Wiener-Hopf factorization via indefinite inner products.

Andrews, Shaun Leigh

Wiener-Hopf Factorization via Indefinite Inner Products

Thesis submitted to the University of Wales Swansea
by

Shaun Leigh Andrews

in candidature for the degree of
Doctor of Philosophy

Department of Mathematics
University of Wales Swansea
Singleton Park
Swansea SA2 8PP
United Kingdom

Winter 2003

Declaration

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

Signed .. .. (candidate)

Date ....5/12/03.............

Statement 1

This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by explicit references. A detailed bibliography is appended.

Signed ... (candidate)

Date ....5/12/03.............

Statement 2

I herby give consent for my thesis, if accepted, to be available for photocopying and for inter-library loan, and for the title and summary to be made available to outside organisations.

Signed ... (candidate)

Date ....5/12/03.............

Abstract

Most of this thesis is concerned with a seemingly simple example, referred to as the two-boundary problem. The problem illustrates (a) that the use of indefinite inner products can illuminate Probabilistic Wiener-Hopf Theory in symmetrizable cases, and (b) that half-winding probabilities should be thought of as branching measures for Ray processes.

The use of the indefinite inner product provides us with an efficient way to tackle the traditionally difficult issue that is duality. A fully rigorous study of time reversal is always a problem for Probability Theory. The analytic approach reveals some results of considerable independent interest.

The thesis is structured as follows:

- Chapter 1 explains the inspiration for the given thesis structure, together with some further important information.
- The two-boundary problem is examined in Chapter 2. We begin with the necessary analysis and then confirm everything with the corresponding probability. The nature in which everything tallies is amazing, chiefly due to the crucial theorem that shows the equivalence of PDE and local martingale properties. Moreover, the way in which the analysis effortlessly provides us with the desired duality results cannot be underestimated.
- In Chapter 3 we look at a one-boundary problem with a drift component. The importance of the duality arguments in Chapter 2 is emphasized. In addition, unlike in the two-boundary problem, continuity of one of the underlying semigroups poses a rather serious problem. This provides motivation for part of Chapter 4. Of particular interest here is a certain mystifying 'independence of drift' result.
- Chapter 4 is concerned with (unorthodox) non-minimal, non-negative solutions of the Riccati equations for both the drift and two-boundary problems. The 'continuity' difficulty mentioned above is resolved in some generality.

Detailed appendices are also included.

Acknowledgements

I would like to express my sincere gratitude to Professor David Williams and Professor Aubrey Truman for suggesting the areas of research and, despite some testing times, their enthusiasm and constant support throughout the preparation of this thesis. In addition, I extend my regards to Professor Daniel Stroock of the Massachusetts Institute of Technology for his generous support concerning continuity properties of functions in Chapters 3 and 4.

Further grateful thanks go to Professor Niels Jacob for many helpful comments. I must also thank Ken Morgan and Nigel Weatherill for inviting David Williams to speak at a conference in honour of Olek Zienkiewicz, which provided motivation for Chapter 2 of this thesis.

Thanks to Brett Reynolds and Chris Reynolds for their patience in teaching me the basics of LaTeX and Mathematica. Also, Sarah Harris and Scott Reasons for whom there was no favour too big nor too small.

Last, but by no means least, I am grateful to the Engineering and Physical Sciences Research Council (EPSRC) for providing financial support during the first three years of my postgraduate studies.

University of Wales Swansea — Shaun Leigh Andrews
December 2003

Contents

Chapter 1

Introduction

Summary

This thesis uses a wide variety of techniques from Probability Theory, Functional Analysis and even Complex Analysis. It is therefore unsurprising that we will appeal to many crucial results from each discipline. However, to give even a brief account of some of the necessary theory would significantly lengthen the thesis and would almost certainly distract the reader from the main crux of each given problem. Consequently, this thesis has been specifically structured to account for this. This chapter is intended to explain the reason for the given structure as well as providing some important additional information.

1.1. Important Remarks

It must be emphasized that indefinite inner products are relevant to Probabilistic Wiener-Hopf Theory only in 'symmetrizable' cases; and by no means all interesting cases are symmetrizable. The kernels Π^{-+} and Π^{+-} and the semigroups $\{P_t^{\pm}\}$ introduced later all have simple probabilistic meanings *without* the assumption of symmetrizability.

It should be stressed that the relevance of indefinite inner products to 'symmetrizable' Wiener-Hopf problems has long been recognized by analysts, particularly by Krein and Gohberg and their schools. For modern references, see, for example, [5] and [8] and the review of the latter by H. Langer in *Mathematical Reviews MR2001m:47001*. It is also true, of course, that absolutely brilliant work has been carried out by applied mathematicians in the Complex Analysis of Wiener-Hopf Theory, with inspired choice of contours. Noble [19] is a fine introduction. A little of the Complex Analysis for our main problem is given towards the end of Chapter 2.

For introductions to various aspects of Probabilistic Wiener-Hopf Theory, see Bingham [3], Bertoin [2], Greenwood and Pitman [12] and Williams [26].

The beauty of using such a wide variety of techniques is that they serve to illuminate each other. We quite frequently see the underlying Analysis and Probability working in conjunction.

1.2. Thesis Structure

Here we give more particular details concerning the thesis structure. Due to the comments given in the above summary, we refrain from beginning with a detailed chapter containing the necessary prerequisite material. After all, this will be a matter of simply providing well-known results. For example results on; Brownian motion, Optional stopping, and various results from Stochastic Calculus. Indefinite inner products and PDEs feature largely throughout this thesis; yet fortunately, no prior knowledge of either of the theories is required. Desired results of particular significance are either deferred to the appropriate appendices or included at the point when they are needed.

In Chapter 2 we immediately begin with the main problem, namely the two-boundary problem. Appendices that are specific to this problem are given at the end of the chapter. However, the main appendix that covers material applicable to the whole thesis is traditionally, and in this case, included at the end of the thesis.

Equations which are particularly important elements of the structure are highlighted amongst the many other equations in this thesis.

1.3. Provenance and Prior Publication

Chapter 2 is based on joint work with David Williams, and Chapter 4 on joint work with Daniel Stroock and David Williams. All this work will be published. Because of the many developments which have occurred recently, we are considering the best form in which to publish the work. One possibility is that a version of Chapter 2 will be submitted to the *Electronic Journal of Probability*, a high-quality journal with the usual refereeing procedure, and that Chapter 4 will appear in a special volume of *Methods and Applications of Analysis* to be produced in tribute to Papanicolaou.

Chapter 2

Two-Boundary Problem

Summary

In this chapter we present an example which, though singular in some respects, seems to convey rather nicely something of the flavour of indefinite inner products in Wiener-Hopf Theory. As previously emphasized, the main theme of the chapter is that Probability Theory and Analysis are working 'hand in hand'. The advantage of this is that each of the subjects can be used to illuminate the other. Eventually, we are often able to cross-check that the subjects tally. Crucial is the fact that we are working with the simple compact space $[0, 1]$. However, it is not until the next chapter that the benefits of this fact are highlighted. Here, we are fortunately able to obtain a precise spectral expansion for one of our underlying semigroups. Amongst other things, this guarantees smoothness properties which cannot be easily established. Chapter 4 deals with such smoothness issues in a more general case.

2.1. The Operator $\mathcal{H}$ and Indefinite Inner Product $\langle \cdot, \cdot \rangle_s$

Notation: within the bounds of reason, we use y to denote a point of the open interval $(0, 1)$, x to denote a point of the boundary $\{0, 1\}$, and z for a point of the compact interval $[0, 1]$.

Throughout this chapter we let $m_0, m_1 \in (0, \infty)$.

2.1A. Definition. *We define the operator $\mathcal{H}$ with domain $\mathcal{D}(\mathcal{H})$ to consist of those $\mathbb{C}$-valued functions in $C^2[0, 1]$ which satisfy the ('reverse Feller') boundary conditions*

$$m_0 f''(0) + f'(0) = 0, \qquad m_1 f''(1) - f'(1) = 0, \tag{1.1}$$

and, for $f \in \mathcal{D}(\mathcal{H})$, define $\mathcal{H}f = \frac{1}{2}f''$.

One reason, based on a discrete approximation, for imposing the boundary conditions in (1.1) is given in Appendix D. Nowhere do we need to extend further the domain of $\mathcal{H}$. For $f \in C[0,1]$, we shall be interested in the equation

$$\partial_\varphi F + \mathcal{H}F = 0, \qquad ((\varphi, z) \in (-\infty, 0) \times [0,1]) \tag{1.2}$$

with final condition

$$F(0-, y) = f(y) \qquad (y \in (0,1)), \tag{1.3}$$

the final condition *not* being imposed at the boundary points 0 and 1. The solution F must belong to $C^{1,2}((-\infty,0) \times [0,1])$ and must satisfy $F(\varphi,\cdot) \in \mathcal{D}(\mathcal{H})$ for $\varphi < 0$.

2.1B. Definition. *For $\mathbb{C}$-valued $f, g \in C^2[0,1]$, and with $\overline{g}$ denoting the complex conjugate of g, define the 'indefinite inner product' $\langle\cdot,\cdot\rangle_s$ (subscript 's' for 'signed') via*

$$\begin{aligned}\langle f, g\rangle_s &= \int_{(0,1)} f(y)\overline{g}(y)\mathrm{d}y - m_0 f(0)\overline{g}(0) - m_1 f(1)\overline{g}(1) \\ &= \int_{[0,1]} f(y)\overline{g}(y)\nu(\mathrm{d}y),\end{aligned} \tag{1.4}$$

where ν is the signed measure Leb $- m$. *An element f of $C^2[0,1]$ will be called* $\langle$positive$\rangle_s$ *if* $\langle f, f\rangle_s > 0$, $\langle$negative$\rangle_s$ *if* $\langle f, f\rangle_s < 0$, $\langle$neutral$\rangle_s$ *if* $\langle f, f\rangle_s = 0$.

2.1C. Lemma. *$\mathcal{H}$ is symmetric relative to $\langle\cdot,\cdot\rangle_s$.*

Proof of Lemma 2.1C. Given Definitions 2.1A and 2.1B, following some elementary integration by parts we find that for $f, g \in \mathcal{D}(\mathcal{H})$,

$$\langle \mathcal{H}f, g\rangle_s = -\tfrac{1}{2}\int_{(0,1)} f'(y)(\overline{g})'(y)\mathrm{d}y = \langle f, \mathcal{H}g\rangle_s. \tag{1.5}$$

Note that the middle expression in (1.5) is minus the classical Dirichlet form. □

We first assume several **Working Hypotheses** and use these to discover the structure of things in some considerable detail. Elements of the discovered structure are then established independently, and used to prove that the Working Hypotheses are indeed true. It is the singular nature of our boundary conditions which makes this 'almost circular' argument appropriate here. It must be emphasized that the Working Hypotheses are essential; and in many other contexts, they may be established directly at the beginning of the story.

In stating our Working Hypotheses, we consider only real-valued functions. This reason for this will soon become clear. As we shall see later, the semigroups $\{P_t^{\pm}\}$ are Ray semigroups.

Working Hypothesis WH1. *For $f \in C[0,1]$ with $f \geq 0$, there exists a* minimal *non-negative solution F of equation* (1.2) *with final condition* (1.3) *in that any other such solution $\tilde{F}$ satisfies $\tilde{F}(\varphi,z) \geq F(\varphi,z)$ for all $(\varphi,z) \in (-\infty,0] \times [0,1]$. Define $(P_t^+ f)(z) = F(-t,z)$ for $t > 0$ and extend P_t^+ (as we may) to $C[0,1]$ by linearity. Then $\{P_t^+ : t > 0\}$ defines a one-parameter semigroup of non-negative operators on $C[0,1]$, so $P_{s+t}^+ = P_s^+ P_t^+$. We will have $P_t^+ \mathbf{1} \leq \mathbf{1}$, where $\mathbf{1}$ is the constant function equal to 1 on $[0,1]$. For $f \in C[0,1]$ and $z \in [0,1]$, the limit*

$$\left(P_0^+ f\right)(z) := \lim_{t\downarrow 0}(P_t^+ f)(z)$$

exists and

$$\begin{aligned}\left(P_0^+ f\right)(y) &= f(y) \quad (y \in (0,1)),\\ \left(P_0^+ f\right)(x) &= \int_{(0,1)} \Pi^{-+}(x,\mathrm{d}y) f(y) \quad (x \in \{0,1\}),\end{aligned}$$

where $\Pi^{-+}(x,\cdot)$ is a measure of total mass at most 1 on (Borel subsets of) the open *interval $(0,1)$. Note that P_0^+ does* not *map $C[0,1]$ into $C[0,1]$. We have $P_0^+ P_t^+ = P_t^+ P_0^+ = P_t^+$.*

2.1D. Important Example. If $f \equiv 1$, then $F \equiv 1$ satisfies the PDE subject to the appropriate conditions, but if $m_0 + m_1 > 1$ it not does arise from a non-negative *semigroup* solution. Furthermore, we shall see that it is the wrong (that is, non-minimal) solution in that case. In fact, the correct (minimal) solution has $F < 1$ on $(-\infty,0) \times [0,1]$, and $F(0-,x) = (P_0^+ f)(x) < 1$ for $x \in \{0,1\}$.

The second Working Hypothesis, much easier to prove for our example, is dual to the first. For $h \in C[0,1]$, we consider the PDE

$$\partial_\varphi H + \mathcal{H} H = 0, \quad \left((\varphi,z) \in (0,\infty) \times [0,1]\right) \tag{1.6}$$

with initial condition

$$H(0+,x) = h(x) \quad (x \in \{0,1\}). \tag{1.7}$$

Note that $H(0+,\cdot)$ *is only specified at the boundary points* $0, 1$.

Working Hypothesis WH2. *For $h \in C[0,1]$ with $h \geq 0$, there exists a* minimal *non-negative solution H of equation* (1.6) *with initial condition* (1.7) *in that any other such solution $\tilde{H}$ satisfies $\tilde{H}(\varphi, z) \geq H(\varphi, z)$ for all $(\varphi, z) \in (0,\infty) \times [0,1]$. Define $(P_t^- h)(z) = H(t,z)$ for $t > 0$ and extend P_t^- (as we may) to $C[0,1]$ by linearity. Then $\{P_t^- : t > 0\}$ defines a one-parameter semigroup of non-negative operators on $C[0,1]$, so $P_{s+t}^- = P_s^- P_t^-$. We have $P_t^- \mathbf{1} \leq \mathbf{1}$. For $h \in C[0,1]$ and $z \in [0,1]$, the limit*

$$\left(P_0^- h\right)(z) := \lim_{t \downarrow 0} (P_t^- h)(z)$$

exists and

$$\begin{aligned} \left(P_0^- h\right)(x) &= h(x) \quad (x \in \{0,1\}), \\ \left(P_0^- h\right)(y) &= \int_{\{0,1\}} \Pi^{+-}(y, \mathrm{d}x) h(x) \quad (y \in (0,1)). \end{aligned}$$

where $\Pi^{+-}(y, \cdot)$ is a measure of total mass at most 1 *on subsets of $\{0,1\}$. This time, P_0^-* does *map $C[0,1]$ into $C[0,1]$. We have $P_0^- P_t^- = P_t^- P_0^- = P_t^-$. For $h \in C[0,1]$, $(P_t^- h)(z)$ depends only on the values of h at the points* 0 *and* 1.

