\|\xi \otimes \eta\| = \|\xi\| \circ \|\eta\|$.

Proposition 1.3.12 *[4] The map* $\hat{\triangleleft} : V \times X \to V$ *defined by* $\xi \hat{\triangleleft} us = (\xi \bar{\triangleleft} u) \hat{\triangleleft} s$ *for* $u \in G$ *and* $s \in M$*, where*

$$\xi \hat{\triangleleft} s = \left((s^L \triangleleft |\xi|^{-1}) \bar{\triangleright} \xi \right) \bar{\triangleleft} \tau(s^L, s)\,,$$

is a right action of the group X *on* V*, for any object* V *in* $\mathcal{D}$*. Moreover* $\|\xi \hat{\triangleleft} us\| = \|\xi\| \tilde{\triangleleft} us$.

Proposition 1.3.13 *[4] The* X*-action on tensor products in the category* $\mathcal{D}$ *is given by*

$$(\xi \otimes \eta) \hat{\triangleleft} x = \left(\xi \hat{\triangleleft} (\|\eta\| \tilde{\triangleright} x) \right) \otimes \eta \hat{\triangleleft} x\,.$$

Proposition 1.3.14 *[4] The braiding* Ψ *, in terms of the* X*-action, is given by*

$$\Psi(\xi \otimes \eta) = \eta \hat{\triangleleft} (\langle\xi\rangle \triangleleft |\eta|)^{-1} \otimes \xi \hat{\triangleleft} |\eta| \quad , \quad \Psi^{-1}(\xi' \otimes \eta') = \eta' \hat{\triangleleft} |\xi' \hat{\triangleleft} \langle\eta'\rangle|^{-1} \otimes \xi' \hat{\triangleleft} \langle\eta'\rangle\,.$$

Next the **Hopf algebra D** in the braided category $\mathcal{D}$ was defined in [4] as the following: Introduce a vector space D with basis $\delta_y \otimes x$ for $y \in Y$ and $x \in X$. Then define

$$\xi \hat{\triangleleft} (\delta_y \otimes x) = \delta_{y,\|\xi\|}\, \xi \hat{\triangleleft} x \,. \tag{1.10}$$

We note that D is an object of $\mathcal{D}$ with grade $y \circ \|\delta_y \otimes x\| = y \tilde{\triangleleft} x$ and action

$$(\delta_y \otimes x) \hat{\triangleleft} z = \delta_{y \tilde{\triangleleft}(a \tilde{\triangleright} z)} \otimes (a \tilde{\triangleright} z)^{-1} x z \,, \tag{1.11}$$

for $z \in X$, where $a = \|\delta_y \otimes x\|$. Then the associative multiplication μ on D consistent with the action is

$$(\delta_y \otimes x)(\delta_w \otimes z) = \delta_{w, y \tilde{\triangleleft} x}\, \delta_{y \tilde{\triangleleft} \tilde{\tau}(a,b)} \otimes \tilde{\tau}(a,b)^{-1} x z \,, \tag{1.12}$$

where $y, w \in Y$, $x, z \in X$ and $b = \|\delta_w \otimes z\|$. Also μ is a morphism in the category $\mathcal{D}$.

This much was done before in $\mathcal{C}$. The additional ingredient we have in $\mathcal{D}$ is the braiding. The braiding can be used to define a coproduct for D which consequently gives the tensor product structure in $\mathcal{D}$.

In the following figure we give, in order, the symbols we shall use for the braiding Ψ, the action $\hat{\triangleleft} : V \otimes D \to V$, the counit, the unit, the product and the coproduct:

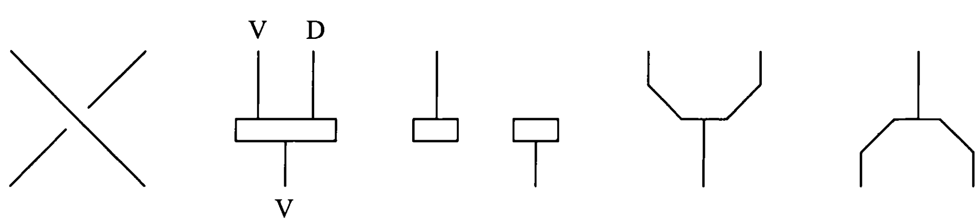


figure 1.7

Definition 1.3.15 *[4] The product μ and the coproduct Δ on D in the category $\mathcal{D}$ are defined by the following diagrams respectively:*

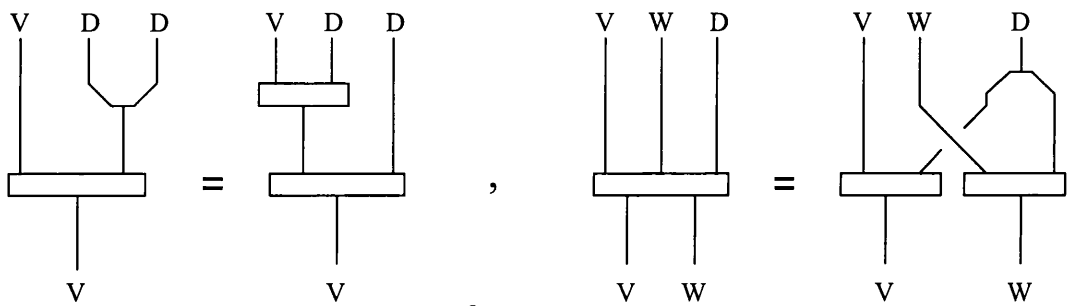


figure 1.8

Using the previous definition it was shown that (D, μ, Δ) is a bialgebra [4]. First, it was proved that Δ is multiplicative i.e.

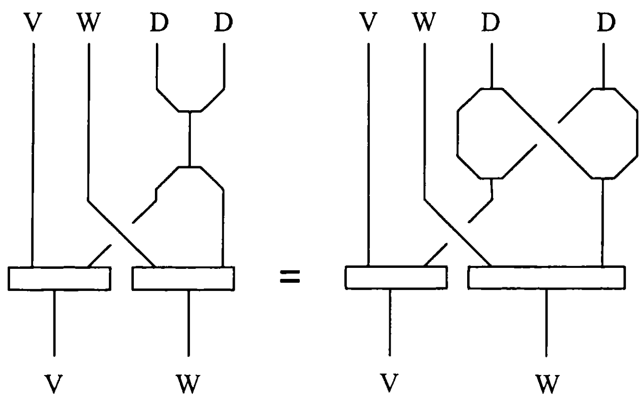


figure 1.9

Next, it was shown that Δ is coassociative by proving that the following equivalence is true:

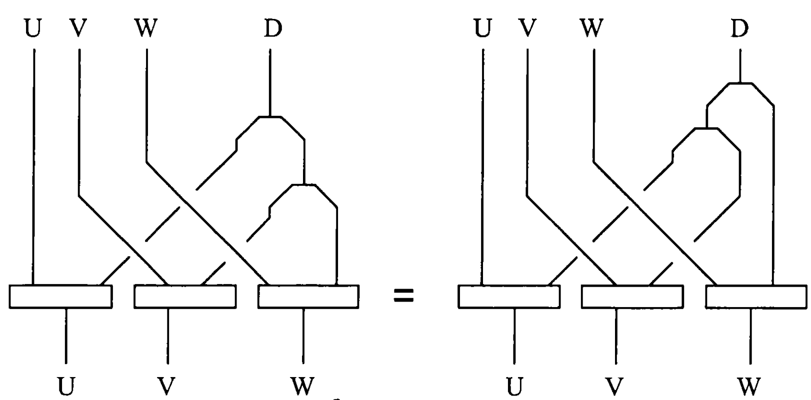


figure 1.10

Finally, a rigid braided category was defined noting that $(Y, \circ)$ has right inverses. The definitions of the dual and the corresponding evaluation and coevaluation maps, considered previously for $\mathcal{C}$, can be also used in $\mathcal{D}$. Recall that the morphisms in $\mathcal{D}$ are required to preserve the actions and the gradings. The following diagrams represent, in order, the evaluation, the coevaluation and the morphism $T : V \to W$ in the category $\mathcal{D}$:

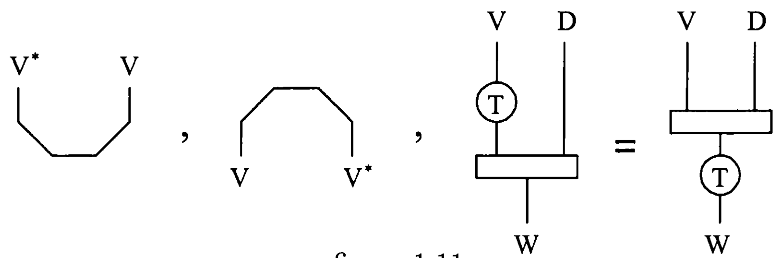


figure 1.11

The last thing was needed to show that D is a braided Hopf algebra in the category $\mathcal{D}$, is the following definition for the antipode:

Definition 1.3.16 *[4] The antipode $S : D \to D$ in $\mathcal{D}$ is defined by the following diagram:*

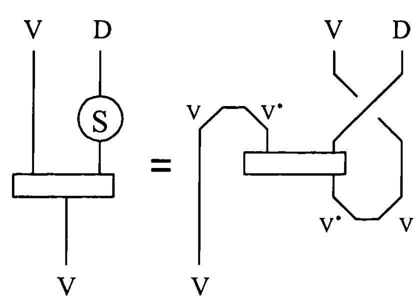


figure 1.12

1.4 Representations and characters of finite groups

In this section we include some important definitions, theorems and results for the representations and characters of the classical finite groups. We will study many of these in the case of Hopf algebras.

Definition 1.4.1 *[23] Let V be a vector space over a field k and let $GL(V)$ be the group of isomorphisms of V onto itself. Suppose G is a finite group with identity element 1 and with composition $(s,t) \mapsto st$ for $s,t \in G$.* **A linear representation** *of G in V is a homomorphism $\rho : G \longrightarrow GL(V)$ defined by $\rho(s) = \rho_s$, i.e. it associates with each element $s \in G$ an element ρ_s of $GL(V)$ in such a way the following equality holds $\rho_{st} = \rho_s \, \rho_t$ for $s,t \in G$.*

If such ρ is given, we say that V is a representation space of G, or simply, a representation of G. In what follows, we restrict ourselves to the case where V has finite dimension. Suppose now we have a finite dimensional vector space V, and let n be its dimension. Then n will be also the degree of the representation under consideration.

Example 1.4.2 *Let G be a finite group with identity element 1. Suppose that G acts on a finite set X, i.e. for each $s \in G$, there is given a permutation $x \mapsto sx$ of X satisfying $1x = x$, $s(tx) = (st)x$ for each $s,t \in G$ and $x \in X$. Let V be a vector space having a basis $(e_x)_{x \in X}$. Let ρ_s be the linear map of V into V which sends e_x to e_{sx} for $s \in G$. Then the linear representation of G, ρ, obtained here is called the* **permutation representation** *associated with X [23].*

Definition 1.4.3 *[23] Let $\rho : G \rightarrow GL(V)$ be a linear representation of G in V. and let W be a vector subspace of V. W is said to be* **a subrepresentation** *of V, if W is stable under the action of G, i.e. if $x \in W$, then $\rho_s(x) \in W$ for all $s \in G$. Thus $\rho|_W : G \rightarrow GL(W)$ is a linear representation of G in W.*

It is proved in [23] that if V is a representation of G and W is a subrepresentation of V, then there is a complement W_o of W which is also a subrepresentation of V.

Definition 1.4.4 *[23] A linear representation $\rho : G \rightarrow GL(V)$ is said to be irreducible (or simple) if V is not 0 and if no vector subspace of V is stable under G, except of course 0 and V itself.*

Note that the second condition is equivalent to saying that V is not a direct sum of two representations, except for the trivial decomposition $V = 0 \oplus V$. Any representation of degree 1 is irreducible. The irreducible representations are used to construct the others by means of direct sum as we can see in the following theorem:

Theorem 1.4.5 *Every representation can be written as a direct sum of irreducible representations.*

This theorem can be proved using the mathematical induction on dim(V) and taking in account what was mentioned after definition 1.4.3 [23].

Now we mention some important definitions and results about the characters of the representations. Let V be a vector space over the feild k. If V has a finite basis (e_i) of n elements, then each linear map $f : V \rightarrow V$ of $GL(V)$, can be defined by a square matrix (f_{ij}) of order n. The coefficients f_{ij} belong to the field k, and they are obtained by the formula $f(e_j) = \sum_i f_{ij}\, e_i$. **The trace of f** is defined by the scalar given by the following formula:

$$\mathrm{Tr(f)} = \sum_{\mathrm{i}} \mathrm{f_{ii}}\,.$$

Definition 1.4.6 *[23] Let $\rho : G \longrightarrow GL(V)$ be a linear representation of G in V and let $s \in G$. Then the character of the representation ρ is given by the following formula:*

$$\chi_\rho(s) = \mathrm{Tr}(\rho_{\mathrm{s}})\,.$$

Proposition 1.4.7 *[12] Let G be finite group with identity element 1, $\rho : G \longrightarrow GL(V)$ be a linear representation of G in V, a vector space over the field $\mathbb{C}$, of order n, and let χ be the character of ρ. Then, for $s, t \in G$, the following equalities holds*

$$\chi(1) = \mathrm{dim(V)} = \mathrm{n} \quad , \quad \chi(\mathrm{tst}^{-1}) = \chi(\mathrm{s}).$$

Proposition 1.4.8 *[23] Let V_1 and V_2 be two representations of the finite group G, and let χ_1 and χ_2 be their characters. Then the character of their direct sum representation, $V_1 \oplus V_2$, is the sum of their characters, $\chi_1 + \chi_2$; and the character of their tensor product, $V_1 \otimes V_2$, is the product of their characters, $\chi_1\, \chi_2$.*

Proposition 1.4.9 **(Schur's Lemma)** *[23] Let $\rho' : G \longrightarrow GL(V_1)$ and $\rho'' : G \longrightarrow GL(V_2)$ be two irreducible representations of G and let $f : V_1 \longrightarrow V_2$ be a linear map satisfying $\rho_s{}'' \circ f = f \circ \rho_s{}'$ for all $s \in G$. Then we have the following:*

1) If ρ' and ρ'' are not isomorphic (i.e. $f : V_1 \longrightarrow V_2$ is a 1-1 correspondence), then $f = 0$.

2) If $V_1 = V_2$ and $\rho' = \rho''$, then f is a scalar multiple of the identity endomorphism Id_{V_1} .

Chapter 2

Further results on the category $\mathcal{D}$

In this chapter we consider some basic facts about the braided category $\mathcal{D}$ which are useful but do not fit in any particular later chapter. We begin with an example of the braided category $\mathcal{D}$ which will be used later. Then the category $\mathcal{D}$ is shown to be a ribbon category. The question of whether the braided Hopf algebra D is braided commutative or cocommutative in the category $\mathcal{D}$ is considered. Finally we look at the construction of integrals in the category $\mathcal{D}$.

2.1 An example of the category $\mathcal{D}$

Take X to be the dihedral group $D_6 = \langle a, b : a^6 = b^2 = e, ab = ba^5 \rangle$, whose elements we list as $\{e, a, a^2, a^3, a^4, a^5, b, ba, ba^2, ba^3, ba^4, ba^5\}$, and G to be the non-abelian normal subgroup of order 6 generated by a^2 and b, i.e. $G = \{e, a^2, a^4, b, ba^2, ba^4\}$. We choose $M = \{e, a\}$. The center of D_6 is the subgroup $\{e, a^3\}$, and it has the following conjugacy classes: $\{e\}$, $\{a^3\}$, $\{a^2, a^4\}$, $\{a, a^5\}$, $\{b, ba^2, ba^4\}$ and $\{ba, ba^3, ba^5\}$.

The category $\mathcal{D}$ consists of right representations of the group $X = D_6$ which are graded by $Y = D_6$ (as a set), using the actions $\tilde{\triangleleft} : Y \times X \longrightarrow Y$ and $\tilde{\triangleright} : Y \times X \longrightarrow X$ which are defined as follows:

$$y \tilde{\triangleleft} x = x^{-1} y x, \quad \text{and} \quad vt \tilde{\triangleright} x = v^{-1} x v' = t x t'^{-1},$$

for $x \in X$, $y \in Y$, $v, v' \in G$ and $t, t' \in M$ where $vt \tilde{\triangleleft} x = v' t'$.

2.2 The ribbon map on the category $\mathcal{D}$

In this section we show that the rigid braided category $\mathcal{D}$ is a ribbon category. This material has been included in [2]. A ribbon category itself is defined as the following:

Definition 2.2.1 *A ribbon category is a rigid braided category equipped with a transformation $\theta \in$Nat(id, id) satisfying the following conditions:*

$$\theta_{V \otimes W} = \Psi^{-1}_{V \otimes W} \circ \Psi^{-1}_{W \otimes V} \circ (\theta_V \otimes \theta_W), \quad \theta_{\underline{1}} = \mathrm{id}, \quad (\theta_{\mathrm{V}})^* = \theta_{\mathrm{V}^*},$$

for any objects V and W in the category. If such a θ exists, then it is called a ribbon transformation (see [19]).

As a simple example, the category of finite dimensional vector spaces over the field k is a ribbon category with trivial ribbon transformation $\theta_V = \mathrm{id}_{\mathrm{V}}$ (see [13]).

Theorem 2.2.2 *The ribbon transformation $\theta_V : V \longrightarrow V$ for any object V in $\mathcal{D}$ can be defined by $\theta_V(\xi) = \xi \hat{\triangleleft} \|\xi\|$.*

Proof. In the following lemmas we show that the required properties hold. □

Lemma 2.2.3 *θ_V is a morphism in the category.*

Proof. Begin by checking the X-grade, for $\xi \in V$

$$\|\theta_V(\xi)\| = \big\|\xi \,\hat{\triangleleft}\, \|\xi\|\big\| = \|\xi\| \tilde{\triangleleft} \|\xi\| = \|\xi\|.$$

Now we check the X-action, i.e. that $\theta_V(\xi \hat{\triangleleft} x) = \theta_V(\xi) \hat{\triangleleft} x$.

$$\theta_V(\xi \hat{\triangleleft} x) = (\xi \hat{\triangleleft} x) \hat{\triangleleft} \|\xi \hat{\triangleleft} x\| = (\xi \hat{\triangleleft} x) \hat{\triangleleft} (\|\xi\| \tilde{\triangleleft} x)$$

$$= \xi \hat{\triangleleft} x x^{-1} \|\xi\| x = (\xi \hat{\triangleleft} \|\xi\|) \hat{\triangleleft} x = \theta_V(\xi) \hat{\triangleleft} x. \qquad \square$$

Lemma 2.2.4 *For any two objects V and W in $\mathcal{D}$,*

$$\theta_{V \otimes W} = \Psi^{-1}_{V \otimes W} \circ \Psi^{-1}_{W \otimes V} \circ (\theta_V \otimes \theta_W) = (\theta_V \otimes \theta_W) \circ \Psi^{-1}_{V \otimes W} \circ \Psi^{-1}_{W \otimes V}$$

This can also be described by figure 2.1:

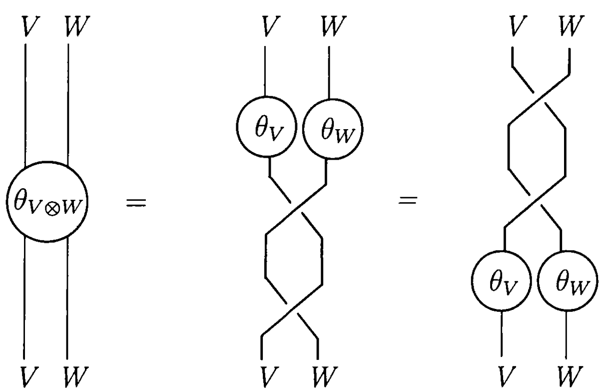


figure 2.1

Proof. First calculate $\Psi\left(\Psi(\xi \otimes \eta)\right)$ for $\xi \in V$ and $\eta \in W$, beginning with

$$\Psi\left(\Psi(\xi \otimes \eta)\right) = \Psi\left(\eta \,\hat{\triangleleft} (\langle \xi \rangle \triangleleft |\eta|)^{-1} \otimes \xi \hat{\triangleleft} |\eta|\,\right). \tag{2.1}$$

To simplify what follows we shall use the substitutions

$$\eta' = \xi \hat{\triangleleft} |\eta| \quad \text{and} \quad \xi' = \eta \hat{\triangleleft} (\langle \xi \rangle \triangleleft |\eta|)^{-1}, \tag{2.2}$$

so equation (2.1) can be rewritten as

$$\begin{aligned} \Psi\big(\Psi(\xi \otimes \eta)\big) &= \Psi(\xi' \otimes \eta') \\ &= \eta' \hat{\triangleleft} (\langle \xi' \rangle \triangleleft |\eta'|)^{-1} \otimes \xi' \hat{\triangleleft} |\eta'|. \end{aligned} \tag{2.3}$$

As $\eta' = \xi \hat{\triangleleft} |\eta| = \xi \bar{\triangleleft} |\eta|$, then $|\eta'| = \big| \xi \bar{\triangleleft} |\eta| \big| = (\langle \xi \rangle \triangleright |\eta|)^{-1} |\xi| |\eta|$, so

$$\begin{aligned} \xi' \hat{\triangleleft} |\eta'| &= \eta \hat{\triangleleft} (\langle \xi \rangle \triangleleft |\eta|)^{-1} (\langle \xi \rangle \triangleright |\eta|)^{-1} |\xi| |\eta| \\ &= \eta \hat{\triangleleft} \big((\langle \xi \rangle \triangleright |\eta|)(\langle \xi \rangle \triangleleft |\eta|) \big)^{-1} |\xi| |\eta| \\ &= \eta \hat{\triangleleft} |\eta|^{-1} \langle \xi \rangle^{-1} |\xi| |\eta|. \end{aligned} \tag{2.4}$$

Hence if we put $y = \|\xi \otimes \eta\| = \|\xi\| \circ \|\eta\| = |\eta|^{-1} |\xi|^{-1} \langle \xi \rangle \langle \eta \rangle$,

$$\Psi\big(\Psi(\xi \otimes \eta)\big) \hat{\triangleleft} \| \xi \otimes \eta \| = \xi \hat{\triangleleft} |\eta| (\langle \xi' \rangle \triangleleft |\eta'|)^{-1} (p \tilde{\triangleright} \|\xi \otimes \eta\|) \otimes \eta \hat{\triangleleft} |\eta|^{-1} \langle \eta \rangle, \tag{2.5}$$

where, using (2.4),

$$p = \|\xi' \hat{\triangleleft} |\eta'| \| = \big|\xi' \bar{\triangleleft} |\eta'| \big|^{-1} \langle \xi' \bar{\triangleleft} |\eta'| \rangle = \|\eta\| \tilde{\triangleleft} \|\eta\| \, y^{-1} = \|\eta\| \tilde{\triangleleft} y^{-1}$$

$$p \tilde{\triangleright} \|\xi \otimes \eta\| = (\|\eta\| \tilde{\triangleleft} y^{-1}) \tilde{\triangleright} y = (\|\eta\| \tilde{\triangleright} y^{-1})^{-1},$$

As $\big\|\xi' \bar{\triangleleft} |\eta'| \big\| = v' t' = \|\eta\| \tilde{\triangleleft} y^{-1}$, by unique factorization, $t' = \langle \xi' \rangle \triangleleft |\eta'|$. Then $\|\eta\| \tilde{\triangleright} y^{-1} = \langle \eta \rangle \, y^{-1} t'^{-1}$, which implies that

$$|\eta| (\langle \xi' \rangle \triangleleft |\eta'|)^{-1} (\|\eta\| \tilde{\triangleright} y^{-1})^{-1} = |\eta| t'^{-1} t' y \langle \eta \rangle^{-1} = \|\xi\| \,. \tag{2.6}$$

Substituting this in (2.5) gives

$$\Psi\big(\Psi(\xi \otimes \eta)\big) \hat{\triangleleft} \| \xi \otimes \eta \| = \xi \hat{\triangleleft} \|\xi\| \otimes \eta \hat{\triangleleft} \|\eta\|. \qquad \square$$

Lemma 2.2.5 *For the unit object $\underline{1} = \mathbb{C}$ in $\mathcal{D}$, $\theta_{\underline{1}}$ is the identity.*

Proof. For any object V in $\mathcal{D}$, $\theta_V : V \longrightarrow V$ is defined by

$$\theta_V(\xi) = \xi \,\hat{\triangleleft}\, \|\xi\| \qquad \text{for } \xi \in V.$$

If we choose $V = \underline{1} = \mathbb{C}$ then $\theta_{\underline{1}}(\xi) = \xi \,\hat{\triangleleft}\, e = \xi$ as $\|\xi\| = e$. □

Lemma 2.2.6 *For any object V in $\mathcal{D}$, $(\theta_V)^* = \theta_{V^*}$ (see figure 2.2).*

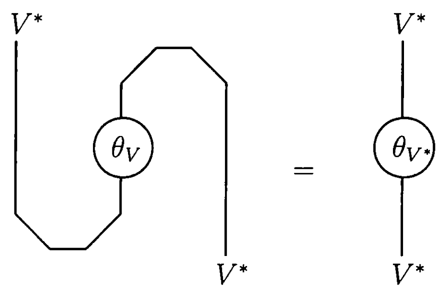


figure 2.2

Proof. Begin with

$$\mathrm{coev}_V(1) = \sum_{\xi \in \text{ basis of } V} \xi \,\hat{\triangleleft}\, \tilde{\tau}(\,\|\xi\|^L, \|\xi\|\,)^{-1} \otimes \hat{\xi} = \sum_{\xi \in \text{ basis of } V} \xi \,\hat{\triangleleft}\, \tau(\,\langle\xi\rangle^L, \langle\xi\rangle\,)^{-1} \otimes \hat{\xi}$$

For $\alpha \in V^*$, we follow figure 2.2 and calculate

$$(\theta_V)^*(\alpha) = (\mathrm{eval}_V \otimes id) \sum_{\xi \in \text{ basis of } V} \Phi^{-1}\Big(\alpha \otimes \Big(\theta_V\big(\xi \hat{\triangleleft} \tau(\langle\xi\rangle^L, \langle\xi\rangle)^{-1}\big) \otimes \hat{\xi}\Big)\Big). \qquad (2.7)$$

Now as $\tau(\langle\xi\rangle^L, \langle\xi\rangle) = \langle\xi\rangle^L\langle\xi\rangle$,

$$\begin{aligned}
\big\|\, \xi \,\hat{\triangleleft}\, \tau(\,\langle\xi\rangle^L, \langle\xi\rangle\,)^{-1} \big\| &= \|\xi\| \,\tilde{\triangleleft}\, (\langle\xi\rangle^L\langle\xi\rangle)^{-1} \\
&= \langle\xi\rangle^L\langle\xi\rangle\, |\xi|^{-1}\, \langle\xi\rangle\langle\xi\rangle^{-1}\langle\xi\rangle^{L-1} \\
&= \langle\xi\rangle^L\langle\xi\rangle\, |\xi|^{-1}\, \langle\xi\rangle^{L-1},
\end{aligned}$$

$$\begin{aligned}
\theta_V\big(\xi \,\hat{\triangleleft}\, \tau(\,\langle\xi\rangle^L, \langle\xi\rangle\,)^{-1}\big) &= \big(\xi \,\hat{\triangleleft}\, \tau(\,\langle\xi\rangle^L, \langle\xi\rangle\,)^{-1}\big) \,\hat{\triangleleft}\, \big\|\, \xi \,\hat{\triangleleft}\, \tilde{\tau}(\,\|\xi\|^L, \|\xi\|\,)^{-1} \big\| \\
&= \xi \,\hat{\triangleleft}\, \langle\xi\rangle^{-1}\langle\xi\rangle^{L-1}\langle\xi\rangle^L\langle\xi\rangle\, |\xi|^{-1}\, \langle\xi\rangle^{L-1} \\
&= \xi \,\hat{\triangleleft}\, |\xi|^{-1}\, \langle\xi\rangle^{L-1}.
\end{aligned}$$

The next step is to find

$$\Phi^{-1}\Big(\alpha\otimes\Big(\big(\xi\hat{\triangleleft}|\xi|^{-1}\langle\xi\rangle^{L-1}\big)\otimes\hat{\xi}\Big)\Big)=\Big(\alpha\,\hat{\triangleleft}\,\tilde{\tau}\big(\|\,\xi\,\hat{\triangleleft}\,|\xi|^{-1}\langle\xi\rangle^{L-1}\|,\|\hat{\xi}\|\,\big)^{-1}$$
$$\otimes\,\big(\xi\,\hat{\triangleleft}\,|\xi|^{-1}\langle\xi\rangle^{L-1}\big)\Big)\otimes\hat{\xi}.$$

As

$$\begin{aligned}\|\,\xi\,\hat{\triangleleft}\,|\xi|^{-1}\langle\xi\rangle^{L-1}\| &= \|\xi\|\,\tilde{\triangleleft}\,|\xi|^{-1}\,\langle\xi\rangle^{L-1}\\ &= \langle\xi\rangle^{L}\,|\xi|\,|\xi|^{-1}\,\langle\xi\rangle\,|\xi|^{-1}\,\langle\xi\rangle^{L-1}\\ &= \tau(\langle\xi\rangle^{L},\langle\xi\rangle)\,|\xi|^{-1}\,\langle\xi\rangle^{L-1}\\ &= \tau(\langle\xi\rangle^{L},\langle\xi\rangle)\,|\xi|^{-1}\,\langle\xi\rangle\,\tau(\langle\xi\rangle^{L},\langle\xi\rangle)^{-1}\\ &= \tau(\langle\xi\rangle^{L},\langle\xi\rangle)\,|\xi|^{-1}\,\big(\,\langle\xi\rangle\triangleright\tau(\langle\xi\rangle^{L},\langle\xi\rangle)^{-1}\,\big)\big(\,\langle\xi\rangle\triangleleft\tau(\langle\xi\rangle^{L},\langle\xi\rangle)^{-1}\,\big)\,,\end{aligned}$$

then as $\|\hat{\xi}\|=\|\xi\|^{L}=|\xi|\tau(\langle\xi\rangle^{L},\langle\xi\rangle)^{-1}\langle\xi\rangle^{L}$,

$$\Phi^{-1}\Big(\alpha\otimes\Big(\big(\xi\hat{\triangleleft}|\xi|^{-1}\langle\xi\rangle^{L-1}\big)\otimes\hat{\xi}\Big)\Big)=\Big(\alpha\,\hat{\triangleleft}\,\tau\,\big(\langle\xi\rangle\triangleleft\tau(\langle\xi\rangle^{L},\langle\xi\rangle)^{-1},\langle\xi\rangle^{L}\big)^{-1}$$
$$\otimes\,\big(\xi\,\hat{\triangleleft}\,|\xi|^{-1}\,\langle\xi\rangle^{L-1}\big)\Big)\otimes\hat{\xi}.$$

Put $v=\tau(\langle\xi\rangle^{L},\langle\xi\rangle)^{-1}=\langle\xi\rangle^{-1}\langle\xi\rangle^{L-1}$ and $w=\tau(\langle\xi\rangle\triangleleft v,\langle\xi\rangle^{L})^{-1}=\big((\langle\xi\rangle\triangleleft v)\langle\xi\rangle^{L}\big)^{-1}$, then substituting in (2.7) gives

$$(\theta_V)^{*}(\alpha)=(\mathrm{eval}_V\otimes\mathrm{id})\sum_{\xi\in\text{ basis of }V}\Big((\alpha\,\hat{\triangleleft}\,w)\otimes\big(\xi\,\hat{\triangleleft}\,|\xi|^{-1}\,\langle\xi\rangle^{L-1}\big)\Big)\otimes\hat{\xi}. \tag{2.8}$$

For a given term in the sum to be non-zero, we require

$$\|\alpha\|=\|\hat{\xi}\|=\|\xi\|^{L}=|\xi|\langle\xi\rangle^{-1}\,, \tag{2.9}$$

and we proceed under this assumption. Now calculate

$$\mathrm{eval}_V\big((\alpha\,\hat{\triangleleft}\,w)\otimes\big(\xi\,\hat{\triangleleft}\,|\xi|^{-1}\,\langle\xi\rangle^{L-1}\big)\big)\;=\;\big(\beta\,\hat{\triangleleft}\,(\,\|\xi\|\,\tilde{\triangleright}\,p)\big)(\xi\,\tilde{\triangleleft}\,p)\;=\;\beta\,(\,\xi\,) \tag{2.10}$$

where $p = |\xi|^{-1}\langle\xi\rangle^{L-1}$ and $\beta = \alpha\,\hat{\triangleleft}\, w\,(\,\|\xi\|\,\tilde{\triangleright}\, p\,)^{-1}$. Next we want to find $\|\xi\|\,\tilde{\triangleright}\, p$. To do this, we first find

$$\begin{aligned}\|\xi\|\,\tilde{\triangleleft}\, p &= \langle\xi\rangle^{L}\,|\xi|\,|\xi|^{-1}\,\langle\xi\rangle\,|\xi|^{-1}\,\langle\xi\rangle^{L-1} \\ &= v^{-1}\,|\xi|^{-1}\,\langle\xi\rangle\, v = v^{-1}\,|\xi|^{-1}\,(\langle\xi\rangle \triangleright v\,)(\langle\xi\rangle \triangleleft v\,),\end{aligned} \tag{2.11}$$

and hence

$$\begin{aligned}\|\xi\|\,\tilde{\triangleright}\, p &= \langle\xi\rangle\, p\,(\langle\xi\rangle \triangleleft v\,)^{-1} \\ &= \langle\xi\rangle\,|\xi|^{-1}\,\langle\xi\rangle\, v\,(\langle\xi\rangle \triangleleft v\,)^{-1} \\ &= \langle\xi\rangle\,|\xi|^{-1}\,(\langle\xi\rangle \triangleright v\,).\end{aligned} \tag{2.12}$$

Thus

$$\begin{aligned}\beta &= \alpha\,\hat{\triangleleft}\, w\,(\langle\xi\rangle \triangleright v\,)^{-1}\,|\xi|\,\langle\xi\rangle^{-1} \\ &= \alpha\,\hat{\triangleleft}\,\langle\xi\rangle^{L-1}\,(\langle\xi\rangle \triangleleft v\,)^{-1}\,(\langle\xi\rangle \triangleright v\,)^{-1}\,|\xi|\,\langle\xi\rangle^{-1} \\ &= \alpha\,\hat{\triangleleft}\,\langle\xi\rangle\, v\,(\langle\xi\rangle\, v\,)^{-1}\,|\xi|\,\langle\xi\rangle^{-1} = \alpha\,\hat{\triangleleft}\,|\xi|\,\langle\xi\rangle^{-1}\,.\end{aligned} \tag{2.13}$$

Now substituting these last equations in (2.8) gives

$$(\theta_V)^*(\alpha) = \sum_{\xi \in \text{basis of } V \text{ with } |\xi|\langle\xi\rangle^{-1} = \|\alpha\,\|} \big(\alpha\,\hat{\triangleleft}\,\|\alpha\,\|\,\big)(\xi)\,.\,\hat{\xi} \tag{2.14}$$

Take a basis $\xi_1\,, \xi_2\,, \ldots, \xi_n$ with $\big(\alpha\,\hat{\triangleleft}\,\|\alpha\,\|\,\big)(\xi_i)$ being 1 if $i = 1$, and 0 otherwise. Then

$$(\theta_V)^*(\alpha) = \hat{\xi}_1 \; + 0 = \alpha\,\hat{\triangleleft}\,\|\alpha\,\| = \theta_{V^*}(\alpha)\ ,$$

where $\hat{\xi}_1\,,\ \hat{\xi}_2\ ,\ \ldots\ ,\hat{\xi}_n$ is the dual basis of V^* defined by $\hat{\xi}_i\,(\,\xi_j\,) = \delta_{i,j}$. □

2.3 The Hopf algebra D is braided cocommutative

We consider a braided Hopf algebra E in a braided category $\mathcal{S}$, in which E has a right action on the objects in $\mathcal{S}$ given by the morphism

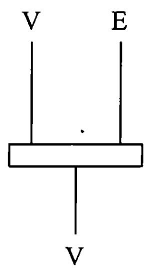


figure 2.3

and the action on tensor product is given by

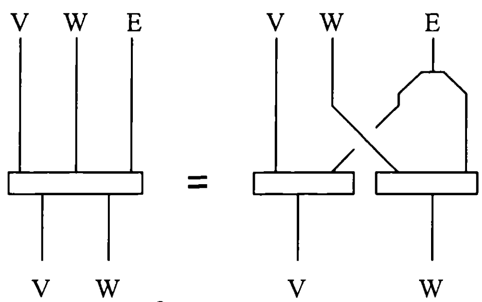


figure 2.4

Definition 2.3.1 *The opposite coproduct, Δ^{op}, for the algebra E in $\mathcal{S}$ can be defined by the following diagram for the representations V and W of E in $\mathcal{S}$:*

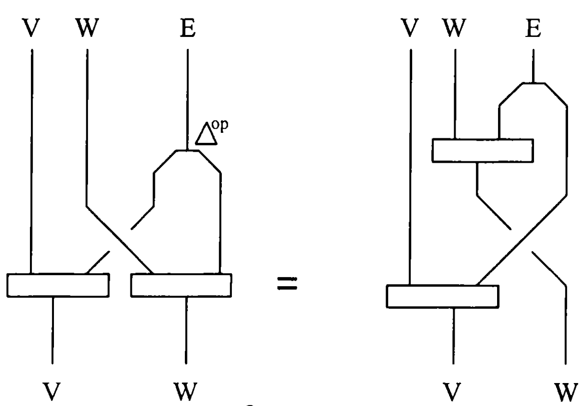


figure 2.5

Lemma 2.3.2 *For the representations V and W of E in $\mathcal{S}$, the opposite coproduct, Δ^{op}, satisfies the following*

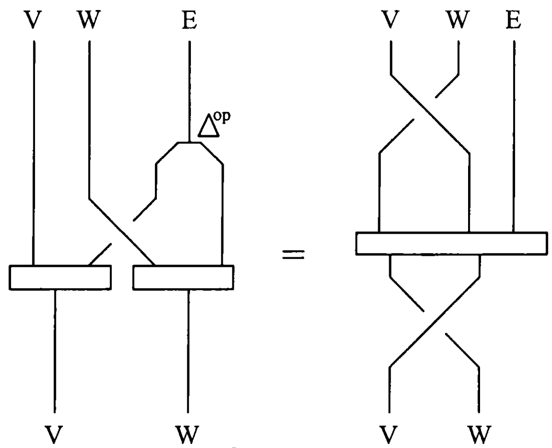
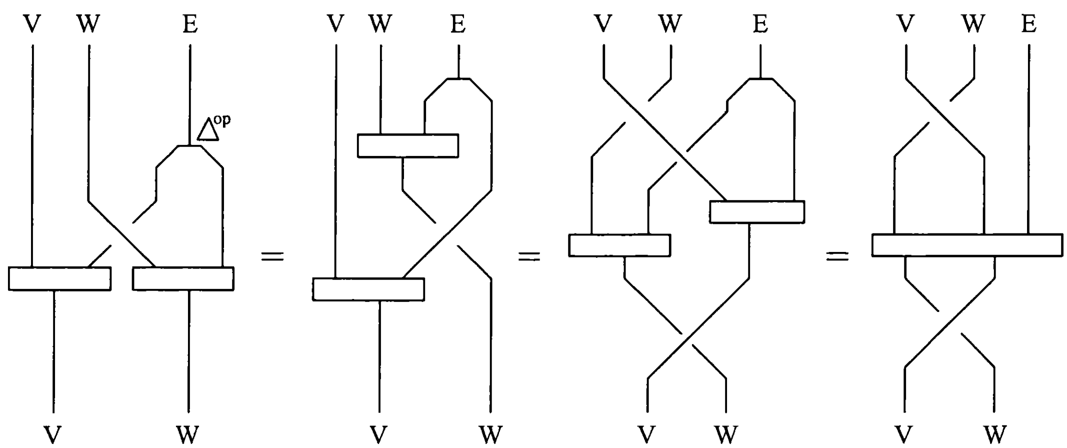


figure 2.6

proof.



Proposition 2.3.3 *For the algebra E the opposite coproduct, Δ^{op}, is coassociative, i.e.*

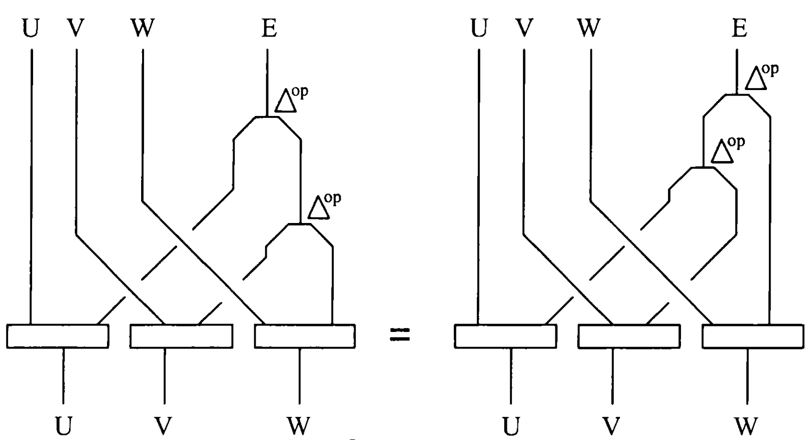
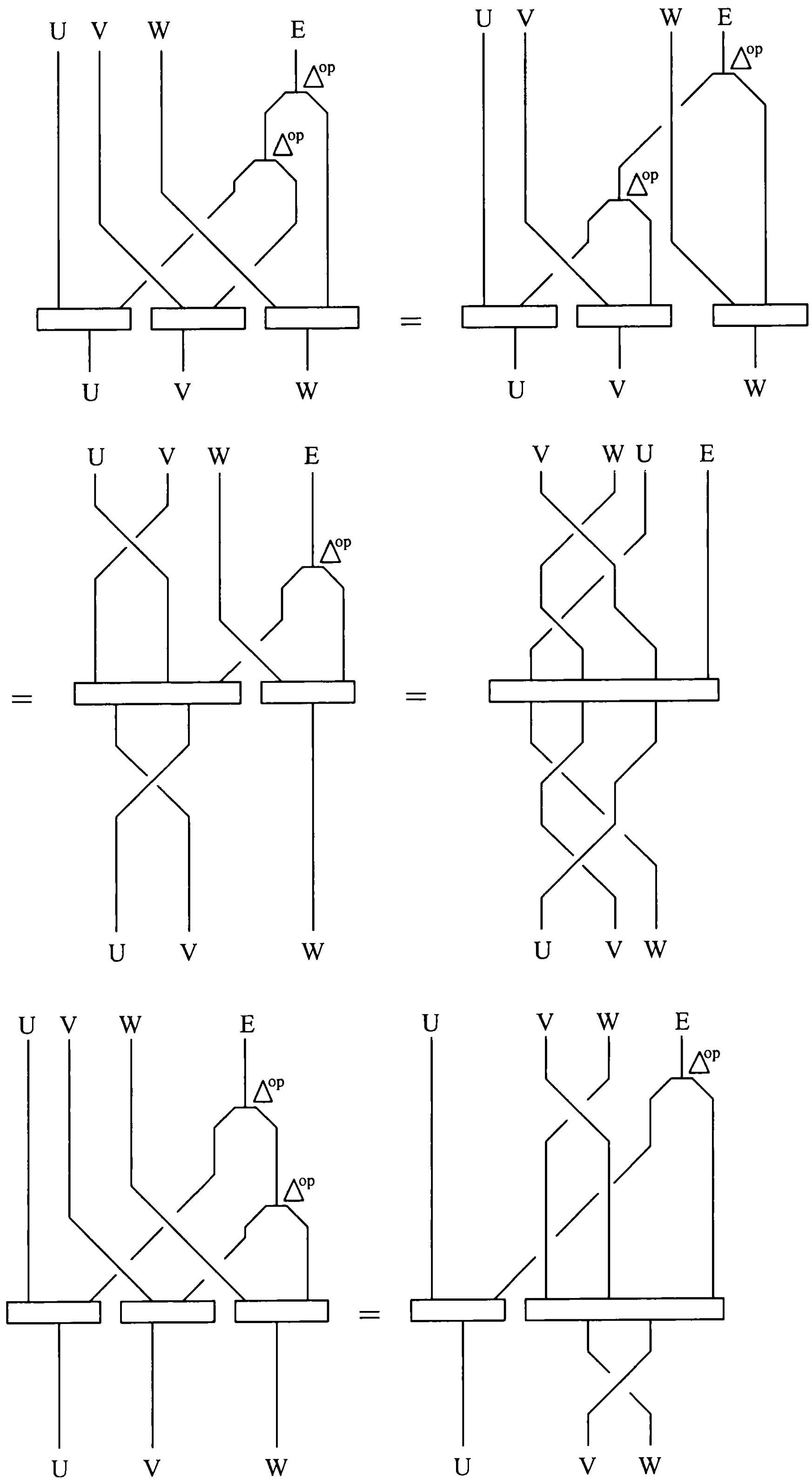
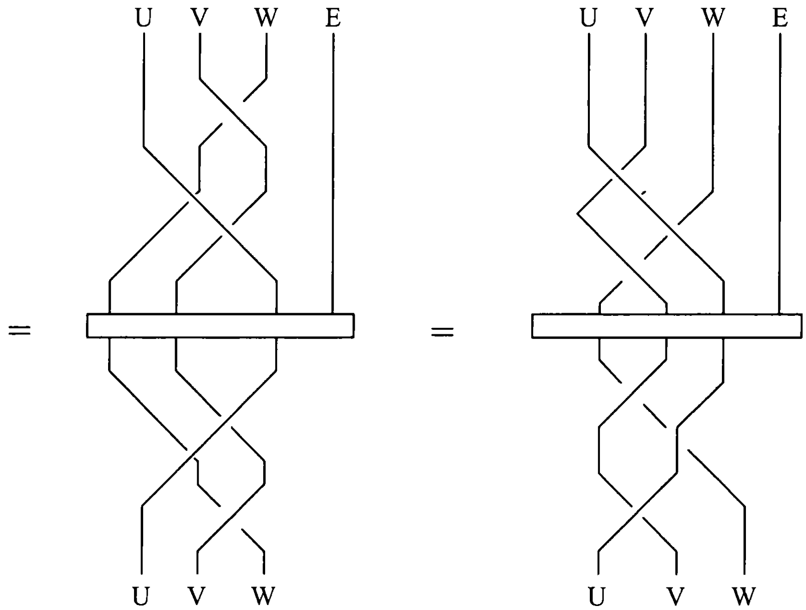


figure 2.7

Proof.

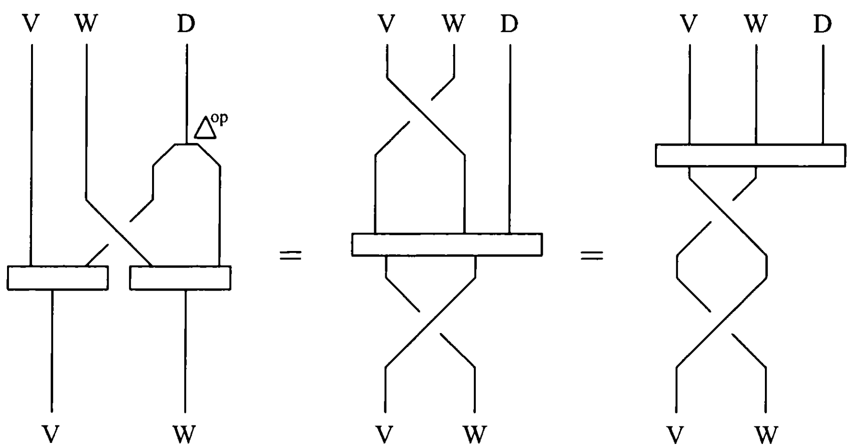


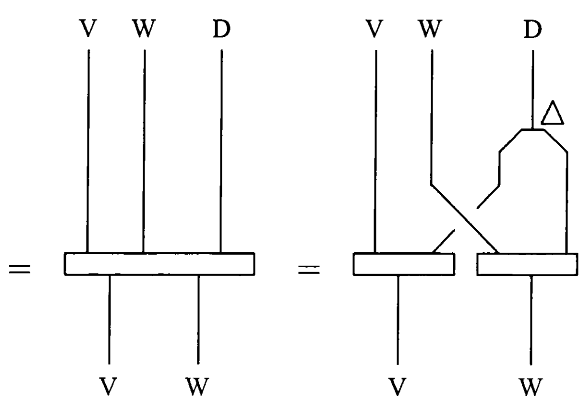


In our case for the category $\mathcal{D}$, we can say more:

Proposition 2.3.4 *Using the definition of the opposite coproduct in 2.3.1, the braided Hopf algebra D in the category $\mathcal{D}$ is cocommutative.*

Proof.





2.4 The Hopf algebra D is not braided commutative

After knowing that the algebra D is braided cocommutative we would like to know whether it is braided commutative or not, i.e. whether for ξ and η in D the following equation is satisfied:

$$\mu(\xi \otimes \eta) = \mu\big(\Psi(\xi \otimes \eta)\big) \ ? \tag{2.15}$$

Put $\xi = \delta_y \otimes x$ and $\eta = \delta_w \otimes z$, then the left hand side of (2.15) becomes

$$(\delta_y \otimes x)(\delta_w \otimes z) = \delta_{w,y\tilde{\triangleleft}x}\, \delta_{y\tilde{\triangleleft}\tilde{\tau}(a,b)} \otimes \tilde{\tau}(a,b)^{-1}xz\,, \tag{2.16}$$

where $a = \|\delta_y \otimes x\| = \|\xi\| = |\xi|^{-1}\langle\xi\rangle$ and $b = \|\delta_w \otimes z\| = \|\eta\| = |\eta|^{-1}\langle\eta\rangle$. On the other hand $\Psi(\xi \otimes \eta) = \eta\hat{\triangleleft}(\langle\xi\rangle\triangleleft|\eta|)^{-1} \otimes \xi\hat{\triangleleft}|\eta| = (\delta_w \otimes z)\hat{\triangleleft}(\langle\xi\rangle\triangleleft|\eta|)^{-1} \otimes (\delta_y \otimes x)\hat{\triangleleft}|\eta|$, so

$$\begin{aligned}
\mu\big(\Psi(\xi \otimes \eta)\big) &= \big(\delta_{w\tilde{\triangleleft}(b\tilde{\triangleright}(\langle\xi\rangle\triangleleft|\eta|)^{-1})} \otimes (b\tilde{\triangleright}(\langle\xi\rangle\triangleleft|\eta|)^{-1})^{-1}z(\langle\xi\rangle\triangleleft|\eta|)^{-1}\big)\ \big(\delta_{y\tilde{\triangleleft}(a\tilde{\triangleright}|\eta|)} \otimes (a\tilde{\triangleright}|\eta|)^{-1}x|\eta|\big) \\
&= \delta_{y\tilde{\triangleleft}(a\tilde{\triangleright}|\eta|)\,,\,w\tilde{\triangleleft}z(\langle\xi\rangle\triangleleft|\eta|)^{-1}}\ \delta_{(w\tilde{\triangleleft}(b\tilde{\triangleright}(\langle\xi\rangle\triangleleft|\eta|)^{-1}))\tilde{\triangleleft}\tau(a\tilde{\triangleleft}(\langle\xi\rangle\triangleleft|\eta|)^{-1},b\tilde{\triangleleft}|\eta|)} \\
&\otimes\ \tau(a\tilde{\triangleleft}(\langle\xi\rangle\triangleleft|\eta|)^{-1}, b\tilde{\triangleleft}|\eta|)^{-1}(b\tilde{\triangleright}(\langle\xi\rangle\triangleleft|\eta|)^{-1})^{-1}z(\langle\xi\rangle\triangleleft|\eta|)^{-1}(a\tilde{\triangleright}|\eta|)^{-1}x|\eta|\,.
\end{aligned} \tag{2.17}$$

To check the δ function the statement $w\tilde{\triangleleft}z(\langle\xi\rangle\triangleleft|\eta|)^{-1}(a\tilde{\triangleright}|\eta|)^{-1} = y$ should be the same as $w\tilde{\triangleleft}x^{-1} = y$, i.e.

$$w\tilde{\triangleleft}z(\langle\xi\rangle\triangleleft|\eta|)^{-1}(a\tilde{\triangleright}|\eta|)^{-1}x = w\,,$$

which means

$$wz(\langle\xi\rangle\triangleleft|\eta|)^{-1}(a\tilde{\triangleright}|\eta|)^{-1}x = z(\langle\xi\rangle\triangleleft|\eta|)^{-1}(a\tilde{\triangleright}|\eta|)^{-1}xw\,. \tag{2.18}$$

Now to calculate $(a\tilde{\triangleright}|\eta|)(\langle\xi\rangle\triangleleft|\eta|)$, put $a = |\xi|^{-1}\langle\xi\rangle = vt$, $\langle\xi\rangle = t$ and $|\eta| = \bar{w}$ then using the fact that $vt\tilde{\triangleright}wp = v^{-1}wpv' = twpt'^{-1}$, where $vt\tilde{\triangleleft}wp = v't'$ we get

$$\bar{w}^{-1}vt\bar{w} = \bar{w}^{-1}v(t\triangleright\bar{w})(t\triangleleft\bar{w}) = v't'\,.$$

So $t' = (t\triangleleft\bar{w}) = (\langle\xi\rangle\triangleleft|\eta|)$ which implies $(a\tilde{\triangleright}|\eta|)(\langle\xi\rangle\triangleleft|\eta|) = t\bar{w} = \langle\xi\rangle|\eta|$. Thus (2.18) becomes

$$wz(\langle\xi\rangle|\eta|)^{-1}x = z(\langle\xi\rangle|\eta|)^{-1}xw\,. \tag{2.19}$$

To check whether this is always true or not we consider the following example:

Example 2.4.1 *Consider example 2.1. Now let $(\delta_y \otimes x) = (\delta_{ba^n} \otimes ba^m)$ and $(\delta_w \otimes z) = (\delta_{ba^p} \otimes ba^q)$. Then we need to check if the following equation holds:*

$$ba^pba^q(\langle\delta_{ba^n} \otimes ba^m\rangle|\delta_{ba^p} \otimes ba^q|)^{-1}ba^m = ba^q(\langle\delta_{ba^n} \otimes ba^m\rangle|\delta_{ba^p} \otimes ba^q|)^{-1}ba^mba^p\,. \tag{2.20}$$

To do so we need to calculate $\|\delta_{ba^n} \otimes ba^m\|$, which we do as follows

$$ba^n \circ \|\delta_{ba^n} \otimes ba^m\| = ba^n\tilde{\triangleleft}ba^m = (ba^m)^{-1}ba^nba^m = ba^{2m-n}\,.$$

Put $\|\delta_{ba^n}\otimes ba^m\| = b^\alpha a^\beta$, where $\alpha = 0, 1$ and β is even, then $b^\alpha a^\beta ba^n = b^{\alpha+1}a^{n-\beta} = ba^{2m-n}$ which implies $\alpha = 0$ and $\beta = 2n - 2m$. Thus

$$\|\delta_{ba^n} \otimes ba^m\| = |\delta_{ba^n} \otimes ba^m|^{-1}\langle\delta_{ba^n} \otimes ba^m\rangle = a^{2n-2m} \in G\,,$$

which implies that $|\delta_{ba^n} \otimes ba^m| = a^{2m-2n}$ and $\langle\delta_{ba^n} \otimes ba^m\rangle = e$. So the left hand side of (2.20) is

$$ba^pba^q|\delta_{ba^p} \otimes ba^q|^{-1}ba^m = ba^pba^qa^{2p-2q}ba^m = ba^{q-p+m}\,,$$

on the other hand the right hand side of (2.20) is

$$ba^q|\delta_{ba^p} \otimes ba^q|^{-1}ba^mba^p = ba^qa^{2p-2q}ba^mba^p = ba^{3p-q-m}$$

which shows that the left hand side of (2.20) is not equal to the right hand side, otherwise $q - p + m \equiv_6 3p - q - m$, i.e. $2q - 4p + 2m$ is a multiple of 6 which is not always true. Therefore, we conclude that D is not braided commutative.

2.5 Integrals in $\mathcal{D}$

In the literature there are two definitions of integral, depending on whether it is viewed as an operator or an element.

Definition 2.5.1 *[19] Let H be a Hopf algebra over the field k. A left integral on H is a non identically zero linear map $\int : H \to k$ satisfying $(\mathrm{id} \otimes \int) \circ \Delta = \eta \circ \int$. In the same way the right integrals are defined. If $\int 1 = 1$, then the integrals are called normalised.*

Definition 2.5.2 *[19, 15] Let H be a Hopf algebra over the field k. A non-zero element $\Lambda \in H$ is called a left integral if $h\Lambda = \epsilon(h)\Lambda$ for all $h \in H$. Similarly, $\Lambda \in H$ is called a right integral if $\Lambda h = \epsilon(h)\Lambda$ for all $h \in H$. An element $\Lambda \in H$ is called integral if it is both right and left integral. Integrals are normalised if $\epsilon(\Lambda) = 1$.*

These definitions are of course connected, for example:

Proposition 2.5.3 *Given a left integral $\int : H \to k$, if we set $\Lambda^* \in H^*$ to be equal to*

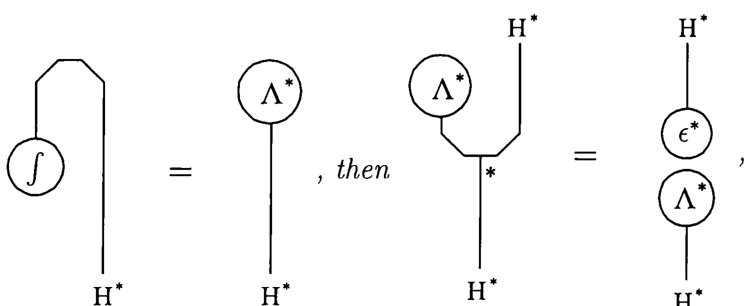
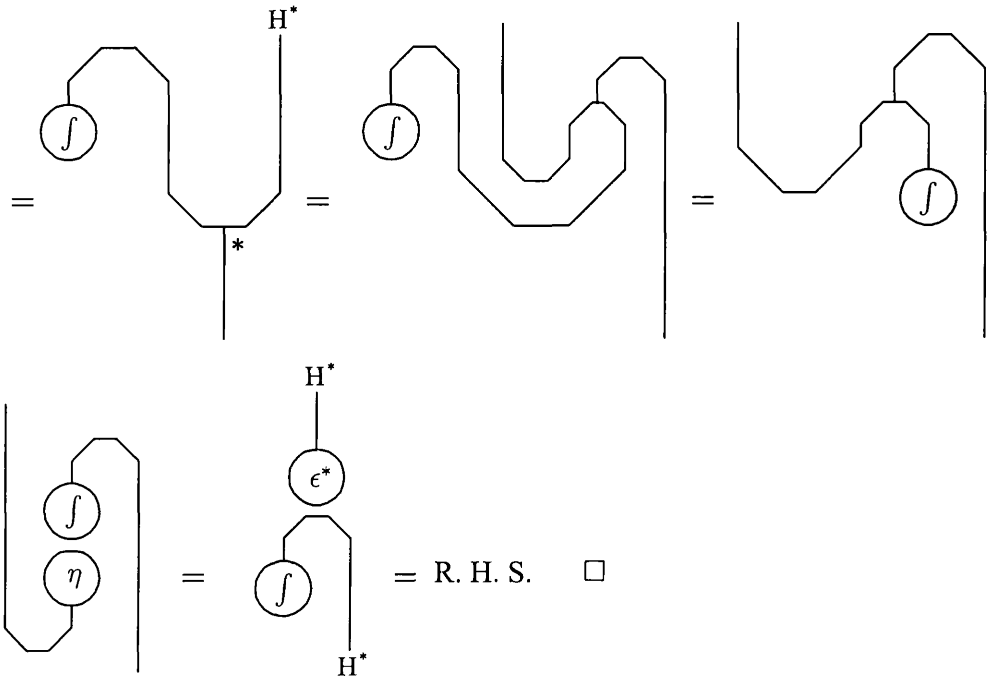


figure 2.8

i.e. $\Lambda^* \in H^*$ is a right integral in H^*.

Proof. We use the standard braided Hopf algebra structure on H^* given in 1.2.14.

L. H. S. =



= R. H. S. □

Now we consider our categories and give specific examples of integrals. First we give a useful lemma from [18].

Lemma 2.5.4

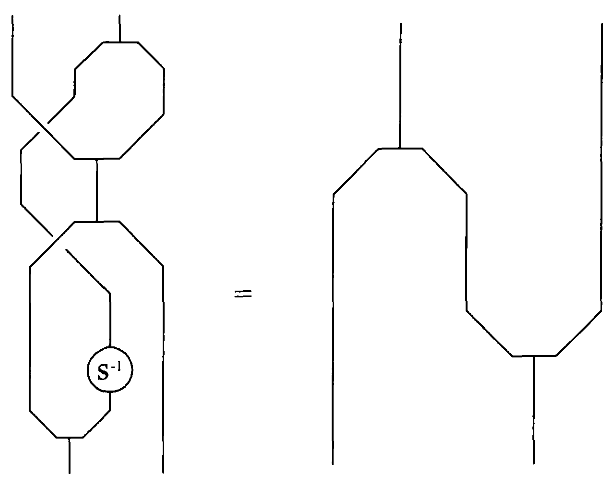
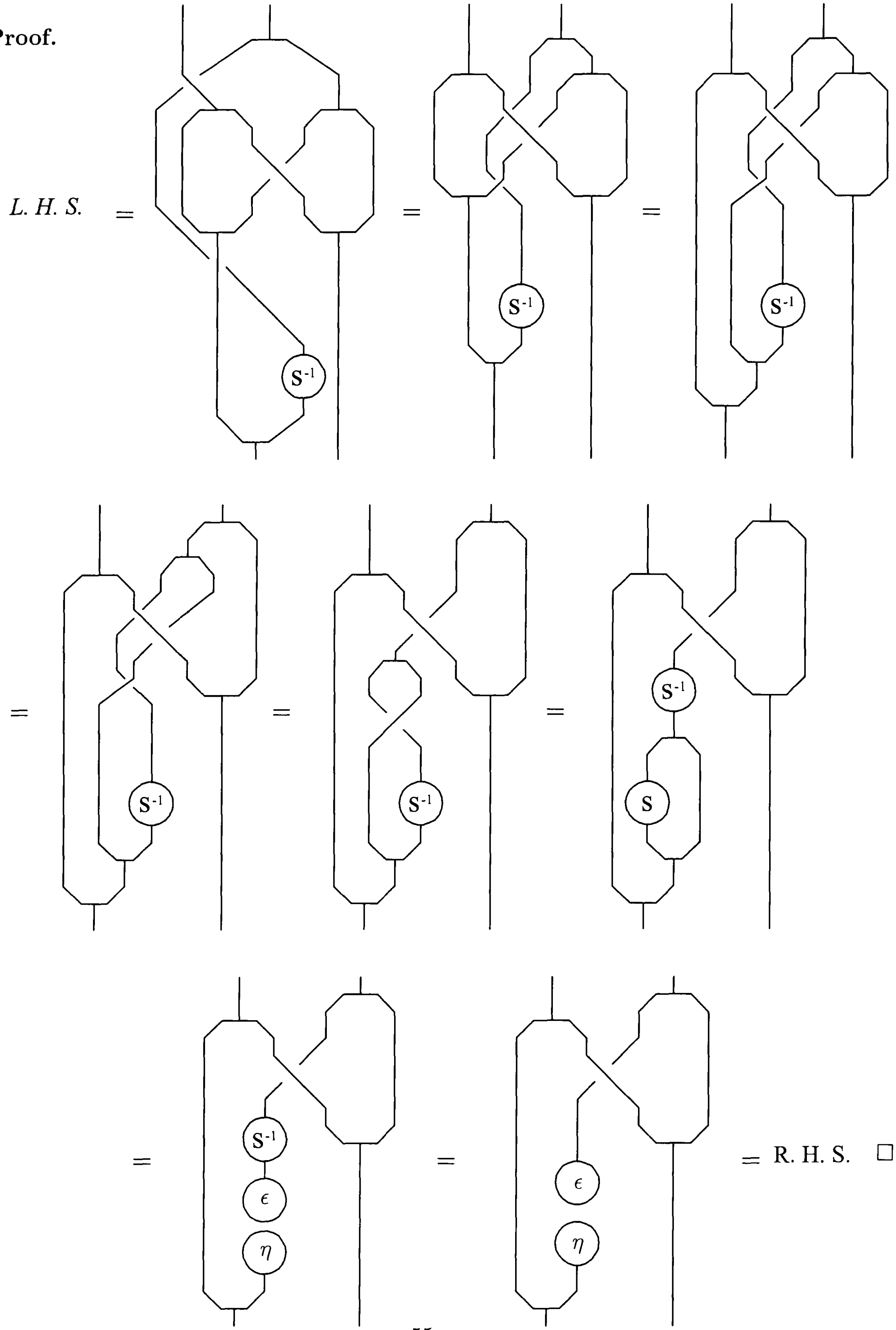


figure 2.9

Proof.



