Glauber and Kawasaki dynamics for permanental point process.

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Abstract

A configuration space over a locally compact Polish space is the set of all locally finite subsets (configurations) in this Polish space. (In most applications, it is sufficient to think of a Polish as a Euclidean space.) A configuration describes positions of indistinguishable particles in an (infinite) system of particles. A probability measure on the configuration space is usually called a point process. This dissertation deals with the so-called Glauber and Kawasaki dynamics on the configuration space. A Glauber dynamics is a stochastic dynamics of an infinite particle system in which particles randomly appear (are born) and disappear (die). A Kawasaki dynamics is a Markov process on the configuration space in which particles randomly hop over the underlying space Polish space. Equilibrium Glauber and Kawasaki dynamics which have a standard Gibbs measure as symmetrizing (and hence invariant) measure have been constructed and actively studied in recent years. Lytvynov and Ohlerich extended this construction to the case of an equilibrium dynamics which has a determinantal (fermion) point process as invariant measure. In 1975 Macchi introduced boson point processes. These point processes are characterized by their correlation functions which have the form of the permanent of a certain matrix constructed through a given correlation kernel. This is why boson point processes are also called permanental point processes. Shirai and Takahashi significantly extended the class of permanental point processes, by including in it the so-called alpha-permanental point processes whose correlation functions have a representation through alpha-permanents introduced by Vere-Jones. All these processes belong to the class of Cox point processes, i.e., Poisson point processes with random intensity. The aim of the dissertation is to show that general criteria of existence of Glauber and Kawasaki dynamics are applicable to a wide class of alpha-permanental point processes. We also consider a diffusion approximation for the Kawasaki dynamics at the level of Dirichlet forms. This leads us to an equilibrium dynamics of interacting Brownian particles for which an alpha-permanental point process is a symmetrizing measure. As a by-product of our considerations, we extend the result of Shirai and Takahashi on the existence of alpha-permanental point process.

DECLARATION

This work has not previously been accepted in substance for any degree and is not concurrently submitted in candidature for any degree.

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Chapter 1

Introduction

Let X be a locally compact Polish space. Let σ be a non-atomic Radon measure on X. In most applications it suffices to think of X as Euclidean space $\mathbb{R}^d$ and $\sigma(dx) = dx$. Let Γ_X denote the space of all locally finite subsets (configurations) in X. A probability measure on Γ_X is usually called a point process.

Starting essentially with papers [1, 2] there have been a lot of activities related to equilibrium and non-equilibrium stochastic dynamics (Markov processes) on Γ_X.

This dissertation deals with the so-called Glauber and Kawasaki dynamics on the configuration space.

A Glauber dynamics (a birth-and-death process of an infinite system of particles in X) is a Markov process on Γ whose formal (pre-)generator has the form

$$(L_{\mathrm{G}}F)(\gamma) = \sum_{x\in\gamma} d(x,\gamma\setminus x)(F(\gamma\setminus x) - F(\gamma)) \\ + \int_X \sigma(dx)\, b(x,\gamma)\,(F(\gamma\cup x) - F(\gamma)), \quad \gamma\in\Gamma_X. \tag{1.1}$$

Here and below, for simplicity of notation we write x instead of $\{x\}$. The coefficient $d(x, \gamma \setminus x)$ describes the rate at which particle x of configuration γ dies, while $b(x, \gamma)$ describes the rate at which, given configuration γ, a new particle is born at x.

A Kawasaki dynamics (a dynamics of hopping particles) is a Markov process on Γ whose formal (pre-)generator is

$$(L_{\mathrm{K}}F)(\gamma) = \sum_{x\in\gamma} c(x, y, \gamma \setminus x) \int_X \sigma(dx)(F(\gamma \setminus x \cup y) - F(\gamma)), \quad \gamma \in \Gamma_X. \tag{1.2}$$

The coefficient $c(x, y, \gamma \setminus x)$ describes the rate at which particle x of configuration γ hops to y, taking the rest of the configuration, $\gamma \setminus x$, into account.

Equilibrium Glauber and Kawasaki dynamics which have a standard Gibbs measure as symmetrizing (and hence invariant) measure were constructed in [27, 28]. In [31], this construction was extended to the case of an equilibrium dynamics which has a determinantal (fermion) point process as invariant measure, For further studies of equilibrium and non-equilibrium Glauber and Kawasaki dynamics, we refer to [8, 12, 13, 14, 15, 18, 20, 21, 23, 24, 25, 26, 29, 40] and the references therein.

In [37] (see also [36]) Macchi introduced boson point processes. These point processes are characterized by their correlation functions which have the form of the permanent of a certain matrix constructed through some given correlation kernel. This is why boson point processes were also called permanental point processes. Shirai and Takahashi [44] significantly extended the class of permanental point processes, by including the so-called α-permanental point processes whose correlation functions have a representation through α-permanents introduced by Vere-Jones in [45] (Shirai and Takahashi call them α-determinants). Such point processes are known to exist for a wide range of correlation kernels if $\alpha^{-1} \in \frac{1}{2}\mathbb{N}$, i.e., $\alpha = \frac{1}{2}\, l$ with $l \in \mathbb{N}$. All these processes belong to the class of Cox point processes (see e.g. [10, 17]), i.e., Poisson point processes with random intensity. In case of permanental point processes, this random intensity is described in terms of a Gaussian random field.

The aim of the dissertation is to show that general criteria of existence of Glauber and Kawasaki dynamics which were developed in [31] are applicable to a wide class of α-permanental point processes. We will also consider a diffusion approximation for the Kawasaki dynamics at the level of Dirichlet forms (compare with [23]). This will lead us to an equilibrium dynamics of interacting Brownian particles for which an α-permanental point process is a symmetrizing measure. As a by-product of our considerations, we will also extend the result of [44] on the existence of α-permanental point process.

The dissertation is organized as follows. In Chapter 2, we will discuss configuration spaces and general Cox point processes. In Chapter 3, we will construct α-permanental point processes. To this end, we will construct a random field which is Gaussian almost surely. In Chapter 4, we will briefly recall some facts related to the general theory of Markov processes and Dirichlet forms. Finally, the main results of the dissertation are in Chapter 5. There we will construct equilibrium Glauber and Kawasaki dynamics for which an α-permanental point process is a symmetrizing (and hence invariant) measure. We will also discuss a diffusion approximation for the corresponding Kawasaki dynamics.

Chapter 2

Configuration space and Cox point processes

In this chapter, we will present an overview of some basic definitions and facts related to configuration spaces and point processes, in particular, Cox point processes.

The standard references for sections 2.1–2.4 below are [10, 22, 35], the references for Cox point processes (section 2.5) include [10, 17, 35, 39].

2.1 Spaces of finite and infinite configurations

Let X be a locally compact Polish space. Recall that a Polish space is a metric space which is complete and separable. That is, there exists a metric ρ on X, generating the topology, such that:
a) every Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in X converges in X (completeness);
b) there exists a countable subset Y of X whose closure $\bar{Y}$ is X (separability).

A set $B \subset X$ is said to be topologically bounded if its closure $\bar{B}$ is compact. Recall that a subset C of X is called compact if from any covering

$\{B_i;\ i \in I\}$ by open balls one can extract a finite subset $\{B_{i_1}, \ldots, B_{i_n}\}$ which is again a covering of C.

Note that an open ball $B(x,r)$ with centre at $x \in X$ and radius $r > 0$ need not be bounded in such a space. However, the condition that X be locally compact means that one can always find $\varepsilon > 0$, small enough, such that the open ball $B(x,\varepsilon)$ is bounded, i.e., the closed ball $\bar{B}(x,\varepsilon)$ is compact.

We denote by $\mathcal{B}(X)$ the Borel σ-algebra in X, and by $\mathcal{B}_0(X)$ the collection of all bounded sets from $\mathcal{B}(X)$.

A configuration space over X, denoted by Γ_X, is defined as the set of all locally finite subsets (configurations) in X:

$$\Gamma_X := \{\gamma \subset X : |\gamma \cap \Delta| < \infty \text{ for each } \Delta \in \mathcal{B}_0(X)\}.$$

Here $|A|$ denotes the cardinality of set A. We will often identify each $\gamma \in \Gamma_X$ with the Radon measure

$$\gamma = \sum_{x \in \gamma} \delta_x.$$

Here δ_x denotes the Dirac measure at x. Thus, Γ_X becomes a subset of the set $\mathcal{M}_0(X)$ of all Randon measures on X, i.e., all measures m on $(X, \mathcal{B}(X))$ such that $m(A) < \infty$ for all $A \in \mathcal{B}_0(X)$. Recall that $\mathcal{M}_0(X)$ has a standard topology, called the vague topology. This is the minimal topology with respect to which each mapping of the form

$$\mathcal{M}_0(X) \ni m \mapsto \langle f, m\rangle \in \mathbb{R}, \quad f \in C_0(X),$$

is continuous. Here $\langle f, m\rangle := \int_X f(x)\, m(dx)$ and $C_0(X)$ denotes the set of all continuous functions on X with compact support. We can now define the vague topology on Γ_X as the relative topology of the vague topology on $\mathcal{M}_0(X)$. Thus, the vague topology on Γ_X is the minimal topology with respect to which each mapping of the form

$$\Gamma_X \ni \gamma \mapsto \langle f, \gamma \rangle = \sum_{x \in \gamma} f(x) \in \mathbb{R}, \quad f \in C_0(X),$$

is continuous.

In the vague topology, Γ_X becomes itself a Polish space (i.e., one can introduce a metric on Γ_X which generates the vague topology and with respect to which Γ_X is complete and separable.)

In fact, one can explicitly describe convergence in the vague topology in Γ_X. So, a sequence $\{\gamma^{(n)}\}_{n=1}^{\infty}$ converges to a $\gamma \in \Gamma_X$ in the vague topology if and only if, for any open, bounded set $O \subset X$ such that $(\overline{O} \setminus O) \cap \gamma = \varnothing$ (here $\overline{O}$ denotes the closure of O), there exist $N \in \mathbb{N}$ and $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ such that

$$|\gamma^{(n)} \cap O| = |\gamma \cap O| = k$$

for all $n \geq N$, and one can introduce a numeration of points of $\gamma^{(n)} \cap O = \{x_1^{(n)}, \dots, x_k^{(n)}\}$, $n \geq N$, and that of $\gamma \cap O = \{x_1, \dots, x_k\}$ such that $x_i^{(n)} \to x_i$ in X as $n \to \infty$, for $i = 1, \dots, k$.

We denote by $\mathcal{B}(\Gamma_X)$ the Borel σ-algebra on Γ_X. In fact, $\mathcal{B}(\Gamma_X)$ is the minimal σ-algebra on Γ_X with respect to which each mapping of the form

$$\Gamma_X \ni \gamma \mapsto |\gamma \cap \Lambda|, \quad \Lambda \in \mathcal{B}_0(X)$$

is measurable. A probability measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ is called a point process in X.

We will also need the space of multiple configurations $\ddot{\Gamma}_X$. This space is defined as the set of all $\mathbb{N}_0 \cup \{+\infty\}$-valued Radon measures. Thus, any element $\ddot{\gamma} \in \ddot{\Gamma}_X$ has a representation

$$\ddot{\gamma} = \sum_{x \in \gamma} n(x) \delta_x, \tag{2.1}$$

where $\gamma \in \Gamma_X$ and for each $x \in \gamma$, $n(x) \in \mathbb{N}$. We will also endow $\ddot{\Gamma}_X$ with the vague topology. (Note that, informally, we may treat $\ddot{\Gamma}_X$ as the configuration space in which different particles may occupy the same position, $n(x)$ in (2.1) being the number of particles at point x.)

Let $\Lambda \in \mathcal{B}(X)$. We denote by $\Gamma_{\Lambda,0}$ the space of all finite configurations in Λ:

$$\Gamma_{\Lambda,0} = \{\gamma \subset \Lambda \mid |\gamma| < \infty\}.$$

(Note that if Λ is bounded, i.e., $\Lambda \in \mathcal{B}_0(X)$, then $\Gamma_{\Lambda,0} = \Gamma_\Lambda$.) Thus,

$$\Gamma_{\Lambda,0} = \bigcup_{n=0}^{\infty} \Gamma_\Lambda^{(n)},$$

where $\Gamma_\Lambda^{(n)}$ is the space of all n-point subsets of Λ (for $n = 0$, $\Gamma_\Lambda^{(0)} = \{\varnothing\}$).

We can, of course, endow $\Gamma_{\Lambda,0}$ with the vague topology, and define $\mathcal{B}(\Gamma_{\Lambda,0})$ as the Borel σ-algebra on $\Gamma_{\Lambda,0}$. There is, however, another equivalent description of this σ-algebra.

For each $n \in \mathbb{N}$, define

$$D_n := \{(x_1, \dots, x_n) \in X^n \mid x_i = x_j \text{ for some } i, j \in \{1, \dots, n\},\ i \neq j\}$$

(i.e., D_n is the collection of all 'diagonals' in X^n). Set

$$\tilde{\Lambda}^n := \Lambda^n \setminus D_n. \tag{2.2}$$

Define $\mathcal{B}(\tilde{\Lambda}^n)$ as the trace σ-algebra of $\mathcal{B}(\Lambda^n)$ on $\tilde{\Lambda}^n$. Define

$$I_n : \tilde{\Lambda}^n \to \Gamma_\Lambda^{(n)}$$

by

$$I_n(x_1, \dots, x_n) = \{x_1, \dots, x_n\}. \tag{2.3}$$

Denote by $\mathcal{B}(\Gamma_\Lambda^{(n)})$ the image of the σ-algebra $\mathcal{B}(\tilde{\Lambda}^n)$ under I_n. Now, we have that $\mathcal{B}(\Gamma_{\Lambda,0})$ is the minimal σ-algebra on $\Gamma_{\Lambda,0}$ which contains all $\mathcal{B}(\Gamma_\Lambda^{(n)})$, $n \in \mathbb{N}$ (note that this implies that $\{\varnothing\} \in \mathcal{B}(\Gamma_{\Lambda,0})$).

Let $\Lambda \in \mathcal{B}_0(X)$. We call a set $A \subset \Gamma_X$ local with respect to Λ if there exits a set $\tilde{A} \subset \Gamma_\Lambda (= \Gamma_{\Lambda,0})$ such that

$$A = \{\gamma \in \Gamma_X \mid \gamma \cap \Lambda \in \tilde{A}\}. \tag{2.4}$$

We denote by $\mathcal{B}_\Lambda(\Gamma_X)$ the σ-algebra of all sets from $\mathcal{B}(\Gamma_X)$ which are local with respect to Λ. By identifying a local set A as in (2.4) with the set $\tilde{A}$ we may identify the σ-algebra $\mathcal{B}_\Lambda(\Gamma_X)$ with a σ-algebra on Γ_Λ. In fact, this σ-algebra coincides with $\mathcal{B}(\Gamma_\Lambda)$. Thus, we have identified the σ-algebras $\mathcal{B}_\Lambda(\Gamma_X)$ and $\mathcal{B}(\Gamma_\Lambda)$.

2.2 Poisson and Lebesgue–Poisson measures in finite volume

Let X be as in Section 2.1. Let σ be a Radon, non-atomic measure on $(X, \mathcal{B}(X))$ (recall that non-atomic means that $\sigma(\{x\}) = 0$ for each $x \in X$). Using the notations of section 2.1, we see that the set $D_n \subset X^n$ is of zero $\sigma^{\otimes n}$ measure. Let $\Lambda \in \mathcal{B}_0(X)$, so that, in particular, $\sigma(\Lambda) < \infty$. Let σ_Λ denote the restriction of the measure σ to $(\Lambda, \mathcal{B}(\Lambda))$. Then the set $\tilde{\Lambda}^n$ is of full measure $\sigma_\Lambda^{\otimes n}$. Therefore, we can consider $\sigma_\Lambda^{\otimes n}$ as a measure on $(\tilde{\Lambda}^n, \mathcal{B}(\tilde{\Lambda}^n))$. We will preserve the notation $\sigma_\Lambda^{\otimes n}$ for the image of the measure $\sigma_\Lambda^{\otimes n}$ under the measurable mapping $I_n : \tilde{\Lambda}^n \to \Gamma_\Lambda^{(n)}$ given through (2.3). Then

$$\sigma_\Lambda^{\otimes n}(\Gamma_\Lambda^{(n)}) = \sigma(\Lambda)^n. \tag{2.5}$$

We will now define the measure $\Pi_\sigma^{(\Lambda)}$ on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ as follows:

$$\begin{aligned} \Pi_\sigma^{(\Lambda)} \restriction \Gamma_\Lambda^{(n)} &:= \frac{1}{n!}\sigma_\Lambda^{\otimes n}, \quad n \in \mathbb{N}, \\ \Pi_\sigma^{(\Lambda)}(\{\varnothing\}) &:= 1. \end{aligned}$$

The measure $\Pi_\sigma^{(\Lambda)}$ is called the Lebesgue–Poisson measure on Γ_Λ with intensity σ. Quite often, one informally writes

$$\Pi_\sigma^{(\Lambda)} = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma_\Lambda^{\otimes n}. \tag{2.6}$$

Clearly, by (2.5),

$$\begin{aligned}\Pi_\sigma^{(\Lambda)}(\Gamma_\Lambda) &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sigma_\Lambda^{\otimes n}(\Gamma_\Lambda^{(n)}) \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sigma(\Lambda)^n \\ &= \exp(\sigma(\Lambda)).\end{aligned}$$

Therefore,

$$\pi_\sigma^{(\Lambda)} := \exp(-\sigma(\Lambda)) \Pi_\sigma^{(\Lambda)}$$

is a probability measure on Γ_Λ. This measure is called Poisson measure on Γ_Λ with intensity σ. Thus, analogously to (2.6), we have

$$\pi_\sigma^{(\Lambda)} = \exp(-\sigma(\Lambda)) \sum_{n=0}^{\infty} \frac{1}{n!} \sigma_\Lambda^{\otimes n}.$$

Let us calculate the Fourier transform of the Poisson measure π_σ. Let $f : \Lambda \to \mathbb{R}$ be a bounded measurable function. Then, using the definition of the Poisson measure, we have

$$\begin{aligned}&\int_{\Gamma_\Lambda} e^{i\langle f, \gamma \rangle} \pi_\sigma^{(\Lambda)}(d\gamma) \\ &= \exp(-\sigma(\Lambda)) \left[1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} e^{i(f(x_1) + \cdots + f(x_n))} \sigma_\Lambda(dx_1) \cdots \sigma_\Lambda(dx_n) \right] \\ &= \exp(-\sigma(\Lambda)) \left[1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\int_\Lambda e^{if(x)} \sigma_\Lambda(dx) \right)^n \right] \\ &= \exp\left(-\sigma(\Lambda) + \int_\Lambda e^{if(x)} \sigma(dx) \right)\end{aligned}$$

$$= \exp\left(\int_\Lambda (e^{if(x)} - 1)\sigma(dx)\right). \tag{2.7}$$

Let $\Delta \in \mathcal{B}_0(X)$, $\Delta \subset \Lambda$, let $u \in \mathbb{R}$, and set $f(x) = u\chi_\Delta(x)$, where χ_Δ denotes the indicator function of Δ. Then, by (2.7),

$$\begin{aligned}
\int_{\Gamma_\Lambda} e^{iu|\gamma\cap\Delta|}\pi_\sigma^{(\Lambda)}(d\gamma) &= \int_{\Gamma_\Lambda} e^{i\langle f,\gamma\rangle}\pi_\sigma^{(\Lambda)}(d\gamma) \\
&= \exp\left(\int_\Lambda (e^{iu\chi_\Delta(x)} - 1)\sigma(dx)\right) \\
&= \exp\left(\int_\Delta (e^{iu} - 1)\sigma(dx)\right) \\
&= \exp\left(\sigma(\Delta)(e^{iu} - 1)\right).
\end{aligned}$$

This shows that, under the probability measure $\pi_\sigma^{(\Lambda)}$ the random variable $|\gamma \cap \Delta|$ has Poisson distribution with intensity $\sigma(\Delta)$. Recall that for any $a > 0$, the Poisson distribution with intensity a is the probability measure on $\mathbb{R}$ given by

$$e^{-a}\sum_{n=0}^{\infty}\frac{a^n}{n!}\delta_n.$$

This is the reason why $\pi_\sigma^{(\Lambda)}$ is called Poisson measure.

Let now $f_1, \dots, f_n : \Lambda \to \mathbb{R}$ be bounded and measurable and let us assume that these functions have mutually disjoint supports. Let $u_1, \dots, u_n \in \mathbb{R}$. Thus, by (2.7),

$$\begin{aligned}
\int_{\Gamma_\Lambda} e^{i(u_1\langle f_1,\gamma\rangle+\cdots+u_n\langle f_n,\gamma\rangle)}\pi_\sigma^{(\Lambda)}(d\gamma) &= \int_{\Gamma_\Lambda} e^{i\langle u_1 f_1+\cdots+u_n f_n,\gamma\rangle}\pi_\sigma^{(\Lambda)}(d\gamma) \\
&= \exp\left(\int_\Lambda (e^{i(u_1 f_1(x)+\cdots+u_n f_n(x))} - 1)\sigma(dx)\right) \\
&= \exp\left(\int_\Lambda \sum_{i=1}^{n}(e^{iu_i f_i(x)} - 1)\sigma(dx)\right) \\
&= \prod_{i=1}^{n}\exp\left(\int_\Lambda (e^{iu_i f_i(x)} - 1)\sigma(dx)\right)
\end{aligned}$$

$$= \prod_{i=1}^{n} \int_{\Gamma_\Lambda} e^{iu_i\langle f_i,\gamma\rangle} \pi_\sigma^{(\Lambda)}(d\gamma). \tag{2.8}$$

Therefore, the random variables $\langle f_1, \gamma\rangle, \dots, \langle f_n, \gamma\rangle$ are independent. In particular, if $\Delta_1, \dots, \Delta_n \in \mathcal{B}_0(X)$, $\Delta_1, \dots, \Delta_n \subset \Lambda$, $\Delta_1, \dots, \Delta_n$ are mutually disjoint, then the random variables $|\gamma \cap \Delta_1|, \dots, |\gamma \cap \Delta_n|$ are independent.

2.3 Poisson measure in infinite volume

Again, let X be as in Section 2.1, and consider the measurable space $(\Gamma_X, \mathcal{B}(\Gamma_X))$. Let σ be a Radon, non-atomic measure on $(X, \mathcal{B}(X))$. Let $\Lambda \in \mathcal{B}_0(X)$. Recall that, in Section 2.1, we have identified the σ-algebra $\mathcal{B}(\Gamma_\Lambda)$ on Γ_Λ with the σ-algebra $\mathcal{B}_\Lambda(\Gamma_X)$ on Γ_X. Hence, by Section 2.2, we have the Poisson measure $\pi_\sigma^{(\Lambda)}$ on $(\Gamma_X, \mathcal{B}_\Lambda(\Gamma_X))$ Thus, we get a family of Poisson measures $(\pi_\sigma^{(\Lambda)})_{\Lambda\in\mathcal{B}_0(X)}$, where each $\pi_\sigma^{(\Lambda)}$ is a probability measure on $\mathcal{B}_\Lambda(\Gamma_X)$

Let Λ, $\Lambda' \in \mathcal{B}_0(X)$, $\Lambda \subset \Lambda'$. Then evidently $\mathcal{B}_\Lambda(\Gamma_X) \subset \mathcal{B}_{\Lambda'}(\Gamma_X)$. Let us check that the family $(\pi_\sigma^{(\Lambda)})_{\Lambda\in\mathcal{B}_0(X)}$ is consistent, i.e., for any Λ, Λ' as above, $\pi_\sigma^{(\Lambda)}$ is equal to the restriction of $\pi_\sigma^{(\Lambda')}$ to $\mathcal{B}_\Lambda(\Gamma_X)$. Indeed, let $f : X \to \mathbb{R}$ be a bounded measurable function with support in Λ. Then, by (2.7),

$$\begin{aligned}
\int_{\Gamma_X} e^{i\langle f,\gamma\rangle} \pi_\sigma^{(\Lambda')}(d\gamma) &= \int_{\Gamma_{\Lambda'}} e^{i\langle f,\gamma\rangle} \pi_\sigma^{(\Lambda')}(d\gamma) \\
&= \exp\left(\int_{\Lambda'} (e^{if(x)} - 1)\sigma(dx)\right) \\
&= \exp\left(\int_{\Lambda} (e^{if(x)} - 1)\sigma(dx) + \int_{\Lambda'\setminus\Lambda} (e^{if(x)} - 1)\sigma(dx)\right) \\
&= \exp\left(\int_{\Lambda} (e^{if(x)} - 1)\sigma(dx)\right) \\
&= \int_{\Gamma_\Lambda} e^{i\langle f,\gamma\rangle} \pi_\sigma^{(\Lambda)}(d\gamma)
\end{aligned}$$

$$= \int_{\Gamma_X} e^{i\langle f,\gamma\rangle} \pi_\sigma^{(\Lambda)}(d\gamma).$$

Therefore, $\pi_\sigma^\Lambda = \pi_\sigma^{\Lambda'} \upharpoonright \mathcal{B}_\Lambda(\Gamma_X)$. The σ-algebra $\mathcal{B}(\Gamma_X)$ is the minimal σ-algebra on Γ_X which contains each $\mathcal{B}_\Lambda(\Gamma_X)$, $\Lambda \in \mathcal{B}_0(X)$. Hence, by a version of Kolmogorov's existence theorem, there exists a unique probability measure π_σ on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ such that $\pi_\sigma \upharpoonright \mathcal{B}_\Lambda(\Gamma_X) = \pi_\sigma^{(\Lambda)}$. This measure π_σ is called the Poisson measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ with intensity σ.