2.1E. Definition. *For $y \in (0,1)$ and $x \in \{0,1\}$ we let*

$$\pi(x,y) := m_x^{-1} \Pi^{+-}(y, \{x\}). \tag{1.8}$$

2.2. Duality

Our PDEs for F (with final value f in $C[0,1]$) and for H (with initial value h in $C[0,1]$) may be written

$$\partial_\varphi F + \mathcal{H} F = 0 \text{ on } (-\infty, 0) \times [0,1], \qquad \partial_\varphi H + \mathcal{H} H = 0 \text{ on } (0,\infty) \times [0,1]. \tag{2.1}$$

Define $G(\varphi, z) := F(-\varphi, z)$ on $(0,\infty) \times [0,1]$, so that

$$-\partial_\varphi G + \mathcal{H} G = 0 \text{ on } (0,\infty) \times [0,1]. \tag{2.2}$$

2.2A. Lemma. *Given the above definitions, we have*

$$\langle P_t^+ f, P_t^- h \rangle_s = \langle P_0^+ f, P_0^- h \rangle_s. \tag{2.3}$$

Proof of Lemma 2.2A. If we multiply the result for H in (2.1) by G, then multiply the corresponding result for G in (2.2) by H, and subtract the consequent equations, we get

$$G \partial_\varphi H + H \partial_\varphi G + G \mathcal{H} H - H \mathcal{H} G = 0.$$

Since $\partial_\varphi(GH) = G\partial_\varphi H + H\partial_\varphi G$, the previous result reduces to

$$\partial_\varphi(GH) + G\mathcal{H}H - H\mathcal{H}G = 0. \tag{2.4}$$

Let ν denote the signed measure $\mathrm{Leb}(0,1) - m$. For $t > \epsilon > 0$, integrate (2.4) over $(\epsilon, t] \times [0,1]$ with respect to $\mathrm{Leb}(\epsilon, t] \times \nu$, to get

$$\int_0^1 \int_\epsilon^t \partial_\varphi(GH)\, \mathrm{d}\varphi \mathrm{d}\nu + \int_\epsilon^t \int_0^1 G\mathcal{H}H\, \mathrm{d}\nu \mathrm{d}\varphi - \int_\epsilon^t \int_0^1 H\mathcal{H}G\, \mathrm{d}\nu \mathrm{d}\varphi = 0. \tag{2.5}$$

Next recall that $\mathcal{H}$ is symmetric relative to $\langle\cdot,\cdot\rangle_s$, so that

$$\langle G, \mathcal{H}H\rangle_s = \langle \mathcal{H}G, H\rangle_s. \tag{2.6}$$

From (2.6) it follows that

$$\int_\epsilon^t \int_0^1 G\mathcal{H}H\, \mathrm{d}\nu \mathrm{d}\varphi - \int_\epsilon^t \int_0^1 H\mathcal{H}G\, \mathrm{d}\nu \mathrm{d}\varphi = \int_\epsilon^t \langle G, \mathcal{H}H\rangle_s\, \mathrm{d}\varphi \int_\epsilon^t \langle \mathcal{H}G, H\rangle_s\, \mathrm{d}\varphi = 0. \tag{2.7}$$

Combining (2.7) with (2.5), things reduce to

$$\begin{aligned}
& \int_0^1 \int_\epsilon^t \partial_\varphi(GH)\, \mathrm{d}\varphi \mathrm{d}\nu = 0 \quad \Leftrightarrow \quad \int_0^1 GH\big|_\epsilon^t\, \mathrm{d}\nu = 0 \\
\Leftrightarrow \quad & \int_0^1 G(t,\cdot)H(t,\cdot)\, \mathrm{d}\nu - \int_0^1 G(\epsilon,\cdot)H(\epsilon,\cdot)\, \mathrm{d}\nu = 0 \\
\Leftrightarrow \quad & \langle G(t,\cdot), H(t,\cdot)\rangle_s = \langle G(\epsilon,\cdot), H(\epsilon,\cdot)\rangle_s.
\end{aligned}$$

Letting $\epsilon \downarrow 0$ in the previous result, we have

$$\langle G(t,\cdot), H(t,\cdot)\rangle_s = \langle G(0+,\cdot), H(0+,\cdot)\rangle_s. \tag{2.8}$$

From Working Hypothesis WH1 and our definition of G we have, for $t > 0$,

$$\begin{aligned}
(P_t^+ f)(\cdot) &= F(-t,\cdot) := G(t,\cdot), \\
(P_0^+ f)(\cdot) &= F(0-,\cdot) := G(0+,\cdot).
\end{aligned} \tag{2.9}$$

Similarly, from Working Hypothesis WH2, we have

$$\begin{aligned}
(P_t^- h)(\cdot) &= H(t,\cdot), \\
(P_0^- h)(\cdot) &= H(0+,\cdot).
\end{aligned} \tag{2.10}$$

Substitution of (2.9) and (2.10) into (2.8) gives the desired result. □

Working Hypothesis WH3. *As $t \to \infty$, the left-hand side of equation* (2.3) *tends to* 0.

2.2B. Remark. The '*minimal* positive' nature of $\{P_t^\pm\}$ is crucial in regard to Working Hypothesis WH3 in general situations. (Look ahead to Important Discussion 3.13D.)

The intuitive probabilistic reason for WH3 is explained at the end of Section 6. However, we shall later give an independent proof of the following result which clearly implies WH3.

2.2C. Corollary. [assuming the current Working Hypotheses]. *For* $f, h \in C[0,1]$, *we have*

$$\langle P_0^+ f, P_0^- h\rangle_s \;=\; 0. \tag{2.11}$$

Hence, for $x \in \{0,1\}$, *we have*

$$\Pi^{-+}(x, \mathrm{d}y) \;=\; \pi(x,y)\mathrm{d}y \qquad \textit{on } (0,1)$$

in the Radon-Nikodým sense.

Proof of Corollary 2.2C. Equation (2.11) is clearly a direct consequence of Working Hypothesis WH3. Moreover, from the definition of $\langle\cdot,\cdot\rangle_s$ and assuming Working Hypotheses WH1 and WH2, (2.11) is equivalent to

$$\int_0^1 (P_0^+ f)(y)(P_0^- h)(y)\,\mathrm{d}y - m_0(P_0^+ f)(0)(P_0^- h)(0) - m_1(P_0^+ f)(1)(P_0^- h)(1) = 0$$

$$\begin{aligned} \Leftrightarrow \qquad \int_0^1 f(y)\int_{\{0,1\}} \Pi^{+-}(y,\mathrm{d}x)h(x)\,\mathrm{d}y - m_0\int_0^1 \Pi^{-+}(0,\mathrm{d}y)f(y)h(0) \\ - m_1\int_0^1 \Pi^{-+}(1,\mathrm{d}y)f(y)h(1) = 0. \end{aligned} \tag{2.12}$$

Substitution of (1.8) into (2.12) now yields

$$\begin{aligned} \int_0^1 f(y)\sum_{x\in\{0,1\}} m_x\pi(x,y)h(x)\,\mathrm{d}y - m_0\int_0^1 \Pi^{-+}(0,\mathrm{d}y)f(y)h(0) \\ - m_1\int_0^1 \Pi^{-+}(1,\mathrm{d}y)f(y)h(1) = 0 \end{aligned}$$

$$\begin{aligned} \Leftrightarrow \qquad m_0\int_0^1 \pi(0,y)f(y)h(0)\,\mathrm{d}y + m_1\int_0^1 \pi(1,y)f(y)h(1)\,\mathrm{d}y \\ - m_0\int_0^1 \Pi^{-+}(0,\mathrm{d}y)f(y)h(0) - m_1\int_0^1 \Pi^{-+}(1,\mathrm{d}y)f(y)h(1) = 0. \end{aligned}$$

Clearly the previous result implies that, for $x \in \{0,1\}$, we have

$$\Pi^{-+}(x,\mathrm{d}y) = \pi(x,y)\,\mathrm{d}y \qquad \text{for } y \in (0,1),$$

in the *Radon-Nikodým* sense. □

We now establish some essential additional duality results. These lead us to a conjecture on a 'new' inner product which contributes to the study of the Hilbert-space structure of $\{P_t^+\}$ examined in Section 11.

Suppose that $u, v \in C[0,1]$. Consider

$$\begin{aligned} \partial_\varphi U + \mathcal{H}U &= 0 \text{ on } (-\infty, 0) \times [0,1], \qquad U(0-, y) = u(y) \quad (y \in (0,1)), \\ \partial_\varphi V + \mathcal{H}V &= 0 \text{ on } (-\infty, 0) \times [0,1], \qquad V(0-, y) = v(y) \quad (y \in (0,1)). \end{aligned}$$

For fixed t such that $-t < \varphi < 0$, define

$$W(\varphi, z) := V(-t-\varphi, z) \text{ on } (-\infty, 0) \times [0,1],$$

so that $-\partial_\varphi W + \mathcal{H}W \;=\; 0.$

2.2D. Lemma. *Given the above definitions, we have the following result;*

$$\big\langle U(0-,\cdot),\, V(-t,\cdot)\big\rangle_s \;=\; \big\langle U(-t,\cdot),\, V(0-,\cdot)\big\rangle_s,$$

so that

$$\langle P_0^+ u, P_0^+ P_t^+ v\rangle_s \;=\; \langle P_0^+ P_t^+ u, P_0^+ v\rangle_s,$$

which is a particularly important duality result.

Proof of Lemma 2.2D. If we multiply the above PDE for W by $-U$, and multiply the corresponding result for U by W, then subtract the consequent equations, we get

$$\begin{aligned} & W\partial_\varphi U + U\partial_\varphi W + W\mathcal{H}U - U\mathcal{H}W = 0 \\ \Leftrightarrow\quad & \partial_\varphi(UW) + W\mathcal{H}U - U\mathcal{H}W = 0. \end{aligned}$$

Once again let ν denote the signed measure $\mathrm{Leb}(0,1) - m$. For $t > \epsilon > 0$, we integrate the previous result over $[-t+\epsilon, -\epsilon] \times [0,1]$ with respect to $\mathrm{Leb}[-t+\epsilon, -\epsilon] \times \nu$. However, due to the symmetry of $\mathcal{H}$ relative to $\langle\cdot,\cdot\rangle_s$, we have

$$\begin{aligned} & \int_0^1 \int_{-t+\epsilon}^{-\epsilon} \partial_\varphi(UW)\, \mathrm{d}\varphi \mathrm{d}\nu = 0 \qquad \Leftrightarrow \qquad \int_0^1 UW\big|_{-t+\epsilon}^{-\epsilon}\, \mathrm{d}\nu = 0 \\ \Leftrightarrow\quad & \int_0^1 U(-\epsilon,\cdot)W(-\epsilon,\cdot)\mathrm{d}\nu - \int_0^1 U(-t+\epsilon,\cdot)W(-t+\epsilon,\cdot)\mathrm{d}\nu = 0 \\ \Leftrightarrow\quad & \langle U(-\epsilon,\cdot), W(-\epsilon,\cdot)\rangle_s = \langle U(-t+\epsilon,\cdot), W(-t+\epsilon,\cdot)\rangle_s. \end{aligned}$$

Next note that

$$W(-\epsilon,\cdot) = V(-t+\epsilon,\cdot) \qquad W(-t+\epsilon,\cdot) = V(-t+t-\epsilon,\cdot) = V(-\epsilon,\cdot),$$

so that we now have

$$\langle U(-\epsilon,\cdot), V(-t+\epsilon,\cdot)\rangle_s = \langle U(-t+\epsilon,\cdot), V(-\epsilon,\cdot)\rangle_s.$$

Letting $\epsilon \downarrow 0$ yields

$$\langle U(0-,\cdot), V(-t,\cdot)\rangle_s = \langle U(-t,\cdot), V(0-,\cdot)\rangle_s. \tag{2.13}$$

Next recall the semigroup property, that is,

$$P^+_{s+t} = P^+_s P^+_t = P^+_t P^+_s \qquad (t, s \geq 0). \tag{2.14}$$

From the final conditions for both U and V given above, we have

$$U(0-, \cdot) = (P^+_0 u)(\cdot), \qquad V(0-, \cdot) = (P^+_0 v)(\cdot). \tag{2.15}$$

Additionally recall from Working Hypothesis WH1 that we define $(P^+_t f)(z) = F(-t, z)$ where F is the solution of the PDE in (1.2), with final condition (1.3). Hence, we make the same definitions for the functions U and V. From (2.14), it follows that

$$V(-t, \cdot) = V(-(t+0), \cdot) = (P^+_{0+t})(\cdot) = (P^+_0 P^+_t)(\cdot). \tag{2.16}$$

Similarly, we have

Similarly, we have

Subs

$$U(-t, \cdot) = (P^+_0 P^+_t)(\cdot). \tag{2.17}$$

Substituting (2.15), (2.16) and (2.17) into (2.13), we have

$$\langle P^+_0 u, P^+_0 P^+_t v\rangle_s = \langle P^+_0 P^+_t u, P^+_0 v\rangle_s,$$

as desired. □

2.2E. Remarks. The various duality results correspond (not surprisingly!) to time-reversal arguments in Probability Theory. See Kennedy [16] and Rogers [23] for uses of time-reversal in Wiener-Hopf Theory similar in type to that which we are considering. An alternative proof of the striking main result in Rogers [23] may be found in [27]. See Chapter 2 of Bertoin [2] for traditional duality. But it is much easier (if perhaps less meaningful?!) to use the analytic arguments described above. Time-reversal in Probability Theory is often plagued by technical difficulties. That Analysis leads so directly to the fact that

$$\langle P^+_t f, P^-_t h\rangle_s \;=\; 0 \qquad (t \geq 0,\; f, h \in C[0,1]) \tag{2.18}$$

is particularly striking.