We can now give the following definition of a left integral from [18] as the following:

Definition 2.5.5 *For a braided Hopf algebra H, define $\int : H \to k$ by*

$$\int(h) = \mathrm{trace}(\mathrm{L_h} \circ \mathrm{S}^2) \quad \forall \mathrm{h} \in \mathrm{H},$$

where L_h is the left multiplication by h. This can be illustrated by the following diagram:

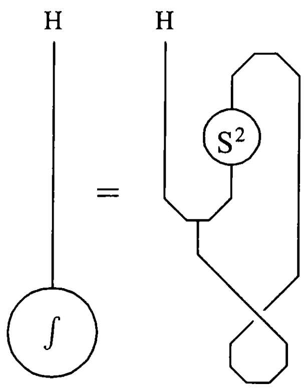


figure 2.10

Proposition 2.5.6 *[18] The integral $\int$ defined in 2.5.5 is a left integral, i.e.*

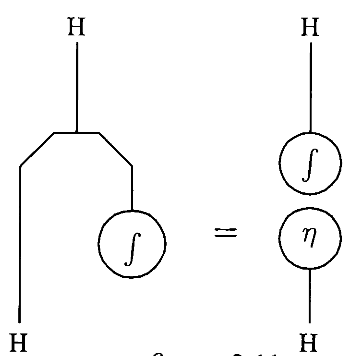
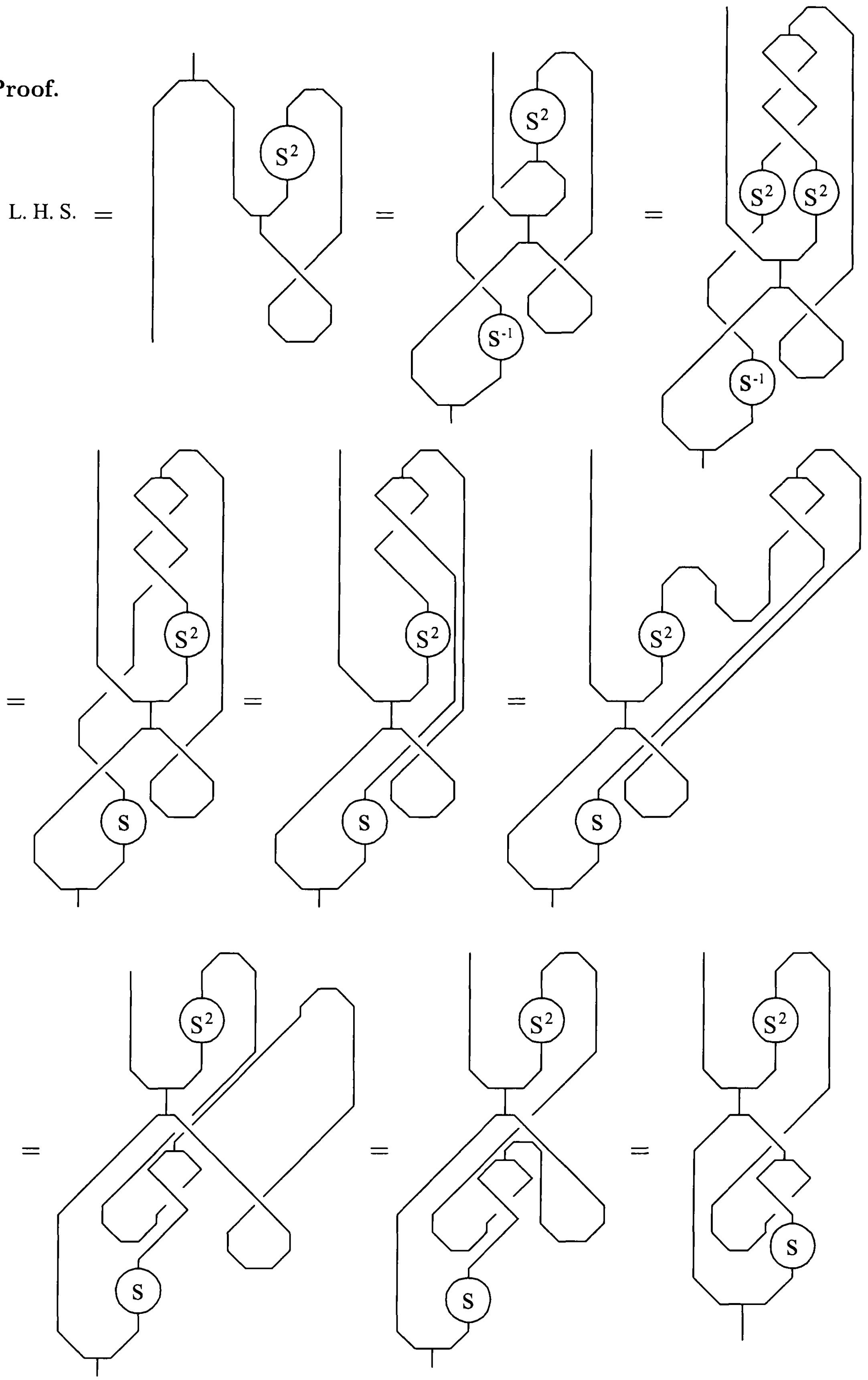
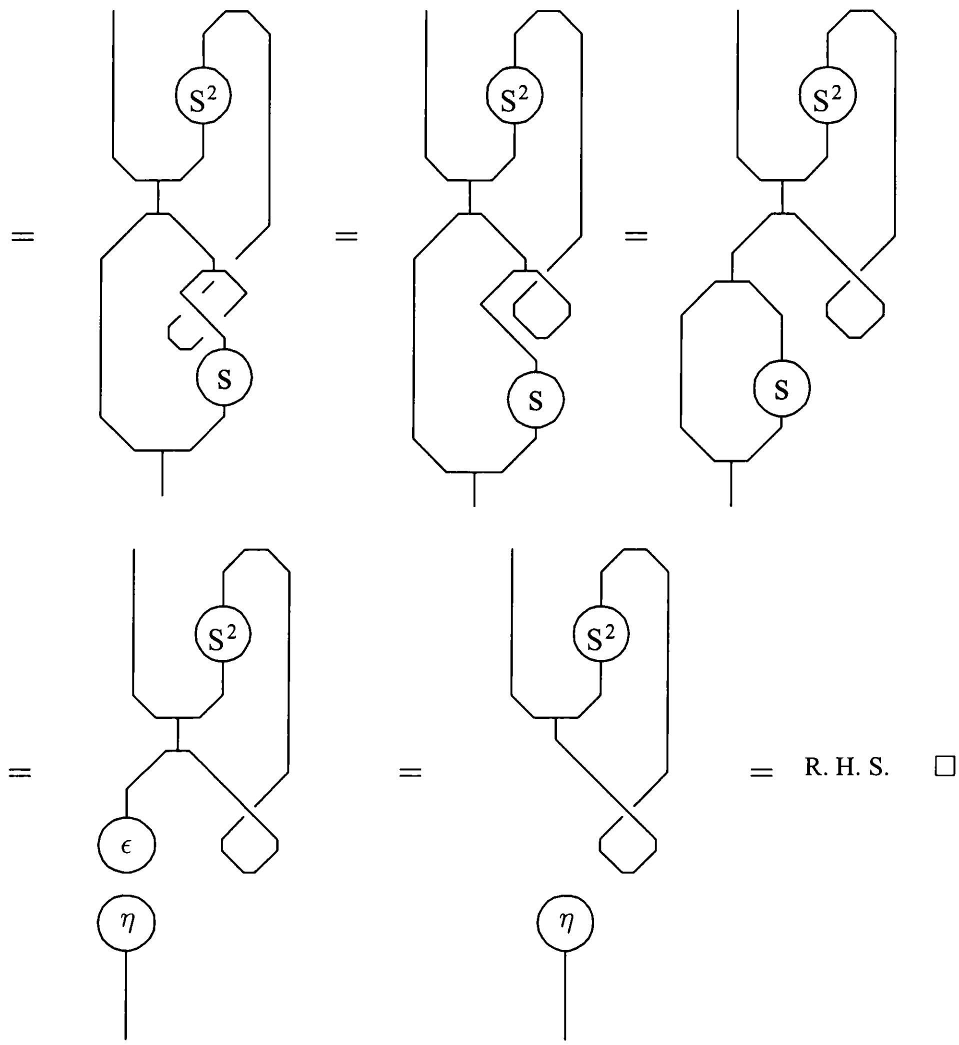


figure 2.11

Proof.





The definition of integral does not require the category to be braided. Here we give an example of an integral in $\mathcal{C}$.

Proposition 2.5.7 *Let A be the algebra in the category $\mathcal{C}$ given in section 1.3, then the element $\Lambda = \sum_u \delta_e \otimes u$ for $u \in G$ is an integral element.*

Proof. We need to prove that $\Lambda = \sum_u \delta_e \otimes u$ is both right and left integral, so for any element $h = (\delta_t \otimes v) \in H$ we have

$$\Lambda h = (\sum_u \delta_e \otimes u)(\delta_t \otimes v) = \sum_u (\delta_e \otimes u)(\delta_t \otimes v) = \sum_u \delta_{t,e\triangleleft u}\, \delta_{e\triangleleft\tau(a,b)} \otimes \tau(a,b)^{-1}uv,$$

where $a = \langle \delta_e \otimes u \rangle$ and $b = \langle \delta_t \otimes v \rangle$. But we know that $e \triangleleft u = e$ and $e \triangleleft \tau(a, b) = e$. Moreover, $e \cdot \langle \delta_e \otimes u \rangle = e \triangleleft u = e$. Also because $a = \langle \delta_e \otimes u \rangle = e$, then $\tau(a, b) = \tau(e, b) = e$. Now as uv is an element in G, then we get

$$\Lambda h = \delta_{t,e} \sum_u \delta_e \otimes uv = \delta_{t,e} \Lambda = \epsilon(h)\Lambda,$$

so $\Lambda = \sum_u \delta_e \otimes u$ is a right integral . Next we want to show that it is also a left integral, so we start with

$$h\Lambda = (\delta_t \otimes v)(\sum_u \delta_e \otimes u) = \sum_u (\delta_t \otimes v)(\delta_e \otimes u) = \sum_u \delta_{e,t \triangleleft v} \, \delta_{t \triangleleft \tau(b,a)} \otimes \tau(b, a)^{-1} vu,$$

where $b = \langle \delta_t \otimes v \rangle$ and $a = \langle \delta_e \otimes u \rangle$. But we know that $t \triangleleft e = t$. Moreover, $\delta_{e,t \triangleleft v} = 1$ implies $e = t \triangleleft v$ or $e \triangleleft v^{-1} = e = t$. Also because $a = \langle \delta_e \otimes u \rangle = e$, then $\tau(b, a) = \tau(b, e) = e$. Now as vu is an element in G, then we get

$$h\Lambda = \delta_{e,t \triangleleft v} \sum_u \delta_t \otimes vu = \delta_{e,t} \sum_u \delta_e \otimes vu = \delta_{t,e} \Lambda = \epsilon(h)\Lambda. \qquad \square$$

Chapter 3

A partial braiding on $\mathcal{C}$

In general the category $\mathcal{C}$ is not braided, but it contains the braided category $\mathcal{D}$ by forgetting the G-grading and the M-action. In fact, the braiding for $\mathcal{D}$ extends to a partially defined braiding on $\mathcal{C}$, as we show in this chapter.

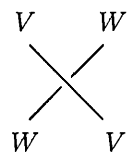
We also look at the algebra $A \in \mathcal{C}$ again, using this partial braiding and find a strange one sided braided counit. This may have some relevance to the work by J. Green, D. Nichols and E. Taft on one sided Hopf structures [8].

The chapter continues by showing that A is isomorphic to A^* as objects in $\mathcal{C}$, and calculate the coproduct on A^*.

There remain the problems of a star structure on the objects in $\mathcal{C}$ and the existence of an antipode on A. While no definitive conclusion was reached on these matters, it is shown in the last section that more morphisms can be added to $\mathcal{C}$ to make a richer structure. Possibly more work in this direction can shed light on the problems mentioned.

3.1 A partial braiding on $\mathcal{C}$

We define $\Psi_{VW} : V \otimes W \to W \otimes V$ for $V \in \mathcal{C}$ and $W \in \mathcal{D}$, and use the usual diagram notation for Ψ_{VW}:



Proposition 3.1.1 *Let $V \in \mathcal{C}$ and $W \in \mathcal{D}$, then the braiding $\Psi : V \otimes W \to W \otimes V$ which is defined by $\Psi(\xi \otimes \eta) = \langle \xi \rangle \bar{\triangleright} \eta \otimes \xi \bar{\triangleleft} |\eta|$ for $\xi \in V$ and $\eta \in W$ is a morphism in $\mathcal{C}$. This is the same as given in 1.3.14.*

Proof. To prove that $\Psi : V \otimes W \to W \otimes V$ is a morphism in $\mathcal{C}$ we need to show that Ψ preserves the M-grade and the G-action. We first check the M-grade as follows

$$\big\langle \Psi(\xi \otimes \eta) \big\rangle = \big\langle \langle \xi \rangle \bar{\triangleright} \eta \otimes \xi \bar{\triangleleft} |\eta| \big\rangle = \big\langle \langle \xi \rangle \bar{\triangleright} \eta \big\rangle \cdot \big\langle \xi \bar{\triangleleft} |\eta| \big\rangle$$

$$= \big\langle \langle \xi \rangle \bar{\triangleright} \eta \big\rangle \cdot \big(\langle \xi \rangle \triangleleft |\eta| \big) = \langle \xi \rangle \cdot \langle \eta \rangle = \langle \xi \otimes \eta \rangle \,,$$

as required. The forth equality according to lemma 3.2.3. Now to check the G-action we need to show that $\Psi\big((\xi \otimes \eta) \bar{\triangleleft} v\big) = \Psi(\xi \otimes \eta) \bar{\triangleleft} v$. To do so we start as follows

$$\begin{aligned}
L.H.S. &= \Psi\big((\xi \otimes \eta) \bar{\triangleleft} v\big) = \Psi\big(\xi \bar{\triangleleft} (\langle \eta \rangle \triangleright v) \otimes \eta \bar{\triangleleft} v\big) \\
&= \big\langle \xi \bar{\triangleleft} (\langle \eta \rangle \triangleright v) \big\rangle \bar{\triangleright} (\eta \bar{\triangleleft} v) \otimes \big(\xi \bar{\triangleleft} (\langle \eta \rangle \triangleright v)\big) \bar{\triangleleft} |\eta \bar{\triangleleft} v| \\
&= \big(\langle \xi \rangle \triangleleft (\langle \eta \rangle \triangleright v)\big) \bar{\triangleright} (\eta \bar{\triangleleft} v) \otimes \xi \bar{\triangleleft} (\langle \eta \rangle \triangleright v) |\eta \bar{\triangleleft} v| \\
&= \big(\langle \xi \rangle \triangleleft (\langle \eta \rangle \triangleright v)\big) \bar{\triangleright} (\eta \bar{\triangleleft} v) \otimes \xi \bar{\triangleleft} (\langle \eta \rangle \triangleright v) (\langle \eta \rangle \triangleright v)^{-1} |\eta| v \\
&= \big(\langle \xi \rangle \triangleleft (\langle \eta \rangle \triangleright v)\big) \bar{\triangleright} (\eta \bar{\triangleleft} v) \otimes \xi \bar{\triangleleft} |\eta| v \,.
\end{aligned}$$

On the other hand

$$R.H.S. = \Psi(\xi \otimes \eta)\bar{\triangleleft}v = (\langle\xi\rangle\bar{\triangleright}\eta \otimes \xi\bar{\triangleleft}|\eta|)\bar{\triangleleft}v$$

$$= ((\langle\xi\rangle\bar{\triangleright}\eta)\bar{\triangleleft}(\langle\xi\bar{\triangleleft}|\eta|\rangle\triangleright v) \otimes (\xi\bar{\triangleleft}|\eta|)\bar{\triangleleft}v$$

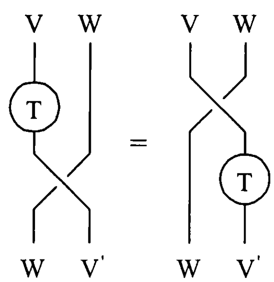
$$= ((\langle\xi\rangle\bar{\triangleright}\eta)\bar{\triangleleft}((\langle\xi\rangle\triangleleft|\eta|)\triangleright v) \otimes \xi\bar{\triangleleft}|\eta|v\,,$$

which is the same as the L. H. S. where

$$\big(\langle\xi\rangle\triangleleft(\langle\eta\rangle\triangleright v)\big)\bar{\triangleright}(\eta\bar{\triangleleft}v) = (\langle\xi\rangle\bar{\triangleright}\eta)\bar{\triangleleft}\big((\langle\xi\rangle\triangleleft|\eta|)\triangleright v\big)\,,$$

according to the cross relation between the actions mentioned in equation (1.7). □

Proposition 3.1.2 *Let $V \in \mathcal{C}$ and $W \in \mathcal{D}$. Then for a morphism $T : V \to V'$ in $\mathcal{C}$ the following equality is true:*



figre 3.1

Proof. We need to prove that $\Psi\big(T(\xi) \otimes \eta\big) = (\mathrm{id} \otimes \mathrm{T})\Psi(\xi \otimes \eta)$. So we start as follows

$$L.H.S. = \Psi\big(T(\xi) \otimes \eta\big) = \big\langle T(\xi)\big\rangle\bar{\triangleright}\eta \otimes T(\xi)\bar{\triangleleft}|\eta| = \langle\xi\rangle\bar{\triangleright}\eta \otimes \big(T(\xi)\big)\bar{\triangleleft}|\eta|\,.$$

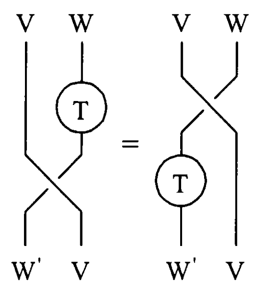
The last equality happens because T is a morphism in $\mathcal{C}$ so it preserves the M-grade. On the other hand

$$R.H.S. = (\mathrm{id} \otimes \mathrm{T})\Psi(\xi \otimes \eta) = (\mathrm{id} \otimes \mathrm{T})(\langle\xi\rangle\bar{\triangleright}\eta \otimes \xi\bar{\triangleleft}|\eta|)$$

$$= \langle\xi\rangle\bar{\triangleright}\eta \otimes T(\xi\bar{\triangleleft}|\eta|) = \langle\xi\rangle\bar{\triangleright}\eta \otimes \big(T(\xi)\big)\bar{\triangleleft}|\eta|\,.$$

The last equality happens because T is a morphism in $\mathcal{C}$ so it preserves the G-action. □

Proposition 3.1.3 *Let $V \in \mathcal{C}$ and $W \in \mathcal{D}$. Then for a morphism $T : W \to W'$ in $\mathcal{D}$ the following equality is true:*



figre 3.2

Proof. We need to prove that $\Psi\big(\xi \otimes T(\eta)\big) = (T \otimes \mathrm{id})\Psi(\xi \otimes \eta)$. So we start as follows

$$L.H.S. = \Psi\big(\xi \otimes T(\eta)\big) = \langle\xi\rangle\bar{\triangleright}T(\eta) \otimes \xi\bar{\triangleleft}|T(\eta)| = \langle\xi\rangle\bar{\triangleright}\big(T(\eta)\big) \otimes \xi\bar{\triangleleft}|\eta|\,.$$

The last equality happens because T is a morphism in $\mathcal{D}$ so it preserves the G-grade. On the other hand

$$R.H.S. = (T \otimes \mathrm{id})\Psi(\xi \otimes \eta) = (\mathrm{T} \otimes \mathrm{id})(\langle\xi\rangle\bar{\triangleright}\eta \otimes \xi\bar{\triangleleft}|\eta|)$$

$$= T(\langle\xi\rangle\bar{\triangleright}\eta) \otimes \xi\bar{\triangleleft}|\eta| = \langle\xi\rangle\bar{\triangleright}\big(T(\eta)\big) \otimes \xi\bar{\triangleleft}|\eta|\,.$$

The last equality happens because T is a morphism in $\mathcal{D}$ so it preserves the M-action. □

Next we see that the algebra A in $\mathcal{C}$ described in [4] is the image of an object in $\mathcal{D}$.

3.2 The algebra A as an object in $\mathcal{D}$

In section (1.3) we defined an algebra A in $\mathcal{C}$ whose representations were exactly the objects of $\mathcal{C}$. In this section we show that it is possible to put a G-grade and M-action on A so that it becomes an object in $\mathcal{D}$. The multiplication and the identity given in (1.3.8) are morphisms in $\mathcal{D}$, so A is an algebra in $\mathcal{D}$. However the action (1.3) of A on objects in $\mathcal{D}$ is shown not to be a morphism in $\mathcal{D}$.

We can now ask if, using the braiding in $\mathcal{D}$, A becomes a braided Hopf algebra. The answer in general is no, but some structure is recovered.

Definition 3.2.1 *Let A be the algebra, given in section (1.3), in the category $\mathcal{C}$. For an element $\delta_s \otimes u \in A$, we define the G-grading by $|\delta_s \otimes u| = u$ and the M-action by*

$$t\bar{\triangleright}(\delta_s \otimes u) = \delta_{((s\triangleleft\tau(t^L,t)^{-1})\cdot t^L)\triangleleft w^{-1}} \otimes w(t\triangleright u)\,,$$

where $w = \tau(b, t\triangleleft u)\tau(t,a)^{-1}, a = \langle \delta_s \otimes u\rangle, b$ satisfies $t \cdot a = b \cdot (t\triangleleft u), u \in G$ and $s,t \in M$.

Theorem 3.2.2 *The algebra A defined in 3.2.1 is an object in the braided tensor category $\mathcal{D}$.*

Proof. The proof is given by the following lemmas. □

Lemma 3.2.3 *The algebra A defined in 3.2.1 satisfies the following connections between the gradings and actions:*

$$|\eta\bar{\triangleleft}v| = (\langle\eta\rangle\triangleright v)^{-1}|\eta|v \ \ , \ \ t\cdot\langle\eta\rangle = \langle t\bar{\triangleright}\eta\rangle \cdot (t\triangleleft|\eta|)\ ,$$

$$\tau(t,\langle\eta\rangle)^{-1}(t\triangleright|\eta|) = \tau(\langle t\bar{\triangleright}\eta\rangle, t\triangleleft|\eta|)^{-1}|t\bar{\triangleright}\eta|\,.$$

Proof. Firstly, to prove that the algebra A satisfies the equation $|\eta\bar{\triangleleft}v| = (\langle\eta\rangle\triangleright v)^{-1}|\eta|v$ we put $\eta = \delta_s \otimes u$ and start with the left hand side as follows

$$L.H.S. = |\eta\bar{\triangleleft}v| = |(\delta_s \otimes u)\bar{\triangleleft}v| = |(\delta_{s\triangleleft(a\triangleright v)} \otimes (a\triangleright v)^{-1}uv)| = (a\triangleright v)^{-1}uv = R.H.S\,,$$

where $a = \langle\eta\rangle = \langle\delta_s \otimes u\rangle$ and $|\eta| = |\delta_s \otimes u| = u$.

Secondly, the equation $t\cdot\langle\eta\rangle = t\cdot a = \langle t\bar{\triangleright}\eta\rangle\cdot(t\triangleleft|\eta|)$ is satisfied directly from the definition of b if we put $b = \langle t\bar{\triangleright}\eta\rangle$.

Finally, to show that the algebra A mentioned in 3.2.1 satisfies the equation $\tau(t, \langle\eta\rangle)^{-1}(t\triangleright|\eta|) = \tau(\langle t\bar{\triangleright}\eta\rangle, t\triangleleft|\eta|)^{-1}|t\bar{\triangleright}\eta|$ we start as follows

$$
\begin{aligned}
R.H.S. &= \tau(\langle t\bar{\triangleright}\eta\rangle, t\triangleleft|\eta|)^{-1}|t\bar{\triangleright}\eta| = \tau(\, b, t\triangleleft u)^{-1}|t\bar{\triangleright}(\delta_s \otimes u)| \\
&= \tau(\, b, t\triangleleft u)^{-1}|\delta_{((s\triangleleft\tau(t^L,t)^{-1})\cdot t^L)\triangleleft w^{-1}} \otimes w(t\triangleright u)| = \tau(\, b, t\triangleleft u)^{-1}w(t\triangleright u) \\
&= \tau(\, b, t\triangleleft u)^{-1}\tau(b, t\triangleleft u)\tau(t, a)^{-1}(t\triangleright u) = \tau(t, a)^{-1}(t\triangleright u) = L.H.S\,,
\end{aligned}
$$

where $w = \tau(b, t\triangleleft u)\tau(t, a)^{-1}$ as mentioned in 3.2.1. □

Lemma 3.2.4 *Let A be the algebra defined in 3.2.1, then for $\kappa \in A$ and $p, t \in M$ the following equation is satisfied where $p' = p\triangleleft\tau(\langle t\bar{\triangleright}\kappa\rangle, t\triangleleft|\kappa|)\tau(t, \langle\kappa\rangle)^{-1}$:*

$$
p\bar{\triangleright}(t\bar{\triangleright}\kappa) = (p' \cdot t\bar{\triangleright}\kappa)\bar{\triangleleft}\tau\left(p'\triangleleft(t\triangleright|\kappa|), t\triangleleft|\kappa|\right)^{-1}.
$$

Proof. We equivalently need to prove that

$$
\left(p\bar{\triangleright}(t\bar{\triangleright}\kappa)\right)\bar{\triangleleft}\tau\left(p'\triangleleft(t\triangleright|\kappa|), t\triangleleft|\kappa|\right) = (p' \cdot t\bar{\triangleright}\kappa)\,.
$$

Put $\kappa = \delta_s \otimes u$, and to calculate the left hand side we calculate the following:

$$
t\bar{\triangleright}(\delta_s \otimes u) = \delta_{s'} \otimes u'\,,
$$

where $u' = w(t\triangleright u)$, $s' = \left((s\triangleleft\tau(t^L, t)^{-1}) \cdot t^L\right)\triangleleft w^{-1}$, $w = \tau(b, t\triangleleft u)\tau(t, a)^{-1}, a = \langle\delta_s \otimes u\rangle, b = \langle t\bar{\triangleright}(\delta_s \otimes u)\rangle, u \in G$ and $s, t \in M$. Note that b satisfies $t \cdot a = b \cdot (t\triangleleft u)$. Then

$$
p\bar{\triangleright}(\delta_{s'} \otimes u') = \delta_{s''} \otimes u''\,,
$$

where $u'' = w'(p\triangleright u')$, $s'' = \left((s'\triangleleft\tau(p^L, p)^{-1}) \cdot p^L\right)\triangleleft w'^{-1}$, $w' = \tau(c, p\triangleleft u')\tau(p, b)^{-1}$, and

$c = \langle \delta_{s''} \otimes u'' \rangle$ satisfies $p \cdot b = c \cdot (p \triangleleft u')$. Now if we set $z = \tau\big(p \triangleleft w(t \triangleright u), t \triangleleft u\big)$ then

$$u'' = w'\big(p \triangleright w(t \triangleright u)\big) = w'(p \triangleright w)\big((p \triangleleft w) \triangleright (t \triangleright u)\big)$$

$$= w'(p \triangleright w)\tau(p \triangleleft w, t)\big(((p \triangleleft w) \cdot t) \triangleright u\big)\tau\big(p \triangleleft w(t \triangleright u), t \triangleleft u\big)^{-1}$$

$$= w'(p \triangleright w)\tau(p \triangleleft w, t)(\bar{t} \triangleright u)z^{-1},$$

where $\bar{t} = (p \triangleleft w) \cdot t$, so the left hand side is given by

$$L.H.S. = (\delta_{s''} \otimes u'')\bar{\triangleleft} z = \delta_{s'' \triangleleft (c \triangleright z)} \otimes (c \triangleright z)^{-1} u'' z$$

$$= \delta_{s'' \triangleleft (c \triangleright z)} \otimes (c \triangleright z)^{-1} w'(p \triangleright w)\tau(p \triangleleft w, t)(\bar{t} \triangleright u) = \delta_{s'''} \otimes u'''.$$

To simplify u''' we need to calculate the following

$$(c \triangleright z)^{-1} w' = \big(c \triangleright \tau(p \triangleleft u', t \triangleleft u)\big)^{-1} \tau(c, p \triangleleft u')\tau(p, b)^{-1}$$

$$= \big(c \triangleright \tau(p \triangleleft u', t \triangleleft u)\big)^{-1} c\,(p \triangleleft u')\big(c \cdot (p \triangleleft u')\big)^{-1}\tau(p, b)^{-1}$$

$$= \big(c \triangleright \tau(p \triangleleft u', t \triangleleft u)\big)^{-1} c\,(p \triangleleft u')\, b^{-1} p^{-1} = \big(c \triangleright \tau(p \triangleleft u', t \triangleleft u)\big)^{-1} c\,(p \triangleleft u')\,(t \triangleleft u)(t \triangleleft u)^{-1} b^{-1} p^{-1}$$

$$= \big(c \triangleright \tau(p \triangleleft u', t \triangleleft u)\big)^{-1} c\,\tau(p \triangleleft u', t \triangleleft u)\big((p \triangleleft u') \cdot (t \triangleleft u)\big)(t \triangleleft u)^{-1} b^{-1} p^{-1}$$

$$= \big(c \triangleleft \tau(p \triangleleft u', t \triangleleft u)\big)\big((p \triangleleft w) \cdot t) \triangleleft u\big)\,(t \triangleleft u)^{-1} b^{-1} p^{-1}$$

$$= \tau\Big(\big(c \triangleleft \tau(p \triangleleft u', t \triangleleft u)\big), \big((p \triangleleft w) \cdot t) \triangleleft u\big)\Big)\Big(\big(c \triangleleft \tau(p \triangleleft u', t \triangleleft u)\big) \cdot \big((p \triangleleft u') \cdot (t \triangleleft u)\big)\Big)(t \triangleleft u)^{-1} b^{-1} p^{-1}$$

$$= \tau(c \triangleleft z, \bar{t} \triangleleft u)\Big(\big(c \cdot (p \triangleleft u')\big) \cdot (t \triangleleft u)\Big)(t \triangleleft u)^{-1} b^{-1} p^{-1} = \tau(c \triangleleft z, \bar{t} \triangleleft u)\Big((p \cdot b) \cdot (t \triangleleft u)\Big)(t \triangleleft u)^{-1} b^{-1} p^{-1}$$

$$= \tau(c \triangleleft z, \bar{t} \triangleleft u)\Big(\big(p \triangleleft \tau(b, t \triangleleft u)\big) \cdot \big(b \cdot (t \triangleleft u)\big)\Big)(t \triangleleft u)^{-1} b^{-1} p^{-1}$$

$$= \tau(c \triangleleft z, \bar{t} \triangleleft u)\Big(\big(p \triangleleft \tau(b, t \triangleleft u)\big) \cdot (t \cdot a)\Big)(t \triangleleft u)^{-1} b^{-1} p^{-1}$$

$$= \tau(c \triangleleft z, \bar{t} \triangleleft u)\Big(\big(p \triangleleft \tau(b, t \triangleleft u)\tau(t, a)^{-1}\big) \cdot t\big) \cdot a\Big)(t \triangleleft u)^{-1} b^{-1} p^{-1}$$

$$= \tau(c \triangleleft z, \bar{t} \triangleleft u)\Big(\big((p \triangleleft w) \cdot t\big) \cdot a\Big)(t \triangleleft u)^{-1} b^{-1} p^{-1}$$

$$= \tau(c \triangleleft z, \bar{t} \triangleleft u)(\bar{t} \cdot a)(t \triangleleft u)^{-1} b^{-1} p^{-1} = \tau(c \triangleleft z, \bar{t} \triangleleft u)\tau(\bar{t}, a)^{-1}\,\bar{t}\,a\,(t \triangleleft u)^{-1} b^{-1} p^{-1}.$$

Now as $b(t \triangleleft u) = \tau(b, t \triangleleft u)\big(b \cdot (t \triangleleft u)\big) = \tau(b, t \triangleleft u)(t \cdot a) = \tau(b, t \triangleleft u)\tau(t, a)^{-1} t\, a = w\, t\, a$ and

$pwt = (p{\triangleright}w)(p{\triangleleft}w)\,t = (p{\triangleright}w)\tau(p{\triangleleft}w,t)\,\bar{t}$ we get

$$(c{\triangleright}z)^{-1}\,w^{'} = \tau(c{\triangleleft}z,\bar{t}{\triangleleft}u)\tau(\bar{t},a)^{-1}\,\bar{t}\,t^{-1}\,w^{-1}\,p^{-1} = \tau(c{\triangleleft}z,\bar{t}{\triangleleft}u)\tau(\bar{t},a)^{-1}\,\tau(p{\triangleleft}w,t)^{-1}(p{\triangleright}w)^{-1}\,.$$

So

$$u^{'''} = \tau(c{\triangleleft}z,\bar{t}{\triangleleft}u)\tau(\bar{t},a)^{-1}(\bar{t}{\triangleright}u)\,.$$

Also $s^{'''}$ can be simplified as

$$s^{'''} = s^{''}{\triangleleft}(c{\triangleright}z) = \big((s^{'}{\triangleleft}\tau(p^{L},p)^{-1})\cdot p^{L}\big){\triangleleft}(p{\triangleright}w)\tau(p{\triangleleft}w,t)\tau(\bar{t},a)\tau(c{\triangleleft}z,\bar{t}{\triangleleft}u)^{-1}\,.$$

On the other hand, note that $p^{'} = p{\triangleleft}w$ and $p^{'}\cdot t = \bar{t}$ then we get

$$R.H.S. = \big((p^{'}\cdot t)\tilde{\triangleright}\kappa\big) = \big(\bar{t}\tilde{\triangleright}(\delta_s\otimes u)\big) = \delta_{\hat{s}}\otimes\hat{u}$$

where $\hat{u} = \bar{w}(\bar{t}{\triangleright}u)$, $\hat{s} = \big((s{\triangleleft}\tau(\bar{t}^{L},\bar{t})^{-1})\cdot\bar{t}^{L}\big){\triangleleft}\bar{w}^{-1}$, $\bar{w} = \tau(\bar{b},\bar{t}{\triangleleft}u)\tau(\bar{t},a)^{-1}, \bar{b} = \langle\bar{t}\tilde{\triangleright}(\delta_s\otimes u)\rangle$ satisfies $\bar{t}\cdot a = \bar{b}\cdot(\bar{t}{\triangleleft}u)$. To show the equivalence between the left and the right hand sides we need to show that $\bar{b} = c{\triangleleft}z$, which we do as follows

$$\begin{aligned}
\bar{b}\cdot(\bar{t}{\triangleleft}u) &= \bar{t}\cdot a = \big((p{\triangleleft}w)\cdot t\big)\cdot a = \big((p{\triangleleft}w\tau(t,a)\big)\cdot(t\cdot a) = \big((p{\triangleleft}\tau(b,t{\triangleleft}u)\big)\cdot\big(b\cdot(t{\triangleleft}u)\big)\\
&= (p\cdot b)\cdot(t{\triangleleft}u) = \big(c\cdot(p{\triangleleft}u^{'})\big)\cdot(t{\triangleleft}u) = \big(c{\triangleleft}\tau(p{\triangleleft}u^{'},t{\triangleleft}u)\big)\cdot\big((p{\triangleleft}u^{'})\cdot(t{\triangleleft}u)\big)\\
&= \big(c{\triangleleft}\tau(p{\triangleleft}u^{'},t{\triangleleft}u)\big)\cdot\big(((p{\triangleleft}w){\triangleleft}(t{\triangleright}u))\cdot(t{\triangleleft}u)\big) = \big(c{\triangleleft}\tau(p{\triangleleft}u^{'},t{\triangleleft}u)\big)\cdot\big(((p{\triangleleft}w)\cdot t){\triangleleft}u\big)\\
&= (c{\triangleleft}z)\cdot(\bar{t}{\triangleleft}u)\,,
\end{aligned}$$

as required. Thus $u^{'''} = \hat{u}$. Now we only need to show that $s^{'''} = \hat{s}$, i.e.

$$\big((s^{'}{\triangleleft}\tau(p^{L},p)^{-1})\cdot p^{L}\big){\triangleleft}(p{\triangleright}w)\tau(p{\triangleleft}w,t) = \big((s{\triangleleft}\tau(\bar{t}^{L},\bar{t})^{-1})\cdot\bar{t}^{L}\big)\,.$$

If we apply $\cdot\bar{t}$ to both sides then we need to prove

$$\Big(\big((s^{'}{\triangleleft}\tau(p^{L},p)^{-1})\cdot p^{L}\big){\triangleleft}(p{\triangleright}w)\tau(p{\triangleleft}w,t)\Big)\cdot\bar{t} = s\,,$$

so we start with the left hand side of the above equation noting that $p'^{L} = (p\triangleleft w)^{L} = p^{L}\triangleleft(p\triangleright w)$ and $\bar{t} = p' \cdot t$ as follows

$$\Big(\big((s'\triangleleft\tau(p^{L},p)^{-1})\cdot p^{L}\big)\triangleleft(p\triangleright w)\tau(p\triangleleft w,t)\Big)\cdot\bar{t} = \bigg(\Big(\big((s'\triangleleft\tau(p^{L},p)^{-1})\cdot p^{L}\big)\triangleleft(p\triangleright w)\Big)\triangleleft\tau(p\triangleleft w,t)\bigg)\cdot\bar{t}$$

$$= \bigg(\Big(\big((s'\triangleleft\tau(p^{L},p)^{-1})\triangleleft\big(p^{L}\triangleright(p\triangleright w)\big)\big)\cdot p'^{L}\Big)\triangleleft\tau(p\triangleleft w,t)\bigg)\cdot\bar{t}$$

$$= \bigg(\Big(s'\triangleleft\tau(p^{L},p)^{-1}\big(p^{L}\triangleright(p\triangleright w)\big)\big(p'^{L}\triangleright\tau(p',t)\big)\Big)\cdot\big(p'^{L}\triangleleft\tau(p',t)\big)\bigg)\cdot(p'\cdot t)$$

$$= \Bigg(\bigg(\Big(s'\triangleleft\tau(p^{L},p)^{-1}\big(p^{L}\triangleright(p\triangleright w)\big)\big(p'^{L}\triangleright\tau(p',t)\big)\Big(\big(p'^{L}\triangleleft\tau(p',t)\big)\triangleright\tau(p',t)^{-1}\Big)\Big)\cdot p'^{L}\bigg)\cdot p'\Bigg)\cdot t$$

$$= \Bigg(\bigg(\Big(s'\triangleleft\tau(p^{L},p)^{-1}\big(p^{L}\triangleright(p\triangleright w)\big)\Big)\cdot p'^{L}\bigg)\cdot p'\Bigg)\cdot t$$

$$= \bigg(\Big(s'\triangleleft\tau(p^{L},p)^{-1}\big(p^{L}\triangleright(p\triangleright w)\big)\tau(p'^{L},p')\Big)\cdot(p'^{L}\cdot p')\bigg)\cdot t$$

$$= \Big(\big((s\triangleleft\tau(t^{L},t)^{-1})\cdot t^{L}\big)\triangleleft w^{-1}\tau(p^{L},p)^{-1}\big(p^{L}\triangleright(p\triangleright w)\big)\tau(p'^{L},p')\Big)\cdot t$$

$$= \big((s\triangleleft\tau(t^{L},t)^{-1})\cdot t^{L}\big)\cdot t = s\,,$$

as required. Note that for the fourth equality we have applied Φ^{-1} and in the equality before the last we have used the following:

$$\tau(p^{L},p)w = p^{L}pw = p^{L}(p\triangleright w)(p\triangleleft w) = \big(p^{L}\triangleright(p\triangleright w)\big)p'^{L}p' = \big(p^{L}\triangleright(p\triangleright w)\big)\tau(p'^{L},p'). \quad \square$$

Lemma 3.2.5 *The algebra A, defined in 3.2.1, satisfies the cross relation between the two actions, i.e. for $\eta \in A$, $t \in M$ and $v \in G$ we have:*

$$(t\bar{\triangleright}\eta)\bar{\triangleleft}\big((t\triangleleft|\eta|)\triangleright v\big) = \big(t\triangleleft(\langle\eta\rangle\triangleright v)\big)\bar{\triangleright}(\eta\bar{\triangleleft}v)\,.$$

Proof. Let $\eta = \delta_s \otimes u$, $a = \langle\eta\rangle = \langle\delta_s \otimes u\rangle$ and $b = \langle t\bar{\triangleright}\eta\rangle = \langle t\bar{\triangleright}(\delta_s\otimes u)\rangle$, which satisfies $t\cdot a = b\cdot(t\triangleleft u)$. Then we start with the left hand side as follows

$$L.H.S. = (t\bar{\triangleright}\eta)\bar{\triangleleft}\big((t\triangleleft|\eta|)\triangleright v\big) = (\delta_{s'}\otimes u')\bar{\triangleleft}\big((t\triangleleft u)\triangleright v\big)$$

$$= \delta_{s'\triangleleft(b\triangleright((t\triangleleft u)\triangleright v))}\otimes\big(b\triangleright\big((t\triangleleft u)\triangleright v\big)\big)^{-1}u'\big((t\triangleleft u)\triangleright v\big) = \delta_{\bar{s}}\otimes\bar{u}\,,$$

where $u^{'} = w(t\triangleright u)$, $s^{'} = \big((s\triangleleft\tau(t^L,t)^{-1})\cdot t^L\big)\triangleleft w^{-1}$ and $w = \tau(b, t\triangleleft u)\tau(t,a)^{-1}$. Now we need to simplify $\bar{u}$ and $\bar{s}$. So first we calculate the following

$$\begin{aligned}\big(b\triangleright((t\triangleleft u)\triangleright v)\big) &= \tau(b,t\triangleleft u)\big((b\cdot(t\triangleleft u))\triangleright v\big)\tau\big(b\triangleleft((t\triangleleft u)\triangleright v)\big),(t\triangleleft u)\triangleleft v\big)^{-1}\\ &= \tau(b,t\triangleleft u)\big((t\cdot a)\triangleright v\big)\tau\big(b\triangleleft((t\triangleleft u)\triangleright v)\big),t\triangleleft uv\big)^{-1},\end{aligned}$$

but $\bar{u}$ is given by

$$\begin{aligned}\bar{u} &= \big(b\triangleright((t\triangleleft u)\triangleright v)\big)^{-1}u^{'}\big((t\triangleleft u)\triangleright v\big)\\ &= \tau\big(b\triangleleft((t\triangleleft u)\triangleright v)\big),t\triangleleft uv\big)\big((t\cdot a)\triangleright v\big)^{-1}\tau(t,a)^{-1}(t\triangleright u)\big((t\triangleleft u)\triangleright v\big)\\ &= \tau\big(b\triangleleft((t\triangleleft u)\triangleright v)\big),t\triangleleft uv\big)\big((t\cdot a)\triangleright v\big)^{-1}\tau(t,a)^{-1}(t\triangleright uv),\end{aligned}$$

and also $\bar{s}$ is given by

$$\begin{aligned}\bar{s} &= s^{'}\triangleleft\big(b\triangleright((t\triangleleft u)\triangleright v)\big) = \big((s\triangleleft\tau(t^L,t)^{-1})\cdot t^L\big)\triangleleft w^{-1}\big(b\triangleright((t\triangleleft u)\triangleright v)\big)\\ &= \big((s\triangleleft\tau(t^L,t)^{-1})\cdot t^L\big)\triangleleft\tau(t,a)\tau(b,t\triangleleft u)^{-1}\big(b\triangleright((t\triangleleft u)\triangleright v)\big)\\ &= \big((s\triangleleft\tau(t^L,t)^{-1})\cdot t^L\big)\triangleleft\tau(t,a)\big((t\cdot a)\triangleright v\big)\tau\big(b\triangleleft((t\triangleleft u)\triangleright v)\big),t\triangleleft uv\big)^{-1}.\end{aligned}$$

On the other hand if we put $\tilde{t} = \big(t\triangleleft(\langle\eta\rangle\triangleright v)\big) = \big(t\triangleleft(a\triangleright v)\big)$ and note that $\langle(\delta_s\otimes u)\bar{\triangleleft}v\rangle = a\triangleleft v$, then we get

$$\begin{aligned}R.H.S. &= \big(t\triangleleft(\langle\eta\rangle\triangleright v)\big)\bar{\triangleright}(\eta\bar{\triangleleft}v) = \big(t\triangleleft(a\triangleright v)\big)\bar{\triangleright}((\delta_s\otimes u)\bar{\triangleleft}v)\\ &= \big(t\triangleleft(a\triangleright v)\big)\bar{\triangleright}\big(\delta_{s\triangleleft(a\triangleright v)}\otimes(a\triangleright v)^{-1}uv\big) = \delta_{\hat{s}}\otimes\hat{u},\end{aligned}$$

where $\hat{u} = \tilde{w}\big((t\triangleleft(a\triangleright v))\triangleright((a\triangleright v)^{-1}uv)\big)$, $\hat{s} = \big((s\triangleleft(a\triangleright v)\tau(\tilde{t}^L,\tilde{t})^{-1})\cdot\tilde{t}^L\big)\triangleleft\tilde{w}^{-1}$ and

$$\begin{aligned}\tilde{w} &= \tau\big(\big\langle(t\triangleleft(a\triangleright v))\bar{\triangleright}(\eta\bar{\triangleleft}v)\big\rangle,\,(t\triangleleft(a\triangleright v))\triangleleft(a\triangleright v)^{-1}uv\big)\tau\big((t\triangleleft(a\triangleright v)),\,(a\triangleleft v)\big)^{-1}\\ &= \tau\big(\big\langle(t\bar{\triangleright}\eta)\bar{\triangleleft}((t\triangleleft u)\triangleright v)\big\rangle,\,t\triangleleft uv\big)\tau\big((t\triangleleft(a\triangleright v)),\,(a\triangleleft v)\big)^{-1}\\ &= \tau\big(\langle(t\bar{\triangleright}\eta)\rangle\triangleleft((t\triangleleft u)\triangleright v),\,t\triangleleft uv\big)\tau\big(\tilde{t},\,(a\triangleleft v)\big)^{-1}\\ &= \tau\big(b\triangleleft((t\triangleleft u)\triangleright v),\,t\triangleleft uv\big)\tau\big(\tilde{t},\,(a\triangleleft v)\big)^{-1}.\end{aligned}$$

To simplify $\hat{u}$ we need to calculate the following:

$$\big(t\triangleleft(a\triangleright v)\big)\triangleright\big((a\triangleright v)^{-1}uv\big) = \Big(\big(t\triangleleft(a\triangleright v)\big)\triangleright(a\triangleright v)^{-1}\Big)(t\triangleright uv) = \big(t\triangleright(a\triangleright v)\big)^{-1}(t\triangleright uv),$$

so $\hat{u}$ and $\hat{s}$ can be rewritten as

$$\hat{u} = \tau\big(b\triangleleft((t\triangleleft u)\triangleright v)\,,\, t\triangleleft uv\big)\tau\big(\tilde{t}\,,\,(a\triangleleft v)\big)^{-1}\big(t\triangleright(a\triangleright v)\big)^{-1}(t\triangleright uv)$$

$$\hat{s} = \big(\big(s\triangleleft(a\triangleright v)\tau(\tilde{t}^L,\tilde{t})^{-1}\big)\cdot\tilde{t}^L\big)\triangleleft\tau\big(\tilde{t}\,,\,(a\triangleleft v)\big)\tau\big(b\triangleleft((t\triangleleft u)\triangleright v)\,,\, t\triangleleft uv\big)^{-1}.$$

So to have $\bar{u} = \hat{u}$ we need the following equation to be true:

$$\tau\big(\tilde{t}\,,\,(a\triangleleft v)\big)^{-1}\big(t\triangleright(a\triangleright v)\big)^{-1} = \big((t\cdot a)\triangleright v\big)^{-1}\tau(t,a)^{-1}$$

which can be rewritten as

$$\big(t\triangleright(a\triangleright v)\big)^{-1} = \tau\big(\,t\triangleleft(\langle\eta\rangle\triangleright v)\,,\,(a\triangleleft v)\big)\big((t\cdot a)\triangleright v\big)^{-1}\tau(t,a)^{-1}$$

which is true according to the identities between $(M,\cdot)$ and τ. So we only need now to show that $\bar{s} = \hat{s}$, which can be shown if we have the following equality true:

$$\big(\big(s\triangleleft(a\triangleright v)\tau(\tilde{t}^L,\tilde{t})^{-1}\big)\cdot\tilde{t}^L\big)\triangleleft\tau\big(\tilde{t}\,,\,(a\triangleleft v)\big) = \big(\big(s\triangleleft\tau(t^L,t)^{-1}\big)\cdot t^L\big)\triangleleft\tau(t,a)\big((t\cdot a)\triangleright v\big)\,,$$

which can be equivalently written as

$$\begin{aligned}
\big(\big(s\triangleleft(a\triangleright v)\tau(\tilde{t}^L,\tilde{t})^{-1}\big)\cdot\tilde{t}^L\big) &= \big(\big(s\triangleleft\tau(t^L,t)^{-1}\big)\cdot t^L\big)\triangleleft\tau(t,a)\big((t\cdot a)\triangleright v\big)\tau\big(\,t\triangleleft(\langle\eta\rangle\triangleright v)\,,\,(a\triangleleft v)\big)^{-1} \\
&= \big(\big(s\triangleleft\tau(t^L,t)^{-1}\big)\cdot t^L\big)\triangleleft\big(t\triangleright(a\triangleright v)\big) \\
&= \Big(\big(s\triangleleft\tau(t^L,t)^{-1}\big)\triangleleft\big(t^L\triangleright(t\triangleright(a\triangleright v))\big)\Big)\cdot\Big(t^L\triangleleft\big(t\triangleright(a\triangleright v)\big)\Big) \\
&= \Big(s\triangleleft(a\triangleright v)\tau\big(t^L\triangleleft(t\triangleright(a\triangleright v))\,,\, t\triangleleft(a\triangleright v)\big)^{-1}\Big)\cdot\big(t^L\triangleleft(t\triangleright(a\triangleright v))\big)\,,
\end{aligned}$$

where $t^L\triangleright(t\triangleright(a\triangleright v)) = \tau(t^L,t)(a\triangleright v)\tau\big(t^L\triangleleft(t\triangleright(a\triangleright v))\,,\, t\triangleleft(a\triangleright v)\big)^{-1}$ was used in the last equality. To show that the above equality is true we only need to show that $\big(t\triangleleft(a\triangleright v)\big)^L =$

$t^L\triangleleft(t\triangleright(a\triangleright v))$. Put $(a\triangleright v) = v'$ then we want to show $(t\triangleleft v')^L = t^L\triangleleft(t\triangleright v')$, i.e. $\big(t^L\triangleleft(t\triangleright v')\big)\cdot(t\triangleleft v') = e$ which is true as

$$t^L\triangleleft(t\triangleright v')\cdot(t\triangleleft v') = (t^L\cdot t)\triangleleft v' = e\triangleleft v' = e\,.$$

Therefore, the $L.H.S. = R.H.S.$ as required. $\square$

Remark 3.2.6 *If $G \neq \{e\}$, then the algebra action $\bar{\triangleleft} : V\otimes A \to V$ is not a morphism in the category $\mathcal{D}$ as it does not preserve the G-grade. The G-grade of $\xi\bar{\triangleleft}(\delta_s\otimes u) = \delta_{s,\langle\xi\rangle}\xi\bar{\triangleleft}u$ is $|\xi\bar{\triangleleft}u| = (\langle\xi\rangle\triangleright u)^{-1}|\xi|u = (s\triangleright u)^{-1}|\xi|u$. On the other hand the G-grade of $\xi\otimes(\delta_s\otimes u)$ is $\tau(\langle\xi\rangle, a)^{-1}|\xi|u$ where $a = \langle\delta_s\otimes u\rangle$. To have a morphism in $\mathcal{D}$ we require that $(s\triangleright u) = \tau(\langle\xi\rangle, a)$. But we know that $s\cdot a = s\triangleleft u$ which implies that $(s\triangleright u)(s\cdot a) = (s\triangleright u)(s\triangleleft u) = su$. On the other hand we know that $\tau(\langle\xi\rangle, a)(s\cdot a) = sa$. So the equality only holds if $a = u \in G\cap M = \{e\}$.*

Proposition 3.2.7 *For $(\delta_s\otimes u), (\delta_t\otimes v)\in A$, the multiplication map $\mu : A\otimes A\to A$ which is defined by (see 1.3.8)*

$$\mu\big((\delta_s\otimes u)\otimes(\delta_t\otimes v)\big) = \delta_{t,s\triangleleft u}\,\delta_{s\triangleleft\tau(a,b)}\otimes\tau(a,b)^{-1}uv\,,$$

is a morphism in $\mathcal{D}$, where $t,s\in M$, $u,v\in G$, $a = \langle\delta_s\otimes u\rangle$ and $b = \langle\delta_t\otimes v\rangle$.

Proof. To prove that the map $\mu : A\otimes A\to A$ is a morphism in $\mathcal{D}$ we only need to prove that it preserves the G-grade and the M-action, as we already know that it is a morphism in $\mathcal{C}$. For the grading we start as

$$\big|(\delta_s\otimes u)\otimes(\delta_t\otimes v)\big| = \tau(\langle\delta_s\otimes u\rangle, \langle\delta_t\otimes v\rangle)^{-1}|\delta_s\otimes u|\,|\delta_t\otimes v| = \tau(a,b)^{-1}\,u\,v\,.$$

On the other hand

$$\left|\delta_{t,s\triangleleft u}\,\delta_{s\triangleleft\tau(a,b)}\otimes\tau(a,b)^{-1}\,u\,v\right| = \tau(a,b)^{-1}\,u\,v\,,$$

as required. So μ does preserve the G-grade. Next to show that μ preserves the M-action we start as follows

$$L.H.S. = \big(p\triangleleft\tau(a,b)\big)\bar{\triangleright}\big((\delta_s\otimes u)\otimes(\delta_t\otimes v)\big) = \big(p\bar{\triangleright}(\delta_s\otimes u)\big)\bar{\triangleleft}z\otimes(p\triangleleft u)\bar{\triangleright}(\delta_t\otimes v)\,,$$

where $z=\tau(p\triangleleft u,b)\tau\big(\langle(p\triangleleft u)\bar{\triangleright}(\delta_t\otimes v)\rangle,p\triangleleft uv\big)^{-1}$. On the other hand

$$R.H.S. = \big(p\triangleleft\tau(a,b)\big)\bar{\triangleright}\delta_{t,s\triangleleft u}\left(\delta_{s\triangleleft\tau(a,b)}\otimes\tau(a,b)^{-1}uv\right) = \delta_{t,s\triangleleft u}\,\delta_{s'''}\otimes u''',$$

where $u'''=w'''\Big(\big(p\triangleleft\tau(a,b)\big)\triangleright\tau(a,b)^{-1}uv\Big)$, $s'''=\bigg(\Big(s\triangleleft\tau(a,b)\tau\big((p\triangleleft\tau(a,b))^{L},(p\triangleleft\tau(a,b))\big)^{-1}\Big)\cdot$ $\big(p\triangleleft\tau(a,b)\big)^{L}\bigg)\triangleleft w'''^{-1}$, $w'''=\tau(c,p\triangleleft uv)\tau\big(p\triangleleft\tau(a,b),a\cdot b\big)^{-1}$, and c satisfies $\big(p\triangleleft\tau(a,b)\big)\cdot(a\cdot b)=$ $c\cdot(p\triangleleft uv)$. We want to get $L.H.S.=R.H.S.$, and to do so we start with the left hand side and do the following calculation

$$(p\triangleleft u)\bar{\triangleright}(\delta_t\otimes v)=(\delta_{t'}\otimes v')$$

where $v'=w'\big((p\triangleleft u)\triangleright v\big)$, $t'=\Big(\big(t\triangleleft\tau\big((p\triangleleft u)^{L},(p\triangleleft u)\big)^{-1}\big)\cdot(p\triangleleft u)^{L}\Big)\triangleleft w'^{-1}$, $w'=\tau(g,p\triangleleft uv)\tau(p\triangleleft u,b)^{-1}$ $=z^{-1}$ and $g=\langle\delta_{t'}\otimes v'\rangle$ which satisfies $(p\triangleleft u)\cdot b=g\cdot(p\triangleleft uv)$. Also

$$p\bar{\triangleright}(\delta_s\otimes u)=(\delta_{s'}\otimes u')$$

where $u'=w(p\triangleright u)$, $s'=\big((s\triangleleft\tau(p^L,p)^{-1})\cdot p^L\big)\triangleleft w^{-1}$, $w=\tau(h,p\triangleleft u)\tau(p,a)^{-1}$ and $h=$ $\langle\delta_{s'}\otimes u'\rangle$ which satisfies $p\cdot a=h\cdot(p\triangleleft u)$. Also we have

$$(\delta_{s'}\otimes u')\bar{\triangleleft}z=\delta_{s'\triangleleft(h\triangleright z)}\otimes(h\triangleright z)^{-1}u'z=\delta_{s''}\otimes u''\,,$$

with $\langle \delta_{s''} \otimes u'' \rangle = f = h \triangleleft z$. So we need to show that

$$\delta_{t,s\triangleleft u}\,\delta_{s'''} \otimes u''' = (\delta_{s''} \otimes u'')(\delta_{t'} \otimes v') = \delta_{t',s''\triangleleft u''}\,\delta_{s''\triangleleft\tau(f,g)} \otimes \tau(f,g)^{-1}u''v'$$

$$= \delta_{t',s'\triangleleft u'z}\,\delta_{s'\triangleleft(h\triangleright z)\tau(h\triangleleft z,g)} \otimes \tau(h\triangleleft z,g)^{-1}(h\triangleright z)^{-1}u'z\,v'$$

$$= \delta_{t',s'\triangleleft u'z}\,\delta_{\hat{s}} \otimes \hat{u}\,.$$

Now we check the δ function as follows $\delta_{t',s'\triangleleft u'z} = 1 \Leftrightarrow \big((s\triangleleft\tau(p^L,p)^{-1})\cdot p^L\big)\triangleleft(p\triangleright u) = \big(t\triangleleft\tau\big((p\triangleleft u)^L,(p\triangleleft u)\big)^{-1}\big)\cdot(p\triangleleft u)^L \Leftrightarrow s\triangleleft\tau(p^L,p)^{-1}\big(p^L\triangleright(p\triangleright u)\big) = t\triangleleft\tau\big((p\triangleleft u)^L,(p\triangleleft u)\big)^{-1} \Leftrightarrow$ $s\triangleleft u = t$, i.e. to have a non-zero answer we must have $s\triangleleft u = t$. This calculation was done knowing that $(p\triangleleft u)^L = p^L\triangleleft(p\triangleright u)$. So we only need now to show that $u''' = \hat{u}$ and $s''' = \hat{s}$. We start with u''' as follows

$$u''' = \tau(c,p\triangleleft uv)\tau\big(p\triangleleft\tau(a,b),a\cdot b\big)^{-1}\Big(\big(p\triangleleft\tau(a,b)\big)\triangleright\tau(a,b)^{-1}uv\Big)$$

$$= \tau(c,p\triangleleft uv)\tau\big(p\triangleleft\tau(a,b),a\cdot b\big)^{-1}\big(p\triangleright\tau(a,b)\big)^{-1}(p\triangleright uv)$$

$$= \tau(c,p\triangleleft uv)\tau(p\cdot a,b)^{-1}\tau(p,a)^{-1}(p\triangleright uv)$$

$$= \tau(c,p\triangleleft uv)\big((p\cdot a)\cdot b\big)b^{-1}(p\cdot a)^{-1}\tau(p,a)^{-1}(p\triangleright uv)\,.$$

On the other hand

$$\hat{u} = \tau(h\triangleleft z,g)^{-1}(h\triangleright z)^{-1}u'z\,v'$$

$$= \tau(h\triangleleft z,g)^{-1}(h\triangleright z)^{-1}u'\big((p\triangleleft u)\triangleright v\big)$$

$$= \tau(h\triangleleft z,g)^{-1}(h\triangleright z)^{-1}w(p\triangleright u)\big((p\triangleleft u)\triangleright v\big)$$

$$= \tau(h\triangleleft z,g)^{-1}(h\triangleright z)^{-1}\tau(h,p\triangleleft u)\tau(p,a)^{-1}(p\triangleright uv)\,.$$

So we need to show that

$$\tau(c,p\triangleleft uv)\big((p\cdot a)\cdot b\big)b^{-1}(p\cdot a)^{-1} = \tau(h\triangleleft z,g)^{-1}(h\triangleright z)^{-1}\tau(h,p\triangleleft u)\,, \tag{3.1}$$

but

$$\tau(h\triangleleft z,g)^{-1}(h\triangleright z)^{-1}\tau(h,p\triangleleft u)=\big((h\triangleleft z)\cdot g\big)g^{-1}(h\triangleleft z)^{-1}(h\triangleright z)^{-1}h(p\triangleleft u)\big(h\cdot(p\triangleleft u)\big)^{-1}$$
$$=\big((h\triangleleft z)\cdot g\big)g^{-1}z^{-1}(p\triangleleft u)(p\cdot a)^{-1}.$$

So we need now to show that

$$\tau(c,p\triangleleft uv)\big((p\cdot a)\cdot b\big)b^{-1}=\big((h\triangleleft z)\cdot g\big)g^{-1}z^{-1}(p\triangleleft u)\,, \tag{3.2}$$

and to do so we do the following calculation

$$\big((h\triangleleft z)\cdot g\big)\cdot(p\triangleleft uv)=\big(h\triangleleft\tau(p\triangleleft u,b)\big)\cdot\big(g\cdot(p\triangleleft uv)\big)=\big(h\triangleleft\tau(p\triangleleft u,b)\big)\cdot\big((p\triangleleft u)\cdot b\big)$$
$$=\big(h\cdot(p\triangleleft u)\big)\cdot b=(p\cdot a)\cdot b=c\cdot(p\triangleleft uv)\,,$$

so $c=(h\triangleleft z)\cdot g$. Now we recalculate the right hand side of (3.2) as follows

$$\big((h\triangleleft z)\cdot g\big)g^{-1}z^{-1}(p\triangleleft u)=c(p\triangleleft uv)(p\triangleleft uv)^{-1}g^{-1}z^{-1}(p\triangleleft u)$$
$$=\tau(c,p\triangleleft uv)\big(c\cdot(p\triangleleft uv)\big)(p\triangleleft uv)^{-1}g^{-1}z^{-1}(p\triangleleft u)$$
$$=\tau(c,p\triangleleft uv)\big(c\cdot(p\triangleleft uv)\big)\big(g\cdot(p\triangleleft uv)\big)^{-1}\tau(g,p\triangleleft uv)^{-1}z^{-1}(p\triangleleft u)$$
$$=\tau(c,p\triangleleft uv)\big(c\cdot(p\triangleleft uv)\big)\big((p\triangleleft u)\cdot b\big)^{-1}\tau(p\triangleleft u,b)^{-1}(p\triangleleft u)$$
$$=\tau(c,p\triangleleft uv)\big(c\cdot(p\triangleleft uv)\big)b^{-1}\,,$$

which is the same as the left hand side of (3.2) as required. The last thing is to show that $s'''=\hat{s}$, i.e. we want to show that

$$\Big(\Big(s\triangleleft\tau(a,b)\tau\big((p\triangleleft\tau(a,b))^{L},(p\triangleleft\tau(a,b))\big)^{-1}\Big)\cdot\big(p\triangleleft\tau(a,b)\big)^{L}\Big)\triangleleft w'''^{-1}=s'\triangleleft(h\triangleright z)\tau(h\triangleleft z,g)\,,$$

or, equivalently

$$\Big(s\triangleleft\tau(a,b)\tau\big((p\triangleleft\tau(a,b))^{L},(p\triangleleft\tau(a,b))\big)^{-1}\Big)\cdot\big(p\triangleleft\tau(a,b)\big)^{L}=s'\triangleleft(h\triangleright z)\tau(h\triangleleft z,g)w'''\,, \tag{3.3}$$

but

$$s'\triangleleft(h\triangleright z)\tau(h\triangleleft z,g)w''' = \big(\big(s\triangleleft\tau(p^L,p)^{-1}\big)\cdot p^L\big)\triangleleft w^{-1}(h\triangleright z)\tau(h\triangleleft z,g)w''' . \tag{3.4}$$

To simplify (3.4) we do the following calculations

$$\begin{aligned}
w^{-1}(h\triangleright z)\tau(h\triangleleft z,g)w''' &= w^{-1}(h\triangleright z)(h\triangleleft z)g\big((h\triangleleft z)\cdot g\big)^{-1}w''' \\
&= w^{-1}hzgc^{-1}\tau(c,p\triangleleft uv)\tau\big(p\triangleleft\tau(a,b),a\cdot b\big)^{-1} \\
&= w^{-1}hzg(p\triangleleft uv)\big(c\cdot(p\triangleleft uv)\big)^{-1}\tau\big(p\triangleleft\tau(a,b),a\cdot b\big)^{-1} \\
&= w^{-1}h\tau(p\triangleleft u,b)\big(g\cdot(p\triangleleft uv)\big)\big((p\triangleleft\tau(a,b))\cdot(a\cdot b)\big)^{-1}\tau\big(p\triangleleft\tau(a,b),a\cdot b\big)^{-1} \\
&= w^{-1}h\tau(p\triangleleft u,b)\big((p\triangleleft u)\cdot b\big)\big((p\triangleleft\tau(a,b))\cdot(a\cdot b)\big)^{-1}\tau\big(p\triangleleft\tau(a,b),a\cdot b\big)^{-1} \\
&= w^{-1}h(p\triangleleft u)b\big((p\triangleleft\tau(a,b))\cdot(a\cdot b)\big)^{-1}\tau\big(p\triangleleft\tau(a,b),a\cdot b\big)^{-1} \\
&= \tau(p,a)\big(h\cdot(p\triangleleft u)\big)b\big((p\cdot a)\cdot b\big)^{-1}\tau\big(p\triangleleft\tau(a,b),a\cdot b\big)^{-1} \\
&= \tau(p,a)(p\cdot a)b\big((p\cdot a)\cdot b\big)^{-1}\tau\big(p\triangleleft\tau(a,b),a\cdot b\big)^{-1} \\
&= \tau(p,a)\tau(p\cdot a\,,\,b)\tau\big(p\triangleleft\tau(a,b),a\cdot b\big)^{-1} \\
&= p\triangleright\tau(a,b)\,.
\end{aligned}$$

So if we substitute in (3.4) then in (3.3) we only need to show that

$$\Big(s\triangleleft\tau(a,b)\tau\big((p\triangleleft\tau(a,b))^L,(p\triangleleft\tau(a,b))\big)^{-1}\Big)\cdot(p\triangleleft\tau(a,b))^L = \big(\big(s\triangleleft\tau(p^L,p)^{-1}\big)\cdot p^L\big)\triangleleft(p\triangleright\tau(a,b))\,,$$

which is true since $\tau(a,b)\tau\big((p\triangleleft\tau(a,b))^L,(p\triangleleft\tau(a,b))\big)^{-1} = \tau(p^L,p)^{-1}\big(p^L\triangleright(p\triangleright\tau(a,b))\big)$, and $\big(p\triangleleft\tau(a,b)\big)^L = \big(p^L\triangleleft(p\triangleright\tau(a,b))\big)$. Therefore $s''' = \hat{s}$ as required. □

It is now natural to ask if A is actually a braided Hopf algebra in $\mathcal{C}$.