Denote by $B_0(X)$ the space of all bounded, measurable function $f : X \to \mathbb{R}$ which have compact support. Fix any $f \in B_0(X)$ and denote $\Lambda := \operatorname{supp} f$, $\Lambda \in \mathcal{B}_0(X)$. Hence the function $\langle f, \gamma\rangle$ on Γ_X is $\mathcal{B}_\Lambda(\Gamma_X)$-measurable. Then, by (2.7)

$$\begin{aligned}
\int_{\Gamma_X} e^{\langle f,\gamma\rangle} \pi_\sigma(d\gamma) &= \int_{\Gamma_X} e^{\langle f,\gamma\rangle} \pi_\sigma^{(\Lambda)}(d\gamma) \\
&= \int_{\Gamma_\Lambda} e^{\langle f,\gamma\rangle} \pi_\sigma^{(\Lambda)}(d\gamma) \\
&= \exp\left(\int_\Lambda (e^{f(x)} - 1)\sigma(dx)\right) \\
&= \exp\left(\int_X (e^{f(x)} - 1)\sigma(dx)\right).
\end{aligned}$$

Thus, the Laplace transform of π_σ is given by

$$\int_{\Gamma_X} e^{\langle f,\gamma\rangle} \pi_\sigma(d\gamma) = \exp\left(\int_X (e^{f(x)} - 1)\sigma(dx)\right), \quad f \in B_0(X). \tag{2.9}$$

Analogously to (2.8), we conclude that for any $f_1, \ldots, f_n \in B_0(X)$ with mutually disjoint supports, under π_σ the random variables $\langle f_1, \gamma\rangle, \ldots, \langle f_n, \gamma\rangle$ are independent, and in particular, for any $\Delta_1, \ldots, \Delta_n \in \mathcal{B}_0(X)$, mutually disjoint, the random variables $|\gamma \cap \Delta_1|, \ldots, |\gamma \cap \Delta_n|$ are independent. Further, for each $\Delta \in \mathcal{B}_0(X)$, the random variable $|\gamma \cap \Delta|$ has Poisson distribution with intensity $\sigma(\Delta)$

The following theorem gives a characterization of Poisson measure.

Theorem 2.1 (Mecke). *For each measurable function $F : X \times \Gamma_X \to [0, +\infty]$, we have*

$$\int_{\Gamma_X} \sum_{x \in \gamma} F(x, \gamma) \pi_\sigma(d\gamma) = \int_{\Gamma_X} \pi_\sigma(d\gamma) \int_X \sigma(dx) F(x, \gamma \cup x). \qquad (2.10)$$

Furthermore, π_σ is uniquely characterized by the above equality, i.e., if μ is a probability measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ which satisfies

$$\int_{\Gamma_X} \sum_{x \in \gamma} F(x, \gamma) \mu(d\gamma) = \int_{\Gamma_X} \mu(d\gamma) \int_X \sigma(dx) F(x, \gamma \cup x) \qquad (2.11)$$

for each measurable $F : X \times \Gamma_X \to [0, +\infty]$, then $\mu = \pi_\sigma$.

Remark 2.1. In formulas (2.10), (2.11) and below, for simplicity of notation, we sometimes write x instead of $\{x\}$.

The equality (2.10) is called the Mecke identity. Since this identity will play a crucial role in our research, we will now briefly recall a proof of (2.10). (Note that the harder part of the proof of the Mecke theorem is, in fact, to prove that the equality (2.11) implies that $\mu = \pi_\sigma$, see [35].)

So, let us first fix any $\Lambda \in \mathcal{B}_0(X)$ and let $F : \Lambda \times \Gamma_\Lambda \to [0, +\infty]$ be a measurable function. Then, using the definition of the Poisson measure in finite volume, we have

$$\begin{aligned} &\int_{\Gamma_\Lambda} \sum_{x \in \gamma_\Lambda} F(x, \gamma_\Lambda) \pi_\sigma^{(\Lambda)}(d\gamma_\Lambda) \\ &= \sum_{n=0}^{\infty} \int_{\Gamma_\Lambda^{(n)}} \sum_{x \in \gamma_\Lambda} F(x, \gamma_\Lambda) \pi_\sigma^{(\Lambda)}(d\gamma_\Lambda) \\ &= e^{-\sigma(\Lambda)} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\widetilde{\Lambda}^n} \sum_{i=1}^{n} F\left(x_i, \{x_1, \dots, x_n\}\right) \sigma(dx_1) \dots \sigma(dx_n). \end{aligned} \qquad (2.12)$$

Note that

$$\int_{\widetilde{\Lambda}^n} F\left(x_1, \{x_1, \dots, x_n\}\right) \sigma(dx_1) \dots \sigma(dx_n)$$

$$= \int_{\widetilde{\Lambda}^n} F(x_2, \{x_1, \dots, x_n\})\, \sigma(dx_1) \dots \sigma(dx_n)$$
$$= \dots = \int_{\widetilde{\Lambda}^n} F(x_n, \{x_1, \dots, x_n\})\, \sigma(dx_1) \dots \sigma(dx_n).$$

Thus, we continue (2.12) as follows:

$$= e^{-\sigma(\Lambda)} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int_{\widetilde{\Lambda}^n} F(x_1, \{x_1, \dots, x_n\})\, \sigma(dx_1) \cdots \sigma(dx_n)$$
$$= e^{-\sigma(\Lambda)} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int_{\Lambda} \sigma(dx) \int_{\Lambda^{n-1}} \sigma(dx_1) \cdots \sigma(dx_{n-1}) F(x, \{x_1, \dots, x_{n-1}\} \cup \{x\})$$
$$= e^{-\sigma(\Lambda)} \int_{\Lambda} \sigma(dx) \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \sigma(dx_1) \cdots \sigma(dx_n) F(x, \{x_1, \dots, x_n\} \cup \{x\})$$
$$= \int_{\Lambda} \sigma(dx) \int_{\Gamma_\Lambda} \pi_\sigma^{(\Lambda)}(d\gamma_\Lambda) F(x, \gamma_\Lambda \cup \{x\}).$$

Next, let $F : X \times \Gamma_X \to [0, +\infty]$ be measurable and such that there exist $\Lambda \in \mathcal{B}_0(X)$ for which $F(x, \gamma) = F(x, \gamma \cap \Lambda)$ for all $\gamma \in \Gamma_X$ and $x \in X$, and $F(x, \gamma) = 0$ if $x \notin \Lambda$. Then

$$\int_{\Gamma_X} \sum_{x \in \gamma} F(x, \gamma) \pi_\sigma(d\gamma) = \int_{\Gamma_X} \sum_{x \in \gamma \cap \Lambda} F(x, \gamma) \pi_\sigma(d\gamma). \tag{2.13}$$

Since for each $x \in X$, $F(\cdot, x)$ is $\mathcal{B}_\Lambda(\Gamma_X)$-measurable, we see that $\sum_{x \in \gamma \cap \Lambda} F(x, \gamma)$ is a $\mathcal{B}_\Lambda(\Gamma_X)$-measurable function. Hence, we continue (2.13) as follows:

$$= \int_{\Gamma_\Lambda} \sum_{x \in \gamma_\Lambda} F(x, \gamma_\Lambda) \pi_\sigma^{(\Lambda)}(d\gamma_\Lambda)$$
$$= \int_{\Gamma_\Lambda} \pi_\sigma^{(\Lambda)}(d\gamma_\Lambda) \int_{\Lambda} \sigma(dx) F(x, \gamma_\Lambda \cup \{x\})$$
$$= \int_{\Gamma_\Lambda} \pi_\sigma^{(\Lambda)}(d\gamma_\Lambda) \int_{X} \sigma(dx) F(x, \gamma_\Lambda \cup \{x\})$$
$$= \int_{\Gamma} \pi_\sigma^{(\Lambda)}(d\gamma) \int_{X} \sigma(dx) F(x, \gamma \cup \{x\}).$$

Hence, we proved the Mecke identity in the special case of F as above.

Let now $(\Lambda_n)_{n=1}^{\infty}$ be such that, $\Lambda_n \in \mathcal{B}_0(X)$, $\Lambda_n \subset \Lambda_{n+1}$, $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} \Lambda_n = X$. Let $F : X \times \Gamma_X \to [0, +\infty]$ be measurable. Take a sequence $(F_n)_{n=1}^{\infty}$ of functions $F_n : X \times \Gamma_X \to [0, +\infty]$ such that

i) For each $x \in X$, $F_n(x, \gamma) = F_n(x, \gamma \cap \Lambda_n)$ for all $\gamma \in \Gamma_x$;

ii) For each $\gamma \in \Gamma_X$, $F_n(x, \gamma) = 0$ if $x \notin \Lambda_n$;

iii) $F_1(x, \gamma) \le F_2(x, \gamma) \le F_3(x, \gamma) \le \cdots$, $\gamma \in \Gamma_X$, $x \in X$;

iv) $F_n(x, \gamma) \to F(x, \gamma)$ as $n \to \infty$ for all $\gamma \in \Gamma_X$ and $x \in X$.

(Note that the existence of such a sequence follows from the fact that $\mathcal{B}(\Gamma_X)$ is the minimal σ-algebra on Γ_X which contains all $\mathcal{B}_{\Lambda_n}(\Gamma_X)$, $n \in \mathbb{N}$.) Then,by the proved above,

$$\int_{\Gamma_X} \sum_{x \in \gamma} F_n(x, \gamma) \pi_\sigma(d\gamma) = \int_{\Gamma_X} \pi_\sigma(d\gamma) \int_X \sigma(dx) F_n(x, \gamma \cup x).$$

Now, letting $n \to \infty$, by the monotone convergence theorem, the Mecke identity follows.

2.4 Correlation measure and correlation functions

We start with a general definition. Let μ be a probability measure (point process) on $(\Gamma_X, \mathcal{B}(\Gamma_X))$. Then the correlation measure ρ_μ of μ is a measure on $(\Gamma_{X,0}, \mathcal{B}(\Gamma_{X,0}))$ which $\int_{\Gamma_X} (KG)(\eta)\mu(d\eta) = \int_{\Gamma_{X,0}} G(\eta)\rho_\mu(d\eta)$

$$\int_{\Gamma_X} (KG)(\eta)\mu(d\eta) = \int_{\Gamma_{X,0}} G(\eta)\rho_\mu(d\eta) \tag{2.14}$$

for any measurable $G : \Gamma_{X,0} \to [0, \infty]$. Here

$$(KG)(\gamma) := \sum_{\eta \Subset \gamma} G(\eta)$$

where the summation is over all finite subset η of γ. It can be easily shown that ρ_μ indeed exists and is uniquely defined by (2.14), see e.g. [22].

Assume that we have fixed a Radon, non-atomic measure σ on $(X, \mathcal{B}(X))$, and let Π_σ denote the corresponding Lebesgue-Poisson measure on $(\Gamma_{X,0}, \mathcal{B}(\Gamma_{X,0}))$, i.e.,

$$\Pi_\sigma = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{\otimes n}$$

(compare with (2.6)). If the correlation measure ρ_μ is absolutely continuous with respect to Π_σ, then its Radon–Nikodym derivative k_μ is called the correlation functional of μ. Thus

$$\int_{\Gamma_X} (KG)(\gamma)\mu(d\gamma) = \int_{\Gamma_{X,0}} G(\eta) k_\mu(\eta) \Pi_\sigma(d\eta).$$

Clearly, the restriction of k_μ to any $\Gamma_X^{(n)}$ may be identified with a symmetric fuction $k_\mu^{(n)}$ on X^n (in fact, on $\widetilde{X}^n$, see (2.2)). The sequence of functions $(k_\mu^{(n)})_{n=1}^\infty$ is called the sequence of correlation functions of μ. (Note that since μ is a probability measure, we always have $k_\mu(\varnothing) = 1$.)

Let us now consider the case where μ is the Poisson measure π_σ.

Proposition 2.1. *The correlation measure of the Poisson measure π_σ is the Lebesgue–Poisson measure Π_σ, so that the correlation functions of π_σ are all identically equal to* 1.

Proof. It suffices to show that for any symmetric measurable function $f^{(n)} : X^n \to [0, +\infty]$, $n \in \mathbb{N}$, we have

$$\int_{\Gamma_X} \sum_{\{x_1, \dots, x_n\} \subset \gamma} f^{(n)}(x_1, \dots, x_n) \pi_\sigma(d\gamma)$$
$$= \frac{1}{n!} \int_{X^n} f^{(n)}(x_1, \dots, x_n) \sigma(dx_1) \cdots \sigma(dx_n).$$

Using the Mecke identity and the Fubini theorem we have:

$$\int_{\Gamma_X} \sum_{\{x_1,\dots,x_n\}\subset\gamma} f^{(n)}(x_1,\dots,x_n)\pi_\sigma(d\gamma)$$

$$= \int_{\Gamma_X} \frac{1}{n!} \sum_{x_1\in\gamma} \sum_{x_2\in\gamma\setminus\{x_1\}} \sum_{x_3\in\gamma\setminus\{x_1,x_2\}} \cdots \sum_{x_n\in\gamma\setminus\{x_1,\dots x_n\}} f^{(n)}(x_1,\dots x_n)\pi_\sigma(d\gamma)$$

$$= \frac{1}{n!}\int_{\Gamma_X} \sum_{x_1\in\gamma}\left(\sum_{y_1\in\gamma\setminus\{x_1\}} \sum_{y_2\in\gamma\setminus\{x_1,y_1\}} \cdots \sum_{y_{n-1}\in\gamma\setminus\{x_1,y_1,\dots,y_{n-2}\}} f^{(n)}(x_1,y_1,\dots y_{n-1})\right)\pi_\sigma(d\gamma)$$

$$= \frac{1}{n!}\int_{\Gamma_X} \pi_\sigma(d\gamma)\int_X \sigma(dx_1) \sum_{y_1\in\gamma} \sum_{y_2\in\gamma\setminus\{y_1\}} \cdots \sum_{y_{n-1}\in\gamma\setminus\{y_1,\dots,y_{n-1}\}} f^{(n)}(x_1,y_1,\dots y_{n-1})$$

$$= \frac{1}{n!}\int_X \sigma(dx_1)\int_{\Gamma_X} \pi_\sigma(d\gamma) \sum_{x_2\in\gamma} \sum_{y_1\in\gamma\setminus\{x_2\}} \cdots \sum_{y_{n-2}\in\gamma\setminus\{x_2,y_1,\dots,y_{n-3}\}} f^{(n)}(x_1,x_2,y_1,\dots,y_{n-2})$$

$$= \frac{1}{n!}\int_X \sigma(dx_1)\int_{\Gamma_X} \pi_\sigma(d\gamma)\int_X \sigma(dx_2) \sum_{y_1\in\gamma} \sum_{y_2\in\gamma\setminus\{y_1\}} \cdots \sum_{y_{n-2}\in\gamma\setminus\{y_1,\dots,y_{n-3}\}} f^{(n)}(x_1,x_2,y_1,\dots,y_{n-2})$$

$$= \frac{1}{n!}\int_X \sigma(dx_1)\dots\int_X \sigma(dx_n)\int_{\Gamma_X} \pi_\sigma(d\gamma) f^n(x_1,\dots,x_n)$$

$$= \dots = \frac{1}{n!}\int_{X^n} f^{(n)}(x_1,\dots,x_n)\sigma(dx_1)\dots\sigma(dx_n). \quad \square$$

Corollary 2.1. *Let ν be a Radon measure on $(X,\mathcal{B}(X))$ which is absolutely continuous with respect to σ, i.e. $\nu(dx) = g(dx)\sigma(dx)$. Then the Poisson measure π_ν has the correlation functions related to the Lebesgue–Poisson measure Π_σ given by*

$$k_{\pi_\nu}^{(n)}(x_1,\dots,x_n) = g(x_1)\cdots g(x_n). \tag{2.15}$$

Proof. By Proposition 2.1, the correlation measure of π_ν is Π_ν, which has the following Radon–Nikodym derivative with respect to Π_σ: $k_{\pi_\nu}(\eta) = \prod_{x\in\eta} g(x)$, from where (2.15) follows. $\square$

2.5 Cox point processes

A Cox point process (also called a doubly stochastic Poisson process) is a Poisson point process with random intensity measure. In the most general case one can consider random Radon non-atomic measure $\sigma(dx, \omega)$ as intencity measure of Poisson processes. We will, however, only treat the case where we have a fixed Radon non-atomic measure σ on X and random intensity $g(x, \omega)$.

So we fix a space X as in Section 2.1. Let σ be fixed Radon, non-atomic measure on $(X, \mathcal{B}(X))$. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $g : X \times \Omega \to [0, \infty]$ be a measurable function on $(X \times \Omega, \mathcal{B}(X) \otimes \mathcal{A})$.

Furthermore, we will assume that, for each $\Delta \in \mathcal{B}_0(X)$ and each $n \in \mathbb{N}$,

$$\int_{\Omega} \left(\int_{\Delta} g(x, \omega)\sigma(dx) \right)^n \mathbb{P}(d\omega) < \infty. \tag{2.16}$$

In particular, for a.a. $\omega \in \Omega$, $g(\cdot, \omega)$ is a locally integrable function with respect to σ. Therefore, for a.a. $\omega \in \Omega$, $g(x, \omega)\sigma(dx)$ is a Radon, non-atomic measure on $(X, \mathcal{B}(X))$. So, we can construct the Poisson measure $\pi_{g(x,\omega)\sigma(dx)}$. Then, we define a probability measure μ on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ by

$$\begin{aligned} \mu(A) :&= \int_{\Omega} \pi_{g(x,\omega)\sigma(dx)}(A)\mathbb{P}(d\omega) \\ &= \mathbb{E}\left(\pi_{g(x,\cdot)\sigma(dx)}(A)\right), \quad A \in \mathcal{B}(\Gamma_X). \end{aligned} \tag{2.17}$$

(Here $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$.) Such a measure is called a Cox point process. For simplicity, we will call it a Cox measure.

Proposition 2.2. *The Cox measure μ has all local moments finite, i.e., for any $\Delta \in \mathcal{B}_0(X)$ and for any $n \in \mathbb{N}$,*

$$\int_{\Gamma_X} |\gamma \cap \Delta|^n \, \mu(d\gamma) < \infty.$$

Proof. We fix $\Delta \in \mathcal{B}_0(X)$ and let χ_Δ denote the indicator function of Δ. Then, for each $n \in \mathbb{N}$,

$$\begin{aligned}\int_{\Gamma_X} |\gamma \cap \Delta|^n \mu(d\gamma) &= \int_{\Gamma_X} \langle \chi_\Delta, \gamma\rangle^n \mu(d\gamma) \\ &= \int_{\Gamma_X} \sum_{x_1\in\gamma} \cdots \sum_{x_n \in \gamma} \chi_\Delta(x_1)\cdots\chi_\Delta(x_n)\mu(d\gamma). \qquad (2.18)\end{aligned}$$

To show that (2.18) is finite, it suffices to prove that, for each $n \in \mathbb{N}$,

$$\int_{\Gamma_X} \sum_{\{x_1,\dots,x_n\}\subset\gamma} \chi_\Delta(x_1)\cdots\chi_\Delta(x_n)\mu(d\gamma) < \infty.$$

But by Corollary 2.1 and (2.16), we have

$$\begin{aligned}&\int_{\Gamma_X} \sum_{\{x_1,\dots,x_n\}\subset\gamma} \chi_\Delta(x_1)\cdots\chi_\Delta(x_n)\mu(d\gamma) \\ &= \int_\Omega \left(\int_{\Gamma_X} \sum_{\{x_1,\dots,x_n\}\subset\gamma} \chi_\Delta(x_1)\cdots\chi_\Delta(x_n)\pi_{g(x,\omega)\sigma(dx)} \right) \mathbb{P}(d\omega) \\ &= \frac{1}{n!}\int_\Omega \left(\int_{X^n} \chi_\Delta(x_1)\cdots\chi_\Delta(x_n) g(x_1,\omega)\dots g(x_n,\omega)\sigma(dx_1)\cdots\sigma(dx_n)\right)\mathbb{P}(d\omega) \\ &= \frac{1}{n!}\int_\Omega \left(\int_\Delta g(x,\omega)\sigma(dx)\right)^n \mathbb{P}(d\omega) < \infty. \quad \square \qquad (2.19)\end{aligned}$$

The following proposition identifies correlation functions of a Cox measure.

Proposition 2.3. *The Cox measure μ has correlation functions*

$$k_\mu^{(n)}(x_1,\dots,x_n) = \int_\Omega g(x_1,\omega)\cdots g(x_n,\omega)\mathbb{P}(d\omega), \quad n \in \mathbb{N} \qquad (2.20)$$

which are $\sigma^{\otimes n}$-a.e. finite.

Proof. Let us take any symmetric measurable function $f^{(n)} : X^n \to [0,\infty]$. Then, by Corollary 2.1,

$$\int_{\Gamma_X} \sum_{\{x_1,\dots,x_n\}\subset\gamma} f^{(n)}(x_1,\dots,x_n)\mu(d\gamma)$$

$$
\begin{aligned}
&= \int_\Omega \mathbb{P}(d\omega) \int_{\Gamma_X} \sum_{\{x_1,\dots,x_n\}\subset\gamma} f^{(n)}(x_1,\dots,x_n)\pi_{g(x,\omega)\sigma(dx)}(d\gamma) \\
&= \int_\Omega \mathbb{P}(d\omega) \int_{X^n} f^{(n)}(x_1,\dots,x_n) g(x_1,\omega)\cdots g(x_n,\omega)\sigma(dx_1)\cdots\sigma(dx_n) \\
&= \int_{X^n} f^{(n)}(x_1,\dots,x_n)\left(\int_\Omega g(x_1,\omega)\cdots g(x_n,\omega)\mathbb{P}(d\omega)\right)\sigma(dx_1)\cdots\sigma(dx_n),
\end{aligned}
$$

which proves (2.20).

Let us choose any $\Delta \in \mathcal{B}_0(X)$. Then, by (2.16)

$$
\begin{aligned}
&\int_{\Delta^n}\int_\Omega g(x_1,\omega)\cdots g(x_n,\omega)\mathbb{P}(d\omega)\sigma(dx_1)\cdots\sigma(dx_n) \\
&= \int_\Omega \left(\int_\Delta g(x_1,\omega)\sigma(dx_1)\right)\left(\int_\Delta g(x_2,\omega)\sigma(dx_2)\right)\cdots\left(\int_\Delta g(x_n,\omega)\sigma(dx_n)\right)\mathbb{P}(d\omega) \\
&= \int_\Omega \left(\int_\Delta g(x,\omega)\sigma(dx)\right)^n \mathbb{P}(d\omega) < \infty.
\end{aligned}
$$

Therefore, for $\sigma^{\otimes n}$-a.a. $(x_1,\dots,x_n)\in\Delta^n$, $\int_\Omega g(x_1,\omega)\cdots g(x_n,\omega)\,\mathbb{P}(d\omega)<\infty$, and hence the above is true for $\sigma^{\otimes n}$-a.a. $(x_1,\dots,x_n)\in X^n$. $\square$

Next, let us find the Laplace transform of μ. If $f \in B_0(X)$, and $f \le 0$, then, by (2.9)

$$
\begin{aligned}
\int_{\Gamma_X} e^{\langle f,\gamma\rangle}\mu(d\gamma) &= \int_\Omega\left(\int_{\Gamma_X} e^{\langle f,\gamma\rangle}\pi_{g(x,\omega)\sigma(dx)}(d\gamma)\right)\mathbb{P}(d\omega) \\
&= \int_\Omega \exp\left(\int_X (e^{f(x)}-1)g(x,\omega)\sigma(dx)\right)\mathbb{P}(d\omega).
\end{aligned}
$$

(Note that due to the condition $f \le 0$, the above integrals are finite.) However, to be able to write down the Laplace transform for a general $f \in B_0(X)$, we need to have a bound on correlation functions. First, we will derive the following fact, which is true for any point process with some upper bound on its correlation functions.

Proposition 2.4. *Assume ρ is a probability measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$. Assume that ρ has correlation functions $(k_\rho^{(n)})_{n=1}^\infty$ which satisfy*

$$
k_\rho^{(n)}(x_1,\dots,x_n) \le C^n (n!)^\varepsilon \tag{2.21}
$$

for $\sigma^{\otimes n}$*-a.a.* $(x_1,\dots,x_n) \in X^n$, $n \in \mathbb{N}$, *where* $C > 0$ *and* $\varepsilon \in [0,1)$. *Then, for each* $f \in B_0(X)$, *we have*

$$\int_{\Gamma_X} e^{\langle f,\gamma\rangle}\rho(d\gamma)$$
$$= 1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{X^n}(e^{f(x_1)}-1)\cdots(e^{f(x_n)}-1)k_\rho^{(n)}(x_1,\dots,x_n)\sigma(dx_1)\cdots\sigma(dx_n), \tag{2.22}$$

the integral on the left hand side of (2.22) *being finite.*

Proof. For a function $u : X \to \mathbb{R}$, we define $e_\lambda(u,\cdot) : \Gamma_{X,0} \to \mathbb{R}$ by

$$e_\lambda(u,\eta) := \prod_{x\in\eta} u(x),$$

where $\prod_{x\in\varnothing} u(x) := 1$. Then, for each $f \in B_0(X)$, we easily have

$$\left(Ke_\lambda(e^f-1,\cdot)\right)(\gamma) = e^{\langle f,\gamma\rangle}. \tag{2.23}$$

Hence, using (2.23) and the upper bound (2.21), we conclude the statement. □

Now, let us assume that

$$\int_\Omega g(x,\omega)^n\mathbb{P}(d\omega) \le C^n(n!)^\varepsilon, \tag{2.24}$$

where $C > 0$, $\varepsilon \in [0,1)$. Then, for any $(x_1,\dots,x_n) \in X^n$, we get for the Cox measure μ,

$$\begin{aligned}
k_\mu^{(n)}(x_1,\dots,x_n) &= \int_\Omega g(x_1,\omega)\cdots g(x_n,\omega)\mathbb{P}(d\omega)\\
&\le \left(\int_\Omega g^n(x_1,\omega)\mathbb{P}(d\omega)\right)^{\frac{1}{n}}\cdots\left(\int_\Omega g^n(x_n,\omega)\mathbb{P}(d\omega)\right)^{\frac{1}{n}}\\
&\le C^n(n!)^\varepsilon.
\end{aligned}$$

Hence, we have

Proposition 2.5. *Assume that* (2.24) *holds. Then, for each* $f \in B_0(X)$, *we have*

$$\int_{\Gamma_X} e^{\langle f,\gamma\rangle}\mu(d\gamma)$$
$$= 1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{X^n}(e^{f(x_1)}-1)\cdots(e^{f(x_n)}-1)k_\mu^{(n)}(x_1,\dots,x_n)\sigma(dx_1)\cdots\sigma(dx_n), \tag{2.25}$$

where $k_\mu^{(n)}$ *is given by* (2.20).