2.2F. Lemma. *We have the following important contraction property;*

$$\langle P^+_0 P^+_t f, P^+_0 P^+_t f\rangle_s \;=\; \langle P^+_t f, P^+_t f\rangle_s \;\leq\; \langle P^+_0 f, P^+_0 f\rangle_s.$$

Proof of Lemma 2.2F. Recalling the definition of $\langle\cdot,\cdot\rangle_s$ in (1.4) we can deduce that

$$\begin{aligned}
\tfrac{1}{2}\partial_\varphi\big\langle F(\varphi,\cdot),\, F(\varphi,\cdot)\big\rangle_s \;&=\; \int_0^1 \tfrac{1}{2}\partial_\varphi\left\{F(\varphi,y)^2\right\}\mathrm{d}y - \tfrac{1}{2}m_0\partial_\varphi\left\{F(\varphi,0)^2\right\} - \tfrac{1}{2}m_1\partial_\varphi\left\{F(\varphi,1)^2\right\} \\
&= \int_0^1 F(\varphi,y)\partial_\varphi\left\{F(\varphi,y)\right\}\mathrm{d}y \\
&\qquad - m_0 F(\varphi,0)\partial_\varphi\left\{F(\varphi,0)\right\} - m_1 F(\varphi,0)\partial_\varphi\left\{F(\varphi,1)\right\}.
\end{aligned}$$

However, from our PDE for F, we have $\partial_\varphi F = -\mathcal{H}F$, so that

$$\begin{aligned}\tfrac{1}{2}\partial_\varphi\langle F(\varphi,\cdot),\, F(\varphi,\cdot)\rangle_s &= \int_0^1 F(\varphi,y)\partial_\varphi\{F(\varphi,y)\}\,\mathrm{d}y \\ &\qquad - m_0F(\varphi,0)\partial_\varphi\{F(\varphi,0)\} - m_1F(\varphi,0)\partial_\varphi\{F(\varphi,1)\} \\ &= -\langle F(\varphi,\cdot),\, \mathcal{H}F(\varphi,\cdot)\rangle_s.\end{aligned}$$

Since $\langle F(\varphi,\cdot),\, \mathcal{H}F(\varphi,\cdot)\rangle_s \le 0$ by (1.5), we have

$$\tfrac{1}{2}\partial_\varphi\langle F(\varphi,\cdot),\, F(\varphi,\cdot)\rangle_s = -\langle F(\varphi,\cdot),\, \mathcal{H}F(\varphi,\cdot)\rangle_s \ge 0. \tag{2.19}$$

For $t > \epsilon > 0$, integrating the extreme LHS of (2.19) over $[-t,-\epsilon)$ w.r.t. Leb$[-t,-\epsilon)$ we have

$$\begin{aligned}&\int_{-t}^{-\epsilon} \tfrac{1}{2}\partial_\varphi\langle F(\varphi,\cdot),\, F(\varphi,\cdot)\rangle_s \mathrm{d}\varphi \ge 0 \qquad \Leftrightarrow \qquad \tfrac{1}{2}\langle F(\varphi,\cdot),\, F(\varphi,\cdot)\rangle_s\Big|_{-t}^{-\epsilon} \ge 0 \\ \Leftrightarrow \quad &\langle F(-\epsilon,\cdot),\, F(-\epsilon,\cdot)\rangle_s - \langle F(-t,\cdot),\, F(-t,\cdot)\rangle_s \ge 0.\end{aligned}$$

Letting $\epsilon \downarrow 0$ in the previous result yields

$$\langle F(0-,\cdot),\, F(0-,\cdot)\rangle_s \ge \langle F(-t,\cdot),\, F(-t,\cdot)\rangle_s. \tag{2.20}$$

From the semigroup property and the definition of F in WH1, we see that

$$F(-t,\cdot) = (P_t^+ f)(\cdot) = F(-(0+t),\cdot) = (P_{0+t}^+ f)(\cdot) = (P_0^+ P_t^+ f)(\cdot).$$

Combining the previous result with (2.20) yields the desired result. □

2.2G. Temporary Conjecture. Lemma 2.2D and Lemma 2.2F make us hope that, at least for suitable m_0, m_1,

$$\langle u,\, v\rangle_+ \;:=\; \langle P_0^+ u, P_0^+ v\rangle_s$$

defines a proper inner product on $L^2[0,1]$ and that, relative to this inner product, $\{P_t^+\}$ is a semigroup of self-adjoint operators of norm at most 1. We shall see that this is true precisely when we have the unbalanced situation when $m_0 + m_1 \neq 1$.

2.3. Eigenfunctions of $\mathcal{H}$

The following example indicates that we must not jump too quickly to conclusions based on our knowledge of the symmetric case for a proper inner product.

2.3A. Instructive example. For column vectors x, y in $\mathbb{C}^2$, define

$$[x,y] := (x_1\, x_2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix}, \qquad Ax := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We will show that A is $[\cdot,\cdot]$-symmetric in that $[Ax, y] = [x, Ay]$, but that the eigenvalues of A are i and $-\mathrm{i}$ and the corresponding eigenvectors are not $[\cdot,\cdot]$-orthogonal.

Firstly note that

$$Ax = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}, \quad \text{and} \quad Ay = \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix}.$$

Thus,

$$[x, Ay] = (x_1\, x_2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \overline{y}_2 \\ -\overline{y}_1 \end{pmatrix} = (x_1\, x_2)\begin{pmatrix} \overline{y}_2 \\ -\overline{y}_1 \end{pmatrix} = x_1\overline{y}_2 + x_2\overline{y}_1.$$

Next,

$$[Ax, y] = (x_2\ -x_1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix} = (x_2\, x_1)\begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix} = x_1\overline{y}_2 + x_2\overline{y}_1.$$

It follows that $[Ax, y] = [x, Ay]$ and so A is $[\cdot,\cdot]$-symmetric.

Eigenvalues/eigenvectors of A. We need to solve

$$\det(\lambda I - A) = 0 \quad \Leftrightarrow \quad \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = 0 \quad \Leftrightarrow \quad \lambda^2 + 1 = 0$$
$$\Leftrightarrow \quad (\lambda + \mathrm{i})(\lambda - \mathrm{i}) = 0.$$

Therefore the eigenvalues of A are $\pm\mathrm{i}$.

Eigenvector corresponding to $+\mathrm{i}$. Consider the matrix

$$A - \mathrm{i}I = \begin{pmatrix} -\mathrm{i} & 1 \\ -1 & -\mathrm{i} \end{pmatrix}.$$

For $(a, b)^T \in \mathbb{C}^2$, we solve

$$\begin{pmatrix} -\mathrm{i} & 1 \\ -1 & -\mathrm{i} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad \begin{array}{l} -\mathrm{i}a + b = 0 \\ -a - \mathrm{i}b = 0 \end{array} \quad \Leftrightarrow \quad b - \mathrm{i}a = 0.$$

Thus the eigenvector corresponding to the eigenvalue $+\mathrm{i}$ is of the form

$$\begin{pmatrix} a \\ \mathrm{i}a \end{pmatrix} \qquad (a \in \mathbb{C},\ a \neq 0), \quad \text{so we take} \quad \underline{u} = \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix},$$

as eigenvectors are undetermined with respect to a scalar multiplier. By considering the matrix $A + \mathrm{i}I$, a similar argument shows that the eigenvector corresponding to the eigenvalue $-\mathrm{i}$ is of the form

$$\underline{v} = \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}.$$

Next consider

$$\begin{aligned}[\underline{u}, \underline{v}] &= (1\ \mathrm{i}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ \overline{(-\mathrm{i})} \end{pmatrix} = (1\ -\mathrm{i}) \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \\ &= 1 - \mathrm{i}^2 = 2 \neq 0.\end{aligned}$$

It follows that the eigenvalues are not orthogonal. □

2.3B. Lemma. *For $\rho \in \mathbb{C}$, the number $\frac{1}{2}\rho^2$ is an eigenvalue of $\mathcal{H}$, and then with corresponding eigenfunction (normalized to be 1 at 0)*

$$y \ \mapsto \ \cosh \rho y - m_0 \rho \sinh \rho y \quad \textit{on } [0, 1],$$

if and only if

$$\epsilon(\rho) := (1 + m_0 m_1 \rho^2) \sinh \rho - (m_0 + m_1) \rho \cosh \rho \ = \ 0. \tag{3.1}$$

Proof of Lemma 2.3B. The result trivially follows from solving $\mathcal{H}h \ = \ \frac{1}{2}\rho^2 h$ subject to the condition that $h \in \mathcal{D}(\mathcal{H})$. □

We shall see later that it is best to think of $\frac{1}{2}\rho\epsilon(\rho) = 0$ as the 'characteristic equation' giving the eigenvalues $\frac{1}{2}\rho^2$ of $\mathcal{H}$. From one point of view, finding the eigenvalues of $\mathcal{H}$ is just a case of solving equation (3.1). (Look ahead to formulae (15.10) and (15.11).)

2.3C. Proposition. *The function $f \in \mathcal{D}(\mathcal{H})$ is not a constant function if and only if $\langle \mathcal{H}f, f\rangle_s = \langle f, \mathcal{H}f\rangle_s < 0$. f is a constant if and only if $\langle \mathcal{H}f, f\rangle_s = 0$.*

Proof of Proposition 2.3C. The result follows directly from (1.5). □

2.3D. Lemma. *All eigenvalues of $\mathcal{H}$ are real.*

Proof of Lemma 2.3D. Modifying a familiar elementary argument, we see that if λ is an eigenvalue of $\mathcal{H}$ with eigenfunction f, then

$$\lambda\langle f, f\rangle_s \ = \ \langle \lambda f, f\rangle_s \ = \ \langle \mathcal{H}f, f\rangle_s \ = \ \langle f, \mathcal{H}f\rangle_s \ = \ \langle f, \lambda f\rangle_s \ = \ \overline{\lambda}\langle f, f\rangle_s, \tag{3.2}$$

so that if $\langle \mathcal{H}f, f\rangle_s \ \neq \ 0$, then λ is real. In addition, we see from Proposition 2.3C that $\langle \mathcal{H}f, f\rangle_s \ = \ 0$ if and only if f is a constant function, in which case f is an eigenfunction corresponding to eigenvalue 0. □

We now need only consider $\mathbb{R}$-valued eigenfunctions. Indeed, from now on,

$C[0, 1]$, $C^2[0, 1]$, *etc, will denote the spaces of $\mathbb{R}$-valued functions.*

2.3E. Remark. Equation (3.2) makes it unsurprising that (as Instructive Example 2.3A demonstrates) the general symmetric operator relative to an indefinite inner product can have non-real eigenvalues, the associated eigenfunctions being $\langle\text{neutral}\rangle_s$.

2.3F. Proposition. *A non-zero eigenfunction corresponding to a negative [respectively, positive] eigenvalue is $\langle$positive$\rangle_s$ [respectively, $\langle$negative$\rangle_s$]. The eigenfunction* $\mathbf{1}$ *is $\langle$positive$\rangle_s$ if $m_0 + m_1 < 1$, $\langle$negative$\rangle_s$ if $m_0 + m_1 > 1$, $\langle$neutral$\rangle_s$ if $m_0 + m_1 = 1$.*

Proof of Proposition 2.3F. Since all eigenvalues of $\mathcal{H}$ are real, suppose that $-\alpha$ ($\alpha \in \mathbb{R}^+$) is an eigenvalue of $\mathcal{H}$. Then the corresponding eigenfunction, f say, will satisfy $\mathcal{H}f = -\alpha f$. Using this fact together with Proposition 2.3C, we have

$$-\alpha\langle f, f\rangle_s = \langle -\alpha f, f\rangle_s = \langle \mathcal{H}f, f\rangle_s < 0,$$

so that, in particular, we have $\langle f, f\rangle_s > 0$, i.e. f is $\langle$positive$\rangle_s$. A similar argument holds to show that the eigenvalue $+\alpha$ leads to the fact that the corresponding eigenfunction is $\langle$negative$\rangle_s$. The remaining points are obvious as we have $\langle 1, 1\rangle_s = 1 - (m_0 + m_1)$. □

The definitions given below of the sets Θ and Γ are motivated by Proposition 2.3F.

2.3G. Lemma. *Eigenfunctions corresponding to distinct eigenvalues of $\mathcal{H}$ are $\langle\cdot,\cdot\rangle_s$-orthogonal.*

Proof of Lemma 2.3G. From Lemma 2.3D, recall that all eigenvalues of $\mathcal{H}$ are real. We therefore suppose that $\lambda_1, \lambda_2 \in \mathbb{R}$ are distinct eigenvalues corresponding to the eigenfunctions f and g respectively. That is, f and g satisfy

$$\mathcal{H}f = \lambda_1 f \quad \text{and} \quad \mathcal{H}g = \lambda_2 g, \quad \text{together with } \lambda_1 \neq \lambda_2. \tag{3.3}$$

Since $\mathcal{H}$ is symmetric relative to $\langle\cdot,\cdot\rangle_s$, we may use (3.3) to deduce the following results

$$\lambda_1\langle f, g\rangle_s = \langle f, \mathcal{H}g\rangle_s, \tag{3.4}$$
$$\lambda_2\langle f, g\rangle_s = \langle f, \mathcal{H}g\rangle_s. \tag{3.5}$$

We now simply subtract equation (3.4) from (3.5) to get $(\lambda_2 - \lambda_1)\langle f, g\rangle_s = 0$. However, $\lambda_1 \neq \lambda_2$ so that we must have $\langle f, g\rangle_s = 0$, so that f and g are $\langle\cdot,\cdot\rangle_s$-orthogonal as desired. Note that the argument still holds if one of the λ's is zero. □

2.3H. Definition (The set Θ). *Let Θ_+ be the infinite set of strictly positive solutions θ of*

$$\tan\theta = \frac{(m_0 + m_1)\theta}{1 - m_0 m_1 \theta^2}. \tag{3.6}$$

(We allow each side to be infinite if it should happen that there exists a θ such that $\cos\theta = 0$ *and* $1 - m_0 m_1\theta^2 = 0$.*) Define*

$$\Theta := \begin{cases} \Theta_+ & \text{if } m_0 + m_1 > 1, \\ \Theta_+ \cup \{0\} & \text{if } m_0 + m_1 \leq 1. \end{cases} \tag{3.7}$$

2.3I. Lemma. *For $\theta \in \Theta$, the number $-\frac{1}{2}\theta^2$ is an eigenvalue of $\mathcal{H}$ with associated eigenfunction*

$$f_\theta(y) \;=\; \cos\theta y + m_0\theta\sin\theta y. \tag{3.8}$$

Proof of Lemma 2.3I. As usual we solve $\mathcal{H}f_\theta \;=\; -\frac{1}{2}\theta^2 f_\theta$ subject to the condition that $f_\theta \in \mathcal{D}(\mathcal{H})$. We see that the fact that $\theta \in \Theta$ is a necessary and sufficient condition for f_θ to be in $\mathcal{D}(\mathcal{H})$ and so be an eigenfunction of $\mathcal{H}$. □

2.3J. Proposition. *Every such eigenfunction f_θ ($\theta \in \Theta$) is $\langle$positive$\rangle_s$, except for the case when $m_0 + m_1 = 1$ in which f_0 is $\langle$neutral$\rangle_s$.*

Proof of Proposition 2.3J. The proof follows from Proposition 2.3F. □

2.3K. Proposition. *If $\Theta_+ = \{\theta_1, \theta_2, \ldots\}$ where $\theta_k < \theta_{k+1}$, then $n^{-1}\theta_n \to \pi$.*

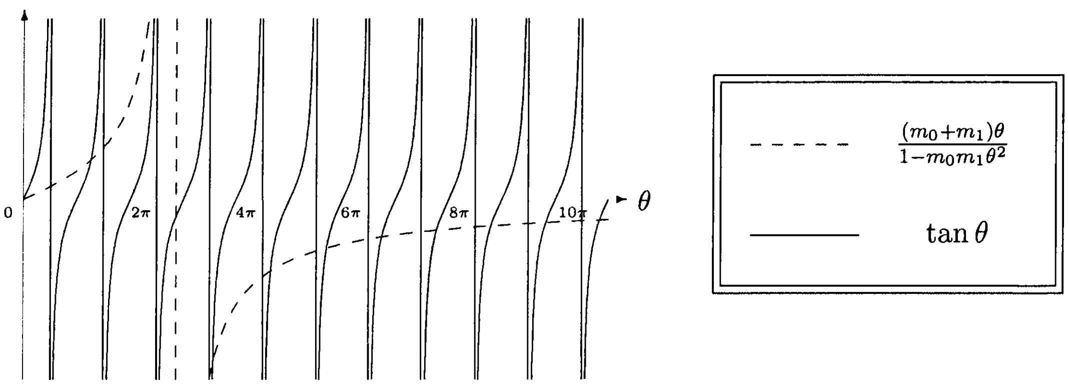

Figure 2.1: An example of the set θ_+.