Proposition 3.2.8 *There is an identity I for the multiplication μ in the category $\mathcal{D}$ given by*

$$I = \sum_{t\in M}\delta_t\otimes e\,.$$

Proof. We already knew (from 1.3.8) that I is a morphism : $k \longrightarrow A$ in the category $\mathcal{C}$ with grade $\langle I \rangle = e$. So we only need to show that it preserves the M-action and the G-grade. For some $s, t \in M$, we check the M-action as

$$s\bar{\triangleright}(\delta_t \otimes e) = \delta_{t'} \otimes u' ,$$

where $u' = w(s{\triangleright}e) = we = w$, $w = \tau(b\,,\, s{\triangleleft}e)\,\tau(s\,,\, \langle \delta_t \otimes e\rangle)^{-1} = \tau(b\,,\, s)\,\tau(s\,,\, e)^{-1}$ and $t' = \big(t{\triangleleft}\tau(s^L\,,\, s)^{-1} \cdot s^L\big){\triangleleft}w^{-1}$ where b satisfies $s \cdot e = b \cdot (s{\triangleleft}e) \Leftrightarrow s = b \cdot s \Leftrightarrow s \cdot s^R = (b \cdot s) \cdot s^R \Leftrightarrow e = b{\triangleleft}\tau(s\,,\, s^R) \Leftrightarrow e{\triangleleft}\tau(s\,,\, s^R)^{-1} = b \Leftrightarrow e = b$. Hence, $w = e$, which, by then, implies that $u' = e$. Thus $t' = \big(t{\triangleleft}\tau(s^L\,,\, s)^{-1} \cdot s^L\big){\triangleleft}w^{-1} = t' = t{\triangleleft}\tau(s^L\,,\, s)^{-1} \cdot s^L$. If we apply $\cdot s$ to both sides we get $t' \cdot s = \big(t{\triangleleft}\tau(s^L\,,\, s)^{-1} \cdot s^L\big) \cdot s$, which implies that $t' \cdot s = t$, which means that there is a $1-1$ correspondence between t and t' which, by then, means that there will be no repeating in the sum over t'. Therefore,

$$s{\triangleright} \sum_{t \in M} \delta_t \otimes e = \sum_{t \in M} \delta_{t'} \otimes e = I\,,$$

which means that the M-action is preserved. Now note that $|\delta_t \otimes e| = |\delta_{t'} \otimes e| = e$, i.e. the morphism preserves the G-grade. $\square$

Proposition 3.2.9 *The multiplication μ on A is braided commutative, i.e. for $(\delta_s \otimes u)$, $(\delta_t \otimes v) \in A$ the following equality is satisfied*

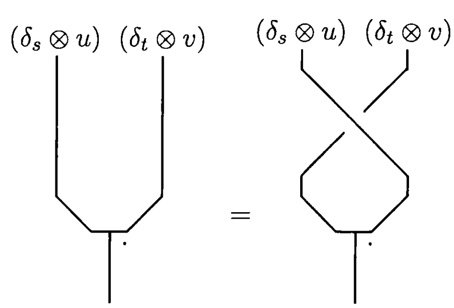


figure 3.3

Proof. Putting $\langle \delta_s \otimes u \rangle = a$ and $\langle \delta_t \otimes v \rangle = b$, we start with the right hand side by calculating the following

$$\begin{aligned}\Psi\big((\delta_s \otimes u) \otimes (\delta_t \otimes v)\big) &= \langle \delta_s \otimes u \rangle \bar{\triangleright} (\delta_t \otimes v) \otimes (\delta_s \otimes u) \bar{\triangleleft} |\delta_t \otimes v| \\ &= a \bar{\triangleright} (\delta_t \otimes v) \otimes (\delta_s \otimes u) \bar{\triangleleft} v \\ &= (\delta_{\bar{t}} \otimes \bar{v}) \otimes \big(\delta_{s \triangleleft (a \triangleright v)} \otimes (a \triangleright v)^{-1} uv\big) ,\end{aligned}$$

where $\bar{v} = w(a \triangleright v)$, $\bar{t} = \Big(\big(t \triangleleft \tau(a^L, a)^{-1}\big) \cdot a^L\Big) \triangleleft w^{-1}$, $w = \tau(c, a \triangleleft v)\tau(a, b)^{-1}$, $c = \langle a \bar{\triangleright} (\delta_t \otimes v) \rangle$ and satisfies $a \cdot b = c \cdot (a \triangleright v)$. So the right hand side is given by

$$\begin{aligned} R.H.S. = \mu\Big(\Psi\big((\delta_s \otimes u) \otimes (\delta_t \otimes v)\big)\Big) &= \delta_{s \triangleleft (a \triangleright v), \bar{t} \triangleleft \bar{v}}\, \delta_{\bar{t} \triangleleft \tau(c, a \triangleleft v)} \otimes \tau(c, a \triangleleft v)^{-1}\, \bar{v}(a \triangleright v)^{-1} uv \\ &= \delta_{s \triangleleft (a \triangleright v), \left(\left(t \triangleleft \tau(a^L, a)^{-1}\right) \cdot a^L\right) \triangleleft (a \triangleright v)}\, \delta_{\left(\left(t \triangleleft \tau(a^L, a)^{-1}\right) \cdot a^L\right) \triangleleft \tau(a, b)} \otimes \tau(a, b)^{-1}\, (a \triangleright v)(a \triangleright v)^{-1} uv \\ &= \delta_{s, \left(\left(t \triangleleft \tau(a^L, a)^{-1}\right) \cdot a^L\right)}\, \delta_{\left(\left(t \triangleleft \tau(a^L, a)^{-1}\right) \cdot a^L\right) \triangleleft \tau(a, b)} \otimes \tau(a, b)^{-1}\, uv \\ &= \delta_{s, \left(\left(t \triangleleft \tau(a^L, a)^{-1}\right) \cdot a^L\right)}\, \delta_{s \triangleleft \tau(a, b)} \otimes \tau(a, b)^{-1}\, uv \\ &= \delta_{s \cdot a, t}\, \delta_{s \triangleleft \tau(a, b)} \otimes \tau(a, b)^{-1}\, uv \\ &= \delta_{s \triangleleft u, t}\, \delta_{s \triangleleft \tau(a, b)} \otimes \tau(a, b)^{-1}\, uv \,. \end{aligned}$$

On the other hand we have

$$L.H.S. = \mu\big((\delta_s \otimes u) \otimes (\delta_t \otimes v)\big) = \delta_{s \triangleleft u, t}\, \delta_{s \triangleleft \tau(a, b)} \otimes \tau(a, b)^{-1}\, uv \,. \quad \square$$

Proposition 3.2.10 *Let A be the algebra defined in 3.2.1, then for $a = (\delta_s \otimes u) \in A$, the non-braided comultiplication $\bar{\Delta} : A \to A \otimes A$ which is defined by $\bar{\Delta}(a) = \Sigma a_{(1)} \otimes a_{(2)}$, where $(\xi \otimes \eta) \bar{\triangleleft} a = \Sigma \xi \bar{\triangleleft} a_{(1)} \otimes \eta \bar{\triangleleft} a_{(2)}$, can be given by the following formula*

$$\bar{\Delta}(a) = \bar{\Delta}(\delta_s \otimes u) = \sum_{s_1, s_2 \in M \text{ and } s_1 \cdot s_2 = s} \delta_{s_1} \otimes (s_2 \triangleright u) \otimes \delta_{s_2} \otimes u \,.$$

Proof. We know that

$$(\xi \otimes \eta)\bar{\triangleleft}(\delta_s \otimes u) = \delta_{s,\langle\xi\rangle\cdot\langle\eta\rangle}\,(\xi \otimes \eta)\bar{\triangleleft} u = \delta_{s,\langle\xi\rangle\cdot\langle\eta\rangle}\,\xi\bar{\triangleleft}(\langle\eta\rangle\triangleright u) \otimes \eta\bar{\triangleleft} u\,.$$

On the other hand for $a_{(1)} \otimes a_{(2)} = \sum \delta_{s_1} \otimes u_1 \otimes \delta_{s_2} \otimes u_2$ where $s_1, s_2 \in M$ and $u_1, u_2 \in G$, we have

$$\xi\bar{\triangleleft}\, a_{(1)} \otimes \eta\bar{\triangleleft}\, a_{(2)} = \sum \delta_{s_1,\langle\xi\rangle}\xi\bar{\triangleleft} u_1 \otimes \delta_{s_2,\langle\eta\rangle}\eta\bar{\triangleleft} u_2\,.$$

To have a non-zero answer we must have $s_1 = \langle\xi\rangle$, $s_2 = \langle\eta\rangle$. Comparing with the first equation, we obtain $s_1 \cdot s_2 = s$, $u_2 = u$ and $s_2 \triangleright u = u_1$ and we get the following

$$a_{(1)} \otimes a_{(2)} = \sum \delta_{s_1} \otimes u_1 \otimes \delta_{s_2} \otimes u_2 = \sum_{s_1, s_2 \in M \text{ and } s_1 \cdot s_2 = s} \delta_{s_1} \otimes (s_2 \triangleright u) \otimes \delta_{s_2} \otimes u\,. \quad \square$$

Proposition 3.2.11 *Let A be the algebra defined in 3.2.1 and let V and W be representations of A. Then for $a = (\delta_s \otimes u) \in A$, the braided coproduct $\Delta : A \to A \otimes A$ which is defined by*

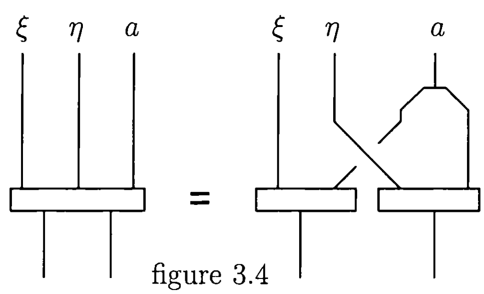


figure 3.4

where $\xi \in V$ and $\eta \in W$, can be given by the following formula, where $I = \sum_t \delta_t \otimes e$:

$$\Delta(a) = \Delta(\delta_s \otimes u) = (\delta_s \otimes u) \otimes I\,.$$

Proof. If we put $\Delta(a) = a_1 \otimes a_2$ where $a_1 = \delta_{t_1} \otimes u_1$ and $a_2 = \delta_{t_2} \otimes u_2$, then

$$
\begin{aligned}
(\xi \otimes \eta) \otimes (a_1 \otimes a_2) &\overset{\Phi^{-1}}{\longmapsto} \big((\xi \otimes \eta)\bar{\triangleleft}\tau(\langle a_1\rangle, \langle a_2\rangle)^{-1} \otimes a_1\big) \otimes a_2 \\
&= \big((\xi\bar{\triangleleft}(\langle\eta\rangle\triangleright w) \otimes (\eta\bar{\triangleleft}w)\big) \otimes a_1\big) \otimes a_2 \\
&\overset{\Phi\otimes\mathrm{id}}{\longmapsto} \big(\xi\bar{\triangleleft}(\langle\eta\rangle\triangleright w)\,\tau(\langle\eta\rangle\triangleleft w, \langle a_1\rangle) \otimes \big((\eta\bar{\triangleleft}w) \otimes a_1\big)\big) \otimes a_2 \\
&\overset{(\mathrm{id}\otimes\Psi)\otimes\mathrm{id}}{\longmapsto} \big(\xi\bar{\triangleleft}(\langle\eta\rangle\triangleright w)\,\tau(\langle\eta\rangle\triangleleft w, \langle a_1\rangle) \otimes \big((\langle\eta\rangle\triangleleft w)\bar{\triangleright}a_1 \otimes \eta\bar{\triangleleft}w|a_1|\big)\big) \otimes a_2 \\
&\overset{\Phi^{-1}\otimes\mathrm{id}}{\longmapsto} \big(\big(\xi\bar{\triangleleft}(\langle\eta\rangle\triangleright w)\,z\,x \otimes (\langle\eta\rangle\triangleleft w)\bar{\triangleright}a_1\big) \otimes \eta\bar{\triangleleft}w|a_1|\big) \otimes a_2 \\
&= \big(\big(\xi\bar{\triangleleft}(\langle\eta\rangle\triangleright w)\,z\,x \otimes a_1'\big) \otimes \eta'\big) \otimes a_2\,,
\end{aligned}
$$

where $w = \tau(\langle a_1\rangle, \langle a_2\rangle)^{-1}, z = \tau(\langle\eta\rangle\triangleleft w, \langle a_1\rangle), x = \tau(\langle a_1'\rangle, \langle\eta'\rangle)^{-1}, a_1' = (\langle\eta\rangle\triangleleft w)\bar{\triangleright}a_1$ and $\eta' = \eta\bar{\triangleleft}w|a_1|$. We first calculate $\big(\xi\bar{\triangleleft}(\langle\eta\rangle\triangleright w)\,z\,x\big)\bar{\triangleleft}a_1'$ starting with calculating a_1' as follows

$$a_1' = (\langle\eta\rangle\triangleleft w)\bar{\triangleright}a_1 = \bar{t}\bar{\triangleright}(\delta_{t_1} \otimes u_1) = \delta_{\hat{t}} \otimes \hat{u}\,,$$

where $\hat{u} = \bar{w}(\bar{t}\triangleright u_1),\ \hat{t} = \big((t_1\triangleleft\tau(\bar{t}^L, \bar{t})^{-1}) \cdot \bar{t}^L\big)\triangleleft\bar{w}^{-1}, \bar{w} = \tau(\langle a_1'\rangle, \bar{t}\triangleleft u_1)\tau(\bar{t}, \langle a_1\rangle)^{-1} = x^{-1}\,z^{-1}$, $\bar{t} = \langle\eta\rangle\triangleleft w$ and $\bar{t} \cdot \langle a_1\rangle = \langle a_1'\rangle \cdot \langle\eta'\rangle$. So $\big(\xi\bar{\triangleleft}(\langle\eta\rangle\triangleright w)\,z\,x\big)\bar{\triangleleft}a_1'$ can be calculated as

$$
\begin{aligned}
\big(\xi\bar{\triangleleft}(\langle\eta\rangle\triangleright w)\,z\,x\big)\bar{\triangleleft}a_1' &= \big(\xi\bar{\triangleleft}(\langle\eta\rangle\triangleright w)\,z\,x\big)\bar{\triangleleft}\big(\delta_{\hat{t}} \otimes \hat{u}\,\big) \\
&= \delta_{\hat{t}\,,\,\langle\xi\rangle\triangleleft(\langle\eta\rangle\triangleright w)\,z\,x}\,\xi\bar{\triangleleft}(\langle\eta\rangle\triangleright w)\,z\,x\hat{u} \\
&= \delta_{\hat{t}\,,\,\langle\xi\rangle\triangleleft(\langle\eta\rangle\triangleright w)\,z\,x}\,\xi\bar{\triangleleft}(\langle\eta\rangle\triangleright w)\,z\,x\bar{w}(\bar{t}\triangleright u_1) \\
&= \delta_{(t_1\triangleleft\tau(\bar{t}^L,\bar{t})^{-1})\cdot\bar{t}^L\,,\,\langle\xi\rangle\triangleleft(\langle\eta\rangle\triangleright w)}\,\xi\bar{\triangleleft}(\langle\eta\rangle\triangleright w)\,(\bar{t}\triangleright u_1) \\
&= \delta_{((t_1\triangleleft\tau(\bar{t}^L,\bar{t})^{-1})\cdot\bar{t}^L)\cdot\bar{t}\,,\,(\langle\xi\rangle\triangleleft(\langle\eta\rangle\triangleright w))\cdot\bar{t}}\ \xi\bar{\triangleleft}(\langle\eta\rangle\triangleright w)\,(\bar{t}\triangleright u_1) \\
&= \delta_{t_1\,,\,(\langle\xi\rangle\triangleleft(\langle\eta\rangle\triangleright w))\cdot(\langle\eta\rangle\triangleleft w)}\,\xi\bar{\triangleleft}(\langle\eta\rangle\triangleright w)\,((\langle\eta\rangle\triangleleft w)\triangleright u_1) \\
&= \delta_{t_1\,,\,(\langle\xi\rangle\cdot\langle\eta\rangle)\triangleleft w}\,\xi\bar{\triangleleft}(\langle\eta\rangle\triangleright wu_1)\ .
\end{aligned}
$$

We now calculate $\eta'\bar{\triangleleft}a_2$ as follows

$$\eta'\bar{\triangleleft}a_2 = \eta\bar{\triangleleft}wu_1\bar{\triangleleft}(\delta_{t_2} \otimes u_2) = \delta_{t_2\,,\,\langle\eta\rangle\triangleleft wu_1}\,\eta\bar{\triangleleft}wu_1u_2\,.$$

On the other hand we know from proposition 3.2.10 that

$$\eta\bar{\triangleleft}(\delta_{s_2}\otimes u)=\delta_{s_2\,,\langle\eta\rangle}\,\eta\bar{\triangleleft}u \quad \text{and} \quad \xi\bar{\triangleleft}(\delta_{s_1}\otimes(s_2\triangleright u))=\delta_{s_1\,,\langle\xi\rangle}\,\xi\bar{\triangleleft}(s_2\triangleright u)\,,$$

so $s_1=\langle\xi\rangle$ and $s_2=\langle\eta\rangle$ and hence from both cases we get

$$t_2=s_2\triangleleft wu_1 \qquad \text{and} \qquad u=wu_1u_2\,, \tag{3.5}$$

also we get

$$t_1=(s_1\cdot s_2)\triangleleft w=s\triangleleft w \qquad \text{and} \qquad s_2\triangleright u=\langle\eta\rangle\triangleright wu_1 \ \text{ or } \ u=wu_1\,. \tag{3.6}$$

Combining (3.5) and (3.6) gives that $u_2=e$. We know that $t_2\triangleleft u_2=t_2\triangleleft e=t_2=t_2\cdot\langle a_2\rangle$, which implies that $\langle a_2\rangle=e$ which by then implies that $w=e$. Therefore

$$\Delta(a)=\Delta(\delta_s\otimes u)=\sum_{s_1\cdot s_2=s}(\delta_s\otimes u)\otimes(\delta_{s_2\triangleleft u}\otimes e)$$
$$=(\delta_s\otimes u)\otimes I\,. \quad \square$$

Definition 3.2.12 *The map $\epsilon:A\to k$ is given by the action of A on the unit object $k\in\mathcal{C}$ and is $\epsilon(\delta_s\otimes u)=\delta_{s,e}$, for $(\delta_s\otimes u)\in A$, $s\in M$ and $u\in G$.*

Proposition 3.2.13 *The map $\epsilon:A\to k$ is a morphism in $\mathcal{C}$ and an algebra map. However it is not a morphism in $\mathcal{D}$ if A is considered in $\mathcal{D}$ according to 3.2.1.*

Proof. This map is in the category $\mathcal{C}$, i.e. it does preserve the M-grade and the G-action which can be shown as

$$\epsilon\big((\delta_s\otimes u)\bar{\triangleleft}v\big)=\epsilon\big((\delta_{s\triangleleft(a\triangleright v)}\otimes(a\triangleright v)^{-1}uv)\big)=\delta_{s\triangleleft(a\triangleright v),e}=\delta_{s,e\triangleleft(a\triangleright v)^{-1}}=\delta_{s,e}\,,$$

where $v\in G$ and $a=\langle\delta_s\otimes u\rangle$. On the other hand

$$\big(\epsilon(\delta_s\otimes u)\big)\triangleleft v=(\delta_{s,e})\triangleleft v=\delta_{s,e}\,,$$

which just means that ϵ does preserve the G-action. Now if $\epsilon(\delta_s \otimes u) \neq 0$, then $\langle \delta_s \otimes u \rangle = e$, which means that the map ϵ preserves the M-grade. It is clear that it does not preserve the G-grade as for all $u \in G$ we have $\epsilon(\delta_s \otimes u) = \delta_{s,e}$. This map is multiplicative as for $(\delta_s \otimes u) \otimes (\delta_t \otimes v) = \delta_{t,s \triangleleft u}\, \delta_{s \triangleleft \tau(a,b)} \otimes \tau(a,b)^{-1}uv$, where $a = \langle \delta_s \otimes u \rangle$ and $b = \langle \delta_t \otimes v \rangle$ we have $\epsilon(\delta_s \otimes u)\, \epsilon(\delta_t \otimes v) = \delta_{s,e}\, \delta_{t,e}$. On the other hand we also have

$$\delta_{t,s \triangleleft u}\, \epsilon(\delta_{s \triangleleft \tau(a,b)} \otimes \tau(a,b)^{-1}uv) = \delta_{t,s \triangleleft u}\, \delta_{s \triangleleft \tau(a,b),e} = \delta_{t,s \triangleleft u}\, \delta_{s,e \triangleleft \tau(a,b)^{-1}} = \delta_{t,s \triangleleft u}\, \delta_{s,e} = \delta_{t,e}\, \delta_{s,e}\,.$$

Proposition 3.2.14 *For* $(\delta_s \otimes u)$, $(\delta_t \otimes v) \in A$ *the following equality is satisfied:*

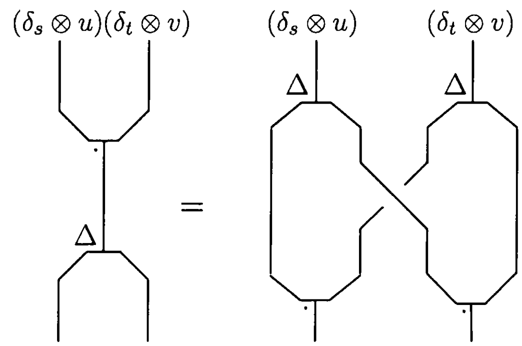


figure 3.5

Proof. We start with the right hand side following the diagram. Consider the following

$$\sum_{c,d \in M} \big((\delta_s \otimes u) \otimes (\delta_c \otimes e)\big) \otimes \big((\delta_t \otimes v) \otimes (\delta_d \otimes e)\big)\,.$$

Putting $\langle \delta_s \otimes u \rangle = a$ and $\langle \delta_t \otimes v \rangle = b$ and applying the associator gives

$$\sum_{c,d \in M} (\delta_s \otimes u) \bar{\triangleleft} \tau(e,b) \otimes \Big((\delta_c \otimes e) \otimes \big((\delta_t \otimes v) \otimes (\delta_d \otimes e)\big)\Big)$$

$$= \sum_{c,d \in M} (\delta_s \otimes u) \otimes \Big((\delta_c \otimes e) \otimes \big((\delta_t \otimes v) \otimes (\delta_d \otimes e)\big)\Big)\,,$$

where $\langle \delta_c \otimes e \rangle = e$, $\langle \delta_d \otimes e \rangle = e$ and $\tau(e,b) = e$. Now we apply $\mathrm{id} \otimes \Phi^{-1}$ to get

$$\begin{aligned} &\sum_{c,d\in M} (\delta_s \otimes u) \otimes \Big(\big((\delta_c \otimes e) \otimes (\delta_t \otimes v) \bar{\triangleleft} \tau(e,b)^{-1} \big) \otimes (\delta_d \otimes e) \Big) \\ &= \sum_{c,d\in M} (\delta_s \otimes u) \otimes \Big(\big((\delta_c \otimes e) \otimes (\delta_t \otimes v) \big) \otimes (\delta_d \otimes e) \Big) , \end{aligned} \tag{3.7}$$

where $\tau(e,b) = e$. Now we calculate

$$\Psi\big((\delta_c \otimes e) \otimes (\delta_t \otimes v) \big) = e \bar{\triangleright} (\delta_t \otimes v) \otimes (\delta_c \otimes e) \bar{\triangleleft} v = (\delta_t \otimes v) \otimes (\delta_{c \triangleleft v} \otimes e) .$$

We now substitute in 3.7 to get

$$\sum_{c,d\in M} (\delta_s \otimes u) \otimes \Big(\big((\delta_t \otimes v) \otimes (\delta_{c\triangleleft v} \otimes e) \big) \otimes (\delta_d \otimes e) \Big) .$$

Next we apply $\mathrm{id} \otimes \Phi$ to get

$$\sum_{c,d\in M} (\delta_s \otimes u) \otimes \Big((\delta_t \otimes v) \otimes \big((\delta_{c\triangleleft v} \otimes e) \otimes (\delta_d \otimes e) \big) \Big) .$$

Applying Φ^{-1} gives

$$\begin{aligned} &\sum_{c,d\in M} \big((\delta_s \otimes u) \bar{\triangleleft} \tau(b,e)^{-1} \otimes (\delta_t \otimes v) \big) \otimes \big((\delta_{c\triangleleft v} \otimes e) \otimes (\delta_d \otimes e) \big) \\ &= \sum_{c,d\in M} \big((\delta_s \otimes u) \otimes (\delta_t \otimes v) \big) \otimes \big((\delta_{c\triangleleft v} \otimes e) \otimes (\delta_d \otimes e) \big) . \end{aligned}$$

Now applying the multiplication map gives

$$\begin{aligned} &\sum_{c,d\in M} \big(\delta_{t,s\triangleleft u}\, \delta_{s\triangleleft\tau(a,b)} \otimes \tau(a,b)^{-1} uv \big) \otimes \big(\delta_{d,c\triangleleft v}\, \delta_{c\triangleleft v} \otimes e \big) \\ &= \sum_{c,d\in M} \big(\delta_{t,s\triangleleft u}\, \delta_{s\triangleleft\tau(a,b)} \otimes \tau(a,b)^{-1} uv \big) \otimes \big(\delta_{d,c}\, \delta_c \otimes e \big) \\ &= \sum_{c\in M} \big(\delta_{t,s\triangleleft u}\, \delta_{s\triangleleft\tau(a,b)} \otimes \tau(a,b)^{-1} uv \big) \otimes \big(\delta_c \otimes e \big) \\ &= \big(\delta_{t,s\triangleleft u}\, \delta_{s\triangleleft\tau(a,b)} \otimes \tau(a,b)^{-1} uv \big) \otimes \sum_{c\in M} \big(\delta_c \otimes e \big) \\ &= \big(\delta_{t,s\triangleleft u}\, \delta_{s\triangleleft\tau(a,b)} \otimes \tau(a,b)^{-1} uv \big) \otimes I . \end{aligned}$$

So the right hand side of the equality is given by

$$R.H.S. = \left(\delta_{t,s\triangleleft u}\, \delta_{s\triangleleft\tau(a,b)} \otimes \tau(a,b)^{-1}uv \right) \otimes I\,.$$

On the other hand the left hand side is given by

$$L.H.S. = \Delta\big((\delta_s \otimes u)(\delta_t \otimes v)\big) = \Delta\left(\delta_{t,s\triangleleft u}\, \delta_{s\triangleleft\tau(a,b)} \otimes \tau(a,b)^{-1}uv \right)$$

$$= \left(\delta_{t,s\triangleleft u}\, \delta_{s\triangleleft\tau(a,b)} \otimes \tau(a,b)^{-1}uv \right) \otimes I\,. \quad \square$$

Remark 3.2.15 *As the proposed counit $\epsilon : A \to k$ is not a morphism in $\mathcal{D}$, the best we can hope for is that A is a braided Hopf algebra in $\mathcal{C}$. However we see that not all the axioms are satisfied. If we apply this map to the coproduct defined in the previous proposition we get the following*

$$(\mathrm{id} \otimes \epsilon)\, \Delta(\delta_s \otimes \mathrm{u}) = (\mathrm{id} \otimes \epsilon)\, \big((\delta_s \otimes \mathrm{u}) \otimes \mathrm{I}\big) = (\delta_s \otimes \mathrm{u}) \otimes\, \epsilon(\mathrm{I}) = (\delta_s \otimes \mathrm{u}) \otimes\, 1 = \delta_s \otimes \mathrm{u}\,,$$

but on the other hand

$$(\epsilon \otimes \mathrm{id})\, \Delta(\delta_s \otimes \mathrm{u}) = (\epsilon \otimes \mathrm{id})\, \big((\delta_s \otimes \mathrm{u}) \otimes \mathrm{I}\big) = \epsilon(\delta_s \otimes \mathrm{u}) \otimes \mathrm{I}\,.$$

In general $\epsilon(\delta_s \otimes u) \otimes I \neq \delta_s \otimes u$. This can be illustrated by the following diagrams:

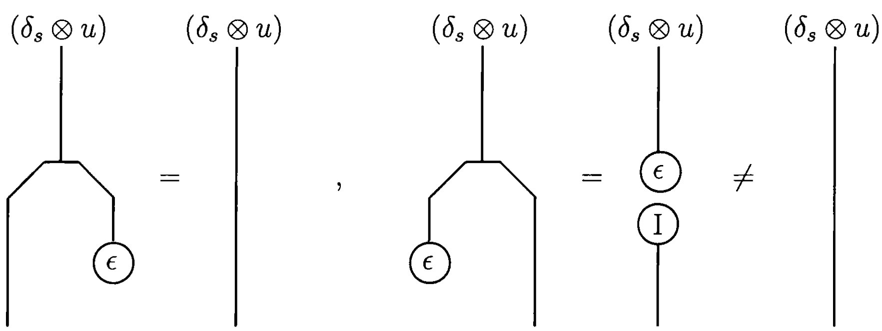


figure 3.6

In [8] there is a definition of a left Hopf algebra. This is an ordinary bialgebra, with a one sided antipode. In the case presented here, we do not get this far, as the counit is already one sided. It remains to be seen if there is even a one-sided antipode for A.

3.3 The dual of the algebra A in $\mathcal{C}$

Proposition 3.3.1 *Define a basis $\delta_u \otimes s$ of A^* with evaluation map given by*

$$\mathrm{ev}\big((\delta_u \otimes s) \otimes (\delta_t \otimes v)\big) = \delta_{s,t}\, \delta_{u,v}\,,$$

for $s, t \in M$ and $u, v \in G$. Then the M-grade and the G-action on A^* are defined as follows: $\langle \delta_u \otimes s \rangle = \langle \delta_s \otimes u \rangle^L$, and for any $w \in G$

$$(\delta_u \otimes s)\bar{\triangleleft}(\langle \delta_u \otimes s \rangle^R \triangleright w) = \delta_{(\langle \delta_u \otimes s \rangle^R \triangleright w)^{-1} uw} \otimes s \triangleleft (\langle \delta_u \otimes s \rangle^R \triangleright w)\,.$$

Proof. Take the algebra $A \in \mathcal{C}$. For $(\delta_s \otimes u) \in A$, $(\delta_u \otimes s) \in A^*$, then for $(\delta_t \otimes v) \in A$ the evaluation map is given by

$$\mathrm{ev}\big((\delta_u \otimes s) \otimes (\delta_t \otimes v)\big) = \delta_{s,t}\, \delta_{u,v}\,,$$

where $s, t \in M$ and $u, v \in G$. The evaluation map will not be affected if for any $w \in G$ we apply $\bar{\triangleleft} w$ as follows

$$\mathrm{ev}\Big(\big((\delta_u \otimes s)\bar{\triangleleft}(\langle \delta_t \otimes v \rangle \triangleright w)\big) \otimes \big((\delta_t \otimes v)\bar{\triangleleft} w\big)\Big) = \delta_{s,t}\, \delta_{u,v}\,.$$

Using the definition of the action of G on the elements of the algebra A, the last equation can be rewritten as

$$\mathrm{ev}\Big(\big((\delta_u \otimes s)\bar{\triangleleft}(\langle \delta_t \otimes v \rangle \triangleright w)\big) \otimes \big(\delta_{t \triangleleft (\langle \delta_t \otimes v \rangle \triangleright w)} \otimes (\langle \delta_t \otimes v \rangle \triangleright w)^{-1} vw\big)\Big) = \delta_{s,t}\, \delta_{u,v}\,.$$

If we take $\delta_t \otimes v = \delta_s \otimes u$ then we get the following equation

$$(\delta_u \otimes s)\bar{\triangleleft}(\langle \delta_s \otimes u \rangle \triangleright w) = \delta_{(\langle \delta_s \otimes u \rangle \triangleright w)^{-1} uw} \otimes s \triangleleft (\langle \delta_s \otimes u \rangle \triangleright w)\,. \tag{3.8}$$

As $\langle\delta_u\otimes s\rangle\cdot\langle\delta_s\otimes u\rangle=e$, so $\langle\delta_u\otimes s\rangle=\langle\delta_s\otimes u\rangle^L$ or $\langle\delta_s\otimes u\rangle=\langle\delta_u\otimes s\rangle^R$. Hence the equation (3.8) can be rewritten as

$$(\delta_u\otimes s)\bar{\triangleleft}(\langle\delta_u\otimes s\rangle^R\triangleright w)=\delta_{(\langle\delta_u\otimes s\rangle^R\triangleright w)^{-1}uw}\otimes s\triangleleft(\langle\delta_u\otimes s\rangle^R\triangleright w)\,. \quad\square$$

Proposition 3.3.2 *There is a morphism $T:A\to A^*$ in the category $\mathcal{C}$ defined by*

$$T(\delta_s\otimes u)=\delta_{u^{-1}\tau(b,b^R)}\otimes s\triangleleft u\,,$$

where $b=\langle\delta_s\otimes u\rangle$.

Proof. We have defined a linear map $T:A\to A^*$. If this map is to be a morphism in the category it should preserve the grading and the action in $\mathcal{C}$ i.e. $\langle T(\delta_s\otimes u)\rangle=\langle\delta_s\otimes u\rangle$ and

$$\big(T(\delta_s\otimes u)\big)\bar{\triangleleft}(b^R\triangleright w)=T\big((\delta_s\otimes u)\bar{\triangleleft}(b^R\triangleright w)\big)\,. \tag{3.9}$$

To prove this we start with

$$T\big((\delta_s\otimes u)\bar{\triangleleft}(b^R\triangleright w)\big)=T\Big(\delta_{s\triangleleft\left(b\triangleright(b^R\triangleright w)\right)}\otimes\big(b\triangleright(b^R\triangleright w)\big)^{-1}u(b^R\triangleright w)\Big)\,.$$

If we put $T(\delta_s\otimes u)=\delta_v\otimes t$ for some $v\in G$ and $t\in M$, then (3.3.1) implies

$$\big(T(\delta_s\otimes u)\big)\bar{\triangleleft}(b^R\triangleright w)=(\delta_v\otimes t)\bar{\triangleleft}(b^R\triangleright w)=\delta_{(b^R\triangleright w)^{-1}vu}\otimes t\triangleleft(b^R\triangleright w)\,.$$

Now we need to find t and v. We know that $\langle\delta_v\otimes t\rangle=\langle\delta_s\otimes u\rangle=b=\langle\delta_t\otimes v\rangle^L$, so $b^R=\langle\delta_t\otimes v\rangle$, then we have

$$t\cdot b^R=t\triangleleft v\quad\text{and}\quad s\cdot b=s\triangleleft u\,,$$

thus $(t\cdot b^R)\cdot b^{RR}=(t\triangleleft v)\cdot b^{RR}$, or $\big(t\triangleleft\tau(b^R,b^{RR})\big)\cdot(b^R\cdot b^{RR})=(t\triangleleft v)\cdot b^{RR}$ which implies

$$t\triangleleft\tau(b^R,b^{RR})=(t\triangleleft v)\cdot b^{RR}\,. \tag{3.10}$$

Now as $b \cdot b^R = e$ and $b^R \cdot b^{RR} = e$, then $(b \cdot b^R) \cdot b^{RR} = b \triangleleft \tau(b^R, b^{RR}) \cdot (b^R \cdot b^{RR})$ implies that $b^{RR} = b \triangleleft \tau(b^R, b^{RR})$. Also we know that

$$b(b^R b^{RR}) = b\tau(b^R, b^{RR}) = \big(b \triangleright \tau(b^R, b^{RR})\big)\big(b \triangleleft \tau(b^R, b^{RR})\big),$$

but on the other hand

$$b(b^R b^{RR}) = (bb^R)b^{RR} = \tau(b, b^R)b^{RR},$$

so by the uniqueness of the factorization $b \triangleright \tau(b^R, b^{RR}) = \tau(b, b^R)$, hence (3.10) can be rewritten as

$$t \triangleleft \tau(b^R, b^{RR}) = (t \triangleleft v) \cdot \big(b \triangleleft \tau(b^R, b^{RR})\big),$$

which implies that

$$\begin{aligned} t &= \Big((t \triangleleft v) \cdot \big(b \triangleleft \tau(b^R, b^{RR})\big)\Big) \triangleleft \tau(b^R, b^{RR})^{-1} \\ &= \Big(t \triangleleft v\big((b \triangleleft \tau(b^R, b^{RR})) \triangleright \tau(b^R, b^{RR})^{-1}\big)\Big) \cdot b \\ &= \Big(t \triangleleft v\big(b \triangleright \tau(b^R, b^{RR})\big)^{-1}\Big) \cdot b \\ &= \big(t \triangleleft v\tau(b, b^R)^{-1}\big) \cdot b\,. \end{aligned}$$

Let $s = t \triangleleft v\tau(b, b^R)^{-1}$, then $u = \tau(b, b^R)v^{-1}$, or $v = u^{-1}\tau(b, b^R)$ and $t = s \triangleleft u$. Therefore,

$$T(\delta_s \otimes u) = \delta_{u^{-1}\tau(b,b^R)} \otimes s \triangleleft u\,. \tag{3.11}$$

Now we want to show that (3.9) is satisfied. Start with the grade as

$$\langle (\delta_s \otimes u) \bar{\triangleleft} (b^R \triangleright w) \rangle = \langle \delta_s \otimes u \rangle \triangleleft (b^R \triangleright w) = b \triangleleft (b^R \triangleright w) = c\,,$$

now to check the action staring with the right hand side of (3.9) as follows

$$\begin{aligned} T\big((\delta_s \otimes u) \bar{\triangleleft} (b^R \triangleright w)\big) &= T\Big(\delta_{s \triangleleft (b \triangleright (b^R \triangleright w))} \otimes \big(b \triangleright (b^R \triangleright w)\big)^{-1} u(b^R \triangleright w)\Big) \\ &= \delta_{(b^R \triangleright w)^{-1} u^{-1} (b \triangleright (b^R \triangleright w))\tau(c, c^R)} \otimes s \triangleleft u(b^R \triangleright w) \\ &= \delta_{(b^R \triangleright w)^{-1} u^{-1} \tau(b, b^R) w} \otimes s \triangleleft u(b^R \triangleright w)\,, \end{aligned}$$

the last equality because

$$b\triangleright(b^R\triangleright w) = \tau(b,b^R)w\tau\big(b\triangleleft(b^R\triangleright w),(b^R\triangleleft w)\big)^{-1} = \tau(b,b^R)w\tau(c,c^R)^{-1}$$

Finally,

$$\big(T(\delta_s\otimes u)\big)\bar{\triangleleft}(b^R\triangleright w) = (\delta_{u^{-1}\tau(b,b^R)}\otimes s\triangleleft u)\bar{\triangleleft}(b^R\triangleright w)$$
$$= \delta_{(b^R\triangleright w)^{-1}u^{-1}\tau(b,b^R)w}\otimes s\triangleleft u(b^R\triangleright w)\,,$$

as required. □

Proposition 3.3.3 *Let the morphism $T : A \longrightarrow A^*$ be as defined in proposition (3.3.2). Then there is an inverse morphism $T^{-1} : A^* \longrightarrow A$ in the category $\mathcal{C}$ defined by*

$$T^{-1}(\delta_v\otimes t) = \delta_{t\triangleleft v\tau(b,b^R)^{-1}}\otimes\tau(b,b^R)\,v^{-1}\,,$$

for $(\delta_v\otimes t)\in A^*$, where $b = \langle\delta_v\otimes t\rangle$.

Proof. From proposition (3.3.2) we know that

$$T(\delta_s\otimes u) = \delta_{u^{-1}\tau(b,b^R)}\otimes s\triangleleft u\,,$$

where $b = \langle\delta_s\otimes u\rangle$. Put $T(\delta_s\otimes u) = \delta_v\otimes t$, then also $\langle\delta_v\otimes t\rangle = b$ as the morphism T preserves the grade. Then also we can get

$$t = s\triangleleft u \qquad \text{and} \qquad v = u^{-1}\tau(b,b^R),$$

which imply that

$$u = \tau(b,b^R)\,v^{-1} \qquad \text{and} \qquad s = t\triangleleft u^{-1} = t\triangleleft v\tau(b,b^R)^{-1}.$$

Therefore

$$T^{-1}(\delta_v\otimes t) = \delta_s\otimes u\,,$$

with s and u as defined above. Note that it is automatic that if T is a morphism, with a linear map inverse T^{-1}, then T^{-1} is also a morphism. $\square$

Proposition 3.3.4 *Let A be the algebra in the category $\mathcal{C}$, then for an element $\alpha = (\delta_u \otimes s)$ in A^* the coproduct, or comultiplication, Δ on A^* can be given by*

$$\Delta(\delta_u \otimes s) = \sum_{v \in G} (\delta_v \otimes s) \otimes (\delta_{\tau(a^L,a)v^{-1}u\,\langle\alpha\rangle^{R-1} a^{L-1} a'} \otimes s \triangleleft v\tau(a^L, a)^{-1}),$$

where $u \in G$, $s \in M$, $a = \langle \delta_s \otimes v \rangle$ and $a' = \left((\langle \alpha \rangle \cdot a) \triangleleft \tau(a^L, a)^{-1} \right)^R$.

Proof. From proposition (1.2.14), we know that

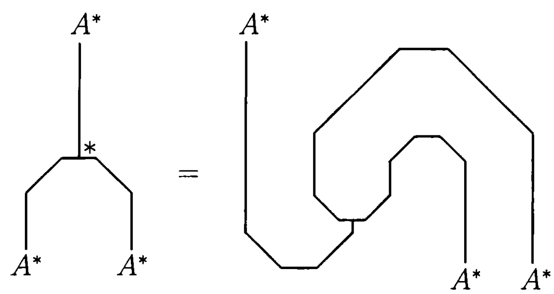


figure 3.7

For $\alpha \in A^*$, we follow the above figure from top to bottom and calculate the following: Put $\mathrm{coev_A}(1) = \beta \otimes \gamma = \beta' \otimes \gamma'$ (we suppress summations as usual) which implies $\langle \beta \rangle \cdot \langle \gamma \rangle = e$ and $\langle \beta' \rangle \cdot \langle \gamma' \rangle = e$. As all parts of the above diagram are morphisms in the category and preserve the grads we should have $\langle \alpha \rangle = \langle \gamma' \rangle \cdot \langle \gamma \rangle$. We start with

$$\alpha \otimes \mathrm{coev_A}(1) = \alpha \otimes (\beta \otimes \gamma).$$

According to the diagram we include $\mathrm{coev_A}(1)$ again to get

$$\alpha \otimes \left((\beta \otimes \mathrm{coev_A}(1)) \otimes \gamma \right) = \alpha \otimes \left((\beta \otimes (\beta' \otimes \gamma')) \otimes \gamma \right).$$

After that we apply the associator inverse Φ^{-1} and then the multiplication to get

$$\alpha \otimes \big(((\beta \bar{\triangleleft} \tau(\langle \beta' \rangle, \langle \gamma' \rangle)^{-1} \otimes \beta') \otimes \gamma') \otimes \gamma \big) \mapsto \alpha \otimes \big((((\beta \bar{\triangleleft} \tau(\langle \beta' \rangle, \langle \gamma' \rangle)^{-1})\beta') \otimes \gamma') \otimes \gamma \big).$$

Now applying the associator Φ gives

$$\alpha \otimes \big(((\beta \bar{\triangleleft} \tau(\langle \beta' \rangle, \langle \gamma' \rangle)^{-1})\beta') \bar{\triangleleft} \tau(\langle \gamma' \rangle, \langle \gamma \rangle) \otimes (\gamma' \otimes \gamma) \big).$$

Applying the associator inverse Φ^{-1} again will give

$$\begin{aligned} \big(\alpha \bar{\triangleleft} \tau(\langle \tilde{\beta} \rangle, \langle \gamma' \otimes \gamma \rangle)^{-1} \otimes \tilde{\beta}\big) \otimes (\gamma' \otimes \gamma) &= \big(\alpha \bar{\triangleleft} \tau(\langle \tilde{\beta} \rangle, \langle \gamma' \rangle \cdot \langle \gamma \rangle)^{-1} \otimes \tilde{\beta}\big) \otimes (\gamma' \otimes \gamma) \\ &= \big(\alpha \bar{\triangleleft} \tau(\langle \tilde{\beta} \rangle, \langle \alpha \rangle)^{-1} \otimes \tilde{\beta}\big) \otimes (\gamma' \otimes \gamma) \end{aligned} \tag{3.12}$$

where

$$\tilde{\beta} = ((\beta \bar{\triangleleft} \tau(\langle \beta' \rangle, \langle \gamma' \rangle)^{-1})\beta') \bar{\triangleleft} \tau(\langle \gamma' \rangle, \langle \gamma \rangle), \tag{3.13}$$

which implies that

$$\langle \tilde{\beta} \rangle = (((\langle \beta \rangle \triangleleft \tau(\langle \beta' \rangle, \langle \gamma' \rangle)^{-1}) \cdot \langle \beta' \rangle) \triangleleft \tau(\langle \gamma' \rangle, \langle \gamma \rangle). \tag{3.14}$$

Now we want to show that $\langle \tilde{\beta} \rangle = \langle \alpha \rangle^L$, which can be proved as follows

$$\begin{aligned} \langle \tilde{\beta} \rangle \cdot \langle \alpha \rangle = \langle \tilde{\beta} \rangle \cdot (\langle \gamma' \rangle \cdot \langle \gamma \rangle) &= (((\langle \beta \rangle \triangleleft \tau(\langle \beta' \rangle, \langle \gamma' \rangle)^{-1}) \cdot \langle \beta' \rangle) \triangleleft \tau(\langle \gamma' \rangle, \langle \gamma \rangle) \cdot (\langle \gamma' \rangle \cdot \langle \gamma \rangle) \\ &= (((\langle \beta \rangle \triangleleft \tau(\langle \beta' \rangle, \langle \gamma' \rangle)^{-1}) \cdot \langle \beta' \rangle) \cdot \langle \gamma' \rangle) \cdot \langle \gamma \rangle \\ &= (\langle \beta \rangle \cdot (\langle \beta' \rangle \cdot \langle \gamma' \rangle)) \cdot \langle \gamma \rangle = (\langle \beta \rangle \cdot e) \cdot \langle \gamma \rangle = \langle \beta \rangle \cdot \langle \gamma \rangle = e\,. \end{aligned}$$

So if we apply the evaluation map to (3.12), it can be rewritten as

$$\begin{aligned} &\big(\alpha \bar{\triangleleft} \tau(\langle \alpha \rangle^L, \langle \alpha \rangle)^{-1}\big)(\tilde{\beta})\,(\gamma' \otimes \gamma) = \\ &\alpha\Big(\tilde{\beta} \bar{\triangleleft} \tau(\langle \tilde{\beta} \rangle^L, \langle \tilde{\beta} \rangle)^{-1}(\langle \tilde{\beta} \rangle^L \triangleright \tau(\langle \alpha \rangle^L, \langle \alpha \rangle))\tau\big(\langle \tilde{\beta} \rangle^L \triangleleft \tau(\langle \alpha \rangle^L, \langle \alpha \rangle), (\langle \tilde{\beta} \rangle^L \triangleleft \tau(\langle \alpha \rangle^L, \langle \alpha \rangle))^R\big)\,(\gamma' \otimes \gamma). \end{aligned} \tag{3.15}$$

To make this equation simpler we need to do the following calculations

$$\big(\langle\alpha\rangle^{LL}\triangleleft\tau(\langle\alpha\rangle^{L},\langle\alpha\rangle)\big)\cdot(\langle\alpha\rangle^{L}\cdot\langle\alpha\rangle)=(\langle\alpha\rangle^{LL}\cdot\langle\alpha\rangle^{L})\cdot\langle\alpha\rangle,$$

which implies that

$$\langle\alpha\rangle^{LL}\triangleleft\tau(\langle\alpha\rangle^{L},\langle\alpha\rangle)=\langle\alpha\rangle.$$

Thus we can consider the following

$$\langle\alpha\rangle^{LL}\langle\alpha\rangle^{L}\langle\alpha\rangle=\langle\alpha\rangle^{LL}\tau(\langle\alpha\rangle^{L},\langle\alpha\rangle)=\big(\langle\alpha\rangle^{LL}\triangleright\tau(\langle\alpha\rangle^{L},\langle\alpha\rangle)\big)\langle\alpha\rangle,$$

which implies that

$$\langle\alpha\rangle^{LL}\langle\alpha\rangle^{L}=\tau(\langle\alpha\rangle^{LL},\langle\alpha\rangle^{L})=\langle\alpha\rangle^{LL}\triangleright\tau(\langle\alpha\rangle^{L},\langle\alpha\rangle).$$

Now substituting in (3.15) gives

$$\Delta_{A^*}(\alpha)=\alpha\big(\tilde{\beta}\bar{\triangleleft}\tau(\langle\alpha\rangle,\langle\alpha\rangle^{R})\big)\,(\gamma^{'}\otimes\gamma),$$

where

$$\tilde{\beta}=((\beta\bar{\triangleleft}\tau(\langle\beta^{'}\rangle,\langle\gamma^{'}\rangle)^{-1})\beta^{'})\bar{\triangleleft}\tau(\langle\gamma^{'}\rangle,\langle\gamma\rangle).$$

From the definition of the coevaluation map we know

$$\mathrm{coev}_A(1)=\sum_{\xi\in\text{ basis of }V}\xi\,\bar{\triangleleft}\tau(\,\langle\xi\rangle^{L},\langle\xi\rangle\,)^{-1}\otimes\hat{\xi},$$

so we put $\beta=\xi\,\bar{\triangleleft}\tau(\,\langle\xi\rangle^{L},\langle\xi\rangle\,)^{-1}$ and $\gamma=\hat{\xi}$, and also in the same way we put $\beta^{'}=\eta\,\bar{\triangleleft}\tau(\,\langle\eta\rangle^{L},\langle\eta\rangle\,)^{-1}$ and $\gamma^{'}=\hat{\eta}$. These imply that

$$\langle\beta\rangle=\langle\xi\rangle\triangleleft\tau(\,\langle\xi\rangle^{L},\langle\xi\rangle\,)^{-1}\qquad\text{and}\qquad\langle\gamma\rangle=\langle\hat{\xi}\rangle=\langle\xi\rangle^{L},$$

$$\langle\beta^{'}\rangle=\langle\eta\rangle\triangleleft\tau(\,\langle\eta\rangle^{L},\langle\eta\rangle\,)^{-1}\qquad\text{and}\qquad\langle\gamma^{'}\rangle=\langle\hat{\eta}\rangle=\langle\eta\rangle^{L}.$$

Now we substitute these in $\tilde{\beta}$ and try to simplify it as follows

$$\tilde{\beta} = \Big(\xi \,\bar{\triangleleft}\, \tau(\, \langle\xi\rangle^{L}, \langle\xi\rangle\,)^{-1}\, \tau(\, \langle\eta\rangle \triangleleft \tau(\, \langle\eta\rangle^{L}, \langle\eta\rangle\,)^{-1}\,, \langle\eta\rangle^{L}\,)^{-1} (\eta \,\bar{\triangleleft}\, \tau(\, \langle\eta\rangle^{L}, \langle\eta\rangle\,)^{-1}) \Big) \bar{\triangleleft}\, \tau(\, \langle\eta\rangle^{L}, \langle\xi\rangle^{L}\,).$$

Put $\pi = \big(\xi \,\bar{\triangleleft}\, \tau(\, \langle\xi\rangle^{L}, \langle\xi\rangle\,)^{-1}\, \tau\big(\, \langle\eta\rangle \triangleleft \tau(\, \langle\eta\rangle^{L}, \langle\eta\rangle\,)^{-1}\,, \langle\eta\rangle^{L}\,\big)^{-1} \big) \big(\eta \,\bar{\triangleleft}\, \tau(\, \langle\eta\rangle^{L}, \langle\eta\rangle\,)^{-1} \big)$, so

$$\pi \bar{\triangleleft} \tau(\, \langle\eta\rangle^{L}, \langle\eta\rangle\,) =$$

$$\Big(\xi \,\bar{\triangleleft}\, \tau(\, \langle\xi\rangle^{L}, \langle\xi\rangle\,)^{-1}\, \tau\big(\, \langle\eta\rangle \triangleleft \tau(\, \langle\eta\rangle^{L}, \langle\eta\rangle\,)^{-1}\,, \langle\eta\rangle^{L}\,\big)^{-1} \big((\langle\eta\rangle \triangleleft \tau(\, \langle\eta\rangle^{L}, \langle\eta\rangle\,)^{-1}) \triangleright \tau(\, \langle\eta\rangle^{L}, \langle\eta\rangle\,) \big) \Big)\, \eta$$

$$= \Big(\xi \,\bar{\triangleleft}\, \tau(\, \langle\xi\rangle^{L}, \langle\xi\rangle\,)^{-1}\, \tau\big(\, \langle\eta\rangle \triangleleft \tau(\, \langle\eta\rangle^{L}, \langle\eta\rangle\,)^{-1}\,, \langle\eta\rangle^{L}\,\big)^{-1} \big(\langle\eta\rangle \triangleright \tau(\, \langle\eta\rangle^{L}, \langle\eta\rangle\,)^{-1} \big)^{-1} \Big)\, \eta$$

$$= \big(\xi \,\bar{\triangleleft}\, \tau(\, \langle\xi\rangle^{L}, \langle\xi\rangle\,)^{-1} \big)\, \eta.$$

The last equivalence is because

$$\tau\big(\, \langle\eta\rangle \triangleleft \tau(\, \langle\eta\rangle^{L}, \langle\eta\rangle\,)^{-1}\,, \langle\eta\rangle^{L}\,\big) = \big(\langle\eta\rangle \triangleleft \tau(\, \langle\eta\rangle^{L}, \langle\eta\rangle\,)^{-1} \big)\, \langle\eta\rangle^{L} =$$

$$\big(\langle\eta\rangle \triangleright \tau(\, \langle\eta\rangle^{L}, \langle\eta\rangle\,)^{-1} \big)^{-1}\, \langle\eta\rangle (\langle\eta\rangle^{L} \langle\eta\rangle)^{-1}\, \langle\eta\rangle^{L} = \big(\langle\eta\rangle \triangleright \tau(\, \langle\eta\rangle^{L}, \langle\eta\rangle\,)^{-1} \big)^{-1}.$$

So

$$\tilde{\beta} = \big(\pi \bar{\triangleleft} \tau(\, \langle\eta\rangle^{L}, \langle\eta\rangle\,) \big) \bar{\triangleleft}\, \tau(\, \langle\eta\rangle^{L}, \langle\eta\rangle\,)^{-1}\, \tau(\, \langle\eta\rangle^{L}, \langle\xi\rangle^{L}\,)$$

$$= \Big(\big(\xi \,\bar{\triangleleft}\, \tau(\, \langle\xi\rangle^{L}, \langle\xi\rangle\,)^{-1} \big)\, \eta \Big) \bar{\triangleleft}\, \tau(\, \langle\eta\rangle^{L}, \langle\eta\rangle\,)^{-1}\, \tau(\, \langle\eta\rangle^{L}, \langle\xi\rangle^{L}\,).$$

Now put $\xi = \delta_t \otimes v$, $a = \langle\xi\rangle = \langle \delta_t \otimes v \rangle$, $\eta = \delta_{t'} \otimes v'$, $a' = \langle\eta\rangle = \langle \delta_{t'} \otimes v' \rangle$ and $w = \tau(\, \langle\xi\rangle^{L}, \langle\xi\rangle\,)^{-1}$ then

$$\xi \bar{\triangleleft}\, w = (\delta_t \otimes v) \bar{\triangleleft}\, w = \delta_{t \triangleleft (a \triangleright w)} \otimes (a \triangleright w)^{-1} v w.$$

Hence

$$(\delta_{t \triangleleft (a \triangleright w)} \otimes (a \triangleright w)^{-1} v w)(\delta_{t'} \otimes v') = \delta_{t', t \triangleleft v w}\, \delta_{t \triangleleft (a \triangleright w) \tau(\, a \triangleleft w, a'\,)} \otimes \tau(\, a \triangleleft w, a'\,)^{-1} (a \triangleright w)^{-1} v w v'.$$

Now put $p = \tau(\, \langle\eta\rangle^{L}, \langle\eta\rangle\,)^{-1}\, \tau(\, \langle\eta\rangle^{L}, \langle\xi\rangle^{L}\,)$, then we get

$$\tilde{\beta} = \Big(\big(\xi \,\bar{\triangleleft}\, \tau(\, \langle\xi\rangle^{L}, \langle\xi\rangle\,)^{-1} \big)\, \eta \Big) \bar{\triangleleft}\, p =$$

$$\delta_{t', t \triangleleft v w}\, \delta_{t \triangleleft (a \triangleright w) \tau(\, a \triangleleft w, a'\,) \left(((a \triangleleft w) \cdot a') \triangleright p \right)} \otimes \big(((a \triangleleft w) \cdot a') \triangleright p \big)^{-1} \tau(\, a \triangleleft w, a'\,)^{-1} (a \triangleright w)^{-1} v w v' p.$$

If we put $q = \tau(\langle\alpha\rangle, \langle\alpha\rangle^R)$, then we get

$$\alpha\big(\tilde{\beta}\bar{\triangleleft}\tau(\langle\alpha\rangle, \langle\alpha\rangle^R)\big) = \delta_{t',t\triangleleft vw}\,\alpha\Big(\delta_{t\triangleleft(a\triangleright w)\tau(a\triangleleft w,a')\big(((a\triangleleft w)\cdot a')\triangleright p\big)\big(\big(((a\triangleleft w)\cdot a')\triangleleft p\big)\triangleright q\big)}$$

$$\otimes\big(\big(((a\triangleleft w)\cdot a')\triangleleft p\big)\triangleright q\big)^{-1}\big(((a\triangleleft w)\cdot a')\triangleright p\big)^{-1}\tau(a\triangleleft w, a')^{-1}(a\triangleright w)^{-1}vwv'pq\Big). \tag{3.16}$$

To make this simpler we do the following

$$\big(((a\triangleleft w)\cdot a')\triangleright p\big)\big(\big(((a\triangleleft w)\cdot a')\triangleleft p\big)\triangleright q\big) = ((a\triangleleft w)\cdot a')\triangleright pq,$$

and also we have $pq = a'^{-1}a'^{L^{-1}}a'^{L}a^{L}\langle\alpha\rangle^{-1}\langle\alpha\rangle\langle\alpha\rangle^{R} = a'^{-1}a^{L}\langle\alpha\rangle^{R}$. If we put $F = (a\triangleright w)\tau(a\triangleleft w, a')\big(((a\triangleleft w)\cdot a')\triangleright pq\big)$, then F will be equal to the G-part of the following unique factorization

$$(a\triangleright w)(a\triangleleft w)a'pq = (a\triangleright w)\tau(a\triangleleft w, a')((a\triangleleft w)\cdot a')pq$$

$$= (a\triangleright w)\tau(a\triangleleft w, a')\big(((a\triangleleft w)\cdot a')\triangleright pq\big)\big(((a\triangleleft w)\cdot a')\triangleleft pq\big),$$

but on the other hand we also have

$$(a\triangleright w)(a\triangleleft w)a'pq = awa'pq = aa^{-1}a^{L^{-1}}a'a'^{-1}a^{L}\langle\alpha\rangle^{R} = \langle\alpha\rangle^{R}.$$

So $F = e$, which means that equation (3.16) can be rewritten as

$$\alpha\big(\tilde{\beta}\bar{\triangleleft}\tau(\langle\alpha\rangle, \langle\alpha\rangle^R)\big) = \delta_{t',t\triangleleft vw}\,\alpha\big(\delta_t\otimes vwv'a'^{-1}a^{L}\langle\alpha\rangle^{R}\big).$$

If we put $\alpha = \delta_u \otimes s$ the R.H.S. of the above equation becomes

$$\delta_{t',t\triangleleft vw}\,\mathrm{ev}\big((\delta_{\mathrm{u}}\otimes \mathrm{s})\otimes(\delta_{\mathrm{t}}\otimes \mathrm{vwv'a'^{-1}a^{L}}\langle\alpha\rangle^{\mathrm{R}})\big) = \delta_{\mathrm{t'},\mathrm{t}\triangleleft \mathrm{vw}}\,\delta_{\mathrm{s},\mathrm{t}}\,\delta_{\mathrm{u},\mathrm{vwv'a'^{-1}a^{L}}\langle\alpha\rangle^{\mathrm{R}}},$$

which implies that $t = s$, $t' = s\triangleleft v\,\tau(a^L, a)^{-1}$ and $u = v\tau(a^L, a)^{-1}v'a'^{-1}a^L\langle\alpha\rangle^R$, or $v' = \tau(a^L, a)v^{-1}u\langle\alpha\rangle^{R^{-1}}a^{L^{-1}}a'$. We know that $\langle\alpha\rangle = \langle\delta_s\otimes u\rangle^L$, or $\langle\alpha\rangle^R = \langle\delta_s\otimes u\rangle$, so we get the following:

$$s \cdot \langle\alpha\rangle^R = s \cdot \langle\delta_s \otimes u\rangle = s\triangleleft u, \quad t \cdot a = t \cdot \langle\delta_t \otimes v\rangle = t\triangleleft v \;\text{ and }\; \mathrm{t}' \cdot \mathrm{a}' = \mathrm{t}' \cdot \langle\delta_{\mathrm{t}'} \otimes \mathrm{v}'\rangle = \mathrm{t}'\triangleleft\mathrm{v}'.$$

To confirm our calculations we prove the last equation substituting by its values as follows

$$t'\triangleleft v' = s\triangleleft u\langle\alpha\rangle^{R^{-1}} a^{L^{-1}} a' = (s \cdot \langle\alpha\rangle^R)\triangleleft \langle\alpha\rangle^{R^{-1}} a^{L^{-1}} a' = \big(s\triangleleft(\langle\alpha\rangle^R\triangleright z)\big) \cdot (\langle\alpha\rangle^R\triangleleft z), \quad (3.17)$$

where $z = \langle\alpha\rangle^{R^{-1}} a^{L^{-1}} a'$. But we know that

$$\langle\alpha\rangle^R z = (\langle\alpha\rangle^R\triangleright z)(\langle\alpha\rangle^R\triangleleft z) = \langle\alpha\rangle^R \langle\alpha\rangle^{R^{-1}} a^{L^{-1}} a' = a^{L^{-1}} a' = \tau(a^{L^{-1}}, a')(a^{L^{-1}} \cdot a').$$

So by the uniqueness of the factorization we get $(\langle\alpha\rangle^R\triangleright z) = \tau(a^{L^{-1}}, a')$ and $(\langle\alpha\rangle^R\triangleleft z) = (a^{L^{-1}} \cdot a')$, then substituting in (3.17) gives

$$t'\triangleleft v' = \big(s\triangleleft\tau(a^{L^{-1}}, a')\big) \cdot (a^{L^{-1}} \cdot a') = (s \cdot a^{L^{-1}}) \cdot a'$$
$$= (s \cdot a\tau(a^L, a)^{-1}) \cdot a' = (s\triangleleft v\tau(a^L, a)^{-1}) \cdot a' = t' \cdot a'.$$

Finally, we calculate a' which we do as the following:

$$\langle\alpha\rangle \cdot a = (a'^L \cdot a^L) \cdot a = (a'^L\triangleleft\tau(a^L, a)) \cdot (a^L \cdot a) = a'^L\triangleleft\tau(a^L, a),$$

so $a'^L = (\langle\alpha\rangle \cdot a)\triangleleft\tau(a^L, a)^{-1}$, or $a' = \big((\langle\alpha\rangle \cdot a)\triangleleft\tau(a^L, a)^{-1}\big)^R$. Therefore,

$$\Delta(\delta_u \otimes s) = \sum_v (\delta_v \otimes s) \otimes (\delta_{\tau(a^L,a)v^{-1}u\,\langle\alpha\rangle^{R^{-1}} a^{L^{-1}} a'} \otimes s\triangleleft v\tau(a^L, a)^{-1}). \qquad \square$$

3.4 Expanding the collection of morphisms in $\mathcal{C}$

Here we consider adding new morphisms to the category $\mathcal{C}$, to make a new category $\bar{\mathcal{C}}$. Consider the linear map $\phi : V \to W$. We call it a type A morphism if it satisfies the following conditions:

$$\langle\phi(\xi)\rangle = \langle\xi\rangle \qquad \text{and} \qquad \phi(\xi\bar{\triangleleft}\mathrm{u}) = \phi(\xi)\bar{\triangleleft}\mathrm{u}\,,$$

for $\xi \in V$ and $u \in G$ (these are just the usual morphism conditions in $\mathcal{C}$). It it is said to be a type B morphism if it satisfies the following conditions:

$$\langle\phi(\xi)\rangle = \langle\xi\rangle^L \qquad \text{and} \qquad \phi(\xi\bar{\triangleleft}\mathrm{u}) = \phi(\xi)\bar{\triangleleft}(\langle\xi\rangle\triangleright\mathrm{u})\,.$$

We will assume for this section, except for proposition 3.4.6, that $s^{LL} = s$ and $s^L\triangleright(s\triangleright u) = u$ for $s \in M$ and $u \in G$. Some more work could be done on the category $\bar{\mathcal{C}}$, which will be mentioned in the last chapter as an idea for more research.

Type B morphisms obey a rather odd order reversing tensor product rule, as we now see.

Proposition 3.4.1 *If $\phi : V \to \tilde{V}$ and $\psi : W \to \tilde{W}$ are type B morphisms, then the map $\phi \boxtimes \psi : V \otimes W \to \tilde{W} \otimes \tilde{V}$ which is defined by*

$$(\phi \boxtimes \psi)(\xi \otimes \eta) = \big(\psi(\eta)\bar{\triangleleft}\tau(a^L, a) \otimes \phi(\xi)\big)\bar{\triangleleft}\tau(a, b)\,,$$

where $\xi \in V$, $\eta \in W$, $a = \langle\xi\rangle$ and $b = \langle\eta\rangle$, is a type B morphism.

Proof. First we need to show that $\langle(\phi \boxtimes \psi)(\xi \otimes \eta)\rangle = \langle\xi \otimes \eta\rangle^L = (a \cdot b)^L$ which we do as the following, taking into account that ϕ and ψ are type B morphisms:

$$\begin{aligned}
\big\langle\big(\psi(\eta)\bar{\triangleleft}\tau(a^L, a) \otimes \phi(\xi)\big)\bar{\triangleleft}\tau(a, b)\big\rangle &= \big\langle\big(\psi(\eta)\bar{\triangleleft}\tau(a^L, a) \otimes \phi(\xi)\big)\big\rangle\triangleleft\tau(a, b)\\
&= \big(\big\langle\psi(\eta)\bar{\triangleleft}\tau(a^L, a)\big\rangle \cdot \big\langle\phi(\xi)\big\rangle\big)\triangleleft\tau(a, b)\\
&= \big(\big(\langle\psi(\eta)\rangle\triangleleft\tau(a^L, a)\big) \cdot \big\langle\phi(\xi)\big\rangle\big)\triangleleft\tau(a, b)\\
&= \big(\big(\langle\eta\rangle^L\triangleleft\tau(a^L, a)\big) \cdot \langle\xi\rangle^L\big)\triangleleft\tau(a, b)\\
&= \big(\big(b^L\triangleleft\tau(a^L, a)\big) \cdot a^L\big)\triangleleft\tau(a, b)\,.
\end{aligned}$$

To show that this is equal to $(a \cdot b)^L$ we dot it by $(a \cdot b)$ to get the identity. So we get

$$\Big(\big((b^L \triangleleft \tau(a^L, a)) \cdot a^L\big) \triangleleft \tau(a,b)\Big) \cdot (a \cdot b) = \Big(\big((b^L \triangleleft \tau(a^L, a)) \cdot a^L\big) \cdot a\Big) \cdot b$$
$$= \Big(\big(b^L \triangleleft \tau(a^L, a)\tau(a^L, a)^{-1}\big) \cdot (a^L \cdot a)\Big) \cdot b$$
$$= b^L \cdot b = e\,.$$

Next we need to show that $(\phi \boxtimes \psi)\big((\xi \otimes \eta)\bar{\triangleleft} u\big) = \big((\phi \boxtimes \psi)(\xi \otimes \eta)\big)\bar{\triangleleft}(\langle \xi \otimes \eta \rangle \triangleright u)$. We start with the left hand side as the following:

$$L.H.S. = (\phi \boxtimes \psi)\big(\xi \bar{\triangleleft}(\langle \eta \rangle \triangleright u) \otimes \eta \bar{\triangleleft} u\big) = (\phi \boxtimes \psi)\big(\xi \bar{\triangleleft}(b \triangleright u) \otimes \eta \bar{\triangleleft} u\big)$$
$$= \Big(\psi(\eta \bar{\triangleleft} u)\bar{\triangleleft}\tau\big((a \triangleleft (b \triangleright u))^L, a \triangleleft (b \triangleright u)\big) \otimes \phi(\xi \bar{\triangleleft}(b \triangleright u))\Big)\bar{\triangleleft}\tau\big(a \triangleleft (b \triangleright u), b \triangleleft u\big)$$
$$= \Big(\big(\psi(\eta)\bar{\triangleleft}(\langle \eta \rangle \triangleright u)\big)\bar{\triangleleft}\tau\big((a \triangleleft (b \triangleright u))^L, a \triangleleft (b \triangleright u)\big) \otimes \phi(\xi)\bar{\triangleleft}(\langle \xi \rangle \triangleright (b \triangleright u))\Big)\bar{\triangleleft}\tau\big(a \triangleleft (b \triangleright u), b \triangleleft u\big)$$
$$= \Big(\big(\psi(\eta)\big)\bar{\triangleleft}(b \triangleright u)\,\tau\big((a \triangleleft (b \triangleright u))^L, a \triangleleft (b \triangleright u)\big) \otimes \phi(\xi)\bar{\triangleleft}(a \triangleright (b \triangleright u))\Big)\bar{\triangleleft}\tau\big(a \triangleleft (b \triangleright u), b \triangleleft u\big)\,.$$

On the other hand

$$R.H.S. = \big((\phi \boxtimes \psi)(\xi \otimes \eta)\big)\bar{\triangleleft}(\langle \xi \otimes \eta \rangle \triangleright u) = \Big(\big(\psi(\eta)\bar{\triangleleft}\tau(a^L, a) \otimes \phi(\xi)\big)\bar{\triangleleft}\tau(a,b)\Big)\bar{\triangleleft}(\langle \xi \otimes \eta \rangle \triangleright u)$$
$$= \big(\psi(\eta)\bar{\triangleleft}\tau(a^L, a) \otimes \phi(\xi)\big)\bar{\triangleleft}\tau(a,b)\,\big((\langle \xi \rangle \cdot \langle \eta \rangle)\triangleright u\big)$$
$$= \big(\psi(\eta)\bar{\triangleleft}\tau(a^L, a) \otimes \phi(\xi)\big)\bar{\triangleleft}\tau(a,b)\,\big((a \cdot b)\triangleright u\big)$$
$$= \big(\psi(\eta)\bar{\triangleleft}\tau(a^L, a) \otimes \phi(\xi)\big)\bar{\triangleleft}\big(a \triangleright (b \triangleright u)\big)\,\tau\big(a \triangleleft (b \triangleright u), b \triangleleft u\big)$$
$$= \Big(\big(\psi(\eta)\big)\bar{\triangleleft}\,\tau(a^L, a)(b \triangleright u) \otimes \phi(\xi)\bar{\triangleleft}(a \triangleright (b \triangleright u))\Big)\bar{\triangleleft}\tau\big(a \triangleleft (b \triangleright u), b \triangleleft u\big)\,,$$

which is the same as the left hand side as $\tau(a^L, a)(b \triangleright u) = (b \triangleright u)\,\tau\big(a^L \triangleleft (a \triangleright (b \triangleright u)), a \triangleleft (b \triangleright u)\big) = (b \triangleright u)\,\tau\big((a \triangleleft (b \triangleright u))^L, a \triangleleft (b \triangleright u)\big)$. Note that we have used $a^L \triangleright \big(a \triangleright (b \triangleright u)\big) = b \triangleright u$ by an assumption for this section. □

Proposition 3.4.2 *The composition of two type B morphisms is a type A morphism.*

Proof. Let $\phi : U \to V$ and $\varphi : V \to W$ be two type B morphisms and for $\xi \in U$ let $\langle \xi \rangle = s$. We first check the grade as the following: As ϕ is a type B morphism then

$\langle\phi(\xi)\rangle = \langle\xi\rangle^L$. So as φ is also a type B morphism then $\langle\varphi(\phi(\xi))\rangle = \langle\xi\rangle^{LL} = s^{LL} = s$ which is the same as type A morphism.

Now to check the G-action, we do the following: As ϕ and φ are type B morphisms then we have the following:

$$\phi(\xi\bar{\triangleleft}u) = \phi(\xi)\bar{\triangleleft}(\langle\xi\rangle\triangleright u) \qquad \text{and} \qquad \varphi(\eta\bar{\triangleleft}\mathrm{u}) = \varphi(\eta)\bar{\triangleleft}(\langle\eta\rangle\triangleright\mathrm{u}),$$

for $\eta \in V$. So their composition can be given as the following:

$$\varphi\big(\phi(\xi\bar{\triangleleft}u)\big) = \varphi\big(\phi(\xi)\bar{\triangleleft}(\langle\xi\rangle\triangleright u)\big) = \varphi\big(\phi(\xi)\big)\bar{\triangleleft}\big(\langle\phi(\xi)\rangle\triangleright(\langle\xi\rangle\triangleright u)\big)$$

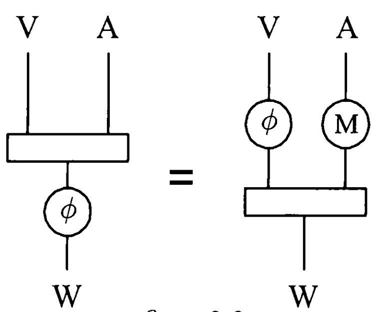
$$= \varphi\big(\phi(\xi)\big)\bar{\triangleleft}\big(\langle\xi\rangle^L\triangleright(\langle\xi\rangle\triangleright u)\big) = \varphi\big(\phi(\xi)\big)\bar{\triangleleft}\, u,$$

which is also the same as type A morphism. □

It is obvious that the composition of a type A morphism and a type B morphism is a type B morphism.

Now we should ask what the effect of a type B morphism is on the action of the algebra A. The answer is given in the following proposition.

Proposition 3.4.3 *If $\phi : V \to W$ is a type B morphism, then*



figre 3.8

where the map $M : A \to A$ is defined by $M(\delta_s \otimes u) = \delta_{s^L} \otimes s\triangleright u$.