Remark 2.2. Assume that the estimate (2.24) holds for $\varepsilon = 1$, i.e.,

$$\int_\Omega g(x,\omega)^n \mathbb{P}(d\omega) \le C^n n!\,.$$

Then, it follows from the above that we still can write down formula (2.25) for $f \in B_0(X)$ which is sufficiently small, more exactly for f satisfying

$$\int_X |e^{f(x)}-1|\sigma(dx) < \frac{1}{C}\,.$$

Note that, in many situations, the knowledge of the Laplace transform in a neighbourhood of 0 is enough.

Another trivial remark about Cox measures is the formula for its local densities. In general, if ρ is a probability measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$, and if $\Lambda \in \mathcal{B}_0(X)$, then we denote by $\rho^{(\Lambda)}$ the image of ρ under the projection mapping $P^{(\Lambda)} : \Gamma_X \to \Gamma_\Lambda$ given by $P^{(\Lambda)}(\gamma) := \gamma \cap \Lambda$, $\gamma \in \Gamma_X$.

In particular, if π_σ is the Poisson measure with intensity σ, then $\pi_\sigma^{(\Lambda)}$ is the projection of π_σ onto Γ_Λ. If the projection $\rho^{(\Lambda)}$ is absolutely continuous with respect to the Lebesgue-Poisson measure $\Pi_\sigma^{(\Lambda)}$, then we call its Radon-Nikodym derivative $\frac{d\rho^{(\Lambda)}}{d\Pi_\sigma^{(\Lambda)}}$ the local density of ρ in Λ. We then define

$$d_{\rho,\Lambda}^{(0)} := \rho^{(\Lambda)}(\varnothing),$$

$$d^{(n)}_{\rho,\Lambda}(x_1,\dots,x_n) = \frac{d\rho^\Lambda}{d\Pi^{(\Lambda)}_\sigma}(\{x_1,\dots,x_n\}), \quad (x_1,\dots,x_n)\in\Lambda^n, n\in\mathbb{N}.$$

In particular, in the case where $\rho = \pi_{g(\cdot)\sigma}$, we have

$$d^{(0)}_{\pi_{g(\cdot)\sigma},\Lambda} = \exp\left(-\int_\Lambda g(x)\sigma(dx)\right),$$
$$d^{(n)}_{\pi_{g(\cdot)\sigma},\Lambda}(x_1,\dots,x_n) = \exp\left(-\int_\Lambda g(x)\sigma(dx)\right) g(x_1)\cdots g(x_n).$$

From here, we conclude:

Proposition 2.6. *Let μ be a Cox measure satisfying* (2.16). *Then, for each $\Lambda\in\mathcal{B}_0(X)$, we have*

$$d^{(0)}_{\mu,\Lambda} = \int_\Omega \exp\left(-\int_\Lambda g(x,\omega)\sigma(dx)\right)\mathbb{P}(d\omega),$$
$$d^{(n)}_{\mu,\Lambda}(x_1,\dots,x_n) = \int_\Omega \exp\left(-\int_\Lambda g(x,\omega)\sigma(dx)\right) g(x_1,\omega)\cdots g(x_n,\omega)\mathbb{P}(d\omega), \quad n\in\mathbb{N}.$$

The integrals are $\sigma^{\otimes n}$ a.e. finite.

Our next aim is to prove an analogue of the Mecke formula for a Cox measure. Consider the space $\Omega\times\Gamma_X$ equipped with the product σ-algebra $\mathcal{A}\otimes\mathcal{B}(\Gamma_X)$. Denote by $\widetilde{\mathbb{P}}$ the probability measure on $(\Omega\times\Gamma_X, \mathcal{A}\otimes\mathcal{B}(\Gamma_X))$ defined by

$$\widetilde{\mathbb{P}}(d\omega, d\gamma) := \mathbb{P}(d\omega)\pi_{g(x,\omega)\sigma(dx)}(d\gamma). \tag{2.26}$$

We will denote by $\widetilde{\mathbb{E}}$ the expectation with respect to $\widetilde{\mathbb{P}}$. Denote by $\mathcal{F}$ the sub-σ-algebra of $\mathcal{A}\otimes\mathcal{B}(\Gamma_X)$ containing all sets of the form $\Omega\times B$, where $B\in\mathcal{B}(\Gamma_X)$, (in particular, $\Omega\times\emptyset=\emptyset$). Thus, an $\mathcal{F}$-measurable random variable $F(\omega,\gamma)$ does not depend on ω.

Theorem 2.2. *Let μ be a Cox measure. We define*

$$r(x,\gamma) := \widetilde{\mathbb{E}}(g(x,\cdot)\mid\mathcal{F})(\gamma). \tag{2.27}$$

Then, for each measurable function $F : X \times \Gamma_X \to [0, +\infty]$, we have

$$\int_{\Gamma_X} \sum_{x \in \gamma} F(x, \gamma) \mu(d\gamma) = \int_{\Gamma_X} \mu(d\gamma) \int_X \sigma(dx)\, r(x, \gamma) F(x, \gamma \cup x). \qquad (2.28)$$

Remark 2.3. Formula (2.28) means, in particular, that a Cox measure μ has property (Σ'_σ), which was introduced by Matthes, Warmuth, and Mecke in [38]. The function $r(x, \gamma)$ is called the Papangelou intensity of a Cox measure. This property may be treated as a weak Gibbsianess of μ, compare with [19].

Proof of Theorem 2.2. Fix any measurable function $F : X \times \Gamma_X \to [0, +\infty]$. Then, using the definition of Cox measure and the Mecke formula, we have:

$$\begin{aligned}
\int_{\Gamma_X} \sum_{x \in \gamma} F(x, \gamma) \mu(d\gamma) &= \int_\Omega \mathbb{P}(d\omega) \int_{\Gamma_X} \pi_{g(\cdot,\omega)d\sigma}(d\gamma) \sum_{x \in \gamma} F(x, \gamma) \\
&= \int_\Omega \mathbb{P}(d\omega) \int_{\Gamma_X} \pi_{g(\cdot,\omega)d\sigma}(d\gamma) \int_X \sigma(dx) g(x, \omega) F(x, \gamma \cup x) \\
&= \int_X \sigma(dx) \int_\Omega \mathbb{P}(d\omega) \int_{\Gamma_X} \pi_{g(\cdot,\omega)d\sigma}(d\gamma) g(x, \omega) F(x, \gamma \cup x) \\
&= \int_X \sigma(dx) \int_{\Omega \times \Gamma_X} \widetilde{\mathbb{P}}(d\omega, d\gamma) g(x, \omega) F(x, \gamma \cup x) \\
&= \int_X \sigma(dx) \int_{\Omega \times \Gamma_X} \widetilde{\mathbb{P}}(d\omega, d\gamma) \widetilde{\mathbb{E}}(g(x, \omega) F(x, \gamma \cup x) \mid \mathcal{F})(\gamma) \\
&= \int_X \sigma(dx) \int_{\Omega \times \Gamma_X} \widetilde{\mathbb{P}}(d\omega, d\gamma) \widetilde{\mathbb{E}}(g(x, \omega) \mid \mathcal{F})(\gamma) F(x, \gamma \cup x) \\
&= \int_X \sigma(dx) \int_{\Omega \times \Gamma_X} \widetilde{\mathbb{P}}(d\omega, d\gamma) r(x, \gamma) F(x, \gamma \cup x) \\
&= \int_X \sigma(dx) \int_\Omega \mathbb{P}(d\omega) \int_{\Gamma_X} \pi_{g(\cdot,\omega)d\sigma}(\gamma) r(x, \gamma) F(x, \gamma \cup x) \\
&= \int_X \sigma(dx) \int_{\Gamma_X} \mu(d\gamma) r(x, \gamma) F(x, \gamma \cup x) \\
&= \int_{\Gamma_X} \mu(d\gamma) \int_X \sigma(dx) r(x, \gamma) F(x, \gamma \cup x). \quad \square
\end{aligned}$$

Chapter 3

Permanental point processes

In this chapter, we will discuss one of the most important classes of Cox point processes — the permanental (or boson) point process [10, 37, 44]. First, we recall some facts from functional analysis.

3.1 Preliminaries

We will first recall some facts related to functional analysis and infinite dimensional analysis. For further details and proofs, see e.g. [5, 6, 7]

3.1.1 Some classes of linear operators

For any Hilbert spaces H_1 and H_2, we denote by $B(H_1, H_2)$ the set of all bounded linear operators from H_1 into H_2. As usual, we denote $B(H) := B(H, H)$.

We will always assume that all the Hilbert spaces we consider are real and separable, i.e., they possess a countable dense subset.

Integral operators: Let $(X, \mathcal{A}, \sigma)$ be a measure space with a σ-finite measure σ. An operator $K \in B(L^2(X, \sigma))$ is called an integral operator if there exists a measurable function $k : X^2 \longrightarrow \mathbb{R}$ such that

$$(KF)(x) = \int_X k(x,y)f(y)\sigma(dy), \quad f \in L^2(X,\sigma).$$

The function $k(x,y)$ is called the integral kernel of the operator K. If the integral operator K is self-adjoint(i.e. $K^* = \mathcal{K}^*$), then its integral kernel is symmetric, i.e,

$$k(y,x) = k(x,y), \quad (x,y) \in X^2.$$

Hilbert–Schmidt operators: An operator $T \in B(H)$ is called a Hilbert–Schmidt operator if there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of H such that

$$\sum_{n=1}^{\infty} \|Te_n\|^2 < \infty, \tag{3.1}$$

where $\|\cdot\|$denotes the norm on the Hilbert space H. In the latter case, the inequality (3.1) holds for any orthonormal basis $\{e_n\}_{n=1}^{\infty}$ in H and furthermore, the value $\sum_{n=1}^{\infty} \|Te_n\|^2$ is independent of the choice of an orthonormal basis $\{e_n\}_{n=1}^{\infty}$.

For $T \in B(H)$, let T^* denote the adjoint operator of T. Then, T is a Hilbert–Schmidt operator if and only if T^* is a Hilbert–Schmidt operator and

$$\sum_{n=1}^{\infty} \|Te_n\|^2 = \sum_{n=1}^{\infty} \|T^* f_n\|^2$$

for any orthonormal bases $\{e_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ of H.

In the case where $H = L^2(X,\sigma)$, an operator K is Hilbert–Schmidt if and only if K is an integral operator and $k \in L^2(X^2, \sigma^{\otimes 2})$, where k is the integral kernel of K. In fact, one has

$$\sum_{n=1}^{\infty} \|Te_n\|^2 = \int_X \int_X k(x,y)^2 \sigma(dx)\sigma(dy)$$

for any orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of $L^2(X,\sigma)$.

An operator $T \in B(H)$ is called a trace class operator if it can be represented as $T = \sum_{k=1}^{n} A_k B_k$, where $n \in \mathbb{N}$ and $A_1, \dots, A_n$, $B_1, \dots, B_n$ are Hilbert–Schmidt operators. If T is a trace class operator and $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis in H, then the series $\sum_{n=1}^{\infty} (Te_n, e_n)_H$ converges absolutely and its value, called the trace of the operator T, is independent of the choice of orthonormal basis.

3.1.2 Functional calculus of a self-adjoint operator

Let T be a bounded operator in a complex Hilbert space H. Then the spectrum of T is the subset $\sigma(T)$ of $\mathbb{C}$ consisting of all $z \in \mathbb{C}$ such that the operator $A - z\mathbf{1}$ is not invertible. If the Hilbert space H is finite-dimensional, then the spectrum of T is just the set of its eigenvalues. The spectrum $\sigma(T)$ is a closed subset of the circle centered at 0 and of radius $\|T\|$. If, additionally, T is a self-adjoint operator, then $\sigma(T) \subset \mathbb{R}$, so that $\sigma(T)$ is a closed subset of the interval $[-\|T\|, \|T\|]$.

A mapping

$$\mathcal{B}(\mathbb{R}) \ni \alpha \mapsto E(\alpha) \in B(H)$$

is called a resolution of the identity if the following conditions are satisfied:

- For each $\alpha \in \mathcal{A}$, $E(\alpha)$ is an orthogonal projection in H.
- $E(\varnothing) = 0$, $E(\mathbb{R}) = \mathbf{1}$.
- If $\{\alpha_n\}_{n=1}^{\infty}$, $\alpha_n \in \mathcal{B}(\mathbb{R})$, $n \in \mathbb{N}$, α_n are mutually disjoint, then for each $f \in H$

$$E\Big(\bigcup_{n=1}^{\infty} \alpha_n\Big) f = \sum_{n=1}^{\infty} E(\alpha_n) f,$$

where the series converges in H.

It follows from the definition of resolution of the identity that, for any vectors $f, g \in H$, the mapping

$$\mathcal{A} \ni \alpha \mapsto (E(\alpha)f, g)_H$$

is a complex-valued measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

To any bounded self-adjoint operator T in H, there corresponds a unique resolution of the identity such that

$$T = \int_{\mathbb{R}} \lambda \, dE(\lambda). \tag{3.2}$$

The equality (3.2) should be understood as follows: for any $f, g \in H$

$$(Tf, g)_H = \int_{\mathbb{R}} \lambda \, d(E(\lambda)f, g). \tag{3.3}$$

Furthermore, the inverse statement holds. If E is a resolution of the identity with compact support in $\mathbb{R}$, that is if there exists a compact set $\Delta \in B(\mathbb{R})$ such that, for any $f, g \in H$, the measure $d(E(\lambda)f, g)$ is concentrated on Δ, then E determines a bounded self-adjoint operator in H through the formulas (3.2) and (3.3).

In fact, the resolution of the identity of any bounded self-adjoint operator T is concentrated on the spectrum of T. That is, for any $f, g \in H$, the support of the measure $d(E(\lambda)f, g)$ is a subset of $\sigma(T)$.

Let F be a real-valued bounded measurable function on the spectrum of T. Then one defines a bounded self-adjoint operator $F(T)$ as

$$F(T) := \int_{\sigma(T)} F(\lambda) d\mathbb{E}(\lambda),$$

that is for any $f, g \in H$.

$$(F(T)f, g)_H = \int_{\sigma(T)} F(\lambda) d\left(\mathbb{E}(\lambda)f, g\right).$$

One can show that, for any bounded measurable functions $F, G : \sigma(T) \to \mathbb{R}$,

$$F(T)G(T) = \int_{\sigma(T)} F(\lambda)G(\lambda)d\mathbb{E}(\lambda).$$

For example, choosing function f to be $f(x) = \sqrt{x}$ and assuming that $T \geq 0$, i.e., $(Tf, f) \geq 0$ for all $f \in H$ (so that $\sigma(T) \subset [0, \|T\|]$), we get the operator $\sqrt{T}$, which is a bounded self-adjoint operator satisfying $\sqrt{T}\sqrt{T} = T$.

3.1.3 Integral kernel of a locally trace-class operator

In this section, we follow [32]. Let X be a topological space as in Chapter 2, and let σ be a non-atomic Radon measure on $(X, \mathcal{B}(X))$. Let K be a bounded, linear, self-adjoint operator in $L^2(X, \sigma)$ which satisfies $K \geq 0$. Denote $K_1 := \sqrt{K}$, (see subsection 3.1.2)

Below, for an integral operator I in $L^2(X, \sigma)$, we will denote by $\mathcal{N}(I)$ the kernel of I.

For each $\Delta \in \mathcal{B}_0(X)$ denote by P_Δ the operator of multiplication by χ_Δ, which is an orthogonal projection in $L^2(X, \sigma)$. We will assume that K is a locally trace class operator, that is, for each $\Delta \in \mathcal{B}_0(X)$ the operator $P_\Delta K P_\Delta$ is of trace class. For each $\Delta \in \mathcal{B}_0(X)$,

$$P_\Delta K_1 (P_\Delta K_1)^* = P_\Delta K_1 K_1 P_\Delta = P_\Delta K P_\Delta.$$

Therefore, if $(e_n)_{n=1}^\infty$ is an orthonormal basis of $L^2(X, \sigma)$, then

$$\begin{aligned}
\sum_{n=1}^\infty (P_\Delta K P_\Delta e_n, e_n) &= \sum_{n=1}^\infty (P_\Delta K_1 (P_\Delta K_1)^* e_n, e_n) \\
&= \sum_{n=1}^\infty ((P_\Delta K_1)^* e_n, (P_\Delta K_1)^* e_n) \\
&= \sum_{n=1}^\infty \|(P_\Delta K_1)^* e_n\|^2 < \infty.
\end{aligned}$$

Hence $(P_\Delta K_1)^*$ is a Hilbert-Schmidt operator, and so $P_\Delta K_1$ is a Hilbert-Schmidt operator. This implies that $P_\Delta K_1$ is an integral operator, whose integral kernel $\mathcal{N}(P_\Delta K_1)$ belongs to $L^2(X^2, \sigma^{\otimes 2})$. Clearly, for any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$ such that $\Delta_1 \subset \Delta_2$, $P_{\Delta_1} K_1 = P_{\Delta_1}(P_{\Delta_2} K_1)$. Therefore, for each $f \in L^2(X, \sigma)$

$$\begin{aligned}(P_{\Delta_1} K_1 f)(x) &= (P_{\Delta_1}(P_{\Delta_2} K_1) f)(x) \\ &= \chi_{\Delta_1}(x) \int_X \mathcal{N}(P_{\Delta_2} K_1)(x, y) f(y) \sigma(dy) \\ &= \int_X \chi_{\Delta_1}(x) \mathcal{N}(P_{\Delta_2} K_1)(x, y) f(y) \sigma(dy).\end{aligned}$$

Hence,

$$\mathcal{N}(P_{\Delta_1} K_1)(x, y) = \chi_{\Delta_1}(x) \mathcal{N}(P_{\Delta_2} K_1)(x, y). \tag{3.4}$$

We now define a function $\mathcal{K}_1(x, y) := \mathcal{N}(P_\Delta K_1)(x, y)$, where $\Delta \in \mathcal{B}_0(X)$ is such that $x \in \Delta$. In view of (3.4), the definition of $\mathcal{K}_1$ is independent of the choice of a set $\Delta \in \mathcal{B}_0(X)$. Hence, for any $x \in X$,

$$(K_1 f)(x) = \int_X \mathcal{K}_1(x, y) f(y) \sigma(dy),$$

so that K_1 is an integral operator and $\mathcal{N}(K_1)(x, y) = \mathcal{K}_1(x, y)$. Furthermore, we have for each $\Delta \in \mathcal{B}_0(X)$,

$$\int_\Delta \int_X \mathcal{K}_1(x, y)^2 \sigma(dx) \sigma(dy) < \infty. \tag{3.5}$$

In particular,

$$\mathcal{K}_1(x, \cdot) \in L^2(X, \sigma) \text{ for } \sigma\text{-a.a. } x \in X. \tag{3.6}$$

Since $K = K_1 K_1$, K is an integral operator whose kernel is given by

$$\mathcal{K}(x, y) := \mathcal{N}(K)(x, y) = \int_X \mathcal{K}_1(x, z) \mathcal{K}_1(z, y) \sigma(dz)$$

$$= \int_X \mathcal{K}_1(x,z)\mathcal{K}_1(y,z)\sigma(dz)$$
$$= (\mathcal{K}_1(x,\cdot), \mathcal{K}_1(y,\cdot))_{L^2(X,\sigma)}. \tag{3.7}$$

By (3.7), for any $\Delta \in \mathcal{B}_0(X)$, we get

$$\int_\Delta \mathcal{K}(x,x)\sigma(dx) = \int_\Delta \left(\int_X \mathcal{K}_1(x,y)\mathcal{K}_1(x,y)\sigma(dy) \right) \sigma(dx)$$
$$= \int_{\Delta \times X} \mathcal{K}_1(x,y)^2 \sigma(dx)\sigma(dy) < \infty. \tag{3.8}$$

Note that the kernel $\mathcal{K}_1(x,y)$ is defined up to a set of $\sigma^{\otimes 2}$-measure 0 in X^2, but the value $\int_\Delta \mathcal{K}(x,x)\sigma(dx)$ is independent of the choice of $\mathcal{K}_1(x,y)$.

3.1.4 Gaussian random fields

Our next aim is to construct a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a Gaussian random field $(\mathfrak{X}(x))_{x\in X}$ on it with zero mean and the covariance $\mathcal{K}(x,y)$ as in subsection 3.1.3. To this end, we will need the notion of a Gaussian white noise measure.

First, let us briefly recall the notion of a space with negative norm, see e.g. [7]. Let H_0 be a real Hilbert space with scalar product $(\cdot,\cdot)_{H_0}$ and norm $\|\cdot\|_{H_0}$, and we suppose that

$$H_+ \subseteq H_0,$$

where H_+ is a dense subset of H_0. We suppose that H_+ is a Hilbert space with respect to another scalar product $(\cdot,\cdot)_{H_+}$ and that the norm $\|\cdot\|_{H_+}$ in H_+ is such that

$$\|\cdot\|_{H_0} \le \|\cdot\|_{H_+}. \tag{3.9}$$

Each element $f \in H_0$ generates a linear continuous functional $\langle f, \cdot\rangle$ on H_+ by the formula

$$\langle f, u\rangle := (f,u)_{H_0}, \quad u \in H_+ \tag{3.10}$$

We introduce a new norm on H_0, denoted by $\|\cdot\|_{H_-}$, by taking the norm of f as the norm of the functional $\langle f, \cdot \rangle$:

$$\|f\|_{H_-} := \sup \left\{ \frac{|(f,u)_{H_0}|}{\|u\|_{H_+}} \mid u \in H_+, \; u \neq 0 \right\}. \tag{3.11}$$

Now we complete H_0 in the norm (3.11) and obtain a Banach space H_-, which is called the space with negative norm. Thus we have constructed the chain

$$H_+ \subseteq H_0 \subseteq H_- \tag{3.12}$$

of spaces with positive, zero and negative norms. One also says that (3.12) is a rigging of the Hilbert space H_0 by spaces H_+ and H_-.

Each element $\alpha \in H_-$ is clearly a linear continuous functional on H_+, so that

$$H_- \subseteq (H_+)', \tag{3.13}$$

where $(H_+)'$ denotes the dual space of H_+. We will write $(\alpha, u)_{H_0}$ or $\langle \alpha, u \rangle$ for the action of the functional α on an element $u \in H_+$. It is obvious that

$$|(\alpha, u)_{H_0}| \leq \|\alpha\|_{H_-} \|u\|_{H_+}, \quad \alpha \in H_-, \; u \in H_+, \tag{3.14}$$

which is a generalization of the Cauchy inequality.

In fact, H_- is a Hilbert space. That is, one can introduce a scalar product $(\cdot,\cdot)_{H_-}$ on H_- so that H_- is a Hilbert space and the norm $\|\cdot\|_{H_-}$ is given through $(\cdot,\cdot)_{H_-}$, i.e., $\|\alpha\|_{H_-} = \sqrt{(\alpha,\alpha)_{H_-}}$, $\alpha \in H_-$. One can show that $H_- = (H_+)'$, i.e, H_- can be thought of as the dual space of H_+. This means that any linear continuous functional $l \in (H_+)'$ is of the form $l(u) = (\alpha, u)_{H_0}$, $u \in H_+$, for some $\alpha \in H_-$.

A rigging $H_+ \subseteq H_0 \subseteq H_-$ is called quasi-nuclear if the inclusion operator $O : H_+ \to H_0$ is a quasi-nuclear (i.e., Hilbert-Schmidt) operator:

$$\sum_{n=1}^{\infty} \|Oe_n\|_{H_0}^2 < \infty,$$

where $(e_n)_{n=1}^{\infty}$ is an orthonormal basis of H_+. In fact, the inclusion operator $O : H_+ \to H_0$ is quasi-nuclear if and only of the inclusion operator $O' : H_0 \to H_-$ is quasi-nuclear.

Let us consider an example of a chain of Hilbert spaces. For any $\tau = (\tau_k)_{k=1}^{\infty}$, $\tau_k > 0$, we define

$$\ell_2(\tau) := \left\{ (f_k)_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} \mid \sum_{k=1}^{\infty} f_k^2 \tau_k < \infty \right\} \tag{3.15}$$

and for any $f = (f_k)_{k=1}^{\infty}$, $g = (g_k)_{k=1}^{\infty} \in \ell_2(\tau)$ we define

$$(f, g)_{\ell_2(\tau)} := \sum_{k=1}^{\infty} f_k g_k \tau_k. \tag{3.16}$$

Then $\ell_2(\tau)$ is a Hilbert space.

Suppose that $\tau_k \geq 1$, $k \in \mathbb{N}$. Then evidently $\ell_2(\tau) \subset \ell_2$ and $\|\cdot\|_{\ell_2} \leq \|\cdot\|_{\ell_2(\tau)}$. Denote by $\ell_{2,0}$ all finite sequences

$$(f_1, f_2, \ldots, f_N, 0, 0, \ldots),$$

where $f_1, \ldots, f_N \in \mathbb{R}$, $N \in \mathbb{N}$. Clearly, $\ell_{2,0} \subset \ell_2(\tau)$ and $\ell_{2,0}$ is dense in ℓ_2. Therefore, $\ell_2(\tau)$ is dense in ℓ_2. Furthermore, by the definition of $\ell_2(\tau)$, for each $f \in \ell_2(\tau)$, $\|f\|_{\ell_2} \leq \|f\|_{\ell_2(\tau)}$. Therefore, we may set $H_0 = \ell_2$ and $H_+ = \ell_2(\tau)$. It can be shown that

$$H_- = \ell_2(\tau^{-1}), \quad \tau^{-1} := (\tau_k^{-1})_{k=1}^{\infty}.$$

Thus, we get the chain

$$\ell_2(\tau) \subset \ell_2 \subset \ell_2(\tau^{-1}).$$

Furthermore, the inclusion $\ell_2(\tau) \subseteq \ell_2$ is quasi-nuclear if and only if

$$\sum_{k=1}^{\infty} \frac{1}{\tau_k} < \infty. \tag{3.17}$$

For example, one can choose $\tau_k = k^2$ in order to get the quasi-nuclear embedding of $\ell_2(\tau)$ into ℓ_2.