Proof of Proposition 2.3K. We appeal to elementary calculus. Now, for all $n \geq 1$, we know that

$$\tan(\theta) \leq 0, \qquad \text{for} \quad \theta \in \left(\frac{\pi(2n-1)}{2}, n\pi\right),$$

together with

$$\lim_{2\theta \downarrow \pi(2n-1)} \tan(\theta) = -\infty \qquad \text{and} \qquad \tan(n\pi) = 0.$$

Moreover, $\tan(\theta)$ is certainly monotone increasing on the given interval. Next define

$$f(\theta) := \frac{(m_0+m_1)\theta}{1 - m_0m_1\theta^2}.$$

Then

$$\lim_{\theta \downarrow c} f(\theta) = -\infty, \qquad \text{where } c = 1/\sqrt{m_0m_1}.$$

In addition, we have

$$f(\theta) < 0,\ f'(\theta) > 0,\ \text{and } f''(\theta) < 0, \quad \text{for all } \theta > c.$$

Finally observe that $f(\theta) \uparrow 0$ as $\theta \to \infty$. Let θ_n be the n^{th} root of Θ_+. Putting the pieces together, it is now clear that, for n sufficiently large, we will have

$$\theta_n \in I := \left(\frac{\pi(2(n+k)-1)}{2}, (n+k)\pi\right), \qquad \text{where } k \in \{0,1\}.$$

Remark: Figure 2.1 relates to the case when $k=1$ as we must 'wait' until $n=3$ so that $\theta_3 \in I$ with $k=1$.

Now,

$$\lim_{n\to\infty} \frac{\pi}{2n}(2(n+k-1)) = \lim_{n\to\infty} \frac{\pi(n+k)}{n} = \pi.$$

Hence, $n^{-1}\theta_n \to \pi$. □

Working Hypothesis WH4. *If $m_0 + m_1 \geq 1$, there is precisely one strictly positive root of*

$$\tanh\gamma = \frac{(m_0+m_1)\gamma}{1+m_0m_1\gamma^2}, \tag{3.9}$$

Otherwise, if $m_0 + m_1 < 1$, then there are precisely two strictly positive roots.

Comments on WH4. Proving WH4 directly seems rather difficult. Ruling out the possibility of double roots is one of the many complications.

2.3L. Definition (The set Γ). [Assuming WH4] *If $m_0+m_1 < 1$, then we define the two strictly positive roots of* (3.9) *as α, β and hence define $\Gamma = \{\alpha, \beta\}$, with $\alpha < \beta$. If $m_0 + m_1 \geq 1$, then we define the only strictly positive root of* (3.9) *as β, and we then define $\alpha = 0$ and $\Gamma = \{\alpha, \beta\} = \{0, \beta\}$.*

Given Definition 2.3L, together with Lemmas 2.3D and 2.3B, we arrive at the following definition.

2.3M. Definition. *For $\gamma \in \{\alpha, \beta\}$, define*

$$h_\gamma(y) = \cosh\gamma y - m_0\gamma\sinh\gamma y, \tag{3.10}$$

the eigenfunction corresponding to $\frac{1}{2}\gamma^2$.

We now know that Lemma 2.3G implies

$$\langle h_\gamma, f_\theta\rangle_s = 0 \qquad (\theta \in \Theta, \gamma \in \Gamma). \tag{3.11}$$

2.3N. Proposition. *Each eigenfunction h_γ ($\gamma \in \Gamma$) is $\langle$negative$\rangle_s$, except for the case when $m_0 + m_1 = 1$ in which h_0 is $\langle$neutral$\rangle_s$.*

Proof of Proposition 2.3N. The result follows from Proposition 2.3F. □

2.3O. Lemma. *For all m_0, m_1, the function $h_\beta(z)$ is monotonic in z and $h_\beta(1) < 0$.*

Proof of Lemma 2.3O. If $\min(m_0^{-1}, m_1^{-1}) \le \rho \le \max(m_0^{-1}, m_1^{-1})$, then

$$\begin{aligned}\epsilon(\rho) &= (1 + m_0 m_1 \rho^2)\sinh\rho - (m_0 + m_1)\rho\cosh\rho \\ &< (\cosh\rho)(1 - m_0\rho)(1 - m_1\rho) \le 0,\end{aligned}$$

and $\rho < \beta$. In particular, we see that $m_0\beta > 1$. Hence

$$h'_\beta(z) = \beta\sinh\beta y - m_0\beta^2\cosh\beta y \le \beta(1 - m_0\beta)\cosh\beta y < 0,$$

so that $h_\beta(z)$ is monotonic in z. Using Definition 2.3L we may obtain an alternative form for $h_\beta(1)$, namely

$$\begin{aligned}m_1\beta h_\beta(1) &= \sinh(\beta) - m_0\beta\cosh(\beta) \\ &< \cosh(\beta)(1 - m_0\beta) < 0,\end{aligned}$$

so that $h_\beta(1) < 0$ as required. □

2.3P. Lemma. *We have $h_\alpha(1) \ne h_\beta(1)$.*

Proof of Lemma 2.3P. If $m_0 + m_1 \ge 1$, then $\alpha = 0$ so that $h_\alpha(1) = 1$. However, in the previous lemma we deduced that $h_\beta(1) < 0$. Thus, $h_\alpha(1) \ne h_\beta(1)$ as desired.

If $m_0 + m_1 < 1$, then $\alpha > 0$ so we proceed differently. For a contradiction, suppose that

$$h_\alpha(1) = h_\beta(1)\ (< 0). \tag{3.12}$$

Suppose further that $m_0 \ne m_1$. Looking forward to (9.1) of this chapter we have

$$1 = h_\alpha^\sharp(0)h_\alpha(1) = h_\beta^\sharp(0)h_\beta(1), \tag{3.13}$$

where $h_\gamma^\sharp(0) = \cosh(\gamma) - m_1\gamma\sinh(\gamma)$. Thus, this result together with our initial supposition in (3.12) imply that

$$h_\alpha^\sharp(0) = h_\beta^\sharp(0). \tag{3.14}$$

Multiplying (3.12) by m_1, (3.14) by m_0 and subtracting the consequent equations, we have

$$(m_0 - m_1)\cosh(\alpha) = (m_0 - m_1)\cosh(\beta) \quad \Leftrightarrow \quad \alpha = \beta \qquad \text{(since } m_0 \ne m_1.)$$

However, from Definition 2.3L we know that $\alpha \ne \beta$, so that we have the desired contraction.

Now suppose that $m_0 = m_1$. It is then clear that $h_\gamma^\sharp(0) = h_\gamma(1)$. From (3.13), we therefore have, for $\gamma \in \Gamma$,

$$h_\gamma(1)^2 = 1, \quad \text{so that} \quad h_\gamma(1) = \pm 1.$$

Conversely, if $\gamma \in (0, \infty)$ and $h_\gamma(1)^2 = 1$, then $\gamma \in \Gamma \cap (0, \infty)$. Consequently, the fact that $h_\gamma(1)^2 = 1$ $(\gamma \in (0, \infty))$ is a necessary and sufficient condition for $\gamma \in \Gamma \cap (0, \infty)$. We now consider $h_\rho(1)$ as a function of $\rho \geq 0$. Recall that $h_\rho(1) := \cosh(\rho) - m_0\rho\sinh(\rho)$. Then, we have the following limits;

$$\lim_{\rho\downarrow 0} h_\rho(1) = h_0(1) = 1, \quad \lim_{\rho\downarrow 0} h'_\rho(1) = h'_0(1) = 0,$$
$$\lim_{\rho\downarrow 0} h''_\rho(1) = h''_0(1) = 1 - 2m_0, \quad \lim_{\rho\to\infty} h_\rho(1) = -\infty. \tag{3.15}$$

If $m_0 + m_1 = 2m_0 < 1$, then $0 < \alpha < \beta$. Here, $1 - 2m_0 > 0$, so the limits in (3.15), together with elementary calculus, impose $h_\alpha(1) = +1$ and $h_\beta(1) = -1$.

Note that even without prior knowledge that $h_\beta(1) < 0$, we would have still been able to establish the desired result. This is because the argument used for $m_0 \neq m_1$ does not require restrictions for $m_0 + m_1$. In addition, for $m_0 + m_1 \geq 1$ and $m_0 = m_1$, we observe that $1 - 2m_0 \leq 0$ and $0 = \alpha < \beta$. Hence, the given limits in (3.15) and Definition 2.3L again force $h_\beta(1) = -1$. □

Remark: The fact that $h_\alpha(1) \neq h_\beta(1)$ is obvious from Probability Theory. See Lemma 2.8J (a).

2.3Q. Lemma. *In the 'perfectly-balanced' case when $m_0 + m_1 = 1$, the function k where*

$$k(z) = z^2 - 2m_0 z = \partial^2_\gamma h_\gamma(z)\big|_{\gamma=0}, \tag{3.16}$$

is a generalized eigenfunction of $\mathcal{H}$ corresponding to eigenvalue 0.

Proof of Lemma 2.3Q. Note that when $m_0 + m_1 = 1$, 0 is a repeated root of equation (3.1). (Look ahead at equation (15.11) and recall that $\frac{1}{2}\rho\epsilon(\rho)$ is the analogue of the characteristic polynomial for eigenvalue $\frac{1}{2}\rho^2$ of $\mathcal{H}$.) We simply solve $\mathcal{H}k = \mathbf{1}$, $\mathcal{H}^2 k = 0$ given that $k \in \mathcal{D}(\mathcal{H})$. □

Observe that our function k shares with the function $\mathbf{1}$ the property that

$$\langle k, h_\beta\rangle_s = 0 = \langle k, f_\theta\rangle_s \qquad (\theta \in \Theta_+). \tag{3.17}$$

2.3R. Proposition. *Suppose that for some constants A, B and some $\theta > 0$,*

$$f(y) = A\cos\theta y + B\sin\theta y.$$

Then $\langle f, h_\gamma\rangle_s = 0$ for all $\gamma \in \Gamma$ if and only f is a multiple of f_θ for some $\theta \in \Theta$.

Proof of Propostion 2.3R. ($\Leftarrow$) Suppose that $f = cf_\theta$ for some $\theta \in \Theta$ and $c \in \mathbb{R} \setminus \{0\}$. Then the desired result trivially follows from (3.11).

($\Rightarrow$) Given the form of f, it is enough to show that the orthogonality condition implies that

$$B - Am_0\theta = 0, \qquad \text{and} \qquad (1 - m_0 m_1\theta^2)\sin(\theta) - (m_0 + m_1)\theta\cos(\theta) = 0.$$

This is best checked on a computer package such *Mathematica* due to the awkward nature of the calculations. See Section 10 for an alternative proof of this result. □

Remark. It is interesting to compare (3.6) and (3.9) with the addition formulae for tan and tanh.

2.4. A Wiener-Hopf Equation

Let us briefly make another (very easily proved) Working Hypothesis.

Working Hypothesis WH5. *For* $\gamma \in \Gamma$, *we have* $P_0^- h_\gamma = h_\gamma$.

2.4A. Corollary. Given Working Hypothesis WH5 we have the following results;

$$\pi(0,y) := \frac{h_\beta(1)h_\alpha(y) - h_\alpha(1)h_\beta(y)}{m_0[h_\beta(1) - h_\alpha(1)]}, \qquad \pi(1,y) := \frac{h_\beta(y) - h_\alpha(y)}{m_1[h_\beta(1) - h_\alpha(1)]}. \tag{4.1}$$

Proof of Corollary 2.4A. Remembering that $h_\gamma(0) = 1$ for $\gamma \in \Gamma$, we simply solve two simultaneous linear equations obtained from WH5. □

In the special case when $m_0 = m_1 = \frac{1}{2}$, we have $\pi(0,y) = 1 + h_\beta(y)$, $\pi(1,y) = 1 - h_\beta(y)$ and $h_\beta(y) + h_\beta(1-y) = 0$.

Working Hypothesis WH5*. *For* $\theta \in \Theta$, *we have* $P_0^+ f_\theta = f_\theta$.

Working Hypothesis WH5* reformulates as the Wiener-Hopf equation for $x \in \{0,1\}$:

$$\int_{(0,1)} \Pi^{-+}(x, \mathrm{d}y) f_\theta(y) \;=\; f_\theta(x) \qquad \text{for every } \theta \in \Theta. \tag{4.2}$$

Proof of Working Hypothesis WH5*. Noting Corollary 2.4A, the desired result follows from Corollary 2.2C and equation (3.11). □

Since (4.2) follows immediately from Probability Theory, a natural question, of which analogues in other contexts have been studied via Complex Analysis, is the following.