Proof. We start with the left hand side as the following: Let $\xi \in V$ and $(\delta_s \otimes u) \in A$, then as $\xi\bar{\triangleleft}(\delta_s \otimes u) = \delta_{s,\langle\xi\rangle}\, \xi\bar{\triangleleft}u$ we get

$$L.H.S. = \phi\big(\xi\bar{\triangleleft}(\delta_s \otimes u)\big) = \phi\big(\delta_{s,\langle\xi\rangle}\, \xi\bar{\triangleleft}u\big) = \delta_{s,\langle\xi\rangle}\phi(\,\xi\bar{\triangleleft}u)\,.$$

To have a non-zero answer we should have $\langle\xi\rangle = s$. As ϕ is a type B morphism then

$$L.H.S. = \delta_{s,\langle\xi\rangle}\phi(\xi)\bar{\triangleleft}(\langle\xi\rangle\triangleright u) = \phi(\xi)\bar{\triangleleft}(s\triangleright u)\,.$$

Now we calculate the right hand side as the following:

$$R.H.S. = \phi(\xi)\bar{\triangleleft}M(\delta_s \otimes u) = \phi(\xi)\bar{\triangleleft}(\delta_{s^L} \otimes s\triangleright u)$$

$$= \delta_{s^L,\langle\phi(\xi)\rangle}\,\phi(\xi)\bar{\triangleleft}(s\triangleright u) = \delta_{s^L,\langle\xi\rangle^L}\,\phi(\xi)\bar{\triangleleft}(s\triangleright u) = \phi(\xi)\bar{\triangleleft}(s\triangleright u)\,. \quad \square$$

From [6], in the case where M is a subgroup of X, there is a $*$ operation defined on A by $(\delta_s \otimes u)^* = \delta_{s\triangleleft u} \otimes u^{-1}$. In our case we have a similar operation, $P : A \longrightarrow A$, given as follows. We have not yet shown that this really is any sort of adjoint operation (see section 7.3).

Proposition 3.4.4 *The map $P : A \longrightarrow A$ which is defined by*

$$P(\delta_s \otimes u) = \delta_{s\triangleleft u\tau(a^L,a)^{-1}} \otimes \tau(a^L, a)\, u^{-1}\,,$$

where $a = \langle\delta_s \otimes u\rangle$, is a type B morphism.

Proof. First we check the grade, i.e. $\langle P(\delta_s \otimes u)\rangle = a^L$. It is known that $s \cdot a = s\triangleleft u$. Now let $\langle P(\delta_s \otimes u)\rangle = b$ and $\tau(a^L, a) = w$, then

$$(s\triangleleft uw^{-1})\triangleleft w\, u^{-1} = s\triangleleft uw^{-1}w\, u^{-1} = s = (s\triangleleft uw^{-1}) \cdot b\,,$$

which implies that

$$s\cdot a=\big((s\triangleleft uw^{-1})\cdot b\big)\cdot a=\big(s\triangleleft uw^{-1}\tau(b,a)\big)\cdot(b\cdot a)\,.$$

But $s\cdot a=s\triangleleft u$, which implies that $b=a^L$ as required.

Now we check the G-action, i.e. $P\big((\delta_s\otimes u)\bar{\triangleleft}v\big)=P(\delta_s\otimes u)\bar{\triangleleft}\big(\langle\delta_s\otimes u\rangle\triangleright v\big)$. We start with the left hand side as the following:

$$\begin{aligned}P\big((\delta_s\otimes u)\bar{\triangleleft}v\big)&=P\big(\delta_{s\triangleleft(a\triangleright v)}\otimes(a\triangleright v)^{-1}uv\big)\\&=\delta_{s\triangleleft uv\tau((a\triangleleft v)^L\,,\,a\triangleleft v)^{-1}}\otimes\tau\big((a\triangleleft v)^L\,,\,a\triangleleft v\big)\,v^{-1}u^{-1}(a\triangleright v)\,.\end{aligned}\tag{3.18}$$

To simplify the last equation we need to do the following calculation: Note that $a^L\cdot a=e$, so $(a^L\cdot a)\triangleleft v=e\triangleleft v=e$ or $\big(a^L\triangleleft(a\triangleright v)\big)\cdot(a\triangleleft v)=e$, which means that $\big(a^L\triangleleft(a\triangleright v)\big)=(a\triangleleft v)^L$. Thus $\tau\big((a\triangleleft v)^L\,,\,a\triangleleft v\big)^{-1}=\tau\big((a^L\triangleleft(a\triangleright v))\,,\,a\triangleleft v\big)^{-1}$, which, from the identities between $(M,\cdot)$ and τ, implies that

$$v\,\tau\big((a\triangleleft v)^L\,,\,a\triangleleft v\big)^{-1}=v\,\tau\big((a^L\triangleleft(a\triangleright v))\,,\,a\triangleleft v\big)^{-1}=\tau(a^L\,,\,a)^{-1}\big(a^L\triangleright(a\triangleright v)\big)\,.$$

So equation (3.18) can be rewritten as

$$P\big((\delta_s\otimes u)\bar{\triangleleft}v\big)=\delta_{s\triangleleft u\tau(a^L\,,\,a)^{-1}(a^L\triangleright(a\triangleright v))}\otimes\big(a^L\triangleright(a\triangleright v)\big)^{-1}\tau(a^L\,,\,a)u^{-1}(a\triangleright v)\,.$$

On the other hand if we put $a\triangleright v=\bar{v}$, then the right hand side is given as the following:

$$\begin{aligned}P(\delta_s\otimes u)\bar{\triangleleft}\big(\langle\delta_s\otimes u\rangle\triangleright v\big)&=P(\delta_s\otimes u)\bar{\triangleleft}\big(a\triangleright v\big)=P(\delta_s\otimes u)\bar{\triangleleft}\bar{v}\\&=\big(\delta_{s\triangleleft u\tau(a^L,a)^{-1}}\otimes\tau(a^L,a)\,u^{-1}\big)\bar{\triangleleft}\bar{v}\\&=\delta_{(s\triangleleft u\tau(a^L,a)^{-1})\triangleleft(a^L\triangleright\bar{v})}\otimes(a^L\triangleright\bar{v})^{-1}\tau(a^L,a)\,u^{-1}\bar{v}\\&=\delta_{s\triangleleft u\tau(a^L,a)^{-1}\,(a^L\triangleright\bar{v})}\otimes(a^L\triangleright\bar{v})^{-1}\tau(a^L,a)\,u^{-1}\bar{v}\\&=\delta_{s\triangleleft u\tau(a^L,a)^{-1}\,(a^L\triangleright(a\triangleright v))}\otimes\big(a^L\triangleright(a\triangleright v)\big)^{-1}\tau(a^L,a)\,u^{-1}(a\triangleright v)\,,\end{aligned}$$

which is the same as the left hand side as required. □

Proposition 3.4.5 *For the algebra A the map $P: A \to A$ defined in 3.4.4 satisfies*

$$P\big(P(\delta_s \otimes u)\bar{\triangleleft}\tau(a, a^L)\big) = \mathrm{id_A}\,,$$

where $(\delta_s \otimes u) \in A$ and $a = \langle \delta_s \otimes u \rangle$.

Proof. First note that $s^{LL} = s$ implies $s^L \triangleleft \tau(s, s^L) = s^L$ and $s^L = s^R$. Now if we put $v = \tau(a, a^L)$ then

$$P(\delta_s \otimes u)\bar{\triangleleft} v = \delta_{s \triangleleft u \tau(a^L, a)^{-1}(a^L \triangleright v)} \otimes (a^L \triangleright v)^{-1} \tau(a^L, a)\, u^{-1} v\,.$$

But $(a^L \triangleright v)^{-1} = \big(a^L \triangleright \tau(a, a^L)\big)^{-1} = \tau\big(a^L \triangleleft \tau(a, a^L)\,,\, a \cdot a^L\big)\tau(a^L \cdot a, a^L)^{-1}\tau(a^L, a)^{-1} = \tau(a^L, a)^{-1}$, so

$$P(\delta_s \otimes u)\bar{\triangleleft}\tau(a, a^L) = \delta_{s \triangleleft u} \otimes u^{-1}\tau(a, a^L)\,.$$

Applying P to this again gives

$$\begin{aligned} P\big(P(\delta_s \otimes u)\bar{\triangleleft}\tau(a, a^L)\big) &= P\big(\delta_{s\triangleleft u} \otimes u^{-1}\tau(a, a^L)\big) \\ &= \delta_{(s\triangleleft u)\triangleleft u^{-1}\tau(a,a^L)\tau\left((a^L\triangleleft v)^L, a^L\triangleleft v\right)^{-1}} \otimes \tau\big((a^L\triangleleft v)^L, a^L \triangleleft v\big)\tau(a, a^L)^{-1}u \\ &= \delta_{s\triangleleft\tau(a,a^L)\tau(a^{LL},a^L)^{-1}} \otimes \tau(a^{LL}, a^L)\tau(a, a^L)^{-1}u \\ &= \delta_{s\triangleleft\tau(a,a^L)\tau(a,a^L)^{-1}} \otimes \tau(a, a^L)\tau(a, a^L)^{-1}u \\ &= \delta_s \otimes u\,. \quad \square \end{aligned}$$

We can also consider the following map, which looks like the formula for the antipode $S(\delta_s \otimes u) = \delta_{(s\triangleleft u)^{-1}} \otimes (s \triangleright u)^{-1}$ given in [6], in the case where M is a subgroup of X. Unfortunately, it is not a type A or a type B morphism. However the following result is true even without our simplifying assumptions that $s^{LL} = s$ and $s^L \triangleright (s \triangleright u) = u$.

Proposition 3.4.6 *For the algebra A there is a map $F : A \to A$ defined by*

$$F(\delta_s \otimes u) = \delta_{(s\triangleleft u)^L} \otimes (s\triangleright u)^{-1}\tau(s, s^R)\,,$$

for $(\delta_s \otimes u) \in A$, satisfying $F^2 = \mathrm{id_A}$.

Proof. First we need to do the following calculation: We know that $su = (s\triangleright u)(s\triangleleft u)$, so

$$u^{-1}s^{-1} = (s\triangleleft u)^{-1}(s\triangleright u)^{-1} = \tau\big((s\triangleleft u)^L, (s\triangleleft u)\big)^{-1}(s\triangleleft u)^L(s\triangleright u)^{-1}\,,$$

which can be rewritten as

$$\tau\big((s\triangleleft u)^L, (s\triangleleft u)\big)u^{-1}\tau(s^L, s)^{-1}s^L = (s\triangleleft u)^L(s\triangleright u)^{-1}\,,$$

but $(s\triangleleft u)^L(s\triangleright u)^{-1} = \big((s\triangleleft u)^L\triangleright(s\triangleright u)^{-1}\big)\big((s\triangleleft u)^L\triangleleft(s\triangleright u)^{-1}\big)$ which implies that

$$(s\triangleleft u)^L\triangleright(s\triangleright u)^{-1} = \tau\big((s\triangleleft u)^L, (s\triangleleft u)\big)u^{-1}\tau(s^L, s)^{-1} \quad \text{and} \quad (s\triangleleft u)^L\triangleleft(s\triangleright u)^{-1} = s^L\,. \tag{3.19}$$

Now we apply F twice to $\delta_s \otimes u$ to get

$$F^2(\delta_s \otimes u) = \delta_{\left((s\triangleleft u)^L\triangleleft(s\triangleleft u)^{-1}\tau(s,s^R)\right)^L} \otimes \big((s\triangleleft u)^L\triangleright(s\triangleright u)^{-1}\tau(s, s^R)\big)^{-1}\tau\big((s\triangleleft u)^L, ((s\triangleleft u)^L)^R\big)\,.$$

Using (3.19) we get

$$\begin{aligned} F^2(\delta_s \otimes u) &= \delta_{\left(s^L\triangleleft\tau(s,s^R)\right)^L} \otimes \Big(\tau\big((s\triangleleft u)^L, s\triangleleft u\big)u^{-1}\tau(s^L, s)^{-1}(s^L\triangleright\tau(s, s^R))\Big)^{-1}\tau\big((s\triangleleft u)^L, s\triangleleft u\big) \\ &= \delta_{\left(s^L\triangleleft\tau(s,s^R)\right)^L} \otimes \big(s^L\triangleright\tau(s, s^R)\big)^{-1}\tau(s^L, s)\, u \\ &= \delta_{\left(s^L\triangleleft\tau(s,s^R)\right)^L} \otimes u\,, \end{aligned}$$

as $\big(s^L\triangleright\tau(s, s^R)\big)^{-1} = \tau\big(s^L\triangleleft\tau(s, s^R)\,,\ s\cdot s^R\big)\tau(s^L\cdot s, s^R)^{-1}\tau(s^L, s)^{-1} = \tau(s^L, s)^{-1}$. Now we know that

$$\big(s^L\triangleleft\tau(s, s^R)\big)\cdot(s\cdot s^R) = s^L\triangleleft\tau(s, s^R)\,,$$

but on the other hand

$$\left(s^L \triangleleft \tau(s, s^R)\right) \cdot (s \cdot s^R) = (s^L \cdot s) \cdot s^R = s^R \,.$$

Thus $s^R = s^L \triangleleft \tau(s, s^R)$, or equivalently $s = \left(s^L \triangleleft \tau(s, s^R)\right)^L$. Therefore,

$$F^2(\delta_s \otimes u) = \delta_s \otimes u \,. \quad \Box$$

Chapter 4

Representations

We begin the chapter by describing the indecomposable objects in $\mathcal{C}$, in a similar manner to that used in [6]. A detailed example is given using the group D_6. Then we show how to find the dual objects in the category, and again illustrate this with the example. Next we explicitly evaluate in $\mathcal{D}$ the standard diagram for trace in a ribbon category [19]. Then we define the character of an object in $\mathcal{D}$ as an element of the dual of the braided Hopf algebra D. This element is shown to be right adjoint invariant. Additionally, we show that the character is multiplicative for the tensor product of objects. A formula is found for the character in $\mathcal{D}$ in terms of characters of group representations. Finally we use integrals to construct abstract projection operators to show that general objects in $\mathcal{D}$ can be split into a sum of simple objects. This chapter, with the exception of the last section, has already been sent for publication as a part of a paper by myself and my supervisor E. J. Beggs [2].

4.1 Indecomposable objects in $\mathcal{C}$

The objects of $\mathcal{C}$ are the right representations of the algebra A described in [4]. We now look at the indecomposable objects in $\mathcal{C}$, or the irreducible representations of A, in a manner similar to that used in [6].

Theorem 4.1.1 *The indecomposable objects in $\mathcal{C}$ are of the form*

$$V = \bigoplus_{s \in \mathcal{O}} V_s$$

where $\mathcal{O}$ is an orbit in M under the G action $\triangleleft$, and each V_s is an irreducible right representation of the stabilizer of s, stab(s). *Every object T in $\mathcal{C}$ can be written as a direct sum of indecomposable objects in $\mathcal{C}$.*

Proof. For an object T in $\mathcal{C}$ we can use the M-grading to write

$$T = \bigoplus_{s \in M} T_s \,, \tag{4.1}$$

but as M is a disjoint union of orbits $\mathcal{O}_s = \{s \triangleleft u : u \in G\}$ for $s \in M$, T can be rewritten as a disjoint sum over orbits,

$$T = \bigoplus_{\mathcal{O}} T_{\mathcal{O}} \,, \tag{4.2}$$

where

$$T_{\mathcal{O}} = \bigoplus_{s \in \mathcal{O}} T_s . \tag{4.3}$$

Now we will define the stabilizer of $s \in \mathcal{O}$, which is a subgroup of G, as

$$\mathrm{stab}(s) = \{u \in G : s \triangleleft u = s\}.$$

As $\langle \eta \bar{\triangleleft} u \rangle = \langle \eta \rangle \triangleleft u$ for all $\eta \in T$, T_s is a representation of the group stab(s). Now fix a base point $t \in \mathcal{O}$. Because stab(t) is a finite group, T_t is a direct sum of irreducible group representations W_i for $i = 1, .., m$, i.e.,

$$T_t = \bigoplus_{i=1}^{m} W_i. \tag{4.4}$$

Suppose that $\mathcal{O} = \{t_1, t_2, ..., t_n\}$ where $t_1 = t$, and take $u_i \in G$ so that $t_i = t \triangleleft u_i$. Define

$$U_i = \bigoplus_{j=1}^{n} W_i \bar{\triangleleft} u_j \subset \bigoplus_{s \in \mathcal{O}} T_s. \tag{4.5}$$

We claim that each U_i is an indecomposable object in $\mathcal{C}$. For any $v \in G$ and $\xi \bar{\triangleleft} u_k \in W_i \bar{\triangleleft} u_k$,

$$(\xi \bar{\triangleleft} u_k) \bar{\triangleleft} v = \big(\xi \bar{\triangleleft} (u_k v u_j{}^{-1})\big) \bar{\triangleleft} u_j,$$

where $u_k v u_j{}^{-1} \in \mathrm{stab}(t)$ for some $u_j \in G$. This shows that U_i is a representation of G. By the definition of U_i, any subrepresentation of U_i which contains W_i must be all of U_i. Thus U_i is an indecomposable object in $\mathcal{C}$, and

$$T_{\mathcal{O}} = \bigoplus_{i=1}^{m} U_i. \qquad \square \tag{4.6}$$

Theorem 4.1.2 {Schur's lemma} *Let V and W be two indecomposable objects in $\mathcal{C}$, and let $\alpha : V \longrightarrow W$ be a morphism. Then α is zero or a scalar multiple of the identity.*

Proof. V and W are associated to orbits $\mathcal{O}$ and $\mathcal{O}'$ so that $V = \bigoplus_{s \in \mathcal{O}} V_s$ and $W = \bigoplus_{s \in \mathcal{O}'} W_s$. As morphisms preserve grade, if $\alpha \neq 0$, then $\mathcal{O} = \mathcal{O}'$. Now if we take $s \in \mathcal{O}$, we find that $\alpha : V_s \longrightarrow W_s$ is a map of irreducible representations of $\mathrm{stab}(s)$, so by Schur's lemma for groups, any non-zero map is a scalar multiple of the identity, and we have $V_s = W_s$ as representations of $\mathrm{stab}(s)$. Now we need to check that the multiple of the identity is the same for each $s \in \mathcal{O}$. Suppose α is a multiplication by λ on V_s. Given $t \in \mathcal{O}$, there is a $u \in G$ so that $t \triangleleft u = s$. Then for $\eta \in V_t$,

$$\alpha(\eta) = \alpha(\eta \bar{\triangleleft} u) \bar{\triangleleft} u^{-1} = \lambda(\eta \bar{\triangleleft} u) \bar{\triangleleft} u^{-1} = \lambda \eta. \qquad \square$$

Lemma 4.1.3 *Let V be an indecomposable object in $\mathcal{C}$ associated to the orbit $\mathcal{O}$. Choose $s, t \in \mathcal{O}$ and $u \in G$ so that $s \triangleleft u = t$. Then V_s and V_t are irreducible representations of* $\mathrm{stab}(s)$ *and* $\mathrm{stab}(t)$ *respectively, and the group characters obey $\chi_{V_t}(v) = \chi_{V_s}(u\,v\,u^{-1})$.*

Proof. Note that $\tilde{\triangleleft}u$ is an invertible map from V_s to V_t. Then we have the commuting diagram

$$\begin{array}{ccc} V_s & \xrightarrow{\tilde{\triangleleft}uvu^{-1}} & V_s \\ \Big\downarrow{\scriptstyle \tilde{\triangleleft}u} & & \Big\downarrow{\scriptstyle \tilde{\triangleleft}u} \\ V_t & \xrightarrow{\tilde{\triangleleft}v} & V_t \end{array}$$

which implies that $\mathrm{trace}(\tilde{\triangleleft}uvu^{-1} : V_s \to V_s) = \mathrm{trace}(\tilde{\triangleleft}v : V_t \to V_t)$. $\square$

4.2 An example of indecomposable objects

We give an example of indecomposable objects in the categories discussed in the last section. As we will later want to have a category with braiding, we use the double construction in [4]. We also use lemma 4.1.3 to list the group characters [9] for every point in the orbit in terms of the given base points.

Take X to be the dihedral group $D_6 = \langle a, b : a^6 = b^2 = e, ab = ba^5 \rangle$, whose elements we list as $\{e, a, a^2, a^3, a^4, a^5, b, ba, ba^2, ba^3, ba^4, ba^5\}$, and G to be the non-abelian normal subgroup of order 6 generated by a^2 and b, i.e. $G = \{e, a^2, a^4, b, ba^2, ba^4\}$. We choose $M = \{e, a\}$. The center of D_6 is the subgroup $\{e, a^3\}$, and it has the following conjugacy classes: $\{e\}$, $\{a^3\}$, $\{a^2, a^4\}$, $\{a, a^5\}$, $\{b, ba^2, ba^4\}$ and $\{ba, ba^3, ba^5\}$.

The category $\mathcal{D}$ consists of right representations of the group $X = D_6$ which are graded by $Y = D_6$ (as a set), using the actions $\tilde{\triangleleft} : Y \times X \to Y$ and $\tilde{\triangleright} : Y \times X \to X$ which are defined as follows:

$$y \tilde{\triangleleft} x = x^{-1} y x, \quad \text{and} \quad vt \tilde{\triangleright} x = v^{-1} x v' = t x t'^{-1},$$

for $x \in X$, $y \in Y$, $v, v' \in G$ and $t, t' \in M$ where $vt \tilde{\triangleleft} x = v' t'$.

Now let V be an indecomposable object in $\mathcal{D}$. We get the following cases:

Case (1): Take the orbit $\{e\}$ with base point e, whose stabilizer is the whole of D_6. There are six possible irreducible group representations of the stabilizer, with their characters given by table(1) [28]:

irreps		$\{e\}$	$\{a^3\}$	$\{b, ba^2, ba^4\}$	$\{ba, ba^3, ba^5\}$	$\{a^2, a^4\}$	$\{a, a^5\}$
1_1	2_1	1	1	1	1	1	1
1_2	2_2	1	-1	-1	1	1	-1
1_3	2_3	1	-1	1	-1	1	-1
1_4	2_4	1	1	-1	-1	1	1
1_5	2_5	2	-2	0	0	-1	1
1_6	2_6	2	2	0	0	-1	-1

table (1)

Case (2): Take the orbit $\{a^3\}$ with base point a^3, whose stabilizer is the whole of D_6. There are six possible irreducible representations $\{2_1, 2_2, 2_3, 2_4, 2_5, 2_6\}$, with characters given by table(1).

Case (3): Take the orbit $\{a^2, a^4\}$ with base point a^2, whose stabilizer is $\{e, a, a^2, a^3, a^4, a^5\}$. There are six irreducible representations $\{3_0, 3_1, 3_2, 3_3, 3_4, 3_5\}$, with characters given by table(2), where $\omega = e^{i\pi/3}$. Applying lemma 4.1.3 gives $\chi_{V_{a^4}}(v) = \chi_{V_{a^2}}(bvb)$.

irreps		e	a	a^2	a^3	a^4	a^5
3_0	4_0	1	1	1	1	1	1
3_1	4_1	1	ω^1	ω^2	ω^3	ω^4	ω^5
3_2	4_2	1	ω^2	ω^4	1	ω^2	ω^4
3_3	4_3	1	ω^3	1	ω^3	1	ω^3
3_4	4_4	1	ω^4	ω^2	1	ω^4	ω^2
3_5	4_5	1	ω^5	ω^4	ω^3	ω^2	ω^1

table (2)

Case (4): Take the orbit $\{a, a^5\}$ with base point a, whose stabilizer is $\{e, a, a^2, a^3, a^4, a^5\}$. There are six irreducible representations $\{4_0,\ 4_1,\ 4_2,\ 4_3,\ 4_4,\ 4_5\ \}$ with characters given in table(2). Applying lemma 4.1.3 gives $\chi_{V_{a^5}}(v) = \chi_{V_a}(ba^2vba^2)$.

Case (5): Take the orbit $\{b, ba^2, ba^4\}$ with base point b, whose stabilizer is $\{e, a^3, b, ba^3\}$. There are four irreducible representations with characters given by table(3). Applying lemma 4.1.3 gives $\chi_{V_{ba^2}}(v) = \chi_{V_b}(a^4va^2)$ and $\chi_{V_{ba^4}}(v) = \chi_{V_b}(a^2va^4)$.

	e	a^3	b	ba^3
5_{++}	1	1	1	1
5_{+-}	1	1	-1	-1
5_{-+}	1	-1	1	-1
5_{--}	1	-1	-1	1

table (3)

Case (6): Take the orbit $\{ba, ba^3, ba^5\}$ with base point ba, whose stabilizer is $\{e, a^3, ba, ba^4\}$. There are four irreducible representations with character given by table(4). Applying lemma 4.1.3 gives $\chi_{V_{ba^3}}(v) = \chi_{V_{ba}}(a^4va^2)$ and $\chi_{V_{ba^5}}(v) = \chi_{V_{ba}}(a^2va^4)$.

	e	a^3	ba	ba^4
6_{++}	1	1	1	1
6_{-+}	1	-1	1	-1
6_{+-}	1	1	-1	-1
6_{--}	1	-1	-1	1

table (4)

4.3 Duals of indecomposable objects in $\mathcal{C}$

Given an irreducible object V with associated orbit $\mathcal{O}$ in $\mathcal{C}$, how do we find its dual V^*? The dual would be described as in section 4.1, by an orbit, a base point in the orbit and a right group representation of the stabilizer of the base point. Using the formula $(s^L \cdot s) \triangleleft u = \left(s^L \triangleleft (s \triangleright u)\right) \cdot (s \triangleleft u) = e$, we see that the left inverse of a point in the orbit containing s is in the orbit containing s^L. By using the evaluation map from $V^* \otimes V$ to the field we can take $(V^*)_{s^L} = (V_s)^*$ as vector spaces. We use $\check{\triangleleft}$ as the action of stab(s) on $(V_s)^*$, i.e. $(\alpha\check{\triangleleft}z)(\xi\bar{\triangleleft}z) = \alpha(\xi)$ for $\alpha \in (V_s)^*$ and $\xi \in V_s$. The action $\bar{\triangleleft}$ of stab(s^L) on $(V^*)_{s^L}$ is given by $\alpha\bar{\triangleleft}(s\triangleright z) = \alpha\check{\triangleleft}z$ for $z \in \text{stab}(s)$. In terms of group characters this gives

$$\chi_{(V^*)_{s^L}}(s\triangleright z) = \chi_{(V_s)^*}(z) , \quad z \in \text{stab}(s) .$$

If we take $\mathcal{O}^L = \{s^L : s \in \mathcal{O}\}$ to have base point p, and choose $u \in G$ so that $p\triangleleft u = s^L$, then using lemma 4.1.3 gives

$$\chi_{(V^*)_{s^L}}(s\triangleright z) = \chi_{(V_s)^*}(z) = \chi_{(V^*)_p}(u(s\triangleright z)u^{-1}) , \quad z \in \text{stab}(s) . \tag{4.7}$$

This formula allows us to find the character of V^* at its base point p as a representation of stab(p) in terms of the character of the dual of V_s as a representation of stab(s).

Lemma 4.3.1 *In $\mathcal{C}$ we can regard the dual $(V \otimes W)^*$ as $W^* \otimes V^*$ with the evaluation*

$$(\alpha \otimes \beta)(\xi \otimes \eta) = \big(\alpha\bar{\triangleleft}\tau(\langle\beta\rangle, \langle\xi\rangle \cdot \langle\eta\rangle)\big)(\eta) \ \big(\beta\bar{\triangleleft}\tau(\langle\xi\rangle, \langle\eta\rangle)^{-1}\big)(\xi).$$

Given a basis $\{\xi\}$ of V and a basis $\{\eta\}$ of W, the dual basis $\{\widehat{\xi \otimes \eta}\}$ of $W^* \otimes V^*$ can be written in terms of the dual basis of V^* and W^* as

$$\widehat{\xi \otimes \eta} = \hat{\eta}\bar{\triangleleft}\tau(\langle\xi\rangle^L \triangleleft\tau(\langle\xi\rangle, \langle\eta\rangle), \langle\xi\rangle \cdot \langle\eta\rangle)^{-1} \otimes \hat{\xi}\bar{\triangleleft}\tau(\langle\xi\rangle, \langle\eta\rangle) .$$

Proof. Applying the associator to $(\alpha \otimes \beta) \otimes (\xi \otimes \eta)$ gives

$$\alpha\bar{\triangleleft}\tau(\langle\beta\rangle, \langle\xi\rangle \cdot \langle\eta\rangle) \otimes \big(\beta \otimes (\xi \otimes \eta)\big) ,$$

and then applying the inverse associator gives

$$\alpha\bar{\triangleleft}\tau(\langle\beta\rangle, \langle\xi\rangle \cdot \langle\eta\rangle) \otimes \Big(\big(\beta\bar{\triangleleft}\tau(\langle\xi\rangle, \langle\eta\rangle)^{-1} \otimes \xi\big) \otimes \eta\Big) .$$

Applying the evaluation map first to $\beta\bar{\triangleleft}\tau(\langle\xi\rangle,\langle\eta\rangle)^{-1}\otimes\xi$, then to $\alpha\bar{\triangleleft}\tau(\langle\beta\rangle,\langle\xi\rangle\cdot\langle\eta\rangle)\otimes\eta$ gives the first equation. For the evaluation to be non-zero, we need $\left(\langle\beta\rangle\triangleleft\tau(\langle\xi\rangle,\langle\eta\rangle)^{-1}\right)\cdot\langle\xi\rangle = e$ which implies $\langle\beta\rangle\triangleleft\tau(\langle\xi\rangle,\langle\eta\rangle)^{-1} = \langle\xi\rangle^{L}$, or equivalently $\langle\beta\rangle = \langle\xi\rangle^{L}\triangleleft\tau(\langle\xi\rangle,\langle\eta\rangle)$. This gives the second equation. $\square$

Example 4.3.2 *Using (4.7) we calculate the duals of the objects given in the last section.*

Case (1): *The orbit $\{e\}$ has left inverse $\{e\}$, so $\chi_{(V^*)_e} = \chi_{(V_e)^*}$. By a calculation with group characters, all the listed irreducible representations of* stab(e) *are self-dual, so $1_r^* = 1_r$ for $r \in \{1,\ldots,6\}$.*

Case (2): *The orbit $\{a^3\}$ has left inverse $\{a^3\}$, so $\chi_{(V^*)_{a^3}} = \chi_{(V_{a^3})^*}$. As in the last case the group representations are self-dual, so $2_r^* = 2_r$ for $r \in \{1,\ldots,6\}$.*

Case (3): *The left inverse of the base point a^2 is a^4, which is still in the orbit. As group representations, the dual of 3_r is 3_{6-r} (mod 6). Applying lemma 4.1.3 to move the base point, we see that the dual of 3_r in the category is 3_r.*

Case (4): *The left inverse of the base point a is a^5, which is still in the orbit. As in the last case, the dual of 4_r in the category is 4_r.*

Case (5): *The left inverse of the base point is itself, and as group representations, all case 5 irreducible representations are self dual. We deduce that in the category the objects are self dual.*

Case (6): *Self dual, as in case (5).*

4.4 Traces in the category $\mathcal{D}$

Definition 4.4.1 *[19] The trace of a morphism $T : V \longrightarrow V$ for any object V in $\mathcal{D}$ is defined by*

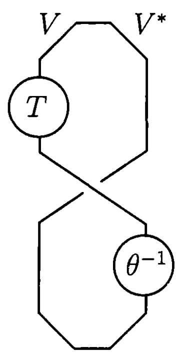


figure 4.1

Theorem 4.4.2 *If we evaluate the diagram of definition 4.4.1 in $\mathcal{D}$, we find*

$$\mathrm{trace}(T) = \sum_{\xi \in \, basis \ of \ V} \hat{\xi}\,(T(\xi))\ .$$

Proof. Begin with

$$\mathrm{coev}_V(1) = \sum_{\xi \in \text{ basis of } V} \xi \,\hat{\triangleleft}\, \tilde{\tau}(\,\|\xi\|^L, \|\xi\|\,)^{-1} \otimes \hat{\xi} = \sum_{\xi \in \text{ basis of } V} \xi \,\hat{\triangleleft}\, \tau(\,\langle\xi\rangle^L, \langle\xi\rangle\,)^{-1} \otimes \hat{\xi},$$

and applying $T \otimes \mathrm{id}$ to this gives

$$\sum_{\xi \in \text{ basis of } V} T\left(\xi \,\hat{\triangleleft}\, \tau(\,\langle\xi\rangle^L, \langle\xi\rangle\,)^{-1}\right) \otimes \hat{\xi} = \sum_{\xi \in \text{ basis of } V} T(\,\xi\,) \,\hat{\triangleleft}\, \tau(\,\langle\xi\rangle^L, \langle\xi\rangle\,)^{-1} \otimes \hat{\xi}.$$

Next apply the braiding map to the last equation to get

$$\sum_{\xi \in \text{ basis of } V} \Psi\left(T(\,\xi\,) \,\hat{\triangleleft}\, \tau(\,\langle\xi\rangle^L, \langle\xi\rangle\,)^{-1} \otimes \hat{\xi}\,\right) = \sum_{\xi \in \text{ basis of } V} \hat{\xi} \,\hat{\triangleleft}\, (\langle\xi'\rangle \triangleleft |\hat{\xi}\,|)^{-1} \otimes \xi' \,\hat{\triangleleft}\, |\,\hat{\xi}\,| \qquad (4.8)$$

where $\xi' = T(\,\xi\,) \,\hat{\triangleleft}\, \tau(\,\langle\xi\rangle^L, \langle\xi\rangle\,)^{-1}$, so

$$\begin{aligned} \langle\,\xi'\,\rangle &= \langle\,T(\,\xi\,) \,\hat{\triangleleft}\, \tau(\,\langle\xi\rangle^L, \langle\xi\rangle\,)^{-1}\rangle = \langle\,T(\,\xi\,) \,\bar{\triangleleft}\, \tau(\,\langle\xi\rangle^L, \langle\xi\rangle\,)^{-1}\rangle \\ &= \langle\,T(\,\xi\,)\,\rangle \triangleleft \tau(\,\langle\xi\rangle^L, \langle\xi\rangle\,)^{-1} = \langle\,\xi\,\rangle \triangleleft \tau(\,\langle\xi\rangle^L, \langle\xi\rangle\,)^{-1}. \end{aligned} \qquad (4.9)$$

To calculate $|\hat{\xi}|$ we start with

$$\|\hat{\xi}\| = \|\xi\|^L = (\,|\xi|^{-1}\langle\xi\rangle\,)^L = |\xi|\,\tau(\,\langle\xi\rangle^L, \langle\xi\rangle\,)^{-1}\langle\xi\rangle^L,$$

which implies that $|\hat{\xi}| = \tau(\,\langle\xi\rangle^L, \langle\xi\rangle\,)\,|\xi|^{-1}$. Then

$$\begin{aligned}\hat{\xi}\,\hat{\triangleleft}\,(\langle\xi'\rangle\triangleleft|\hat{\xi}|\,)^{-1} &= \hat{\xi}\,\hat{\triangleleft}\,\big(\langle\,\xi\,\rangle\triangleleft\tau(\,\langle\xi\rangle^L,\langle\xi\rangle\,)^{-1}\tau(\,\langle\xi\rangle^L,\langle\xi\rangle\,)\,|\xi|^{-1}\,\big)^{-1}\\ &= \hat{\xi}\,\hat{\triangleleft}\,\big(\langle\,\xi\,\rangle\triangleleft|\xi|^{-1}\,\big)^{-1},\end{aligned}$$

$$\xi'\,\hat{\triangleleft}\,|\hat{\xi}| = \big(T(\xi)\,\hat{\triangleleft}\,\tau(\,\langle\xi\rangle^L,\langle\xi\rangle\,)^{-1}\big)\,\hat{\triangleleft}\big(\tau(\,\langle\xi\rangle^L,\langle\xi\rangle)|\xi|^{-1}\,\big) = T(\xi)\,\hat{\triangleleft}\,|\xi|^{-1},$$

which gives

$$\sum_{\xi\in\text{ basis of }V}\hat{\xi}\hat{\triangleleft}(\langle\xi'\rangle\triangleleft|\hat{\xi}|\,)^{-1}\otimes\xi'\hat{\triangleleft}|\hat{\xi}| = \sum_{\xi\in\text{ basis of }V}\hat{\xi}\hat{\triangleleft}\,\big(\langle\xi\,\rangle\triangleleft|\xi|^{-1}\,\big)^{-1}\otimes T(\xi)\hat{\triangleleft}|\xi|^{-1}. \quad (4.10)$$

Next

$$\begin{aligned}\theta^{-1}\big(T(\,\xi\,)\,\hat{\triangleleft}\,|\xi|^{-1}\,\big) &= \big(T(\,\xi\,)\,\hat{\triangleleft}\,|\xi|^{-1}\,\big)\hat{\triangleleft}\,\|\,T(\,\xi\,)\,\hat{\triangleleft}\,|\xi|^{-1}\,\|^{-1}\\ &= \big(T(\,\xi\,)\,\hat{\triangleleft}\,|\xi|^{-1}\,\big)\hat{\triangleleft}\,\big(\|\,T(\,\xi\,)\,\|\tilde{\triangleleft}\,|\xi|^{-1}\,\big)^{-1}\\ &= T(\,\xi\,)\,\hat{\triangleleft}\,|\xi|^{-1}\,\big(\|\,\xi\,\|\tilde{\triangleleft}\,|\xi|^{-1}\,\big)^{-1} \qquad (4.11)\\ &= T(\,\xi\,)\,\hat{\triangleleft}\,|\xi|^{-1}\,\big(\,|\xi|\,|\xi|^{-1}\,\langle\xi\rangle\,|\xi|^{-1}\big)^{-1}\\ &= T(\,\xi\,)\,\hat{\triangleleft}\,|\xi|^{-1}\,|\xi|\,\langle\xi\rangle^{-1} = T(\,\xi\,)\,\hat{\triangleleft}\,\langle\xi\rangle^{-1},\end{aligned}$$

and finally we need to calculate

$$\mathrm{eval}\big(\hat{\xi}\hat{\triangleleft}\big(\langle\xi\rangle\triangleleft|\xi|^{-1}\big)^{-1}\otimes T(\xi)\hat{\triangleleft}\langle\xi\rangle^{-1}\big) = \big(\hat{\xi}\hat{\triangleleft}\big(\langle\xi\rangle\triangleleft|\xi|^{-1}\big)^{-1}\,\big)\,\big(T(\xi)\hat{\triangleleft}\langle\xi\rangle^{-1}\big)\,. \qquad (4.12)$$

We know, from the definition of the action on V^*, that

$$\Big(\hat{\xi}\hat{\triangleleft}\big(\|T(\xi)\|\tilde{\triangleright}x\big)\Big)\big(T(\xi)\hat{\triangleleft}x\big) = \hat{\xi}\big(T(\xi)\big). \qquad (4.13)$$

If we put $x = \langle\xi\rangle^{-1}$, we want to show that $\|T(\xi)\|\tilde{\triangleright}x = \big(\langle\xi\rangle\triangleleft|\xi|^{-1}\big)^{-1}$, so

$$\|\xi\|\tilde{\triangleleft}x = |\xi|^{-1}\langle\xi\rangle\tilde{\triangleleft}\langle\xi\rangle^{-1} = \langle\xi\rangle|\xi|^{-1} = (\langle\xi\rangle\triangleright|\xi|^{-1})(\langle\xi\rangle\triangleleft|\xi|^{-1}) = v't',$$

which implies that $t' = \langle\xi\rangle \triangleleft |\xi|^{-1}$, and hence

$$\|T(\xi)\| \tilde{\triangleright} x = \|\xi\| \tilde{\triangleright} x = |\xi|^{-1} \langle\xi\rangle \tilde{\triangleright} \langle\xi\rangle^{-1} = t \langle\xi\rangle^{-1} t'^{-1} \qquad \square \qquad (4.14)$$
$$= \langle\xi\rangle \langle\xi\rangle^{-1} (\langle\xi\rangle \triangleleft |\xi|^{-1})^{-1} = (\langle\xi\rangle \triangleleft |\xi|^{-1})^{-1}.$$

4.5 Characters in the category $\mathcal{D}$

The braided Hopf algebra D acts on the objects of $\mathcal{D}$. We define characters for these actions, and show that they have similar properties to the characters given by representations of a group.

Definition 4.5.1 *[19] The right adjoint action in $\mathcal{D}$ of the algebra D on itself is defined by*

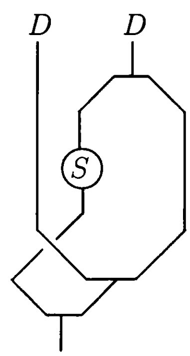


figure 4.2

Proposition 4.5.2 *[19] The right adjoint action given in 4.5.1 really is a right action, i.e.*

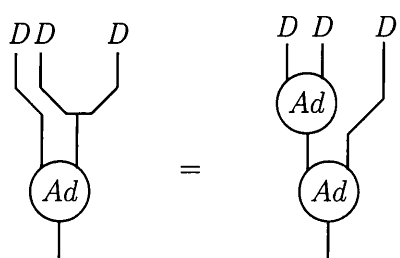
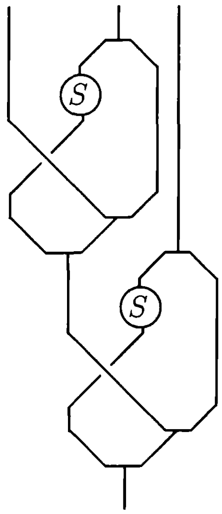
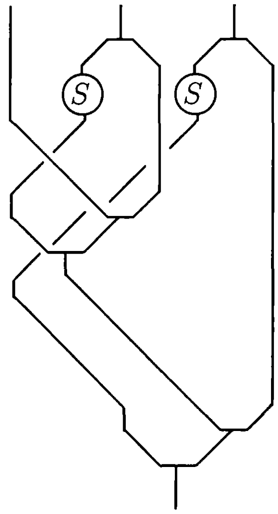
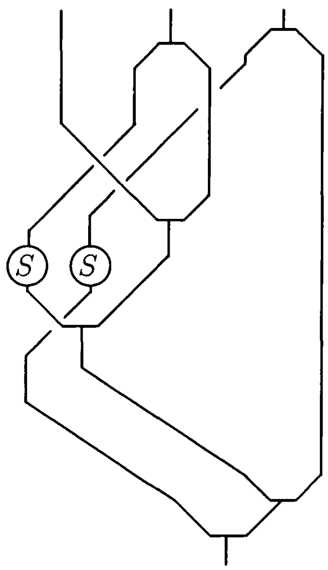


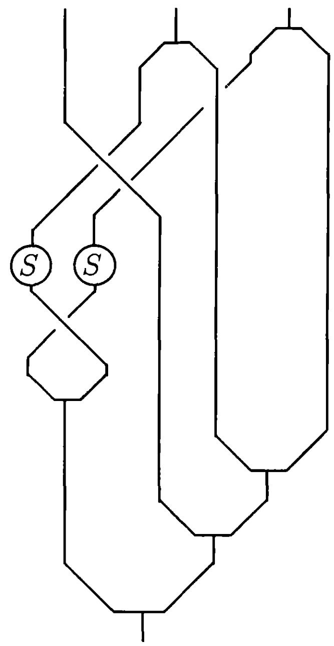
figure 4.3

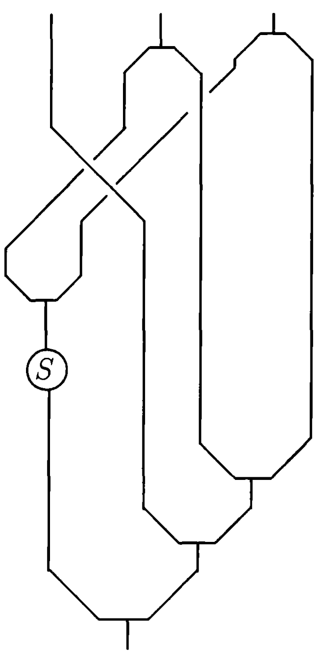
Proof.

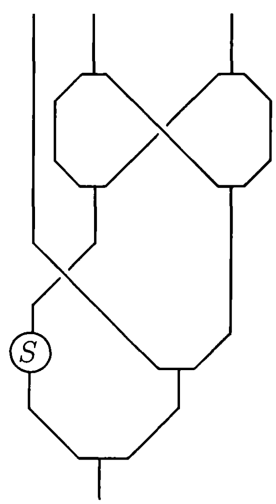
R.H.S. = 

= 

= 

= 

= 

= 

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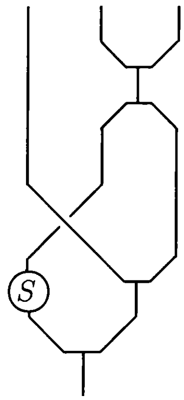

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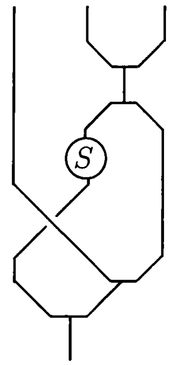


$= L.H.S.$ □

Definition 4.5.3 *The character χ_V of an object V in $\mathcal{D}$ is defined by*

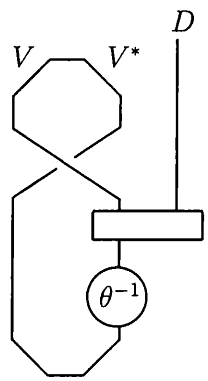


figure 4.4

To prove some properties of the character we will find it convenient to use the following result:

Lemma 4.5.4 *For an object V in $\mathcal{D}$ we have*

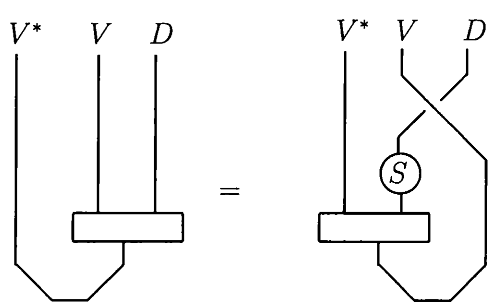
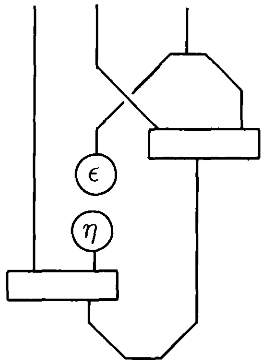
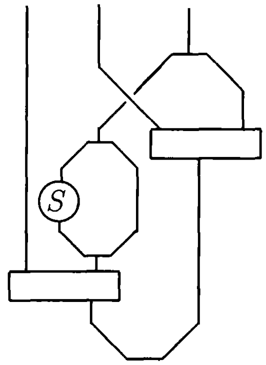
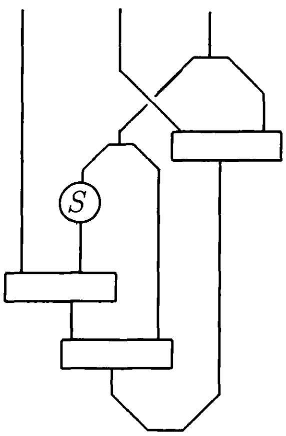
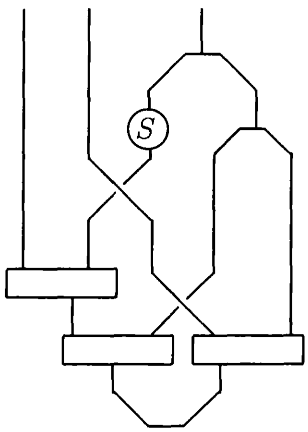
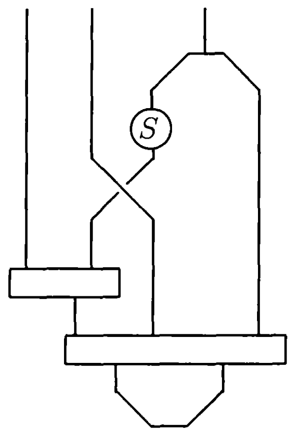
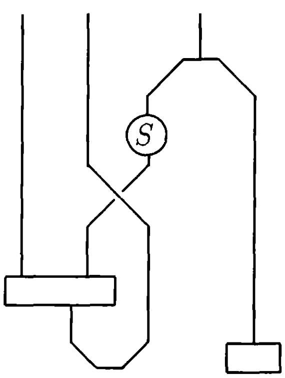
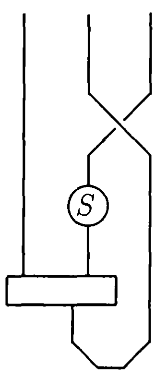


figure 4.5

Proof.

L.H.S. =  = 

=  =  = 

=  =  = R.H.S □

Proposition 4.5.5 *The character is right adjoint invariant, i.e. for an object V in $\mathcal{D}$*

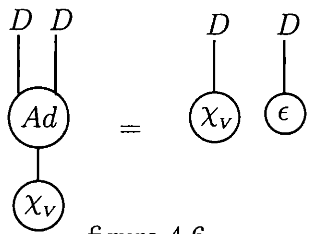
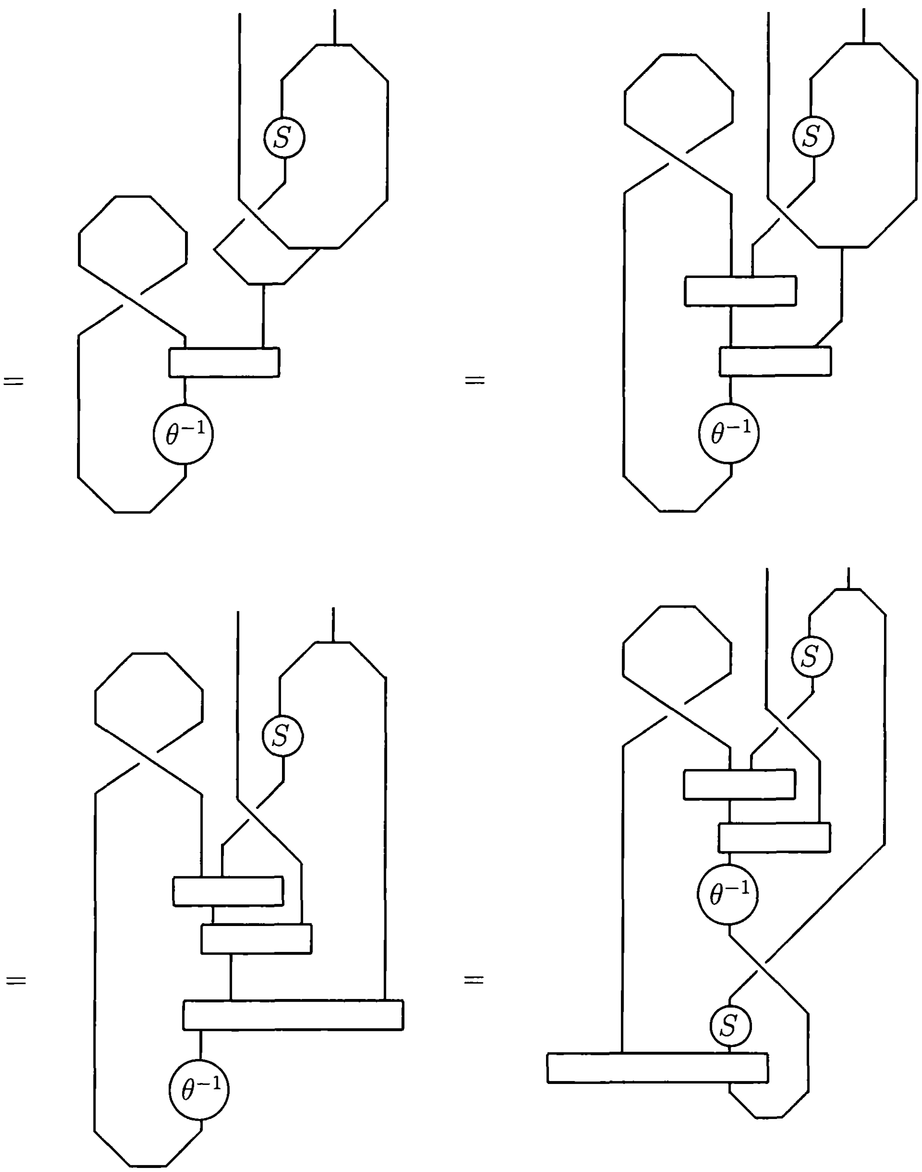


figure 4.6

Proof.

L.H.S. =



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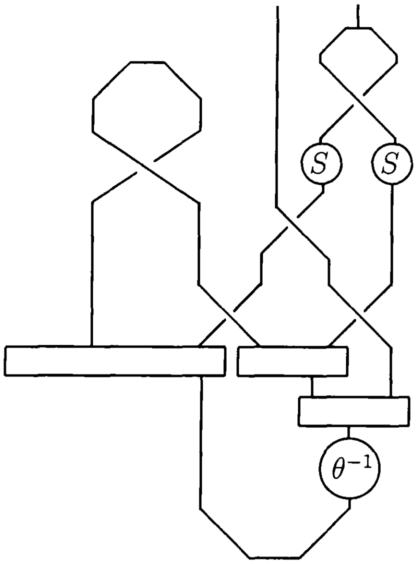

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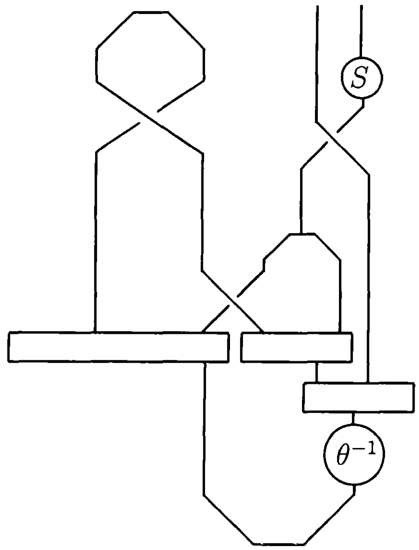

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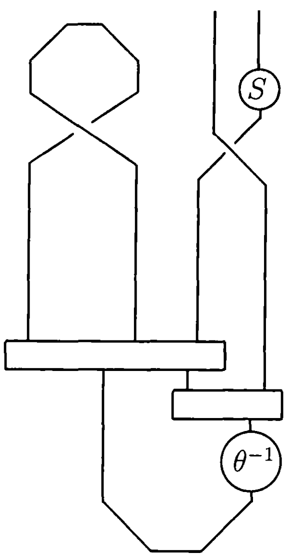

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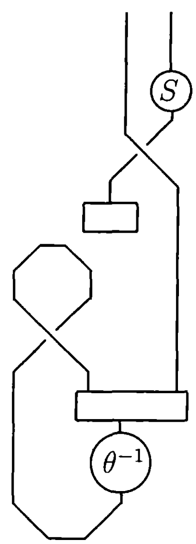

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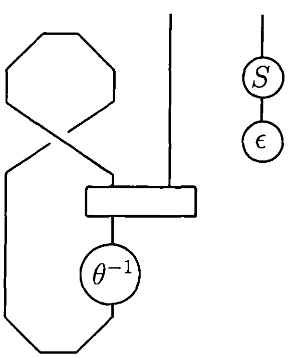

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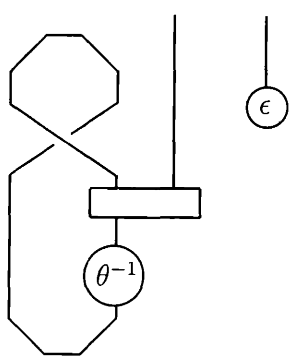


= R.H.S □

Proposition 4.5.6 *The character of a tensor product of representations is the product of the characters, i.e. for two objects V and W in $\mathcal{D}$*

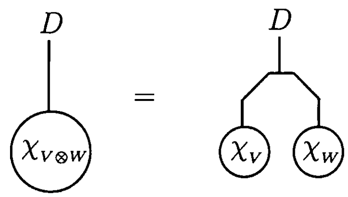
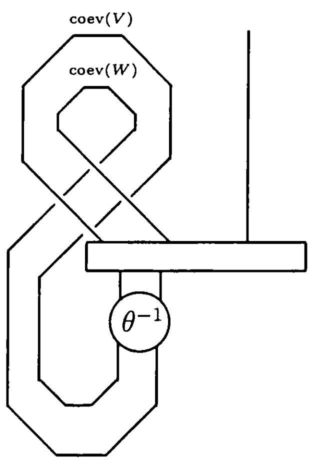


figure 4.7

Proof.

L.H.S. =



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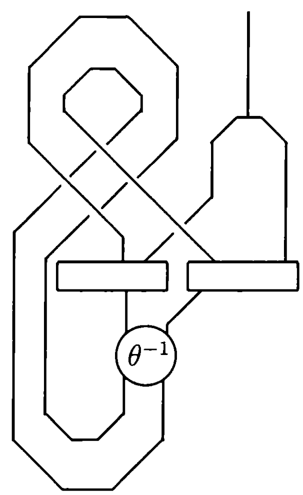

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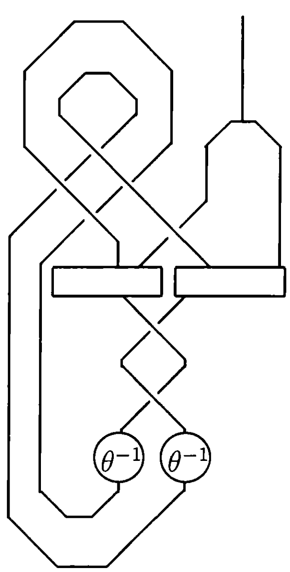

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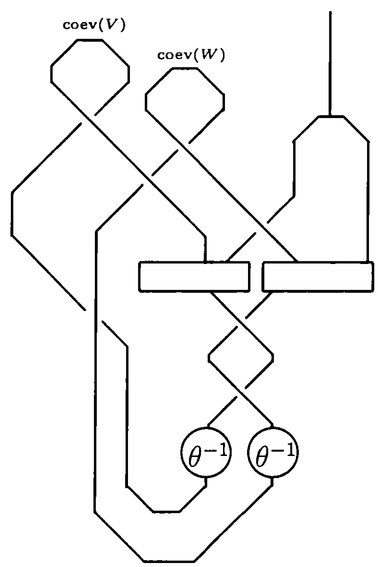
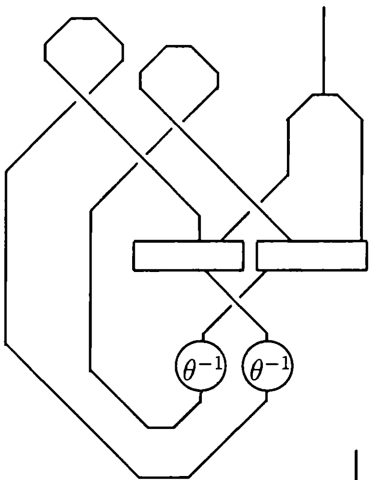
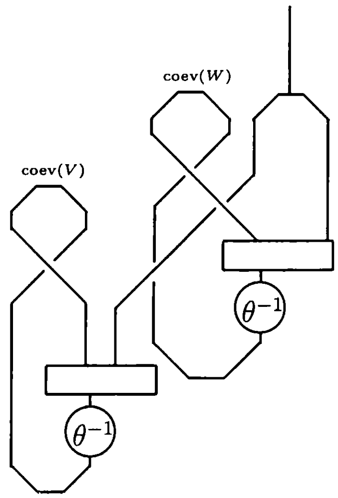
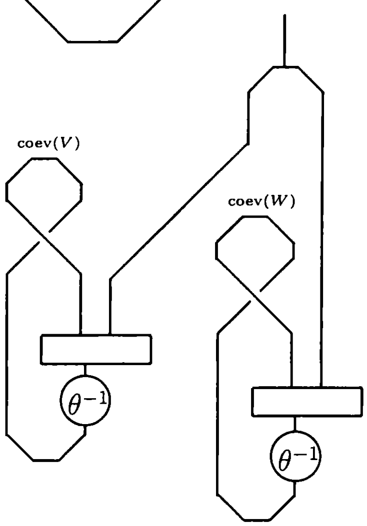


$=$  $=$ 

$=$ 

$= R.H.S.$ $\square$

Theorem 4.5.7 *In terms of the standard basis of D, we have the following formula for the character;*

$$\chi_V(\delta_y \otimes x) = \sum_{\xi \in \text{ basis of } V \text{ with } y = \langle\xi\rangle|\xi|^{-1}} \hat{\xi}\,(\,\xi \hat{\triangleleft} \langle\xi\rangle^{-1} x \,\langle\xi\rangle\,),$$

for $xy = yx$, otherwise $\chi_V(\delta_y \otimes x) = 0$.

Proof. Set $a = \delta_y \otimes x$. To have $\chi_V(a) \neq 0$ we must have $\|a\| = e$, i.e. $y = y \tilde{\triangleleft} x$ which implies that x and y commute. Assuming this, we continue with the diagrammatic definition of the character, starting with

$$\Big(\sum_{\xi \in \text{ basis of } V} \xi \hat{\triangleleft} \tilde{\tau}(\|\xi\|^L, \|\xi\|)^{-1} \otimes \hat{\xi} \Big) \otimes a = \sum_{\xi \in \text{ basis of } V} \Big(\xi \hat{\triangleleft} \tau(\langle\xi\rangle^L, \langle\xi\rangle)^{-1} \otimes \hat{\xi} \Big) \otimes a\,.$$

Next we calculate

$$\Psi\left(\xi\,\hat{\triangleleft}\,\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\otimes\hat{\xi}\right)=\hat{\xi}\,\hat{\triangleleft}\,(\langle\xi'\rangle\triangleleft|\hat{\xi}|)^{-1}\otimes\xi'\hat{\triangleleft}|\hat{\xi}| \tag{4.15}$$

where $\xi' = \xi\,\hat{\triangleleft}\,\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}$, so

$$\langle\xi'\rangle=\langle\xi\hat{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\rangle=\langle\xi\bar{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\rangle=\langle\xi\rangle\triangleleft\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}. \tag{4.16}$$

From a previous calculation we know that $|\hat{\xi}|=\tau(\langle\xi\rangle^L,\langle\xi\rangle)\,|\xi|^{-1}$, so

$$\begin{aligned}\hat{\xi}\,\hat{\triangleleft}\,(\langle\xi'\rangle\triangleleft|\hat{\xi}|)^{-1}&=\hat{\xi}\,\hat{\triangleleft}\left(\langle\xi\rangle\triangleleft\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\tau(\langle\xi\rangle^L,\langle\xi\rangle)\,|\xi|^{-1}\right)^{-1}\\&=\hat{\xi}\,\hat{\triangleleft}\left(\langle\xi\rangle\triangleleft|\xi|^{-1}\right)^{-1}\\ \xi'\,\hat{\triangleleft}|\hat{\xi}|=\left(\xi\,\hat{\triangleleft}\,\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\right)\hat{\triangleleft}&\left(\tau(\langle\xi\rangle^L,\langle\xi\rangle)\,|\xi|^{-1}\right)=\xi\,\hat{\triangleleft}|\xi|^{-1},\end{aligned}$$

which gives the next stage in the evaluation of the diagram:

$$\begin{aligned}&\sum_{\xi\in\text{ basis of }V}\Psi\left(\xi\hat{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\otimes\hat{\xi}\right)\otimes a\\&\qquad=\sum_{\xi\in\text{ basis of }V}\left(\hat{\xi}\hat{\triangleleft}\left(\langle\xi\rangle\triangleleft|\xi|^{-1}\right)^{-1}\otimes\xi\,\hat{\triangleleft}|\xi|^{-1}\right)\otimes a.\end{aligned} \tag{4.17}$$

Now we apply the associator to the last equation to get

$$\begin{aligned}&\sum_{\xi\in\text{ basis of }V}\Phi\left(\left(\hat{\xi}\hat{\triangleleft}(\langle\xi\rangle\triangleleft|\xi|^{-1})^{-1}\otimes\xi\,\hat{\triangleleft}|\xi|^{-1}\right)\otimes a\right)\\&=\sum_{\xi\in\text{ basis of }V}\hat{\xi}\hat{\triangleleft}(\langle\xi\rangle\triangleleft|\xi|^{-1})^{-1}\,\tilde{\tau}(\|\xi\,\hat{\triangleleft}|\xi|^{-1}\|^L,\|a\|)\otimes\left(\xi\,\hat{\triangleleft}|\xi|^{-1}\otimes a\right)\\&=\sum_{\xi\in\text{ basis of }V}\hat{\xi}\,\hat{\triangleleft}(\langle\xi\rangle\triangleleft|\xi|^{-1})^{-1}\,\tau(\langle\xi\,\hat{\triangleleft}|\xi|^{-1}\rangle,e)\otimes\left(\xi\,\hat{\triangleleft}|\xi|^{-1}\otimes a\right)\\&=\sum_{\xi\in\text{ basis of }V}\hat{\xi}\,\hat{\triangleleft}(\langle\xi\rangle\triangleleft|\xi|^{-1})^{-1}\otimes\left(\xi\,\hat{\triangleleft}|\xi|^{-1}\otimes(\delta_y\otimes x)\right)\end{aligned}$$

as $\tau(\langle\xi\hat{\triangleleft}|\xi|^{-1}\rangle,e)=e$. Now apply the action $\hat{\triangleleft}$ to $\xi\,\hat{\triangleleft}|\xi|^{-1}\otimes(\delta_y\otimes x)$ to get

$$(\xi\,\hat{\triangleleft}|\xi|^{-1})\hat{\triangleleft}(\delta_y\otimes x)=\delta_{y,\|\xi\,\hat{\triangleleft}|\xi|^{-1}\|}\,(\xi\,\hat{\triangleleft}|\xi|^{-1})\hat{\triangleleft}x=\delta_{y,\|\xi\|\hat{\triangleleft}|\xi|^{-1}}\,\xi\,\hat{\triangleleft}|\xi|^{-1}x, \tag{4.18}$$

and to get a non-zero answer we must have

$$y = \| \xi \| \tilde{\triangleleft} |\xi|^{-1} = |\xi|^{-1} \langle \xi \rangle \tilde{\triangleleft} |\xi|^{-1} = |\xi| \, |\xi|^{-1} \langle \xi \rangle \, |\xi|^{-1} = \langle \xi \rangle \, |\xi|^{-1}. \tag{4.19}$$

Thus the character of V is given by

$$\chi_V(\delta_y \otimes x) = \sum_{\xi \in \text{basis of } V \text{ with } y = \langle \xi \rangle |\xi|^{-1}} \operatorname{eval}\Big(\hat{\xi} \hat{\triangleleft} (\langle \xi \rangle \triangleleft |\xi|^{-1})^{-1} \otimes \theta^{-1}(\xi \hat{\triangleleft} |\xi|^{-1} x) \Big) .$$

Next

$$\begin{aligned}
\theta^{-1}(\xi \hat{\triangleleft} |\xi|^{-1} x) &= (\xi \hat{\triangleleft} |\xi|^{-1} x) \hat{\triangleleft} \big\| \xi \hat{\triangleleft} |\xi|^{-1} x \big\|^{-1} \\
&= (\xi \hat{\triangleleft} |\xi|^{-1} x) \hat{\triangleleft} \big(\| \xi \| \tilde{\triangleleft} |\xi|^{-1} x \big)^{-1} \\
&= (\xi \hat{\triangleleft} |\xi|^{-1} x) \hat{\triangleleft} \big(x^{-1} |\xi| \, |\xi|^{-1} \langle \xi \rangle \, |\xi|^{-1} x \big)^{-1} \\
&= \xi \hat{\triangleleft} |\xi|^{-1} x x^{-1} |\xi| \, \langle \xi \rangle^{-1} x = \xi \hat{\triangleleft} \langle \xi \rangle^{-1} x.
\end{aligned}$$

Now we need to calculate $\operatorname{eval}\big(\hat{\xi} \hat{\triangleleft} (\langle \xi \rangle \triangleleft |\xi|^{-1})^{-1} \otimes \xi \hat{\triangleleft} \langle \xi \rangle^{-1} x \big)$. Start with $\|\xi\| \tilde{\triangleleft} \langle \xi \rangle^{-1} x = \langle \xi \rangle |\xi|^{-1} \tilde{\triangleleft} x = \langle \xi \rangle |\xi|^{-1}$, as we only have nonzero summands for $y = \langle \xi \rangle |\xi|^{-1}$. Then

$$\begin{aligned}
&\operatorname{eval}\big(\hat{\xi} \hat{\triangleleft} (\langle \xi \rangle \triangleleft |\xi|^{-1})^{-1} \otimes \xi \hat{\triangleleft} \langle \xi \rangle^{-1} x \big) \\
&= \operatorname{eval}\big(\big(\hat{\xi} \hat{\triangleleft} (\langle \xi \rangle \triangleleft |\xi|^{-1})^{-1} \otimes \xi \hat{\triangleleft} \langle \xi \rangle^{-1} x \big) \hat{\triangleleft} \langle \xi \rangle \big) \\
&= \operatorname{eval}\big(\hat{\xi} \hat{\triangleleft} (\langle \xi \rangle \triangleleft |\xi|^{-1})^{-1} (\langle \xi \rangle |\xi|^{-1} \tilde{\triangleright} \langle \xi \rangle) \otimes \xi \hat{\triangleleft} \langle \xi \rangle^{-1} x \langle \xi \rangle \big) .
\end{aligned}$$

To find $\langle \xi \rangle |\xi|^{-1} \tilde{\triangleright} \langle \xi \rangle$, first find $\langle \xi \rangle |\xi|^{-1} \tilde{\triangleleft} \langle \xi \rangle = |\xi|^{-1} \langle \xi \rangle$, so

$$\langle \xi \rangle |\xi|^{-1} \tilde{\triangleright} \langle \xi \rangle = (\langle \xi \rangle \triangleright |\xi|^{-1})(\langle \xi \rangle \triangleleft |\xi|^{-1}) \tilde{\triangleright} \langle \xi \rangle = (\langle \xi \rangle \triangleleft |\xi|^{-1}) \langle \xi \rangle \langle \xi \rangle^{-1} = \langle \xi \rangle \triangleleft |\xi|^{-1} . \quad \square$$

Lemma 4.5.8 *Let V be an object in $\mathcal{D}$. For $\delta_y \otimes x \in D$ the character of V is given by the following formula, where $y = su^{-1}$ with $s \in M$ and $u \in G$:*

$$\chi_V(\delta_y \otimes x) = \sum_{\xi \in \text{basis of } V_{u^{-1}s}} \hat{\xi}(\xi \hat{\triangleleft} s^{-1} x s) = \chi_{V_{u^{-1}s}}(s^{-1} x s)$$

where $xy = yx$, *otherwise* $\chi_V(\delta_y \otimes x) = 0$. *Here* $\chi_{V_{u^{-1}s}}$ *is the group representation character of the representation* $V_{u^{-1}s}$ *of the group* $stab(u^{-1}s)$.