Let now H_0 be a separable Hilbert space. Thus, H_0 has an orthonormal basis $(e_n)_{n=1}^{\infty}$. We define a unitary operator

$$I : H_0 \to \ell_2$$

by setting for every $f = \sum_{k=1}^{\infty} f_k e_k \in H_0$, $If := (f_1, f_2, f_3, \cdots) \in \ell_2$.

Then, we fix a sequence τ as above and define a Hilbert space $H_+ = I^{-1}\ell_2(\tau)$ with scalar product $(u, v)_{H_+} = (Iu, Iv)_{\ell_2(\tau)}$, $u, v \in H_+$. Thus, for any $f = \sum_{k=1}^{\infty} f_k e_k$, $g = \sum_{k=1}^{\infty} g_k e_k$ from H_+, we have

$$(f, g)_{H_+} = \sum_{k=1}^{\infty} f_k g_k \tau_k$$

.

Then, H_+ is densely and continuous embedded into H_0, and in fact I may be extended by continuity to a unitary operator $I : H_- \to \ell_2(\tau^{-1})$. Thus, we get a rigging $H_+ \subset H_0 \subset H_-$ and it is quasi-nuclear if and only if condition (3.17) is satisfied.

Let now $H_+ \subset H_0 \subset H_-$ be a quasi-nuclear rigging. We denote by $\mathscr{C}(H_-)$ the cylinder σ-algebra on H_-, i.e., the minimal σ-algebra on H_- which contains all cylinder sets of the form

$$C(f_1, \ldots, f_n, A) = \{\omega \in H_- : (\langle \omega, f_1 \rangle, \cdots, \langle \omega, f_n \rangle) \in A\}$$

for some $f_1, \ldots, f_n \in H_+$ and $A \in \mathcal{B}(\mathbb{R}^n)$. The following theorem is an infinite dimensional generalization of the classical Bochner theorem on the Fourier transform of a probability measure.

Theorem 3.1 (Minlos). *Let $H_+ \subset H_0 \subset H_-$ be a quasi-nuclear rigging. Suppose $F : H_+ \to \mathbb{C}$. Then F is the Fourier transform of a unique probability* *measure* $\mathbb{P}$ *on* $(H_-, \mathscr{C}_\sigma(H_-))$, *i.e.*

$$F(f) = \int_{H_-} e^{i\langle \omega, f\rangle} d\mathbb{P}(\omega), \quad f \in H_+, \tag{3.18}$$

if and only if

- $F(0) = 1$,
- *F is positive definite, i.e., for all* $c_1, \ldots, c_n \in \mathbb{C}$, $n \in \mathbb{N}$, $f_1, \ldots, f_n \in H_+$:

$$\sum_{i,j=1}^{n} c_i \overline{c_j} F(f_i - f_j) \geq 0,$$

- *F is continuous on* H_+.

Let again $H_+ \subset H_0 \subset H_-$ be a quasi-nuclear rigging and we define

$$F(f) = \exp\left(-\frac{1}{2}\|f\|_{H_0}^2\right), \quad f \in H_+.$$

One can easily verify that the function $F : H_+ \to \mathbb{C}$ satisfies the conditions of the Minlos theorem. Therefore, there exist a unique probability measure $\mathbb{P}$ on $(H_-, \mathscr{C}_\sigma(H_-))$ such that

$$\int_{H_-} e^{i\langle \omega, f\rangle} d\mathbb{P}(\omega) = \exp\left(-\frac{1}{2}\|f\|_{H_0}^2\right), \quad f \in H_+. \tag{3.19}$$

Let us fix any $f \in H_+$. Then for each $a \in \mathbb{R}$

$$\begin{aligned} \int_{H_-} e^{ia\langle \omega, f\rangle} d\mathbb{P}(\omega) &= \int_{H_-} e^{i\langle \omega, af\rangle} d\mathbb{P}(\omega) \\ &= \exp\left(-\frac{1}{2}a^2\|f\|_{H_0}^2\right). \end{aligned}$$

Therefore, under $\mathbb{P}$ the random variable $\langle \cdot, f\rangle$ has Gaussian distribution with mean 0 and variance $\|f\|_{H_0}^2$. Hence, for each $f \in H_+$, we have $\int_{H_-} \langle \omega, f\rangle^2 \mathbb{P}(d\omega) = \|f\|_{H_0}^2$.

Therefore, the linear operator $I : H_+ \to L^2(H_-, d\mathbb{P})$ given by $(If)(\omega) = \langle \omega, f \rangle$, $\omega \in H_-$, may be extended by continuity to an isometric operator $I : H_0 \to L^2(H_-, d\mathbb{P})$.

Now for each $f \in H_0$, we denote by $\langle \omega, f \rangle$ the random variable on H_- which is the image of f under I, i.e., $\langle \omega, f \rangle := (If)(\omega)$. It can be easily checked that formula (3.19) remains true for each $f \in H_0$. Thus, we get a family $(\langle \omega, f \rangle)_{f \in H_0}$ of random variables, i.e., a random field, such that each $\langle \omega, f \rangle$ is a Gaussian random variable with mean 0, i.e.,

$$\int_{H_-} \langle \omega, f \rangle d\mathbb{P}(\omega) = 0$$

and

$$\begin{aligned} \operatorname{Cov}(\langle \omega, f \rangle, \langle \omega, g \rangle) &= \int_{H_-} \langle \omega, f \rangle, \langle \omega, g \rangle d\mathbb{P}(\omega) \\ &= (f, g)_{H_0}. \end{aligned}$$

The probability measure $\mathbb{P}$ is called the Gaussian white noise measure.

As a conclusion, we may state that, for any separable Hilbert space H_0 there exist a probability space $(\Omega, \mathcal{A}, P)$ and a family $(Y(f))_{f \in H_0}$ of random variables on it such that each $Y(f)$ is Gaussian random variable with mean 0 and $\mathbb{E}(Y(f)Y(g)) = (f, g)_{H_0}$.

Indeed we only need to construct a quasi-nuclear rigging of $H_0 : H_+ \subset H_0 \subset H_-$, then we define the Gaussian white noise measure μ on H_-, and then set $Y(f) = \langle \cdot, f \rangle$, $f \in H_0$.

Theorem 3.2. *There exists a random field $(Y(x))_{x \in X}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that the mapping*

$$X \times \Omega \ni (x, \omega) \mapsto Y(x, \omega) \tag{3.20}$$

is measurable, and for σ-a.a. $x \in X$, $Y(x)$ is a Gaussian random variable with mean 0 *and such that*

$$\begin{aligned} \mathbb{E}(Y(x)^2) &= \mathcal{K}(x,x) \quad \textit{for } \sigma\textit{-a.a. } x \in X, \\ \mathbb{E}\left(Y(x)Y(y)\right) &= \mathcal{K}(x,y) \quad \textit{for } \sigma^{\otimes 2}\textit{-a.a. } (x,y) \in X^2. \end{aligned} \tag{3.21}$$

Remark 3.1. The statement of Theorem 3.2 is well-known if the integral kernel of the operator K admits a continuous version (see e.g. Theorem 1.8 and p. 456 in [44]). In the latter case, $(Y(x))_{x\in X}$ is a Gaussian random field and formula (3.21) holds for all $(x,y) \in X^2$.

Proof of Theorem 3.2. Consider a standard triple of real Hilbert spaces

$$H_+ \subset H_0 = L^2(X,\sigma) \subset H_- .$$

Here the Hilbert space H_+ is densely and continuously embedded into H_0, the inclusion operator $H_+ \hookrightarrow H_0$ is of Hilbert–Schmidt class, and the Hilbert space H_- is the dual space of H_+ with respect to the center space H_0.

Let $\mathbb{P}$ be the standard Gaussian measure on H_-, i.e., the probability measure on the cylinder σ-algebra $\mathscr{C}(H_-)$ which has Fourier transform

$$\int_{H_-} e^{i\langle\omega,f\rangle}\,\mathbb{P}(d\omega) = \exp\left[-\frac{1}{2}\|f\|^2_{H_0}\right], \quad f \in H_+.$$

By (3.6), we set for σ-a.a. $x \in X$, $\widetilde{Y}(x,\omega) := \langle\omega, \mathcal{K}_1(x,\cdot)\rangle$, where $\mathcal{K}(x,\cdot)$ denotes the function of y given by $\|(x,y)$ for a fixed x. Hence $\widetilde{Y}(x)$ is a Gaussian random variable and by (3.7), (3.21) holds.

Hence, it remains to prove that there exists a random field $Y = (Y(x))_{x\in X}$ for which the mapping (3.20) is measurable and such that $Y(x,\omega) = \widetilde{Y}(x,\omega)$ for $\sigma \otimes \mathbb{P}$-a.a. (x,ω). To this end, we fix any $\Lambda \in \mathcal{B}_0(X)$ and denote by $\mathcal{B}(\Lambda)$ the trace σ-algebra of $\mathcal{B}(X)$ on Λ. We define a set $\mathcal{D}_\Lambda$ of the functions $u : \Lambda \times X \to \mathbb{R}$ of the form

$$u(x,y) = \sum_{i=1}^{n} \chi_{\Delta_i}(x) f_i(y), \tag{3.22}$$

where $\Delta_i \in \mathcal{B}(\Lambda)$, $f_i \in H_+$, $i = 1, \dots, n$. Define a linear mapping

$$I_\Lambda : \mathcal{D}_\Lambda \to L^2(\Lambda \times H_-, \sigma \otimes \mathbb{P}) \tag{3.23}$$

by setting, for each $u \in \mathcal{D}_\Lambda$ of the form (3.22),

$$(I_\Lambda u)(x, \omega) = \sum_{i=1}^{n} \chi_{\Delta_i}(x) \langle \omega, f_i \rangle, \quad (x, \omega) \in \Lambda \times H_- .$$

Clearly, I_Λ can be extended to an isometry

$$I_\Lambda : L^2(\Lambda \times X, \sigma^{\otimes 2}) \to L^2(\Lambda \times H_-, \sigma \otimes \mathbb{P}),$$

and we have $I_\Lambda = \mathbf{1}_\Lambda \otimes I$, where $\mathbf{1}_\Lambda$ is the identity operator in $L^2(\Lambda, \sigma)$ and the operator I is as above.

Fix any $u \in L^2(\Lambda \times X, \sigma^{\otimes 2})$. As easily seen, there exist a sequence $(u_n)_{n=1}^\infty \subset \mathcal{D}_\Lambda$ such that $u_n \to u$ in $L^2(\Lambda \times X, \sigma^{\otimes 2})$ and for σ-a.a. $x \in \Lambda$, $u_n(x, \cdot) \to u(x, \cdot)$ in $L^2(X, \sigma)$ Hence, for σ-a.a. $x \in \Lambda$, $I_\Lambda u_n(x, \cdot) \to I_\Lambda u(x, \cdot)$ in $L^2(H_-, \mathbb{P})$, which implies:

$$(I_\Lambda u)(x, \omega) = \langle \omega, u(x, \cdot) \rangle \quad \text{for } \mathbb{P}\text{-a.a. } \omega \in H_- . \tag{3.24}$$

Now, denote by $\mathcal{K}_1^\Lambda$ the restriction of $\mathcal{K}_1$ to the set $\Lambda \times X$. For σ-a.a. $x \in \Lambda$, we define $Y_\Lambda(x) := (I_\Lambda \mathcal{K}_1^\Lambda)(x, \cdot)$. Hence, by (3.24), for σ-a.a. $x \in \Lambda$, $Y_\Lambda(x) = \widetilde{Y}(x)$ $\mathbb{P}$-a.e. Finally, let $(\Lambda_n)_{n=1}^\infty \subset \mathcal{B}_0(X)$ be such that $\Lambda_n \cap \Lambda_m = \emptyset$ if $n \neq m$ and $\bigcup_{n=1}^\infty \Lambda_n = X$. Setting $Y(x) := Y_{\Lambda_n}(x)$ for σ-a.a. $x \in \Lambda_n, n \in \mathbb{N}$, we conclude the statement. Evidently, the definition of $Y(x)$ is independent of the choice of the sets $Lambda_n$. □

3.1.5 Moments of Gaussian white noise measure

In this section, we will present the classical result about the moments of Gaussian white noise measure. Our presentation is based on [7] and we will give a much more detailed proof of the moments theorem.

Let again $H_+ \subset H_0 \subset H_-$ be a quasi-nuclear rigging of a real separable Hilbert space H_0 and let $\mathbb{P}$ be the Gaussian white noise measure on H_-. Recall that, for each $f \in H_0$ we defined a random variable $\langle \omega, f \rangle$ from $L^2(H_-, d\mathbb{P})$.

For each $n \in \mathbb{N}$, the n-th moment of $\mathbb{P}$ is defined by

$$C_n(f_1, \dots, f_n) := \int_{H_-} \langle \omega, f_1 \rangle \cdots \langle \omega, f_n \rangle \mathbb{P}(d\omega), \quad f_1, \dots, f_n \in H_0 \tag{3.25}$$

(Later on we will see that the integral indeed exists)

Theorem 3.3. *(Moments of Gaussian white noise measure) We have*

$$C_{2n+1}(f_1, \dots, f_{2n+1}) = 0 \tag{3.26}$$

$$C_{2n}(f_1, \dots, f_{2n}) = \sum (f_{k_1}, f_{l_1})_{H_0} \cdots (f_{k_n}, f_{l_n})_{H_0}, \tag{3.27}$$

where $f_1, \dots, f_{2n+1} \in H_0$ *and the summation in* (3.27) *is over all possible* $\frac{(2n)!}{2^n n!}$ *pairing of the numbers* $1, 2, \dots, 2n$*, i.e., all possible partition of* $1, 2, \dots, 2n$ *into* n *pairs* $\{k_1, l_1\}, \cdots, \{k_n, l_n\}$

Proof. We first consider the special case where $f_1 = f_2 = \ldots = f_n = f \in H_0$. Then,

$$\begin{aligned}
\int_{H_-} \langle \omega, f \rangle^n \mathbb{P}(d\omega) &= i^{-n} \left(\frac{d}{dt} \right)^n \Big|_{t=0} \int_{H_-} e^{it\langle \omega, f \rangle} \mathbb{P}(d\omega) \\
&= i^{-n} \left(\frac{d}{dt} \right)^n \Big|_{t=0} \int_{H_-} e^{i\langle \omega, tf \rangle} \mathbb{P}(d\omega) \\
&= i^{-n} \left(\frac{d}{dt} \right)^n \Big|_{t=0} \exp\left(-\frac{1}{2} t^2 \|f\|_{H_0}^2 \right)
\end{aligned}$$

For a smooth function $f : \mathbb{R} \to \mathbb{R}$, we have the following well-known formula

$$\frac{d^n}{dt^n} e^{f(t)} = \sum_{\substack{m_1, \dots, m_k \in \mathbb{Z}_+ \\ m_1 + 2m_2 + \cdots + km_k = n}} \frac{n! (f^{(1)}(t))^{m_1} (f^{(2)}(t))^{m_2} \ldots (f^{(n)}(t))^{m_k}}{m_1! (2!)^{m_2} m_2! \ldots (k!)^{m_k} m_k!} e^{f(t)}. \tag{3.28}$$

Setting

$$f(t) = -\frac{1}{2}t^2\|f\|_{H_0}^2, \tag{3.29}$$

we have

$$f'(t) = -t\|f\|_{H_0}^2,$$
$$f''(t) = -\|f\|_{H_0}^2, \qquad f^{(n)}(t) = 0,\ n \geq 3.$$

Hence,

$$f''(0) = -\|f\|_{H_0}^2$$
$$f^{(n)}(0) = 0 \quad if \quad n > 2.$$

Thus, now (3.28) reduces to

$$\frac{d^n}{dt^n}e^{f(t)}|_{t=0} = \sum_{m_2\in\mathbb{Z}_+:\,2m_2=n} \frac{n!(-\|f\|_{H_0}^2)^{m_2}}{(2!)^{m_2}m_2!}.$$

Therefore, it follows that

$$C_{2n+1}(f,\dots,f) = 0,\ C_{2n}(f,\dots,f) = \frac{(2n)!}{2^n n!}\|f\|_{H_0}^{2n}.$$

Let $B : H_0^n \to \mathbb{R}$ be an n-linear mapping, that is, for any $i \in \{1,\dots,n\}$ and any fixed $f_1,\dots,f_{i-1},f_{i+1},\dots,f_n$, the mapping

$$B(f_1,\dots,f_{i-1},\cdot,f_{i+1},\dots,f_n) : H_0 \to \mathbb{R}$$

is linear. For each $f \in H_0$, we denote $\widetilde{B}(f) := B(f,\dots,f)$. Then the following polarization identity holds (see e.g. [5]). For any $f_1,\dots,f_n \in H_0$,

$$B(f_1,\dots,f_n) = \frac{1}{n!}\sum_{p=1}^{n}(-1)^{n-p}\sum_{1\leq k_1<k_2<\cdots<k_p\leq n}\widetilde{B}(f_{k_1}+f_{k_2}+\cdots+f_{k_p}).$$

Thus, the n-linear mapping B is completely identified by $\widetilde{B}$.

Now for any $f_1, \ldots, f_n \in H_0$, by Hölder's inequality

$$\int_{H_-} |\langle \omega, f_1 \rangle \cdots \langle \omega, f_n \rangle| \mathbb{P}(d\omega)$$
$$\leq \left(\int_{H_-} |\langle \omega, f_1 \rangle|^n \mu(d\omega) \cdots \int_{H_-} |\langle \omega, f_n \rangle|^n \mathbb{P}(d\omega) \right)^{\frac{1}{n}}$$
$$\leq \left(\int_{H_-} |\langle \omega, f_1 \rangle|^{2n} \mathbb{P}(d\omega) \cdots \int_{H_-} |\langle \omega, f_n \rangle|^{2n} \mathbb{P}(d\omega) \right)^{\frac{1}{2n}}.$$

Thus, the moments of Gaussian white noise measure are well defined. We note that, for each $n \in N$ the mapping $H_0^n \ni (f_1, \ldots, f_n) \to C_n(f_1, \ldots, f_n) \in \mathbb{R}$ is n-linear. On the other hand, the right hand sides of (3.26) and (3.27) are also $(2n+1)-$linear and $(2n)-$linear mappings, respectively. Therefore, the theorem holds in the general case. □

3.2 Permanental point processes

In this section we will discuss a class of Cox point processes which are called permanental (or boson-like). The kernel $\mathcal{K}(x, y)$ from subsection 3.1.3 will play the role of a correlation kernel (to be explained below). Compared with the available literature (see [10, 44]), in our presentation of these processes we will not assume that the correlation kernel is continuous.

Following [45, 44], we introduce the notion of α-permanent, also called α-determinant. Let $\alpha \in \mathbb{R}$ be fixed. For a square matrix $A = (a_{ij})_{i,j=1}^n$, we define its α-permanent as follows:

$$\operatorname{per}_\alpha A = \sum_{\xi \in S_n} \alpha^{n-n(\xi)} \prod_{i=1}^n a_{i\xi(i)}. \tag{3.30}$$

Here S_n denotes the group of all permutations of $\{1, 2, \ldots, n\}$, and for $\xi \in S_n$ $n(\xi)$ denotes the number of cycles in ξ. In particular, for $\alpha = -1$,

$$\operatorname{per}_{-1} A = \sum_{\xi \in S_n} (-1)^{n-n(\xi)} \prod_{i=1}^n a_{i\xi(i)},$$

so that $\text{per}_{-1} A$ is the usual determinant of A. For $\alpha = 1$,

$$\text{per}_1 A = \sum_{\xi \in S_n} \prod_{i=1}^{n} a_{i\xi(i)},$$

so that $\text{per}_1(A)$ is the usual permanent of A.

Let the operator K acting in $L^2(X, \sigma)$ be as in subsec. 3.1.3. Let the kernel $\mathcal{K}(x, y)$ of K be also chosen as in subsection 3.1.3. Let a random field $(Y(x))_{x \in X}$ be as in Theorem 3.2.

Proposition 3.1. *For each $n \in \mathbb{N}$, we have*

$$\mathbb{E}\left(Y^2(x_1) \cdots Y^2(x_n)\right) = \text{per}_2 \left(\mathcal{K}(x_i, x_j)\right)_{i,j=1}^{n}, \quad (x_1, \dots, x_n) \in X^n.$$

Proof. By the definition of α-permanent, we have:

$$\text{per}_2 \left(\mathcal{K}(x_i, x_j)\right)_{i,j=1}^{n} = \sum_{\xi \in S_n} 2^{n - n(\xi)} \prod_{i=1}^{n} \mathcal{K}(x_i, x_{\xi(i)}).$$

Note that a permutation ξ can be identified with the sequence of corresponding cycles

$$\xi = \{(k_1, k_2, \dots, k_{m_1}), (k_{m_1+1}, \dots, k_{m_1+m_2}), \dots, (k_{m_1+m_2+\dots+m_{n(\xi)-1}+1}, \dots, k_n)\}. \tag{3.31}$$

Here, $\xi(k_1) = k_2$, $\xi(k_2) = k_3, \dots, \xi(k_{m_1-1}) = k_{m_1}$, $\xi(k_{m_1}) = k_1$, and analogously for the other cycles. Without loss of generality, we may assume that

$$k_1 = 1,$$
$$k_{m_1+1} = \min\{k_{m_1+1}, k_{m_1+2}, \dots, k_{m_1+m_2}\},$$
$$\vdots$$
$$k_{m_1+m_2+\dots+m_{n(\xi)-1}+1} = \min\{k_{m_1+m_2+\dots+m_{n(\xi)-1}+1}, \dots, k_n\}$$

and

$$k_1 < k_{m_1+1} < k_{m_1+m_2+1} < \dots < k_{m_1+m_2+\dots+m_{n(\xi)-1}+1}.$$

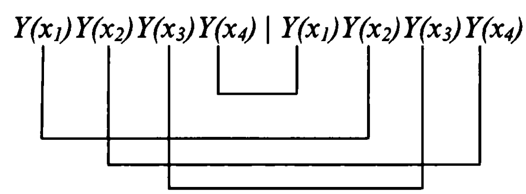
In (3.31), the first cycle contains m_1 elements, the second cycle contains m_2 elements, etc. Hence,

$$2^{n-n(\xi)} = 2^{m_1-1} \cdot 2^{m_2-1} \cdots 2^{m_{n(\xi)}-1}. \tag{3.32}$$

Next, let us write down the product $Y^2(x_1) \cdots Y^2(x_n)$ as

$$Y(x_1) \cdots Y(x_n) \,\Big|\, Y(x_1) \cdots Y(x_n) \tag{3.33}$$

with imaginary bar in the middle, which divides the product into the left and right hand side parts. Consider a collection of pairings of (3.33) which connect elements on the left with elements on the right. Let us call such a collection of pairings a *standard collection.* For example,

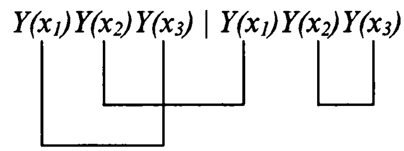


Then, every standard collection identifies a permutation ξ which maps the number of every variable on the left to the number of the variable on the right which is paired with the left one. For example, the pairings in the above formula identify the permutation

$$\xi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}.$$

For every standard collection of pairings, the corresponding term in the expectation of $Y^2(x_1) \cdots Y^2(x_n)$ is clearly $\prod_{i=1}^{n} \mathcal{K}(x_i, x_{\sigma(i)})$, see Theorem 3.3. However, a general collection of pairings of (3.33) should not necessarily be standard. We will now describe a procedure of the change of a collection of pairings such that the resulting collection will always be a standard collection.

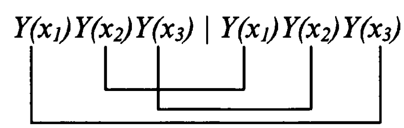
Let us first consider a simple example:



which identifies the expression

$$\mathcal{K}(x_1, x_3)\mathcal{K}(x_2, x_1)\mathcal{K}(x_2, x_3)$$

in the expectation. We start with the left x_1. It is paired with the left x_3. So, instead of this pairing we will take the pairing which connects the left x_1 with the right x_3, which gives us $\xi(1) = 3$, while the x_3 which was on the right is moved to the left. Next, the left x_3 is paired with the right x_2, which requires no changes and gives us $\xi(3) = 2$. Finally, the left x_2 is connected to the right x_1, which again requires no changes and gives us $\xi(2) = 1$. Thus, the required standard collection is



It is easily seen that this procedure can be carried out in the general case. Just start with the left x_1 and look at the pairing. If it connects x_1 with a right element $x_{\xi(1)}$, then leave it without changes, otherwise swap the left and right $x_{\xi(1)}$. Next take $x_{\xi(1)}$ on the left and look at the pairing. If it connects $x_{\xi(1)}$ with $x_{\xi(\xi(1))}$ on the right, then leave it without changes, otherwise swap $x_{\xi^2(1)}$ on the left and on the right, etc, until we close the cycle $(1 = k_1, k_2, \ldots, k_{m_1})$. Then we start with the left variable which has number $\min(\{1, 2, \ldots, n\} \setminus \{k_1, k_2, \ldots, k_{m_1}\})$ and continue this procedure until we close the second cycle $(k_{m_1+1}, k_{m_1+2}, \ldots, k_{m_1+m_2})$. Next we start with the left variable which has number

$$\min(\{1, 2, \ldots, n\} \setminus \{k_1, k_2, \ldots, k_{m_1}, k_{m_1+1}, \ldots, k_{m_1+m_2}\})$$

and continue as before.

Finally we have to calculate how many collections of pairings lead to the same standard collection which identifies a permutation ξ as in (3.31). It is evident that we need to calculate the number of collections of pairings which identify each separable cycle and then multiply these numbers. Let us look at the number of the pairings which identify the cycle $(1 = k_1, k_2, \ldots, k_{m_1})$. So, we start with the left $x_1 = x_{k_1}$ and then we have two possibilities: either x_{k_2} is on the left or it is on the right. After modification of the first pairing, if it is necessary, we look at the left x_{k_2} and again have two possibilities: either x_{k_3} is on the right or it is on the left. Again, if necessary, we modify the pairing and look at the left x_{k_3} and so on. Thus, the total number of possibilities is 2^{m_1-1}. Hence, the total number of different collections of pairings is $2^{m_1-1}2^{m_2-1}\cdots 2^{m_{n(\xi)}-1}$. Comparing this with (3.32) we conclude the proposition. $\square$

Next, let us fix $l \in \mathbb{N}$ and let us take l independent copies of the random field $(Y(x))_{x\in X}$ as above. We call these copies $(Y_i(x))_{x\in X}$, $i = 1, 2, \ldots, l$. We define a new random field

$$g(x) = Y_1^2(x) + Y_2^2(x) + \ldots + Y_l^2(x), \quad x \in X. \tag{3.34}$$

The following theorem generalizes Proposition 3.1.