2.4B. Question. *For* $x \in \{0,1\}$, *does equation* (4.2) *specify the measure* $\Pi^{-+}(x,\cdot)$ *on* $(0,1)$ *uniquely?*

We shall later prove that

$$\mathrm{cls}\{f_\theta : \theta \in \Theta\} \;=\; \{f \in C[0,1] : \langle h_\gamma, f\rangle_s = 0 \text{ for } \gamma \in \Gamma\}, \tag{4.3}$$

where 'cls' stands for closed linear span in $C[0,1]$. Note that this result implies that the closed linear span of $\{f_\theta : \theta \in \Theta\}$ in $L^2(0,1)$ is $L^2(0,1)$.

2.4C. Theorem. *Assume result (4.3). Let* $\Pi^{-+}(0,\cdot)$ *satisfy equation (4.2) with* $x=0$*, and let* δ_0 *be the unit mass at* 0*. Let* ℓ *be the bounded linear functional* $\delta_0 - \Pi^{-+}(0,\cdot)$ *acting on* $C[0,1]$*. Then, for* $f \in C[0,1]$*,*

$$\ell(f) = 0 \textit{ whenever both } \langle h_\alpha, f\rangle_s = 0 \textit{ and } \langle h_\beta, f\rangle_s = 0.$$

It follows that for some constants $c_{0\alpha}$ *and* $c_{0\beta}$ *we have*

$$\ell(f) \;=\; c_{0\alpha}\langle h_\alpha, f\rangle_s + c_{0\beta}\langle h_\beta, f\rangle_s \qquad (f \in C[0,1]).$$

Result (4.3) therefore gives a 'Yes' answer to Question 2.4B in that $\Pi^{-+}(x,\cdot)$ *must have density* $\pi(x,\cdot)$*.*

Proof of Theorem 2.4C. Assume result (4.3). It is then obvious that (4.2) holds for any function in the closed linear span (cls). Let $f \in$ cls, then from (4.2) with $x=0$, we have

$$\int_0^1 \Pi^{-+}(0,\mathrm{d}y)f(y) = f(0). \tag{4.4}$$

If we let ℓ be the bounded linear functional $\delta_0 - \Pi^{-+}(0,\cdot)$ acting on $C[0,1]$ then (4.4) is equivalent to

$$\ell(f) = 0.$$

However, $f \in$ cls if and only if $\langle h_\gamma, f\rangle_s = 0$, so that

$$\ell(f) = 0 \text{ whenever both } \langle h_\alpha, f\rangle_s = 0 \text{ and } \langle h_\beta, f\rangle_s = 0.$$

Next we follow a familiar argument from Dunford & Schwartz. Consider the map $\psi : C[0,1] \to \mathbb{R}^2$ defined by

$$\psi(f) := (\langle h_\alpha, f\rangle_s, \langle h_\beta, f\rangle_s).$$

Next, by a standard argument, we can define a linear map $L : \mathbb{R}^2 \to \mathbb{R}$, *uniquely* via

$$L(\psi(f)) = \ell(f). \tag{4.5}$$

Note that

$$\psi(f) = (0,0) \quad\Rightarrow\quad L(\psi(f)) = \ell(f) = 0 \quad\text{and}\quad \psi(f) = \psi(g) \quad\Rightarrow\quad \ell(f) = \ell(g),$$

where the latter result can be explained via linearity. It is well-known that a linear functional on $\mathbb{R}^2$ has the form $(v_\alpha, v_\beta) \mapsto c_{0\alpha}v_\alpha + c_{0\beta}v_\beta$ for some constants $c_{0\alpha}$ and $c_{0\beta}$. Hence using (4.5), in our case we have

$$\ell(f) = c_{0\alpha}\langle h_\alpha, f\rangle_s + c_{0\beta}\langle h_\beta, f\rangle_s \qquad (f \in C[0,1]),$$

for some constants $c_{0\alpha}$ and $c_{0\beta}$.

We can now find the density $\pi(x,\cdot)$. From the previous result, we have

$$\begin{aligned} f(0) - \int_0^1 \Pi^{-+}(0,\mathrm{d}y)f(y) &= c_{0\alpha}\langle h_\alpha, f\rangle_s + c_{0\beta}\langle h_\beta, f\rangle_s \\ &= c_{0\alpha}\int_0^1 h_\alpha(y)f(y)\mathrm{d}y + c_{0\beta}\int_0^1 h_\beta(y)f(y)\mathrm{d}y \\ &\quad - c_{0\alpha}m_0 h_\alpha(0)f(0) - c_{0\beta}m_0 h_\beta(0)f(0) \\ &\quad - c_{0\alpha}m_1 h_\alpha(1)f(1) - c_{0\beta}m_1 h_\beta(1)f(1). \end{aligned}$$

Note that $h_\gamma(0) = 1$, and that the above result is true *for all* $f \in C[0,1]$. In addition, both sides are signed measures on f and consequently the masses at zero must agree. Thus, taking $f \equiv 1$ on $[0,1]$, we have

$$\begin{aligned} 1 &= -c_{0\alpha}m_0 - c_{0\beta}m_0, \\ 0 &= c_{0\alpha}h_\alpha(1) + c_{0\beta}h_\beta(1). \end{aligned}$$

Multiplying the first equation by $h_\alpha(1)$, the second by m_0, and adding, we have

$$h_\alpha(1) = c_{0\beta}m_0[h_\beta(1) - h_\alpha(1)] \qquad \Leftrightarrow \qquad c_{0\beta} = \frac{h_\alpha(1)}{m_0[h_\beta(1) - h_\alpha(1)]}.$$

Similarly, we have

$$h_\beta(1) = c_{0\alpha}m_0[h_\alpha(1) - h_\beta(1)] \qquad \Leftrightarrow \qquad c_{0\alpha} = -\frac{h_\beta(1)}{m_0[h_\beta(1) - h_\alpha(1)]}.$$

We now have

$$\begin{aligned} \int_0^1 \pi(0,y)f(y)\mathrm{d}y &= \frac{h_\alpha(1)}{m_0[h_\beta(1) - h_\alpha(1)]}\int_0^1 h_\alpha(y)f(y)\mathrm{d}y \\ &\quad - \frac{h_\beta(1)}{m_0[h_\beta(1) - h_\alpha(1)]}\int_0^1 h_\beta(y)f(y)\mathrm{d}y \\ &= \int_0^1 \left\{\frac{h_\alpha(y)h_\beta(1) - h_\alpha(1)h_\beta(y)}{m_0[h_\beta(1) - h_\alpha(1)]}\right\} f(y)\mathrm{d}y, \end{aligned}$$

as expected. A similar argument holds for the density $\pi(1,\cdot)$ if we consider $x = 1$ in (4.2), together with the linear functional $\delta_1 - \Pi^{-+}(1,\cdot)$. □

2.5. The Processes Z and Φ

The necessary stochastic calculus may be found, for example, in Durrett [7], Karatzas & Schreve [15], Revuz & Yor [22], Rogers & Williams [24]. The last of these also contains the 'Markovian' results and results on resolvents which we need.

Let $Z = \{Z(t) : t \geq 0\}$ be Brownian motion on $[0,1]$ reflected at the boundary points. We therefore have

$$\mathrm{d}Z(t) \;=\; \mathrm{d}B(t) + \mathrm{d}L_0(t) - \mathrm{d}L_1(t),$$

for some Brownian motion B on $\mathbb{R}$ and continuous non-decreasing processes L_x ($x \in \{0,1\}$) with

$$\int_0^t I_{\{x\}}(Z(s))\mathrm{d}L_x(s) \;=\; L_x(t),$$

so that L_x grows only when Z is at x. The process L_x is called the local-time process at x.

The fluctuating additive functional Φ. We define Φ via the equation

$$\mathrm{d}\Phi(t) \;=\; \mathrm{d}t - 2m_0\mathrm{d}L_0(t) - 2m_1\mathrm{d}L_1(t).$$

For the moment, we concentrate on the situation when $\Phi(0) = 0$.

For $z \in [0,1]$, we write $\mathbb{P}^z$ for the law of the (Markov) process (Φ, Z) when $\Phi(0) = 0$ and $Z(0) = z$. As usual, $\mathbb{E}^z$ denotes the expectation associated with $\mathbb{P}^z$. A statement about Z will be said to hold almost surely (a.s.) if it has $\mathbb{P}^z$ probability 1 for every z.

2.5A. Theorem (Long-term behaviour of Φ). *We have the following situation:*

- *if $m_0 + m_1 < 1$, then (a.s.) $\Phi(t) \to +\infty$ as $t \to \infty$,*
- *if $m_0 + m_1 > 1$, then (a.s.) $\Phi(t) \to -\infty$ as $t \to \infty$.*

In addition, if $m_0 + m_1 = 1$, then (a.s.) Φ fluctuates infinitely in that

$$\limsup \Phi(t) = +\infty, \qquad \liminf \Phi(t) = -\infty. \tag{5.1}$$

Proof of Theorem 2.5A. It is well-known (see for example, Section 6.8 of Itô & McKean [14]) that (a.s.) $t^{-1}L_x(t) \to \frac{1}{2}$ for $x \in \{0,1\}$. This secures the former result when $m_0 + m_1 \neq 1$. We will prove the remaining result for $m_0 + m_1 = 1$ at the end of Section 7. □

Short-term behaviour of Φ. If $Z_0 = x \in \{0,1\}$, then initially $L_x(t)$ will grow faster than t so that there will be a (random) non-empty time-interval $(0,\delta)$ on which $\Phi < 0$. See the Instructive Example at the end of Appendix C.

2.6. The Processes Z^+ and Z^-

2.6A. Definition (The time-substitutions $\tau^{\pm}$). For $t \geq 0$, we define (with the strict '>' conditions being important)

$$\tau_t^+ := \inf\{u : \Phi(u) > t\}, \qquad \tau_t^- := \inf\{u : -\Phi(u) > t\},$$

with the usual convention that $\inf(\emptyset) = \infty$.

2.6B. Lemma. *The following results hold.*

(a) $\mathbb{P}^z(\tau_t^+ < \infty) = 1$ *if and only if either* $m_0 + m_1 \leq 1$ *or both* $t = 0$ *and* $z \in (0, 1)$.

(b) $\mathbb{P}^z(\tau_t^- < \infty) = 1$ *if and only if either* $m_0 + m_1 \geq 1$ *or both* $t = 0$ *and* $z \in \{0, 1\}$.

Proof of Lemma 2.6B. We simply appeal to Theorem 2.5A and Lemma 2.16B of Section 16. □

2.6C. Definition (The processes $Z^{\pm}$). For $t \geq 0$, we define

$$Z^+(t) := Z(\tau_t^+), \qquad Z^-(t) := Z(\tau_t^-),$$

with the usual convention that $Z^{\pm}(t) = \partial$ if $\tau_t^{\pm} = \infty$, where ∂ is a 'coffin state'.

2.6D. Hypothesis. *For the process Z^+, we have the following situation*

(a) if $m_0 + m_1 \leq 1$, then Z^+ is positive recurrent, and (a.s.) for any interval I, $\mathbb{P}(Z_t^+ \in I) \to \int_I \eta(y)\mathrm{d}y$ as $t \to \infty$, where η is the invariant density for Z^+;

(b) if $m_0 + m_1 > 1$, then (a.s.) Z^+ has finite lifetime.

Comments on Hypothesis 2.6D. The above result has been deliberately labelled an hypothesis as we are not in a position to prove it until the end of the chapter. Despite it being unnecessary, the result is given simply to inform the reader.

2.6E. Lemma. *Z^+ and Z^- are strong Markov processes.*

Proof of Lemma 2.6E. We firstly deduce that Z^+ has right continuous sample paths, that is,

$$\lim_{\delta \downarrow 0} Z_{t+\delta}^+ = Z_t^+.$$

Now $Z_{t+\delta}^+ = Z(\tau_{t+\delta}^+)$. However, Z is continuous so that Z^+ inherits its continuity properties from τ_t^+, in that,

$$\lim_{\delta \downarrow 0} Z_{t+\delta}^+ = Z\left(\lim_{\delta \downarrow 0} \tau_{t+\delta}^+\right) = Z(\tau_t^+) = Z_t^+.$$

Here Z is associated with the filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^z)$ satisfying the usual conditions.

We know that τ is right continuous. Since $\{\tau_s^+ < t\} = \{\Phi_t > s\}$, and $\{\mathcal{F}_t\}$ is right continuous, for each s, τ_s^+ is an $\{\mathcal{F}_t\}$ stopping time. From III.21 of Rogers & Williams [24], we need the fact that if t is an $\{\mathcal{F}_{\tau_t^+}\}$ stopping time, then τ_t^+ is an $\{\mathcal{F}_t\}$ stopping time. Furthermore, the main point is that

$$\tau_{t+u}^+ = \tau_t^+ + \tau_u^+ \circ \theta_{\tau_t^+}, \tag{6.1}$$

where $\theta_{\tau_t^+}$ 'shifts paths' through time τ_t^+ (see (A.5) of Appendix A). Now,

$$\begin{aligned}
\mathbb{E}^z\left[Z_{t+u}^+ | \mathcal{F}_{\tau_t^+}\right] &= \mathbb{E}^z\left[Z(\tau_{t+u}^+) | \mathcal{F}_{\tau_t^+}\right] \\
&= \mathbb{E}^z\left[Z(\tau_t^+ + \tau_u^+ \circ \theta_{\tau_t^+}) | \mathcal{F}_{\tau_t^+}\right] && \text{(from (6.1))} \\
&= \mathbb{E}^{Z(\tau_t^+)}\left[Z(\tau_u^+)\right] = \mathbb{E}^{Z_t^+}(Z_u^+) && \text{(by the SMP)}
\end{aligned}$$

which is exactly the required result. □

It is clear that Z^- is a Markov chain on $\{0,1\} \cup \{\partial\}$. Under $\mathbb{P}^x$ where $x \in \{0,1\}$, the value τ_0^+ will (a.s.) be strictly positive and Z_0^+ will belong to $(0,1)$. We see that Z^+ is therefore a process which behaves like Brownian motion inside $(0,1)$ but which, on approaching a point x of $\{0,1\}$, jumps into $(0,1)$ according to some measure $\Pi^{-+}(x, \cdot)$ (of total mass at most 1) on (Borel subsets of) $(0,1)$ (and jumps to ∂ with probability $1 - \Pi^{-+}(x, (0,1))$.

2.6F. Definition. The 'half-winding' probabilities are defined as follows:

$$\begin{aligned}
\Pi^{-+}(x, J) &:= \mathbb{P}^x(Z_0^+ \in J) && (x \in \{0,1\},\ J \in \mathcal{B}(0,1)), \\
\Pi^{+-}(y, J) &:= \mathbb{P}^y(Z_0^- \in J) && (y \in (0,1),\ J \subseteq \{0,1\}).
\end{aligned}$$

The 'half-winding' terminology is natural when one looks at the phase-space path $t \mapsto (\Phi_t, Z_t)$.