Proof. From theorem 4.5.7, we know that

$$\chi_V(\delta_y \otimes x) = \sum_{\xi \in \text{ basis of } V \text{ with } y = \langle\xi\rangle|\xi|^{-1}} \hat{\xi}\,(\,\xi \,\hat{\triangleleft}\, \langle\xi\rangle^{-1}\, x\, \langle\xi\rangle\,),$$

for with $xy = yx$. Set $s = \langle\xi\rangle$ and $u = |\xi|$, so $y = su^{-1}$. We note that $s^{-1}xs$ is in $stab(u^{-1}s)$, because

$$u^{-1}s\tilde{\triangleleft}s^{-1}xs = s^{-1}x^{-1}su^{-1}ss^{-1}xs = s^{-1}x^{-1}xsu^{-1}s = u^{-1}s.$$

It just remains to note that $\|\xi\| = |\xi|^{-1}\langle\xi\rangle = u^{-1}s$. $\square$

4.6 Projections on representations in $\mathcal{D}$ using integrals

Before going further, we recall some concepts and results from ordinary group representations to make things more comprehensible. We use [23] as a reference for group representations.

Let V be a vector space, and let W and W_o be two subspaces of V. Then for the direct sum $V = W \oplus W_o$, W_o is called a **complement** of W in V. The map p which sends each $x \in V$ to its component $w \in W$ is called the **projection** of V onto W associated with the decomposition $V = W \oplus W_o$. The image of p is W, and $p(x) = x$ for all $x \in W$. Conversely, if p is a linear map of V into itself satisfying the above two properties, then we can show that V is the direct sum of W and the kernel W_o of p.

Theorem 4.6.1 *[23] Let ρ be a linear representation of G in V and let W be a vector subspace of V stable under G. Then there exists a complement W_o of W in V which is stable under G.*

Proof. Let W_o be an arbitrary complement of W in V, and let p be the corresponding projection of V onto W. We know that from the definition of the average p_o of the conjugates of p by the elements of G:

$$p_o = \frac{1}{n} \sum_{t \in G} \rho_t \cdot p \cdot \rho_t^{-1},$$

where n is the order of G. Since p maps V into W and ρ_t preserves W we see that p_o maps V into W. We have $\rho_t^{-1}(x) \in W$, If $x \in W$ hence

$$p \cdot \rho_t^{-1}(x) = \rho_t^{-1}(x), \qquad \rho_t \cdot p \cdot \rho_t^{-1}(x) = x, \qquad \text{and} \qquad p_o(x) = x.$$

Thus p_o is a projection of V onto W, corresponding to some complement W_o of W. Moreover, we have

$$\rho_t \, p_o = p_o \, \rho_t, \qquad \text{for all} s \in G.$$

And if we compute $\rho_s \cdot p_o \cdot \rho_s^{-1}$, we find:

$$\rho_s \cdot p_o \cdot \rho_s^{-1} = \frac{1}{n} \sum_{t \in G} \rho_s \cdot \rho_t \cdot p \cdot \rho_t^{-1} \cdot \rho_s^{-1} = \frac{1}{n} \sum_{t \in G} \rho_{st} \cdot p \cdot \rho_{st}^{-1} = p_o.$$

Now for $x \in W_o$ and $s \in G$, we have $p_o(x) = 0$, hence

$$p_o \cdot \rho_s(x) = \rho_s \cdot p_o(x) = 0,$$

that is $(\rho_s(x)) \in W_o$, which shows that W_o is stable under G. $\square$

We return now to the right representation of the Hopf algebra D in the braided category $\mathcal{D}$ supposing that $\Lambda \in D$ is a right integral, i.e.

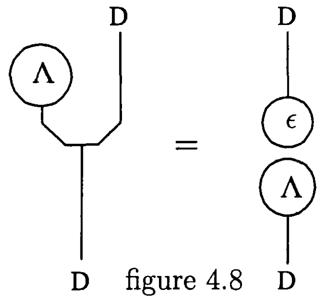


figure 4.8

Lemma 4.6.2

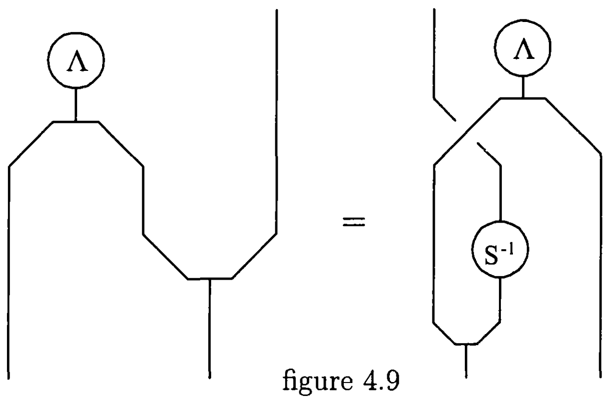


figure 4.9

Proof. Using Lemma 2.5.4:

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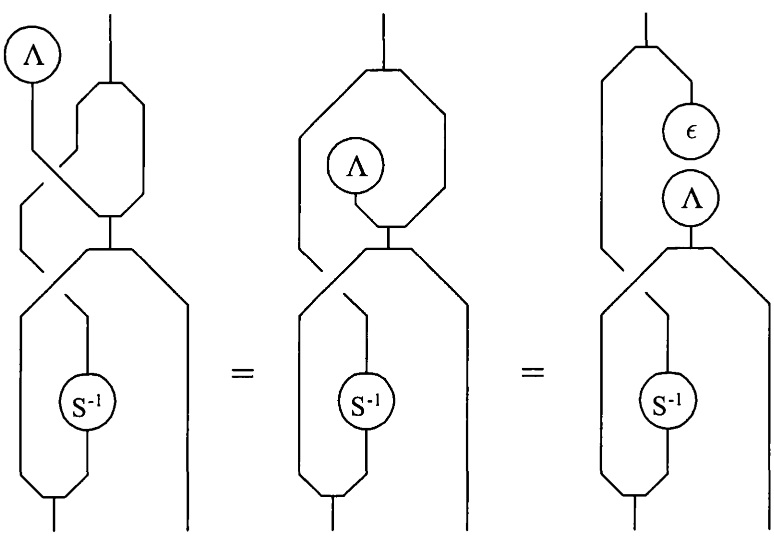


= R. H. S.

Definition 4.6.3 *For the right representations V and U of D, and a linear map (not necessarily a morphism) $t : V \longrightarrow U$, we define $t_{\circ} : V \longrightarrow U$ by*

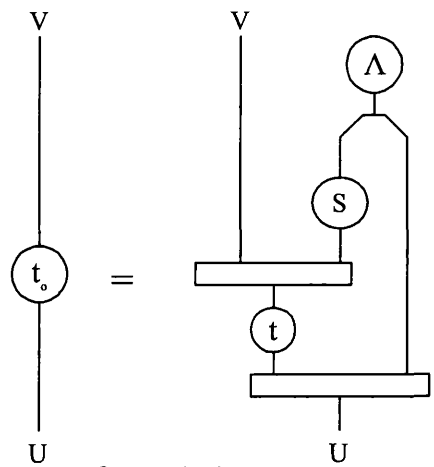


figure 4.10

Proposition 4.6.4 *The map t_o is a morphism in the category $\mathcal{D}$, i.e.*

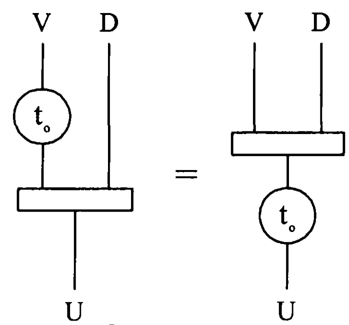
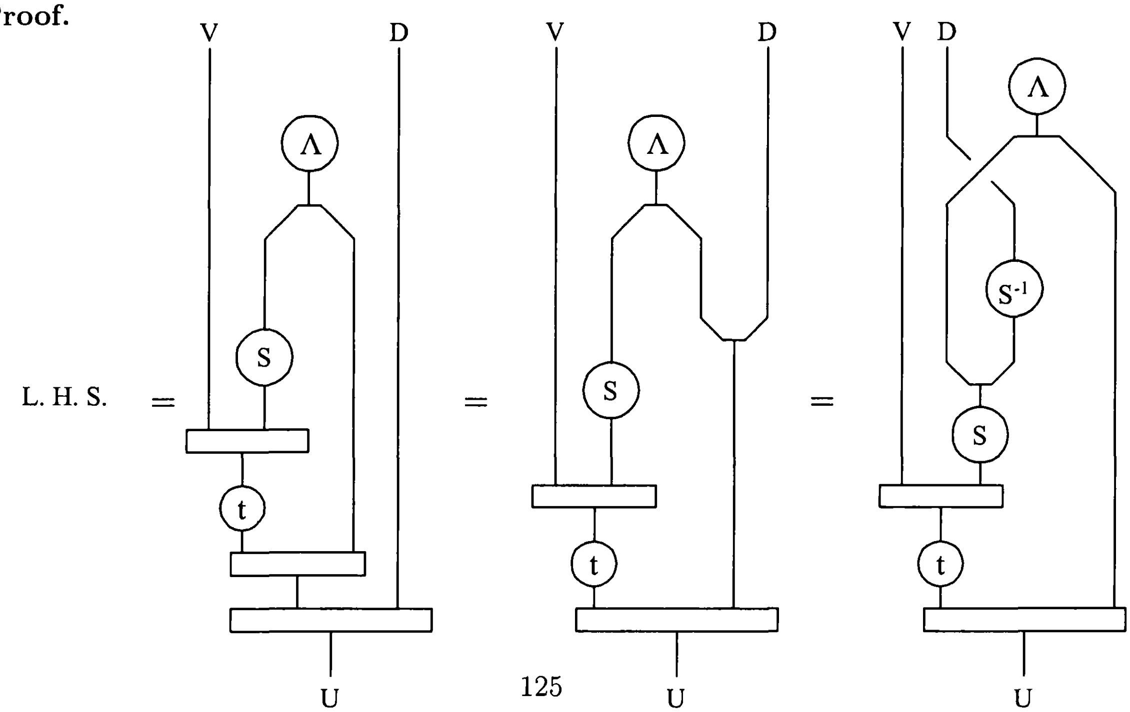
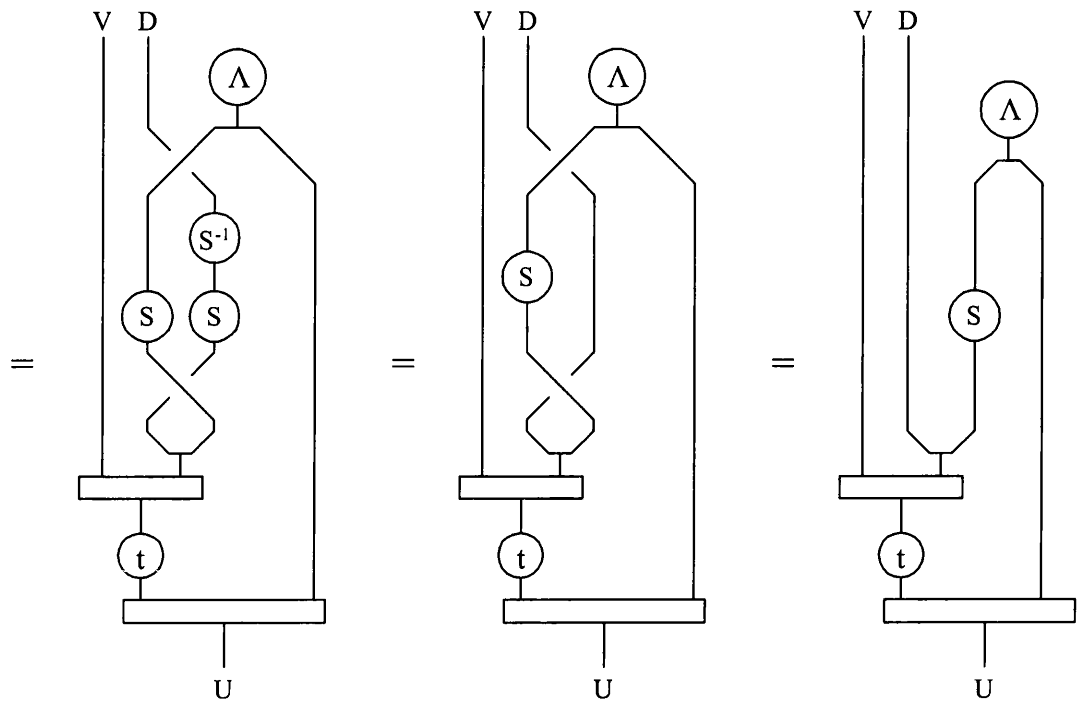
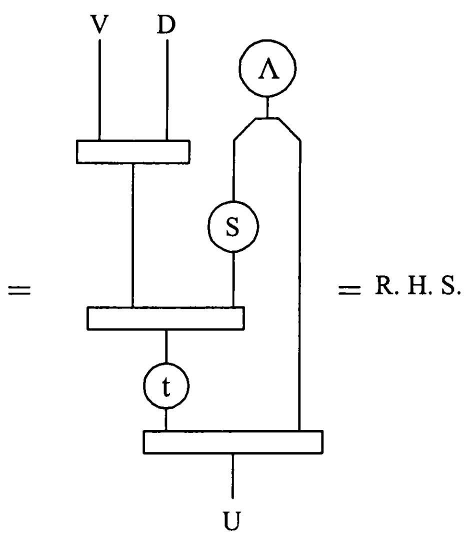


figure 4.11

Proof.



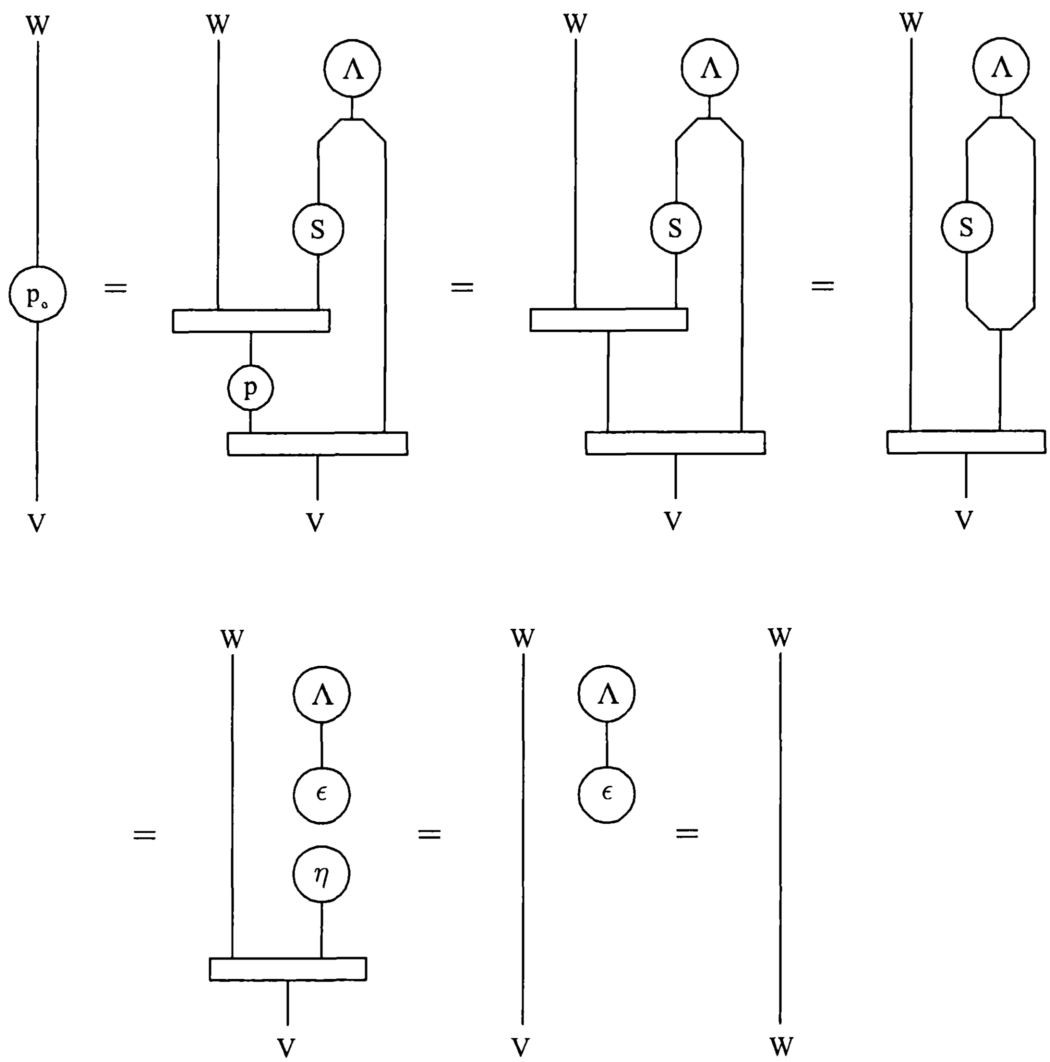




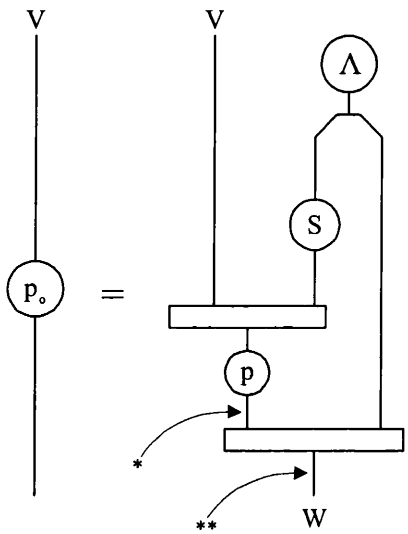
Proposition 4.6.5 *Suppose $\epsilon(\Lambda) = 1$. Let V be a right representation of D, and $W \subset V$ be a subrepresentation. Then there is a complement W_o of W which is also a right representation of D.*

Proof. Take any projection $p : V \rightarrow V$ with image W. By 4.6.3 we also get a morphism $p_o : V \rightarrow V$. Then the proof is given as follows:

a) Show that $p_o|_W$ is the identity.



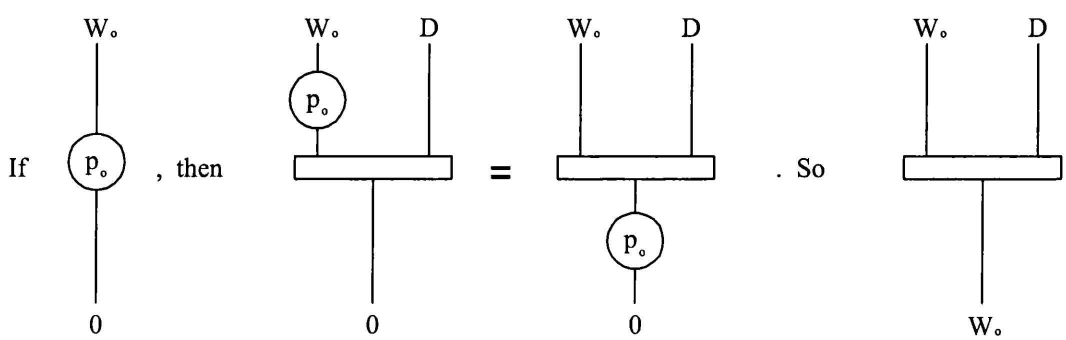
b) Show that the image of $p_o : V \to V$ is contained in W.



As $p(V) \subset W$, the elements in the diagram at position $*$ is in W. But as W is a subrepresentation of V, the output at $**$ is also in W.

Combining a) and b) shows that p_o is a projection.

c) Show that W_o =ker p_o is a subrepresentation.

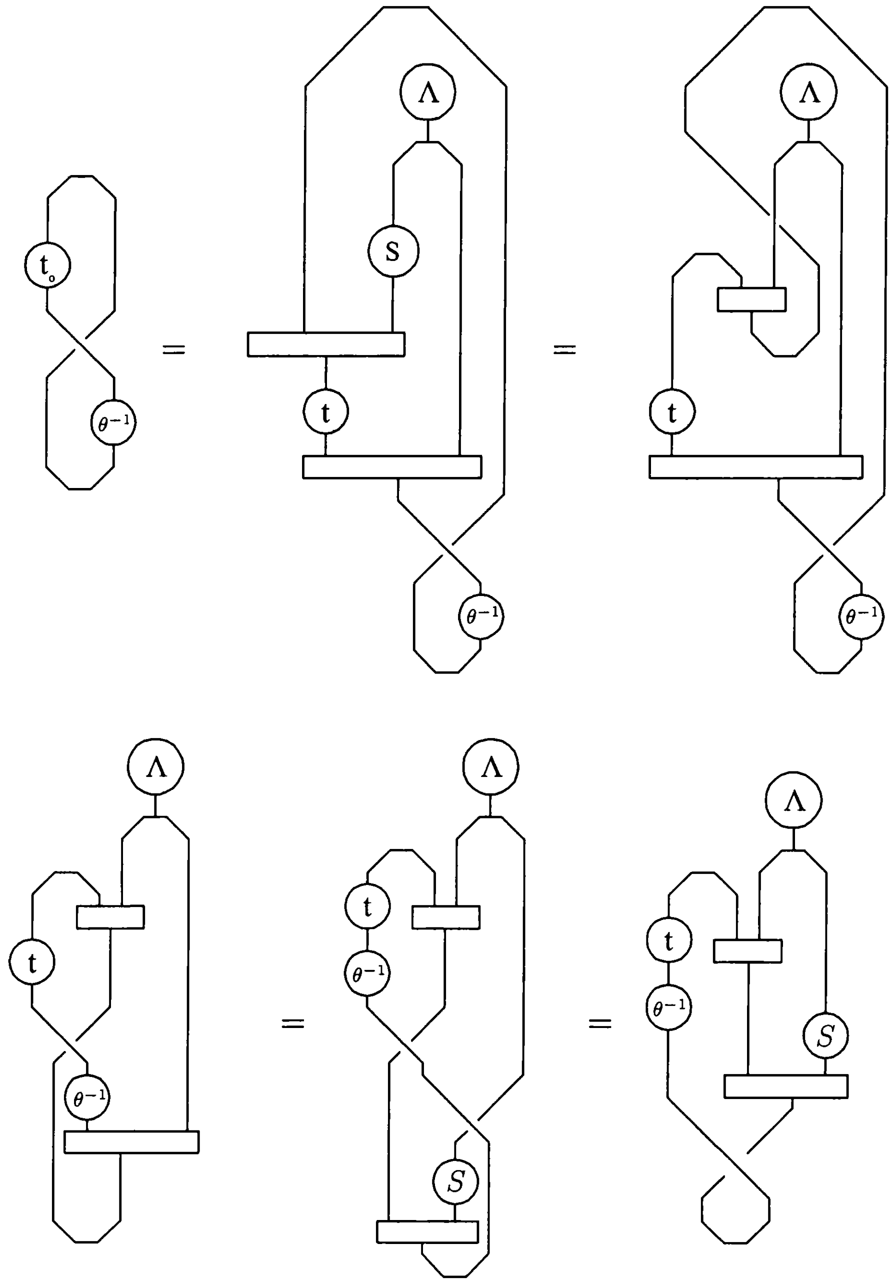


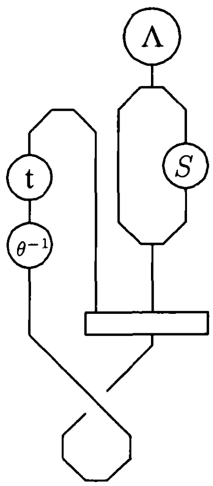
i.e. W_o is a right representation of D.

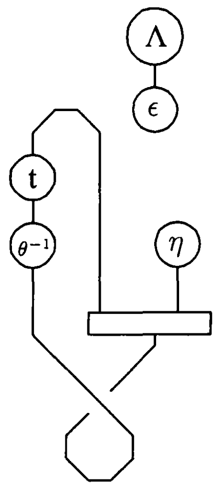
Proposition 4.6.6 *Suppose $\epsilon(\Lambda) = 1$. Let V and W be two right irreducible representations of D. For a linear map $t : V \to W$, by Schur's Lemma we have $t_{_o} = 0$ if V is not isomorphic to W, and if $V = W$ then $t_{_o} = c\ \mathrm{id}_V$. The value of c is given by* $c = \frac{\mathrm{trace(t)}}{\dim_V}$.

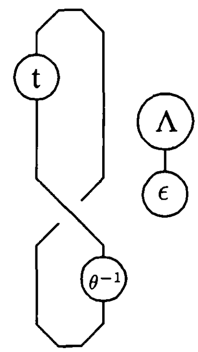
Proof.

$c\,\dim(V) =$



$=$ 

$=$ 

$=$ 

$= \text{trace}(\text{t})\, \epsilon(\Lambda) = \text{trace}(\text{t})\,.$ $\square$

Chapter 5

$\mathcal{D}$ as a modular category

In this chapter we show that the category $\mathcal{D}$ of the representations of the non-trivially associated algebra D has a modular structure in the same way that the category of representations of the double of a group has a modular structure. This chapter has already been sent for publication as a part of a paper by myself and my supervisor E. J. Beggs [2].

We begin the chapter by giving the definition of modular categories and some other important definitions and results. The ribbon maps are calculated for the indecomposable objects in our example category of section 4.2. The last ingredient needed for a modular category is the trace of the double braiding, and this is calculated in $\mathcal{D}$ in terms of group characters. Then the matrices S, T and C implementing the modular representation are calculated explicitly for our example.

5.1 General theory of modular categories

Let $\mathcal{M}$ be a semisimple ribbon category. For objects V and W in $\mathcal{M}$ define $\tilde{S}_{VW} \in \underline{1}$ by

$\tilde{S}_{VW} =$

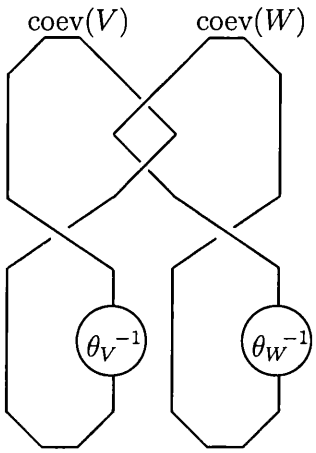


figure 5.1

There are standard results [3, 29]:

$$\tilde{S}_{VW} = \tilde{S}_{WV} = \tilde{S}_{V^*W^*} = \tilde{S}_{W^*V^*}, \;\; \tilde{S}_{V\underline{1}} = \dim(V) \; .$$

Here $\dim(V)$ is the trace in $\mathcal{M}$ of the identity map on V, which can be illustrated by the following diagram:

$\dim(V) =$

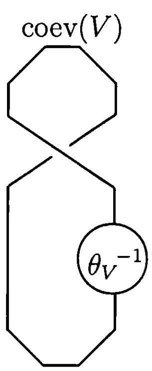


figure 5.2

Definition 5.1.1 *[3] We call an object U in an abelian category $\mathcal{M}$ simple if, for any V in $\mathcal{M}$, any injection $V \hookrightarrow U$ is either 0 or an isomorphism.*

Definition 5.1.2 *[29] A semisimple category is an abelian category whose objects split as a direct sums of simple objects.*

Definition 5.1.3 *[3] A modular category is a semisimple ribbon category $\mathcal{M}$ satisfying the following properties:*

1- There are only a finite number of isomorphism classes of simple objects in $\mathcal{M}$.

2- Schur's lemma holds, i.e. the morphisms between simple objects are zero unless they are isomorphic, in which case the morphisms are a multiple of the identity.

3- The matrix $\tilde{S}_{VW}$ with indices in isomorphism classes of simple objects is invertible.

Remark 5.1.4 *If $\mathcal{M}$ is symmetric (i.e. $\Psi = \Psi^{-1}$), then overcrossing and undercrossing can be interchanged. Hence $\tilde{S}_{VW} = \dim(V)\dim(W)$. Therefore $\mathcal{M}$ is not modular [3].*

Definition 5.1.5 *[3] For a simple object V, the ribbon map on V is a multiple of the identity, and we use Θ_V for the scalar multiple. The numbers $P^{\pm}$ are defined as the following sums over simple isomorphism classes:*

$$P^{\pm} = \sum_V {\Theta_V}^{\pm 1} (\dim(V))^2 ,$$

and the matrices T and C are defined using the Kronecker delta function by

$$T_{VW} = \delta_{VW}\, \Theta_V \ , \quad C_{VW} = \delta_{VW^*} \ ,$$

where the indices V and W are representatives of the isomorphism classes of simple objects.

Theorem 5.1.6 *[3] In a modular category, if we define the matrix S by*

$$S = \frac{\tilde{S}}{\sqrt{P^+ P^-}} ,$$

then we have the following matrix equations:

$$(ST)^3 = \sqrt{\frac{P^+}{P^-}}\, S^2 \ , \qquad S^2 = C \ , \qquad CT = TC \ , \qquad C^2 = 1 \ .$$

5.2 Calculating the ribbon map - an example

The ribbon map $\theta_V : V \to V$, for a simple object V, must be a multiple of the identity by Schur's Lemma. Here we calculate the multiple, Θ_V, for the example of section 4.2.

Example 5.2.1 *First we calculate the value of the ribbon map on the indecomposable objects. For an irreducible representation V, we have $\theta_V : V \to V$ defined by $\theta_V(\xi) = \xi \,\hat{\triangleleft}\, \|\xi\|$ for $\xi \in V$. At the base point $s \in \mathcal{O}$, we have $\theta_V(\xi) = \xi \,\bar{\triangleleft}\, s$ for $\xi \in V$ and $\theta : V_s \to V_s$ is a multiple, Θ_V, of the identity or, more explicitly,* $\mathrm{trace}\,(\theta : V_s \to V_s) = \Theta_V \dim_{\mathbb{C}}(V_s)$, i.e.,

$$\Theta_V = \frac{\text{group character}\,(s)}{\dim_{\mathbb{C}}(V_s)} \ . \tag{5.1}$$

And then, for the different cases we will get the following table :

irreps	Θ_V	irreps	Θ_V
1_1	1	3_4	ω^2
1_2	1	3_5	ω^4
1_3	1	4_0	1
1_4	1	4_1	ω^1
1_5	1	4_2	ω^2
1_6	1	4_3	-1
2_1	1	4_4	ω^4
2_2	-1	4_5	ω^5
2_3	-1	5_{++}	1
2_4	1	5_{+-}	-1
2_5	-1	5_{-+}	1
2_6	1	5_{--}	-1
3_0	1	6_{++}	1
3_1	ω^2	6_{-+}	1
3_2	ω^4	6_{+-}	-1
3_3	1	6_{--}	-1

table (5)

5.3 The double braiding

We now give some results which allow us to calculate the matrix $\tilde{S}$ in $\mathcal{D}$.

Lemma 5.3.1

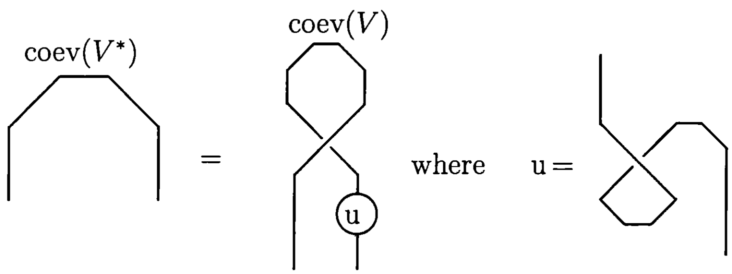
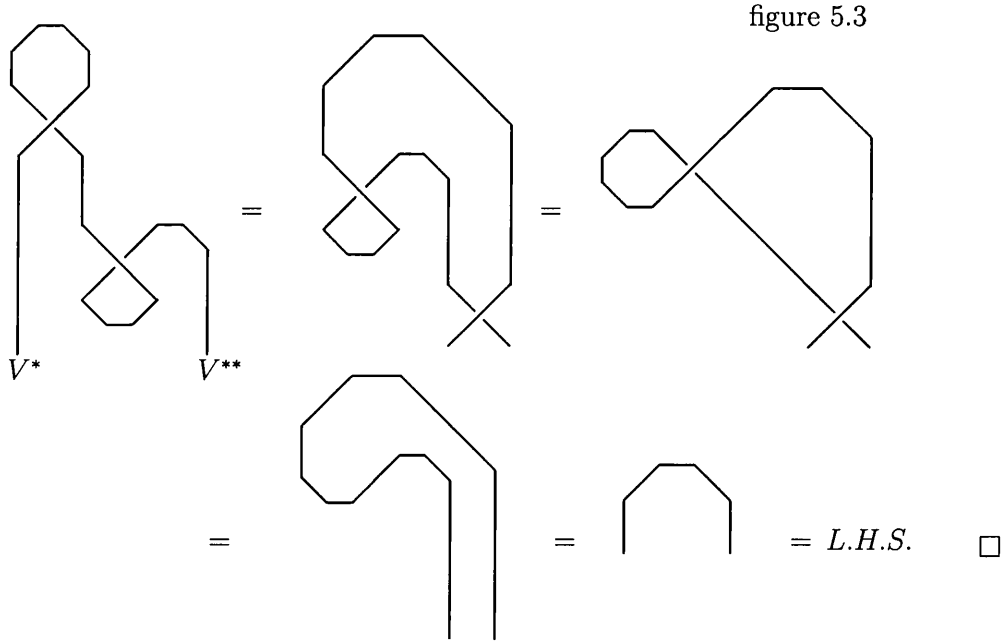


figure 5.3

Proof.

R. H. S.=



□

Lemma 5.3.2

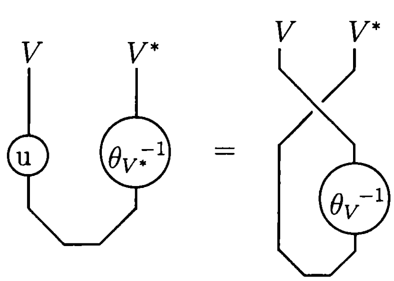
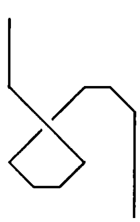
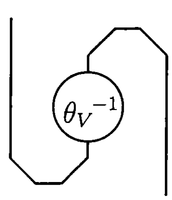
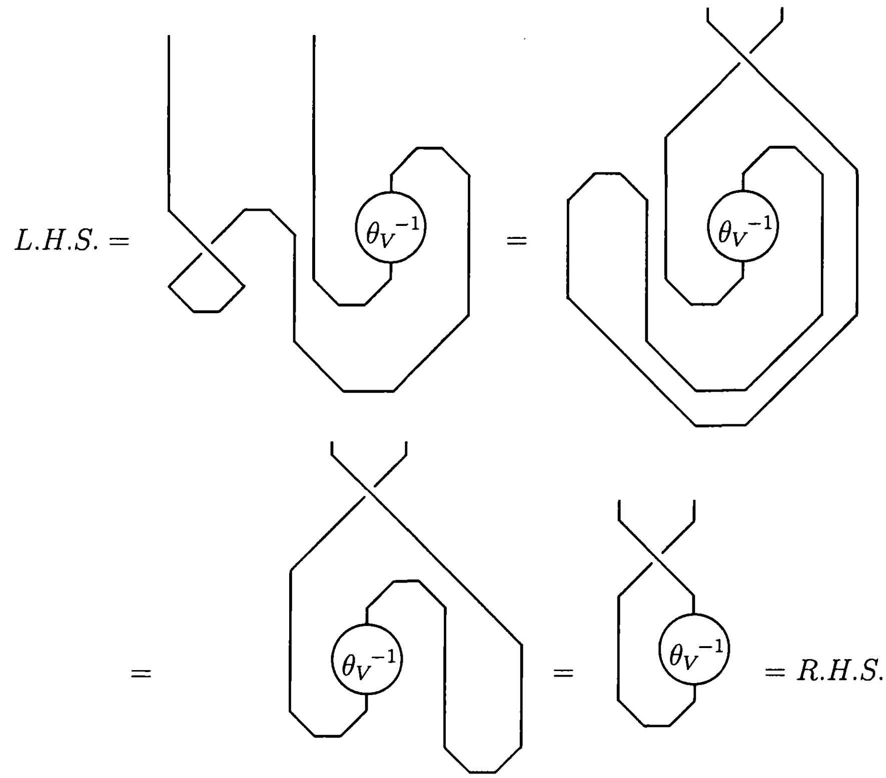


figure 5.4

where u=  and $\theta_{V^*}^{-1}$ = 

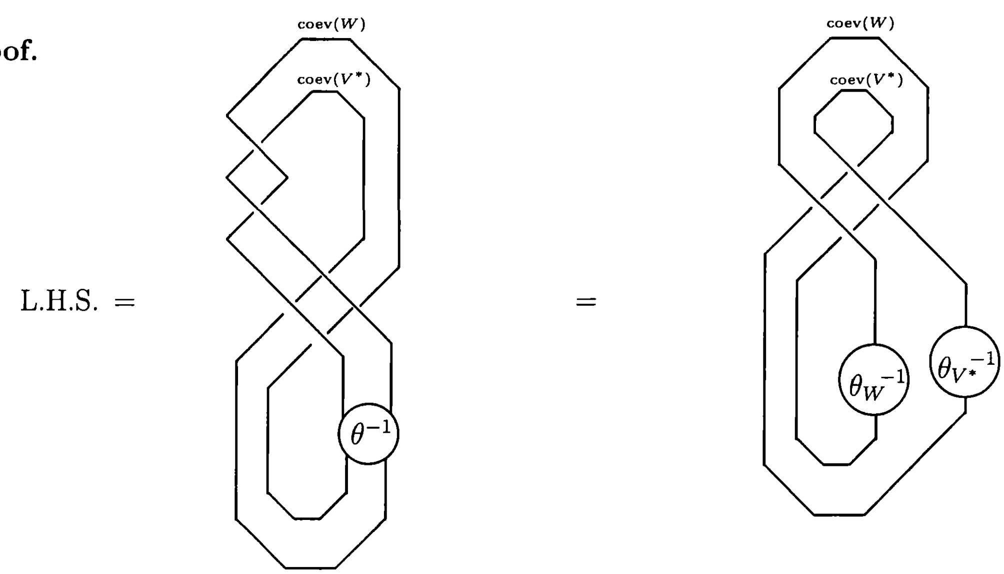
Proof.

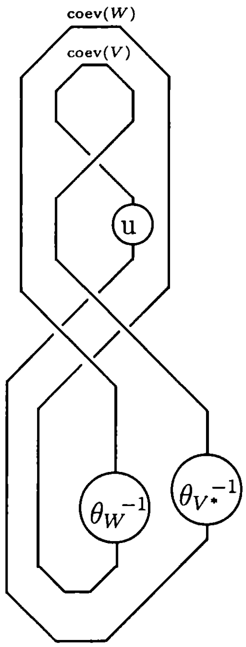


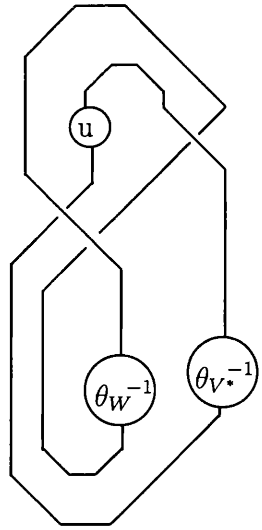
□

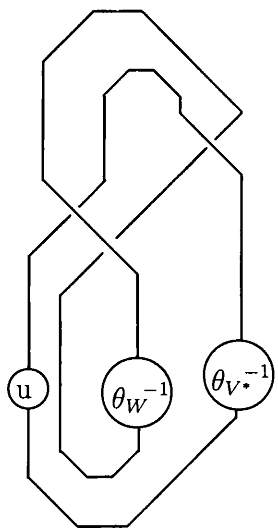
Lemma 5.3.3 *For V, W indecomposable objects in $\mathcal{D}$,* $\text{trace}(\Psi_{V^*W} \circ \Psi_{WV^*}) = \tilde{S}_{VW}$.

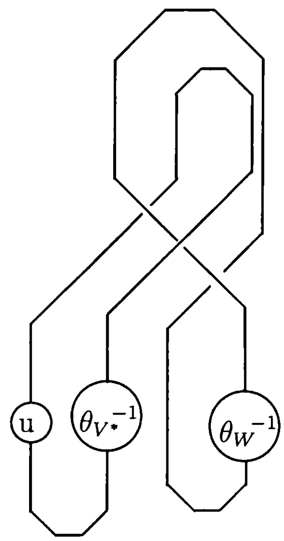
Proof.

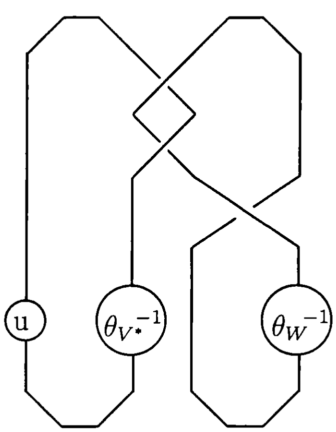


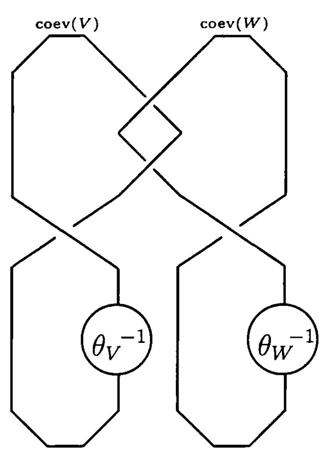
$=$ 
coev(W)
coev(V)
u
$\theta_W{}^{-1}$
$\theta_{V^*}{}^{-1}$

$=$ 
u
$\theta_W{}^{-1}$
$\theta_{V^*}{}^{-1}$

$=$ 
u
$\theta_W{}^{-1}$
$\theta_{V^*}{}^{-1}$

$=$ 
u
$\theta_{V^*}{}^{-1}$
$\theta_W{}^{-1}$

$=$ 
u
$\theta_{V^*}{}^{-1}$
$\theta_W{}^{-1}$

$=$ 
coev(V)
coev(W)
$\theta_V{}^{-1}$
$\theta_W{}^{-1}$

$=$ R. H. S. □

Lemma 5.3.4 *For two objects V and W in $\mathcal{D}$,*

$$\mathrm{trace}\big(\Psi_{W\otimes V}\circ\Psi_{V\otimes W}\big) = \sum_{\substack{\xi\otimes\eta\in\ basis\ of\ V\otimes W\ and\\ |\xi|^{-1}\langle\xi\rangle\ commutes\ with\ |\eta|\langle\eta\rangle^{-1}}} \hat{\eta}(\eta\hat{\triangleleft}|\eta|^{-1}\langle\xi\rangle^{-1}|\xi||\eta|)\ \hat{\xi}(\xi\hat{\triangleleft}|\eta|\langle\eta\rangle^{-1})$$

Proof. From theorem 4.4.2 , we know that

$$\mathrm{trace}\big(\Psi_{W\otimes V}\circ\Psi_{V\otimes W}\big) = \sum_{(\xi\otimes\eta)\in\ \text{basis of}\ V\otimes W} \widehat{(\xi\otimes\eta)}\big(\Psi^2(\xi\otimes\eta)\big)\ . \tag{5.2}$$

From the definition of the ribbon map, we know that $\Psi\left(\Psi(\xi\otimes\eta)\right)\hat{\triangleleft}\|\xi\otimes\eta\| = \xi\hat{\triangleleft}\|\xi\|\otimes \eta\hat{\triangleleft}\|\eta\|$, so

$$\begin{aligned}\Psi\big(\Psi(\xi\otimes\eta)\big) &= \big(\xi\hat{\triangleleft}\|\xi\|\otimes\eta\hat{\triangleleft}\|\eta\|\big)\hat{\triangleleft}\|\xi\otimes\eta\|^{-1}\\
&= (\xi\hat{\triangleleft}|\xi|^{-1}\langle\xi\rangle\otimes\eta\hat{\triangleleft}|\eta|^{-1}\langle\eta\rangle)\hat{\triangleleft}\langle\eta\rangle^{-1}\langle\xi\rangle^{-1}|\xi||\eta|\\
&= (\xi\hat{\triangleleft}|\xi|^{-1}\langle\xi\rangle)\hat{\triangleleft}\big(\|\eta\hat{\triangleleft}\|\eta\|\|\tilde{\triangleright}\langle\eta\rangle^{-1}\langle\xi\rangle^{-1}|\xi||\eta|\big)\otimes\eta\hat{\triangleleft}|\eta|^{-1}\langle\eta\rangle\langle\eta\rangle^{-1}\langle\xi\rangle^{-1}|\xi||\eta|\\
&= \xi\hat{\triangleleft}|\xi|^{-1}\langle\xi\rangle\left(\|\eta\|\tilde{\triangleright}\langle\eta\rangle^{-1}\langle\xi\rangle^{-1}|\xi||\eta|\right)\otimes\eta\hat{\triangleleft}|\eta|^{-1}\langle\xi\rangle^{-1}|\xi||\eta|.\end{aligned}$$

Put $\Psi\big(\Psi(\xi\otimes\eta)\big) = \xi'\otimes\eta'$ and $\widehat{\xi\otimes\eta} = \alpha\otimes\beta$, and then from lemma 4.3.1, we get

$$\widehat{(\xi\otimes\eta)}(\xi'\otimes\eta') = \big(\alpha\bar{\triangleleft}\tau(\langle\beta\rangle,\langle\xi'\rangle\cdot\langle\eta'\rangle)\big)(\eta')\ \big(\beta\bar{\triangleleft}\tau(\langle\xi'\rangle,\langle\eta'\rangle)^{-1}\big)(\xi')\,.$$

As $\widehat{\xi\otimes\eta}$ is part of a dual basis, the last expression can only be non-zero if $\|\xi'\| = \|\xi\|$ and $\|\eta'\| = \|\eta\|$. A simple calculation shows that $\|\eta'\| = \|\eta\|$ if and only if $|\xi|^{-1}\langle\xi\rangle$ commutes with $|\eta|\langle\eta\rangle^{-1}$. We use this to find

$$\begin{aligned}\|\eta\|\tilde{\triangleleft}\langle\eta\rangle^{-1}\langle\xi\rangle^{-1}|\xi||\eta| &= |\eta|^{-1}|\xi|^{-1}\langle\xi\rangle\langle\eta\rangle|\eta|^{-1}\langle\eta\rangle\langle\eta\rangle^{-1}\langle\xi\rangle^{-1}|\xi||\eta|\\
&= |\eta|^{-1}\langle\eta\rangle|\eta|^{-1}|\xi|^{-1}\langle\xi\rangle\langle\xi\rangle^{-1}|\xi||\eta| = |\eta|^{-1}\langle\eta\rangle\ ,\end{aligned}$$

and then

$$\|\eta\|\tilde{\triangleright}\langle\eta\rangle^{-1}\langle\xi\rangle^{-1}|\xi||\eta| = \langle\eta\rangle\langle\eta\rangle^{-1}\langle\xi\rangle^{-1}|\xi||\eta|\langle\eta\rangle^{-1} = \langle\xi\rangle^{-1}|\xi||\eta|\langle\eta\rangle^{-1}\ .$$

Now using the formula for $\widehat{\xi\otimes\eta} = \alpha\otimes\beta$ from lemma 4.3.1 gives the result. $\square$

Lemma 5.3.5 *Let V and W be objects in $\mathcal{D}$. Then in terms of group characters:*

$$\mathrm{trace}(\Psi^2{}_{V\otimes W}) = \sum_{\substack{u,v\in G,\ s,t\in M \text{ and} \\ su \text{ commutes with } vt}} \chi_{W_{us}}(s^{-1}t^{-1}v^{-1}s)\ \chi_{V_{vt}}(u^{-1}s^{-1})\ .$$

Proof. This is more or less immediate from lemma 5.3.4. Put $\|\eta\| = u^{-1}s$ and $\|\xi\| = v^{-1}t$, and sum over basis elements of constant degree first. We then use the summation condition that su commutes with vt to say $ut^{-1}v^{-1}u^{-1} = s^{-1}t^{-1}v^{-1}s$. $\square$

5.4 Calculating the trace of the double braiding - an example

To find S, we calculate the trace of the double braiding trace$(\Psi_{VW} \circ \Psi_{WV})$. We do this using the result from 5.3.5, and split into different cases for the object V:

case (1) The orbit is $\{e\}$, then for $vt = e$, $su \in \{e, a, a^2, a^3, a^4, a^5, b, ba, ba^2, ba^3, ba^4, ba^5\}$.

So

$$\begin{aligned}\mathrm{trace}(\Psi^2) = {} & \chi_{W_e}(e)\chi_{V_e}(e) + \chi_{W_{a^2}}(e)\chi_{V_e}(a^4) + \chi_{W_{a^4}}(e)\chi_{V_e}(a^2) + \chi_{W_b}(e)\chi_{V_e}(b) \\ & + \chi_{W_{ba^2}}(e)\chi_{V_e}(ba^2) + \chi_{W_{ba^4}}(e)\chi_{V_e}(ba^4) + \chi_{W_a}(e)\chi_{V_e}(a^5) + \chi_{W_{a^3}}(e)\chi_{V_e}(a^3) \\ & + \chi_{W_{a^5}}(e)\chi_{V_e}(a) + \chi_{W_{ba}}(e)\chi_{V_e}(ba^5) + \chi_{W_{ba^3}}(e)\chi_{V_e}(ba) + \chi_{W_{ba^5}}(e)\chi_{V_e}(ba^3)\end{aligned}$$

case (2) The orbit is $\{a^3\}$, then for $vt = a^3$, $su \in \{e, a, a^2, a^3, a^4, a^5, b, ba, ba^2, ba^3, ba^4, ba^5\}$.

So

$$\begin{aligned}\mathrm{trace}(\Psi^2) = {} & \chi_{W_e}(a^3)\chi_{V_{a^3}}(e) + \chi_{W_{a^2}}(a^3)\chi_{V_{a^3}}(a^4) + \chi_{W_{a^4}}(a^3)\chi_{V_{a^3}}(a^2) \\ & + \chi_{W_b}(a^3)\chi_{V_{a^3}}(b) + \chi_{W_{ba^2}}(a^3)\chi_{V_{a^3}}(ba^2) + \chi_{W_{ba^4}}(a^3)\chi_{V_{a^3}}(ba^4) \\ & + \chi_{W_a}(a^3)\chi_{V_{a^3}}(a^5) + \chi_{W_{a^3}}(a^3)\chi_{V_{a^3}}(a^3) + \chi_{W_{a^5}}(a^3)\chi_{V_{a^3}}(a) \\ & + \chi_{W_{ba}}(a^3)\chi_{V_{a^3}}(ba^5) + \chi_{W_{ba^3}}(a^3)\chi_{V_{a^3}}(ba) + \chi_{W_{ba^5}}(a^3)\chi_{V_{a^3}}(ba^3)\end{aligned}$$

case (3) The orbit is $\{a^2, a^4\}$, then for $vt = a^2$ and $vt = a^4$, $su \in \{e, a, a^2, a^3, a^4, a^5\}$. So

$$\text{trace}(\Psi^2) = \chi_{W_e}(a^4)\chi_{V_{a^2}}(e) + \chi_{W_a}(a^4)\chi_{V_{a^2}}(a^5) + \chi_{W_{a^2}}(a^4)\chi_{V_{a^2}}(a^4)$$
$$+ \chi_{W_{a^3}}(a^4)\chi_{V_{a^2}}(a^3) + \chi_{W_{a^4}}(a^4)\chi_{V_{a^2}}(a^2) + \chi_{W_{a^5}}(a^4)\chi_{V_{a^2}}(a)$$
$$+ \chi_{W_e}(a^2)\chi_{V_{a^4}}(e) + \chi_{W_a}(a^2)\chi_{V_{a^4}}(a^5) + \chi_{W_{a^2}}(a^2)\chi_{V_{a^4}}(a^4)$$
$$+ \chi_{W_{a^3}}(a^2)\chi_{V_{a^4}}(ba^3) + \chi_{W_{a^4}}(a^2)\chi_{V_{a^4}}(a^2) + \chi_{W_{a^5}}(a^2)\chi_{V_{a^4}}(a)$$

case (4) The orbit is $\{a, a^5\}$, then for $vt = a$ and $vt = a^5$, $su \in \{e, a, a^2, a^3, a^4, a^5\}$. So

$$\text{trace}(\Psi^2) = \chi_{W_e}(a^5)\chi_{V_a}(e) + \chi_{W_a}(a^5)\chi_{V_a}(a^5) + \chi_{W_{a^2}}(a^5)\chi_{V_a}(a^4)$$
$$+ \chi_{W_{a^3}}(a^5)\chi_{V_a}(a^3) + \chi_{W_{a^4}}(a^5)\chi_{V_a}(a^2) + \chi_{W_{a^5}}(a^5)\chi_{V_a}(a)$$
$$+ \chi_{W_e}(a)\chi_{V_{a^5}}(e) + \chi_{W_a}(a)\chi_{V_{a^5}}(a^5) + \chi_{W_{a^2}}(a)\chi_{V_{a^5}}(a^4)$$
$$+ \chi_{W_{a^3}}(a)\chi_{V_{a^5}}(ba^3) + \chi_{W_{a^4}}(a)\chi_{V_{a^5}}(a^2) + \chi_{W_{a^5}}(a)\chi_{V_{a^5}}(a)$$

case (5) The orbit is $\{b, ba^2, ba^4\}$. Then for $vt = b$, $su \in \{e, a^3, b, ba^3\}$, for $vt = ba^2$, $su \in \{e, a^3, ba^2, ba^5\}$ and for $vt = ba^4$, $su \in \{e, a^3, ba^4, ba\}$. So

$$\text{trace}(\Psi^2) = \chi_{W_e}(b)\chi_{V_b}(e) + \chi_{W_{a^3}}(ba^2)\chi_{V_b}(a^3) + \chi_{W_b}(b)\chi_{V_b}(b)$$
$$+ \chi_{W_{ba^5}}(ba^2)\chi_{V_b}(ba^3) + \chi_{W_e}(ba^2)\chi_{V_{ba^2}}(e) + \chi_{W_{a^3}}(ba^4)\chi_{V_{ba^2}}(a^3)$$
$$+ \chi_{W_{ba^2}}(ba^2)\chi_{V_{ba^2}}(ba^2) + \chi_{W_{ba}}(ba^4)\chi_{V_{ba^2}}(ba^5) + \chi_{W_e}(ba^4)\chi_{V_{ba^4}}(e)$$
$$+ \chi_{W_{a^3}}(b)\chi_{V_{ba^4}}(a^3) + \chi_{W_{ba^4}}(ba^4)\chi_{V_{ba^4}}(ba^4) + \chi_{W_{ba^3}}(b)\chi_{V_{ba^4}}(ba)$$

case (6) The orbit is $\{ba, ba^3, ba^5\}$. Then for $vt = ba$, $su \in \{e, a^3, ba^4, ba\}$, for $vt = ba^3$, $su \in \{e, a^3, b, ba^3\}$ and for $vt = ba^5$, $su \in \{e, a^3, ba^2, ba^5\}$. So

$$\text{trace}(\Psi^2) = \chi_{W_e}(ba)\chi_{V_{ba}}(e) + \chi_{W_{a^3}}(ba^3)\chi_{V_{ba}}(a^3) + \chi_{W_{ba^4}}(ba)\chi_{V_{ba}}(ba^4)$$
$$+ \chi_{W_{ba^3}}(ba^3)\chi_{V_{ba}}(ba) + \chi_{W_e}(ba^3)\chi_{V_{ba^3}}(e) + \chi_{W_{a^3}}(ba^5)\chi_{V_{ba^3}}(a^3)$$
$$+ \chi_{W_b}(ba^3)\chi_{V_{ba^3}}(b) + \chi_{W_{ba^5}}(ba^5)\chi_{V_{ba^3}}(ba^3) + \chi_{W_e}(ba^5)\chi_{V_{ba^5}}(e)$$
$$+ \chi_{W_{a^3}}(ba)\chi_{V_{ba^5}}(a^3) + \chi_{W_{ba^2}}(ba^5)\chi_{V_{ba^5}}(ba^2) + \chi_{W_{ba}}(ba)\chi_{V_{ba^5}}(ba^5).$$

Now split each of these cases into the six orbit cases for W, and move the points the characters are evaluated at to the base points for each orbit using 4.1.3. We use 4.1.3 in the case of the category $\mathcal{D}$, using the actions $\tilde{\triangleright}$ and $\tilde{\triangleleft}$.

Case (1) $\otimes$ Case (1): (i.e. the orbit of W is $\{e\}$ and the orbit of V is $\{e\}$)

$$\text{trace}(\Psi^2) = \chi_{W_e}(e)\chi_{V_e}(e).$$

Case (1) $\otimes$ Case (2): (i.e. the orbit of W is $\{e\}$ and the orbit of V is $\{a^3\}$)

$$\text{trace}(\Psi^2) = \chi_{W_e}(a^3)\chi_{V_{a^3}}(e).$$

Case (1) $\otimes$ Case (3): (i.e. the orbit of W is $\{e\}$ and the orbit of V is $\{a^2, a^4\}$)

$$\text{trace}(\Psi^2) = \Big(\chi_{W_e}(a^4) + \chi_{W_e}(a^2)\Big)\chi_{V_{a^2}}(e).$$

Case (1) $\otimes$ Case (4): (i.e. the orbit of W is $\{e\}$ and the orbit of V is $\{a, a^5\}$)

$$\text{trace}(\Psi^2) = \Big(\chi_{W_e}(a^5) + \chi_{W_e}(a)\Big)\chi_{V_a}(e).$$

Case (1) $\otimes$ Case (5): (i.e. the orbit of W is $\{e\}$ and the orbit of V is $\{b, ba^2, ba^4\}$)

$$\text{trace}(\Psi^2) = \Big(\chi_{W_e}(b) + \chi_{W_e}(ba^2) + \chi_{W_e}(ba^4)\Big)\chi_{V_b}(e).$$

Case (1) $\otimes$ Case (6): (i.e. the orbit of W is $\{e\}$ and the orbit of V is $\{ba, ba^3, ba^5\}$)

$$\text{trace}(\Psi^2) = \Big(\chi_{W_e}(ba) + \chi_{W_e}(ba^3) + \chi_{W_e}(ba^5)\Big)\chi_{V_{ba}}(e).$$

Case (2) $\otimes$ Case (1): (i.e. the orbit of W is $\{a^3\}$ and the orbit of V is $\{e\}$)

$$\text{trace}(\Psi^2) = \chi_{W_{a^3}}(e)\chi_{V_e}(a^3).$$

Case (2) $\otimes$ Case (2): (i.e. the orbit of W is $\{a^3\}$ and the orbit of V is $\{a^3\}$)

$$\text{trace}(\Psi^2) = \chi_{W_{a^3}}(a^3)\chi_{V_{a^3}}(a^3).$$

Case (2) $\otimes$ Case (3):(i.e. the orbit of W is $\{a^3\}$ and the orbit of V is $\{a^2, a^4\}$)

$$\text{trace}(\Psi^2) = \Big(\chi_{W_{a^3}}(a^4) + \chi_{W_{a^3}}(a^2)\Big)\chi_{V_{a^2}}(a^3).$$

Case (2) $\otimes$ Case (4): (i.e. the orbit of W is $\{a^3\}$ and the orbit of V is $\{a, a^5\}$)

$$\text{trace}(\Psi^2) = \Big(\chi_{W_{a^3}}(a^5) + \chi_{W_{a^3}}(a)\Big)\chi_{V_a}(a^3).$$

Case (2) $\otimes$ Case (5): (i.e. the orbit of W is $\{a^3\}$ and the orbit of V is $\{b, ba^2, ba^4\}$)

$$\text{trace}(\Psi^2) = \Big(\chi_{W_{a^3}}(ba^2) + \chi_{W_{a^3}}(ba^4) + \chi_{W_{a^3}}(b)\Big)\chi_{V_b}(a^3).$$

Case (2) $\otimes$ Case (6): (i.e. the orbit of W is $\{a^3\}$ and the orbit of V is $\{ba, ba^3, ba^5\}$)

$$\text{trace}(\Psi^2) = \Big(\chi_{W_{a^3}}(ba^3) + \chi_{W_{a^3}}(ba^5) + \chi_{W_{a^3}}(ba)\Big)\chi_{V_{ba}}(a^3).$$

Case (3) $\otimes$ Case (1): (i.e. the orbit of W is $\{a^2, a^4\}$ and the orbit of V is $\{e\}$)

$$\text{trace}(\Psi^2) = \chi_{W_{a^2}}(e)\Big(\chi_{V_e}(a^4) + \chi_{V_e}(a^2)\Big).$$

Case (3) $\otimes$ Case (2): (i.e. the orbit of W is $\{a^2, a^4\}$ and the orbit of V is $\{a^3\}$)

$$\text{trace}(\Psi^2) = \chi_{W_{a^2}}(a^3)\Big(\chi_{V_{a^3}}(a^4) + \chi_{V_{a^3}}(a^2)\Big).$$

Case (3) $\otimes$ Case (3): (i.e. the orbit of W is $\{a^2, a^4\}$ and the orbit of V is $\{a^2, a^4\}$)

$$\text{trace}(\Psi^2) = 2\Big(\chi_{W_{a^2}}(a^4)\chi_{V_{a^2}}(a^4) + \chi_{W_{a^2}}(a^2)\chi_{V_{a^2}}(a^2)\Big).$$

Case (3) $\otimes$ Case (4): (i.e. the orbit of W is $\{a^2, a^4\}$ and the orbit of V is $\{a, a^5\}$)

$$\text{trace}(\Psi^2) = 2\Big(\chi_{W_{a^2}}(a^5)\chi_{V_a}(a^4) + \chi_{W_{a^2}}(a)\chi_{V_a}(a^2)\Big).$$

Case (3) $\otimes$ Case (5): (i.e. the orbit of W is $\{a^2, a^4\}$ and the orbit of V is $\{b, ba^2, ba^4\}$)

$$\text{trace}(\Psi^2) = 0.$$

Case (3) $\otimes$ Case (6): (i.e. the orbit of W is $\{a^2, a^4\}$ and the orbit of V is $\{ba, ba^3, ba^5\}$)

$$\text{trace}(\Psi^2) = 0.$$

Case (4) $\otimes$ Case (1): (i.e. the orbit of W is $\{a, a^5\}$ and the orbit of V is $\{e\}$)

$$\text{trace}(\Psi^2) = \chi_{W_a}(e)\Big(\chi_{V_e}(a^5) + \chi_{V_e}(a)\Big).$$

Case (4) $\otimes$ Case (2): (i.e. the orbit of W is $\{a, a^5\}$ and the orbit of V is $\{a^3\}$)

$$\text{trace}(\Psi^2) = \chi_{W_a}(a^3)\Big(\chi_{V_{a^3}}(a^5) + \chi_{V_{a^3}}(a)\Big).$$

Case (4) $\otimes$ Case (3): (i.e. the orbit of W is $\{a, a^5\}$ and the orbit of V is $\{a^2, a^4\}$)

$$\text{trace}(\Psi^2) = 2\Big(\chi_{W_a}(a^4)\chi_{V_{a^2}}(a^5) + \chi_{W_a}(a^2)\chi_{V_{a^2}}(a)\Big).$$

Case (4) $\otimes$ Case (4): (i.e. the orbit of W is $\{a, a^5\}$ and the orbit of V is $\{a, a^5\}$)

$$\text{trace}(\Psi^2) = 2\Big(\chi_{W_a}(a^5)\chi_{V_a}(a^5) + \chi_{W_a}(a)\chi_{V_a}(a)\Big).$$

Case (4) $\otimes$ Case (5): (i.e. the orbit of W is $\{a, a^5\}$ and the orbit of V is $\{b, ba^2, ba^4\}$)

$$\text{trace}(\Psi^2) = 0.$$

Case (4) $\otimes$ Case (6): (i.e. the orbit of W is $\{a, a^5\}$ and the orbit of V is $\{ba, ba^3, ba^5\}$)

$$\text{trace}(\Psi^2) = 0.$$

Case (5) $\otimes$ Case (1): (i.e. the orbit of W is $\{b, ba^2, ba^4\}$ and the orbit of V is $\{e\}$)

$$\text{trace}(\Psi^2) = \chi_{W_b}(e)\Big(\chi_{V_e}(b) + \chi_{V_e}(ba^2) + \chi_{V_e}(ba^4)\Big).$$

Case (5) $\otimes$ Case (2): (i.e. the orbit of W is $\{b, ba^2, ba^4\}$ and the orbit of V is $\{a^3\}$)

$$\text{trace}(\Psi^2) = \chi_{W_b}(a^3)\Big(\chi_{V_{a^3}}(b) + \chi_{V_{a^3}}(ba^2) + \chi_{V_{a^3}}(ba^4)\Big).$$

Case (5) $\otimes$ Case (3): (i.e. the orbit of W is $\{b, ba^2, ba^4\}$ and the orbit of V is $\{a^2, a^4\}$)

$$\text{trace}(\Psi^2) = 0.$$

Case (5) $\otimes$ Case (4): (i.e. the orbit of W is $\{b, ba^2, ba^4\}$ and the orbit of V is $\{a, a^5\}$)

$$\text{trace}(\Psi^2) = 0.$$

Case (5) $\otimes$ Case (5): (i.e. the orbit of W is $\{b, ba^2, ba^4\}$ and the orbit of V is $\{b, ba^2, ba^4\}$)

$$\text{trace}(\Psi^2) = 3\Big(\chi_{W_b}(b)\chi_{V_b}(b)\Big).$$

Case (5) $\otimes$ Case (6): (i.e. the orbit of W is $\{b, ba^2, ba^4\}$ and the orbit of V is $\{ba, ba^3, ba^5\}$)

$$\text{trace}(\Psi^2) = 3\Big(\chi_{W_b}(ba^3)\chi_{V_{ba}}(ba^4)\Big).$$

Case (6) $\otimes$ Case (1): (i.e. the orbit of W is $\{ba, ba^3, ba^5\}$ and the orbit of V is $\{e\}$)

$$\text{trace}(\Psi^2) = \chi_{W_{ba}}(e)\Big(\chi_{V_e}(ba^5) + \chi_{V_e}(ba) + \chi_{V_e}(ba^3)\Big).$$

Case (6) $\otimes$ Case (2): (i.e. the orbit of W is $\{ba, ba^3, ba^5\}$ and the orbit of V is $\{a^3\}$)

$$\text{trace}(\Psi^2) = \chi_{W_{ba}}(a^3)\Big(\chi_{V_{a^3}}(ba^5) + \chi_{V_{a^3}}(ba) + \chi_{V_{a^3}}(ba^3)\Big).$$

Case (6) $\otimes$ Case (3): (i.e. the orbit of W is $\{ba, ba^3, ba^5\}$ and the orbit of V is $\{a^2, a^4\}$)

$$\text{trace}(\Psi^2) = 0.$$

Case (6) $\otimes$ Case (4): (i.e. the orbit of W is $\{ba, ba^3, ba^5\}$ and the orbit of V is $\{a, a^5\}$)

$$\text{trace}(\Psi^2) = 0.$$

Case (6) $\otimes$ Case (5): (i.e. the orbit of W is $\{ba, ba^3, ba^5\}$ and the orbit of V is $\{b, ba^2, ba^4\}$)

$$\text{trace}(\Psi^2) = 3\Big(\chi_{W_{ba}}(ba^4)\chi_{V_b}(ba^3)\Big).$$

Case (6) $\otimes$ Case (6): (i.e. the orbit of W is $\{ba, ba^3, ba^5\}$ and the orbit of V is $\{ba, ba^3, ba^5\}$)

$$\text{trace}(\Psi^2) = 3\Big(\chi_{W_{ba}}(ba)\chi_{V_{ba}}(ba)\Big).$$

5.5 The matrices

From the cases of section 5.4 we get $\tilde{S}$ to be the following 32×32 symmetric matrix:

$$
\tilde{S} = \left(\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 2 & 2 & -1 & -1 & -1 & -1 & -2 & -2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 2 & 2 & -1 & -1 & -1 & -1 & -2 & -2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 4 & 4 & -2 & -2 & -2 & -2 & -4 & -4 & -2 & -2 & -2 & -2 \\
2 & 2 & 2 & 2 & 4 & 4 & 2 & 2 & 2 & 2 & 4 & 4 & -2 & -2 & -2 & -2 \\
1 & -1 & -1 & 1 & -2 & 2 & 1 & -1 & -1 & 1 & -2 & 2 & 2 & -2 & 2 & -2 \\
1 & -1 & -1 & 1 & -2 & 2 & -1 & 1 & 1 & -1 & 2 & -2 & 2 & -2 & 2 & -2 \\
1 & -1 & -1 & 1 & -2 & 2 & -1 & 1 & 1 & -1 & 2 & -2 & 2 & -2 & 2 & -2 \\
1 & -1 & -1 & 1 & -2 & 2 & 1 & -1 & -1 & 1 & -2 & 2 & 2 & -2 & 2 & -2 \\
2 & -2 & -2 & 2 & -4 & 4 & -2 & 2 & 2 & -2 & 4 & -4 & -2 & 2 & -2 & 2 \\
2 & -2 & -2 & 2 & -4 & 4 & 2 & -2 & -2 & 2 & -4 & 4 & -2 & 2 & -2 & 2 \\
2 & 2 & 2 & 2 & -2 & -2 & 2 & 2 & 2 & 2 & -2 & -2 & 4 & -2 & -2 & 4 \\
2 & 2 & 2 & 2 & -2 & -2 & -2 & -2 & -2 & -2 & 2 & 2 & -2 & -2 & 4 & -2 \\
2 & 2 & 2 & 2 & -2 & -2 & 2 & 2 & 2 & 2 & -2 & -2 & -2 & 4 & -2 & -2 \\
2 & 2 & 2 & 2 & -2 & -2 & -2 & -2 & -2 & -2 & 2 & 2 & 4 & -2 & -2 & 4 \\
2 & 2 & 2 & 2 & -2 & -2 & 2 & 2 & 2 & 2 & -2 & -2 & -2 & -2 & 4 & -2 \\
2 & 2 & 2 & 2 & -2 & -2 & -2 & -2 & -2 & -2 & 2 & 2 & -2 & 4 & -2 & -2 \\
2 & -2 & -2 & 2 & 2 & -2 & 2 & -2 & -2 & 2 & 2 & -2 & 4 & 2 & -2 & -4 \\
2 & -2 & -2 & 2 & 2 & -2 & -2 & 2 & 2 & -2 & -2 & 2 & -2 & -4 & -2 & 2 \\
2 & -2 & -2 & 2 & 2 & -2 & 2 & -2 & -2 & 2 & 2 & -2 & -2 & 2 & 4 & 2 \\
2 & -2 & -2 & 2 & 2 & -2 & -2 & 2 & 2 & -2 & -2 & 2 & 4 & 2 & -2 & -4 \\
2 & -2 & -2 & 2 & 2 & -2 & 2 & -2 & -2 & 2 & 2 & -2 & -2 & -4 & -2 & 2 \\
2 & -2 & -2 & 2 & 2 & -2 & -2 & 2 & 2 & -2 & -2 & 2 & -2 & 2 & 4 & 2 \\
3 & -3 & 3 & -3 & 0 & 0 & 3 & -3 & 3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -3 & 3 & -3 & 0 & 0 & 3 & -3 & 3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -3 & 3 & -3 & 0 & 0 & -3 & 3 & -3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -3 & 3 & -3 & 0 & 0 & -3 & 3 & -3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & -3 & -3 & 0 & 0 & 3 & 3 & -3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & -3 & -3 & 0 & 0 & -3 & -3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & -3 & -3 & 0 & 0 & 3 & 3 & -3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & -3 & -3 & 0 & 0 & -3 & -3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \right.
$$

$$
\left. \begin{array}{cccccccccccccccc}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
2 & 2 & -2 & -2 & -2 & -2 & -2 & -2 & -3 & -3 & -3 & -3 & 3 & 3 & 3 & 3 \\
2 & 2 & -2 & -2 & -2 & -2 & -2 & -2 & 3 & 3 & 3 & 3 & -3 & -3 & -3 & -3 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & -3 & -3 & -3 & -3 & -3 & -3 & -3 & -3 \\
-2 & -2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & 2 & -2 & 2 & -2 & 2 & -2 & 3 & 3 & -3 & -3 & 3 & -3 & 3 & -3 \\
2 & -2 & -2 & 2 & -2 & 2 & -2 & 2 & -3 & -3 & 3 & 3 & 3 & -3 & 3 & -3 \\
2 & -2 & -2 & 2 & -2 & 2 & -2 & 2 & 3 & 3 & -3 & -3 & -3 & 3 & -3 & 3 \\
2 & -2 & 2 & -2 & 2 & -2 & 2 & -2 & -3 & -3 & 3 & 3 & -3 & 3 & -3 & 3 \\
-2 & 2 & 2 & -2 & 2 & -2 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 2 & -2 & 2 & -2 & 2 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & -2 & 4 & -2 & -2 & 4 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 4 & 2 & -4 & 2 & 2 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & -2 & -2 & -2 & 4 & -2 & -2 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & -2 & -4 & 2 & 2 & -4 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 4 & -2 & 4 & -2 & -2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & -2 & 2 & 2 & -4 & 2 & 2 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 2 & 4 & 2 & -2 & -4 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 2 & 2 & -2 & -4 & -2 & 2 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & -4 & -2 & -4 & -2 & 2 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 2 & -4 & -2 & 2 & 4 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 2 & -2 & 2 & 4 & 2 & -2 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & -4 & 2 & 4 & 2 & -2 & -4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3 & 3 & -3 & 3 & -3 & -3 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 3 & -3 & 3 & -3 & 3 & 3 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3 & 3 & -3 & -3 & 3 & 3 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 3 & -3 & 3 & 3 & -3 & -3 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3 & -3 & 3 & 3 & 3 & -3 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 3 & 3 & -3 & 3 & 3 & -3 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 3 & 3 & -3 & -3 & -3 & 3 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3 & -3 & 3 & -3 & -3 & 3 & 3
\end{array} \right)
$$

Now to find S we need to calculate $P^{\pm} = \sum_V {\Theta_V}^{\pm 1}(\dim(V))^2$, which we do as the following noting that the dimension of V in $\mathcal{D}$ is just its usual dimension:

$$P^{+} = \sum_V {\Theta_V}^{+1}(\dim(V))^2 = 1(1)^2 + 1(1)^2 + 1(1)^2 + 1(1)^2 + 1(2)^2 + 1(2)^2 + 1(1)^2 - 1(1)^2$$

$$- 1(1)^2 + 1(1)^2 - 1(2)^2 + 1(2)^2 + 1(1)^2 + \omega^2(1)^2 + \omega^4(1)^2 + 1(1)^2 + \omega^2(1)^2 + \omega^4(1)^2$$

$$+ 1(1)^2 + \omega^1(1)^2 + \omega^2(1)^2 - 1(1)^2 + \omega^4(1)^2 + \omega^5(1)^2 + 1(1)^2 - 1(1)^2 + 1(1)^2 - 1(1)^2$$

$$+ 1(1)^2 + 1(1)^2 - 1(1)^2 - 1(1)^2$$

$$= 1 + 1 + 1 + 1 + 4 + 4 + 1 - 1 - 1 + 1 - 4 + 4 + 1 + \omega^2 + \omega^4 + 1 + \omega^2 + \omega^4 + 1 + \omega^1$$

$$+ \omega^2 - 1 + \omega^4 + \omega^5 + 1 - 1 + 1 - 1 + 1 + 1 - 1 - 1$$

$$= 14 + 3(\omega^2 + \omega^4) + \omega^1 + \omega^5$$

$$= 14 + 3\big((-\frac{1}{2} + i\frac{\sqrt{3}}{2}) + (-\frac{1}{2} - i\frac{\sqrt{3}}{2})\big) + (\frac{1}{2} + i\frac{\sqrt{3}}{2}) + (\frac{1}{2} - i\frac{\sqrt{3}}{2}) = 14 - 3 + 1 = 12.$$

In the same way we can calculate $P^{-} = \sum_V {\Theta_V}^{-1}(\dim(V))^2$ to get the following

$$P^{-} = \sum_V {\Theta_V}^{-1}(\dim(V))^2 = \sum_V \frac{1}{\Theta_V}(\dim(V))^2 = 14 + 3(\omega^{-2} + \omega^{-4}) + \omega^{-1} + \omega^{-5}$$

$$= 14 + 3\big((-\frac{1}{2} - i\frac{\sqrt{3}}{2}) + (-\frac{1}{2} + i\frac{\sqrt{3}}{2})\big) + (\frac{1}{2} - i\frac{\sqrt{3}}{2}) + (\frac{1}{2} + i\frac{\sqrt{3}}{2})$$

$$= 14 - 3 + 1 = 12.$$

If we substitute these values in theorem 5.1.6 we get

$$S = \frac{\tilde{S}}{\sqrt{P^{+}P^{-}}} = \frac{\tilde{S}}{12}.$$

Using the order of the indecomposable objects in table (5), we get T to be a diagonal 32×32 matrix whose diagonal entries are taken from the table as follows

$$
\left(\begin{array}{cccccccccccccccccccccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right)
$$

As every indecomposable object in our example is self-dual, the matrix C is the 32×32 identity matrix :

$$C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Now it is possible to check that the matrices S, T and C satisfy the following relations:

$$(ST)^3 = \sqrt{\frac{P^+}{P^-}}\, S^2 \,, \qquad S^2 = C \,, \qquad CT = TC \,, \qquad C^2 = 1 \,.$$

For more details about the matrices see the appendix.