Theorem 3.4. *For each $n \in \mathbb{N}$, we have*

$$\mathbb{E}\left(g(x_1)\cdots g(x_n)\right) = \mathrm{per}_{\frac{2}{l}}[l\mathcal{K}(x_i, x_j)]_{i,j=1}^n, \quad (x_1, \ldots, x_n) \in X^n.$$

Proof. We have

$$\begin{aligned}
&\mathrm{per}_{\frac{2}{l}}\left(l\mathcal{K}(x_i, x_j)\right)_{i,j=1}^n \\
&= \sum_{\xi\in S_n} \left(\frac{2}{l}\right)^{n-n(\xi)} l^n \prod_{i=1}^n \mathcal{K}(x_i, x_{\xi(i)}) \\
&= \sum_{\xi\in S_n} 2^{n-n(\xi)} l^{n(\xi)} \prod_{i=1}^n \mathcal{K}(x_i, x_{\xi(i)}).
\end{aligned} \tag{3.35}$$

Now, by the definition of $g(x)$

$$\begin{aligned}
&\mathbb{E}\left(g(x_1)\cdots g(x_n)\right) \\
&= \mathbb{E}\left(\sum_{i_1=1}^l \sum_{i_2=1}^l \cdots \sum_{i_n=1}^l Y_{i_1}^2(x_1) Y_{i_2}^2(x_2)\cdots Y_{i_n}^2(x_n)\right) \\
&= \mathbb{E}\left(\sum_{(\eta_1,\eta_2,\ldots,\eta_l)\in P_l(\{1,2,\ldots,n\})} (\prod_{i_1\in\eta_1} Y_1^2(x_{i_1}))(\prod_{i_2\in\eta_2} Y_2^2(x_{i_2}))\cdots(\prod_{i_l\in\eta_l} Y_l^2(x_{i_l}))\right),
\end{aligned} \tag{3.36}$$

where $P_l(\{1,2,\ldots,n\})$ denotes the collection of all ordered partitions of $\{1,2,\ldots,n\}$ into l parts, and $\prod_{i\in\emptyset} Y_k^2(x_i) := 1$. Hence,by (3.36),

$$\begin{aligned}
&\mathbb{E}\left(g(x_1)\cdots g(x_n)\right) \\
&= \sum_{(\eta_1,\eta_2,\ldots,\eta_l)\in P_l(\{1,2,\ldots,n\})} \mathbb{E}\left((\prod_{i_1\in\eta_1} Y_1^2(x_{i_1}))(\prod_{i_2\in\eta_2} Y_2^2(x_{i_2}))\cdots(\prod_{i_l\in\eta_l} Y_l^2(x_{i_l}))\right).
\end{aligned}$$

Since, the random field $(Y_1(x))_{x\in X}, (Y_2(x))_{x\in X}, \ldots, (Y_l(x))_{x\in X}$ are independent, we have get

$$\mathbb{E}\left(g(x_1)\cdots g(x_n)\right) = \sum_{(\eta_1,\eta_2,\ldots,\eta_l)\in P_l(\{1,2,\ldots,n\})} \mathbb{E}(\prod_{i_1\in\eta_1} Y_1^2(x_{i_1}))\mathbb{E}(\prod_{i_2\in\eta_2} Y_2^2(x_{i_2})) \cdots \mathbb{E}(\prod_{i_l\in\eta_l} Y_l^2(x_{i_l})). \tag{3.37}$$

For any subset $\eta = \{j_1, j_2, \ldots, j_k\}$ of $\{1, 2, \ldots, n\}$, we define

$$F(x_1, x_2, \ldots, x_n; \eta) := \mathrm{per}_2 \begin{pmatrix} \mathcal{K}(x_{j_1}, x_{j_1}) & \mathcal{K}(x_{j_1}, x_{j_2}) & \cdots & \mathcal{K}(x_{j_1}, x_{j_k}) \\ \mathcal{K}(x_{j_2}, x_{j_1}) & \mathcal{K}(x_{j_2}, x_{j_2}) & \cdots & \mathcal{K}(x_{j_2}, x_{j_k}) \\ \vdots & \vdots & & \vdots \\ \mathcal{K}(x_{j_k}, x_{j_1}) & \mathcal{K}(x_{j_k}, x_{j_2}) & \cdots & \mathcal{K}(x_{j_k}, x_{j_k}) \end{pmatrix}$$

Then, by (3.37) and Proposition 3.1,

$$\begin{aligned} &\mathbb{E}\left(g(x_1)\cdots g(x_n)\right) \\ &= \sum_{(\eta_1,\eta_2,\ldots,\eta_l)\in P_l(\{1,2,\ldots,n\})} F(x_1, x_2, \ldots, x_n; \eta_1) F(x_1, x_2, \ldots, x_n; \eta_2) \\ &\quad \cdots F(x_1, x_2, \ldots, x_n; \eta_l). \end{aligned}$$

Fix any $(\eta_1, \eta_2, \ldots, \eta_l) \in P_l(\{1, 2, \ldots, n\})$. We note that any collection $\xi_1, \xi_2, \ldots, \xi_l$ of permutations of $\eta_1, \eta_2, \ldots, \eta_l$, respectively, yields a permutation ξ of $\{1, 2, \ldots, n\}$. Furthermore, by (3.32), we have

$$2^{n-n(\xi)} = 2^{|\eta_1|-n(\xi_1)} 2^{|\eta_2|-n(\xi_2)} \cdots 2^{|\eta_l|-n(\xi_l)}$$

Hence,by (3.37),

$$\begin{aligned} &\mathbb{E}\left(g(x_1)\cdots g(x_n)\right) \\ &= \sum_{\xi\in S_n} 2^{n-n(\xi)} N(l,\xi) \mathcal{K}(x_1, x_{\xi(1)}) \mathcal{K}(x_2, x_{\xi(2)}) \cdots \mathcal{K}(x_n, x_{\xi(n)}) \end{aligned} \tag{3.38}$$

Here $N(l,\xi)$ denotes the number of all possible collections of permutations $\xi_1, \xi_2, \dots, \xi_l$ of some $(\eta_1, \eta_2, \dots, \eta_l) \in P_l(\{1,2,\dots,n\})$ which yields the permutation ξ of $\{1,2,\dots,n\}$.

Let $\varphi_1, \varphi_2, \dots, \varphi_{n(\xi)}$ be the cycles of a permutation $\xi \in S_n$. Then each cycles φ_i must be a cycle in one of the permutation $\xi_1, \xi_2, \dots, \xi_l$ of $\eta_1, \eta_2, \dots, \eta_l$, respectively, where $\eta_1, \eta_2, \dots, \eta_l \in P_l(\{1,2,\dots,n\})$. Therefore, the number $N(l,\xi)$ is equal to the number of all ordered partitions of the set $\{1,2,\dots,n(\xi)\}$ into l parts. Hence,

$$\begin{aligned} N(l,\xi) &= \sum_{(\eta_1,\eta_2,\dots,\eta_l)\in P_l(\{1,2,\dots,n(\xi)\})} 1 \\ &= \sum_{\substack{k_1,k_2,\dots,k_l=0,1,\dots,n(\sigma\xi) \\ k_1+k_2+\dots+k_l=n(\sigma)}} \frac{n(\xi)!}{k_1!k_2!\cdots k_l!} \\ &= \sum_{\substack{k_1,k_2,\dots,k_l=0,1,\dots,n(\xi) \\ k_1+k_2+\dots+k_l=n(\xi)}} \frac{n(\xi)!}{k_1!k_2!\cdots k_l!} 1^{k_1} 1^{k_2} \dots^{k_l} \\ &= (\underbrace{1+1+\cdots+1}_{l \text{ times}})^{n(\xi)} \\ &= l^{n(\xi)} \end{aligned}$$

Therefore, by(3.38)

$$\mathbb{E}\left(g(x_1)\cdots g(x_n)\right) = \sum_{\xi\in S_n} 2^{n-n(\xi)} l^{n(\xi)} \mathcal{K}(x_1, x_{\xi(1)})\mathcal{K}(x_2, x_{\xi(2)}) \cdots \mathcal{K}(x_n, x_{\xi(n)})$$

Thus, by (3.35), the theorem is proved. □

Thus, using the results of Section 2.5 and Theorem 3.4, we conclude

Theorem 3.5. *Let the operator K acting in $L^2(X,\sigma)$ be as in subsec. 3.1.3, and let the kernel $\mathcal{K}(x,y)$ of K be also chosen as in subsection 3.1.3. Then, for each $l \in \mathbb{N}$, there exists a point process $\mu^{(l)}$ in X whose correlation func-*

tions are given by

$$k^{(n)}_{\mu^{(l)}}(x_1, \dots, x_n) = \operatorname{per}_{\frac{l}{2}}(l\mathcal{K}(x_i, x_j))^n_{i,j=1} \quad \textit{for } \sigma^{\otimes n}\textit{-a.a. } (x_1, \dots, x_n) \in X^n. \tag{3.39}$$

The $\mu^{(l)}$ satisfies condition (Σ'_σ) and its Papangelou intensity is given by

$$r^{(l)}(x, \gamma) = \widetilde{\mathbb{E}}\left(\sum_{i=1}^{l} |Y_i(x)|^2 \mid \mathcal{F}\right)(\gamma). \tag{3.40}$$

Chapter 4

Markov process and Dirichlet forms

In this chapter, we will briefly recall some definitions and theorems concerning unbounded linear operators, Markov processes, their generators, and construction of Markov processes through Dirichlet forms. For more details and proofs, see [6, 16, 33, 41, 42].

4.1 Unbounded linear operators and quadratic forms

Let H be a real, separable Hilbert space. Let $\mathcal{D}(A)$ be a linear subspace of H, i.e., $\mathcal{D}(A)$ is a subset of H and for any $a_1, a_2 \in \mathbb{R}$, $h_1, h_2 \in \mathcal{D}(A)$, $a_1h_1 + a_2h_2 \in \mathcal{D}(A)$. A linear operator A in H with domain $\mathcal{D}(A)$ is a mapping $A : \mathcal{D}(A) \to H$ such that

$$A(a_1h_1 + a_2h_2) = a_1Ah_1 + a_2Ah_2, \quad h_1, h_2 \in \mathcal{D}(A), \quad a_1, a_2 \in \mathbb{R}$$

Thus, a linear operator A is characterized by its domain $\mathcal{D}(A)$ and by the action of A on $\mathcal{D}(A)$. We will write a linear operator as a pair $(A, \mathcal{D}(A))$ to stress the domain of A.

Define

$$\Gamma_A := \{(f, Af) \mid f \in \mathcal{D}(A)\} \subset H \times H.$$

Then Γ_A is called the graph of the operator A. The operator A is called closed if Γ_A is a closed set in $H \times H$

If an operator $(A, \mathcal{D}(A))$ is not closed, we may take the closure of Γ_A in $H \times H$, denoted by $\tilde{\Gamma}_A$. However, this closure is not necessarily a graph of a linear operator. Indeed, the set $\tilde{\Gamma}_A$ may contain vectors of the form (f, g_1) and (f, g_2) with $g_1 \neq g_2$, so that $\tilde{\Gamma}_A$ is not a graph of a mapping. We say that a linear operator $(A, \mathcal{D}(A))$ is closable if $\tilde{\Gamma}_A$ is a graph of a mapping, i.e., there do not exist (f, g_1) and (f, g_2) in $\tilde{\Gamma}_A$ with $g_1 \neq g_2$. The linear operator $(\tilde{A}, \mathcal{D}(\tilde{A}))$ whose graph is $\tilde{\Gamma}_A$ is called the closure of $(A, \mathcal{D}(A))$. (Indeed, one may check that if $(A, \mathcal{D}(A))$ is closable, then the corresponding closure is a linear operator.)

If $\mathcal{D}(A)$ is a dense set in H, then we say that a linear operator $(A, \mathcal{D}(A))$ is densely defined.

Let $(A, \mathcal{D}(A))$ be a densely defined linear operator. Assume that a vector $g \in H$ is such that there exists $g^* \in H$ for which

$$(Af, g)_H = (f, g^*) \quad \text{for all} \quad f \in \mathcal{D}(A), \tag{4.1}$$

Then we say that $g \in \mathcal{D}(A^*)$ and define $A^*g := g^*$. (Since $\mathcal{D}(A)$ is dense in H, if g and g^* as above exist, then for each $g \in H$ there exists a unique $g^* \in H$ which satisfies (4.1).) $(A^*, \mathcal{D}(A^*))$ is a linear operator, and it is called the adjoint operator of $(A, \mathcal{D}(A))$. It might happen that $\mathcal{D}(A^*) = \{0\}$ and $A^*0 = 0$.

A densely defined operator $(A, \mathcal{D}(A))$ is called symmetric if for any $f, g \in \mathcal{D}(A)$,

$$(Af, g) = (f, Ag).$$

Clearly, for such an operator, $\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and $A^*f = Af$ for any $f \in \mathcal{D}(A)$, so that $(A^*, \mathcal{D}(A^*))$ is an extension of $(\mathcal{D}(A), A)$. An operator $(A, \mathcal{D}(A))$ is called self-adjoint if $(A^*, \mathcal{D}(A^*)) = (A, \mathcal{D}(A))$. A self-adjoint operator is a symmetric operator for which $\mathcal{D}(A) = \mathcal{D}(A^*)$. Every symmetric operator $(A, \mathcal{D}(A))$ is closable and its closure $(\tilde{A}, \mathcal{D}(\tilde{A}))$ is also a symmetric operator. A symmetric operator $(A, \mathcal{D}(A))$ is called essentially self-adjoint if its closure $(\tilde{A}, \mathcal{D}(\tilde{A}))$ is self-adjoint operator. In the latter case, the set $\mathcal{D}(A)$ is called a core or a domain of essential self-adjointness of $(\tilde{A}, \mathcal{D}(\tilde{A}))$.

An operator $(A, \mathcal{D}(A))$ is called non-negative if, for any $f \in \mathcal{D}(A)$, $(Af, f) \geq 0$. A non-negative operator is symmetric.

Now, let $\mathcal{D}(\mathcal{E})$ be a dense linear subspace of H. A mapping

$$\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{R}$$

is called a quadratic form if $\mathcal{E}$ is linear in both argument. The set $\mathcal{D}(\mathcal{E})$ is called the domain of the quadratic form $\mathcal{E}$, and we will write $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ for this quadratic form. A quadratic form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is called symmetric if, for any $f, g \in \mathcal{D}(\mathcal{E})$,

$$\mathcal{E}(f, g) = \mathcal{E}(g, f).$$

$(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is called non-negative if $\mathcal{E}(f, f) \geq 0$ for any $f \in \mathcal{E}$. A non-negative quadratic form is symmetric.

A symmetric quadratic form $(\mathcal{E}, \mathcal{D}(\mathcal{E})$ is completely characterized by the values $\mathcal{E}(f, f)$, $f \in \mathcal{E}$. Indeed, for any $f, g \in \mathcal{D}(\mathcal{E})$,

$$\mathcal{E}(f, g) = \frac{1}{4}(\mathcal{E}(f + g, f + g) - \mathcal{E}(f - g, f - g)).$$

Sometimes, for $f \in \mathcal{D}(\mathcal{E})$, we will just write $\mathcal{E}(f)$ instead of $\mathcal{E}(f, f)$.

Assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is non-negative and define a scalar product on $\mathcal{D}(\mathcal{E})$ by

$$(f, g)_+ := (f, g) + \mathcal{E}(f, g).$$

A densely defined (i.e., $\mathcal{D}(\mathcal{E})$ being dense in H), non-negative quadratic form is called closed if $\mathcal{D}(\mathcal{E})$ is a Hilbert space with scalar product $(\cdot, \cdot)_+$, i.e., if $\mathcal{D}(\mathcal{E})$ is complete in the norm $\|\cdot\|_+$ generated by the scalar product $(\cdot, \cdot)_+$. A non-negative quadratic form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is called closable if for any sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{D}(\mathcal{E})$ such that $f_n \to 0$ is H as $n \to \infty$, and $\mathcal{E}(f_n - f_m) \to 0$ as $n, m \to +\infty$, we have $\mathcal{E}(f_n, f_n) \to 0$ as $n \to \infty$.

Assume that a non-negative quadratic form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is closable. Let $\{f_n\}_{n=1}^{\infty} \subset \mathcal{D}(\mathcal{E})$ be a Cauchy sequence with respect to the norm $\|\cdot\|_+$. Then $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in H and let $f \in H$ be the limit of $\{f_n\}_{n=1}^{\infty}$ in H. Let $\mathcal{D}(\tilde{\mathcal{E}})$ be the set of all such $f \in H$. Furthermore, for any $f, g \in \mathcal{D}(\tilde{\mathcal{E}})$, set

$$\tilde{\mathcal{E}}(f, g) = \lim_{n \to \infty} \mathcal{E}(f_n, g_n),$$

where $\{f_n\}_{n=1}^{\infty} \subset \mathcal{D}(\mathcal{E})$, $\{g_n\}_{n=1}^{\infty} \subset \mathcal{D}(\mathcal{E})$ and $f_n \to f$ and $g_n \to g$ with respect to the norm $\|\cdot\|_+$ as $n \to \infty$. The definition of a closable quadratic form ensures that the value $\tilde{\mathcal{E}}(f, g)$ is independent of the choice of $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$. In fact, $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$ is a closed non-negative quadratic form, which is called the closure of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Theorem 4.1 (Friedrichs' theorem). *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a non-negative quadratic form. Assume that there exists a linear operator $A : \mathcal{D}(\mathcal{E}) \to H$ such that*

$$\mathcal{E}(f, g) = (Af, g), \quad f, g \in \mathcal{D}(\mathcal{E}).$$

Then the quadratic form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is closable.

Theorem 4.2. *Assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a non-negative closed quadratic form. Then, there exists a self-adjoint operator $(A, \mathcal{D}(A))$ such that $\mathcal{D}(A)$ is dense in $\mathcal{D}(\mathcal{E})$ with respect to $||\cdot||_+$ norm and, for any $f \in \mathcal{D}(A)$, $g \in \mathcal{D}(\mathcal{E})$,*

$$(Af, g) = \mathcal{E}(f, g).$$

The operator $(A, \mathcal{D}(A))$ as in Theorem 4.2 is called the generator of the closed quadratic form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Let us consider a linear symmetric oerator $(A, \mathcal{D}(A))$ in H, Assume that A is not self-adjoint, that is, $\mathcal{D}(A)$ is smaller then $\mathcal{D}(A^*)$. Assume that A is non-negative. Define a quadratic form

$$\mathcal{E}(f, g) = (Af, g), \quad f, g \in \mathcal{D}(\mathcal{E}) := \mathcal{D}(A).$$

Then $\mathcal{E}$ is a non-negative quadratic form and by the Friedrichs' theorem, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is closable. Let $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$ be the closure of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Then, by Theorem 4.2, $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$ has generator $(\tilde{A}, \mathcal{D}(\tilde{A}))$, which is a self-adjoint operator. Clearly, $\mathcal{D}(A) \subset \mathcal{D}(\tilde{A})$ and for each $f \in \mathcal{D}(A)$ $Af = \tilde{A}f$. Thus $(\tilde{A}, \mathcal{D}(\tilde{A}))$ is an extension of $(A, \mathcal{D}(A))$. The operator $(\tilde{A}, \mathcal{D}(\tilde{A}))$ is called the Friedrichs' extension of $(A, \mathcal{D}(A))$.

4.2 Markov processes

Let X be a Polish space. Consider a family $(P_{x,t})_{x \in X, t \geq 0}$ of probability measures on $(X, \mathcal{B}(X))$ such that, for each $A \in \mathcal{B}(X)$, the mapping

$$X \ni x \mapsto P_x(A) \in \mathbb{R}$$

is measurable. Let us assume that for each $x \in X$ $P_{x,0} = \delta_x$, the Dirac measure with mass at x, and furthermore, for any $s, t \geq 0$ and $A \in \mathcal{B}(X)$,

$$P_{x,s+t}(A) = \int_X P_{x,s}(dy) P_{y,t}(A).$$

Then, for each $x \in X$, by Kolmogorov's existence theorem, there exists a stochastic process $(M_x(t))_{t\geq 0}$ taking values in X such that $M(0) = x$ a.s. (x being called the starting point of the process), and for any $0 < t_1 < t_2 < \cdots < t_n < t_{n+1}$, $A_1, \dots, A_n \in \mathcal{B}(X)$,

$$
\begin{aligned}
&P(M_x(t_1) \in A_1, \dots, M_x(t_n) \in A_n) \\
&= \int_{A_1} P_{x,t_1}(dx_1) \int_{A_2} P_{x_1,t_2-t_1}(dx_2) \cdots \int_{A_n} P_{x_{n-1},t_n-t_{n-1}}(dx_n).
\end{aligned}
$$

Such a process is called a time homogeneous Markov process. This process has 'no memory', which can be mathematically written as follows: for any $0 \leq t_1 < t_2 < \ldots < t_n$ and any $A_1, A_2, \dots, A_n, A_{n+1} \in \mathcal{B}(X)$,

$$
\begin{aligned}
&P(M_x(t_{n+1}) \in A_{n+1} \mid M_x(t_1) \in A_1,\ M_x(t_2) \in A_2, \dots,\ M_x(t_n) \in A_n) \\
&= P_x(M_x(t_{n+1}) \in A_{n+1} \mid M_x(t_n) \in A_n).
\end{aligned}
$$

Thus, each probability measure $P_{x,t}$ describes the distribution of a stochastic process M_y at time $s + t$ ($s \geq 0$) given that at time s, the process is at x. This is why these measures are called transition probabilities.

Recall that a mapping $[0, +\infty) \ni t \mapsto R(t) \in X$ is called cadlag, or right continuous with left limits if, for any $t \geq 0$ and for any sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \geq t$ and $t_n \to t$ as $n \to \infty$, we have $R(t_n) \to R(t)$ as $n \to \infty$ (right continuity) and for any $t > 0$ and any sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \leq t$ and $t_n \to t$ as $n \to \infty$, the sequence $\{R(t_n)\}_{n=1}^{\infty}$ converges in X.

Often we will drop the lower index x in the notation of a Markov process M_x and just write $(M(t))_{t\geq 0}$. Under some additional conditions on transition probabilities, one may choose a version of a Markov process such that each sample path becomes a cadlag function, (or in some cases even a continuous function). From now on we will assume that such a version of a Markov process has been choosen.

Denote by $D([0,+\infty),X)$ the set of all cadlag function on $[0,+\infty)$ with values in X. So, by our assumption, each sample path of $(M(t))_{t\geq 0}$ belongs to $D([0,+\infty),X)$. We define a σ-algebra $\mathcal{C}$ on $D([0,+\infty),X)$ as the minimal σ-algebra with respect to which all mappings of the form

$$D([0,+\infty),X) \ni \omega \longmapsto (\omega(t_1),\omega(t_2),\ldots,\omega(t_n)) \in X^n,$$
$$0 < t_1 < t_2 < \cdots < t_n, \quad n \in \mathbb{N},$$

are measurable.

Recall that the law of a stochastic process is the distribution of sample paths of this process. So, in our case, for each starting point $x \in X$ we have a probability measure P_x on $(D([0,+\infty),X),\mathcal{C})$. So, we may always assume that, for each $x \in X$, the corresponding probability space is $(D([0,+\infty),\mathcal{C},P_x)$ and the Markov process is defined by

$$M(t,\omega) = \omega(t).$$

This is called a canonical realization of the process.

Clearly, every probability measure P on $(D([0,+\infty),X),\mathcal{C})$ defines a stochastic process $(Y(t))_{t\geq 0}$ by

$$Y(t,\omega) = \omega(t).$$

So, let μ be a probability measure on X and define a probability measure P on $(D([0,+\infty),X),\mathcal{C})$ by

$$P(C) = \int_X P_x(C)\mu(dx), \quad C \in \mathcal{C}.$$

This measure yields a Markov process $M = (M(t))_{t\geq 0}$ with a random starting point. The measure μ is called the initial distribution of M. If it happens that, for any $t > 0$, the distribution of this process at time t, i.e., the distribution of $M(t)$ is μ, then we say that the Markov process is stationary.

Assume that for any bounded measurable functions $f, g : X \to \mathbb{R}$, we have

$$\int_X \mu(dx) \left(\int_X P_{t,x}(dy) f(y) \right) g(x) = \int_X \mu(dx) f(x) \left(\int_X P_{t,x}(dy) g(y) \right).$$

Then the corresponding Markov process M with initial distribution μ is called an equilibrium Markov process and the measure μ is called a symmetrizing measure for M. The measure μ is stationary for M.

4.3 Generator of a Markov process

Let $(T_t)_{t\geq 0}$ be a family of bounded linear operators in H. $(T_t)_{t\geq 0}$ is called a semigroup if $T_0 = \mathbf{1}$ and $T_{s+t} = T_s T_t$ for any $s, t \geq 0$. A semigroup $(T_t)_{t\geq 0}$ is called strongly continuous if for any $f \in H$, the mapping $t \mapsto T_t f \in H$ is continuous. A semigroup $(T_t)_{t\geq 0}$ is called a contraction semigroup if it is a strongly continuous semigroup and furthermore, $\|T_t\| \leq 1$ for all $t \geq 0$.

Define

$$\mathcal{D}(L) = \left\{ \phi \in H \mid \text{there exists a limit } \lim_{t\to 0} \frac{1}{t}(T_t\phi - \phi) \text{ in } H \right\}.$$

Then, $\mathcal{D}(L)$ is a dense, linear subset of H. Define,

$$L\phi := \lim_{t\to 0} \frac{1}{t} (T_t\phi - \phi), \quad \text{for each } \phi \in \mathcal{D}(L).$$

Then, $(L, \mathcal{D}(L))$ is a closed operator in H.