We also recall Definition 2.1E

$$\pi(x,y) := \Pi^{+-}(y, \{x\})/m_x \qquad (y \in (0,1),\ x \in \{0,1\}). \tag{6.2}$$

2.6G. Definition (The transition semigroups $P_t^\pm$). For $t \geq 0$, we now *define* the map $P_t^\pm$ on $C[0,1]$ via

$$\left(P_t^\pm f\right)(z) := \mathbb{E}^z\left(f(Z_t^\pm);\ \tau_t^\pm < \infty\right) \quad (f \in C[0,1],\ z \in [0,1]).$$

Of course, we shall need to prove that the $P^\pm$ semigroups satisfy Working Hypotheses WH1 and WH2.

Intuitive discussion of Working Hypothesis WH3. If $m_0 + m_1 \neq 1$, then one of the processes $Z^\pm$ will have finite lifetime, so the fact that for $f, h \in C[0,1]$, we have

$$\langle P_t^+ f, P_t^- h\rangle_s \to 0 \qquad (t \to \infty)$$

is highly plausible.

Suppose now that $m_0 + m_1 = 1$. Then each of the processes Z^+ and Z^- lives for ever. However, we then expect $(P_t^+ f)(z)$ to converge as $t \to \infty$ to a constant $c^+(f)$ independent of z (and likewise for $(P_t^- h)(z)$). The desired result will then follow from the fact that $\langle \mathbf{1}, \mathbf{1} \rangle_s = 0$ (when $m_0 + m_1 = 1$). Pitman (see, for example, Pitman [21]) taught us that the best way to prove that $(P_t^+ f)(z)$ converges as $t \to \infty$ to a constant $c^+(f)$ is by the probabilistic-coupling method. See Section 13 for a more detailed study of the probabilistic coupling for this case. For a concise account of the general probabilistic-coupling method and its wide usage, see Lindvall [17].

2.7. The Probabilistic Significance of the PDE for F

If $\Phi(0) = 0$ and $Z_0 \in (0,1)$ then $\tau_0^+ = 0$. Consequently, it is now convenient to let $\Phi(0)$ take an initial value $\varphi < 0$ and to let $\mathbb{P}^{\varphi,z}$ denote the law of (Φ, Z) for this new situation: it is the $\mathbb{P}^z$ law of $(\Phi + \varphi, Z)$. This is beneficial in part (a) of the following theorem. For part (b), we may make a similar remark for $Z_0 \in \{0,1\}$ and so consider $\varphi > 0$. It is clear that we are simply horizontally shifting the process. For $\varphi \leq 0$, our Definition 2.6G for P^+ can now be expressed as follows

$$(P_{-\varphi}^+ f)(z) = \mathbb{E}^z \left[f(Z_{-\varphi}^+); \tau_{-\varphi}^+ < \infty \right] = \mathbb{E}^{\varphi,z} \left[f(Z_0^+); \tau_0^+ < \infty \right]. \tag{7.1}$$

Hence, for $\varphi < 0$, it is clear that Lemma 2.6B (a) is equivalent to

$$\mathbb{P}^{\varphi,z}(\tau_0^+ < \infty) = 1 \text{ if and only if } m_0 + m_1 \leq 1. \tag{7.2}$$

Of course, we may state analogous results to (7.1) and (7.2) for P_φ^- and τ_0^- respectively.

2.7A. Theorem.

(a) *Suppose that* $F \in C^{1,2}\big((-\infty,0) \times [0,1]\big)$ *with continuous extension to* $\{0\} \times (0,1)$, *and define*

$$M_t := F\big(\Phi(t \wedge \tau_0^+), Z(t \wedge \tau_0^+)\big).$$

Then our PDE

$$\partial_\varphi F + \mathcal{H} F = 0$$

holds on $(-\infty, 0) \times [0,1]$ *if and only if*

M is a local martingale under each $\mathbb{P}^{\varphi,z}$ with $z \in [0,1]$, $\varphi < 0$.

(b) *Suppose that* $H \in C^{1,2}\big((0,\infty) \times [0,1]\big)$ *with continuous extension to* $\{0\} \times \{0,1\}$, *and define*

$$N_t := H\big(\Phi(t \wedge \tau_0^-), Z(t \wedge \tau_0^-)\big).$$

Then our PDE

$$\partial_\varphi H + \mathcal{H} H = 0$$

holds on $(0, \infty) \times [0,1]$ *if and only if*

N is a local martingale under each $\mathbb{P}^{\varphi,z}$ with $z \in [0,1]$, $\varphi > 0$.

2.7B. Reminder. When we say that our PDE for F holds, it is automatically inferred that the necessary $\mathcal{D}(\mathcal{H})$ condition is satisfied and the required final condition for F holds (as in (1.3)). However, the final condition will be verified separately due to the 'dual rôle' of the stochastic integral at 0. Again, a similar remark may be made about the PDE for H and its initial condition.

Proof of Theorem 2.7A (a). We shall write Φ_t for $\Phi(t)$, etc, when convenient. When $\Phi(0) < 0$, Itô's formula gives for $t < \tau_0^+$,

$$\begin{aligned} \mathrm{d}M_t &= (\partial_\varphi F)(\Phi_t, Z_t)\mathrm{d}\Phi + (\partial_z F)(\Phi_t, Z_t)\mathrm{d}Z + \tfrac{1}{2}(\partial_z^2 F)(\Phi_t, Z_t)\mathrm{d}t \\ &= (\partial_\varphi F + \tfrac{1}{2}\partial_z^2 F)\mathrm{d}t + (-2m_0\partial_\varphi F + \partial_z F)\mathrm{d}L_0(t) + (-2m_1\partial_\varphi F - \partial_z F)\mathrm{d}L_1(t) \\ &\quad + (\partial_z F)\mathrm{d}B. \end{aligned}$$

Hence M is a local martingale under $\mathbb{P}^{\varphi,z}$ if and only if we have $\mathbb{P}^{\varphi,z}$ probability 1,

$$\begin{aligned} (\partial_\varphi F + \tfrac{1}{2}\partial_z^2 F)(\Phi_t, Z_t) &= 0 \text{ for Lebesgue almost all } t < \tau_0^+, \\ (-2m_0\partial_\varphi F + \partial_z F)(\Phi_t, 0) &= 0 \text{ for '}\mathrm{d}L_0\text{' almost all } t < \tau_0^+, \\ (-2m_1\partial_\varphi F - \partial_z F)(\Phi_t, 1) &= 0 \text{ for '}\mathrm{d}L_1\text{' almost all } t < \tau_0^+. \end{aligned}$$

The 'only if' part of the theorem now follows immediately, since if F satisfies (1.2), so that in particular $F(\varphi, \cdot) \in \mathcal{D}(\mathcal{H})$ for $\varphi < 0$, then, for $\varphi < 0$, we have

$$(-2m_0\partial_\varphi F + \partial_z F)(\varphi, 0) = \left(m_0\partial_z^2 F + \partial_z F\right)(\varphi, 0) = 0.$$

For the 'if' part, we suppose now that F is $C^{1,2}$ as stated and that for *some* $(\varphi, z) \in (-\infty, 0) \times [0, 1]$, M is a $\mathbb{P}^{\varphi,z}$ local martingale. Then

$$\mathbb{P}^{\varphi,z}\left(\int_a^b (\partial_\varphi F + \tfrac{1}{2}\partial_z^2 F)(\Phi_t, Z_t)\mathrm{d}t = 0 \text{ whenever } 0 < a < b < \tau_0^+\right) = 1, \tag{7.3}$$

and for $x \in \{0, 1\}$,

$$\mathbb{P}^{\varphi,z}\left(\int_a^b (-2m_x\partial_\varphi F + s(x)\partial_z F)(\Phi_t, x)\, \mathrm{d}L_x(t) = 0 \text{ if } 0 < a < b < \tau_0^+\right) = 1, \tag{7.4}$$

where

$$s(x) = \begin{cases} +1 & \text{if } x = 0, \\ -1 & \text{if } x = 1. \end{cases}$$

We can now prove that the PDE for F holds. For a contradiction to (7.3), suppose that there exists some $(u, y) \in (-\infty, 0) \times (0, 1)$ such that

$$\partial_\varphi F(u, y) + \tfrac{1}{2}\partial_z^2 F(u, y) > 0.$$

(The modification with '< 0' replacing '> 0' is obvious.) Given that $F \in C^{1,2}$, there exists a neighbourhood N of (u, y) and an $\epsilon_0 > 0$ such that

$$\partial_\varphi F(u_0, y_0) + \tfrac{1}{2}\partial_z^2 F(u_0, y_0) > \epsilon_0, \quad \text{for all } (u_0, y_0) \in N.$$

From Probability Theory we can say:
1°.

$$\mathbb{P}^{\varphi,z}\left((\Phi_t, Z_t) \in N \text{ for some } t < \tau_0^+\right) > 0,$$

and hence (by continuity) there is a positive $\mathbb{P}^{\varphi,z}$ probability that there exist times c, d with $0 < c < d < \tau_0^+$ such that $(\Phi_t, Z_t) \in N$ for $c \le t \le d$, and so (also with positive probability)

$$\int_c^d (\partial_\varphi F + \tfrac{1}{2}\partial_z^2 F)(\Phi_t, Z_t)\, \mathrm{d}t > \epsilon_0(d - c) > 0.$$

This contradicts the local martingale property in (7.3).

For a contradiction to (7.4) this time, suppose that there exists some $u \in (-\infty, 0)$ such that

$$-2m_0\partial_\varphi F(u, 0) + \partial_z F(u, 0) > 0.$$

Continuity properties of F guarantee the existence of a relatively open neighbourhood N^* of $(u, 0)$ in $(-\infty, 0) \times [0, 1]$ such that

$$-2m_0\partial_\varphi F(u_0, x_0) + \partial_z F(u_0, x_0) > 0, \quad \text{for all } (u_0, x_0) \in N^*.$$

Here Probability Theory tells us that the following statement is true:
2°.

$$\mathbb{P}^{\varphi,z}\left(\int_0^{\tau_0^+} I_{N^*}(\Phi_t, Z_t)\mathrm{d}L_x(t) \;>\; 0\right) > 0,$$

so that, for some (ω-dependent) e, f with $0 \le e < f < \tau_0^+$, we have $(\Phi_t, Z_t) \in N^*$ for $e \le t \le f$ and $L_0(f) - L_0(e) > 0$, whence (with positive $\mathbb{P}^{\varphi,z}$ probability)

$$\int_e^f (-2m_0\partial_\varphi F + \partial_z F)(\Phi_t, Z_t)\, \mathrm{d}L_0(t) \;=\; \int_e^f (-2m_0\partial_\varphi F + \partial_z F)(\Phi_t, 0)\, \mathrm{d}L_0(t) \;>\; 0;$$

and this contradicts the local-martingale property in (7.4). A similar argument holds for the situation at 1.

Proof of results 1° and 2° are deferred to Appendix C, since some of the arguments presented there are useful elsewhere. □

Proof of Theorem 2.7A (b). The proof of the second part of the theorem follows similar arguments to that for part (a), but with some obvious 'minus' modifications. □

The following Corollary to Theorem 2.7A turns out to be crucial to the whole account.

2.7C. Corollary. *For $u \ge 0$, the following facts are true:*

for $\gamma \in \Gamma$, $t \mapsto \exp(-\frac{1}{2}\gamma^2\Phi_t)h_\gamma(Z_t)$ is a local martingale bounded on $[0, \tau_u^-]$,
for $\theta \in \Theta$, $t \mapsto \exp(+\frac{1}{2}\theta^2\Phi_t)f_\theta(Z_t)$ is a local martingale bounded on $[0, \tau_u^+]$.

Proof of Corollary 2.7C. Since both functions are $C^{1,2}$ on the appropriate space, the facts follow immediately from the 'only if' parts of Theorem 2.7A (a) and (b). □

The 'minimal non-negative' result. Suppose that $f \in C[0,1]$ and that a non-negative function $\tilde{F}$ satisfies the PDE for F with final value f on $(0,1)$. Then, under each $\mathbb{P}^{\varphi,z}$, $\tilde{M}$ (obvious notation!) is a non-negative local martingale, hence a supermartingale, and so

$$\tilde{F}(\varphi,z) = \mathbb{E}^{\varphi,z}\tilde{M}(0) \geq \mathbb{E}^{\varphi,z}\left[\tilde{M}(\tau_0^+);\tau_0^+ < \infty\right] = (P^+_{-\varphi}f)(z) = F(\varphi,z). \tag{7.5}$$

Moreover, if we *define* $F(\psi,w) := (P^+_{-\psi}f)(w)$ for $(\psi,w) \in (-\infty,0] \times [0,1]$, then, by the Strong Markov Theorem, for any $(\varphi,z) \in (-\infty,0) \times [0,1]$,

$$\mathbb{E}^{\varphi,z}\big(f(Z_0^+);\tau_0^+ < \infty \,\big|\, \mathcal{F}_t\big) = F\big(\Phi(t\wedge\tau_0^+), Z(t\wedge\tau_0^+)\big),$$

where $\{\mathcal{F}_t\}$ is the filtration determined by Z, so that the M corresponding to this F is a true martingale. If we knew that this present F, constructed from the P^+ semigroup, is $C^{1,2}$, then we could conclude that it is indeed the minimal non-negative solution of our PDE.

2.7D. Important discussion. The 'honest' way to proceed would be to prove directly that the transition semigroup of $Z^\pm$ does have the required $C^{1,2}$ properties. Then, via the 'if' part of Theorem 2.7A, we could deduce the duality results as indicated in Section 2. It is easy to establish the smoothness results for $\{P_t^-\}$, but rather more tricky to do so for $\{P_t^+\}$. (Probabilists often skip such details, and are then surprised that even for a nice 'Feller-minimal' Markov chain with all states stable, a transition probability p_{ij} need not be twice-differentiable.) For our problem, the main difficulty concerns boundary behaviour, since everything 'internal' is 'well-mollified'. To deal with the boundaries, we would need to look carefully at 'first-entrance last-exit' decompositions, well-known concepts for Markov chains and reflected to some extent in our treatment of the Kolmogorov forward equation in Section 15. There is a general theory of 'first-entrance last-exit' decompositions as part of Maissonneuve's theory of *incursions*. See Maisonneuve [18].

However, we shall follow a different route. Since we can easily find P_0^-, and hence $\pi(\cdot,\cdot)$, the duality idea allows us to *guess* the exact form of $\{P_t^+\}$. Recall that we believe that at least when $m_0 + m_1 \neq 1$, the P^+ semigroup is a strongly continuous semigroup of self-adjoint operators on the L^2 space associated with the $\langle\cdot,\cdot\rangle_+$ inner product, and is therefore likely to be wonderfully smooth. We can check that our conjectured version $\{\hat{P}_t^+\}$ of $\{P_t^+\}$ is self-adjoint relative to the appropriate inner product, and is correctly related to $\mathcal{H}$. The Optional-Stopping Theorem then allows us to deduce that $\{P_t^+\}$ does equal $\{\hat{P}_t^+\}$. The Hilbert-space structure of $\{P_t^+\}$ is of independent interest.