Chapter 6

An equivalence of tensor categories

In this chapter we generalize some results of [5] which considered group doublecross products, i.e. a group X factoring into two subgroups G and M, to the case where M is not a subgroup and the algebra D here is that defined in 1.3. In fact, we show an equivalence of categories between the double $\mathcal{D}$ of the non-trivially associated tensor category, constructed from left coset representatives of a subgroup of a finite group X, and the category of representations of the Drinfeld double of the group, $D(X)$.

This chapter has also already been sent for publication as a part of a paper by myself and my supervisor E. J. Beggs [2].

6.1 The definition of $D(X)$

The double of a group is well known. We give a definition of $D(X)$ and its representations from [5]. The case we are interested in is where $X = GM$ as previously discussed in 1.3.

Definition 6.1.1 *[5] For the doublecross product group $X = GM$ there is a quantum double $D(X) = k(X) \bowtie kX$, with basis $\delta_y \otimes x$ for $x, y \in X$ and which has the following operations*

$$(\delta_y \otimes x)(\delta_{y'} \otimes x') = \delta_{x^{-1}yx,y'}(\delta_y \otimes xx'), \;\; \Delta(\delta_y \otimes x) = \sum_{ab=y} \delta_a \otimes x \otimes \delta_b \otimes x$$

$$1 = \sum_y \delta_y \otimes e, \;\; \epsilon(\delta_y \otimes x) = \delta_{y,e}, \;\; S(\delta_y \otimes x) = \delta_{x^{-1}y^{-1}x} \otimes x^{-1},$$

$$(\delta_y \otimes x)^* = \delta_{x^{-1}yx} \otimes x^{-1}, \;\; R = \sum_{x,z} \delta_x \otimes e \otimes \delta_z \otimes x.$$

The representations of $D(X)$ are given by X-graded left kX-modules. The kX action will be denoted by $\dot{\triangleright}$ and the grading by $\|\!|.\|\!|$. The grading and X action are related by

$$\|\!|x \dot{\triangleright} \xi\|\!| = x \|\!|\xi\|\!| x^{-1} \quad x \in X, \;\; \xi \in V, \tag{6.1}$$

and the action of $(\delta_y \otimes x) \in D(X)$ is given by

$$(\delta_y \otimes x) \dot{\triangleright} \xi = \delta_{y, \|\!|x\dot{\triangleright}\xi\|\!|} x \dot{\triangleright} \xi. \tag{6.2}$$

6.2 The algebra structure

Proposition 6.2.1 *There is a functor χ from $\mathcal{D}$ to the category of representations of $D(X)$ given by the following: As vector spaces, $\chi(V)$ is the same as V, and χ is the identity map. The X-grading $\|\!|.\|\!|$ on $\chi(V)$ and the action of $us \in kX$ are defined by*

$$\begin{aligned} \|\!|\chi(\eta)\|\!| &= \langle \eta \rangle^{-1} |\eta| \quad \textit{for } \eta \in V, \\ us \dot{\triangleright} \chi(\eta) &= \chi\Big(((s \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta) \bar{\triangleleft} u^{-1} \Big), \quad s \in M \qquad u \in G\,. \end{aligned}$$

A morphism $\phi : V \to W$ in $\mathcal{D}$ is sent to the morphism $\chi(\phi) : \chi(V) \to \chi(W)$ defined by $\chi(\phi)(\chi(\xi)) = \chi(\phi(\xi))$.

Proof. First we show that $\dot{\triangleright}$ is an action, i.e. $vt\dot{\triangleright}\big(us\dot{\triangleright}\chi(\eta)\big) = vtus\dot{\triangleright}\chi(\eta)$ for all $s, t \in M$ and $u, v \in G$. Note that

$$\begin{aligned} vt\dot{\triangleright}\big(us\dot{\triangleright}\chi(\eta)\big) &= vt\dot{\triangleright}\chi\big(((s\triangleleft|\eta|^{-1})\bar{\triangleright}\eta)\bar{\triangleleft}u^{-1}\big) \\ &= \chi\big(((t\triangleleft|\bar{\eta}|^{-1})\bar{\triangleright}\bar{\eta})\bar{\triangleleft}v^{-1}\big), \end{aligned}$$

where $\bar{\eta} = \big((s\triangleleft|\eta|^{-1})\bar{\triangleright}\eta\big)\bar{\triangleleft}u^{-1}$. On the other hand we have

$$vtus = v(t\triangleright u)\tau(t\triangleleft u, s)\big((t\triangleleft u)\cdot s\big),$$

where $v(t\triangleright u)\tau(t\triangleleft u, s) \in G$ and $(t\triangleleft u)\cdot s \in M$, so

$$\begin{aligned} vtus\dot{\triangleright}\chi(\eta) &= \chi\Big(\Big((((t\triangleleft u)\cdot s)\triangleleft|\eta|^{-1})\bar{\triangleright}\eta\Big)\bar{\triangleleft}\tau(t\triangleleft u, s)^{-1}(t\triangleright u)^{-1}v^{-1}\Big) \\ &= \chi\left(\left(\Big((((t\triangleleft u)\cdot s)\triangleleft|\eta|^{-1})\bar{\triangleright}\eta\Big)\bar{\triangleleft}\tau(t\triangleleft u, s)^{-1}(t\triangleright u)^{-1}\right)\bar{\triangleleft}v^{-1}\right). \end{aligned}$$

We need to show that

$$\begin{aligned} (t\triangleleft|\bar{\eta}|^{-1})\bar{\triangleright}\bar{\eta} &= \Big((((t\triangleleft u)\cdot s)\triangleleft|\eta|^{-1})\bar{\triangleright}\eta\Big)\bar{\triangleleft}\tau(t\triangleleft u, s)^{-1}(t\triangleright u)^{-1} \\ &= \Big(((t\triangleleft u(s\triangleright|\eta|^{-1}))\cdot(s\triangleleft|\eta|^{-1}))\bar{\triangleright}\eta\Big)\bar{\triangleleft}\tau(t\triangleleft u, s)^{-1}(t\triangleright u)^{-1} \end{aligned} \tag{6.3}$$

Put $\bar{s} = s\triangleleft|\eta|^{-1}$ and $\eta' = \bar{s}\,\bar{\triangleright}\eta$ which give $\bar{\eta} = \eta'\bar{\triangleleft}u^{-1}$. Then using the connections between the gradings and actions,

$$|\bar{\eta}| = |\eta'\bar{\triangleleft}u^{-1}| = (\langle\eta'\rangle\triangleright u^{-1})^{-1}|\eta'|u^{-1}.$$

Putting $\bar{t} = t\triangleleft u|\eta'|^{-1}$, the left hand side of (6.3) will become

$$\begin{aligned} (t\triangleleft|\bar{\eta}|^{-1})\bar{\triangleright}\bar{\eta} &= \Big(t\triangleleft u|\eta'|^{-1}(\langle\eta'\rangle\triangleright u^{-1})\Big)\bar{\triangleright}(\eta'\bar{\triangleleft}u^{-1}) \\ &= \Big(\bar{t}\triangleleft(\langle\eta'\rangle\triangleright u^{-1})\Big)\bar{\triangleright}(\eta'\bar{\triangleleft}u^{-1}) \\ &= (\,\bar{t}\,\bar{\triangleright}\eta')\bar{\triangleleft}\Big((\bar{t}\triangleleft|\eta'|)\triangleright u^{-1}\Big)\ . \end{aligned}$$

Note that we have used the cross relation between the actions in the last equivalence. Now, from (6.3) and the fact that $(t\triangleright u)^{-1} = (\bar{t}\triangleleft|\eta'|)\triangleright u^{-1}$, we only need to show that

$$\bar{t}\,\bar{\triangleright}\eta' = \Big(((t\triangleleft u(s\triangleright|\eta|^{-1}))\cdot(s\triangleleft|\eta|^{-1}))\bar{\triangleright}\eta\Big)\bar{\triangleleft}\tau(t\triangleleft u, s)^{-1}\ . \tag{6.4}$$

From the formula for the composition of the M 'action' the right hand side of (6.4) becomes $\bar{p}\,\bar{\triangleright}(\bar{s}\,\bar{\triangleright}\eta) = \bar{p}\,\bar{\triangleright}\eta'$, where $\bar{p}' = t\triangleleft u(s\triangleright|\eta|^{-1})$ and $\bar{p} = \bar{p}'\triangleleft\tau(\bar{s},\langle\eta\rangle)\tau(\langle\bar{s}\,\bar{\triangleright}\eta\rangle, \bar{s}\triangleleft|\eta|)^{-1}$. We have used the fact that $\tau(t\triangleleft u, s) = \tau(\bar{p}'\triangleleft(\bar{s}\triangleright|\eta|), \bar{s}\triangleleft|\eta|)$. Now we just have to prove that $\bar{p} = \bar{t}$. Because $\tau(\bar{s},\langle\eta\rangle)^{-1}(\bar{s}\triangleright|\eta|) = \tau(\langle\bar{s}\,\bar{\triangleright}\eta\rangle, \bar{s}\triangleleft|\eta|)^{-1}|\bar{s}\,\bar{\triangleright}\eta|$ and knowing that $(\bar{s}\triangleright|\eta|) = (s\triangleright|\eta|^{-1})^{-1}$, we can write $\bar{p}$ as follows

$$\begin{aligned}
\bar{p} &= \bar{p}'\triangleleft(\bar{s}\triangleright|\eta|)|\bar{s}\triangleright\eta|^{-1}\\
&= t\triangleleft u(s\triangleright|\eta|^{-1})(s\triangleright|\eta|^{-1})^{-1}|\eta'|^{-1}\\
&= t\triangleleft u|\eta'|^{-1} = \bar{t}\ .
\end{aligned}$$

Next we show that $\|us\dot{\triangleright}\chi(\eta)\| = us\,\|\chi(\eta)\|\,(us)^{-1}$ where $u\in G$ and $s\in M$.

$$\begin{aligned}
\|us\dot{\triangleright}\chi(\eta)\| &= \|\chi\Big(((s\triangleleft|\eta|^{-1})\bar{\triangleright}\eta)\bar{\triangleleft}u^{-1}\Big)\|\\
&= \langle\eta'\bar{\triangleleft}u^{-1}\rangle^{-1}|\eta'\bar{\triangleleft}u^{-1}| \ =\ u\langle\eta'\rangle^{-1}|\eta'|u^{-1}\\
&= u\langle\bar{s}\,\bar{\triangleright}\eta\rangle^{-1}|\bar{s}\,\bar{\triangleright}\eta|u^{-1} \ =\ u(\bar{s}\triangleleft|\eta|)\langle\eta\rangle^{-1}|\eta|(\bar{s}\triangleleft|\eta|)^{-1}u^{-1}\\
&= us\langle\eta\rangle^{-1}|\eta|s^{-1}u^{-1}\ . \quad \square
\end{aligned}$$

Theorem 6.2.2 *The functor χ is invertible.*

Proof. We have already proved in the previous proposition that the X-grading $\|.\|$ and the action $\dot{\triangleright}$ give a representation of $D(X)$, so we only need to show that χ is an isomorphism, which we do by giving its inverse χ^{-1} as the following: Let W be a representation of $D(X)$, with kX action $\dot{\triangleright}$ and X-grading $\|.\|$. Define a D representation as follows: $\chi^{-1}(W)$ will be the same as W as a vector space. There will be G and M gradings given by the factorization

$$|\!|\!|\xi|\!|\!|^{-1} = |\chi^{-1}(\xi)|^{-1}\langle\chi^{-1}(\xi)\rangle, \;\; \xi \in W, \; \langle\chi^{-1}(\xi)\rangle \in M, \;\; |\chi^{-1}(\xi)| \in G.$$

The action of $s \in M$ and $u \in G$ are given by

$$s\bar{\triangleright}\chi^{-1}(\xi) = \chi^{-1}\big((s{\triangleleft}|\chi^{-1}(\xi)|)\dot{\triangleright}\xi\big)\;,\;\; \chi^{-1}(\xi)\bar{\triangleleft}u = \chi^{-1}(u^{-1}\dot{\triangleright}\xi) \tag{6.5}$$

We now check that this preserves the X-grading and action. We first check the X-grading as follows

$$|\!|\!|\chi(\xi)|\!|\!| = \langle\chi^{-1}(\chi(\xi))\rangle^{-1}|\chi^{-1}(\chi(\xi))| = \langle\xi\rangle^{-1}|\xi|, \;\text{ and}$$

$$|\!|\!|\chi(\chi^{-1}(\xi))|\!|\!| = \langle\chi^{-1}(\xi)\rangle^{-1}|\chi^{-1}(\xi)| = |\!|\!|\xi|\!|\!|.$$

Next we check it for the M-action

$$\begin{aligned} s\bar{\triangleright}\chi^{-1}(\chi(\xi)) &= \chi^{-1}\Big((s{\triangleleft}|\chi^{-1}\chi(\xi)|)\dot{\triangleright}\chi(\xi)\Big) \\ &= \chi^{-1}\Big(\bar{s}\dot{\triangleright}\chi(\xi)\Big) \\ &= \chi^{-1}\chi\Big((\bar{s}{\triangleleft}|\xi|^{-1})\bar{\triangleright}\xi\Big) \\ &= s\bar{\triangleright}\xi, \end{aligned}$$

where $\bar{s} = s{\triangleleft}|\xi|$. For the M-action, we have

$$\begin{aligned} s\dot{\triangleright}\chi(\chi^{-1}(\xi)) &= \chi\Big((s{\triangleleft}|\chi^{-1}(\xi)|^{-1})\bar{\triangleright}\chi^{-1}(\xi)\Big) \\ &= \chi\Big(\,\tilde{s}\,\bar{\triangleright}\chi^{-1}(\xi)\Big) \\ &= \chi\chi^{-1}\Big(\,(\tilde{s}\,{\triangleleft}|\chi^{-1}(\xi)|)\dot{\triangleright}\xi\Big) \\ &= s\dot{\triangleright}\xi, \end{aligned}$$

where $\tilde{s} = s{\triangleleft}|\chi^{-1}(\xi)|^{-1}$.

Finally, for the G-action we have

$$\begin{aligned}\chi^{-1}(\chi(\xi))\bar{\triangleleft}u &= \chi^{-1}\big(u^{-1}\dot{\triangleright}\chi(\xi)\big)\\ &= \chi^{-1}\big(\chi(\xi\bar{\triangleleft}u)\big)\\ &= \xi\bar{\triangleleft}u,\end{aligned}$$

and

$$\begin{aligned}u\dot{\triangleright}\chi(\chi^{-1}(\xi)) &= \chi\big(\chi^{-1}(\xi)\bar{\triangleleft}u^{-1}\big)\\ &= \chi\big(\chi^{-1}(u\dot{\triangleright}\xi)\big)\\ &= u\dot{\triangleright}\xi.\end{aligned}$$

Note that we have used 6.2.1 and (6.5). $\square$

Proposition 6.2.3 *For an element $(\delta_y \otimes x)$ of the algebra D in the category $\mathcal{D}$,*

$$\chi\big(\xi\hat{\triangleleft}(\delta_y \otimes x)\big) = \delta_{y,\|\xi\|}\, x^{-1}\dot{\triangleright}\chi(\xi).$$

Proof. Starting with the left hand side,

$$\chi\big(\xi\hat{\triangleleft}(\delta_y \otimes x)\big) = \chi(\delta_{y,\|\xi\|}\,\xi\hat{\triangleleft}x) \;=\; \delta_{y,\|\xi\|}\,\chi(\xi\hat{\triangleleft}x).$$

Putting $x = us$ for $u \in G$ and $s \in M$, then using proposition 1.3.12 we get

$$\xi\hat{\triangleleft}x = \xi\hat{\triangleleft}us = (\xi\bar{\triangleleft}u)\hat{\triangleleft}s = \bar{\xi}\hat{\triangleleft}s = \big((s^L\triangleleft|\bar{\xi}|^{-1})\bar{\triangleright}\bar{\xi}\big)\bar{\triangleleft}\tau(s^L, s).$$

where $\bar{\xi} = \xi\bar{\triangleleft}u$. But from the connection between the gradings and the actions we know that $|\xi\bar{\triangleleft}u|^{-1} = u^{-1}|\xi|^{-1}(\langle\xi\rangle\triangleright u)$. So if we put $\tilde{s} = s^L\triangleleft u^{-1}|\xi|^{-1}$ and substitute in the above equation we get

$$\xi\hat{\triangleleft}x = (\xi\bar{\triangleleft}u)\hat{\triangleleft}s = \big((s^L\triangleleft u^{-1}|\xi|^{-1}(\langle\xi\rangle\triangleright u))\bar{\triangleright}(\xi\bar{\triangleleft}u)\big)\bar{\triangleleft}\tau(s^L, s) = \big(\tilde{s}\triangleleft(\langle\xi\rangle\triangleright u)\bar{\triangleright}(\xi\bar{\triangleleft}u)\big)\bar{\triangleleft}\tau(s^L, s).$$

Now using the cross relation and knowing that $(s^L \triangleleft u^{-1}) \triangleright u = (s^L \triangleright u^{-1})^{-1}$ we get

$$\xi \hat{\triangleleft} x = \big((\tilde{s} \bar{\triangleright} \xi) \bar{\triangleleft} ((\tilde{s} \triangleleft |\xi|) \triangleright u)\big) \bar{\triangleleft} \tau(s^L, s) = \big((\tilde{s} \bar{\triangleright} \xi) \bar{\triangleleft} ((s^L \triangleleft u^{-1}) \triangleright u)\big) \bar{\triangleleft} \tau(s^L, s)$$

$$= \big(((s^L \triangleleft u^{-1}|\xi|^{-1}) \bar{\triangleright} \xi) \bar{\triangleleft} (s^L \triangleright u^{-1})^{-1}\big) \bar{\triangleleft} \tau(s^L, s) = \big((s^L \triangleleft u^{-1}|\xi|^{-1}) \bar{\triangleright} \xi\big) \bar{\triangleleft} (s^L \triangleright u^{-1})^{-1} \tau(s^L, s).$$

Now put $\bar{u} = \tau(s^L, s)^{-1}(s^L \triangleright u^{-1})$ and $\bar{s} = s^L \triangleleft u^{-1}$. Then

$$\chi\big(\xi \hat{\triangleleft} (\delta_y \otimes x)\big) = \delta_{y, \|\xi\|}\, \chi\Big(\big((\bar{s} \triangleleft |\xi|^{-1}) \bar{\triangleright} \xi\big) \bar{\triangleleft} \bar{u}^{-1}\Big)$$

$$= \delta_{y, \|\xi\|}\, \bar{u}\bar{s} \dot{\triangleright} \chi(\xi)$$

$$= \delta_{y, \|\xi\|}\, \tau(s^L, s)^{-1}(s^L \triangleright u^{-1})(s^L \triangleleft u^{-1}) \dot{\triangleright} \chi(\xi)$$

$$= \delta_{y, \|\xi\|}\, s^{-1} s^{L^{-1}} s^L u^{-1} \dot{\triangleright} \chi(\xi)$$

$$= \delta_{y, \|\xi\|}\, (us)^{-1} \dot{\triangleright} \chi(\xi) \;=\; \delta_{y, \|\xi\|}\, x^{-1} \dot{\triangleright} \chi(\xi). \quad \square$$

Proposition 6.2.4 *Define a map* $\psi : D \longrightarrow D(X)$ *by* $\psi(\delta_y \otimes x) = \delta_{x^{-1}yx} \otimes x^{-1}$ *. Then* ψ *satisfies the equation* $\chi(\xi \hat{\triangleleft} (\delta_y \otimes x)) = \psi(\delta_y \otimes x) \dot{\triangleright} \chi(\xi)$ *.*

Proof. Let $\delta_y \otimes x$ be an element of the algebra D in the category $\mathcal{D}$. As $\psi(a)$ is in $D(X)$, we put $\psi(\delta_y \otimes x) = \delta_{\bar{y}} \otimes \bar{x}$. Then by definition 6.1.1

$$(\delta_{\bar{y}} \otimes \bar{x}) \dot{\triangleright} \chi(\xi) = \delta_{\bar{y}, \|\bar{x} \dot{\triangleright} \chi(\xi)\|}\, \bar{x} \dot{\triangleright} \chi(\xi). \tag{6.6}$$

As $\|\bar{x} \dot{\triangleright} \chi(\xi)\| = \bar{x} \|\chi(\xi)\| \bar{x}^{-1} = \bar{x} \langle \xi \rangle^{-1} |\xi| \bar{x}^{-1}$, we get the following

$$\delta_{\bar{y}, \|\bar{x} \dot{\triangleright} \chi(\xi)\|} = \delta_{\bar{x}^{-1}\bar{y}\bar{x}, \langle \xi \rangle^{-1}|\xi|} = \delta_{\bar{x}^{-1}\bar{y}^{-1}\bar{x}, |\xi|^{-1}\langle \xi \rangle} = \delta_{\bar{x}^{-1}\bar{y}^{-1}\bar{x}, \|\xi\|} \;.$$

But, from the previous proposition, we know that

$$\chi\big(\xi \hat{\triangleleft} (\delta_y \otimes x)\big) = \delta_{y, \|\xi\|}\, x^{-1} \dot{\triangleright} \chi(\xi).$$

So we conclude that 6.6 is true if and only if $\bar{x} = x^{-1}$ and $\bar{y} = \bar{x} y^{-1} \bar{x}^{-1} = x^{-1} y^{-1} x$. Hence, if we define

$$\psi(\delta_y \otimes x) = \delta_{x^{-1}yx} \otimes x^{-1},$$

then we obtain

$$\chi(\xi\hat{\triangleleft}(\delta_y\otimes x)) = \psi(\delta_y\otimes x)\dot{\triangleright}\chi(\xi)\ .\quad\square$$

The reader will recall that D is in general a non-trivially associated algebra (i.e. it is only associative in the category $\mathcal{D}$ with its non-trivial associator). Thus, in general, it can not be isomorphic to $D(X)$, which really is associative. In general, ψ can not be an algebra map.

Proposition 6.2.5 *For a and b elements of the algebra D in the category $\mathcal{D}$,*

$$\psi(b)\psi(a) = \psi(ab)\Big(\sum_{y\in Y}\delta_y\otimes\tau(\langle a\rangle,\langle b\rangle)^{-1}\Big).$$

Proof. by 6.2.4 we have

$$\begin{aligned}\chi\big((\xi\hat{\triangleleft}a)\hat{\triangleleft}b\big) &= \psi(b)\dot{\triangleright}\chi(\xi\hat{\triangleleft}a)\\ &= \psi(b)\dot{\triangleright}\big(\psi(a)\dot{\triangleright}\chi(\xi)\big)\\ &= \psi(b)\psi(a)\dot{\triangleright}\chi(\xi).\end{aligned}$$

But also

$$\begin{aligned}\chi\big((\xi\hat{\triangleleft}a)\hat{\triangleleft}b\big) &= \chi\Big(\big(\xi\hat{\triangleleft}\tilde{\tau}(\|a\|,\|b\|)\big)\hat{\triangleleft}ab\Big)\\ &= \psi(ab)\dot{\triangleright}\chi\big(\xi\hat{\triangleleft}\tilde{\tau}(\|a\|,\|b\|)\big) = \psi(ab)\dot{\triangleright}\chi\big(\xi\hat{\triangleleft}\tau(\langle a\rangle,\langle b\rangle)\big)\\ &= \psi(ab)\psi(f)\dot{\triangleright}\chi(\xi),\end{aligned}$$

where $f = \sum_y\delta_y\otimes\tau(\langle a\rangle,\langle b\rangle)$. $\square$

6.3 The coalgebra structure

For the coalgebra to be examined, we turn to tensor products of representations. It is known that for a general Hopf algebra H with representations V and W, the tensor product representation $V \otimes W$ is defined by the action

$$h \triangleright (\eta \otimes \xi) = \sum h_{(1)} \triangleright \eta \otimes h_{(2)} \triangleright \xi \,,$$

for $\eta \in V$ and $\xi \in W$.

In the case of representations V' and W' of $D(X)$, according to this formula, we get the following equations:

$$x \dot{\triangleright} (\eta' \otimes \xi') = x \dot{\triangleright} \eta' \otimes x \dot{\triangleright} \xi' \,,$$

for $\eta' \in V'$, $\xi' \in W'$ and $x \in X$.

For the representations V and W of D, we get the following actions and gradings on the tensor product $V \otimes W$:

$$|\xi \otimes \eta| = \tau(\langle \xi \rangle, \langle \eta \rangle)^{-1} |\xi||\eta| \,,$$

$$\big(s \triangleleft \tau(\langle \xi \rangle, \langle \eta \rangle)\big) \bar{\triangleright} (\xi \otimes \eta) = (s \bar{\triangleright} \xi) \tau(s \triangleleft |\xi|, \langle \eta \rangle) \tau\big(\langle (s \triangleleft |\xi|) \bar{\triangleright} \eta \rangle, s \triangleleft |\xi||\eta|\big)^{-1} \otimes (s \triangleleft |\xi|) \bar{\triangleright} \eta \,,$$

$$\langle \xi \otimes \eta \rangle = \langle \xi \rangle \cdot \langle \eta \rangle \quad \text{and} \quad (\xi \otimes \eta) \bar{\triangleleft} u = \xi \bar{\triangleleft} (\langle \eta \rangle \triangleright u) \otimes \eta \bar{\triangleleft} u \,,$$

where $\eta \in V$, $\xi \in W$, $s \in M$ and $u \in G$.

It is found that χ does not preserve the tensor product of representations. For this to be corrected we introduce the map given in the following definition :

Definition 6.3.1 *Let V and W be objects of the category $\mathcal{D}$. The map $c : \chi(V) \otimes \chi(W) \longrightarrow \chi(V \otimes W)$ is defined by:*

$$c\big(\chi(\eta) \otimes \chi(\xi)\big) = \chi\Big(\big((\langle \xi \rangle \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta\big) \otimes \xi\Big) \,,$$

where $\eta \in V$ and $\xi \in W$.

Proposition 6.3.2 *The map c, defined above, is a $D(X)$ module map, i.e.*

$$\| c(\chi(\eta) \otimes \chi(\xi)) \| = \| \chi(\eta) \otimes \chi(\xi) \| \ ,$$

$$x \dot{\triangleright} c(\chi(\eta) \otimes \chi(\xi)) = c\Big(x \dot{\triangleright} (\chi(\eta) \otimes \chi(\xi))\Big) \qquad \forall x \in X \ .$$

Proof. We will begin with the grading first. It is known that

$$\| \chi(\eta) \otimes \chi(\xi) \| = \| \chi(\eta) \| \| \chi(\xi) \| = \langle \eta \rangle^{-1} |\eta| \langle \xi \rangle^{-1} |\xi|.$$

But on the other hand we know, from the definition of c, that

$$\begin{aligned} \| c(\chi(\eta) \otimes \chi(\xi)) \| &= \| \chi(((\langle \xi \rangle \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta \otimes \xi) \| \\ &= \big\langle ((\langle \xi \rangle \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta \otimes \xi \big\rangle^{-1} \big| ((\langle \xi \rangle \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta \otimes \xi \big| \\ &= \langle \xi \rangle^{-1} \langle \bar{\eta} \rangle^{-1} |\bar{\eta}| |\xi| \\ &= \langle \xi \rangle^{-1} \langle \bar{s} \bar{\triangleright} \eta \rangle^{-1} |\bar{s} \bar{\triangleright} \eta| |\xi| \\ &= \langle \xi \rangle^{-1} (\bar{s} \triangleleft |\eta|) \langle \eta \rangle^{-1} |\eta| (\bar{s} \triangleleft |\eta|)^{-1} |\xi| \\ &= \langle \eta \rangle^{-1} |\eta| \langle \xi \rangle^{-1} |\xi|, \end{aligned}$$

where $\bar{s} = \langle \xi \rangle \triangleleft |\eta|^{-1}$ and $\bar{\eta} = (\langle \xi \rangle \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta = \bar{s} \bar{\triangleright} \eta$, which gives the result.

For the G action, we need to show that

$$c\Big(u \dot{\triangleright} (\chi(\eta) \otimes \chi(\xi))\Big) = u \dot{\triangleright} c(\chi(\eta) \otimes \chi(\xi)) \, .$$

We know from the definitions that

$$\begin{aligned} u \dot{\triangleright} (\chi(\eta) \otimes \chi(\xi)) &= \chi(\eta \bar{\triangleleft} u^{-1}) \otimes \chi(\xi \bar{\triangleleft} u^{-1}), \\ c\Big(u \dot{\triangleright} (\chi(\eta) \otimes \chi(\xi))\Big) &= \chi\Big((((\langle \xi \bar{\triangleleft} u^{-1} \rangle \triangleleft |\eta \bar{\triangleleft} u^{-1}|^{-1}) \bar{\triangleright} (\eta \bar{\triangleleft} u^{-1})) \otimes (\xi \bar{\triangleleft} u^{-1}) \Big). \end{aligned}$$

By using the properties of the G and M gradings,

$$
\begin{aligned}
\langle \xi \bar{\triangleleft} u^{-1} \rangle \triangleleft |\eta \bar{\triangleleft} u^{-1}|^{-1} &= ((\langle \xi \rangle \triangleleft u^{-1}) \triangleleft u |\eta|^{-1} (\langle \eta \rangle \triangleright u^{-1}) \\
&= \langle \xi \rangle \triangleleft |\eta|^{-1} (\langle \eta \rangle \triangleright u^{-1}) \\
(\langle \xi \bar{\triangleleft} u^{-1} \rangle \triangleleft |\eta \bar{\triangleleft} u^{-1}|^{-1}) \bar{\triangleright} (\eta \bar{\triangleleft} u^{-1}) &= \big(((\langle \xi \rangle \triangleleft |\eta|^{-1}) \triangleleft (\langle \eta \rangle \triangleright u^{-1}) \big) \bar{\triangleright} (\eta \bar{\triangleleft} u^{-1}) \\
&= \big(((\langle \xi \rangle \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta \big) \bar{\triangleleft} \Big(((\langle \xi \rangle \triangleleft |\eta|^{-1}) \triangleleft |\eta|) \triangleright u^{-1} \Big) \\
&= \big(((\langle \xi \rangle \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta \big) \bar{\triangleleft} (\langle \xi \rangle \triangleright u^{-1}).
\end{aligned}
$$

Now we can write

$$
c\Big(u \dot{\triangleright} (\chi(\eta) \otimes \chi(\xi)) \Big) = \chi\Big(((\langle \xi \rangle \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta \big) \bar{\triangleleft} (\langle \xi \rangle \triangleright u^{-1}) \otimes (\xi \bar{\triangleleft} u^{-1}) \Big). \tag{6.7}
$$

On the other hand,

$$
\begin{aligned}
u \dot{\triangleright} c\big(\chi(\eta) \otimes \chi(\xi)\big) &= u \dot{\triangleright} \chi\Big(((\langle \xi \rangle \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta \big) \otimes \xi \Big) \\
&= \chi\Big(\Big(((\langle \xi \rangle \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta \big) \otimes \xi \Big) \bar{\triangleleft} u^{-1} \Big) \\
&= \chi\Big(((\langle \xi \rangle \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta \big) \bar{\triangleleft} (\langle \xi \rangle \triangleright u^{-1}) \otimes (\xi \bar{\triangleleft} u^{-1}) \Big),
\end{aligned}
$$

which is the same as (6.7).

Now we show that c preserves the M action, i.e. for $s \in M$,

$$
c\Big(s \dot{\triangleright} (\chi(\eta) \otimes \chi(\xi)) \Big) = s \dot{\triangleright} c\big(\chi(\eta) \otimes \chi(\xi)\big).
$$

We know from the definitions that

$$
\begin{aligned}
s \dot{\triangleright} \big(\chi(\eta) \otimes \chi(\xi)\big) &= \chi\big((s \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta \big) \otimes \chi\big((s \triangleleft |\xi|^{-1}) \bar{\triangleright} \xi \big) \\
c\Big(s \dot{\triangleright} (\chi(\eta) \otimes \chi(\xi)) \Big) &= \chi\Big((\langle (s \triangleleft |\xi|^{-1}) \bar{\triangleright} \xi \rangle \triangleleft \big| (s \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta \big|^{-1}) \bar{\triangleright} \big((s \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta \big) \\
&\qquad \otimes \big((s \triangleleft |\xi|^{-1}) \bar{\triangleright} \xi \big) \Big).
\end{aligned}
$$

Using the 'action' property for $\bar{\triangleright}$, we get

$$
\big(\big\langle (s \triangleleft |\xi|^{-1}) \bar{\triangleright} \xi \big\rangle \triangleleft \big| (s \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta \big|^{-1} \big) \bar{\triangleright} \big((s \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta \big) = \big((p' \cdot \bar{t}) \bar{\triangleright} \eta \big) \bar{\triangleleft} \tau \big(p' \triangleleft (\bar{t} \triangleright |\eta|), \bar{t} \triangleleft |\eta| \big)^{-1} ,
$$

where $\bar{t} = s\triangleleft|\eta|^{-1}$ and

$$p' = \big\langle (s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\big\rangle\triangleleft\big|\bar{t}\bar{\triangleright}\eta\big|^{-1}\,\tau\big(\langle\bar{t}\bar{\triangleright}\eta\rangle, \bar{t}\triangleleft|\eta|\big)\;\tau\big(\bar{t}, \langle\eta\rangle\big)^{-1}.$$

But using the connections between the grading and the actions, we know that $\big|\bar{t}\bar{\triangleright}\eta\big|^{-1} = (\bar{t}\triangleright|\eta|)^{-1}\,\tau\big(\bar{t}, \langle\eta\rangle\big)\tau\big(\langle\bar{t}\bar{\triangleright}\eta\rangle, \bar{t}\triangleleft|\eta|\big)^{-1}$, so

$$\begin{aligned}
p' &= \big\langle (s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\big\rangle\triangleleft(\bar{t}\triangleright|\eta|)^{-1}\\
&= \big\langle (s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\big\rangle\triangleleft\big((s\triangleleft|\eta|^{-1})\triangleright|\eta|\big)^{-1}\\
&= \big\langle (s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\big\rangle\triangleleft(s\triangleright|\eta|^{-1}).
\end{aligned}$$

Substituting in the equation above gives

$$\begin{aligned}
&\Big(\big\langle (s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\big\rangle\triangleleft\big|(s\triangleleft|\eta|^{-1})\bar{\triangleright}\eta\big|^{-1}\Big)\bar{\triangleright}\big((s\triangleleft|\eta|^{-1})\bar{\triangleright}\eta\big)\\
&\quad= \Big(\Big(\big(\big\langle (s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\big\rangle\triangleleft(s\triangleright|\eta|^{-1})\big)\cdot(s\triangleleft|\eta|^{-1})\Big)\bar{\triangleright}\eta\Big)\bar{\triangleleft}\tau\big(\big\langle (s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\big\rangle, s\big)^{-1}\\
&\quad= \Big(\Big(\big(\big\langle (s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\big\rangle\cdot s\big)\triangleleft|\eta|^{-1}\Big)\bar{\triangleright}\eta\Big)\bar{\triangleleft}\tau\big(\big\langle (s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\big\rangle, s\big)^{-1}\\
&\quad= \Big(\Big(\big((s\triangleleft|\xi|^{-1})\cdot\langle\xi\rangle\big)\triangleleft|\eta|^{-1}\Big)\bar{\triangleright}\eta\Big)\bar{\triangleleft}\tau\big(\big\langle (s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\big\rangle, s\big)^{-1}.
\end{aligned}$$

On the other hand, we know that

$$\begin{aligned}
s\dot{\triangleright}c\big(\chi(\eta)\otimes\chi(\xi)\big) &= s\dot{\triangleright}\chi\Big(\big((\langle\xi\rangle\triangleleft|\eta|^{-1})\bar{\triangleright}\eta\big)\otimes\xi\Big)\\
&= s\dot{\triangleright}\chi\big(\bar{\eta}\otimes\xi\big) = \chi\big((s\triangleleft|\bar{\eta}\otimes\xi|^{-1})\bar{\triangleright}(\bar{\eta}\otimes\xi)\big),
\end{aligned}$$

where $\bar{\eta} = (\langle\xi\rangle\triangleleft|\eta|^{-1})\bar{\triangleright}\eta$. Next we calculate

$$|\bar{\eta}\otimes\xi| = \tau(\langle\bar{\eta}\rangle, \langle\xi\rangle)^{-1}|\bar{\eta}||\xi|\ ,$$

$$s\triangleleft|\bar{\eta}\otimes\xi|^{-1} = s\triangleleft|\xi|^{-1}|\bar{\eta}|^{-1}\tau(\langle\bar{\eta}\rangle, \langle\xi\rangle)\ .$$

If we put $\bar{s} = s\triangleleft|\xi|^{-1}|\bar{\eta}|^{-1}$, then

$$\begin{aligned}(s\triangleleft|\bar{\eta}\otimes\xi|^{-1})\bar{\triangleright}(\bar{\eta}\otimes\xi) &= \big(\bar{s}\triangleleft\tau(\langle\bar{\eta}\rangle,\langle\xi\rangle)\big)\bar{\triangleright}(\bar{\eta}\otimes\xi)\\ &= (\bar{s}\bar{\triangleright}\bar{\eta})\bar{\triangleleft}\,\tau(\bar{s}\triangleleft|\bar{\eta}|,\langle\xi\rangle)\,\tau(\langle(\bar{s}\triangleleft|\bar{\eta}|)\bar{\triangleright}\xi\rangle,\bar{s}\triangleleft|\bar{\eta}||\xi|)^{-1}\\ &\qquad\otimes(\bar{s}\triangleleft|\bar{\eta}|)\bar{\triangleright}\xi\\ &= (\bar{s}\bar{\triangleright}\bar{\eta})\bar{\triangleleft}\,\tau(s\triangleleft|\xi|^{-1},\langle\xi\rangle)\,\tau(\langle(s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\rangle,s)^{-1}\\ &\qquad\otimes(s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\ .\end{aligned}$$

Using the ‘action’ property again,

$$\begin{aligned}\bar{s}\bar{\triangleright}\bar{\eta} &= (s\triangleleft|\xi|^{-1}|\bar{\eta}|^{-1})\bar{\triangleright}\big((\langle\xi\rangle\triangleleft|\eta|^{-1})\bar{\triangleright}\eta\big)\\ &= \Big(\big(q'\cdot(\langle\xi\rangle\triangleleft|\eta|^{-1})\big)\bar{\triangleright}\eta\Big)\bar{\triangleleft}\,\tau\big(q'\triangleleft((\langle\xi\rangle\triangleleft|\eta|^{-1})\triangleright|\eta|),\langle\xi\rangle\big)^{-1}\\ &= \Big(\big(q'\cdot(\langle\xi\rangle\triangleleft|\eta|^{-1})\big)\bar{\triangleright}\eta\Big)\bar{\triangleleft}\,\tau\big(q'\triangleleft(\langle\xi\rangle\triangleright|\eta|^{-1})^{-1},\langle\xi\rangle\big)^{-1}\ ,\end{aligned}$$

where

$$\begin{aligned}q' &= (s\triangleleft|\xi|^{-1}|\bar{\eta}|^{-1})\triangleleft\tau\big(\langle((\langle\xi\rangle\triangleleft|\eta|^{-1})\triangleright\eta\rangle,\langle\xi\rangle\big)\,\tau\big(\langle\xi\rangle\triangleleft|\eta|^{-1},\langle\eta\rangle\big)^{-1}\\ &= (s\triangleleft|\xi|^{-1})\triangleleft(\langle\xi\rangle\triangleright|\eta|^{-1})\ ,\end{aligned}$$

as

$$|\bar{\eta}|^{-1} = \big((\langle\xi\rangle\triangleleft|\eta|^{-1})\triangleright|\eta|\big)^{-1}\tau\big(\langle\xi\rangle\triangleleft|\eta|^{-1},\langle\eta\rangle\big)\,\tau\big(\langle((\langle\xi\rangle\triangleleft|\eta|^{-1})\bar{\triangleright}\eta\rangle,\langle\xi\rangle\big)^{-1}\ .$$

Hence substituting with the value of q' we get

$$\begin{aligned}\bar{s}\bar{\triangleright}\bar{\eta} &= \Big(\Big(\big((s\triangleleft|\xi|^{-1})\triangleleft(\langle\xi\rangle\triangleright|\eta|^{-1})\big)\cdot(\langle\xi\rangle\triangleleft|\eta|^{-1})\Big)\bar{\triangleright}\eta\Big)\bar{\triangleleft}\,\tau\big((s\triangleleft|\xi|^{-1}),\langle\xi\rangle\big)^{-1}\\ &= \Big(\Big(\big((s\triangleleft|\xi|^{-1})\cdot\langle\xi\rangle\big)\triangleleft|\eta|^{-1}\Big)\bar{\triangleright}\eta\Big)\bar{\triangleleft}\,\tau\big(s\triangleleft|\xi|^{-1},\langle\xi\rangle\big)^{-1}\ ,\end{aligned}$$

giving the required result

$$\begin{aligned}&(\bar{s}\bar{\triangleright}\bar{\eta})\bar{\triangleleft}\,\tau(s\triangleleft|\xi|^{-1},\langle\xi\rangle)\,\tau\big(\langle(s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\rangle,s\big)^{-1}\\ &\qquad= \Big(\Big(\big((s\triangleleft|\xi|^{-1})\cdot\langle\xi\rangle\big)\triangleleft|\eta|^{-1}\Big)\bar{\triangleright}\eta\Big)\bar{\triangleleft}\tau\big(\langle(s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\rangle,s\big)^{-1}\ .\quad\square\end{aligned}$$

Chapter 7

Ideas for further research

In this chapter we introduce some ideas for further research. We will not discuss them in much detail, although some detail will be introduced in the second section.

In the first section we show that for different algebras A and $\underline{A}$ associated with different coset representatives M and $\underline{M}$ there is a morphism $\theta : \underline{A} \to A$ in the category $\mathcal{C}$ which is proved to be an algebra map.

In the second section we try to study the algebraic structure described in section 1.3 in the case where our group X is an infinite topological group. We give a case where we can not give a single choice of coset representatives on $G \setminus X$ which is continuous. We are then forced to pick several choices, and these are related on the overlaps by the material discussed in section 7.1. This is not yet understood.

In the third section, some more ideas are included for those who are interested in this kind of research.

7.1 Different choices of coset representatives

In [4] it was stated that for a given subgroup G of a group X, different sets of representatives M and $\underline{M}$ for the left cosets can be chosen and these are related by an arbitrary function $\gamma : G\setminus X \to G$, so that if $s \in M$ then $\gamma([s])s \in \underline{M}$. Also the binary operations $\underline{\cdot} : G\setminus X \times G\setminus X \to G\setminus X$, $\underline{\tau} : G\setminus X \times G\setminus X \to G$ and $\underline{\triangleright} : G\setminus X \times G \to G$ for $\underline{M}$ were shown to be the following:

$$s\underline{\cdot}t = (s\triangleleft\gamma(t)\cdot t), \quad \underline{\tau}(s,t) = \gamma(s)(s\triangleright\gamma(t))\tau(s\triangleleft\gamma(t),t)\gamma((s\triangleleft\gamma(t))\cdot t)^{-1},$$

$$t\underline{\triangleright}u = \gamma(t)(t\triangleright u)\gamma(t\triangleleft u)^{-1}.$$

Proposition 7.1.1 *For the algebras $\underline{A}$ and A, the map $\theta : \underline{A} \to A$ defined by*

$$\theta_{\underline{A}}(\delta_s\otimes u) = \delta_{s\triangleleft\gamma(\underline{a})}\otimes\gamma(\underline{a})^{-1}u\,,$$

where $(\delta_s\otimes u)\in\underline{A}$ and $\underline{a} = \langle\delta_s\otimes u\rangle$, is a morphism in the category $\mathcal{C}$.

Proof. We should first show that $\theta_{\underline{A}}$ preserves the grades. Let $\underline{a} = \langle\delta_s\otimes u\rangle$, then $\underline{a}$ is defined by

$$s\,\underline{\cdot}\,\underline{a} = s\underline{\triangleleft}u = s\triangleleft u\,,$$

but also we have

$$\big(s\triangleleft\gamma(\underline{a})\big)\cdot\langle\theta_{\underline{A}}(\delta_s\otimes u)\rangle = \big(s\triangleleft\gamma(\underline{a})\big)\triangleleft\gamma(\underline{a})^{-1}u = s\triangleleft\gamma(\underline{a})\gamma(\underline{a})^{-1}u = s\triangleleft u\,,$$

so it preserves the grades. Now we need to check that it also preserves the actions, i.e. $\theta_{\underline{A}}\big((\delta_s\otimes u)\underline{\bar\triangleleft}v\big) = \theta_{\underline{A}}\big(\delta_s\otimes u\big)\bar\triangleleft v$. To calculate the left hand side we need to calculate the following:

$$(\delta_s\otimes u)\underline{\bar\triangleleft}v = \delta_{s\underline{\triangleleft}(\underline{a}\,\underline{\triangleright}v)}\otimes(\underline{a}\,\underline{\triangleright}v)^{-1}uv = \delta_{s\triangleleft\gamma(\underline{a})\,(\underline{a}\triangleright v)\gamma(\underline{a}\triangleleft v)^{-1}}\otimes\gamma(\underline{a}\triangleleft v)(\underline{a}\triangleright v)^{-1}\gamma(\underline{a})^{-1}uv,$$

so

$$L.H.S. = \theta_{\underline{A}}\big((\delta_s \otimes u)\underline{\triangleleft} v\big)$$

$$= \delta_{s\triangleleft\gamma(\underline{a})\,(\underline{a}\triangleright v)\gamma(\underline{a}\triangleleft v)^{-1}\gamma(\underline{a}\triangleleft v)} \otimes\ \gamma(\underline{a}\triangleleft v)^{-1}\gamma(\underline{a}\triangleleft v)(\underline{a}\triangleright v)^{-1}\gamma(\underline{a})^{-1}uv$$

$$= \delta_{s\triangleleft\gamma(\underline{a})\,(\underline{a}\triangleright v)} \otimes\ (\underline{a}\triangleright v)^{-1}\gamma(\underline{a})^{-1}uv\,.$$

Now we calculate the right hand side

$$R.H.S. = \theta_{\underline{A}}\big(\delta_s \otimes u\big)\bar{\triangleleft} v = \big(\delta_{s\triangleleft\gamma(\,\underline{a}\,)} \otimes \gamma(\,\underline{a}\,)^{-1}u\big)\bar{\triangleleft} v$$

$$= \delta_{s\triangleleft\gamma(\underline{a})\,(\underline{a}\triangleright v)} \otimes\ (\underline{a}\triangleright v)^{-1}\gamma(\underline{a})^{-1}uv\,,$$

which shows that θ preserves the actions. $\square$

In [4], the morphism $F_{V,W} : V\underline{\otimes}W \to V \otimes W$ for $V, W \in \mathcal{C}$ and $\underline{\otimes}$ is the tensor structure given by $\underline{M}$, was defined by $F(\xi\underline{\otimes}\eta) = \xi\bar{\triangleleft}\gamma(\langle\eta\rangle) \otimes \eta$. This morphism will be used in the next proposition.

Proposition 7.1.2 *For the algebras $\underline{A}$ and A, the morphism $\theta : \underline{A} \to A$ is an algebra map. Note that we have to be careful about just what this means, as there are two different tensor products. We mean that the following diagram commutes:*

$$\begin{array}{ccc}
\underline{A}\underline{\otimes}\underline{A} & \xrightarrow{\ \ \underline{\mu}\ \ } & \underline{A} \\
\downarrow{\scriptstyle \theta\underline{\otimes}\theta} & & \\
A\underline{\otimes}A & & \downarrow{\scriptstyle \theta} \\
\downarrow{\scriptstyle F_{AA}} & & \\
A \otimes A & \xrightarrow{\ \ \mu\ \ } & A
\end{array}$$

Proof. For the elements $(\delta_s \otimes u)$ and $(\delta_t \otimes v)$ in the algebra $\underline{A}$ we have

$$\underline{\mu}\big((\delta_s \otimes u) \otimes \delta_t \otimes v)\big) = \delta_{t,s\underline{\triangleleft}u}\, \delta_{s\underline{\triangleleft}\underline{\tau}(\underline{a},\underline{b})} \otimes \underline{\tau}(\underline{a},\underline{b})^{-1}uv\,,$$

where $\underline{a} = \langle\delta_s \otimes u\rangle$ and $\underline{b} = \langle\delta_t \otimes v\rangle$. So

$$\theta\big(\underline{\mu}\big((\delta_s \otimes u) \otimes \delta_t \otimes v)\big)\big) = \delta_{t,s\underline{\triangleleft}u}\, \delta_{s\underline{\triangleleft}\underline{\tau}(\underline{a},\underline{b})\,\gamma(\underline{a}\cdot\underline{b})} \otimes \gamma(\underline{a}\cdot\underline{b})^{-1}\underline{\tau}(\underline{a},\underline{b})^{-1}uv\,.$$

But we know that

$$\underline{\tau}(\underline{a},\underline{b}) = \gamma(\underline{a})(\underline{a}\triangleright\gamma(\underline{b}))\,\tau(\underline{a}\triangleleft\gamma(\underline{b}),\underline{b})\,\gamma\big((\underline{a}\triangleleft\gamma(\underline{b}))\cdot\underline{b}\big)^{-1} \quad \text{and} \quad \underline{a}\,\underline{:}\,\underline{b} = (\underline{a}\triangleleft\gamma(\underline{b}))\cdot\underline{b}\,,$$

So

$$\underline{\tau}(\underline{a},\underline{b})\gamma(\underline{a}\,\underline{:}\,\underline{b}) = \gamma(\underline{a})(\underline{a}\triangleright\gamma(\underline{b}))\,\tau(\underline{a}\triangleleft\gamma(\underline{b}),\underline{b})\,.$$

Thus, one direction of the diagram is given by the following equation:

$$\theta\big(\underline{\mu}\big((\delta_s\otimes u)\otimes\delta_t\otimes v)\big)\big) = \delta_{t,s\triangleleft u}\,\delta_{s\triangleleft\gamma(\underline{a})(\underline{a}\triangleright\gamma(\underline{b}))\,\tau(\underline{a}\triangleleft\gamma(\underline{b}),\underline{b})}\otimes\tau(\underline{a}\triangleleft\gamma(\underline{b}),\underline{b})^{-1}(\underline{a}\triangleright\gamma(\underline{b}))^{-1}\gamma(\underline{a})^{-1}uv\,.$$

Now to calculate the other direction of the diagram we do the following calculation:

$$\theta(\delta_s\otimes u)\underline{\otimes}\theta(\delta_t\otimes v) = (\delta_{s\triangleleft\gamma(\underline{a})}\otimes\gamma(\underline{a})^{-1}u)\underline{\otimes}(\delta_{t\triangleleft\gamma(\underline{b})}\otimes\gamma(\underline{b})^{-1}v)\,.$$

Applying the map F_{AA} to the above equation gives

$$\begin{aligned} F_{AA}\Big(\theta(\delta_s\otimes u)\underline{\otimes}\theta(\delta_t\otimes v)\Big) &= (\delta_{s\triangleleft\gamma(\underline{a})}\otimes\gamma(\underline{a})^{-1}u)\bar{\triangleleft}\gamma(\underline{b})\otimes(\delta_{t\triangleleft\gamma(\underline{b})}\otimes\gamma(\underline{b})^{-1}v) \\ &= \big(\delta_{s\triangleleft\gamma(\underline{a})(\underline{a}\triangleright\gamma(\underline{b}))}\otimes(\underline{a}\triangleright\gamma(\underline{b}))^{-1}\gamma(\underline{a})^{-1}u\,\gamma(\underline{b})\big)\otimes\big(\delta_{t\triangleleft\gamma(\underline{b})}\otimes\gamma(\underline{b})^{-1}v\big)\,. \end{aligned}$$

Therefore, the other direction of the diagram is given by the following equation:

$$\begin{aligned} \mu\Big(F_{AA}\Big(\theta(\delta_s\otimes u)\underline{\otimes}\theta(\delta_t\otimes v)\Big)\Big) &= \delta_{t\triangleleft\gamma(\underline{b}),s\triangleleft u\gamma(\underline{b})}\,\delta_{s\triangleleft\gamma(\underline{a})(\underline{a}\triangleright\gamma(\underline{b}))\,\tau(\underline{a}\triangleleft\gamma(\underline{b}),\underline{b})} \\ &\qquad\otimes\tau(\underline{a}\triangleleft\gamma(\underline{b}),\underline{b})^{-1}(\underline{a}\triangleright\gamma(\underline{b}))^{-1}\gamma(\underline{a})^{-1}u\gamma(\underline{b})\gamma(\underline{b})^{-1}v \\ &= \delta_{t,s\triangleleft u}\,\delta_{s\triangleleft\gamma(\underline{a})(\underline{a}\triangleright\gamma(\underline{b}))\,\tau(\underline{a}\triangleleft\gamma(\underline{b}),\underline{b})} \\ &\qquad\otimes\tau(\underline{a}\triangleleft\gamma(\underline{b}),\underline{b})^{-1}(\underline{a}\triangleright\gamma(\underline{b}))^{-1}\gamma(\underline{a})^{-1}uv\,, \end{aligned}$$

which is the same as the first direction which by then completes the proof. □

7.2 The Hopf fibration and local choice of coset representatives

In the case where X is a topological group, we would like to include continuity in the algebraic structure we have described. Unfortunately there can be problems. Take $X = SU_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A \,:\, a,b,c,d \in \mathbb{C}, \quad ad - bc = 1 \quad \text{and} \quad A\bar{A}^T = \mathrm{id} \right\}.$

SU_2 acts on $\mathbb{P}\mathbb{C}^2$ by

$$\left[\begin{pmatrix} x & y \end{pmatrix}\right] \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left[\begin{pmatrix} xa + yc & xb + yd \end{pmatrix}\right].$$

We want to write $SU_2 = GM$. Let $G = \mathrm{stab}([\begin{pmatrix} 1 & 0 \end{pmatrix}])$ which can be calculated as the following:

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} x & 0 \end{pmatrix},$$

so $b = 0$ and then $ad = 1$ which implies that $d = \frac{1}{a}$. We also need

$$\begin{pmatrix} a & 0 \\ c & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ 0 & \frac{1}{\bar{a}} \end{pmatrix} = \begin{pmatrix} |a|^2 & a\bar{c} \\ c\bar{a} & |c|^2 + \frac{1}{|a|^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which implies $c = 0$ and $|a| = 1$. So

$$G = \mathrm{stab}([\begin{pmatrix} 1 & 0 \end{pmatrix}]) = \left\{ \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} : |\mathrm{a}| = 1 \right\}.$$

Note that cosets of G in SU_2 are in $1-1$ correspondence with $\mathbb{P}\mathbb{C}^2 \cong \mathbb{C} \cup \{\infty\}$ by $[(1, z)] \leftrightarrow z$. There is no continuous choice of coset representative for G, as the map

$m \in SU_2 \mapsto [(1\ \ 0)]m$ from SU_2 to $\mathbb{C} \cup \{\infty\}$ is the Hopf fibration. It is however possible to continuously choose coset representatives over two open sets $\mathbb{C}$ and $\mathbb{C}^* \cup \{\infty\}$ of $\mathbb{C} \cup \{\infty\}$.

Case(1): To find coset representatives, M_1, over $\mathbb{C}$ we need to do the following calculation. We take $z = \frac{y}{x}$ where $x, y, z \in \mathbb{C}$, and find a representatives for $z \in \mathbb{C}$:

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix},$$

so $z = \frac{b}{a}$ or $b = za$. For $\begin{pmatrix} a & za \\ c & d \end{pmatrix}$ to be in SU_2 we need the following:

$$\begin{pmatrix} a & za \\ c & d \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{z}\bar{a} & \bar{d} \end{pmatrix} = \begin{pmatrix} |a|^2 + |z|^2|a|^2 & a\bar{c} + za\bar{d} \\ c\bar{a} + d\bar{a}\bar{z} & |c|^2 + |d|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{7.1}$$

which implies $|a|^2(1 + |z|^2) = 1$ or $|a|^2 = \frac{1}{1+|z|^2}$ and $\bar{c} = -z\bar{d}$. Also we get $|c|^2 + |d|^2 = |z|^2|d|^2 + |d|^2 = 1$ which implies that $|d|^2 = |a|^2 = \frac{1}{1+|z|^2}$. In addition we require

$$ad + |z|^2 ad = (1 + |z|^2)ad = 1,$$

which implies $ad = a\bar{a}$, so $d = \bar{a}$. Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & za \\ -\bar{z}\,\bar{a} & \bar{a} \end{pmatrix}. \tag{7.2}$$

Putting $a = \frac{1}{\sqrt{1+|z|^2}}$, the first choice of coset representatives can be given as the following:

$$M_1 = \left\{ c(z) = \frac{1}{\sqrt{1 + |z|^2}} \begin{pmatrix} 1 & z \\ -\bar{z} & 1 \end{pmatrix}, \qquad z \in \mathbb{C} \right\}.$$

Note that if we put $z = Re^{i\theta}$ in M_1, where $R \in [0, \infty)$, then we get

$$c(Re^{i\theta}) = \frac{1}{\sqrt{1+R^2}} \begin{pmatrix} 1 & Re^{i\theta} \\ -Re^{-i\theta} & 1 \end{pmatrix}.$$

If R is large then

$$c(Re^{i\theta}) \simeq \begin{pmatrix} 0 & e^{i\theta} \\ -e^{-i\theta} & 0 \end{pmatrix},$$

which is not well-defined at $\infty \in \mathbb{C}_\infty$.

Case(2): To find coset representatives, M_2, over $\mathbb{C}^* \cup \{\infty\}$ we put $a = \frac{\bar{z}}{|z|\sqrt{1+|z|^2}}$ and substitute in (7.2). Thus the second choice of coset representatives can be given as the following:

$$M_2 = \left\{ \underline{c}(z) = \frac{1}{|z|\sqrt{1+|z|^2}} \begin{pmatrix} \bar{z} & |z|^2 \\ -|z|^2 & z \end{pmatrix}, \qquad z \in \Big(\mathbb{C} \cup \{\infty\}\Big) \setminus \{0\} \right\}.$$

or equivalently

$$M_2 = \left\{ \underline{c}(z) = \frac{|z|}{\sqrt{1+|z|^2}} \begin{pmatrix} \frac{1}{z} & 1 \\ -1 & \frac{1}{\bar{z}} \end{pmatrix}, \qquad z \in \Big(\mathbb{C} \cup \{\infty\}\Big) \setminus \{0\} \right\}.$$

$$= \left\{ \underline{c}(z) = \frac{1}{\sqrt{1+\frac{1}{|z|^2}}} \begin{pmatrix} \frac{1}{z} & 1 \\ -1 & \frac{1}{\bar{z}} \end{pmatrix}, \qquad z \in \Big(\mathbb{C} \cup \{\infty\}\Big) \setminus \{0\} \right\}.$$

Note that we identify G with the group of the unit circle, via the identification $a \in S^1 = \{a \in \mathbb{C} : |a| = 1\}$ corresponding to the matrix $\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \in G$.