This operator $(L, \mathcal{D}(L))$ is called the generator of the contraction semigroup $(T_t)_{t\geq 0}$, and one writes,

$$T_t = e^{tL}.$$

Remark 4.1. For $\phi \in \mathcal{D}(L)$, at least heuristically, we have:

$$L\phi = \left.\frac{d}{dt}\right|_{t=0} e^{tL}\phi$$

$$= \lim_{t \to 0} \frac{1}{t}(e^{tL} - e^{-0L})\phi$$
$$= \lim_{t \to 0} \frac{1}{t}(T_t\phi - \phi).$$

If each operator T_t of a contraction semigroup $(T_t)_{t\geq 0}$ is self-adjoint, then the generator $(L, \mathcal{D}(L))$ of this semigroup is a self-adjoint, non-positive definite operator, the latter meaning that $(Lf, f) \leq 0$ for any $f \in \mathcal{D}(L)$.

Assume that we have a probability measure μ on $(X, \mathcal{B}(X))$, and we consider the Hilbert space $L^2(X, \mu)$. Let $(M(t))_{t\geq 0}$ be an equilibrium Markov process on X with stationary distribution μ and transition probabilities $(P_{t,x})_{t\geq 0, x\in X}$.

Define for each $f \in L^2(X, \mu)$ and $t \geq 0$,

$$(T_t f)(x) = \int_X f(\omega) P_{t,x}(d\omega), \quad x \in X. \tag{4.2}$$

Then, $(T_t)_{t\geq 0}$ is a contraction semigroup in $L^2(X, \mu)$ and the generator L of this semigroup is called the generator of the Markov procress. L is a self-adjoint operator in $L^2(X, \mu)$. In many cases, study of a Markov process reduces to study of its generator L in $L^2(X, \mu)$.

Let X be a Polish space, let $\mathcal{B}(X)$ be the Borel σ-algebra on X, and let m be a probability measure on $(X, \mathcal{B}(X))$. Let $(\mathcal{E}, D(\mathcal{E}))$ be a closed, symmetric, non-negative, quadratic form on $L^2(X, m)$. Then $(\mathcal{E}, D(\mathcal{E}))$ is called a Dirichlet form if, for each $F \in D(\mathcal{E})$, we have $(F \vee 0) \wedge 1 \in D(\mathcal{E})$ and

$$\mathcal{E}((F \vee 0) \wedge 1) \leq \mathcal{E}(F).$$

Let $(\mathcal{E}, D(\mathcal{E}))$ be a Dirichlet form on $L^2(X, m)$. For a subset $A \subset X$, we define

$$D(\mathcal{E})_A := \{F \in D(\mathcal{E}) \mid F = 0 \text{ on } X \setminus A\}.$$

A sequence $(A_n)_{n\in\mathbb{N}}$ of closed subsets of X is called an $\mathcal{E}$-nest if

$$\bigcup_{n\in\mathbb{N}} D(\mathcal{E})_{A_n}$$

is dense in $D(\mathcal{E})$ with respect to the norm

$$\|\cdot\|_+ := \left(\mathcal{E}(\cdot) + \|\cdot\|_{L^2(X,m)}\right)^{1/2}.$$

A subset $N \subset X$ is called $\mathcal{E}$-exceptional if

$$N \subset \bigcap_{n\in\mathbb{N}} A_n^c$$

for some $\mathcal{E}$-nest $(A_n)_{n\in\mathbb{N}}$. Note that every Borel $\mathcal{E}$-exceptional set has m measure zero. A property of points in X holds $\mathcal{E}$-quasi-everywhere (abbreviated $\mathcal{E}$-q.e.) if the property holds outside some $\mathcal{E}$-exceptional set. Evidently, if a property holds $\mathcal{E}$-q.e., then it holds m-m.e.

Assume that there exists a subset $\mathcal{F}$ of $D(\mathcal{E}) \cap C(X)$, where $C(X)$ is the set of all continuous functions on E, which is dense in $D(\mathcal{E})$ with respect to the norm $\|\cdot\|_+$ and such that the functions from $\mathcal{F}$ separate points of X. The latter means that for any $x, y \in X$, $x \neq y$, there exist $f \in \mathcal{F}$ such that $f(x) \neq f(y)$. Then, the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is called quasi-regular if there exists an $\mathcal{E}$-nest $(A_n)_{n\in\mathbb{N}}$ consisting of compact sets in X.

Finally, let us define the notion of quasi-continuity. Let $F : A \to \mathbb{R}, A \subset X$. And let $(\mathcal{E}, D(\mathcal{E}))$ be a Dirichlet form on $L^2(X,m)$. Then the function F is called $\mathcal{E}$-quasi continuous if there exist an increasing sequence $(A_n)_{n\in\mathbb{N}}$ of closed subsets of X which form an $\mathcal{E}$-nest, $\bigcup_{n\in\mathbb{N}} A_n \subset A$ and for each $n \in \mathbb{N}$ the restriction of F to A_n is a continuous function.

4.4 Conservative Hunt processes

Let us consider a special class of Markov processes which is called conservative Hunt processes.

So let X be a Polish space, and let $(M(t))_{t\geq 0}$ be a Markov process taking values in X and defined on sample space Ω equipped with a σ-algebra $\mathcal{F}$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by $(M(t))_{t\geq 0}$. For each $x \in X$ we denote by P_x the probability measure on $(\Omega, \mathcal{F})$ which corresponds to the process $(M_x(t))_{t\geq 0}$, i.e., to our Markov process starting at x. We also assume that, for each $t \geq 0$, there exist a mapping $\theta_t : \Omega \to \Omega$, called time shift, which satisfies

$$M_s(\theta_t \omega) = M_{s+t}(\omega).$$

For all $s \geq 0$, we may now write our Markov process as follows:

$$M = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (M(t))_{t\geq 0}, (P_x)_{x\in X}).$$

Such a process is called a conservative Hunt process if it satisfies the following additional properties:

- (*Normal property*) $P_x(M(0) = x) = 1$ for all $x \in X$.
- (*Infinite life-time*) $P_x(M(t) \in X \text{ for all } t > 0) = 1$ for all $x \in X$.
- (*Right continuity*) For each $\omega \in \Omega$, $[0, \infty) \ni t \mapsto M(t, \omega) \in X$ is right continuous.
- (*Strong Markov property*) For every $(\mathcal{F}_t)_{t\geq 0}$-stopping time τ and every probability measure m on $(X, \mathcal{B}(X))$,

  $$P_m(M(\tau + t) \in A \mid \mathcal{F}_\tau) = P_{X(\tau)}(X(t) \in A) \quad P_m\text{-a.s.}$$

  for all $A \in \mathcal{B}(X)$ and $t \geq 0$. Here, $P_m(\cdot) := \int_X m(dx) P_x(\cdot)$.
- (*Left limits*) For every probability measure m on $(X, \mathcal{B}(X))$, $\lim_{s\uparrow t} M(s)$ exists in X for all $t > 0$ P_m-a.s.

- (*quasi-left continuity*) For every probability measure m on $(X, \mathcal{B}(X))$, if τ, τ_n, $n \in \mathbb{N}$, are $(\mathcal{F}_t^{P_m})_{t\geq 0}$-stopping times such that $\tau_n \uparrow \tau$, then $X(\tau_n) \to X(\tau)$ as $n \to \infty$ P_m-a.s. Here, $\mathcal{F}_t^{P_m}$ denotes the completion of the σ-algebra $\mathcal{F}_t$ with respect to the probability measure P_m(i.e. we add to $\mathcal{F}_t$ all sets from $\mathcal{F}$ which are of zero measure P_m.)

Let m be a probability measure on $(X, \mathcal{B})$. Let M and M' be two Hunt processes with state space X. Denote by $(T_t)_{t\geq 0}$ and $(T_t')_{t\geq 0}$, their corresponding semigroups(see (4.2)). Then M and M' are called m-equivalent if there exist a set $S \in \mathcal{B}(X)$, such that

- $m(X \setminus S) = 0$;
- S is both M-invariant are M'-invariant;
- $(T_t f)(x) = (T_t' f)(x)$ for each $t > 0$, each bounded measurable function $f : X \to \mathbb{R}$ and each $x \in S$.

Recall that S being M-invariant means that there exist $\widetilde{\Omega} \in \mathcal{F}$ such that $\{\overline{M_0^t} \bigcap (X \setminus S) \neq 0 \text{ for some } t > 0\ \} \subset \widetilde{\Omega}$ and $P_x(\widetilde{\Omega}) = 0$ for all $x \in S$. Here $\overline{M_0^t}$ is the closure of $\{M_s(\omega) \mid s \in [0, t]\}$ in X.

Chapter 5

Glauber and Kawasaki dynamics for permanental point processes

In this chapter we will study two types of equilibrium dynamics of an infinite particle system which leave a permanental point process invariant. These dynamics are Glauber (spatial birth-and-death) and Kawasaki (a dynamic of hopping particles).

5.1 Equilibrium Glauber and Kawasaki dynamics — general results

Let a space X and a measure σ on $(X, \mathcal{B}(X))$ be as in subsection 3.1.3.

Let us start with an informal description of Glauber and Kawasaki dynamics. A Glauber dynamics is a stochastic dynamics of an infinite particle system in which particles randomly appear (are born) and disappear (die).

It is a Markov process whose formal (pre-)generator has the form

$$(L_G F)(\gamma) = \sum_{x\in\gamma} d(x,\gamma\backslash x)(F(\gamma\backslash x)-F(\gamma)) + \int_X \sigma(dx) b(x,\gamma)(F(\gamma\cup x)-F(\gamma)). \tag{5.1}$$

The coefficient $d(x, \gamma \setminus x)$ describes the rate at which particle x of configuration γ dies, while $b(x, \gamma)$ describes the rate at which, given configuration γ, a new particle is born at x.

A Kawasaki dynamics is a Markov process on Γ in which particles randomly hop over the space X. The Markov (pre-)generator of such a dynamics is given by

$$(L_K F)(\gamma) = \sum_{x\in\gamma} \int_X c(x, y, \gamma \setminus x)(F(\gamma \setminus x \cup y) - F(\gamma))\sigma(dy). \tag{5.2}$$

The coefficient $c(x, y, \gamma \setminus x)$ describes the rate at which particle x of configuration γ hops to y, taking the rest of the configuration, $\gamma \setminus x$, into account.

Generally speaking, both Glauber and Kawasaki dynamics are not equilibrium dynamics. Let μ be a point process on X which satisfies the (Σ'_σ) property, i.e., there exists Papangelou intensity $r(x, \gamma)$ of the measure μ: for each measurable function $F : X \times \Gamma_X \to [0, +\infty]$, we have

$$\int_{\Gamma_X} \sum_{x\in\gamma} F(x, \gamma)\mu(dy) = \int_{\Gamma_X} \mu(dy) \int_X \sigma(dx) r(x, \gamma) F(x, \gamma \cup x). \tag{5.3}$$

The condition (Σ'_σ) can be thought of as a kind of weak Gibbsianess of μ. Intuitively, we may treat the Papangelou intensity as

$$r(x, \gamma) = \exp[-E(x, \gamma)], \tag{5.4}$$

where $E(x, \gamma)$ is the relative energy of interaction between particle x and configuration γ.

Then it is easy to see which conditions the coefficients b and d, respectively c, must satisfy in order that the operators L_G and L_K be symmetric in

$L^2(\Gamma_X, \mu)$ (and so μ is a symmetrizing, and so invariant measure of the process). These conditions (called balance conditions) look as follows, see [20, 21]:

- $b(x, \gamma) = r(x, \gamma)d(x, \gamma)$ (for the Glauber dynamics)
- $r(x, \gamma)c(x, y, \gamma) = r(y, \gamma)c(y, x, \gamma)$ (for the Kawasaki dynamics)

So, let μ be a point process in X which satisfies the condition (Σ'_σ), i.e., (5.3) holds.

To define an equilibrium Glauber dynamics for which μ is a symmetrizing measure, we fix a death coefficient as a measurable function $d : X \times \Gamma_X \to [0, +\infty]$, and then define a birth coefficient $b : X \times \Gamma_X \to [0, +\infty]$ by

$$b(x, \gamma) = d(x, \gamma)r(x, \gamma), \quad (x, \gamma) \in X \times \Gamma_X \tag{5.5}$$

To define a Kawasaki dynamics, we fix a measurable function $c : X^2 \times \Gamma_X^2 \to [0, +\infty]$ which satisfies

$$r(x, \gamma)c(x, y, \gamma) = r(y, \gamma)c(y, x, \gamma), \quad (x, y, \gamma) \in X^2 \times \Gamma_X. \tag{5.6}$$

We will also assume that the function $c(x, y, \gamma)$ vanishes if at least one of the functions $r(x, \gamma)$ and $r(y, \gamma)$ vanishes, i.e.,

$$c(x, y, \gamma) = c(x, y, \gamma)\chi_{\{r>0\}}(x, \gamma)\chi_{\{r>0\}}(y, \gamma). \tag{5.7}$$

Here, for a set A, χ_A denotes the indicator function of A. We refer to [31, Remark 3.1] for a justification of this assumption. Let us recall this argumentation.

As we discussed, the coefficient $c(x, y, \gamma \setminus x)$ describes the rate of the jump of particle $x \in \gamma$ to y. For each $\gamma \in \Gamma_X$ and $x \in X \setminus \gamma$, we interpreted $r(x, \gamma)$ as $\exp[-E(x, \gamma)]$, where $E(x, \gamma)$ is the relative energy of interaction between configuration γ and particle x. Hence, if $r(y, \gamma \setminus x) = 0$, then the relative energy of interaction between the configuration $\gamma \setminus x$ and particle y is $+\infty$. Hence, it is intuitively clear that the particle x cannot hop to y, i.e., $c(x, y, \gamma \setminus x)$ should be equal to zero. A symmetry reason also implies that we should have $c(x, y, \gamma \setminus x) = 0$ if $r(x, \gamma \setminus x) = 0$, i.e., if the relative energy of interaction between $x \in \gamma$ and the rest of configuration is ∞.

We denote by $\mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma_X)$ the space of all functions of the form

$$\Gamma_X \ni \gamma \mapsto F(\gamma) = g(\langle \varphi_1, \gamma \rangle, \dots, \langle \varphi_N, \gamma \rangle), \tag{5.8}$$

where $N \in \mathbb{N}$, $\varphi_1, \dots, \varphi_N \in C_0(X)$ and $g \in C_{\mathrm{b}}(\mathbb{R}^N)$. Here, $C_{\mathrm{b}}(\mathbb{R}^N)$ denotes the space of all continuous bounded functions on $\mathbb{R}^N$. We assume that, for each $\Lambda \in \mathcal{B}_0(X)$,

$$\int_{\Gamma_X} \mu(d\gamma) \int_\Lambda \gamma(dx)\, d(x, \gamma \setminus x) < \infty, \tag{5.9}$$

$$\int_{\Gamma_X} \mu(d\gamma) \int_X \gamma(dx) \int_X \sigma(dy)\, c(x, y, \gamma \setminus x)(\chi_\Lambda(x) + \chi_\Lambda(y)) < \infty. \tag{5.10}$$

As easily seen, conditions (5.9) and (5.10) are sufficient in order to define quadratic forms

$$\begin{aligned}
\mathcal{E}_{\mathrm{G}}(F, G) &:= \int_{\Gamma_X} \mu(d\gamma) \int_X \gamma(dx)\, d(x, \gamma \setminus x)(F(\gamma \setminus x) - F(\gamma))(G(\gamma \setminus x) - G(\gamma)), \\
\mathcal{E}_{\mathrm{K}}(F, G) &:= \frac{1}{2} \int_{\Gamma_X} \mu(d\gamma) \int_X \gamma(dx) \int_X \sigma(dy)\, c(x, y, \gamma \setminus x)(F(\gamma \setminus x \cup y) - F(\gamma)) \\
&\qquad \times (G(\gamma \setminus x \cup y) - G(\gamma)),
\end{aligned}$$

where $F, G \in \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma_X)$.

For the construction of the Kawasaki dynamics, we will also assume that the following technical assumptions holds:

$$\exists u, v \in \mathbb{R} \quad \forall \Lambda \in \mathcal{B}_0(X):$$

$$\int_\Lambda \gamma(dx) \int_\Lambda \sigma(dy)\, r(x,\gamma\setminus x)^u r(y,\gamma\setminus x)^v c(x,y,\gamma\setminus x) \in L^2(\Gamma_X,\mu) < \infty. \tag{5.11}$$

Note that in formula (5.11) and below, we use the convention $\frac{0}{0} := 0$.

The following theorem was essentially proved in [31].

Theorem 5.1. *(i) Assume that a point process μ satisfies* (5.3)*. Assume that conditions* (3.1), (5.9), *respectively* (5.6), (5.7), (5.10), *and* (5.11) *are satisfied. Let $\sharp = \mathrm{G}, \mathrm{K}$. Then the quadratic form $(\mathcal{E}_\sharp, \mathcal{F}C_\mathrm{b}(C_0(x), \Gamma_X))$ is closable in $L^2(\Gamma_X,\mu)$ and its closure will be denoted by $(\mathcal{E}_\sharp, D(\mathcal{E}_\sharp))$. Further there exists a conservative Hunt process (Glauber, respectively Kawasaki dynamics)*

$$M^\sharp = \left(\Omega^\sharp, \mathcal{F}^\sharp, (\mathcal{F}^\sharp_t)_{t\ge 0}, (\Theta^\sharp_t)_{t\ge 0}, (X^\sharp(t))_{t\ge 0}, (P^\sharp_\gamma)_{\gamma\in\Gamma_X}\right)$$

on Γ_X which is properly associated with $(\mathcal{E}_\sharp, D(\mathcal{E}_\sharp))$, i.e., for all ($\mu$-version of) $F \in L^2(\Gamma_X,\mu)$ and $t > 0$

$$\Gamma_X \ni \gamma \mapsto p^\sharp_t F(\gamma) := \int_{\Omega^\sharp} F(X^\sharp(t))\, dP^\sharp_\gamma$$

is an $\mathcal{E}^\sharp$-quasi continuous version of $\exp(tL_\sharp)F$, where $(-L_\sharp, D(L_\sharp))$ is the generator of $(\mathcal{E}_\sharp, D(\mathcal{E}_\sharp))$. $M^\sharp$ is up-to μ-equivalence unique. In particular, $M^\sharp$ is μ-symmetric and has μ as invariant measure.

(ii) Further assume that, for each $\Lambda \in \mathcal{B}_0(X)$,

$$\int_\Lambda \gamma(dx)\, d(x,\gamma\setminus x) \in L^2(\Gamma_X,\mu), \qquad \int_\Lambda \sigma(dx)\, b(x,\gamma) \in L^2(\Gamma_X,\mu), \tag{5.12}$$

in the Glauber case, and

$$\int_X \gamma(dx) \int_X \sigma(dy)\, c(x,y,\gamma\setminus x)(\chi_\Lambda(x) + \chi_\Lambda(y)) \in L^2(\Gamma_X,\mu) \tag{5.13}$$

in the Kawasaki case. Then $\mathcal{F}C_\mathrm{b}(C_0(X),\Gamma_X) \subset D(L_\sharp)$, and for each $F \in \mathcal{F}C_\mathrm{b}(C_0(X),\Gamma_X)$, $L_\sharp F$ is given by formulas (5.1) *and* (5.2)*, respectively.*

Proof. The statement follows from Theorems 3.1 and 3.2 in [31]. Note that, although these theorems are formulated for determinantal point processes only, their proof only uses the (Σ'_σ) property of these point processes. Note also that condition (5.11) is formulated in [31] only for $v = 1$, however the proof of Lemma 3.2 in [31] admits a straightforward generalization to the case of an arbitrary $v \in \mathbb{R}$ [30]. $\square$

Remark 5.1. Part (ii) of Theorem 5.1 states that the operator $(-L_\sharp, D(L_\sharp))$ is the Friedrichs' extention of the operator $(-L_\sharp, \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma_X))$ defined by formulas (5.1), (5.2), respectively (see Section 4.1).

Let us fix a parameter $s \in [0,1]$ and define

$$d(x,\gamma) := r(x,\gamma)^{s-1}\chi_{\{r>0\}}(x,\gamma), \quad (x,\gamma) \in X \times \Gamma_X, \tag{5.14}$$

$$b(x,\gamma) := r(x,\gamma)^{s}\chi_{\{r>0\}}(x,\gamma), \quad (x,\gamma) \in X \times \Gamma_X, \tag{5.15}$$

$$c(x,y,\gamma) := a(x,y)r(x,\gamma)^{s-1}r(y,\gamma)^{s}\chi_{\{r>0\}}(x,\gamma)\chi_{\{r>0\}}(y,\gamma), \\ (x,y,\gamma) \in X^2 \times \Gamma_X. \tag{5.16}$$

Here the function $a : X^2 \to [0, +\infty)$ is bounded, measurable, symmetric (i.e., $a(x,y) = a(y,x)$), and satisfies

$$\sup_{x\in X} \int_X a(x,y)\,\sigma(dy) < \infty. \tag{5.17}$$

Note that the balance conditions (3.1) and (5.6) are satisfied for these coefficients, and so is condition (5.7).

Remark 5.2. Note that, if $X = \mathbb{R}^d$ and $a(x,y)$ has the form $a(x-y)$ for a function $a : \mathbb{R}^d \to [0,\infty)$, then condition (5.17) means that $a \in L^1(\mathbb{R}^d, dx)$. (Here and below, in the case $X = \mathbb{R}^d$, we use an obvious abuse of notation.)

Remark 5.3. Using representation (5.4), we can rewrite formulas (5.14)–(5.16) as follows:

$$d(x,\gamma\setminus x) = \exp[(1-s)E(x,\gamma\setminus x)]\chi_{\{E<+\infty\}}(x,\gamma\setminus x),$$

$$b(x,\gamma\setminus x)=\exp[-sE(x,\gamma\setminus x)]\chi_{\{E<+\infty\}}(x,\gamma\setminus x),$$

$$c(x,y,\gamma\setminus x)=a(x,y)\exp[(1-s)E(x,\gamma\setminus x)-sE(y,\gamma\setminus x)]$$
$$\times\,\chi_{\{E<+\infty\}}(x,\gamma\setminus x)\chi_{\{E<+\infty\}}(y,\gamma\setminus x).$$

So, if the corresponding dynamics exist, one can give the following heuristic description of them: Both dynamics are concentrated on configurations $\gamma\in\Gamma_X$ such that, for each $x\in\gamma$, the relative energy of interaction between x and the rest of configuration, $\gamma\setminus x$, is finite; those particles tend to die, respectively hop, which have a high energy of interaction with the rest of the configuration, while it is more probable that a new particle is born at y, respectively x hops to y, if the energy of interaction between y and the rest of the configuration is low.

Let us assume that the point process μ satisfies:

$$\forall\Lambda\in\mathcal{B}_0(X):\quad\int_\Lambda\gamma(dx)\in L^2(\Gamma_X,\mu).$$

Then, by choosing $u=1-s$ and $v=-s$ in (5.11), we conclude from (5.17) that the coefficient c given by (5.16) satisfies (5.11).

5.2 Equilibrium Glauber and Kawasaki dynamics for permanental point processes

We will now prove that, for a point process $\mu^{(l)}$ as in Theorem 3.5, Glauber and Kawasaki dynamics with coefficients (5.14), (5.15) and (5.16), respectively exist.

Theorem 5.2. *Let the operator K acting in $L^2(X,\sigma)$ be as in subsec. 3.1.3, and let the kernel $\mathcal{K}(x,y)$ of K be also chosen as in subsection 3.1.3. Let $\mu^{(l)}$, $l\in\mathbb{N}$, be a point process as in Theorem 3.5. Then:*

(i) For $\mu^{(l)}$, the coefficients $d(x,\gamma)$ and $b(x,\gamma)$ defined by (5.14) *and* (5.15), *satisfy conditions* (3.1) *and* (5.9) *and so statements (i) and (ii) of Theorem 5.1 hold, in particular, a corresponding Glauber dynamics exists.*

(ii) Assume additionally that $\mathcal{K}(x,x)$ is bounded outside a set $\Delta \in \mathcal{B}_0(X)$. Then for $\mu^{(l)}$, the coefficient $c(x,y,\gamma)$ defined by (5.16), *satisfies* (5.6), (5.7), (5.10) *and* (5.11), *and so statements (i) and (ii) of Theorem 5.1 hold, in particular, a corresponding Kawasaki dynamics exists.*

Proof. We will only prove statement (ii) of Theorem 5.2, as the proof of statement (i) is similar and simper. Also, for simplicity of notation, we will only consider the case $l = 1$ (for $l > 1$ the proof being completely analogous). We will also omit the upper index $^{(1)}$ from our notation.

We already know that the coefficient $c(x,y,\gamma)$ defined by (5.16) satisfies (5.6). By the very definition of $c(x,y,\gamma)$, formula (5.16), condition (5.7) is satisfied. Furthermore, at the end of Section 5.1 we have already explained why condition (5.11) is satisfied. Thus, we only need to prove (5.10).

We start with the following

Lemma 5.1. *For the permanental point process μ under consideration, for each $n \in \mathbb{N}$ and for σ-a.a. $x \in X$*

$$\int_\Gamma r(x,\gamma)^n \,\mu(d\gamma) \le \frac{(2n)!}{2^n\, n!}\,\mathcal{K}(x,x)^n. \tag{5.18}$$

Proof. Recall that by $\widetilde{\mathbb{E}}$ we denote the expectation with respect to the probability measure $\widetilde{\mathbb{P}}$ defined by (2.26) with $g(x)$ given by (3.34) (recall that $l = 1$ now). Recall also that we denote by $\mathcal{K}_1(x,y)$ the integral kernel of the operator $\sqrt{K}$.