Bounded solutions. If $m_0 + m_1 \leq 1$, so that $\mathbb{P}^{\varphi,z}(\tau_0 < \infty) = 1$, and $f \in C[0,1]$ is given, then (assuming suitable smoothness on the part of $\{P_t^+\}$, $F(\varphi,z) = (P^+_{-\varphi}f)(z)$ is the unique bounded solution of (1.2) subject to condition (1.3).

Infinite fluctuation of Φ when $m_0 + m_1 = 1$. We skip some 'almost surely' qualifications here. Suppose that $m_0 + m_1 = 1$. We appeal to the concept of *quadratic variation* of a stochastic process.

2.7E. Corollary. *The process M is a local martingale, where $M_t := F\big(\Phi(t \wedge \tau_0^+), Z(t \wedge \tau_0^+)\big) = k(Z(t \wedge \tau_0^+)) - \Phi(t \wedge \tau_0^+)$, k being our generalized eigenfunction $z^2 - 2m_0 z$.*

Proof of Corollary 2.7E. Since F is $C^{1,2}$ on $(-\infty, 0) \times [0, 1]$ and the necessary PDE for F holds, the result is a consequence of the 'only if' part of Theorem 2.7A (a). □

2.7F. Lemma. *The quadratic variation of our local martingale M is given by*

$$\langle M \rangle_t = \int_0^t 4(Z_s - m_0)^2 \, \mathrm{d}s.$$

Therefore, if $\sigma_n := \inf\{t : L_0(t) > n\}$ for $n = 0, 1, 2, \ldots$, then the $\langle M \rangle_{\sigma_{n+1}} - \langle M \rangle_{\sigma_n}$ are independent, identically distributed positive random variables (hereafter referred to as IID's).

Proof of Lemma 2.7F. First, we know that each σ_n is a.s. finite by the ergodic result that $t^{-1} L_0(t) \to \frac{1}{2}$. Next, we know that

$$\mathrm{d}M_t = (\partial_z F)(\mathrm{d}B_t) \qquad \Leftrightarrow \qquad M = H \bullet B,$$

where $H_t = 2(Z_t - m_0)$ so that the notation conforms with that given in Rogers & Williams [24]. We now find an expression for the quadratic variation of M. By the rule

$$\langle C \bullet U, D \bullet V \rangle = (CD) \bullet \langle U, V \rangle,$$

we have

$$\begin{aligned} \langle M \rangle_t &= \langle M, M \rangle_t = \langle H \bullet B, H \bullet B \rangle_t \\ &= (H^2 \bullet \langle B, B \rangle_t) = \int_0^t H_s^2 \mathrm{d}s = \int_0^t 4(Z_s - m_0)^2 \mathrm{d}s. \end{aligned}$$

It is clear that $\sigma_{n+1} > \sigma_n$ for each n, so that $\langle M \rangle_{\sigma_{n+1}} > \langle M \rangle_{\sigma_n}$ and the fact that the random variables are positive is obvious. Here we are only concerned with the local time at zero. Hence, if we start in $(0, 1]$, then we have to wait until the process hits zero in order to accumulate local time there. However, the process is strong Markov, so the history until we hit level zero has no bearing. We may therefore start the process from zero without any significant loss of generality. Thus, strictly speaking, we are working in terms of the law $\mathbb{P}^0$. The benefit of this is that $\sigma_0 = 0$. Next suppose that θ_u is the familiar 'time-shift' map, shifting time through u. Then, by the strong Markov property and our particular form for $\langle M \rangle_{(\cdot)}$, we have for any bounded Borel function b on $[0, \infty)$,

$$\begin{aligned} \mathbb{E}\left[b(\langle M \rangle_{\sigma_{n+1}} - \langle M \rangle_{\sigma_n}) \middle| \mathcal{F}_{\sigma_n}\right] &= \mathbb{E}\left[b(\langle M \rangle_{\sigma_1}) \circ \theta_{\sigma_n} \middle| \mathcal{F}_{\sigma_n}\right] \\ &= \mathbb{E}^0\left[b(\langle M \rangle_{\sigma_1})\right], \qquad \text{for all } n \in \mathbb{N}. \end{aligned}$$

The fact that $\langle M \rangle_{\sigma_{n+1}} - \langle M \rangle_{\sigma_n}$, for $n \in \mathbb{N}$, are IID's is now clear, since M_{σ_k} is $\mathcal{F}_{\sigma_n}$ measurable for $k \leq n$. □

2.7G. Corollary. *We now have $\langle M \rangle_t \to \infty$ as $t \to \infty$ almost surely, so that Φ must fluctuate infinitely in the sense of* (5.1).

Proof of Corollary 2.7G. From the proof of Lemma 2.7F, we may legitimately restrict our attention to the case in which $Z_0 = 0$. For $n \in \mathbb{N}$, we begin by defining the IID's as follows

$$V_n := \langle M \rangle_{\sigma_{n+1}} - \langle M \rangle_{\sigma_n}.$$

Define

$$\mu = \mathbb{E}\left(e^{-V_n}\right) \in (0,1), \quad \text{for each } n \in \mathbb{N}. \tag{7.6}$$

Recall that $\sigma_0 = 0$. Thus, considering the obvious cancellation, we now have

$$V_0 + V_1 + \ldots + V_n = \langle M \rangle_{\sigma_{n+1}}. \tag{7.7}$$

For $K \in \mathbb{R}^+$, we find that

$$\begin{aligned}
\mathbb{P}\left[\langle M \rangle_{\sigma_{n+1}} < K\right] &= \mathbb{P}\left[V_0 + V_1 + \ldots + V_n < K\right] \\
&= \mathbb{P}\left[e^{-(V_0+V_1+\ldots+V_n)} > e^{-K}\right] \\
&\le e^K \mathbb{E}\left[e^{-(V_0+V_1+\ldots+V_n)}\right] && \text{(by the Markov inequality)} \\
&= e^K \prod_{i=0}^{n} \mathbb{E}\left[e^{-V_i}\right] = e^K \mu^n && \text{(by independence and (7.6))}
\end{aligned}$$

In particular, for $K \in \mathbb{R}^+$, we have

$$\mathbb{P}\left[\langle M \rangle_{\sigma_{n+1}} < K\right] \le e^K \mu^n, \qquad \text{for all } n \in \mathbb{N}, \tag{7.8}$$

so

$$\mathbb{P}\left(\bigcap \{\langle M \rangle_{\sigma_n} < K\}\right) = 0 \quad \text{for every } K,$$

and (a.s.) $\langle M \rangle_{\sigma_n} \to \infty$, whence, by monotonicity, $\langle M \rangle_t \to \infty$. Then, by Corollary IV.34.13 of Rogers & Williams [24], M fluctuates infinitely, and since $k(Z)$ is bounded, Φ (which is continuous) must fluctuate infinitely. We have therefore confirmed (5.1). □

2.8. $\{P_t^-\}$ and Positive Eigenvalues of $\mathcal{H}$

Without a proof of Working Hypothesis WH4, nothing about positive eigenvalues of $\mathcal{H}$ is yet proved. Probabilistic definitions of two numbers α, β with $0 \le \alpha < \beta$ will be given. It will then be proved that $\frac{1}{2}\alpha^2$ and $\frac{1}{2}\beta^2$ are indeed eigenvalues of $\mathcal{H}$ and that for no strictly positive γ with $\gamma \notin \{\alpha, \beta\}$ is $\frac{1}{2}\gamma^2$ an eigenvalue of $\mathcal{H}$. All of this will be achieved by firstly obtaining an explicit form for the $\{P_t^-\}$ semigroup.

Let $\mathcal{G}^-$ denote the Q-matrix of Z^- when Z^- is considered as a Markov chain on $\{0,1\}$. Then $\mathcal{G}^-$ will be of the form

$$\mathcal{G}^- = \begin{pmatrix} q_{0,0} & q_{0,1} \\ q_{1,0} & q_{1,1} \end{pmatrix},$$

where $q_{i,j} \in \mathbb{R}$.

2.8A. Proposition. *The off-diagonal elements $q_{i,j}$ $(i \neq j)$ of $\mathcal{G}^-$ are strictly positive. Consequently, for $i \in \{0,1\}$, $q_{i,i} \le -q_{i,1-i} < 0$, with equality in the foremost terms if and only if $m_0 + m_1 \ge 1$.*

Proof of Proposition 2.8A. The former result, which is intuitively obvious, may be proved by the method in Appendix C. The remaining point follows from a Q-matrix property, namely

$$S_i := \sum_{j\in\{0,1\}} q_{i,j} + c_i = 0, \qquad (i \in \{0,1\}) \tag{8.1}$$

where c_i (≥ 0) represents the 'rate of entering' the cemetery state starting from state i. We know that $c_i > 0$ if and only if Z_t^- has a finite lifetime, in which case $m_0 + m_1 < 1$. The result is now obvious. □

2.8B. Corollary. *The Q-matrix $\mathcal{G}^-$ has distinct real, non-positive, eigenvalues. Then $\mathcal{G}^-$ is diagonalizable.*

Proof of Corollary 2.8B. The proof relies on a standard argument that begins by solving $\det(\lambda I - \mathcal{G}^-) = 0$. Additionally, in such an argument, it can immediately be seen that the case when $m_0 + m_1 \geq 1$ leads to a zero eigenvalue. The fact that $\mathcal{G}^-$ is diagonalizable is elementary linear algebra. □

2.8C. Definition. *We define the distinct eigenvalues of $\mathcal{G}^-$ as $-\frac{1}{2}\alpha^2$ and $-\frac{1}{2}\beta^2$, where $0 \leq \alpha < \beta$ and $\alpha = 0$ if and only if $m_0 + m_1 \geq 1$.*

2.8D. Remark. Suppose that the eigenvector corresponding to the eigenvalue $-\frac{1}{2}\gamma^2$ of $\mathcal{G}^-$ is $\underline{v}_\gamma$, where $\underline{v}_\gamma = (v_\gamma(0), v_\gamma(1))^T$. If WH4 had already been proved, then we could show that eigenvectors of $\mathcal{G}^-$ are related to eigenfunctions of $\mathcal{H}$ so that everything would feature terms involving h_γ for $\gamma \in \{\alpha, \beta\}$.

2.8E. Lemma. *If we consider P_t^- just as the transition function* $\exp(t\mathcal{G}^-)$ *on* $\{0,1\}$, *then*

$$P_t^- \;=\; \mathrm{e}^{-\frac{1}{2}\alpha^2 t} V_\alpha + \mathrm{e}^{-\frac{1}{2}\beta^2 t} V_\beta, \tag{8.2}$$

where V_α, V_β are $\{0,1\} \times \{0,1\}$ matrices with explicit forms;

$$V_\alpha = \frac{1}{\Delta v_\gamma}\begin{pmatrix} v_\alpha(0)v_\beta(1) & -v_\alpha(0)v_\beta(0) \\ v_\alpha(1)v_\beta(1) & -v_\alpha(1)v_\beta(0) \end{pmatrix}, \quad \textit{and} \quad V_\beta = \frac{1}{\Delta v_\gamma}\begin{pmatrix} -v_\alpha(1)v_\beta(0) & v_\alpha(0)v_\beta(0) \\ -v_\alpha(1)v_\beta(1) & v_\alpha(0)v_\beta(1) \end{pmatrix},$$

with $\Delta v_\gamma = v_\alpha(0)v_\beta(1) - v_\alpha(1)v_\beta(0)$.

Proof of Lemma 2.8E. We shall firstly express the underlying Q-matrix $\mathcal{G}^-$ in terms of its eigenvectors. Although this is not strictly needed, it is certainly useful in order to check things later. Using the eigenvectors of $\mathcal{G}^-$, let

$$U = (\underline{v}_\alpha, \underline{v}_\beta) = \begin{pmatrix} v_\alpha(0) & v_\beta(0) \\ v_\alpha(1) & v_\beta(1) \end{pmatrix}, \quad \text{so that} \quad U^{-1} = \frac{1}{\Delta v_\gamma}\begin{pmatrix} v_\beta(1) & -v_\beta(0) \\ -v_\alpha(1) & v_\alpha(0) \end{pmatrix}.$$

Since $\mathcal{G}^-$ is diagonalizable, we have

$$\begin{aligned}
\mathcal{G}^- &= U\begin{pmatrix} -\frac{1}{2}\alpha^2 & 0 \\ 0 & -\frac{1}{2}\beta^2 \end{pmatrix}U^{-1} \\
&= \frac{1}{\Delta v_\gamma}\begin{pmatrix} v_\alpha(0) & v_\beta(0) \\ v_\alpha(1) & v_\beta(1) \end{pmatrix}\begin{pmatrix} -\frac{1}{2}\alpha^2 & 0 \\ 0 & -\frac{1}{2}\beta^2 \end{pmatrix}\begin{pmatrix} v_\beta(1) & -v_\beta(0) \\ -v_\alpha(1) & v_\alpha(0) \end{pmatrix} \\
&= \frac{1}{\Delta v_\gamma}\begin{pmatrix} -\frac{1}{2}\alpha^2 v_\alpha(0)v_\beta(1) + \frac{1}{2}\beta^2 v_\alpha(1)v_\beta(0) & \frac{1}{2}\alpha^2 v_\alpha(0)v_\beta(0) - \frac{1}{2}\beta^2 v_\alpha(0)v_\beta(0) \\ -\frac{1}{2}\alpha^2 v_\alpha(1)v_\beta(1) + \frac{1}{2}\beta^2 v_\alpha(1)v_\beta(1) & \frac{1}{2}\alpha^2 v_\alpha(1)v_\beta(0) - \frac{1}{2}\beta^2 v_\alpha(0)v_\beta(1) \end{pmatrix}.
\end{aligned} \tag{8.3}$$

We now construct the matrices V_α and V_β in our alternative form for P_t^- as follows.

$$\begin{aligned} P_t^- = e^{t\mathcal{G}^-} &= \sum_{k=0}^{\infty} \frac{(t\mathcal{G}^-)^k}{k!} \\ &= U \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} (-\frac{1}{2}\alpha^2 t)^k & 0 \\ 0 & (-\frac{1}{2}\beta^2 t)^k \end{pmatrix} U^{-1} \\ &= U \begin{pmatrix} e^{-\frac{1}{2}\alpha^2 t} & 0 \\ 0 & e^{-\frac{1}{2}\beta^2 t} \end{pmatrix} U^{-1}. \end{aligned}$$

From the forms of U and U^{-1}, we now have

$$P_t^- = \frac{1}{\Delta v_\gamma} e^{-\frac{1}{2}\alpha^2 t} \begin{pmatrix} v_\alpha(0)v_\beta(1) & -v_\alpha(0)v_\beta(0) \\ v_\alpha(1)v_\beta(1) & -v_\alpha(1)v_\beta(0) \end{pmatrix} + \frac{1}{\Delta v_\gamma} e^{-\frac{1}{2}\beta^2 t} \begin{pmatrix} -v_\alpha(1)v_\beta(0) & v_\alpha(0)v_\beta(0) \\ -v_\alpha(1)v_\beta(1) & v_\alpha(0)v_\beta(1) \end{pmatrix},$$

so that simple comparison with (8.2) gives the desired result. Note that we have $V_\alpha + V_\beta = I_2$. This is expected since $P_0^- = e^{t\mathcal{G}^-}|_{t=0} = I_2$. □

2.8F. Important discussion. We have constructed the matrices V_α and V_β via eigenvectors of $\mathcal{G}^-$. However, as already pointed out, we do not yet know that they are related to eigenfunctions of $\mathcal{H}$. In fact, we shall later see that they are actually related, in that, $\underline{v}_\gamma = \underline{h}_\gamma$, where $\underline{h}_\gamma = (h_\gamma(0), h_\gamma(1))^T$. Using this fact in (8.3), we see that $\sum_j g_{ij} \leq 0$ and equals zero if and only if $m_0 + m_1 \geq 1$. The fact that $\mathcal{G}^-$ is an improper Q-matrix when $m_0 + m_1 < 1$ is apparent in light of (8.1), due to the omission of the cemetery state.