For our case, γ can be defined as the following

Proposition 7.2.1 *The function $\gamma : \mathbb{C} \to S^1$ is defined by*

$$\gamma(z) = |z| \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & \frac{1}{\bar{z}} \end{pmatrix} \qquad \text{for} \quad z \in \mathbb{C}.$$

Proof. From [4] we know that $\gamma(z)\, c(z) = \underline{c}(z)$ which implies that

$$\gamma(z) = \underline{c}(z)\, c(z)^{-1}\,. \tag{7.3}$$

If $c(z) = \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 & z \\ -\bar{z} & 1 \end{pmatrix}$ and $\underline{c}(z) = \frac{|z|}{\sqrt{1+|z|^2}} \begin{pmatrix} \frac{1}{z} & 1 \\ -1 & \frac{1}{\bar{z}} \end{pmatrix}$, then

$$c(z)^{-1} = \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 & -z \\ \bar{z} & 1 \end{pmatrix},$$

where $\det(c(z))=1$. So equation (7.3) can be rewritten as the following

$$\begin{aligned}
\gamma(z) &= \frac{|z|}{\sqrt{1+|z|^2}} \begin{pmatrix} \frac{1}{z} & 1 \\ -1 & \frac{1}{\bar{z}} \end{pmatrix} \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 & -z \\ \bar{z} & 1 \end{pmatrix} \\
&= \frac{|z|}{1+|z|^2} \begin{pmatrix} \frac{1}{z}+\bar{z} & -\frac{z}{z}+1 \\ -1+\frac{\bar{z}}{\bar{z}} & z+\frac{1}{\bar{z}} \end{pmatrix} \\
&= \frac{|z|}{1+|z|^2} \begin{pmatrix} \frac{1+|z|^2}{z} & -1+1 \\ -1+1 & \frac{|z|^2+1}{\bar{z}} \end{pmatrix} = |z| \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & \frac{1}{\bar{z}} \end{pmatrix}. \quad \square
\end{aligned}$$

Proposition 7.2.2 *For case(1) where $t, s \in \mathbb{C}$, we have*

$$t \triangleright a = a \quad , \quad t \triangleleft a = \frac{t}{a^2} \quad , \quad t \cdot s = \frac{s+t}{1-t\bar{s}} \quad \text{and} \quad \tau(t,s) = \frac{1-t\bar{s}}{|1-t\bar{s}|}\,.$$

Proof. For case(1) we have $c(t) = \frac{1}{\sqrt{1+|t|^2}} \begin{pmatrix} 1 & t \\ -\bar{t} & 1 \end{pmatrix}$ and $u(a) = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$ where $t, a \in \mathbb{C}$ and $|a| = 1$, then we need to find a' and t', with $|a'| = 1$, that satisfy $c(t)\, u(a) = u(a')\, c(t')$. We start with the left hand side as the following:

$$c(t)\, u(a) = \frac{1}{\sqrt{1+|t|^2}} \begin{pmatrix} 1 & t \\ -\bar{t} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} = \frac{1}{\sqrt{1+|t|^2}} \begin{pmatrix} a & \frac{t}{a} \\ -\bar{t}a & \frac{1}{a} \end{pmatrix}.$$

On the other hand

$$u(a^{'})\,c(t^{'}) = \frac{1}{\sqrt{1+|t^{'}|^2}} \begin{pmatrix} a^{'} & 0 \\ 0 & \frac{1}{a^{'}} \end{pmatrix} \begin{pmatrix} 1 & t^{'} \\ -\bar{t^{'}} & 1 \end{pmatrix} = \frac{1}{\sqrt{1+|t^{'}|^2}} \begin{pmatrix} a^{'} & a^{'}t^{'} \\ \frac{-\bar{t^{'}}}{a^{'}} & \frac{1}{a^{'}} \end{pmatrix}.$$

So $a^{'} = a$ and $\frac{t}{a} = a^{'}t^{'}$, which implies that $t^{'} = \frac{t}{a^2}$. These can be rewritten as

$$t \triangleright a = a^{'} = a \quad \text{and} \quad \mathrm{t} \triangleleft \mathrm{a} = \mathrm{t}^{'} = \frac{\mathrm{t}}{\mathrm{a}^2}.$$

Now if $c(t^{''}) = \frac{1}{\sqrt{1+|t^{''}|^2}} \begin{pmatrix} 1 & t^{''} \\ -\bar{t^{''}} & 1 \end{pmatrix}$, $c(t^{'''}) = \frac{1}{\sqrt{1+|t^{'''}|^2}} \begin{pmatrix} 1 & t^{'''} \\ -\bar{t^{'''}} & 1 \end{pmatrix}$ and $u(b) = \begin{pmatrix} b & 0 \\ 0 & \frac{1}{b} \end{pmatrix}$ where $t^{''}, t^{'''}, b \in \mathbb{C}$ and $|\mathrm{b}| = 1$, then we need to find $t^{'''}$ and b that satisfying $c(t)\,c(t^{''}) = u(b)\,c(t^{'''})$. We start with the left hand side as the following:

$$c(t)\,c(t^{''}) = \frac{1}{\sqrt{1+|t|^2}} \begin{pmatrix} 1 & t \\ -\bar{t} & 1 \end{pmatrix} \frac{1}{\sqrt{1+|t^{''}|^2}} \begin{pmatrix} 1 & t^{''} \\ -\bar{t^{''}} & 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{1+|t|^2}\sqrt{1+|t^{''}|^2}} \begin{pmatrix} 1 - t\bar{t^{''}} & t^{''} + t \\ -\bar{t} - \bar{t^{''}} & -\bar{t}t^{''} + 1 \end{pmatrix}.$$

On the other hand

$$u(b)\,c(t^{'''}) = \frac{1}{\sqrt{1+|t^{'''}|^2}} \begin{pmatrix} b & 0 \\ 0 & \frac{1}{b} \end{pmatrix} \begin{pmatrix} 1 & t^{'''} \\ -\bar{t^{'''}} & 1 \end{pmatrix} = \frac{1}{\sqrt{1+|t^{'''}|^2}} \begin{pmatrix} b & bt^{'''} \\ \frac{-\bar{t^{'''}}}{b} & \frac{1}{b} \end{pmatrix}.$$

So $t^{'''} = \frac{t^{''}+t}{1-t\bar{t^{''}}}$. If we put $s = t^{''}$, then $t^{'''}$ can be rewritten as

$$t \cdot s = \frac{s+t}{1 - t\bar{s}},$$

which is not in $\mathbb{C}$ if $\bar{s} = t^{-1}$. Also we get

$$b = \frac{\sqrt{1+|t^{'''}|^2}}{\sqrt{1+|t|^2}\sqrt{1+|t^{''}|^2}}\,(1 - t\bar{t^{''}}).$$

To calculate b put $t^{''} = s$, $a = s + t$ and $c = 1 - t\bar{s}$, then $1 + |t^{'''}|^2 = 1 + \frac{|a|^2}{|c|^2} = \frac{|a|^2+|c|^2}{|c|^2}$.

But $|a|^2 = |s|^2 + |t|^2 + s\bar{t} + \bar{s}t$ and $|c|^2 = 1 + |s|^2|t|^2 - t\bar{s} - \bar{t}s$. So $|a|^2 + |c|^2 =$ $1 + |s|^2 + |t|^2 + |s|^2|t|^2 = (1 + |s|^2)(1 + |t|^2)$, which implies that

$$b = \sqrt{\frac{|a|^2 + |c|^2}{|c|^2(1 + |t|^2)(1 + |s|^2)}}\,(1 - t\bar{s}) = \frac{1 - t\bar{s}}{|1 - t\bar{s}|}.$$

We can also rewrite b as

$$\tau(t, s) = \frac{1 - t\bar{s}}{|1 - t\bar{s}|}. \quad \square$$

Proposition 7.2.3 *For case(2) where $s, t \in \mathbb{C}^* \cup \{\infty\}$, we have*

$$t \underline{\triangleright} a = \frac{1}{a} \quad , \quad t \underline{\triangleleft} a = t \triangleleft a = \frac{t}{a^2} \quad , \quad t \underline{\cdot} s = \frac{|s|^2 + st}{\bar{s} - t|s|^2} \quad \text{and} \quad \underline{\tau}(t, s) = \frac{|t||s|}{|\bar{s} + t|}\,\frac{\bar{s} + t}{t\bar{s}}.$$

Proof. For case(2) we have $\underline{c}(t) = \frac{|t|}{\sqrt{1+|t|^2}} \begin{pmatrix} \frac{1}{t} & 1 \\ -1 & \frac{1}{\bar{t}} \end{pmatrix}$ and $u(a) = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$ where $t, a \in \mathbb{C}$ and $|a| = 1$, then we need to find a' and t', with $|a'| = 1$, that satisfying $\underline{c}(t)\, u(a) = u(a')\, \underline{c}(t')$. We start with the left hand side as the following:

$$\underline{c}(t)\, u(a) = \frac{|t|}{\sqrt{1 + |t|^2}} \begin{pmatrix} \frac{1}{t} & 1 \\ -1 & \frac{1}{\bar{t}} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} = \frac{|t|}{\sqrt{1 + |t|^2}} \begin{pmatrix} \frac{a}{t} & \frac{1}{a} \\ -a & \frac{1}{a\bar{t}} \end{pmatrix}.$$

On the other hand

$$u(a')\, \underline{c}(t') = \frac{|t'|}{\sqrt{1 + |t'|^2}} \begin{pmatrix} a' & 0 \\ 0 & \frac{1}{a'} \end{pmatrix} \begin{pmatrix} \frac{1}{t'} & 1 \\ -1 & \frac{1}{\bar{t}'} \end{pmatrix} = \frac{|t'|}{\sqrt{1 + |t'|^2}} \begin{pmatrix} \frac{a'}{t'} & a' \\ \frac{-1}{a'} & \frac{1}{a'\bar{t}'} \end{pmatrix}.$$

So $a' = \frac{1}{a}$ and $\frac{a}{t} = \frac{a'}{t'}$, which implies that $t' = \frac{a' t}{a} = \frac{t}{a^2}$. These can be rewritten as

$$t \underline{\triangleright} a = a' = \frac{1}{a} \quad \text{and} \quad t \underline{\triangleleft} a = t' = \frac{t}{a^2}.$$

Now if $\underline{c}(t'') = \frac{|t''|}{\sqrt{1+|t''|^2}} \begin{pmatrix} \frac{1}{t''} & 1 \\ -1 & \frac{1}{\bar{t''}} \end{pmatrix}$, $\underline{c}(t''') = \frac{|t'''|}{\sqrt{1+|t'''|^2}} \begin{pmatrix} \frac{1}{t'''} & 1 \\ -1 & \frac{1}{\bar{t'''}} \end{pmatrix}$ and $u(b) = \begin{pmatrix} b & 0 \\ 0 & \frac{1}{b} \end{pmatrix}$ where $t'', t''', b \in \mathbb{C}$ and |b| = 1, then we need to find t''' and b that satisfying $\underline{c}(t)\,\underline{c}(t'') = u(b)\,\underline{c}(t''')$. We start with the left hand side as the following:

$$\underline{c}(t)\,\underline{c}(t'') = \frac{|t|}{\sqrt{1+|t|^2}} \begin{pmatrix} \frac{1}{t} & 1 \\ -1 & \frac{1}{\bar{t}} \end{pmatrix} \frac{|t''|}{\sqrt{1+|t''|^2}} \begin{pmatrix} \frac{1}{t''} & 1 \\ -1 & \frac{1}{\bar{t''}} \end{pmatrix}$$

$$= \frac{|t||t''|}{\sqrt{1+|t|^2}\sqrt{1+|t''|^2}} \begin{pmatrix} \frac{1}{tt''} - 1 & \frac{1}{t} + \frac{1}{\bar{t''}} \\ -\frac{1}{t''} - \frac{1}{\bar{t}} & -1 + \frac{1}{\bar{t}\bar{t''}} \end{pmatrix}.$$

On the other hand

$$u(b)\,\underline{c}(t''') = \frac{|t'''|}{\sqrt{1+|t'''|^2}} \begin{pmatrix} b & 0 \\ 0 & \frac{1}{b} \end{pmatrix} \begin{pmatrix} \frac{1}{t'''} & 1 \\ -1 & \frac{1}{\bar{t'''}} \end{pmatrix} = \frac{|t'''|}{\sqrt{1+|t'''|^2}} \begin{pmatrix} \frac{b}{t'''} & b \\ \frac{-1}{b} & \frac{1}{b\bar{t'''}} \end{pmatrix}.$$

So

$$t''' = \frac{\frac{1}{t} + \frac{1}{\bar{t''}}}{\frac{1}{tt''} - 1} = \frac{\frac{\bar{t''}+t}{t\bar{t''}}}{\frac{1-tt''}{tt''}} = \frac{t''(\bar{t''}+t)}{\bar{t''}(1-tt'')}.$$

If we put $s = t''$, then t''' can be rewritten as

$$t \underline{\cdot} s = \frac{s(\bar{s}+t)}{\bar{s}(1-ts)} = \frac{|s|^2 + st}{\bar{s} - t|s|^2}.$$

Also we get

$$b = \frac{|t||s|\sqrt{1+|t'''|^2}}{|t'''|\sqrt{1+|t|^2}\sqrt{1+|t''|^2}} \left(\frac{1}{t} + \frac{1}{\bar{t''}}\right).$$

To calculate b put $t'' = s$, $a = \bar{s} + t$ and $c = 1 - ts$, then $|t'''|^2 = \frac{|s|^2|\bar{s}+t|^2}{|\bar{s}|^2|1-ts|^2} = \frac{|a|^2}{|c|^2}$ and $1 + |t'''|^2 = 1 + \frac{|a|^2}{|c|^2} = \frac{|a|^2+|c|^2}{|c|^2}$. But $|a|^2 = (\bar{s}+t)(s+\bar{t}) = |s|^2 + |t|^2 + st + \bar{s}\bar{t}$ and

$|c|^2 = (1-ts)(1-\bar{t}\bar{s}) = 1 + |t|^2|s|^2 - ts - \bar{t}\bar{s}$. So $|a|^2 + |c|^2 = 1 + |s|^2 + |t|^2 + |s|^2|t|^2 =$ $(1+|s|^2)(1+|t|^2)$, which implies that

$$b = \frac{|t||s|}{|t'''|}\sqrt{\frac{|a|^2+|c|^2}{|c|^2(1+|t|^2)(1+|s|^2)}}\,(\frac{1}{t}+\frac{1}{\bar{s}}) = \frac{|t||s||1-ts|}{|\bar{s}+t||1-ts|}\,\frac{\bar{s}+t}{t\bar{s}} = \frac{|t||s|}{|\bar{s}+t|}\,\frac{\bar{s}+t}{t\bar{s}}\,.$$

We can also rewrite b as

$$\underline{\tau}(t,s) = \frac{|t||s|}{|\bar{s}+t|}\,\frac{\bar{s}+t}{t\bar{s}}\,. \quad \square$$

Note for 7.2.2 $(t, s \in \mathbb{C})$ that $t \cdot s$ and $\tau(t,s)$ are not defined if $t\bar{s} = 1$. Also for 7.2.3 $(s, t \in \mathbb{C}^* \cup \{\infty\})$ we see that $t \,\underline{\cdot}\, s$ and $\underline{\tau}(t,s)$ are not defined if $\frac{1}{t} + \frac{1}{\bar{s}} = 0$. If $s = \pm i$ and $t = \frac{1}{\bar{s}}$, then none of the above formulae work. Probably this means that we need to introduce more open sets.

We can construct an "algebra" A with generators $\delta_{c(s)} \otimes u$ for $s \in \mathbb{C}$ and $u \in S^1$ in the same manner as before.

Proposition 7.2.4 *For an element $\delta_{c(s)} \otimes u \in A$, the M-grade can be given by the following formula:*

$$\langle \delta_{c(s)} \otimes u \rangle = \frac{s}{1-|s|^2}\,(\frac{1}{u^2-1})\,.$$

Proof. Put $c(s) = s$ for short and put $\langle \delta_{c(s)} \otimes u \rangle = \langle \delta_s \otimes u \rangle = a$, then from 7.2.2 we have

$$\frac{s+a}{1-s\bar{a}} = s \cdot a = s \triangleleft u = \frac{s}{u^2}\,.$$

If we put $s \cdot a = t = \frac{s}{u^2}$, then $s + a = t(1 - s\bar{a})$ or, equivalently,

$$a + ts\bar{a} = t - s\,.$$

If we put $ts = z$ and $t - s = w$, then the equation $a + ts\bar{a} = t - s$ can be rewritten as

$$a + z\bar{a} = w\,. \tag{7.4}$$

As a, z and w are complex numbers they can be written as $z = p + iq$, $w = n + im$ and $a = x + iy$ where p, q, n, m, x, and y are real numbers. Substituting these in equation (7.4) gives the following:

$$x + iy + (p + iq)(x - iy) = x + iy + px + qy + iqx - iyp = n + im\,,$$

so we get the following equations for the real part and the imaginary part : $x + px + qy = n$ and $y + qx - yp = m$ or equivalently, $(1+p)x + qy = n$ and $qx + y(1-p) = m$ which can be solved as a system of equations in x and y as the following:

$$\begin{pmatrix} 1+p & q \\ q & 1-p \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} n \\ m \end{pmatrix}.$$

Now we find

$$det(\begin{pmatrix} 1+p & q \\ q & 1-p \end{pmatrix}) = 1 - p^2 - q^2 = 1 - |z|^2\,.$$

So

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1-|z|^2} \begin{pmatrix} 1-p & -q \\ -q & 1+p \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} = \frac{1}{1-|z|^2} \begin{pmatrix} (1-p)n - qm \\ -qn + (1+p)m \end{pmatrix},$$

which implies

$$(x+iy)(1-|z|^2) = n - pn - qm + im + ipm - iqn = w - (p+iq)n + qi^2m + ipm$$

$$= w - (p+iq)n + im(p+iq) = w - (p+iq)(n-im) = w - z\bar{w}\,,$$

so substituting the values of w, t and z gives

$$a = \frac{w - z\bar{w}}{1-|z|^2} = \frac{t - s - ts(\bar{t} - \bar{s})}{1 - |t|^2|s|^2} = \frac{t - s - s|t|^2 + t|s|^2}{1 - |t|^2|s|^2} = \frac{t(1+|s|^2) - s(1+|t|^2)}{1 - |t|^2|s|^2}$$

$$= \frac{(\frac{s}{u^2})(1+|s|^2) - s(1+|s|^2)}{1-|s|^4} = \frac{(\frac{s}{u^2} - s)(1+|s|^2)}{(1+|s|^2)(1-|s|^2)} = \frac{s(\frac{1}{u^2} - 1)}{(1-|s|^2)} = \frac{s}{(1-|s|^2)}(\frac{1}{u^2} - 1)\,.$$

Note that $|u|^2$ and $|u|^4$ did not appear since $|u| = 1$. □

We do not understand what the following proposition means

Proposition 7.2.5 *There is no* $\frac{1}{\underline{a}} \in \mathbb{C}$ *so that* $\underline{a} = \langle \delta_{\underline{c}(s)} \otimes u \rangle$ *unless* $\frac{u}{s}$ *is pure imaginary.*

Proof. Put $\underline{c}(s) = s$ for short, then from proposition 7.2.3 we have

$$\frac{\frac{1}{s} + \frac{1}{\bar{\underline{a}}}}{\frac{1}{s\underline{a}} - 1} = s \,\underline{\cdot}\, \underline{a} = s \underline{\triangleleft} u = s \triangleleft u = \frac{s}{u^2}\,.$$

If we put $p = \frac{1}{s}$, $q = \frac{1}{\underline{a}}$, $s \,\underline{\cdot}\, \underline{a} = t = \frac{s}{u^2}$, then the above equation can be rewritten as

$$\frac{p + \bar{q}}{pq - 1} = t\,,$$

which implies that $p + \bar{q} = tpq - t$ or, equivalently,

$$\bar{q} - tpq = -p - t\,.$$

If we put $tp = z$ and $-p - t = w$, then the equation $\bar{q} - tpq = -p - t$ can be rewritten as

$$\bar{q} - zq = w\,. \tag{7.5}$$

As q, z and w are complex numbers they can be written as $z = c + id$, $w = n + im$ and $q = x + iy$ where c, d, n, m, x, and y are real numbers. Substituting these in equation (7.5) gives the following:

$$x - iy \;-\; (c + id)(x + iy) = x - iy - cx + dy - idx - icy = n + im\,,$$

so we get the following equations for the real part and the imaginary part : $x - cx + dy = n$ and $-y - dx - cy = m$ or equivalently, $(1 - c)x + dy = n$ and $-dx + y(-1 - c) = m$ which can be solved as a system of equations in x and y as the following:

$$\begin{pmatrix} 1 - c & d \\ -d & -1 - c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} n \\ m \end{pmatrix}.$$

Now we find

$$det(\begin{pmatrix} 1-c & d \\ -d & -1-c \end{pmatrix}) = (1-c)(-1-c)+d^2 = -1-c+c+c^2+d^2 = -1+|z|^2 .$$

So

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-1+|z|^2} \begin{pmatrix} -1-c & -d \\ d & 1-c \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} = \frac{1}{-1+|z|^2} \begin{pmatrix} (-1-c)n-dm \\ dn+(1-c)m \end{pmatrix},$$

which implies

$$(x+iy)(-1+|z|^2) = -n-cn-dm+im-icm+idn .$$

If we multiply both sides of the above equation by -1 we get

$$(x+iy)(1-|z|^2) = n+cn+dm-im+icm-idn = \bar{w}+(n+im)c-i^2dm-idn$$

$$= \bar{w}+(n+im)c-id(n+im) = \bar{w}+(n+im)(c-id) = \bar{w}+w\bar{z} ,$$

so we get $q = \frac{\bar{w}+w\bar{z}}{1-|z|^2}$ which implies that

$$\underline{a} = \frac{1-|z|^2}{\bar{w}+w\bar{z}} .$$

Substituting the value of z in the above equation gives $\underline{a}=0$ as $z = tp = \frac{1}{u^2}$. There is no chance of the bottom line having a factor cancelling with the top line as

$$\bar{w}+w\bar{z} = -\bar{p}-\bar{t}+\bar{t}\bar{p}(-p-t) = -\bar{p}-\bar{t}-\bar{t}|p|^2-\bar{p}|t|^2 = -\bar{p}(1+|t|^2)-\bar{t}(1+|p|^2)$$

$$= -\bar{p}(1+|s|^2)-\bar{t}(1+\frac{1}{|s|^2}) = -(1+|s|^2)(\bar{p}+\frac{\bar{t}}{|s|^2}) = -(1+|s|^2)(\frac{1}{\bar{s}}+\frac{\bar{s}u^2}{s\bar{s}})$$

$$= -(1+|s|^2)(\frac{1}{\bar{s}}+\frac{u^2}{s}) = -(1+|s|^2)u(\frac{1}{\bar{s}u}+\frac{u}{s}) . \quad \square$$

Proposition 7.2.6 *The product* μ *on the algebra* A *is given by*

$$(\delta_{c(s)} \otimes u)(\delta_{c(t)} \otimes v) = \delta_{t,\frac{s}{u^2}} \delta_{\frac{s}{\tau(a,b)^2}} \otimes \tau(a,b)^{-1}uv ,$$

where $a = \langle \delta_{c(s)} \otimes u \rangle = \frac{s}{(1-|s|^2)}\left(\frac{1}{u^2}-1\right)$, $b = \langle \delta_{c(t)} \otimes v \rangle = \frac{t}{(1-|t|^2)}\left(\frac{1}{v^2}-1\right)$ *and* $\tau(a,b) = \frac{1-a\bar{b}}{|1-a\bar{b}|}$.

Proof. If we put $c(s) = s$ and $c(t) = t$ for short, then we have

$$(\delta_s \otimes u)(\delta_t \otimes v) = \delta_{t,s\triangleleft u}\delta_{s\triangleleft\tau(a,b)} \otimes \tau(a,b)^{-1}uv\,.$$

But we know from the previous propositions that $s\triangleleft u = \frac{s}{u^2}$, $s\triangleleft\tau(a,b) = \frac{s}{\tau(a,b)^2}$ and $\tau(a,b) = \frac{1-a\bar{b}}{|1-a\bar{b}|}$ where $a = \langle\delta_{c(s)} \otimes u\rangle = \frac{s}{(1-|s|^2)}\left(\frac{1}{u^2} - 1\right)$ and $b = \langle\delta_{c(t)} \otimes v\rangle = \frac{t}{(1-|t|^2)}\left(\frac{1}{v^2} - 1\right)$. □

7.3 Further research

1– What about factorizing a group into two sets ?

For the group $Q_6 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ defined by the following table [28] :

1	2	3	4	5	6	7	8	9	10	11	12
2	3	4	5	6	1	8	9	10	11	12	7
3	4	5	6	1	2	9	10	11	12	7	8
4	5	6	1	2	3	10	11	12	7	8	9
5	6	1	2	3	4	11	12	7	8	9	10
6	1	2	3	4	5	12	7	8	9	10	11
7	12	11	10	9	8	4	3	2	1	6	5
8	7	12	11	10	9	5	4	3	2	1	6
9	8	7	12	11	10	6	5	4	3	2	1
10	9	8	7	12	11	1	6	5	4	3	2
11	10	9	8	7	12	2	1	6	5	4	3
12	11	10	9	8	7	3	2	1	6	5	4

we can have the following factorizations into two sets, none of which are subgroups and both contain the identity:

a) If $G_1 = \{1, 3, 5, 7, 9, 11\}$ and $M_1 = \{1, 2\}$ then $Q_6 = G_1\, M_1$.

b) If $G_2 = \{1, 3, 5, 8, 10, 12\}$ and $M_2 = \{1, 2\}$ then $Q_6 = G_2\, M_2$.

c) If $G_3 = \{1, 3, 5, 7, 9, 11\}$ and $M_3 = \{1, 6\}$ then $Q_6 = G_3\, M_3$.

d) If $G_4 = \{1, 3, 5, 8, 10, 12\}$ and $M_4 = \{1, 6\}$ then $Q_6 = G_4\, M_4$.

What sort of algebraic structures can be made from this data?

2–Consider the differential structures on the braided Hopf algebras given by the coset representatives. It is likely that this could be done using similar methods to [6].

3–What is the meaning of having different tensor products over different open subsets of $\mathbb{C}_\infty$? This is the case in section 7.2, but many other examples could be constructed.

4– Complete the work on type A and type B morphisms in the category $\bar{\mathcal{C}}$. This should include a study of inner products : $V \otimes V \to k$. Possibly this would allow the construction of some sort of "antipode" (may be one sided) for the algebra A in $\bar{\mathcal{C}}$. Also to be investigated is whether 3.4.4 or similar definition gives an 'adjoint' operation on A.

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Appendix (The Mathematica files)

In this appendix we include the main part of the Mathematica files that show the modularity of the category $\mathcal{D}$ for the example discussed in chapter five.

The Matrix S

Here we simplify the matrix S and show that it is symmetric and invertible.

In[1]:= **w = Exp[I Pi / 3]**

Out[1]= $e^{\frac{i\pi}{3}}$

s̃ =

1	1	1	1	2	2	1	1	1	1	2	2	2	2	2	2
1	1	1	1	2	2	-1	-1	-1	-1	-2	-2	2	2	2	2
1	1	1	1	2	2	-1	-1	-1	-1	-2	-2	2	2	2	2
1	1	1	1	2	2	1	1	1	1	2	2	2	2	2	2
2	2	2	2	4	4	-2	-2	-2	-2	-4	-4	-2	-2	-2	-2
2	2	2	2	4	4	2	2	2	2	4	4	-2	-2	-2	-2
1	-1	-1	1	-2	2	1	-1	-1	1	-2	2	2	2 w^3	2 w^6	2 w^9
1	-1	-1	1	-2	2	-1	1	1	-1	2	-2	2	2 w^3	2 w^6	2 w^9
1	-1	-1	1	-2	2	-1	1	1	-1	2	-2	2	2 w^3	2 w^6	2 w^9
1	-1	-1	1	-2	2	1	-1	-1	1	-2	2	2	2 w^3	2 w^6	2 w^9
2	-2	-2	2	-4	4	-2	2	2	-2	4	-4	-2	-2 w^3	-2 w^6	-2 w^9
2	-2	-2	2	-4	4	2	-2	-2	2	-4	4	-2	-2 w^3	-2 w^6	-2 w^9
2	2	2	2	-2	-2	2	2	2	2	-2	-2	4	2 (w^4 + w^2)	2 (w^8 + w^4)	2 (w^12 + w^6)
2	2	2	2	-2	-2	2 w^3	2 w^3	2 w^3	2 w^3	-2 w^3	-2 w^3	2 (w^4 + w^2)	2 (w^8 + w^4)	2 (w^12 + w^6)	2 (w^16 + w^8)
2	2	2	2	-2	-2	2 w^6	2 w^6	2 w^6	2 w^6	-2 w^6	-2 w^6	2 (w^8 + w^4)	2 (w^12 + w^6)	2 (w^16 + w^8)	2 (w^20 + w^10)
2	2	2	2	-2	-2	2 w^9	2 w^9	2 w^9	2 w^9	-2 w^9	-2 w^9	2 (w^12 + w^6)	2 (w^16 + w^8)	2 (w^20 + w^10)	2 (w^24 + w^12)
2	2	2	2	-2	-2	2 w^12	2 w^12	2 w^12	2 w^12	-2 w^12	-2 w^12	2 (w^16 + w^8)	2 (w^20 + w^10)	2 (w^24 + w^12)	2 (w^28 + w^14)
2	2	2	2	-2	-2	2 w^15	2 w^15	2 w^15	2 w^15	-2 w^15	-2 w^15	2 (w^20 + w^10)	2 (w^24 + w^12)	2 (w^28 + w^14)	2 (w^32 + w^16)
2	-2	-2	2	2	-2	2	-2	-2	2	2	-2	4	2 (w^5 + w^1)	2 (w^10 + w^2)	2 (w^15 + w^3)
2	-2	-2	2	2	-2	2 w^3	-2 w^3	-2 w^3	2 w^3	2 w^3	-2 w^3	2 (w^4 + w^2)	2 (w^9 + w^3)	2 (w^14 + w^4)	2 (w^19 + w^5)
2	-2	-2	2	2	-2	2 w^6	-2 w^6	-2 w^6	2 w^6	2 w^6	-2 w^6	2 (w^8 + w^4)	2 (w^13 + w^5)	2 (w^18 + w^6)	2 (w^23 + w^7)
2	-2	-2	2	2	-2	2 w^9	-2 w^9	-2 w^9	2 w^9	2 w^9	-2 w^9	2 (w^12 + w^6)	2 (w^17 + w^7)	2 (w^22 + w^8)	2 (w^27 + w^9)
2	-2	-2	2	2	-2	2 w^12	-2 w^12	-2 w^12	2 w^12	2 w^12	-2 w^12	2 (w^16 + w^8)	2 (w^21 + w^9)	2 (w^26 + w^10)	2 (w^31 + w^11)
2	-2	-2	2	2	-2	2 w^15	-2 w^15	-2 w^15	2 w^15	2 w^15	-2 w^15	2 (w^20 + w^10)	2 (w^25 + w^11)	2 (w^30 + w^12)	2 (w^35 + w^13)
3	-3	3	-3	0	0	3	-3	3	-3	0	0	0	0	0	0
3	-3	3	-3	0	0	3	-3	3	-3	0	0	0	0	0	0
3	-3	3	-3	0	0	-3	3	-3	3	0	0	0	0	0	0
3	-3	3	-3	0	0	-3	3	-3	3	0	0	0	0	0	0
3	3	-3	-3	0	0	3	3	-3	-3	0	0	0	0	0	0
3	3	-3	-3	0	0	-3	-3	3	3	0	0	0	0	0	0
3	3	-3	-3	0	0	3	3	-3	-3	0	0	0	0	0	0
3	3	-3	-3	0	0	-3	-3	3	3	0	0	0	0	0	0

2	2	2	2	2	2	2	2	3	3	3	3	3	3	3	3
2	2	-2	-2	-2	-2	-2	-2	-3	-3	-3	-3	3	3	3	3
2	2	-2	-2	-2	-2	-2	-2	3	3	3	3	-3	-3	-3	-3
2	2	2	2	2	2	2	2	-3	-3	-3	-3	-3	-3	-3	-3
-2	-2	2	2	2	2	2	2	0	0	0	0	0	0	0	0
-2	-2	-2	-2	-2	-2	-2	-2	0	0	0	0	0	0	0	0
$2w^{12}$	$2w^{15}$	2	$2w^{3}$	$2w^{6}$	$2w^{9}$	$2w^{12}$	$2w^{15}$	3	3	-3	-3	3	-3	3	-3
$2w^{12}$	$2w^{15}$	-2	$-2w^{3}$	$-2w^{6}$	$-2w^{9}$	$-2w^{12}$	$-2w^{15}$	-3	-3	3	3	3	-3	3	-3
$2w^{12}$	$2w^{15}$	-2	$-2w^{3}$	$-2w^{6}$	$-2w^{9}$	$-2w^{12}$	$-2w^{15}$	3	3	-3	-3	-3	3	-3	3
$2w^{12}$	$2w^{15}$	2	$2w^{3}$	$2w^{6}$	$2w^{9}$	$2w^{12}$	$2w^{15}$	-3	-3	3	3	-3	3	-3	3
$-2w^{12}$	$-2w^{15}$	2	$2w^{3}$	$2w^{6}$	$2w^{9}$	$2w^{12}$	$2w^{15}$	0	0	0	0	0	0	0	0
$-2w^{12}$	$-2w^{15}$	-2	$-2w^{3}$	$-2w^{6}$	$-2w^{9}$	$-2w^{12}$	$-2w^{15}$	0	0	0	0	0	0	0	0
$2(w^{16}+w^{8})$	$2(w^{20}+w^{10})$	4	$2(w^{4}+w^{2})$	$2(w^{8}+w^{4})$	$2(w^{12}+w^{6})$	$2(w^{16}+w^{8})$	$2(w^{20}+w^{10})$	0	0	0	0	0	0	0	0
$2(w^{20}+w^{10})$	$2(w^{24}+w^{12})$	$2(w^{5}+w^{1})$	$2(w^{9}+w^{3})$	$2(w^{13}+w^{5})$	$2(w^{17}+w^{7})$	$2(w^{21}+w^{9})$	$2(w^{25}+w^{11})$	0	0	0	0	0	0	0	0
$2(w^{24}+w^{12})$	$2(w^{28}+w^{14})$	$2(w^{10}+w^{2})$	$2(w^{14}+w^{4})$	$2(w^{18}+w^{6})$	$2(w^{22}+w^{8})$	$2(w^{26}+w^{10})$	$2(w^{30}+w^{12})$	0	0	0	0	0	0	0	0
$2(w^{28}+w^{14})$	$2(w^{32}+w^{16})$	$2(w^{15}+w^{3})$	$2(w^{19}+w^{5})$	$2(w^{23}+w^{7})$	$2(w^{27}+w^{9})$	$2(w^{31}+w^{11})$	$2(w^{35}+w^{13})$	0	0	0	0	0	0	0	0
$2(w^{32}+w^{16})$	$2(w^{36}+w^{18})$	$2(w^{20}+w^{4})$	$2(w^{24}+w^{6})$	$2(w^{28}+w^{8})$	$2(w^{32}+w^{10})$	$2(w^{36}+w^{12})$	$2(w^{40}+w^{14})$	0	0	0	0	0	0	0	0
$2(w^{36}+w^{18})$	$2(w^{40}+w^{20})$	$2(w^{25}+w^{5})$	$2(w^{29}+w^{7})$	$2(w^{33}+w^{9})$	$2(w^{37}+w^{11})$	$2(w^{41}+w^{13})$	$2(w^{45}+w^{15})$	0	0	0	0	0	0	0	0
$2(w^{20}+w^{4})$	$2(w^{25}+w^{5})$	4	$2(w^{5}+w^{1})$	$2(w^{10}+w^{2})$	$2(w^{15}+w^{3})$	$2(w^{20}+w^{4})$	$2(w^{25}+w^{5})$	0	0	0	0	0	0	0	0
$2(w^{24}+w^{6})$	$2(w^{29}+w^{7})$	$2(w^{5}+w^{1})$	$2(w^{10}+w^{2})$	$2(w^{15}+w^{3})$	$2(w^{20}+w^{4})$	$2(w^{25}+w^{5})$	$2(w^{30}+w^{6})$	0	0	0	0	0	0	0	0
$2(w^{28}+w^{8})$	$2(w^{33}+w^{9})$	$2(w^{10}+w^{2})$	$2(w^{15}+w^{3})$	$2(w^{20}+w^{4})$	$2(w^{25}+w^{5})$	$2(w^{30}+w^{6})$	$2(w^{35}+w^{7})$	0	0	0	0	0	0	0	0
$2(w^{32}+w^{10})$	$2(w^{37}+w^{11})$	$2(w^{15}+w^{3})$	$2(w^{20}+w^{4})$	$2(w^{25}+w^{5})$	$2(w^{30}+w^{6})$	$2(w^{35}+w^{7})$	$2(w^{40}+w^{8})$	0	0	0	0	0	0	0	0
$2(w^{36}+w^{12})$	$2(w^{41}+w^{13})$	$2(w^{20}+w^{4})$	$2(w^{25}+w^{5})$	$2(w^{30}+w^{6})$	$2(w^{35}+w^{7})$	$2(w^{40}+w^{8})$	$2(w^{45}+w^{9})$	0	0	0	0	0	0	0	0
$2(w^{40}+w^{14})$	$2(w^{45}+w^{15})$	$2(w^{25}+w^{5})$	$2(w^{30}+w^{6})$	$2(w^{35}+w^{7})$	$2(w^{40}+w^{8})$	$2(w^{45}+w^{9})$	$2(w^{50}+w^{10})$	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	3	-3	3	-3	3	-3	-3	3
0	0	0	0	0	0	0	0	-3	3	-3	3	-3	3	3	-3
0	0	0	0	0	0	0	0	3	-3	3	-3	-3	3	3	-3
0	0	0	0	0	0	0	0	-3	3	-3	3	3	-3	-3	3
0	0	0	0	0	0	0	0	3	-3	-3	3	3	3	-3	-3
0	0	0	0	0	0	0	0	-3	3	3	-3	3	3	-3	-3
0	0	0	0	0	0	0	0	-3	3	3	-3	-3	-3	3	3
0	0	0	0	0	0	0	0	3	-3	-3	3	-3	-3	3	3

= {{1, 1, 1, 1, 2, 2, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3},
{1, 1, 1, 1, 2, 2, -1, -1, -1, -1, -2, -2, 2, 2, 2, 2, 2, 2, 2, 2, -2, -2, -2,
-2, -2, -2, -3, -3, -3, -3, 3, 3, 3, 3}, {1, 1, 1, 1, 2, 2, -1, -1, -1, -1,
-2, -2, 2, 2, 2, 2, 2, 2, -2, -2, -2, -2, -2, -2, 3, 3, 3, 3, -3, -3, -3, -3},
{1, 1, 1, 1, 2, 2, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, -3, -3, -3, -3, -3, -3, -3, -3},
{2, 2, 2, 2, 4, 4, -2, -2, -2, -2, -4, -4, -2, -2, -2, -2, -2, -2, 2, 2, 2, 2, 2, 2, 0,
0, 0, 0, 0, 0, 0, 0}, {2, 2, 2, 2, 4, 4, 2, 2, 2, 2, 4, 4, -2, -2, -2, -2, -2, -2, -2, -2,
-2, -2, -2, -2, 0, 0, 0, 0, 0, 0, 0, 0}, {1, -1, -1, 1, -2, 2, 1, -1, -1, 1, -2, 2, 2, 2, -2,
2, -2, 2, -2, 2, -2, 2, -2, 2, -2, 3, 3, -3, -3, 3, -3, 3, -3}, {1, -1, -1, 1, -2, 2, -1,
1, 1, -1, 2, -2, 2, -2, 2, -2, 2, -2, -2, 2, -2, 2, -2, 2, -3, -3, 3, 3, 3, -3, 3, -3},
{1, -1, -1, 1, -2, 2, -1, 1, 1, -1, 2, -2, 2, -2, 2, -2, 2, -2, -2, 2, -2, 2, -2, 2, 3,
3, -3, -3, -3, 3, -3, 3}, {1, -1, -1, 1, -2, 2, 1, -1, -1, 1, -2, 2, 2, -2, 2, -2,
2, -2, 2, -2, 2, -2, 2, -2, -3, -3, 3, 3, -3, 3, -3, 3}, {2, -2, -2, 2, -4, 4, -2,
2, 2, -2, 4, -4, -2, 2, -2, 2, -2, 2, 2, -2, 2, -2, 2, -2, 0, 0, 0, 0, 0, 0, 0, 0},
{2, -2, -2, 2, -4, 4, 2, -2, -2, 2, -4, 4, -2, 2, -2, 2, -2, 2, -2, 2, -2, 2, -2, 2, 0,
0, 0, 0, 0, 0, 0, 0}, {2, 2, 2, 2, -2, -2, 2, 2, 2, 2, -2, -2, 4, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$,
$2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, 4, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, 4, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$,
$2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, 4, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, 0, 0, 0, 0, 0, 0, 0, 0},
{2, 2, 2, 2, -2, -2, -2, -2, -2, -2, 2, 2, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, 4,
$2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, 4, $2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right)$, -4, $2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right)$, $2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right)$,
-4, $2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right)$, 0, 0, 0, 0, 0, 0, 0, 0}, {2, 2, 2, 2, -2, -2, 2, 2, 2, 2, -2, -2,
$2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, 4, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, 4, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$,
$2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, 4, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, 4, 0, 0, 0, 0, 0, 0, 0, 0},
{2, 2, 2, 2, -2, -2, -2, -2, -2, -2, 2, 2, 4, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, 4,
$2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, -4, $2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right)$, $2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right)$, -4, $2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right)$,
$2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right)$, 0, 0, 0, 0, 0, 0, 0, 0}, {2, 2, 2, 2, -2, -2, 2, 2, 2, 2, -2, -2, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$,
$2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, 4, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, 4, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, 4,
$2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, 4, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, 0, 0, 0, 0, 0, 0, 0, 0},
{2, 2, 2, 2, -2, -2, -2, -2, -2, -2, 2, 2, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, 4, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$,
$2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, 4, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, $2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right)$, $2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right)$, -4, $2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right)$,
$2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right)$, -4, 0, 0, 0, 0, 0, 0, 0, 0}, {2, -2, -2, 2, 2, -2, 2, -2, -2, 2, 2, -2, 4,
$2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right)$, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, -4, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, $2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right)$, 4, $2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right)$,
$2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, -4, $2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right)$, $2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right)$, 0, 0, 0, 0, 0, 0, 0, 0},

$$\{2, -2, -2, 2, 2, -2, -2, 2, 2, -2, -2, 2, 2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right), -4, 2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right),$$

$$2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right), 4, 2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right), 2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right), 2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right), -4, 2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right),$$

$$2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right), 4, 0, 0, 0, 0, 0, 0, 0, 0\}, \{2, -2, -2, 2, 2, -2, 2, -2, -2, 2, 2, -2,$$

$$2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right), 2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right), 4, 2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right), 2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right), -4, 2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right),$$

$$-4, 2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right), 2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right), 4, 2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right), 0, 0, 0, 0, 0, 0, 0, 0\},$$

$$\{2, -2, -2, 2, 2, -2, -2, 2, 2, -2, -2, 2, 4, 2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right), 2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right), -4,$$

$$2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right), 2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right), -4, 2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right), 2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right), 4, 2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right),$$

$$2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right), 0, 0, 0, 0, 0, 0, 0, 0\}, \{2, -2, -2, 2, 2, -2, 2, -2, -2, 2, 2, -2,$$

$$2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right), -4, 2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right), 2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right), 4, 2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right), 2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right),$$

$$2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right), 4, 2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right), 2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right), -4, 0, 0, 0, 0, 0, 0, 0, 0\},$$

$$\{2, -2, -2, 2, 2, -2, -2, 2, 2, -2, -2, 2, 2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right), 2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right), 4,$$

$$2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right), 2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right), -4, 2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right), 4, 2\left(e^{-\frac{i \pi}{3}}+e^{\frac{i \pi}{3}}\right),$$

$$2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right), -4, 2\left(e^{-\frac{2 i \pi}{3}}+e^{\frac{2 i \pi}{3}}\right), 0, 0, 0, 0, 0, 0, 0, 0\},$$

{3, -3, 3, -3, 0, 0, 3, -3, 3, -3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, -3, 3, -3, 3, -3, -3, 3},
{3, -3, 3, -3, 0, 0, 3, -3, 3, -3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3, 3, -3, 3, -3, 3, 3, -3},
{3, -3, 3, -3, 0, 0, -3, 3, -3, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, -3, 3, -3, -3, 3, 3, -3},
{3, -3, 3, -3, 0, 0, -3, 3, -3, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3, 3, -3, 3, 3, -3, -3, 3},
{3, 3, -3, -3, 0, 0, 3, 3, -3, -3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, -3, -3, 3, 3, 3, -3, -3},
{3, 3, -3, -3, 0, 0, -3, -3, 3, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3, 3, 3, -3, 3, 3, -3, -3},
{3, 3, -3, -3, 0, 0, 3, 3, -3, -3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3, 3, 3, -3, -3, -3, 3, 3},
{3, 3, -3, -3, 0, 0, -3, -3, 3, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, -3, -3, 3, -3, -3, 3, 3}}

If we substitute in the value of $e^{\frac{i\pi}{3}}$ we get

```
In[18]:= s̃ = {{1, 1, 1, 1, 2, 2, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3},
          {1, 1, 1, 1, 2, 2, -1, -1, -1, -1, -2, -2, 2, 2, 2, 2, 2, 2, -2, -2, -2, -2,
           -2, -2, -3, -3, -3, -3, 3, 3, 3, 3}, {1, 1, 1, 1, 2, 2, -1, -1, -1, -1, -2,
           -2, 2, 2, 2, 2, 2, 2, -2, -2, -2, -2, -2, -2, 3, 3, 3, 3, -3, -3, -3, -3},
          {1, 1, 1, 1, 2, 2, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, -3, -3, -3, -3, -3, -3, -3, -3},
          {2, 2, 2, 2, 4, 4, -2, -2, -2, -2, -4, -4, -2, -2, -2, -2, -2, -2, 2, 2, 2, 2, 2,
           2, 0, 0, 0, 0, 0, 0, 0, 0}, {2, 2, 2, 2, 4, 4, 2, 2, 2, 2, 4, 4, -2, -2, -2, -2, -2,
           -2, -2, -2, -2, -2, -2, -2, 0, 0, 0, 0, 0, 0, 0, 0}, {1, -1, -1, 1, -2, 2, 1, -1,
           -1, 1, -2, 2, 2, -2, 2, -2, 2, -2, 2, -2, 2, -2, 2, -2, 3, 3, -3, -3, 3, -3, 3, -3},
          {1, -1, -1, 1, -2, 2, -1, 1, 1, -1, 2, -2, 2, -2, 2, -2, 2, -2, -2, 2, -2, 2, -2, 2, -3,
           -3, 3, 3, 3, -3, 3, -3}, {1, -1, -1, 1, -2, 2, -1, 1, 1, -1, 2, -2, 2, -2, 2, -2, 2,
           -2, -2, 2, -2, 2, -2, 2, 3, 3, -3, -3, -3, 3, -3, 3}, {1, -1, -1, 1, -2, 2, 1, -1,
           -1, 1, -2, 2, 2, -2, 2, -2, 2, -2, 2, -2, 2, -2, 2, -2, -3, -3, 3, 3, -3, 3, -3, 3},
          {2, -2, -2, 2, -4, 4, -2, 2, 2, -2, 4, -4, -2, 2, -2, 2, -2, 2, 2, -2, 2, -2, 2, -2, 0,
           0, 0, 0, 0, 0, 0, 0}, {2, -2, -2, 2, -4, 4, 2, -2, -2, 2, -4, 4, -2, 2, -2, 2, -2, 2,
           -2, 2, -2, 2, -2, 2, 0, 0, 0, 0, 0, 0, 0, 0}, {2, 2, 2, 2, -2, -2, 2, 2, 2, 2, -2, -2, 4,
           -2, -2, 4, -2, -2, 4, -2, -2, 4, -2, -2, 0, 0, 0, 0, 0, 0, 0, 0}, {2, 2, 2, 2, -2, -2,
           -2, -2, -2, -2, 2, 2, -2, -2, 4, -2, -2, 4, 2, -4, 2, 2, -4, 2, 0, 0, 0, 0, 0, 0, 0, 0},
          {2, 2, 2, 2, -2, -2, 2, 2, 2, 2, -2, -2, -2, 4, -2, -2, 4, -2, -2, -2, 4, -2, -2, 4, 0,
           0, 0, 0, 0, 0, 0, 0}, {2, 2, 2, 2, -2, -2, -2, -2, -2, -2, 2, 2, 4, -2, -2, 4, -2, -2,
           -4, 2, 2, -4, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0}, {2, 2, 2, 2, -2, -2, 2, 2, 2, 2, -2, -2, -2,
           -2, 4, -2, -2, 4, -2, 4, -2, -2, 4, -2, 0, 0, 0, 0, 0, 0, 0, 0}, {2, 2, 2, 2, -2, -2,
           -2, -2, -2, -2, 2, 2, -2, 4, -2, -2, 4, -2, 2, 2, -4, 2, 2, -4, 0, 0, 0, 0, 0, 0, 0, 0},
          {2, -2, -2, 2, 2, -2, 2, -2, -2, 2, 2, -2, 4, 2, -2, -4, -2, 2, 4, 2, -2, -4, -2, 2,
           0, 0, 0, 0, 0, 0, 0, 0}, {2, -2, -2, 2, 2, -2, -2, 2, 2, -2, -2, 2, -2, -4, -2, 2,
           4, 2, 2, -2, -4, -2, 2, 4, 0, 0, 0, 0, 0, 0, 0, 0}, {2, -2, -2, 2, 2, -2, 2, -2,
           -2, 2, 2, -2, -2, 2, 4, 2, -2, -4, -2, -4, -2, 2, 4, 2, 0, 0, 0, 0, 0, 0, 0, 0},
          {2, -2, -2, 2, 2, -2, -2, 2, 2, -2, -2, 2, 4, 2, -2, -4, -2, 2, -4, -2, 2, 4, 2, -2,
           0, 0, 0, 0, 0, 0, 0, 0}, {2, -2, -2, 2, 2, -2, 2, -2, -2, 2, 2, -2, -2, -4, -2, 2,
           4, 2, -2, 2, 4, 2, -2, -4, 0, 0, 0, 0, 0, 0, 0, 0}, {2, -2, -2, 2, 2, -2, -2, 2,
           2, -2, -2, 2, -2, 2, 4, 2, -2, -4, 2, 4, 2, -2, -4, -2, 0, 0, 0, 0, 0, 0, 0, 0},
          {3, -3, 3, -3, 0, 0, 3, -3, 3, -3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, -3, 3, -3, 3, -3, -3, 3},
          {3, -3, 3, -3, 0, 0, 3, -3, 3, -3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3, 3, -3, 3, -3, 3, 3, -3},
          {3, -3, 3, -3, 0, 0, -3, 3, -3, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, -3, 3, -3, -3, 3, 3, -3},
          {3, -3, 3, -3, 0, 0, -3, 3, -3, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3, 3, -3, 3, 3, -3, -3, 3},
          {3, 3, -3, -3, 0, 0, 3, 3, -3, -3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, -3, -3, 3, 3, 3, -3, -3},
          {3, 3, -3, -3, 0, 0, -3, -3, 3, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3, 3, 3, -3, 3, 3, -3, -3},
          {3, 3, -3, -3, 0, 0, 3, 3, -3, -3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3, 3, 3, -3, -3, -3, 3, 3},
          {3, 3, -3, -3, 0, 0, -3, -3, 3, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, -3, -3, 3, -3, -3, 3, 3}}
```

```
In[3]:=
        j1 = Det[s̃]

Out[3]= 34182189187166852111368841966125056

In[4]:= FactorInteger[j1]

Out[4]= {{2, 64}, {3, 32}}
```

~ ~

In[19]:=

s = s̃ / 12

Out[19]= $\{\{\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6},$

$\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\},$

$\{\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6},$

$-\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}, \{\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12},$

$-\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\},$

$\{\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6},$

$\frac{1}{6}, \frac{1}{6}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\}, \{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6},$

$-\frac{1}{6}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6},$

$-\frac{1}{6}, -\frac{1}{6}, 0, 0, 0, 0, 0, 0, 0, 0\}, \{\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, \frac{1}{12}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, \frac{1}{12}, -\frac{1}{6},$

$\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\},$

$\{\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, \frac{1}{12}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, -\frac{1}{12}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6},$

$\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\}, \{\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, \frac{1}{12}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, -\frac{1}{12},$

$\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\},$

$\{\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, \frac{1}{12}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, \frac{1}{12}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6},$

$\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\}, \{\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{6},$

$\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\{\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6},$

$-\frac{1}{6}, \frac{1}{6}, 0, 0, 0, 0, 0, 0, 0, 0\}, \{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, -\frac{1}{6},$

$\frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}, 0, 0, 0, 0, 0, 0, 0, 0\}, \{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6},$

$-\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{6}, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6},$

$\frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}, 0, 0, 0, 0, 0, 0, 0, 0\}, \{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6},$

$\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6},$

$-\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, 0, 0, 0, 0, 0, 0, 0, 0\}, \{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6},$

$\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{3}, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\{\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{3}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6},$

$-\frac{1}{6}, -\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}, 0, 0, 0, 0, 0, 0, 0, 0\}, \{\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6},$

$-\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\{\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{3}, -\frac{1}{6}, -\frac{1}{3},$

$-\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, 0, 0, 0, 0, 0, 0, 0, 0\}, \{\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6},$

$-\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, -\frac{1}{6}, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\{\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6},$

$\frac{1}{3}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{3}, 0, 0, 0, 0, 0, 0, 0, 0\}, \{\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6},$

$-\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{3}, -\frac{1}{6}, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\{\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, 0, 0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\},$

$\{\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, 0, 0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\},$

$\{\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, 0, 0, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\},$

$\{\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, 0, 0, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\},$

$\{\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\},$

$\{\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, 0, 0, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\},$

$\{\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\},$

$\{\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, 0, 0, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}\}$

```
In[6]:= r = Transpose[s]

Out[6]= {{1/12, 1/12, 1/12, 1/12, 1/6, 1/6, 1/12, 1/12, 1/12, 1/12, 1/6, 1/6, 1/6,
  1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 1/4, 1/4, 1/4, 1/4, 1/4, 1/4, 1/4, 1/4},
 {1/12, 1/12, 1/12, 1/12, 1/6, 1/6, -1/12, -1/12, -1/12, -1/12, -1/6, -1/6, 1/6, 1/6, 1/6, 1/6,
  1/6, 1/6, -1/6, -1/6, -1/6, -1/6, -1/6, -1/6, -1/4, -1/4, -1/4, -1/4, 1/4, 1/4, 1/4, 1/4},
 {1/12, 1/12, 1/12, 1/12, 1/6, 1/6, -1/12, -1/12, -1/12, -1/12, -1/6, -1/6, 1/6, 1/6, 1/6, 1/6,
  1/6, 1/6, -1/6, -1/6, -1/6, -1/6, -1/6, -1/6, 1/4, 1/4, 1/4, 1/4, -1/4, -1/4, -1/4, -1/4},
 {1/12, 1/12, 1/12, 1/12, 1/6, 1/6, 1/12, 1/12, 1/12, 1/12, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6,
  1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, -1/4, -1/4, -1/4, -1/4, -1/4, -1/4, -1/4, -1/4},
 {1/6, 1/6, 1/6, 1/6, 1/3, 1/3, -1/6, -1/6, -1/6, -1/6, -1/3, -1/3, -1/6, -1/6, -1/6, -1/6, -1/6, -1/6, 1/6,
  1/6, 1/6, 1/6, 1/6, 1/6, 0, 0, 0, 0, 0, 0, 0, 0}, {1/6, 1/6, 1/6, 1/6, 1/3, 1/3, 1/6, 1/6, 1/6, 1/6, 1/3,
  1/3, -1/6, -1/6, -1/6, -1/6, -1/6, -1/6, -1/6, -1/6, -1/6, -1/6, -1/6, -1/6, 0, 0, 0, 0, 0, 0, 0, 0},
 {1/12, -1/12, -1/12, 1/12, -1/6, 1/6, 1/12, -1/12, -1/12, 1/12, -1/6, 1/6, 1/6, -1/6, 1/6, -1/6,
  1/6, -1/6, 1/6, -1/6, 1/6, -1/6, 1/6, -1/6, 1/4, 1/4, -1/4, -1/4, 1/4, -1/4, 1/4, -1/4},
 {1/12, -1/12, -1/12, 1/12, -1/6, 1/6, -1/12, 1/12, 1/12, -1/12, 1/6, -1/6, 1/6, -1/6, 1/6, -1/6,
  1/6, -1/6, -1/6, 1/6, -1/6, 1/6, -1/6, 1/6, -1/4, -1/4, 1/4, 1/4, 1/4, -1/4, 1/4, -1/4},
 {1/12, -1/12, -1/12, 1/12, -1/6, 1/6, -1/12, 1/12, 1/12, -1/12, 1/6, -1/6, 1/6, -1/6, 1/6, -1/6,
  1/6, -1/6, -1/6, 1/6, -1/6, 1/6, -1/6, 1/6, 1/4, 1/4, -1/4, -1/4, -1/4, 1/4, -1/4, 1/4},
 {1/12, -1/12, -1/12, 1/12, -1/6, 1/6, 1/12, -1/12, -1/12, 1/12, -1/6, 1/6, 1/6, -1/6, 1/6, -1/6,
  1/6, -1/6, 1/6, -1/6, 1/6, -1/6, 1/6, -1/6, -1/4, -1/4, 1/4, 1/4, -1/4, 1/4, -1/4, 1/4},
 {1/6, -1/6, -1/6, 1/6, -1/3, 1/3, -1/6, 1/6, 1/6, -1/6, 1/3, -1/3, -1/6, 1/6, -1/6, 1/6, -1/6, 1/6, 1/6, -1/6,
  1/6, -1/6, 1/6, -1/6, 0, 0, 0, 0, 0, 0, 0, 0}, {1/6, -1/6, -1/6, 1/6, -1/3, 1/3, 1/6, -1/6, -1/6, 1/6,
  -1/3, 1/3, -1/6, 1/6, -1/6, 1/6, -1/6, 1/6, -1/6, 1/6, -1/6, 1/6, -1/6, 1/6, 0, 0, 0, 0, 0, 0, 0, 0},
 {1/6, 1/6, 1/6, 1/6, -1/6, -1/6, 1/6, 1/6, 1/6, 1/6, -1/6, -1/6, 1/3, -1/6, -1/6, 1/3, -1/6, -1/6, 1/3, -1/6,
  -1/6, 1/3, -1/6, -1/6, 0, 0, 0, 0, 0, 0, 0, 0}, {1/6, 1/6, 1/6, 1/6, -1/6, -1/6, -1/6, -1/6, -1/6, -1/6,
  1/6, 1/6, -1/6, -1/6, 1/3, -1/6, -1/6, 1/3, 1/6, -1/3, 1/6, 1/6, -1/3, 1/6, 0, 0, 0, 0, 0, 0, 0, 0},
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  1/3, -1/6, -1/6, 1/3, 0, 0, 0, 0, 0, 0, 0, 0}, {1/6, 1/6, 1/6, 1/6, -1/6, -1/6, -1/6, -1/6, -1/6, -1/6,
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In[7]:= **s - r**

```
Out[7]= {{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
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```

which shows that the matrix S is symmetric. Now we show that the matrix S is invertible by calculating its determinant.

In[8]:= **j2 = Det[s]**

Out[8]= 1

** **

The relations between the matrices S,T and C

Here we just want to show that $S^2 = (ST)^3$ and $S^2=C$ as the other relations mentioned in the last section of chapter five come directly as C is the identity matrix. Note that $\sqrt{p+/p-}=1$.

```
In[9]:= ••{{1/12, 1/12, 1/12, 1/12, 1/6, 1/6, 1/12, 1/12, 1/12, 1/12, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 1/4, 1/4, 1/4, 1/4, 1/4, 1/4, 1/4, 1/4},
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  {1/4, -1/4, 1/4, -1/4, 0, 0, 1/4, -1/4, 1/4, -1/4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/4, -1/4, 1/4, -1/4, 1/4, -1/4, -1/4, 1/4},
  {1/4, -1/4, 1/4, -1/4, 0, 0, 1/4, -1/4, 1/4, -1/4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1/4, 1/4, -1/4, 1/4, -1/4, 1/4, 1/4, -1/4},
  {1/4, -1/4, 1/4, -1/4, 0, 0, -1/4, 1/4, -1/4, 1/4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/4, -1/4, 1/4, -1/4, -1/4, 1/4, 1/4, -1/4},
  {1/4, -1/4, 1/4, -1/4, 0, 0, -1/4, 1/4, -1/4, 1/4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1/4, 1/4, -1/4, 1/4, 1/4, -1/4, -1/4, 1/4},
  {1/4, 1/4, -1/4, -1/4, 0, 0, 1/4, 1/4, -1/4, -1/4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/4, -1/4, -1/4, 1/4, 1/4, 1/4, -1/4, -1/4},
  {1/4, 1/4, -1/4, -1/4, 0, 0, -1/4, -1/4, 1/4, 1/4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1/4, 1/4, 1/4, -1/4, 1/4, 1/4, -1/4, -1/4},
  {1/4, 1/4, -1/4, -1/4, 0, 0, 1/4, 1/4, -1/4, -1/4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1/4, 1/4, 1/4, -1/4, -1/4, -1/4, 1/4, 1/4},
  {1/4, 1/4, -1/4, -1/4, 0, 0, -1/4, -1/4, 1/4, 1/4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/4, -1/4, -1/4, 1/4, -1/4, -1/4, 1/4, 1/4}}
```

```
In[10]:=  t = {{1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0,
   1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0,
   0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, e^(2 ⅈ π/3), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, e^(-2 ⅈ π/3),
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, e^(2 ⅈ π/3), 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, e^(-2 ⅈ π/3), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, e^(ⅈ π/3), 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, e^(2 ⅈ π/3), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, e^(-2 ⅈ π/3), 0,
   0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, e^(-ⅈ π/3), 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1,
   0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1}}
```

In[11]:= **s2 = MatrixPower[s, 2]**

Out[11]=
```
{{1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 1, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1}}
```

In[12]:= **l = s.t**

Out[12]=

$$\left\{\left\{\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, \frac{1}{12},\right.\right.$$
$$-\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6}, \frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6}, \frac{1}{6} e^{\frac{i \pi}{3}},$$
$$\left.\frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{i \pi}{3}}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right\},$$
$$\left\{\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, -\frac{1}{12}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6},\right.$$
$$\frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6}, \frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6}, -\frac{1}{6} e^{\frac{i \pi}{3}}, -\frac{1}{6} e^{\frac{2 i \pi}{3}},$$

$\frac{1}{6}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{i \pi}{3}}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\},$

$\{\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, -\frac{1}{12}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6},$

$\frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6}, \frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6}, -\frac{1}{6} e^{\frac{i \pi}{3}}, -\frac{1}{6} e^{\frac{2 i \pi}{3}},$

$\frac{1}{6}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{i \pi}{3}}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\},$

$\{\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, \frac{1}{12}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6},$

$\frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6}, \frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6}, \frac{1}{6} e^{\frac{i \pi}{3}}, \frac{1}{6} e^{\frac{2 i \pi}{3}},$

$-\frac{1}{6}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{i \pi}{3}}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\},$

$\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{6}, -\frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6},$

$-\frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6}, \frac{1}{6} e^{\frac{i \pi}{3}}, \frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{i \pi}{3}}, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{3}, -\frac{1}{6}, -\frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}},$

$-\frac{1}{6}, -\frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6}, -\frac{1}{6} e^{\frac{i \pi}{3}}, -\frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{i \pi}{3}},$

$0, 0, 0, 0, 0, 0, 0, 0\}, \{\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, \frac{1}{12}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12},$

$\frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6}, \frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6}, -\frac{1}{6} e^{\frac{i \pi}{3}},$

$\frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{i \pi}{3}}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\},$

$\{\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, \frac{1}{12}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6},$

$-\frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6}, \frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6}, \frac{1}{6} e^{\frac{i \pi}{3}}, -\frac{1}{6} e^{\frac{2 i \pi}{3}},$

$-\frac{1}{6}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{i \pi}{3}}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\},$

$\{\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, \frac{1}{12}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6},$

$-\frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6}, \frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6}, \frac{1}{6} e^{\frac{i \pi}{3}}, -\frac{1}{6} e^{\frac{2 i \pi}{3}},$

$-\frac{1}{6}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{i \pi}{3}}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\},$

$\{\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, \frac{1}{12}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6},$

$-\frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6}, \frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6}, -\frac{1}{6} e^{\frac{i \pi}{3}}, \frac{1}{6} e^{\frac{2 i \pi}{3}},$

$\frac{1}{6}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{i \pi}{3}}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\},$

$\{\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{6}, \frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6},$

$-\frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6}, -\frac{1}{6} e^{\frac{i \pi}{3}}, \frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{i \pi}{3}}, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\{\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6}, -\frac{1}{6} e^{\frac{2 i \pi}{3}},$

$\frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6}, \frac{1}{6} e^{\frac{i \pi}{3}}, -\frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{i \pi}{3}}, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{3},$

$-\frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{3}, -\frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{3}, -\frac{1}{6} e^{\frac{i \pi}{3}},$

$-\frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{3}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{i \pi}{3}}, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{3} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6},$

$-\frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{3} e^{-\frac{2 i \pi}{3}}, \frac{1}{6}, -\frac{1}{3} e^{\frac{i \pi}{3}}, \frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6}, -\frac{1}{3} e^{-\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{i \pi}{3}}, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{3} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6},$

$\frac{1}{3} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6}, -\frac{1}{6} e^{\frac{i \pi}{3}}, \frac{1}{3} e^{\frac{2 i \pi}{3}}, \frac{1}{6}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{3} e^{-\frac{i \pi}{3}}, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, -\frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{3},$

$-\frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{3}, \frac{1}{6} e^{\frac{i \pi}{3}}, \frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{3}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{i \pi}{3}}, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{3} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6},$

$-\frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{3} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6}, \frac{1}{3} e^{\frac{i \pi}{3}}, -\frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6}, \frac{1}{3} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{i \pi}{3}}, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{3} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6},$

$\frac{1}{3} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6}, \frac{1}{6} e^{\frac{i \pi}{3}}, -\frac{1}{3} e^{\frac{2 i \pi}{3}}, -\frac{1}{6}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{3} e^{-\frac{i \pi}{3}}, 0, 0, 0, 0, 0, 0, 0, 0\},$

$$\left\{\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}, \frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{3},\right.$$
$$\left. -\frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{3}, \frac{1}{6} e^{\frac{i \pi}{3}}, -\frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{3}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{i \pi}{3}}, 0, 0, 0, 0, 0, 0, 0, 0\right\},$$
$$\left\{\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{3} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6},\right.$$
$$\left. \frac{1}{3} e^{\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6}, -\frac{1}{6} e^{\frac{i \pi}{3}}, -\frac{1}{3} e^{\frac{2 i \pi}{3}}, \frac{1}{6}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{3} e^{-\frac{i \pi}{3}}, 0, 0, 0, 0, 0, 0, 0, 0\right\},$$
$$\left\{\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{3} e^{-\frac{2 i \pi}{3}}, \frac{1}{6}, -\frac{1}{6} e^{\frac{2 i \pi}{3}},\right.$$
$$\left. -\frac{1}{3} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6}, -\frac{1}{3} e^{\frac{i \pi}{3}}, -\frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6}, \frac{1}{3} e^{-\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{i \pi}{3}}, 0, 0, 0, 0, 0, 0, 0, 0\right\},$$
$$\left\{\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3},\right.$$
$$\frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{3}, -\frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{3}, -\frac{1}{6} e^{\frac{i \pi}{3}},$$
$$\left. \frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{3}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{i \pi}{3}}, 0, 0, 0, 0, 0, 0, 0, 0\right\},$$
$$\left\{\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{3} e^{\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, \frac{1}{6},\right.$$
$$\left. \frac{1}{3} e^{\frac{2 i \pi}{3}}, \frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6}, \frac{1}{6} e^{\frac{i \pi}{3}}, \frac{1}{3} e^{\frac{2 i \pi}{3}}, -\frac{1}{6}, -\frac{1}{6} e^{-\frac{2 i \pi}{3}}, -\frac{1}{3} e^{-\frac{i \pi}{3}}, 0, 0, 0, 0, 0, 0, 0, 0\right\},$$
$$\left\{\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{3} e^{-\frac{2 i \pi}{3}}, \frac{1}{6},\right.$$
$$\left. -\frac{1}{6} e^{\frac{2 i \pi}{3}}, -\frac{1}{3} e^{-\frac{2 i \pi}{3}}, \frac{1}{6}, \frac{1}{3} e^{\frac{i \pi}{3}}, \frac{1}{6} e^{\frac{2 i \pi}{3}}, \frac{1}{6}, -\frac{1}{3} e^{-\frac{2 i \pi}{3}}, -\frac{1}{6} e^{-\frac{i \pi}{3}}, 0, 0, 0, 0, 0, 0, 0, 0\right\},$$
$$\left\{\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,\right.$$
$$\left. 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\right\}, \left\{\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4},\right.$$
$$\left. 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\right\},$$
$$\left\{\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, 0, 0, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,\right.$$
$$\left. 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\right\}, \left\{\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, 0, 0, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4},\right.$$
$$\left. 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\right\},$$
$$\left\{\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, 0, 0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,\right.$$
$$\left. 0, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}, \left\{\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, 0, 0, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4},\right.$$
$$\left. \frac{1}{4}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\},$$
$$\left\{\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, 0, 0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,\right.$$
$$\left. 0, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right\}, \left\{\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, 0, 0, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4},\right.$$
$$\left. \frac{1}{4}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right\}\right\}$$

```
In[13]:= f = MatrixPower[l, 3]
```

```
In[15]:=
         rf = f // N
```