Using Jensen's inequality for conditional expectation (e.g. [3, 15.3 Theorem]), (3.40), and Theorem 3.2, we have:

$$\int_\Gamma r(x,\gamma)^n \,\mu(d\gamma) = \int_\Omega \mathbb{P}(d\omega) \int_\Gamma \pi_{g(\cdot,\omega)\sigma}(d\gamma) r(x,\gamma)^n$$

$$
\begin{aligned}
&= \int_{\Omega\times\Gamma} \widetilde{\mathbb{P}}(d\omega, d\gamma) r(x,\gamma)^n \\
&= \widetilde{\mathbb{E}}\left(r(x,\gamma)^n\right) \\
&= \widetilde{\mathbb{E}}\left(\widetilde{\mathbb{E}}(Y(x)^2 \mid \mathcal{F})^n\right) \\
&\le \widetilde{\mathbb{E}}\left(\widetilde{\mathbb{E}}(Y(x)^{2n} | \mathcal{F})\right) \\
&= \widetilde{\mathbb{E}}(Y(x)^{2n}) \\
&= \int_\Omega \mathbb{P}(d\omega) \int_\Gamma \pi_{g(\cdot,\omega)\sigma}(d\gamma) Y(x)^{2n} \\
&= \int_\Omega \mathbb{P}(d\omega) Y(x)^{2n} \int_\Gamma \pi_{g(\cdot,\omega)\sigma}(d\gamma) \\
&= \int_\Omega \mathbb{P}(d\omega) Y(x)^{2n} \\
&= \frac{(2n)!}{2^n n!} \mathcal{K}(x,x)^n,
\end{aligned}
$$

for σ-a.a. $x \in X$. □

By (5.3) we have, for each $\Lambda \in \mathcal{B}_0(X)$,

$$
\begin{aligned}
&\int_\Gamma \mu(d\gamma) \int_X \gamma(dx) \int_X \sigma(dy)\, c(x,y,\gamma\setminus x)(\chi_\Lambda(x) + \chi_\Lambda(y)) \\
&\quad = \int_\Gamma \mu(d\gamma) \int_X \sigma(dx) \int_X \sigma(dy)\, r(x,\gamma) c(x,y,\gamma)(\chi_\Lambda(x) + \chi_\Lambda(y)) \\
&\quad = \int_\Gamma \mu(d\gamma) \int_X \sigma(dx) \int_X \sigma(dy)\, a(x,y) r(x,\gamma)^s r(y,\gamma)^s \chi_{\{r>0\}}(x,\gamma) \\
&\qquad \times \chi_{\{r>0\}}(y,\gamma)(\chi_\Lambda(x) + \chi_\Lambda(y)) \\
&\le \int_\Gamma \mu(d\gamma) \int_X \sigma(dx) \int_X \sigma(dy)\, a(x,y) r(x,\gamma)^s r(y,\gamma)^s (\chi_\Lambda(x) + \chi_\Lambda(y)) \\
&= 2 \int_\Gamma \mu(d\gamma) \int_\Lambda \sigma(dx) \int_X \sigma(dy)\, a(x,y) r(x,\gamma)^s r(y,\gamma)^s \\
&\le 2 \int_\Gamma \mu(d\gamma) \int_\Lambda \sigma(dx) \int_X \sigma(dy)\, a(x,y)(1 + r(x,\gamma))(1 + r(y,\gamma)). \qquad (5.19)
\end{aligned}
$$

Here, we used that, for each $s \in [0,1]$ and $a > 0$,

$$
a^s \le 1 + a, \qquad (5.20)
$$

Indeed, if $a \in (0,1]$, then $a^s \le 1 \le 1+a$, and if $a > 1$, then $a^s \le a^1 \le 1+a$. By (5.17)

$$\int_\Gamma \mu(d\gamma) \int_\Lambda \sigma(dx) \int_X \sigma(dy)\, a(x,y) < \infty. \tag{5.21}$$

Below, C_i, $i = 1, 2, 3, \ldots$, will denote positive constants whose explicit values are not important for us. We have, by (5.17) and (3.39),

$$\begin{aligned}
&\int_\Gamma \mu(d\gamma) \int_\Lambda \sigma(dx) \int_X \sigma(dy)\, a(x,y) r(x,\gamma) \\
&\quad = \int_\Gamma \mu(d\gamma) \int_\Lambda \sigma(dx)\, r(x,\gamma) \left(\int_X \sigma(dy)\, a(x,y) \right) \\
&\quad \le C_1 \int_\Gamma \mu(d\gamma) \int_\Lambda \sigma(dx) r(x,\gamma) \\
&\quad = C_1 \int_\Gamma \mu(d\gamma) \int_\Lambda \gamma(dx) \\
&\quad = C_1 \int_\Lambda \mathcal{K}(x,x)\, \sigma(dx) < \infty.
\end{aligned} \tag{5.22}$$

Here we used that the first correlation function of μ, $k_\mu^{(1)}(x)$ is $\mathcal{K}(x,x)$.

Next, by (3.40) and (3.8),

$$\begin{aligned}
&\int_\Gamma \mu(d\gamma) \int_\Lambda \sigma(dx) \int_X \sigma(dy)\, a(x,y) r(y,\gamma) \\
&\quad = \int_\Lambda \sigma(dx) \int_X \sigma(dy)\, a(x,y) \int_\Gamma \mu(d\gamma) r(y,\gamma) \\
&\quad = \int_\Lambda \sigma(dx) \int_X \sigma(dy)\, a(x,y) \mathcal{K}(y,y) \\
&\quad = \int_\Lambda \sigma(dx) \int_\Delta \sigma(dy)\, a(x,y) \mathcal{K}(y,y) + \int_\Lambda \sigma(dx) \int_{\Delta^c} \sigma(dy)\, a(x,y) \mathcal{K}(y,y) \\
&\quad \le C_2 \int_\Lambda \sigma(dx) \int_\Delta \sigma(dy) \mathcal{K}(y,y) + C_3 \int_\Lambda \sigma(dx) \int_{\Delta^c} \sigma(dy)\, a(x,y) < \infty, \\
&\quad \le C_2 \int_\Lambda \sigma(dx) \int_\Delta \sigma(dy) \mathcal{K}(y,y) + C_3 \int_\Lambda \sigma(dx) \int_X \sigma(dy)\, a(x,y) < \infty,
\end{aligned} \tag{5.23}$$

where we used that the function a is bounded and $\mathcal{K}(y,y)$ is bounded on Δ^c.

Analogously, using Lemma 5.1 and the Cauchy inequality, we have:

$$\begin{aligned}
&\int_\Gamma \mu(d\gamma)\int_\Lambda \sigma(dx)\int_X \sigma(dy)a(x,y)r(x,\gamma)r(y,\gamma)\\
&\quad= \int_\Lambda \sigma(dx)\int_X \sigma(dy)a(x,y)\int_\Gamma \mu(d\gamma)r(x,\gamma)r(y,\gamma)\\
&\quad\le \int_\Lambda \sigma(dx)\int_X \sigma(dy)\,a(x,y)\left(\int_\Gamma r(x,\gamma)^2\,\mu(d\gamma)\right)^{1/2}\left(\int_\Gamma r(y,\gamma)^2\,\mu(d\gamma)\right)^{1/2}\\
&\quad\le C_4\int_\Lambda \sigma(dx)\int_X \sigma(dy)\,a(x,y)\mathcal{K}(x,x)\mathcal{K}(y,y)\\
&\quad= C_4\int_\Lambda \sigma(dx)\int_\Delta \sigma(dy)\,a(x,y)\mathcal{K}(x,x)\mathcal{K}(y,y)\\
&\qquad+ C_4\int_\Lambda \sigma(dx)\int_{\Delta^c} \sigma(dy)\,a(x,y)\mathcal{K}(x,x)\mathcal{K}(y,y)\\
&\quad\le C_5\int_\Lambda \sigma(dx)\,\mathcal{K}(x,x)\int_\Delta \sigma(dy)\,\mathcal{K}(y,y)\\
&\qquad+ C_6\int_\Lambda \sigma(dx)\mathcal{K}(x,x)\int_{\Delta^c} \sigma(dy)\,a(x,y)\\
&\quad\le C_5\int_\Lambda \sigma(dx)\,\mathcal{K}(x,x)\int_\Delta \sigma(dy)\,\mathcal{K}(y,y)\\
&\qquad+ C_6\int_\Lambda \sigma(dx)\mathcal{K}(x,x)\int_X \sigma(dy)\,a(x,y) < \infty. \qquad (5.24)
\end{aligned}$$

Thus, by (5.19)–(5.24), the theorem is proven. □

Theorem 5.3. *(i) Let* $s \in \left[\frac{1}{2}, 1\right]$, *and let the conditions of Theorem 5.2 (i) be satisfied. Then the coefficients* $d(x,\gamma)$ *and* $b(x,\gamma)$ *defined by* (5.14) *and* (5.15), *satisfy condition* (5.12). *Thus,* $\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma) \subset D(L_{\mathrm{G}})$, *and for each* $F \in \mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$, $L_{\mathrm{G}}F$ *is given by formula* (5.1).

(ii) Let $s \in \left[\frac{1}{2}, 1\right]$, *and let the conditions of Theorem 5.2 (ii) be satisfied. Further assume that either*

$$\forall \Lambda \in \mathcal{B}_0(X)\ \exists \Lambda' \in \mathcal{B}_0(X)\ \forall x \in \Lambda\ \forall y \in (\Lambda')^c: \quad a(x,y) = 0, \qquad (5.25)$$

or

$$\int_\Delta \mathcal{K}(x,x)^2\,\sigma(dx) < \infty, \qquad (5.26)$$

where Δ is as in Theorem 5.2 (ii). Then the coefficient $c(x,y,\gamma)$ *defined by* (5.16)*, satisfies condition* (5.13)*. Thus,* $\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)\subset D(L_{\mathrm{K}})$*, and for each* $F\in\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$*,* $L_{\mathrm{K}}F$ *is given by formula* (5.2).

Remark 5.4. If $X=\mathbb{R}^d$ and the function a is as in Remark 5.2, then condition (5.25) means that the function a has a compact support.

Proof of Theorem 5.3. We again prove only the part related to Kawasaki dynamics and only in the case $l=1$, omitting the upper index $^{(1)}$ from our notation.

We first assume that (5.25) is satisfied. Since the function a is bounded and satisfies (5.25), it suffices to show that, for each $\Lambda\in\mathcal{B}_0(X)$,

$$\int_\Lambda\gamma(dx)\int_\Lambda\sigma(dy)r(x,\gamma\backslash x)^{s-1}r(y,\gamma\backslash x)^s\chi_{\{r>0\}}(x,\gamma\backslash x)\chi_{\{r>0\}}(y,\gamma\backslash x)\in L^2(\mu). \tag{5.27}$$

We note that, for $s\in\left[\frac{1}{2},1\right]$, $2s-1\in[0,1]$. Therefore, by the Cauchy inequality and (5.20), we have

$$\begin{aligned}
&\int_\Gamma\mu(d\gamma)\left(\int_\Lambda\gamma(dx)\,r(x,\gamma\setminus x)^{s-1}\chi_{\{r>0\}}(x,\gamma\setminus x)\right.\\
&\qquad\times\left.\int_\Lambda\sigma(dy)\,r(y,\gamma\setminus x)^s\chi_{\{r>0\}}(y,\gamma\setminus x)\right)^2\\
&\quad\le\int_\Gamma\mu(d\gamma)\int_\Lambda\gamma(dx)\,r(x,\gamma\setminus x)^{2(s-1)}\chi_{\{r>0\}}(x,\gamma\setminus x)\\
&\qquad\times\left(\int_\Lambda\sigma(dy)\,r(y,\gamma\setminus x)^s\chi_{\{r>0\}}(y,\gamma\setminus x)\right)^2\gamma(\Lambda)\\
&\quad=\int_\Gamma\mu(d\gamma)\int_\Lambda\sigma(dx)\,r(x,\gamma)^{2s-1}\chi_{\{r>0\}}(x,\gamma)\\
&\qquad\times\left(\int_\Lambda\sigma(dy)r(y,\gamma)^s\chi_{\{r>0\}}(y,\gamma)\right)^2(\gamma(\Lambda)+1)\\
&\quad\le\int_\Gamma\mu(d\gamma)\int_\Lambda\sigma(dx)\,(1+r(x,\gamma))\chi_{\{r>0\}}(x,\gamma)\\
&\qquad\times\left(\int_\Lambda\sigma(dy)(1+r(y,\gamma))\chi_{\{r>0\}}(y,\gamma)\right)^2(\gamma(\Lambda)+1)
\end{aligned}\tag{5.28}$$

$$\le \int_\Gamma \mu(d\gamma) \left(\int_\Lambda \sigma(dx)(1 + r(x,\gamma)) \right)^3 (\gamma(\Lambda) + 1)$$

$$\le \left(\int_\Gamma \mu(d\gamma) \left(\int_\Lambda \sigma(dx)(1 + r(x,\gamma)) \right)^6 \right)^{1/2} \left(\int_\Gamma \mu(d\gamma)(\gamma(\Lambda) + 1)^2 \right)^{1/2}$$

$$= \left(\int_\Gamma \mu(d\gamma) \left(\sigma(\Lambda) + \int_\Lambda \sigma(dx) r(x,\gamma) \right)^6 \right)^{1/2} \left(\int_\Gamma \mu(d\gamma)(\gamma(\Lambda) + 1)^2 \right)^{1/2}. \tag{5.29}$$

We have, for each $n \in \mathbb{N}$,

$$\int_\Gamma \mu(d\gamma) \left(\int_\Lambda \sigma(dx)\, r(x,\gamma) \right)^n$$

$$= \int_\Lambda \sigma(dx_1) \cdots \int_\Lambda \sigma(dx_n) \int_\Gamma \mu(d\gamma)\, r(x_1,\gamma) \cdots r(x_n,\gamma)$$

$$\le \int_\Lambda \sigma(dx_1) \cdots \int_\Lambda \sigma(dx_n) \|r(x_1,\cdot)\|_{L^n(\mu)} \cdots \|r(x_n,\cdot)\|_{L^n(\mu)}$$

$$\le \left(\int_\Lambda \sigma(dx) \|r(x,\cdot)\|_{L^n(\mu)} \right)^n$$

By Lemma 5.1,

$$\|r(x,\cdot)\|_{L^n(\mu)} = \left(\int_\Gamma \mu(d\gamma) r(x,\gamma)^n \right)^{\frac{1}{n}}$$

$$\le \left(\frac{(2n)!}{2^n n!} \mathcal{K}(x,x)^n \right)^{\frac{1}{n}}$$

$$= \left(\frac{(2n)!}{2^n n!} \right)^{\frac{1}{n}} \mathcal{K}(x,x).$$

Hence,

$$\int_\Gamma \mu(d\gamma) \left(\int_\Lambda \sigma(dx)\, r(x,\gamma) \right)^n$$

$$\le \left(\int_\Lambda \sigma(dx) \left(\frac{(2n)!}{2^n n!} \right)^{\frac{1}{n}} \mathcal{K}(x,x) \right)^n$$

$$\le \frac{(2n)!}{2^n n!} \left(\int_\Lambda \sigma(dx) \mathcal{K}(x,x) \right)^n < \infty. \tag{5.30}$$

Now, (5.27) follows from (5.29) and (5.30).

Next, we assume that (5.26) is satisfied. We fix $\Lambda \in \mathcal{B}_0(X)$ and denote

$$u(x, y) := a(x, y)(\chi_\Lambda(x) + \chi_\Lambda(y)).$$

Then, by the Cauchy inequality and (5.17),

$$\begin{aligned}
&\int_\Gamma \mu(d\gamma) \left(\int_X \gamma(dx) \int_X \sigma(dy)\, u(x, y) r(x, \gamma \setminus x)^{s-1} \chi_{\{r>0\}}(x, \gamma \setminus x) \right. \\
&\qquad \left. \times\, r(y, \gamma \setminus x)^s \chi_{\{r>0\}}(y, \gamma \setminus x) \right)^2 \\
&\quad \le \int_\Gamma \mu(d\gamma) \int_X \gamma(dx) \int_X \sigma(dy)\, u(x, y) r(x, \gamma \setminus x)^{2(s-1)} \chi_{\{r>0\}}(x, \gamma \setminus x) \\
&\quad \times r(y, \gamma \setminus x)^{2s} \chi_{\{r>0\}}(y, \gamma \setminus x) \int_X \gamma(dx') \int_X \sigma(dy')\, u(x', y') \\
&\quad = \int_\Gamma \mu(d\gamma) \int_X \sigma(dx) \int_X \sigma(dy)\, u(x, y) r(x, \gamma)^{2s-1} \chi_{\{r>0\}}(x, \gamma) \\
&\quad \times r(y, \gamma)^{2s} \chi_{\{r>0\}}(y, \gamma) \int_X (\gamma + \varepsilon_x)(dx') \int_X \sigma(dy')\, u(x', y') \\
&\quad \le \int_\Gamma \mu(d\gamma) \int_X \sigma(dx) \int_X \sigma(dy)\, u(x, y)(1 + r(x, \gamma))(1 + r(y, \gamma)^2) \\
&\quad \times \left(\int_X \gamma(dx') \int_X \sigma(dy') u(x', y') + \int_X \sigma(dy') u(x, y') \right) \\
&\quad \le \int_\Gamma \mu(d\gamma) \left(\int_X \sigma(dx) \int_X \sigma(dy)\, u(x, y)(1 + r(x, \gamma))(1 + r(y, \gamma)^2) \right) \\
&\quad \times \left(\int_X \gamma(dx') \int_X \sigma(dy') u(x', y') + C_7 \right).
\end{aligned}$$

Hence, by the Cauchy inequality, it suffices to prove that

$$\int_\Gamma \mu(d\gamma) \left(\int_X \sigma(dx) \int_X \sigma(dy)\, u(x, y)(1 + r(x, \gamma))(1 + r(y, \gamma)^2) \right)^2 < \infty, \tag{5.31}$$

$$\int_\Gamma \mu(d\gamma) \left(\int_X \gamma(dx) \int_X \sigma(dy)\, u(x, y) \right)^2 < \infty. \tag{5.32}$$

We first to prove (5.32). We have, by Theorem 3.5,

$$\int_\Gamma \mu(d\gamma) \left(\int_X \gamma(dx) \int_X \sigma(dy) u(x, y) \right)^2$$

$$
\begin{aligned}
&= \int_\Gamma \mu(d\gamma) \int_X \gamma(dx) \int_X \sigma(dy)\, u(x,y) \int_X \gamma(dx') \int_X \sigma(dy')\, u(x',y') \\
&= \int_X \sigma(dy) \int_X \sigma(dy') \int_\Gamma \mu(d\gamma) \int_X \gamma(dx) \int_X \gamma(dx')\, u(x,y)u(x',y') \\
&= \int_X \sigma(dy) \int_X \sigma(dy') \int_\Gamma \mu(d\gamma) \bigg(\int_X \gamma(dx)\, u(x,y)u(x,y') \\
&\quad + \int_X \gamma(dx) \int_X (\gamma - \varepsilon_x)(dx')\, u(x,y)u(x',y') \bigg) \\
&= \int_X \sigma(dy) \int_X \sigma(dy') \bigg(\int_X \sigma(dx)\, \mathcal{K}(x,x)u(x,y)u(x,y') \\
&+ \int_X \sigma(dx) \int_X \sigma(dx') \left(\frac{1}{2}\mathcal{K}(x,x')^2 + \mathcal{K}(x,x)\mathcal{K}(x',x') \right) u(x,y)u(x',y') \bigg) .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_\Gamma \mu(d\gamma) \left(\int_X \gamma(dx) \int_X \sigma(dy) u(x,y) \right)^2 \\
&\quad \le \int_X \sigma(dy) \int_X \sigma(dy') \bigg(\int_X \sigma(dx)\, \mathcal{K}(x,x)u(x,y)u(x,y') \\
&\qquad + \int_X \sigma(dx) \int_X \sigma(dx')\, \frac{3}{2}\mathcal{K}(x,x)\mathcal{K}(x',x')u(x,y)u(x',y') \bigg) \\
&\quad = \int_X \sigma(dy) \int_X \sigma(dy') \int_X \sigma(dx)\, \mathcal{K}(x,x)u(x,y)u(x,y') \\
&\qquad + \frac{3}{2} \left(\int_X \sigma(dy) \int_X \sigma(dx)\, \mathcal{K}(x,x)u(x,y) \right)^2 \\
&\quad \le \int_\Delta \sigma(dx)\, \mathcal{K}(x,x) \left(\int_X \sigma(dy)\, u(x,y) \right)^2 \\
&\qquad + C_8 \int_X \sigma(dy) \int_X \sigma(dy') \int_{\Delta^c} \sigma(dx)\, u(x,y)u(x,y') \\
&\qquad + \frac{3}{2} \left(\int_\Delta \sigma(dx)\, \mathcal{K}(x,x) \int_X \sigma(dy)\, u(x,y) + C_8 \int_X \sigma(dy) \int_{\Delta^c} \sigma(dx)\, u(x,y) \right)^2 < \infty.
\end{aligned}
$$

Next, we prove (5.31). By Lemma 5.1 and Hölder's inequality, we have

$$
\begin{aligned}
&\int_\Gamma \mu(d\gamma) \left(\int_X \sigma(dx) \int_X \sigma(dy)\, u(x,y)(1 + r(x,\gamma))(1 + r(y,\gamma)^2) \right)^2 \\
&\quad = \int_X \sigma(dx) \int_X \sigma(dx') \int_X \sigma(dy) \int_X \sigma(dy')\, u(x,y)u(x',y')
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_\Gamma \mu(d\gamma)(1+r(x,\gamma))(1+r(x',\gamma))(1+r(y,\gamma)^2)(1+r(y',\gamma)^2) \\
&\le \int_X \sigma(dx) \int_X \sigma(dx') \int_X \sigma(dy) \int_X \sigma(dy')\, u(x,y)u(x',y')\left(1+\|r(x,\cdot)\|_{L^4(\mu)}\right) \\
&\times \left(1+\|r(x',\cdot)\|_{L^4(\mu)}\right)\left(1+\|r(y,\cdot)^2\|_{L^4(\mu)}\right)\left(1+\|r(y',\cdot)^2\|_{L^4(\mu)}\right) \\
&= \left(\int_X \sigma(dx) \int_X \sigma(dy) u(x,y)\left(1+\|r(x,\cdot)\|_{L^4(\mu)}\right)\left(1+\|r(y,\cdot)^2\|_{L^4(\mu)}\right)\right)^2 \\
&\le C_9 \left(\int_X \sigma(dx) \int_X \sigma(dy)\, u(x,y)(1+\mathcal{K}(x,x))(1+\mathcal{K}(y,y)^2)\right)^2 < \infty.
\end{aligned}
$$

Indeed, since the function u is bounded, by (5.26)

$$
\begin{aligned}
&\int_\Delta \sigma(dx) \int_\Delta \sigma(dy) u(x,y)(1+\mathcal{K}(x,x))(1+\mathcal{K}(y,y)^2) \\
&\le C_{10} \int_\Delta \sigma(dx)(1+\mathcal{K}(x,x)) \int_\Delta \sigma(dy)(1+\mathcal{K}(y,y)^2) \\
&< \infty.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\int_{\Delta^c} \sigma(dx) \int_\Delta \sigma(dy) u(x,y)(1+\mathcal{K}(x,x))(1+\mathcal{K}(y,y)^2) \\
&\le C_{11} \int_{\Delta^c} \sigma(dx) \int_\Delta \sigma(dy) u(x,y)(1+\mathcal{K}(y,y)^2) \\
&\le C_{11} \int_\Delta \sigma(dy)(1+\mathcal{K}(y,y)^2) \int_X \sigma(dx) u(x,y) \\
&\le C_{12} \int_\Delta \sigma(dy)(1+\mathcal{K}(y,y)^2) \\
&< \infty.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\int_\Delta \sigma(dx) \int_{\Delta^c} \sigma(dy) u(x,y)(1+\mathcal{K}(x,x))(1+\mathcal{K}(y,y)^2) \\
&\le C_{13} \int_\Delta \sigma(dx) \int_{\Delta^c} \sigma(dy) u(x,y)(1+\mathcal{K}(x,x)) \\
&\le C_{13} \int_\Delta \sigma(dx)(1+\mathcal{K}(x,x)) \int_X \sigma(dy) u(x,y)
\end{aligned}
$$

$$\leq C_{14} \int_{\Delta} \sigma(dx)(1 + \mathcal{K}(x, x))$$
$$< \infty.$$

Finally,

$$\int_{\Delta^c} \sigma(dx) \int_{\Delta^c} \sigma(dy) u(x, y)(1 + \mathcal{K}(x, x))(1 + \mathcal{K}(y, y)^2)$$
$$\leq C_{15} \int_{\Delta^c} \sigma(dx) \int_{\Delta^c} \sigma(dy) u(x, y)$$
$$\leq C_{15} \int_{X} \sigma(dx) \int_{X} \sigma(dy) u(x, y)$$
$$= C_{15} \int_{X} \sigma(dx) \int_{X} \sigma(dy) a(x, y)(\chi_\Lambda(x) + \chi_\Lambda(y))$$
$$< \infty.$$

Thus, the theorem is proven. □

5.3 Diffusion approximation

From now on, we set $X = \mathbb{R}^d$, $d \in \mathbb{N}$, and σ to be Lebesgue measure. We will show that, under an appropriate scaling, the Dirichlet form of the Kawasaki dynamics converges to a Dirichlet form which identifies a diffusion process on Γ having a permanental point process $\mu^{(l)}$ as a symmetrizing measure. The way we scale the Kawasaki dynamics will be similar to the ansatz of [23].

We denote by $\mathcal{F}C_{\mathrm{b}}^{\infty}(C_0^{\infty}(\mathbb{R}^d), \Gamma)$ the space of all functions of the form (5.8) where $N \in \mathbb{N}$, $\varphi_1, \dots, \varphi_N \in C_0^{\infty}(\mathbb{R}^d)$ and $g \in C_{\mathrm{b}}^{\infty}(\mathbb{R}^N)$. Here, $C_0^{\infty}(\mathbb{R}^d)$ denotes the space of smooth functions on $\mathbb{R}^d$ with compact support, and $C_{\mathrm{b}}^{\infty}(\mathbb{R}^N)$ denotes the space of all smooth bounded functions on $\mathbb{R}^N$ whose all derivatives are bounded. Clearly,

$$\mathcal{F}C_{\mathrm{b}}^{\infty}(C_0^{\infty}(\mathbb{R}^d), \Gamma) \subset \mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d), \Gamma),$$

and the set $\mathcal{F}C_{\mathrm{b}}^{\infty}(C_0^{\infty}(\mathbb{R}^d),\Gamma)$ is a core for the Dirichlet form $(\mathcal{E}_{\mathrm{K}}, D(\mathcal{E}_{\mathrm{K}}))$.

We fix $s = 1/2$. Let us assume that the function $a(x,y)$ is as in Remark 5.2. Thus, the coefficient $c(x,y,\gamma)$ has the form

$$c(x,y,\gamma) = a(x-y)r(x,\gamma)^{-1/2}r(y,\gamma)^{1/2}\chi_{\{r>0\}}(x,\gamma)\chi_{\{r>0\}}(y,\gamma). \tag{5.33}$$

Note that $y-x$ describes the change of the position of a particle which hops from x to y. We now scale the function a as follows: for each $\varepsilon > 0$, we denote

$$a_\varepsilon(x) := \varepsilon^{-d-2}a(x/\varepsilon), \quad x \in \mathbb{R}^d. \tag{5.34}$$

The Dirichlet form $(\mathcal{E}_{\mathrm{K}}, D(\mathcal{E}_{\mathrm{K}}))$ which corresponds to the choice of function a as in (5.34) will be denoted by $(\mathcal{E}_\varepsilon, D(\mathcal{E}_\varepsilon))$.

Theorem 5.4. *Assume that the function a has compact support, and the value $a(x)$ only depends on $|x|$, i.e., $a(x) = \tilde{a}(|x|)$ for some function $\tilde{a} : [0,\infty) \to \mathbb{R}$. Further assume that the function $\mathcal{K}_1(x,y)$ has the form $\mathcal{K}_1(x-y)$ for some $\mathcal{K}_1 : \mathbb{R}^d \to \mathbb{R}$, and*

$$\lim_{y\to 0}\int_{\mathbb{R}^d}(\mathcal{K}_1(x) - \mathcal{K}_1(x+y))^2\,dx = 0. \tag{5.35}$$

For each $l \in \mathbb{N}$, define a quadratic form $(\mathcal{E}_0, \mathcal{F}C_{\mathrm{b}}^{\infty}(C_0^{\infty}(\mathbb{R}^d),\Gamma))$ by

$$\mathcal{E}_0(F,G) := c\int_\Gamma \mu^{(l)}(d\gamma)\int_{\mathbb{R}^d} dx\, r(x,\gamma)\langle \nabla_x F(\gamma\cup x), \nabla_x G(\gamma \cup x)\rangle. \tag{5.36}$$

Here

$$c := \frac{1}{2}\int_{\mathbb{R}^d} a(x)x_1^2\,dx$$

(x_1 denoting the first coordinate of $x \in \mathbb{R}^d$), ∇_x denotes the gradient in the x variable, and $\langle\cdot,\cdot\rangle$ stands for the scalar product in $\mathbb{R}^d$. Then, for any $F, G \in \mathcal{F}C_{\mathrm{b}}^{\infty}(C_0^{\infty}(\mathbb{R}^d),\Gamma)$,

$$\mathcal{E}_\varepsilon(F,G) \to \mathcal{E}_0(F,G) \quad \text{as } \varepsilon \to 0.$$

Remark 5.5. Assume that the function $\mathcal{K}_1$ is differentiable on $\mathbb{R}^d$. Denote

$$K(x,\delta) := \sup_{y\in B(x,\delta)} |\nabla\mathcal{K}_1(y)|, \quad x\in\mathbb{R}^d,\ \delta>0.$$

Here $B(x,\delta)$ denotes the closed ball in $\mathbb{R}^d$ centered at x and of radius δ. Assume that, for some $\delta > 0$,

$$K(\cdot,\delta)\in L^2(\mathbb{R}^d, dx). \tag{5.37}$$

Then, by Taylor's formula condition (5.35) is clearly satisfied. Note that condition (5.37) is slightly stronger than the condition $|\nabla\mathcal{K}_1|\in L^2(\mathbb{R}^d, dx)$.

Proof of Theorem 5.4. Again we will only present the proof in the case $l = 1$, omitting the upper index $^{(1)}$. We start with the following

Lemma 5.2. *Fix any $\Lambda\in\mathcal{B}_0(\mathbb{R}^d)$ and $\alpha\in(0,1]$. Then, under the conditions of Theorem 5.4,*

$$r(x+\varepsilon y,\gamma)^\alpha \to r(x,\gamma)^\alpha \text{ in } L^2(\Gamma\times\Lambda\times\mathbb{R}^d, \mu(d\gamma)\,dx\,dy\,a(y)) \text{ as } \varepsilon\to 0.$$

Proof. We first prove the statement for $\alpha = 1$. Thus, equivalently we have to prove that

$$r(x+\varepsilon y,\gamma)\to r(x,\gamma) \text{ in } L^2(\Omega\times\Gamma\times\Lambda\times\mathbb{R}^d, \tilde{\mathbb{P}}(d\omega,d\gamma)\,dx\,dy\,a(y)) \text{ as } \varepsilon\to 0. \tag{5.38}$$

We have, using Jensen's inequality for conditional expactation,

$$\begin{aligned}
&\int_\Lambda dx\int_{\mathbb{R}^d} dy\,a(y)\int_{\Omega\times\Gamma}\tilde{\mathbb{P}}(d\omega,d\gamma)\,(r(x+\varepsilon y)-r(x,\gamma))^2\\
&\quad=\int_\Lambda dx\int_{\mathbb{R}^d} dy\,a(y)\int_{\Omega\times\Gamma}\tilde{\mathbb{P}}(d\omega,d\gamma)\tilde{\mathbb{E}}(Y(x+\varepsilon y)^2-Y(x)^2\mid\mathcal{F})^2\\
&\quad\le\int_\Lambda dx\int_{\mathbb{R}^d} dy\,a(y)\int_{\Omega\times\Gamma}\tilde{\mathbb{P}}(d\omega,d\gamma)(Y(x+\varepsilon y)^2-Y(x)^2)^2\\
&\quad=\int_\Lambda dx\int_{\mathbb{R}^d} dy\,a(y)\int_\Omega d\mathbb{P}\,(Y(x+\varepsilon y)^4+Y(x)^4-2Y(x+\varepsilon y)^2Y(x)^2).
\end{aligned} \tag{5.39}$$

We have

$$
\begin{aligned}
&\int_\Omega Y(x+\varepsilon y)^4 \, d\mathbb{P} \\
&\quad = 3\Big(\int_{\mathbb{R}^d} \mathcal{K}_1(x+\varepsilon y-u)^2 \, du\Big)^2 \\
&\quad = 3\Big(\int_{\mathbb{R}^d} \mathcal{K}_1(x-u)^2 \, du\Big)^2 \\
&\quad = \int_\Omega Y(x)^4 \, d\mathbb{P}.
\end{aligned} \tag{5.40}
$$

We have:

$$
\begin{aligned}
&\int_\Lambda dx \int_{\mathbb{R}^d} dy \, a(y) \int_\Omega d\tilde{\mathbb{P}} \, Y(x+\varepsilon y)^2 Y(x)^2 \\
&\quad = \int_\Lambda dx \int_{\mathbb{R}^d} dy \, a(y) \Big[\int_{\mathbb{R}^d} \mathcal{K}_1(x+\varepsilon y-u)^2 \, du \cdot \int_{\mathbb{R}^d} \mathcal{K}_1(x-u')^2 \, du' \\
&\qquad + 2\Big(\int_{\mathbb{R}^d} \mathcal{K}_1(x+\varepsilon y-u)\, \mathcal{K}_1(x-u) \, du\Big)^2 \Big] \\
&\quad = \int_\Lambda dx \int_{\mathbb{R}^d} dy \, a(y) \Big[\Big(\int_{\mathbb{R}^d} \mathcal{K}_1(u)^2 du\Big)^2 + 2\Big(\int_{\mathbb{R}^d} \mathcal{K}_1(u+\varepsilon y)\mathcal{K}_1(u) du\Big)^2 \Big].
\end{aligned} \tag{5.41}
$$

By (5.35), for a fixed $y \in \mathbb{R}^d$,

$$
\int_{\mathbb{R}^d} \mathcal{K}_1(u+\varepsilon y)\mathcal{K}_1(u) du \to \int_{\mathbb{R}^d} \mathcal{K}_1(u)^2 du,
$$

as $\varepsilon \to 0$. On the other hand for each $y \in \mathbb{R}^d$, by the Cauchy inequality,

$$
\begin{aligned}
\Big(\int_{\mathbb{R}^d} \mathcal{K}_1(u+\varepsilon y)\mathcal{K}_1(u) du\Big)^2 &\le \int_{\mathbb{R}^d} \mathcal{K}_1(u+\varepsilon y)^2 du \cdot \int_{\mathbb{R}^d} \mathcal{K}_1(u)^2 du \\
&= \Big(\int_{\mathbb{R}^d} \mathcal{K}_1(u)^2 du\Big)^2.
\end{aligned}
$$

Hence, by the dominated convergence theorem,

$$
\begin{aligned}
&\int_\Lambda dx \int_{\mathbb{R}^d} dy \, a(y) \int_\Omega d\mathbb{P} \, Y(x+\varepsilon y)^2 Y(x)^2 \\
&\quad \to 3 \int_\Lambda dx \int_{\mathbb{R}^d} dy \, a(y) \Big(\int_{\mathbb{R}^d} \mathcal{K}_1(u)^2 du\Big)^2
\end{aligned}
$$

$$= \int_\Lambda dx \int_{\mathbb{R}^d} dy\, a(y) \int_\Omega d\mathbb{P} Y(x)^4$$

as $\varepsilon \to 0$

By (5.39)–(5.41), statement (5.38) follows.

To prove the result for $\alpha \in (0,1)$, it is now sufficient to show the following

Claim. Let $(\mathbf{A}, \mathcal{A}, m)$ be a measure space and let $m(\mathbf{A}) < \infty$. Let $f_\varepsilon \in L^2(m)$, $f_\varepsilon \geq 0$, $\varepsilon \in [-1,1]$, and let $f_\varepsilon \to f_0$ in $L^2(m)$ as $\varepsilon \to 0$. Then, for each $\alpha \in (0,1)$, $f_\varepsilon^\alpha \to f_0^\alpha$ in $L^2(m)$ as $\varepsilon \to 0$.

By e.g. [4, Theorems 21.2 and 21.4], $f_\varepsilon \to f_0$ in $L^2(m)$ is equivalent to:

(i) $f_\varepsilon \to f_0$ in measure;

(ii) $\sup_{\varepsilon \in [-1,1]} \int f_\varepsilon^2\, dm < \infty$;

(iii) (*uniform integrability*) For each $\theta > 0$ there exist $h \in L^1(m)$ and $\delta > 0$ such that, for all $0 < |\varepsilon| \leq 1$ and for each $A \in \mathcal{A}$

$$\int_A h\, dm \leq \delta \Rightarrow \int_A f_\varepsilon^2\, dm \leq \theta.$$

Hence, for $\alpha \in (0,1)$, we get:

a) $f_\varepsilon^\alpha \to f_0^\alpha$ in measure;

b) $\sup_{\varepsilon \in [-1,1]} \int f_\varepsilon^{2\alpha}\, dm \leq \sup_{\varepsilon \in [-1,1]} \int (1 + f_\varepsilon^2)\, dm < \infty$;

c) Let θ, h, and δ be as in (iii). Set $h' := h + \frac{\delta}{\theta}$. Clearly, $h \in L^1(m)$. Assume that, for some $A \in \mathcal{A}$, $\int_A h'\, dm \leq \delta$. Hence $\int_A h\, dm \leq \delta$, and therefore $\int_A f_\varepsilon^2\, dm \leq \theta$ for all $0 < |\varepsilon| \leq 1$. Furthermore, we get $\int_A \frac{\delta}{\theta}\, dm \leq \delta$, and therefore $m(A) \leq \theta$. Now

$$\int_A f_\varepsilon^{2\alpha}\, dm \leq \int_A (1 + f_\varepsilon^2)\, dm \leq 2\theta.$$

Applying again [4, Theorems 21.2 and 21.4], we conclude the claim. □

Fix any $F \in \mathcal{F}C_{\mathrm{b}}^{\infty}(C_0^{\infty}(\mathbb{R}^d), \Gamma)$. We have

$$\begin{aligned}
&\mathcal{E}_\varepsilon(F, F)\\
&= \frac{1}{2}\int_\Gamma \mu(d\gamma)\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \varepsilon^{-d-2} a((x-y)/\varepsilon)\\
&\quad \times r(x,\gamma)^{1/2} r(y,\gamma)^{1/2} (F(\gamma\cup x) - F(\gamma \cup y))^2\\
&= \frac{1}{2}\int_\Gamma \mu(d\gamma)\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a(y) r(x+\varepsilon y,\gamma)^{1/2} r(x,\gamma)^{1/2}\\
&\quad \times \left(\frac{F(\gamma\cup\{x+\varepsilon y\}) - F(\gamma\cup x)}{\varepsilon}\right)^2.
\end{aligned}$$

Assume that $0 < |\varepsilon| \le 1$. Note that the function F is local, i.e., there exists $\Delta \in \mathcal{B}_0(\mathbb{R}^d)$ such that $F(\gamma) = F(\gamma_\Delta)$ for all $\gamma \in \Gamma$. The function a has compact support. Hence, there exists $R > 0$ such that $a(y) = 0$ if $|y| > R$. Choose $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$ to be the collection of all points x in $\mathbb{R}^d$ such that the distance from x to the set Δ is $\le R$. Then, for each $x \notin \Lambda$ and all $y \in \mathbb{R}^d$ with $|y| \le R$, we have $x \notin \Delta$, and $x + \varepsilon y \notin \Delta$. Therefore

$$F(\gamma\cup\{x+\varepsilon y\}) - F(\gamma\cup x) = F(\gamma) - F(\gamma) = 0.$$

Hence,

$$\begin{aligned}
\mathcal{E}_\varepsilon(F, F) = \frac{1}{2}\int_\Gamma \mu(d\gamma)\int_{\Lambda} dx \int_{\mathbb{R}^d} dy\, a(y) r(x+\varepsilon y,\gamma)^{1/2} r(x,\gamma)^{1/2}&\\
\times \left(\frac{F(\gamma\cup\{x+\varepsilon y\}) - F(\gamma\cup x)}{\varepsilon}\right)^2.& \qquad (5.42)
\end{aligned}$$

By the dominated convergence theorem

$$r(x,\gamma)^{1/2}\left(\frac{F(\gamma\cup\{x+\varepsilon y\}) - F(\gamma\cup x)}{\varepsilon}\right)^2 \to r(x,\gamma)^{1/2}\langle \nabla_x F(\gamma\cup x), y\rangle^2 \qquad (5.43)$$

in $L^2(\Gamma\times\Lambda\times\mathbb{R}^d, \mu(d\gamma)\,dx\,dy\,a(y))$ as $\varepsilon\to 0$. By Lemma 5.2 with $\alpha = 1/2$, (5.42) and (5.43)

$$\mathcal{E}_\varepsilon(F,F) \to \frac{1}{2}\int_\Gamma \mu(d\gamma)\int_\Lambda dx \int_{\mathbb{R}^d} dy\, a(y) r(x,\gamma)\langle \nabla_x F(\gamma\cup x), y\rangle^2. \tag{5.44}$$

Since $a(y) = \tilde{a}(|y|)$, for any $i,j\in\{1,\dots,d\}$, $i\neq j$, we have

$$\int_{\mathbb{R}^d} a(y) y_i y_j\, dy = 0$$

and

$$c = \frac{1}{2}\int_{\mathbb{R}^d} a(y) y_i^2\, dy, \quad i = 1,\dots,d.$$

Therefore, by (5.44),

$$\mathcal{E}_\varepsilon(F,F) \to c\int_\Gamma \mu(d\gamma)\int_{\mathbb{R}^d} dx\, r(x,\gamma)|\nabla_x F(\gamma\cup x)|^2.$$

From here the theorem follows by the polarization identity for quadratic forms. □

We will now show that the limiting form $(\mathcal{E}_0, \mathcal{F}C_b^\infty(C_0^\infty(\mathbb{R}^d),\Gamma))$ is closable and its closure identifies a diffusion process.

In what follows, we will assume that the conditions of Theorem 5.4 are satisfied. We have

$$\begin{aligned}\mathcal{K}(x,y) &= \int_{\mathbb{R}^d} \mathcal{K}_1(x-u)\mathcal{K}_1(y-u)\,du\\ &= \int_{\mathbb{R}^d} \mathcal{K}_1(u-y)\mathcal{K}_1(u-x)\,du\\ &= \int_{\mathbb{R}^d} \mathcal{K}_1(u)\mathcal{K}_1(u+y-x)\,du.\end{aligned}$$

Hence, by (5.35), the function $\mathcal{K}(x,y)$ is continuous on $(\mathbb{R}^d)^2$. Indeed, assume that $(x_n,y_n)\to(x,y)$ on $n\to\infty$.Then

$$\mathcal{K}(x_n,y_n) - \mathcal{K}(x,y)$$

$$= \int_{\mathbb{R}^l} \mathcal{K}_1(u)[\mathcal{K}_1(u + y_n - x_n) - \mathcal{K}_1(u + y - x)]du.$$

From here

$$\begin{aligned}
&|\mathcal{K}(x_n, y_n) - \mathcal{K}(x, y)| \\
&\leq (\int_{\mathbb{R}^d} \mathcal{K}_1(u)^2 du)^{\frac{1}{2}} \times (\int_{\mathbb{R}^d} (\mathcal{K}_1(u + y_n - x_n) - \mathcal{K}(u + y - x))^2 du)^{\frac{1}{2}} \\
&= (\int_{\mathbb{R}^d} \mathcal{K}_1(u)^2 du)^{\frac{1}{2}} \times (\int_{\mathbb{R}^d} (\mathcal{K}_1(u - y + x + y_n - x_n) - \mathcal{K}_1(u))^2 du)^{\frac{1}{2}} \\
&\to 0,
\end{aligned}$$

as $n \to \infty$. Since $-y + x + y_n - x_n \to 0$ as $n \to \infty$. Thus, by Remark 3.1, $(Y(x))_{x \in X}$ is a Gaussian random field and formula (3.21) holds for all $(x, y) \in (\mathbb{R}^d)^2$.

Remark 5.6. Recall that a function $D : (\mathbb{R}^d)^2 \to [0, \infty)$ is called a metric on $\mathbb{R}^d$ if it satisfies the following axioms:

1. $D(x, y) = 0$ if and only if $x = y$;
2. $D(x, y) = D(y, x)$ for all $x, y \in \mathbb{R}^d$;
3. $D(x, z) \leq D(x, y) + D(y, z)$ for all $x, y, z \in \mathbb{R}^d$.

If a function $D : (\mathbb{R}^d)^2 \to [0, \infty)$ satisfies the first and second axioms, but not necessarily the third one (the triangle inequality), then D is called a semimetric.

Following [9], we consider the semimetric on $\mathbb{R}^d$:

$$\begin{aligned}
D(x, y) :&= \frac{1}{2} \left(\int_\Omega (Y(x) - Y(y))^2 \, d\mathbb{P} \right)^{1/2} \\
&= \frac{1}{2} (\mathcal{K}(x, x) + \mathcal{K}(y, y) - 2\mathcal{K}(x, y))^{1/2} \\
&= \frac{1}{\sqrt{2}} \left(\int_{\mathbb{R}^d} \mathcal{K}_1(u) \big(\mathcal{K}_1(u) - \mathcal{K}_1(u + y - x)\big) \, du \right)^{1/2}, \quad x, y \in \mathbb{R}^d.
\end{aligned} \tag{5.45}$$

The associated metric entropy $H(\delta)$ is defined as

$$H(\delta) := \log N(\delta),$$

where $N(\delta)$ is the minimal number of points in a δ-net in

$$B(0,1) = \{x \in \mathbb{R}^d \mid |x| \le 1\}$$

with respect to the semimetric D, i.e., points x_i such that the open balls centered at x_i and of radius δ (with respect to D) cover $B(0,1)$. The expression

$$J := \int_0^1 \sqrt{H(\delta)}\, d\delta$$

is called the Dudley integral. The following result holds, see e.g. [9, Corollary 7.1.4] and the references therein.

Theorem 5.5. *Assume that $J < \infty$. Then the Gaussian random field $(Y(x))_{x\in\mathbb{R}^d}$ has a continuous modification, i.e., for each $x \in \mathbb{R}^d$, there exists a version of the random variable $Y(x)$ such that, for each $\omega \in \Omega$, the function*

$$\mathbb{R}^d \ni x \mapsto Y(x,\omega) \in \mathbb{R}$$

is continuous.

Remark 5.7. Let $\mathcal{K}_1$ be as in Remark 5.5. Then, by (5.45), for any $x, y \in B(0,1)$

$$\begin{aligned} D(x,y)^2 &\le \|\mathcal{K}_1(\cdot)\|_{L^2(\mathbb{R}^d,dx)} \left(\int_{\mathbb{R}^d} (\mathcal{K}_1(u) - \mathcal{K}_1(u+y-x))^2\, du \right)^{1/2} \\ &\le \|\mathcal{K}_1(\cdot)\|_{L^2(\mathbb{R}^d,dx)} \|K(\cdot,2)\|_{L^2(\mathbb{R}^d,dx)} |y-x|, \end{aligned}$$

where we assumed that $K(\cdot,2) \in L^2(\mathbb{R}^d, dx)$. Then $J < \infty$, see e.g. [9, Example 7.1.5].

Denote by $\ddot{\Gamma}$ the space of all multiple configurations in $\mathbb{R}^d$. Thus, $\ddot{\Gamma}$ is the set of all Radon $\mathbb{Z}_+ \cup \{+\infty\}$-valued measures on $\mathbb{R}^d$, In particular, $\Gamma \subset \ddot{\Gamma}$. Analogously to the case of Γ, we define the vague topology on $\ddot{\Gamma}$ and the corresponding Borel σ-algebra $\mathcal{B}(\ddot{\Gamma})$.

Theorem 5.6. *Let $\mathcal{K}_1(x, y)$ be of the form $\mathcal{K}_1(x-y)$ for some $\mathcal{K}_1 \in L^2(\mathbb{R}^d, dx)$. Let $J(D) < \infty$. Let $l \in \mathbb{N}$ and $c > 0$. Then*

(i) The quadratic form $(\mathcal{E}_0, \mathcal{F}C_{\mathrm{b}}^{\infty}(C_0^{\infty}(\mathbb{R}^d), \Gamma))$ defined by (5.36) is closable on $L^2(\Gamma, \mu^{(l)})$ and its closure will be denoted by $(\mathcal{E}_0, D(\mathcal{E}_0))$.

(ii) There exists a conservative diffusion process

$$M^0 = \left(\Omega^0, \mathcal{F}^0, (\mathcal{F}_t^0)_{t\geq 0}, (\Theta_t^0)_{t\geq 0}, (X^0(t))_{t\geq 0}, (P_\gamma^0)_{\gamma\in\ddot{\Gamma}}\right)$$

on $\ddot{\Gamma}$ which is properly associated with $(\mathcal{E}_0, D(\mathcal{E}_0))$. In particular, M^0 is $\mu^{(l)}$-symmetric and has $\mu^{(l)}$ as invariant measure. In the case $d \geq 2$, the set $\ddot{\Gamma} \setminus \Gamma$ is $\mathcal{E}^0$-exceptional, so that $\ddot{\Gamma}$ may be replaced with Γ in the above statement.

Proof. We again discuss only the case $l = 1$, omitting the upper index $^{(1)}$. By (5.36), for any $F, G \in \mathcal{F}C_{\mathrm{b}}^{\infty}(C_0^{\infty}(\mathbb{R}^d), \Gamma)$,

$$\begin{aligned}\mathcal{E}_0(F, G) &= c \int_{\Omega\times\Gamma} \tilde{\mathbb{P}}(d\omega, d\gamma) \int_{\mathbb{R}^d} dx\, \tilde{\mathbb{E}}(Y(x, \omega)^2 \mid \mathcal{F})\langle \nabla_x F(\gamma \cup x), \nabla_x G(\gamma \cup x)\rangle \\ &= \int_{\Omega\times\Gamma} \tilde{\mathbb{P}}(d\omega, d\gamma) \int_{\mathbb{R}^d} dx\, Y(x, \omega)^2 \\ &\quad \times \langle \nabla_x (F(\gamma \cup x) - F(\gamma)), \nabla_x (G(\gamma \cup x) - G(\gamma))\rangle. \end{aligned} \tag{5.46}$$

Fix $(\omega, \gamma) \in \Omega \times \Gamma$. Denote

$$f(x) := F(\gamma \cup x) - F(\gamma), \quad g(x) := G(\gamma \cup x) - G(\gamma).$$

Clearly, $f, g \in C_0^{\infty}(\mathbb{R}^d)$. In view of Theorem 5.5, $Y(x, \omega)^2$ is a continuous function of $x \in \mathbb{R}^d$. Hence, by [11, Theorem 6.2], the quadratic form

$$\mathcal{E}(f, g) := \int_{\mathbb{R}^d} \langle \nabla f(x), \nabla g(x)\rangle Y(x, \omega)^2\, dx, \quad f, g \in C_0^{\infty}(\mathbb{R}^d),$$

is closable on $L^2(\mathbb{R}^d, Y(x,\omega)^2\,dx)$. Now the closability of $(\mathcal{E}_0, \mathcal{F}C_{\mathrm{b}}^{\infty}(C_0^{\infty}(\mathbb{R}^d), \Gamma))$ on $L^2(\Gamma, \mu^{(l)})$ follows by a straightforward generalization of the proof of [11, Theorem 6.3].

Due to [30], Part (ii) of the theorem can be shown completely analogously to [28], see also [34, 43]. □

Remark 5.8. Heuristically, the generator of $(\mathcal{E}_0, D(\mathcal{E}_0))$ has the form

$$(LF)(\gamma) = \sum_{x\in\gamma}\left(\Delta_x F(\gamma) + \left\langle \frac{\nabla_x r(x, \gamma\setminus x)}{r(x, \gamma\setminus x)}, \nabla_x F(\gamma)\right\rangle\right).$$

Here, for $x \in \gamma$, $\nabla_x F(\gamma) := \nabla_y F(\gamma\setminus x \cup y)\big|_{y=x}$ and analogously Δ_x is defined. However, we should not expect that $r(x,\gamma)$ is differentiable in x.

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