Out[15]=

$$\{\{1. + 0.\,i, -2.77556\times10^{-17} + 0.\,i, -2.77556\times10^{-17} + 0.\,i, -6.93889\times10^{-18} + 0.\,i, 2.77556\times10^{-17} + 0.\,i,$$
$$4.16334\times10^{-17} + 0.\,i, -2.77556\times10^{-17} + 0.\,i, 3.46945\times10^{-18} + 0.\,i, 3.46945\times10^{-18} + 0.\,i,$$
$$-2.77556\times10^{-17} + 0.\,i, 6.93889\times10^{-18} + 0.\,i, 2.77556\times10^{-17} + 0.\,i, 4.16334\times10^{-17} + 0.\,i,$$
$$2.42861\times10^{-17} + 4.51867\times10^{-17}\,i, -3.46945\times10^{-17} - 6.03774\times10^{-17}\,i, 2.77556\times10^{-17} + 0.\,i,$$
$$-4.51028\times10^{-17} + 5.73721\times10^{-17}\,i, 3.46945\times10^{-17} - 3.98701\times10^{-17}\,i, 2.77556\times10^{-17} + 0.\,i,$$
$$4.16334\times10^{-17} + 3.38604\times10^{-17}\,i, 2.42861\times10^{-17} + 4.51867\times10^{-17}\,i, 6.93889\times10^{-18} + 0.\,i,$$
$$3.46945\times10^{-17} - 3.98701\times10^{-17}\,i, 2.77556\times10^{-17} - 3.0856\times10^{-17}\,i, 0., 0., 0., 0., 0., 0., 0., 0.\},$$
$$\{-3.46945\times10^{-17} + 0.\,i, 1. + 0.\,i, -5.55112\times10^{-17} + 0.\,i, -3.46945\times10^{-17} + 0.\,i, 4.85723\times10^{-17} + 0.\,i,$$
$$2.77556\times10^{-17} + 0.\,i, 2.77556\times10^{-17} + 0.\,i, 2.08167\times10^{-17} + 0.\,i, 2.08167\times10^{-17} + 0.\,i,$$
$$2.77556\times10^{-17} + 0.\,i, -2.08167\times10^{-17} + 0.\,i, -1.38778\times10^{-17} + 0.\,i, 2.77556\times10^{-17} + 0.\,i,$$
$$-4.51028\times10^{-17} + 5.73721\times10^{-17}\,i, -6.93889\times10^{-18} - 3.98701\times10^{-17}\,i, 4.85723\times10^{-17} + 0.\,i,$$
$$2.42861\times10^{-17} + 4.51867\times10^{-17}\,i, -2.08167\times10^{-17} - 6.03774\times10^{-17}\,i, -1.38778\times10^{-17} + 0.\,i,$$
$$-1.38778\times10^{-17} - 3.38604\times10^{-17}\,i, -3.46945\times10^{-17} - 4.51867\times10^{-17}\,i, -2.08167\times10^{-17} + 0.\,i,$$
$$-2.77556\times10^{-17} + 3.98701\times10^{-17}\,i, -1.04083\times10^{-17} + 3.0856\times10^{-17}\,i, 0., 0., 0., 0., 0., 0., 0., 0.\},$$
$$\{-3.46945\times10^{-17} + 0.\,i, -5.55112\times10^{-17} + 0.\,i, 1. + 0.\,i, -3.46945\times10^{-17} + 0.\,i, 4.85723\times10^{-17} + 0.\,i,$$

$2.77556\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,$
$2.77556\times10^{-17}+0.\,i,\ -2.08167\times10^{-17}+0.\,i,\ -1.38778\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,$
$-4.51028\times10^{-17}+5.73721\times10^{-17}\,i,\ -6.93889\times10^{-18}-3.98701\times10^{-17}\,i,\ 4.85723\times10^{-17}+0.\,i,$
$2.42861\times10^{-17}+4.51867\times10^{-17}\,i,\ -2.08167\times10^{-17}-6.03774\times10^{-17}\,i,\ -1.38778\times10^{-17}+0.\,i,$
$-1.38778\times10^{-17}-3.38604\times10^{-17}\,i,\ -3.46945\times10^{-17}-4.51867\times10^{-17}\,i,\ -2.08167\times10^{-17}+0.\,i,$
$-2.77556\times10^{-17}+3.98701\times10^{-17}\,i,\ -1.04083\times10^{-17}+3.0856\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-6.93889\times10^{-18}+0.\,i,\ -2.77556\times10^{-17}+0.\,i,\ -2.77556\times10^{-17}+0.\,i,\ 1.+0.\,i,\ 2.77556\times10^{-17}+0.\,i,$
$4.16334\times10^{-17}+0.\,i,\ -2.77556\times10^{-17}+0.\,i,\ 3.46945\times10^{-18}+0.\,i,\ 3.46945\times10^{-18}+0.\,i,$
$-2.77556\times10^{-17}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,\ 4.16334\times10^{-17}+0.\,i,$
$2.42861\times10^{-17}+4.51867\times10^{-17}\,i,\ -3.46945\times10^{-17}-6.03774\times10^{-17}\,i,\ 2.77556\times10^{-17}+0.\,i,$
$-4.51028\times10^{-17}+5.73721\times10^{-17}\,i,\ 3.46945\times10^{-17}-3.98701\times10^{-17}\,i,\ 2.77556\times10^{-17}+0.\,i,$
$4.16334\times10^{-17}+3.38604\times10^{-17}\,i,\ 2.42861\times10^{-17}+4.51867\times10^{-17}\,i,\ 6.93889\times10^{-18}+0.\,i,$
$3.46945\times10^{-17}-3.98701\times10^{-17}\,i,\ 2.77556\times10^{-17}-3.0856\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{2.08167\times10^{-17}+0.\,i,\ 6.245\times10^{-17}+0.\,i,\ 6.245\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,$
$1.+0.\,i,\ -5.55112\times10^{-17}+0.\,i,\ 3.1225\times10^{-17}+0.\,i,\ 0.+0.\,i,\ 0.+0.\,i,$
$3.1225\times10^{-17}+0.\,i,\ 1.73472\times10^{-17}+0.\,i,\ 4.16334\times10^{-17}+0.\,i,\ -2.77556\times10^{-17}+0.\,i,$
$4.51028\times10^{-17}-5.73721\times10^{-17}\,i,\ 1.04083\times10^{-17}+3.98701\times10^{-17}\,i,\ -3.46945\times10^{-17}+0.\,i,$
$-1.04083\times10^{-17}-4.51867\times10^{-17}\,i,\ 3.46945\times10^{-17}+6.03774\times10^{-17}\,i,\ 2.77556\times10^{-17}+0.\,i,$
$-3.46945\times10^{-18}+3.38604\times10^{-17}\,i,\ 6.93889\times10^{-18}+4.51867\times10^{-17}\,i,\ 1.38778\times10^{-17}+0.\,i,$
$-2.08167\times10^{-17}-3.98701\times10^{-17}\,i,\ -6.93889\times10^{-18}-3.0856\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{1.04083\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,\ 1.04083\times10^{-17}+0.\,i,$
$-5.55112\times10^{-17}+0.\,i,\ 1.+0.\,i,\ 2.08167\times10^{-17}+0.\,i,\ -2.42861\times10^{-17}+0.\,i,\ -2.42861\times10^{-17}+0.\,i,$
$2.08167\times10^{-17}+0.\,i,\ -4.51028\times10^{-17}+0.\,i,\ -5.55112\times10^{-17}+0.\,i,\ -3.46945\times10^{-17}+0.\,i,$
$-1.04083\times10^{-17}-4.51867\times10^{-17}\,i,\ 2.77556\times10^{-17}+6.03774\times10^{-17}\,i,\ -2.08167\times10^{-17}+0.\,i,$
$4.51028\times10^{-17}-5.73721\times10^{-17}\,i,\ 6.93889\times10^{-18}+3.98701\times10^{-17}\,i,\ -2.08167\times10^{-17}+0.\,i,$
$-1.38778\times10^{-17}-3.38604\times10^{-17}\,i,\ -1.04083\times10^{-17}-4.51867\times10^{-17}\,i,\ -1.73472\times10^{-17}+0.\,i,$
$6.93889\times10^{-18}+3.98701\times10^{-17}\,i,\ 0.+3.0856\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-3.46945\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,\ -3.46945\times10^{-17}+0.\,i,$
$-1.38778\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,\ 1.+0.\,i,\ 2.08167\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,$
$-5.55112\times10^{-17}+0.\,i,\ -2.08167\times10^{-17}+0.\,i,\ 4.85723\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,$
$-3.46945\times10^{-17}-4.51867\times10^{-17}\,i,\ -6.93889\times10^{-18}-3.98701\times10^{-17}\,i,\ -1.38778\times10^{-17}+0.\,i,$
$2.42861\times10^{-17}+4.51867\times10^{-17}\,i,\ -2.77556\times10^{-17}+3.98701\times10^{-17}\,i,\ 4.85723\times10^{-17}+0.\,i,$
$-1.38778\times10^{-17}-3.38604\times10^{-17}\,i,\ -4.51028\times10^{-17}+5.73721\times10^{-17}\,i,\ -2.08167\times10^{-17}+0.\,i,$
$-2.08167\times10^{-17}-6.03774\times10^{-17}\,i,\ -1.04083\times10^{-17}+3.0856\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-1.38778\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,\ -1.38778\times10^{-17}+0.\,i,$
$-6.93889\times10^{-18}+0.\,i,\ 1.38778\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,\ 1.+0.\,i,\ -1.04083\times10^{-17}+0.\,i,$
$2.77556\times10^{-17}+0.\,i,\ 4.85723\times10^{-17}+0.\,i,\ -6.93889\times10^{-18}+0.\,i,\ 1.38778\times10^{-17}+0.\,i,$
$-3.46945\times10^{-17}+1.21354\times10^{-17}\,i,\ 2.08167\times10^{-17}+1.74537\times10^{-17}\,i,\ -6.93889\times10^{-18}+0.\,i,$
$2.42861\times10^{-17}-1.21354\times10^{-17}\,i,\ -2.77556\times10^{-17}-1.74537\times10^{-17}\,i,\ -6.93889\times10^{-18}+0.\,i,$
$-1.38778\times10^{-17}-4.62802\times10^{-17}\,i,\ -3.46945\times10^{-17}+1.21354\times10^{-17}\,i,\ 4.85723\times10^{-17}+0.\,i,$
$-2.77556\times10^{-17}-1.74537\times10^{-17}\,i,\ -1.38778\times10^{-17}+4.92846\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-1.38778\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,\ -1.38778\times10^{-17}+0.\,i,$
$-6.93889\times10^{-18}+0.\,i,\ 1.38778\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,\ -1.04083\times10^{-17}+0.\,i,$
$1.+0.\,i,\ 2.77556\times10^{-17}+0.\,i,\ 4.85723\times10^{-17}+0.\,i,\ -6.93889\times10^{-18}+0.\,i,\ 1.38778\times10^{-17}+0.\,i,$
$-3.46945\times10^{-17}+1.21354\times10^{-17}\,i,\ 2.08167\times10^{-17}+1.74537\times10^{-17}\,i,\ -6.93889\times10^{-18}+0.\,i,$
$2.42861\times10^{-17}-1.21354\times10^{-17}\,i,\ -2.77556\times10^{-17}-1.74537\times10^{-17}\,i,\ -6.93889\times10^{-18}+0.\,i,$
$-1.38778\times10^{-17}-4.62802\times10^{-17}\,i,\ -3.46945\times10^{-17}+1.21354\times10^{-17}\,i,\ 4.85723\times10^{-17}+0.\,i,$
$-2.77556\times10^{-17}-1.74537\times10^{-17}\,i,\ -1.38778\times10^{-17}+4.92846\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-3.46945\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,\ -3.46945\times10^{-17}+0.\,i,$
$-1.38778\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,\ -5.55112\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,$
$2.08167\times10^{-17}+0.\,i,\ 1.+0.\,i,\ -2.08167\times10^{-17}+0.\,i,\ 4.85723\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,$
$-3.46945\times10^{-17}-4.51867\times10^{-17}\,i,\ -6.93889\times10^{-18}-3.98701\times10^{-17}\,i,\ -1.38778\times10^{-17}+0.\,i,$
$2.42861\times10^{-17}+4.51867\times10^{-17}\,i,\ -2.77556\times10^{-17}+3.98701\times10^{-17}\,i,\ 4.85723\times10^{-17}+0.\,i,$
$-1.38778\times10^{-17}-3.38604\times10^{-17}\,i,\ -4.51028\times10^{-17}+5.73721\times10^{-17}\,i,\ -2.08167\times10^{-17}+0.\,i,$
$-2.08167\times10^{-17}-6.03774\times10^{-17}\,i,\ -1.04083\times10^{-17}+3.0856\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-1.38778\times10^{-17}+0.\,i,\ 2.42861\times10^{-17}+0.\,i,\ 2.42861\times10^{-17}+0.\,i,\ -1.38778\times10^{-17}+0.\,i,$
$1.38778\times10^{-17}+0.\,i,\ 5.55112\times10^{-17}+0.\,i,\ 2.42861\times10^{-17}+0.\,i,\ 7.28584\times10^{-17}+0.\,i,$
$7.28584\times10^{-17}+0.\,i,\ 2.42861\times10^{-17}+0.\,i,\ 1.+0.\,i,\ 1.38778\times10^{-17}+0.\,i,\ -1.38778\times10^{-17}+0.\,i,$
$6.93889\times10^{-18}-1.21354\times10^{-17}\,i,\ -1.38778\times10^{-17}-1.74537\times10^{-17}\,i,\ 2.77556\times10^{-17}+0.\,i,$
$-1.04083\times10^{-17}+1.21354\times10^{-17}\,i,\ -6.93889\times10^{-18}+1.74537\times10^{-17}\,i,\ 2.77556\times10^{-17}+0.\,i,$
$1.38778\times10^{-17}+4.62802\times10^{-17}\,i,\ 6.93889\times10^{-18}-1.21354\times10^{-17}\,i,\ -4.16334\times10^{-17}+0.\,i,$
$-6.93889\times10^{-18}+1.74537\times10^{-17}\,i,\ 2.77556\times10^{-17}-4.92846\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{2.08167\times10^{-17}+0.\,i,\ 3.1225\times10^{-17}+0.\,i,\ 3.1225\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,$
$4.16334\times10^{-17}+0.\,i,\ -5.55112\times10^{-17}+0.\,i,\ 6.245\times10^{-17}+0.\,i,\ 0.+0.\,i,$
$0.+0.\,i,\ 6.245\times10^{-17}+0.\,i,\ 1.73472\times10^{-17}+0.\,i,\ 1.+0.\,i,\ -2.77556\times10^{-17}+0.\,i,$
$6.93889\times10^{-18}+4.51867\times10^{-17}\,i,\ 1.04083\times10^{-17}+3.98701\times10^{-17}\,i,\ 2.77556\times10^{-17}+0.\,i,$
$-1.04083\times10^{-17}-4.51867\times10^{-17}\,i,\ -2.08167\times10^{-17}-3.98701\times10^{-17}\,i,\ -3.46945\times10^{-17}+0.\,i,$
$-3.46945\times10^{-18}+3.38604\times10^{-17}\,i,\ 4.51028\times10^{-17}-5.73721\times10^{-17}\,i,\ 1.38778\times10^{-17}+0.\,i,$
$3.46945\times10^{-17}+6.03774\times10^{-17}\,i,\ -6.93889\times10^{-18}-3.0856\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{1.04083\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,\ 1.04083\times10^{-17}+0.\,i,$
$-2.08167\times10^{-17}+0.\,i,\ -3.46945\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,\ -2.42861\times10^{-17}+0.\,i,$
$-2.42861\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,\ -1.73472\times10^{-17}+0.\,i,\ -2.08167\times10^{-17}+0.\,i,\ 1.+0.\,i,$
$-1.04083\times10^{-17}-4.51867\times10^{-17}\,i,\ 2.77556\times10^{-17}+6.03774\times10^{-17}\,i,\ -5.55112\times10^{-17}+0.\,i,$
$4.51028\times10^{-17}-5.73721\times10^{-17}\,i,\ 3.46945\times10^{-17}+3.98701\times10^{-17}\,i,\ -5.55112\times10^{-17}+0.\,i,$
$-1.38778\times10^{-17}-3.38604\times10^{-17}\,i,\ -1.04083\times10^{-17}-4.51867\times10^{-17}\,i,\ -4.51028\times10^{-17}+0.\,i,$
$3.46945\times10^{-17}+3.98701\times10^{-17}\,i,\ 0.+3.0856\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$

$\{5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ 4.16334\times10^{-17}+3.93438\times10^{-17}\,i,\ 4.16334\times10^{-17}+3.93438\times10^{-17}\,i,$
$5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ 2.08167\times10^{-17}-3.93438\times10^{-17}\,i,\ 6.93889\times10^{-18}-1.51424\times10^{-17}\,i,$
$-4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ -5.20417\times10^{-17}+4.21822\times10^{-17}\,i,$
$-5.20417\times10^{-17}+4.21822\times10^{-17}\,i,\ -4.16334\times10^{-17}-1.51424\times10^{-17}\,i,$
$6.245\times10^{-17}-4.21822\times10^{-17}\,i,\ 2.77556\times10^{-17}+1.51424\times10^{-17}\,i,\ 6.93889\times10^{-18}-1.51424\times10^{-17}\,i,$
$1.-3.27641\times10^{-17}\,i,\ -2.77556\times10^{-17}+1.15479\times10^{-18}\,i,\ 2.08167\times10^{-17}-3.93438\times10^{-17}\,i,$
$-5.20417\times10^{-17}+3.91778\times10^{-17}\,i,\ 2.77556\times10^{-17}-8.57384\times10^{-17}\,i,\ 2.77556\times10^{-17}+1.51424\times10^{-17}\,i,$
$1.38778\times10^{-17}-1.52607\times10^{-17}\,i,\ 6.93889\times10^{-17}-3.91778\times10^{-17}\,i,\ 6.245\times10^{-17}-4.21822\times10^{-17}\,i,$
$-6.93889\times10^{-18}-1.15479\times10^{-18}\,i,\ 3.46945\times10^{-18}+2.68083\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{4.85723\times10^{-17}-3.93438\times10^{-17}\,i,\ 4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ 4.16334\times10^{-17}-1.51424\times10^{-17}\,i,$
$4.85723\times10^{-17}-3.93438\times10^{-17}\,i,\ -5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ 1.38778\times10^{-17}+3.93438\times10^{-17}\,i,$
$4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ 4.16334\times10^{-17}+4.21822\times10^{-17}\,i,\ 4.16334\times10^{-17}+4.21822\times10^{-17}\,i,$
$4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ -6.59195\times10^{-17}-4.21822\times10^{-17}\,i,\ -5.55112\times10^{-17}+1.51424\times10^{-17}\,i,$
$1.38778\times10^{-17}+3.93438\times10^{-17}\,i,\ -3.46945\times10^{-17}+6.00885\times10^{-18}\,i,\ 1.+4.09735\times10^{-17}\,i,$
$-5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ -6.93889\times10^{-18}+9.01362\times10^{-17}\,i,$
$6.93889\times10^{-18}-4.62291\times10^{-17}\,i,\ -5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ 0.+2.38039\times10^{-17}\,i,$
$-3.46945\times10^{-17}+6.00885\times10^{-18}\,i,\ -6.59195\times10^{-17}-4.21822\times10^{-17}\,i,$
$6.93889\times10^{-18}-4.62291\times10^{-17}\,i,\ -4.16334\times10^{-17}-9.25186\times10^{-18}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{2.08167\times10^{-17}+0.\,i,\ 6.245\times10^{-17}+0.\,i,\ 6.245\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,$
$-3.46945\times10^{-17}+0.\,i,\ -2.77556\times10^{-17}+0.\,i,\ 3.1225\times10^{-17}+0.\,i,\ 0.+0.\,i,\ 0.+0.\,i,$
$3.1225\times10^{-17}+0.\,i,\ 1.38778\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,\ -5.55112\times10^{-17}+0.\,i,$
$4.51028\times10^{-17}-5.73721\times10^{-17}\,i,\ 1.38778\times10^{-17}+3.98701\times10^{-17}\,i,\ 1.+0.\,i,$
$-1.04083\times10^{-17}-4.51867\times10^{-17}\,i,\ 3.46945\times10^{-17}+6.03774\times10^{-17}\,i,\ 4.16334\times10^{-17}+0.\,i,$
$-2.77556\times10^{-17}+3.38604\times10^{-17}\,i,\ -4.85723\times10^{-17}+4.51867\times10^{-17}\,i,\ 1.73472\times10^{-17}+0.\,i,$
$-6.245\times10^{-17}-3.98701\times10^{-17}\,i,\ -3.1225\times10^{-17}-3.0856\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{5.55112\times10^{-17}+3.93438\times10^{-17}\,i,\ 5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ 5.55112\times10^{-17}+1.51424\times10^{-17}\,i,$
$5.55112\times10^{-17}+3.93438\times10^{-17}\,i,\ -2.77556\times10^{-17}-1.51424\times10^{-17}\,i,\ 4.16334\times10^{-17}-3.93438\times10^{-17}\,i,$
$5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ 5.20417\times10^{-17}-4.21822\times10^{-17}\,i,\ 5.20417\times10^{-17}-4.21822\times10^{-17}\,i,$
$5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ -6.245\times10^{-17}+4.21822\times10^{-17}\,i,\ -2.77556\times10^{-17}-1.51424\times10^{-17}\,i,$
$4.16334\times10^{-17}-3.93438\times10^{-17}\,i,\ -1.04083\times10^{-17}+3.91778\times10^{-17}\,i,$
$2.08167\times10^{-17}-8.57384\times10^{-17}\,i,\ -2.77556\times10^{-17}-1.51424\times10^{-17}\,i,\ 1.-3.27641\times10^{-17}\,i,$
$-3.46945\times10^{-17}+1.15479\times10^{-18}\,i,\ -2.77556\times10^{-17}-1.51424\times10^{-17}\,i,$
$-4.16334\times10^{-17}+1.52607\times10^{-17}\,i,\ -1.04083\times10^{-17}+3.91778\times10^{-17}\,i,\ -6.245\times10^{-17}+4.21822\times10^{-17}\,i,$
$-3.46945\times10^{-17}+1.15479\times10^{-18}\,i,\ 0.-2.68083\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ 4.16334\times10^{-17}-3.93438\times10^{-17}\,i,\ 4.16334\times10^{-17}-3.93438\times10^{-17}\,i,$
$4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ 5.55112\times10^{-17}+3.93438\times10^{-17}\,i,\ -5.55112\times10^{-17}+1.51424\times10^{-17}\,i,$
$-5.20417\times10^{-17}+1.51424\times10^{-17}\,i,\ -6.245\times10^{-17}-4.21822\times10^{-17}\,i,\ -6.245\times10^{-17}-4.21822\times10^{-17}\,i,$
$-5.20417\times10^{-17}+1.51424\times10^{-17}\,i,\ 4.51028\times10^{-17}+4.21822\times10^{-17}\,i,$
$4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ -5.55112\times10^{-17}+1.51424\times10^{-17}\,i,$
$2.08167\times10^{-17}+9.01362\times10^{-17}\,i,\ -6.93889\times10^{-18}-4.62291\times10^{-17}\,i,\ 5.55112\times10^{-17}+3.93438\times10^{-17}\,i,$
$-5.55112\times10^{-17}+6.00885\times10^{-18}\,i,\ 1.+4.09735\times10^{-17}\,i,\ 4.16334\times10^{-17}-1.51424\times10^{-17}\,i,$
$6.93889\times10^{-18}-2.38039\times10^{-17}\,i,\ -1.38778\times10^{-17}-6.00885\times10^{-18}\,i,\ 4.51028\times10^{-17}+4.21822\times10^{-17}\,i,$
$-4.85723\times10^{-17}+4.62291\times10^{-17}\,i,\ 2.08167\times10^{-17}+9.25186\times10^{-18}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{2.08167\times10^{-17}+0.\,i,\ 3.1225\times10^{-17}+0.\,i,\ 3.1225\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,$
$2.77556\times10^{-17}+0.\,i,\ -2.77556\times10^{-17}+0.\,i,\ 6.245\times10^{-17}+0.\,i,\ 0.+0.\,i,\ 0.+0.\,i,$
$6.245\times10^{-17}+0.\,i,\ 1.38778\times10^{-17}+0.\,i,\ -3.46945\times10^{-17}+0.\,i,\ -5.55112\times10^{-17}+0.\,i,$
$-4.85723\times10^{-17}+4.51867\times10^{-17}\,i,\ 1.38778\times10^{-17}+3.98701\times10^{-17}\,i,\ 4.16334\times10^{-17}+0.\,i,$
$-1.04083\times10^{-17}-4.51867\times10^{-17}\,i,\ -6.245\times10^{-17}-3.98701\times10^{-17}\,i,\ 1.+0.\,i,$
$-2.77556\times10^{-17}+3.38604\times10^{-17}\,i,\ 4.51028\times10^{-17}-5.73721\times10^{-17}\,i,\ 1.73472\times10^{-17}+0.\,i,$
$3.46945\times10^{-17}+6.03774\times10^{-17}\,i,\ -3.1225\times10^{-17}-3.0856\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{4.16334\times10^{-17}+4.21848\times10^{-17}\,i,\ -3.81639\times10^{-17}-4.21848\times10^{-17}\,i,\ -3.81639\times10^{-17}-4.21848\times10^{-17}\,i,$
$4.16334\times10^{-17}+4.21848\times10^{-17}\,i,\ 6.93889\times10^{-17}+4.21848\times10^{-17}\,i,\ -4.16334\times10^{-17}-4.21848\times10^{-17}\,i,$
$-3.81639\times10^{-17}-4.21848\times10^{-17}\,i,\ 5.89806\times10^{-17}-3.93464\times10^{-17}\,i,$
$5.89806\times10^{-17}-3.93464\times10^{-17}\,i,\ -3.81639\times10^{-17}-4.21848\times10^{-17}\,i,$
$3.46945\times10^{-18}+3.93464\times10^{-17}\,i,\ 6.93889\times10^{-17}+4.21848\times10^{-17}\,i,\ -4.16334\times10^{-17}-4.21848\times10^{-17}\,i,$
$6.93889\times10^{-17}-6.00885\times10^{-18}\,i,\ 3.46945\times10^{-17}+1.10913\times10^{-17}\,i,\ 6.93889\times10^{-17}+4.21848\times10^{-17}\,i,$
$-6.245\times10^{-17}+6.00885\times10^{-18}\,i,\ -2.08167\times10^{-17}-1.10913\times10^{-17}\,i,\ 6.93889\times10^{-17}+4.21848\times10^{-17}\,i,$
$1.-1.02095\times10^{-16}\,i,\ 6.93889\times10^{-17}-6.00885\times10^{-18}\,i,\ 3.46945\times10^{-18}+3.93464\times10^{-17}\,i,$
$-2.08167\times10^{-17}-1.10913\times10^{-17}\,i,\ 1.52656\times10^{-16}-3.4193\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ -4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ -4.16334\times10^{-17}-1.51424\times10^{-17}\,i,$
$5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ 2.77556\times10^{-17}+1.51424\times10^{-17}\,i,\ 6.93889\times10^{-18}-1.51424\times10^{-17}\,i,$
$4.16334\times10^{-17}+3.93438\times10^{-17}\,i,\ -5.20417\times10^{-17}+4.21822\times10^{-17}\,i,$
$-5.20417\times10^{-17}+4.21822\times10^{-17}\,i,\ 4.16334\times10^{-17}+3.93438\times10^{-17}\,i,\ 6.245\times10^{-17}-4.21822\times10^{-17}\,i,$
$2.08167\times10^{-17}-3.93438\times10^{-17}\,i,\ 6.93889\times10^{-18}-1.51424\times10^{-17}\,i,\ 6.93889\times10^{-17}-3.91778\times10^{-17}\,i,$
$-2.77556\times10^{-17}+1.15479\times10^{-18}\,i,\ 2.77556\times10^{-17}+1.51424\times10^{-17}\,i,$
$-5.20417\times10^{-17}+3.91778\times10^{-17}\,i,\ -6.93889\times10^{-18}-1.15479\times10^{-18}\,i,\ 2.08167\times10^{-17}-3.93438\times10^{-17}\,i,$
$1.38778\times10^{-17}-1.52607\times10^{-17}\,i,\ 1.-3.27641\times10^{-17}\,i,\ 6.245\times10^{-17}-4.21822\times10^{-17}\,i,$
$2.77556\times10^{-17}-8.57384\times10^{-17}\,i,\ 3.46945\times10^{-18}+2.68083\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-1.38778\times10^{-17}+0.\,i,\ 2.42861\times10^{-17}+0.\,i,\ 2.42861\times10^{-17}+0.\,i,\ -1.38778\times10^{-17}+0.\,i,$
$2.77556\times10^{-17}+0.\,i,\ -1.38778\times10^{-17}+0.\,i,\ 2.42861\times10^{-17}+0.\,i,\ 7.28584\times10^{-17}+0.\,i,$
$7.28584\times10^{-17}+0.\,i,\ 2.42861\times10^{-17}+0.\,i,\ -4.16334\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,$
$5.55112\times10^{-17}+0.\,i,\ -6.93889\times10^{-18}-1.21354\times10^{-17}\,i,\ 4.16334\times10^{-17}-1.74537\times10^{-17}\,i,$
$1.38778\times10^{-17}+0.\,i,\ 3.46945\times10^{-18}+1.21354\times10^{-17}\,i,\ -6.93889\times10^{-18}+1.74537\times10^{-17}\,i,$
$1.38778\times10^{-17}+0.\,i,\ 1.38778\times10^{-17}+4.62802\times10^{-17}\,i,\ -6.93889\times10^{-18}-1.21354\times10^{-17}\,i,\ 1.+0.\,i,$
$-6.93889\times10^{-18}+1.74537\times10^{-17}\,i,\ 2.77556\times10^{-17}-4.92846\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ -5.20417\times10^{-17}+1.51424\times10^{-17}\,i,\ -5.20417\times10^{-17}+1.51424\times10^{-17}\,i,$

$4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ 4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ -5.55112\times10^{-17}+1.51424\times10^{-17}\,i,$
$4.16334\times10^{-17}-3.93438\times10^{-17}\,i,\ -6.245\times10^{-17}-4.21822\times10^{-17}\,i,\ -6.245\times10^{-17}-4.21822\times10^{-17}\,i,$
$4.16334\times10^{-17}-3.93438\times10^{-17}\,i,\ 4.51028\times10^{-17}+4.21822\times10^{-17}\,i,\ 5.55112\times10^{-17}+3.93438\times10^{-17}\,i,$
$-5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ -1.38778\times10^{-17}-6.00885\times10^{-18}\,i,$
$-6.93889\times10^{-18}-4.62291\times10^{-17}\,i,\ 4.16334\times10^{-17}-1.51424\times10^{-17}\,i,$
$-5.55112\times10^{-17}+6.00885\times10^{-18}\,i,\ -4.85723\times10^{-17}+4.62291\times10^{-17}\,i,\ 5.55112\times10^{-17}+3.93438\times10^{-17}\,i,$
$6.93889\times10^{-18}-2.38039\times10^{-17}\,i,\ 2.08167\times10^{-17}+9.01362\times10^{-17}\,i,\ 4.51028\times10^{-17}+4.21822\times10^{-17}\,i,$
$1.+4.09735\times10^{-17}\,i,\ 2.08167\times10^{-17}+9.25186\times10^{-18}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{5.55112\times10^{-17}-4.21848\times10^{-17}\,i,\ -1.38778\times10^{-17}+4.21848\times10^{-17}\,i,\ -1.38778\times10^{-17}+4.21848\times10^{-17}\,i,$
$5.55112\times10^{-17}-4.21848\times10^{-17}\,i,\ 5.55112\times10^{-17}-4.21848\times10^{-17}\,i,\ -4.85723\times10^{-17}+4.21848\times10^{-17}\,i,$
$-1.38778\times10^{-17}+4.21848\times10^{-17}\,i,\ 3.81639\times10^{-17}+3.93464\times10^{-17}\,i,\ 3.81639\times10^{-17}+3.93464\times10^{-17}\,i,$
$-1.38778\times10^{-17}+4.21848\times10^{-17}\,i,\ -6.93889\times10^{-18}-3.93464\times10^{-17}\,i,$
$5.55112\times10^{-17}-4.21848\times10^{-17}\,i,\ -4.85723\times10^{-17}+4.21848\times10^{-17}\,i,$
$6.93889\times10^{-18}+1.81443\times10^{-17}\,i,\ -6.245\times10^{-17}+1.15818\times10^{-18}\,i,\ 5.55112\times10^{-17}-4.21848\times10^{-17}\,i,$
$1.73472\times10^{-17}-1.81443\times10^{-17}\,i,\ 8.32667\times10^{-17}-1.15818\times10^{-18}\,i,\ 5.55112\times10^{-17}-4.21848\times10^{-17}\,i,$
$1.66533\times10^{-16}+4.02019\times10^{-17}\,i,\ 6.93889\times10^{-18}+1.81443\times10^{-17}\,i,\ -6.93889\times10^{-18}-3.93464\times10^{-17}\,i,$
$8.32667\times10^{-17}-1.15818\times10^{-18}\,i,\ 1.+9.90901\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 0.+0.\,i,\ 1.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 0.+0.\,i,\ 0.,\ 1.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 0.+0.\,i,\ 0.,\ 0.,\ 1.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 0.+0.\,i,\ 0.,\ 0.,\ 0.,\ 1.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 0.+0.\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 1.,\ 0.,\ 0.,\ 0.\},$
$\{-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 0.+0.\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 1.,\ 0.,\ 0.\},$
$\{-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 0.+0.\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 1.,\ 0.\},$
$\{-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 0.+0.\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 1.\}\}$

~ ~

Here we give $S^2 - (ST)^3$ as a numerical approximation and include the result directly as the detail needs a huge number of pages. One can check one entry for the exact solution using hand calculation. Note that all the entries are zero up to rounding errors, typically 10^{17} .

```
In[16]:= r = rf - s2 // N

Out[16]= {{0. + 0. i, -2.77556×10^-17 + 0. i, -2.77556×10^-17 + 0. i, -6.93889×10^-18 + 0. i, 2.77556×10^-17 + 0. i,
   4.16334×10^-17 + 0. i, -2.77556×10^-17 + 0. i, 3.46945×10^-18 + 0. i, 3.46945×10^-18 + 0. i,
   -2.77556×10^-17 + 0. i, 6.93889×10^-18 + 0. i, 2.77556×10^-17 + 0. i, 4.16334×10^-17 + 0. i,
   2.42861×10^-17 + 4.51867×10^-17 i, -3.46945×10^-17 - 6.03774×10^-17 i, 2.77556×10^-17 + 0. i,
   -4.51028×10^-17 + 5.73721×10^-17 i, 3.46945×10^-17 - 3.98701×10^-17 i, 2.77556×10^-17 + 0. i,
   4.16334×10^-17 + 3.38604×10^-17 i, 2.42861×10^-17 + 4.51867×10^-17 i, 6.93889×10^-18 + 0. i,
   3.46945×10^-17 - 3.98701×10^-17 i, 2.77556×10^-17 - 3.0856×10^-17 i, 0., 0., 0., 0., 0., 0., 0., 0.},
  {-3.46945×10^-17 + 0. i, 0. + 0. i, -5.55112×10^-17 + 0. i, -3.46945×10^-17 + 0. i, 4.85723×10^-17 + 0. i,
   2.77556×10^-17 + 0. i, 2.77556×10^-17 + 0. i, 2.08167×10^-17 + 0. i, 2.08167×10^-17 + 0. i,
   2.77556×10^-17 + 0. i, -2.08167×10^-17 + 0. i, -1.38778×10^-17 + 0. i, 2.77556×10^-17 + 0. i,
   -4.51028×10^-17 + 5.73721×10^-17 i, -6.93889×10^-18 - 3.98701×10^-17 i, 4.85723×10^-17 + 0. i,
   2.42861×10^-17 + 4.51867×10^-17 i, -2.08167×10^-17 - 6.03774×10^-17 i, -1.38778×10^-17 + 0. i,
   -1.38778×10^-17 - 3.38604×10^-17 i, -3.46945×10^-17 - 4.51867×10^-17 i, -2.08167×10^-17 + 0. i,
   -2.77556×10^-17 + 3.98701×10^-17 i, -1.04083×10^-17 + 3.0856×10^-17 i, 0., 0., 0., 0., 0., 0., 0., 0.},
  {-3.46945×10^-17 + 0. i, -5.55112×10^-17 + 0. i, 0. + 0. i, -3.46945×10^-17 + 0. i, 4.85723×10^-17 + 0. i,
   2.77556×10^-17 + 0. i, 2.77556×10^-17 + 0. i, 2.08167×10^-17 + 0. i, 2.08167×10^-17 + 0. i,
   2.77556×10^-17 + 0. i, -2.08167×10^-17 + 0. i, -1.38778×10^-17 + 0. i, 2.77556×10^-17 + 0. i,
   -4.51028×10^-17 + 5.73721×10^-17 i, -6.93889×10^-18 - 3.98701×10^-17 i, 4.85723×10^-17 + 0. i,
   2.42861×10^-17 + 4.51867×10^-17 i, -2.08167×10^-17 - 6.03774×10^-17 i, -1.38778×10^-17 + 0. i,
   -1.38778×10^-17 - 3.38604×10^-17 i, -3.46945×10^-17 - 4.51867×10^-17 i, -2.08167×10^-17 + 0. i,
   -2.77556×10^-17 + 3.98701×10^-17 i, -1.04083×10^-17 + 3.0856×10^-17 i, 0., 0., 0., 0., 0., 0., 0., 0.},
  {-6.93889×10^-18 + 0. i, -2.77556×10^-17 + 0. i, -2.77556×10^-17 + 0. i, 0. + 0. i, 2.77556×10^-17 + 0. i,
   4.16334×10^-17 + 0. i, -2.77556×10^-17 + 0. i, 3.46945×10^-18 + 0. i, 3.46945×10^-18 + 0. i,
   -2.77556×10^-17 + 0. i, 6.93889×10^-18 + 0. i, 2.77556×10^-17 + 0. i, 4.16334×10^-17 + 0. i,
   2.42861×10^-17 + 4.51867×10^-17 i, -3.46945×10^-17 - 6.03774×10^-17 i, 2.77556×10^-17 + 0. i,
   -4.51028×10^-17 + 5.73721×10^-17 i, 3.46945×10^-17 - 3.98701×10^-17 i, 2.77556×10^-17 + 0. i,
   4.16334×10^-17 + 3.38604×10^-17 i, 2.42861×10^-17 + 4.51867×10^-17 i, 6.93889×10^-18 + 0. i,
   3.46945×10^-17 - 3.98701×10^-17 i, 2.77556×10^-17 - 3.0856×10^-17 i, 0., 0., 0., 0., 0., 0., 0., 0.},
  {2.08167×10^-17 + 0. i, 6.245×10^-17 + 0. i, 6.245×10^-17 + 0. i, 2.08167×10^-17 + 0. i,
   0. + 0. i, -5.55112×10^-17 + 0. i, 3.1225×10^-17 + 0. i, 0. + 0. i, 0. + 0. i,
   3.1225×10^-17 + 0. i, 1.73472×10^-17 + 0. i, 4.16334×10^-17 + 0. i, -2.77556×10^-17 + 0. i,
   4.51028×10^-17 - 5.73721×10^-17 i, 1.04083×10^-17 + 3.98701×10^-17 i, -3.46945×10^-17 + 0. i,
   -1.04083×10^-17 - 4.51867×10^-17 i, 3.46945×10^-17 + 6.03774×10^-17 i, 2.77556×10^-17 + 0. i,
   -3.46945×10^-18 + 3.38604×10^-17 i, 6.93889×10^-18 + 4.51867×10^-17 i, 1.38778×10^-17 + 0. i,
   -2.08167×10^-17 - 3.98701×10^-17 i, -6.93889×10^-18 - 3.0856×10^-17 i, 0., 0., 0., 0., 0., 0., 0., 0.},
  {1.04083×10^-17 + 0. i, 2.08167×10^-17 + 0. i, 2.08167×10^-17 + 0. i, 1.04083×10^-17 + 0. i,
   -5.55112×10^-17 + 0. i, 0. + 0. i, 2.08167×10^-17 + 0. i, -2.42861×10^-17 + 0. i, -2.42861×10^-17 + 0. i,
   2.08167×10^-17 + 0. i, -4.51028×10^-17 + 0. i, -5.55112×10^-17 + 0. i, -3.46945×10^-17 + 0. i,
   -1.04083×10^-17 - 4.51867×10^-17 i, 2.77556×10^-17 + 6.03774×10^-17 i, -2.08167×10^-17 + 0. i,
   4.51028×10^-17 - 5.73721×10^-17 i, 6.93889×10^-18 + 3.98701×10^-17 i, -2.08167×10^-17 + 0. i,
   -1.38778×10^-17 - 3.38604×10^-17 i, -1.04083×10^-17 - 4.51867×10^-17 i, -1.73472×10^-17 + 0. i,
   6.93889×10^-18 + 3.98701×10^-17 i, 0. + 3.0856×10^-17 i, 0., 0., 0., 0., 0., 0., 0., 0.},
  {-3.46945×10^-17 + 0. i, 2.77556×10^-17 + 0. i, 2.77556×10^-17 + 0. i, -3.46945×10^-17 + 0. i,
   -1.38778×10^-17 + 0. i, 2.77556×10^-17 + 0. i, 0. + 0. i, 2.08167×10^-17 + 0. i, 2.08167×10^-17 + 0. i,
   -5.55112×10^-17 + 0. i, -2.08167×10^-17 + 0. i, 4.85723×10^-17 + 0. i, 2.77556×10^-17 + 0. i,
   -3.46945×10^-17 - 4.51867×10^-17 i, -6.93889×10^-18 - 3.98701×10^-17 i, -1.38778×10^-17 + 0. i,
   2.42861×10^-17 + 4.51867×10^-17 i, -2.77556×10^-17 + 3.98701×10^-17 i, 4.85723×10^-17 + 0. i,
   -1.38778×10^-17 - 3.38604×10^-17 i, -4.51028×10^-17 + 5.73721×10^-17 i, -2.08167×10^-17 + 0. i,
   -2.08167×10^-17 - 6.03774×10^-17 i, -1.04083×10^-17 + 3.0856×10^-17 i, 0., 0., 0., 0., 0., 0., 0., 0.},
  {-1.38778×10^-17 + 0. i, 2.77556×10^-17 + 0. i, 2.77556×10^-17 + 0. i, -1.38778×10^-17 + 0. i,
   -6.93889×10^-18 + 0. i, 1.38778×10^-17 + 0. i, 2.77556×10^-17 + 0. i, 0. + 0. i, -1.04083×10^-17 + 0. i,
   2.77556×10^-17 + 0. i, 4.85723×10^-17 + 0. i, -6.93889×10^-18 + 0. i, 1.38778×10^-17 + 0. i,
   -3.46945×10^-17 + 1.21354×10^-17 i, 2.08167×10^-17 + 1.74537×10^-17 i, -6.93889×10^-18 + 0. i,
   2.42861×10^-17 - 1.21354×10^-17 i, -2.77556×10^-17 - 1.74537×10^-17 i, -6.93889×10^-18 + 0. i,
   -1.38778×10^-17 - 4.62802×10^-17 i, -3.46945×10^-17 + 1.21354×10^-17 i, 4.85723×10^-17 + 0. i,
   -2.77556×10^-17 - 1.74537×10^-17 i, -1.38778×10^-17 + 4.92846×10^-17 i, 0., 0., 0., 0., 0., 0., 0., 0.},
  {-1.38778×10^-17 + 0. i, 2.77556×10^-17 + 0. i, 2.77556×10^-17 + 0. i, -1.38778×10^-17 + 0. i,
   -6.93889×10^-18 + 0. i, 1.38778×10^-17 + 0. i, 2.77556×10^-17 + 0. i, -1.04083×10^-17 + 0. i,
   0. + 0. i, 2.77556×10^-17 + 0. i, 4.85723×10^-17 + 0. i, -6.93889×10^-18 + 0. i, 1.38778×10^-17 + 0. i,
   -3.46945×10^-17 + 1.21354×10^-17 i, 2.08167×10^-17 + 1.74537×10^-17 i, -6.93889×10^-18 + 0. i,
   2.42861×10^-17 - 1.21354×10^-17 i, -2.77556×10^-17 - 1.74537×10^-17 i, -6.93889×10^-18 + 0. i,
   -1.38778×10^-17 - 4.62802×10^-17 i, -3.46945×10^-17 + 1.21354×10^-17 i, 4.85723×10^-17 + 0. i,
   -2.77556×10^-17 - 1.74537×10^-17 i, -1.38778×10^-17 + 4.92846×10^-17 i, 0., 0., 0., 0., 0., 0., 0., 0.},
  {-3.46945×10^-17 + 0. i, 2.77556×10^-17 + 0. i, 2.77556×10^-17 + 0. i, -3.46945×10^-17 + 0. i,
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$-1.38778\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,\ -5.55112\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,$
$2.08167\times10^{-17}+0.\,i,\ 0.+0.\,i,\ -2.08167\times10^{-17}+0.\,i,\ 4.85723\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,$
$-3.46945\times10^{-17}-4.51867\times10^{-17}\,i,\ -6.93889\times10^{-18}-3.98701\times10^{-17}\,i,\ -1.38778\times10^{-17}+0.\,i,$
$2.42861\times10^{-17}+4.51867\times10^{-17}\,i,\ -2.77556\times10^{-17}+3.98701\times10^{-17}\,i,\ 4.85723\times10^{-17}+0.\,i,$
$-1.38778\times10^{-17}-3.38604\times10^{-17}\,i,\ -4.51028\times10^{-17}+5.73721\times10^{-17}\,i,\ -2.08167\times10^{-17}+0.\,i,$
$-2.08167\times10^{-17}-6.03774\times10^{-17}\,i,\ -1.04083\times10^{-17}+3.0856\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-1.38778\times10^{-17}+0.\,i,\ 2.42861\times10^{-17}+0.\,i,\ 2.42861\times10^{-17}+0.\,i,\ -1.38778\times10^{-17}+0.\,i,$
$1.38778\times10^{-17}+0.\,i,\ 5.55112\times10^{-17}+0.\,i,\ 2.42861\times10^{-17}+0.\,i,\ 7.28584\times10^{-17}+0.\,i,$
$7.28584\times10^{-17}+0.\,i,\ 2.42861\times10^{-17}+0.\,i,\ 0.+0.\,i,\ 1.38778\times10^{-17}+0.\,i,\ -1.38778\times10^{-17}+0.\,i,$
$6.93889\times10^{-18}-1.21354\times10^{-17}\,i,\ -1.38778\times10^{-17}-1.74537\times10^{-17}\,i,\ 2.77556\times10^{-17}+0.\,i,$
$-1.04083\times10^{-17}+1.21354\times10^{-17}\,i,\ -6.93889\times10^{-18}+1.74537\times10^{-17}\,i,\ 2.77556\times10^{-17}+0.\,i,$
$1.38778\times10^{-17}+4.62802\times10^{-17}\,i,\ 6.93889\times10^{-18}-1.21354\times10^{-17}\,i,\ -4.16334\times10^{-17}+0.\,i,$
$-6.93889\times10^{-18}+1.74537\times10^{-17}\,i,\ 2.77556\times10^{-17}-4.92846\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{2.08167\times10^{-17}+0.\,i,\ 3.1225\times10^{-17}+0.\,i,\ 3.1225\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,$
$4.16334\times10^{-17}+0.\,i,\ -5.55112\times10^{-17}+0.\,i,\ 6.245\times10^{-17}+0.\,i,\ 0.+0.\,i,$
$0.+0.\,i,\ 6.245\times10^{-17}+0.\,i,\ 1.73472\times10^{-17}+0.\,i,\ 0.+0.\,i,\ -2.77556\times10^{-17}+0.\,i,$
$6.93889\times10^{-18}+4.51867\times10^{-17}\,i,\ 1.04083\times10^{-17}+3.98701\times10^{-17}\,i,\ 2.77556\times10^{-17}+0.\,i,$
$-1.04083\times10^{-17}-4.51867\times10^{-17}\,i,\ -2.08167\times10^{-17}-3.98701\times10^{-17}\,i,\ -3.46945\times10^{-17}+0.\,i,$
$-3.46945\times10^{-18}+3.38604\times10^{-17}\,i,\ 4.51028\times10^{-17}-5.73721\times10^{-17}\,i,\ 1.38778\times10^{-17}+0.\,i,$
$3.46945\times10^{-17}+6.03774\times10^{-17}\,i,\ -6.93889\times10^{-18}-3.0856\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{1.04083\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,\ 1.04083\times10^{-17}+0.\,i,$
$-2.08167\times10^{-17}+0.\,i,\ -3.46945\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,\ -2.42861\times10^{-17}+0.\,i,$
$-2.42861\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,\ -1.73472\times10^{-17}+0.\,i,\ -2.08167\times10^{-17}+0.\,i,\ 0.+0.\,i,$
$-1.04083\times10^{-17}-4.51867\times10^{-17}\,i,\ 2.77556\times10^{-17}+6.03774\times10^{-17}\,i,\ -5.55112\times10^{-17}+0.\,i,$
$4.51028\times10^{-17}-5.73721\times10^{-17}\,i,\ 3.46945\times10^{-17}+3.98701\times10^{-17}\,i,\ -5.55112\times10^{-17}+0.\,i,$
$-1.38778\times10^{-17}-3.38604\times10^{-17}\,i,\ -1.04083\times10^{-17}-4.51867\times10^{-17}\,i,\ -4.51028\times10^{-17}+0.\,i,$
$3.46945\times10^{-17}+3.98701\times10^{-17}\,i,\ 0.+3.0856\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ 4.16334\times10^{-17}+3.93438\times10^{-17}\,i,\ 4.16334\times10^{-17}+3.93438\times10^{-17}\,i,$
$5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ 2.08167\times10^{-17}-3.93438\times10^{-17}\,i,\ 6.93889\times10^{-18}-1.51424\times10^{-17}\,i,$
$-4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ -5.20417\times10^{-17}+4.21822\times10^{-17}\,i,$
$-5.20417\times10^{-17}+4.21822\times10^{-17}\,i,\ -4.16334\times10^{-17}-1.51424\times10^{-17}\,i,$
$6.245\times10^{-17}-4.21822\times10^{-17}\,i,\ 2.77556\times10^{-17}+1.51424\times10^{-17}\,i,\ 6.93889\times10^{-18}-1.51424\times10^{-17}\,i,$
$-2.22045\times10^{-16}-3.27641\times10^{-17}\,i,\ -2.77556\times10^{-17}+1.15479\times10^{-18}\,i,\ 2.08167\times10^{-17}-3.93438\times10^{-17}\,i,$
$-5.20417\times10^{-17}+3.91778\times10^{-17}\,i,\ 2.77556\times10^{-17}-8.57384\times10^{-17}\,i,\ 2.77556\times10^{-17}+1.51424\times10^{-17}\,i,$
$1.38778\times10^{-17}-1.52607\times10^{-17}\,i,\ 6.93889\times10^{-17}-3.91778\times10^{-17}\,i,\ 6.245\times10^{-17}-4.21822\times10^{-17}\,i,$
$-6.93889\times10^{-18}-1.15479\times10^{-18}\,i,\ 3.46945\times10^{-18}+2.68083\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{4.85723\times10^{-17}-3.93438\times10^{-17}\,i,\ 4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ 4.16334\times10^{-17}-1.51424\times10^{-17}\,i,$
$4.85723\times10^{-17}-3.93438\times10^{-17}\,i,\ -5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ 1.38778\times10^{-17}+3.93438\times10^{-17}\,i,$
$4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ 4.16334\times10^{-17}+4.21822\times10^{-17}\,i,\ 4.16334\times10^{-17}+4.21822\times10^{-17}\,i,$
$4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ -6.59195\times10^{-17}-4.21822\times10^{-17}\,i,\ -5.55112\times10^{-17}+1.51424\times10^{-17}\,i,$
$1.38778\times10^{-17}+3.93438\times10^{-17}\,i,\ -3.46945\times10^{-17}+6.00885\times10^{-18}\,i,\ -1.11022\times10^{-16}+4.09735\times10^{-17}\,i,$
$-5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ -6.93889\times10^{-18}+9.01362\times10^{-17}\,i,$
$6.93889\times10^{-18}-4.62291\times10^{-17}\,i,\ -5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ 0.+2.38039\times10^{-17}\,i,$
$-3.46945\times10^{-17}+6.00885\times10^{-18}\,i,\ -6.59195\times10^{-17}-4.21822\times10^{-17}\,i,$
$6.93889\times10^{-18}-4.62291\times10^{-17}\,i,\ -4.16334\times10^{-17}-9.25186\times10^{-18}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{2.08167\times10^{-17}+0.\,i,\ 6.245\times10^{-17}+0.\,i,\ 6.245\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,$
$-3.46945\times10^{-17}+0.\,i,\ -2.77556\times10^{-17}+0.\,i,\ 3.1225\times10^{-17}+0.\,i,\ 0.+0.\,i,\ 0.+0.\,i,$
$3.1225\times10^{-17}+0.\,i,\ 1.38778\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,\ -5.55112\times10^{-17}+0.\,i,$
$4.51028\times10^{-17}-5.73721\times10^{-17}\,i,\ 1.38778\times10^{-17}+3.98701\times10^{-17}\,i,\ 0.+0.\,i,$
$-1.04083\times10^{-17}-4.51867\times10^{-17}\,i,\ 3.46945\times10^{-17}+6.03774\times10^{-17}\,i,\ 4.16334\times10^{-17}+0.\,i,$
$-2.77556\times10^{-17}+3.38604\times10^{-17}\,i,\ -4.85723\times10^{-17}+4.51867\times10^{-17}\,i,\ 1.73472\times10^{-17}+0.\,i,$
$-6.245\times10^{-17}-3.98701\times10^{-17}\,i,\ -3.1225\times10^{-17}-3.0856\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{5.55112\times10^{-17}+3.93438\times10^{-17}\,i,\ 5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ 5.55112\times10^{-17}+1.51424\times10^{-17}\,i,$
$5.55112\times10^{-17}+3.93438\times10^{-17}\,i,\ -2.77556\times10^{-17}-1.51424\times10^{-17}\,i,\ 4.16334\times10^{-17}-3.93438\times10^{-17}\,i,$
$5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ 5.20417\times10^{-17}-4.21822\times10^{-17}\,i,\ 5.20417\times10^{-17}-4.21822\times10^{-17}\,i,$
$5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ -6.245\times10^{-17}+4.21822\times10^{-17}\,i,\ -2.77556\times10^{-17}-1.51424\times10^{-17}\,i,$
$4.16334\times10^{-17}-3.93438\times10^{-17}\,i,\ -1.04083\times10^{-17}+3.91778\times10^{-17}\,i,\ 2.08167\times10^{-17}-8.57384\times10^{-17}\,i,$
$-2.77556\times10^{-17}-1.51424\times10^{-17}\,i,\ -1.11022\times10^{-16}-3.27641\times10^{-17}\,i,$
$-3.46945\times10^{-17}+1.15479\times10^{-18}\,i,\ -2.77556\times10^{-17}-1.51424\times10^{-17}\,i,$
$-4.16334\times10^{-17}+1.52607\times10^{-17}\,i,\ -1.04083\times10^{-17}+3.91778\times10^{-17}\,i,\ -6.245\times10^{-17}+4.21822\times10^{-17}\,i,$
$-3.46945\times10^{-17}+1.15479\times10^{-18}\,i,\ 0.-2.68083\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ 4.16334\times10^{-17}-3.93438\times10^{-17}\,i,\ 4.16334\times10^{-17}-3.93438\times10^{-17}\,i,$
$4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ 5.55112\times10^{-17}+3.93438\times10^{-17}\,i,\ -5.55112\times10^{-17}+1.51424\times10^{-17}\,i,$
$-5.20417\times10^{-17}+1.51424\times10^{-17}\,i,\ -6.245\times10^{-17}-4.21822\times10^{-17}\,i,\ -6.245\times10^{-17}-4.21822\times10^{-17}\,i,$
$-5.20417\times10^{-17}+1.51424\times10^{-17}\,i,\ 4.51028\times10^{-17}+4.21822\times10^{-17}\,i,$
$4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ -5.55112\times10^{-17}+1.51424\times10^{-17}\,i,$
$2.08167\times10^{-17}+9.01362\times10^{-17}\,i,\ -6.93889\times10^{-18}-4.62291\times10^{-17}\,i,\ 5.55112\times10^{-17}+3.93438\times10^{-17}\,i,$
$-5.55112\times10^{-17}+6.00885\times10^{-18}\,i,\ -2.22045\times10^{-16}+4.09735\times10^{-17}\,i,\ 4.16334\times10^{-17}-1.51424\times10^{-17}\,i,$
$6.93889\times10^{-18}-2.38039\times10^{-17}\,i,\ -1.38778\times10^{-17}-6.00885\times10^{-18}\,i,\ 4.51028\times10^{-17}+4.21822\times10^{-17}\,i,$
$-4.85723\times10^{-17}+4.62291\times10^{-17}\,i,\ 2.08167\times10^{-17}+9.25186\times10^{-18}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{2.08167\times10^{-17}+0.\,i,\ 3.1225\times10^{-17}+0.\,i,\ 3.1225\times10^{-17}+0.\,i,\ 2.08167\times10^{-17}+0.\,i,$
$2.77556\times10^{-17}+0.\,i,\ -2.77556\times10^{-17}+0.\,i,\ 6.245\times10^{-17}+0.\,i,\ 0.+0.\,i,\ 0.+0.\,i,$
$6.245\times10^{-17}+0.\,i,\ 1.38778\times10^{-17}+0.\,i,\ -3.46945\times10^{-17}+0.\,i,\ -5.55112\times10^{-17}+0.\,i,$
$-4.85723\times10^{-17}+4.51867\times10^{-17}\,i,\ 1.38778\times10^{-17}+3.98701\times10^{-17}\,i,\ 4.16334\times10^{-17}+0.\,i,$
$-1.04083\times10^{-17}-4.51867\times10^{-17}\,i,\ -6.245\times10^{-17}-3.98701\times10^{-17}\,i,\ 0.+0.\,i,$
$-2.77556\times10^{-17}+3.38604\times10^{-17}\,i,\ 4.51028\times10^{-17}-5.73721\times10^{-17}\,i,\ 1.73472\times10^{-17}+0.\,i,$

$3.46945\times10^{-17}+6.03774\times10^{-17}\,i,\ -3.1225\times10^{-17}-3.0856\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{4.16334\times10^{-17}+4.21848\times10^{-17}\,i,\ -3.81639\times10^{-17}-4.21848\times10^{-17}\,i,\ -3.81639\times10^{-17}-4.21848\times10^{-17}\,i,$
$4.16334\times10^{-17}+4.21848\times10^{-17}\,i,\ 6.93889\times10^{-17}+4.21848\times10^{-17}\,i,\ -4.16334\times10^{-17}-4.21848\times10^{-17}\,i,$
$-3.81639\times10^{-17}-4.21848\times10^{-17}\,i,\ 5.89806\times10^{-17}-3.93464\times10^{-17}\,i,$
$5.89806\times10^{-17}-3.93464\times10^{-17}\,i,\ -3.81639\times10^{-17}-4.21848\times10^{-17}\,i,$
$3.46945\times10^{-18}+3.93464\times10^{-17}\,i,\ 6.93889\times10^{-17}+4.21848\times10^{-17}\,i,\ -4.16334\times10^{-17}-4.21848\times10^{-17}\,i,$
$6.93889\times10^{-17}-6.00885\times10^{-18}\,i,\ 3.46945\times10^{-17}+1.10913\times10^{-17}\,i,\ 6.93889\times10^{-17}+4.21848\times10^{-17}\,i,$
$-6.245\times10^{-17}+6.00885\times10^{-18}\,i,\ -2.08167\times10^{-17}-1.10913\times10^{-17}\,i,\ 6.93889\times10^{-17}+4.21848\times10^{-17}\,i,$
$0.-1.02095\times10^{-16}\,i,\ 6.93889\times10^{-17}-6.00885\times10^{-18}\,i,\ 3.46945\times10^{-18}+3.93464\times10^{-17}\,i,$
$-2.08167\times10^{-17}-1.10913\times10^{-17}\,i,\ 1.52656\times10^{-16}-3.4193\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ -4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ -4.16334\times10^{-17}-1.51424\times10^{-17}\,i,$
$5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ 2.77556\times10^{-17}+1.51424\times10^{-17}\,i,\ 6.93889\times10^{-18}-1.51424\times10^{-17}\,i,$
$4.16334\times10^{-17}+3.93438\times10^{-17}\,i,\ -5.20417\times10^{-17}+4.21822\times10^{-17}\,i,$
$-5.20417\times10^{-17}+4.21822\times10^{-17}\,i,\ 4.16334\times10^{-17}+3.93438\times10^{-17}\,i,\ 6.245\times10^{-17}-4.21822\times10^{-17}\,i,$
$2.08167\times10^{-17}-3.93438\times10^{-17}\,i,\ 6.93889\times10^{-18}-1.51424\times10^{-17}\,i,\ 6.93889\times10^{-17}-3.91778\times10^{-17}\,i,$
$-2.77556\times10^{-17}+1.15479\times10^{-18}\,i,\ 2.77556\times10^{-17}+1.51424\times10^{-17}\,i,$
$-5.20417\times10^{-17}+3.91778\times10^{-17}\,i,\ -6.93889\times10^{-18}-1.15479\times10^{-18}\,i,\ 2.08167\times10^{-17}-3.93438\times10^{-17}\,i,$
$1.38778\times10^{-17}-1.52607\times10^{-17}\,i,\ -2.22045\times10^{-16}-3.27641\times10^{-17}\,i,\ 6.245\times10^{-17}-4.21822\times10^{-17}\,i,$
$2.77556\times10^{-17}-8.57384\times10^{-17}\,i,\ 3.46945\times10^{-18}+2.68083\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-1.38778\times10^{-17}+0.\,i,\ 2.42861\times10^{-17}+0.\,i,\ 2.42861\times10^{-17}+0.\,i,\ -1.38778\times10^{-17}+0.\,i,$
$2.77556\times10^{-17}+0.\,i,\ -1.38778\times10^{-17}+0.\,i,\ 2.42861\times10^{-17}+0.\,i,\ 7.28584\times10^{-17}+0.\,i,$
$7.28584\times10^{-17}+0.\,i,\ 2.42861\times10^{-17}+0.\,i,\ -4.16334\times10^{-17}+0.\,i,\ 2.77556\times10^{-17}+0.\,i,$
$5.55112\times10^{-17}+0.\,i,\ -6.93889\times10^{-18}-1.21354\times10^{-17}\,i,\ 4.16334\times10^{-17}-1.74537\times10^{-17}\,i,$
$1.38778\times10^{-17}+0.\,i,\ 3.46945\times10^{-18}+1.21354\times10^{-17}\,i,\ -6.93889\times10^{-18}+1.74537\times10^{-17}\,i,$
$1.38778\times10^{-17}+0.\,i,\ 1.38778\times10^{-17}+4.62802\times10^{-17}\,i,\ -6.93889\times10^{-18}-1.21354\times10^{-17}\,i,\ 0.+0.\,i,$
$-6.93889\times10^{-18}+1.74537\times10^{-17}\,i,\ 2.77556\times10^{-17}-4.92846\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ -5.20417\times10^{-17}+1.51424\times10^{-17}\,i,\ -5.20417\times10^{-17}+1.51424\times10^{-17}\,i,$
$4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ 4.16334\times10^{-17}-1.51424\times10^{-17}\,i,\ -5.55112\times10^{-17}+1.51424\times10^{-17}\,i,$
$4.16334\times10^{-17}-3.93438\times10^{-17}\,i,\ -6.245\times10^{-17}-4.21822\times10^{-17}\,i,\ -6.245\times10^{-17}-4.21822\times10^{-17}\,i,$
$4.16334\times10^{-17}-3.93438\times10^{-17}\,i,\ 4.51028\times10^{-17}+4.21822\times10^{-17}\,i,\ 5.55112\times10^{-17}+3.93438\times10^{-17}\,i,$
$-5.55112\times10^{-17}+1.51424\times10^{-17}\,i,\ -1.38778\times10^{-17}-6.00885\times10^{-18}\,i,$
$-6.93889\times10^{-18}-4.62291\times10^{-17}\,i,\ 4.16334\times10^{-17}-1.51424\times10^{-17}\,i,$
$-5.55112\times10^{-17}+6.00885\times10^{-18}\,i,\ -4.85723\times10^{-17}+4.62291\times10^{-17}\,i,\ 5.55112\times10^{-17}+3.93438\times10^{-17}\,i,$
$6.93889\times10^{-18}-2.38039\times10^{-17}\,i,\ 2.08167\times10^{-17}+9.01362\times10^{-17}\,i,\ 4.51028\times10^{-17}+4.21822\times10^{-17}\,i,$
$-2.22045\times10^{-16}+4.09735\times10^{-17}\,i,\ 2.08167\times10^{-17}+9.25186\times10^{-18}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{5.55112\times10^{-17}-4.21848\times10^{-17}\,i,\ -1.38778\times10^{-17}+4.21848\times10^{-17}\,i,\ -1.38778\times10^{-17}+4.21848\times10^{-17}\,i,$
$5.55112\times10^{-17}-4.21848\times10^{-17}\,i,\ 5.55112\times10^{-17}-4.21848\times10^{-17}\,i,\ -4.85723\times10^{-17}+4.21848\times10^{-17}\,i,$
$-1.38778\times10^{-17}+4.21848\times10^{-17}\,i,\ 3.81639\times10^{-17}+3.93464\times10^{-17}\,i,\ 3.81639\times10^{-17}+3.93464\times10^{-17}\,i,$
$-1.38778\times10^{-17}+4.21848\times10^{-17}\,i,\ -6.93889\times10^{-18}-3.93464\times10^{-17}\,i,$
$5.55112\times10^{-17}-4.21848\times10^{-17}\,i,\ -4.85723\times10^{-17}+4.21848\times10^{-17}\,i,$
$6.93889\times10^{-18}+1.81443\times10^{-17}\,i,\ -6.245\times10^{-17}+1.15818\times10^{-18}\,i,\ 5.55112\times10^{-17}-4.21848\times10^{-17}\,i,$
$1.73472\times10^{-17}-1.81443\times10^{-17}\,i,\ 8.32667\times10^{-17}-1.15818\times10^{-18}\,i,\ 5.55112\times10^{-17}-4.21848\times10^{-17}\,i,$
$1.66533\times10^{-16}+4.02019\times10^{-17}\,i,\ 6.93889\times10^{-18}+1.81443\times10^{-17}\,i,\ -6.93889\times10^{-18}-3.93464\times10^{-17}\,i,$
$8.32667\times10^{-17}-1.15818\times10^{-18}\,i,\ 0.+9.90901\times10^{-17}\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 0.+0.\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 0.+0.\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 0.+0.\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 0.+0.\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 6.93889\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 0.+0.\,i,\ 0.+0.\,i,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.\},$
$\{-3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$
$6.93889\times10^{-18}+0.\,i,\ 6.93889\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,\ -3.46945\times10^{-18}+0.\,i,$

-3.46945×10^{-18} + 0. i, -3.46945×10^{-18} + 0. i, 6.93889×10^{-18} + 0. i, 6.93889×10^{-18} + 0. i,
6.93889×10^{-18} + 0. i, 6.93889×10^{-18} + 0. i, 0. + 0. i, 6.93889×10^{-18} + 0. i,
6.93889×10^{-18} + 0. i, 0. + 0. i, 6.93889×10^{-18} + 0. i, 0. + 0. i, 6.93889×10^{-18} + 0. i,
6.93889×10^{-18} + 0. i, 0. + 0. i, 0. + 0. i, 0., 0., 0., 0., 0., 0., 0., 0.},
{-3.46945×10^{-18} + 0. i, -3.46945×10^{-18} + 0. i, -3.46945×10^{-18} + 0. i, -3.46945×10^{-18} + 0. i,
6.93889×10^{-18} + 0. i, 6.93889×10^{-18} + 0. i, -3.46945×10^{-18} + 0. i, -3.46945×10^{-18} + 0. i,
-3.46945×10^{-18} + 0. i, -3.46945×10^{-18} + 0. i, 6.93889×10^{-18} + 0. i, 6.93889×10^{-18} + 0. i,
6.93889×10^{-18} + 0. i, 6.93889×10^{-18} + 0. i, 0. + 0. i, 6.93889×10^{-18} + 0. i,
6.93889×10^{-18} + 0. i, 0. + 0. i, 6.93889×10^{-18} + 0. i, 0. + 0. i, 6.93889×10^{-18} + 0. i,
6.93889×10^{-18} + 0. i, 0. + 0. i, 0. + 0. i, 0., 0., 0., 0., 0., 0., 0., 0.},
{-3.46945×10^{-18} + 0. i, -3.46945×10^{-18} + 0. i, -3.46945×10^{-18} + 0. i, -3.46945×10^{-18} + 0. i,
6.93889×10^{-18} + 0. i, 6.93889×10^{-18} + 0. i, -3.46945×10^{-18} + 0. i, -3.46945×10^{-18} + 0. i,
-3.46945×10^{-18} + 0. i, -3.46945×10^{-18} + 0. i, 6.93889×10^{-18} + 0. i, 6.93889×10^{-18} + 0. i,
6.93889×10^{-18} + 0. i, 6.93889×10^{-18} + 0. i, 0. + 0. i, 6.93889×10^{-18} + 0. i,
6.93889×10^{-18} + 0. i, 0. + 0. i, 6.93889×10^{-18} + 0. i, 0. + 0. i, 6.93889×10^{-18} + 0. i,
6.93889×10^{-18} + 0. i, 0. + 0. i, 0. + 0. i, 0., 0., 0., 0., 0., 0., 0., 0.}}