Now consider the extension of $\{P_t^-\}$. For $x \in \{0, 1\}$ let $T_x := \inf\{t : Z_t = x\}$.

2.8G. Theorem. *For $z \in [0,1]$, and $h \in C[0,1]$,*

$$(P_t^- h)(z) = e^{-\frac{1}{2}\alpha^2 t}(S_\alpha h)(z) + e^{-\frac{1}{2}\beta^2 t}(S_\beta h)(z),$$

where for $\gamma \in \{\alpha, \beta\}$,

$$(S_\gamma h)(z) := \mathbb{E}^z\left(e^{-\frac{1}{2}\gamma^2 T_0}; T_0 < T_1\right)(V_\gamma h)(0) + \mathbb{E}^z\left(e^{-\frac{1}{2}\gamma^2 T_1}; T_1 < T_0\right)(V_\gamma h)(1).$$

In particular,

$$V_\gamma h = \begin{pmatrix} V_\gamma(0,0)h(0) + V_\gamma(0,1)h(1) \\ V_\gamma(1,0)h(0) + V_\gamma(1,1)h(1) \end{pmatrix} = \begin{pmatrix} (V_\gamma h)(0) \\ (V_\gamma h)(1) \end{pmatrix}, \tag{8.4}$$

so that the form of $(V_\gamma h)(x)$ in Theorem 2.8G is clear. Observe that $P_t^- h$ is defined for $h \in C[0,1]$, yet we only rely on its value at the boundaries.

Proof of Theorem 2.8G. Let $T := T_0 \wedge T_1$. By the tower property of conditional expectation we have

$$\begin{aligned}(P_t^- h)(z) :&= \mathbb{E}^z\left[h(Z_t^-);\ \tau_t^- < \infty\right]\\ &= \mathbb{E}^z\mathbb{E}^z\left[h(Z_t^-)\middle|\text{behaviour up to } T\right]\\ &= \mathbb{E}^z\left[\mathbb{E}^z\left[h(Z_t^-)\middle|\text{behaviour up to } T_0\right]; T_0 < T_1\right]\\ &\qquad + \mathbb{E}^z\left[\mathbb{E}^z\left[h(Z_t^-)\middle|\text{behaviour up to } T_1\right]; T_1 < T_0\right].\end{aligned} \tag{8.5}$$

To deal with the above we consider an application of the strong Markov theorem. We have for $i \in \{0, 1\}$,

$$\begin{aligned}&\mathbb{E}^z\left[h(Z_t^-)\middle|\text{behaviour up to } T \text{ on the set where } T_i < T_{1-i}\right]\\ &\qquad = \mathbb{E}^{Z_T^-}\left[h(Z_{T+t}^-)\right] = (P_{T+t}^- h)(i).\end{aligned} \tag{8.6}$$

On the set where $T_0 < T_1$ (i.e. $T = T_0$), from (8.6) we have

$$\begin{aligned}\mathbb{E}^z\left[h(Z_t^-)\middle|\text{behaviour up to } T_0\right] &= \mathbb{E}^{Z_{T_0}^-}\left[h(Z_{T_0+t}^-)\right]\\ &= \mathbb{E}^0\left[h(Z_{T_0+t}^-)\right]\\ &= (P_{T_0+t}^- h)(0)\\ &= e^{-\frac{1}{2}\alpha^2(T_0+t)}(V_\alpha h)(0) + e^{-\frac{1}{2}\beta^2(T_0+t)}(V_\beta h)(0).\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}^z\left[(P_{T_0+t}^- h)(0); T_0 < T_1\right] &= e^{-\frac{1}{2}\alpha^2 t}\,\mathbb{E}^z\left(e^{-\frac{1}{2}\alpha^2 T_0}; T_0 < T_1\right)(V_\alpha h)(0)\\ &\quad + e^{-\frac{1}{2}\beta^2 t}\,\mathbb{E}^z\left(e^{-\frac{1}{2}\beta^2 T_0}; T_0 < T_1\right)(V_\beta h)(0).\end{aligned} \tag{8.7}$$

Similarly on the set where $T_1 < T_0$ (i.e. $T = T_1$), we have

$$\begin{aligned}\mathbb{E}^z\left[h(Z_t^-)\middle|\text{behaviour up to } T_1\right] &= \mathbb{E}^{Z_{T_1}^-}\left[h(Z_{T_1+t}^-)\right]\\ &= \mathbb{E}^1\left[h(Z_{T_1+t}^-)\right]\\ &= (P_{T_1+t}^- h)(1)\\ &= e^{-\frac{1}{2}\alpha^2(T_1+t)}(V_\alpha h)(1) + e^{-\frac{1}{2}\beta^2(T_1+t)}(V_\beta h)(1).\end{aligned}$$

This time it follows that

$$\begin{aligned}\mathbb{E}^z\left[(P_{T_1+t}^- h)(1); T_1 < T_0\right] &= e^{-\frac{1}{2}\alpha^2 t}\,\mathbb{E}^z\left(e^{-\frac{1}{2}\alpha^2 T_1}; T_1 < T_0\right)(V_\alpha h)(1)\\ &\quad + e^{-\frac{1}{2}\beta^2 t}\,\mathbb{E}^z\left(e^{-\frac{1}{2}\beta^2 T_1}; T_1 < T_0\right)(V_\beta h)(1).\end{aligned} \tag{8.8}$$

Finally, from (8.5), (8.7) and (8.8), we have

$$\begin{aligned}(P_t^- h)(z) &= \mathbb{E}^z\left[(P_{T_0+t}^- h)(0); T_0 < T_1\right] + \mathbb{E}^z\left[(P_{T_1+t}^- h)(1); T_1 < T_0\right]\\ &= e^{-\frac{1}{2}\alpha^2 t}\,(S_\alpha h)(z) + e^{-\frac{1}{2}\beta^2 t}\,(S_\beta h)(z),\end{aligned}$$

where

$$(S_\gamma h)(z) = \mathbb{E}^z\left(e^{-\frac{1}{2}\gamma^2 T_0}; T_0 < T_1\right)(V_\gamma h)(0) + \mathbb{E}^z\left(e^{-\frac{1}{2}\gamma^2 T_1}; T_1 < T_0\right)(V_\gamma h)(1).$$

□

It is elementary that the $\mathbb{E}^z$ expectations define C^2 functions of z. Indeed

$$\mathbb{E}^z\left(\mathrm{e}^{-\frac{1}{2}\gamma^2 T_0}; T_0 < T_1\right) = \frac{\sinh\gamma(1-z)}{\sinh\gamma}, \tag{8.9}$$

with the obvious modification for the other expectation.

We now know that $(t, z) \mapsto (P_t^- h)(z)$ is $C^{1,2}$. This crucial fact allows us to legitimately use Theorem 2.7A in order to prove some of what we have previously hypothesized. We begin by proving the following useful result which will be heavily relied upon in the remainder of the section.

2.8H. Lemma. *Let $\gamma > 0$ be such that $\frac{1}{2}\gamma^2$ is an eigenvalue of $\mathcal{H}$ with corresponding eigenfunction h_γ. Then, for $t \geq 0$, we have*

$$(P_t^- h_\gamma)(z) = \mathbb{E}^z\left[h_\gamma(Z_t^-)\right] = e^{-\frac{1}{2}\gamma^2 t} h_\gamma(z).$$

Proof of Lemma 2.8H. From Corollary 2.7C, recall that,

$$\text{for } t \geq 0, \quad t \mapsto \exp(-\tfrac{1}{2}\gamma^2 \Phi_t) h_\gamma(Z_t) \text{ defines a local martingale bounded on } [0, \tau_t^-]. \tag{8.10}$$

Furthermore, if $\tau_t^- = \infty$, then by the long-term behaviour of Φ we must have $\Phi_t \to +\infty$ as $t \to \infty$ (this applies only to the case in which $m_0 + m_1 < 1$ or if $t = 0$ then $z \neq 0$ or 1), so that

$$\Phi(\tau_t^-) = \infty \quad \text{on the set } \{\tau_t^- = \infty\}.$$

Thus, $\exp\{-\frac{1}{2}\gamma^2\Phi(\tau_t^-)\} = 0$ on $\{\tau_t^- = \infty\}$, so such a case gives no contribution to the underlying expectation, in that

$$\mathbb{E}^z\left[\exp\{-\tfrac{1}{2}\gamma^2\Phi(\tau_t^-)\} h_\gamma(Z_t^-); \tau_t^- < \infty\right] = \mathbb{E}^z\left[\exp\{-\tfrac{1}{2}\gamma^2\Phi(\tau_t^-)\} h_\gamma(Z_t^-)\right]. \tag{8.11}$$

Note further that $\Phi(\tau_t^-) = -t$ when $\tau_t^- < \infty$ (see Proposition A.3 of Appendix A) and $\Phi(0) = 0$. Since τ_t^- is a valid stopping time, we can legitimately apply the Optional-Stopping Theorem to get

$$\begin{aligned}
\mathbb{E}^z\left[\exp\{-\tfrac{1}{2}\gamma^2\Phi(\tau_t^-)\} h_\gamma(Z_t^-); \tau_t^- < \infty\right] &= \mathbb{E}^z\left[\exp\{\tfrac{1}{2}\gamma^2 t\} h_\gamma(Z_t^-); \tau_t^- < \infty\right] \\
&= \exp\{\tfrac{1}{2}\gamma^2 t\}\,(P_t^- h_\gamma)(z) \\
&= \mathbb{E}^z\left[\exp\{\tfrac{1}{2}\gamma^2 t\} h_\gamma(Z_t^-)\right] \qquad \text{(by (8.11))} \\
&= \exp\{\tfrac{1}{2}\gamma^2 t\}\,\mathbb{E}^z\left[h_\gamma(Z_t^-)\right] \\
&= \mathbb{E}^z\left[h_\gamma(Z_0)\right] \\
&= \mathbb{E}^z\left[h_\gamma(z)\right] \\
&= h_\gamma(z).
\end{aligned}$$

In particular, we have $\exp\{\frac{1}{2}\gamma^2 t\}\ (P_t^- h_\gamma)(z) = \exp\{\frac{1}{2}\gamma^2 t\}\ \mathbb{E}^z\left[h_\gamma(Z_t^-)\right] = h_\gamma(z)$, from which the desired result follows. Note that this result can only become a 'weapon' when we know that $\frac{1}{2}\gamma^2$ is a eigenvalue of $\mathcal{H}$. □

The following Theorem is clearly an amalgamation of Working Hypothesis WH4 and Definitions 2.3L and 2.3M.

> **2.8I. Theorem.** *$+\frac{1}{2}\alpha^2$ and $+\frac{1}{2}\beta^2$ are eigenvalues of $\mathcal{H}$, and for no $\gamma \in (0,\infty)\setminus\{\alpha,\beta\}$ is $\frac{1}{2}\gamma^2$ an eigenvalue of $\mathcal{H}$.*

Proof of Theorem 2.8I. Staying consistent with the notation used in Theorem 2.7A (b), define the function H via $H(t,z) := (P_t^- h)(z)$ for $t > 0$. Again, we know that H is $C^{1,2}$ and, by mirroring the sentence following (7.5) for the '-' situation, $H\left(\Phi(t\wedge\tau_0^-), Z(t\wedge\tau_0^-)\right)$ is a martingale. Hence by the 'if' part of Theorem 2.7A (b),

$$(\partial_t + \mathcal{H})H(t,z) = 0,$$

whence $S_\alpha h$ and $S_\beta h$ are in $\mathcal{D}(\mathcal{H})$ and

$$(\mathcal{H} - \tfrac{1}{2}\alpha^2)\mathrm{e}^{-\frac{1}{2}\alpha^2 t} S_\alpha h + (\mathcal{H} - \tfrac{1}{2}\beta^2)\mathrm{e}^{-\frac{1}{2}\beta^2 t} S_\beta h \;=\; 0.$$

Now we see that $\frac{1}{2}\alpha^2$ and $\frac{1}{2}\beta^2$ are eigenvalues of $\mathcal{H}$ and that $S_\alpha h$ and $S_\beta h$ are associated eigenfunctions. Thus, recalling Definition 2.3M, $S_\gamma h$ must be a multiple of h_γ ($\gamma \in \Gamma$). This is fine as eigenvalues are undetermined with respect to scalar multipliers.

Let $\gamma > 0$ be such that $\frac{1}{2}\gamma^2$ is an eigenvalue of $\mathcal{H}$ with corresponding eigenfunction h_γ. Then, from Lemma 2.8H we may conclude that, $P_t^- h_\gamma = \mathrm{e}^{-\frac{1}{2}\gamma^2 t} h_\gamma$ and that $\mathcal{G}^- \underline{h}_\gamma = -\frac{1}{2}\gamma^2 \underline{h}_\gamma$, since P_t^- is viewed as the transition function $\exp\{-t\mathcal{G}^-\}$. We have therefore shown that if $\frac{1}{2}\gamma^2$ is an eigenvalue of $\mathcal{H}$, then $-\frac{1}{2}\gamma^2$ is an eigenvalue of $\mathcal{G}^-$. Hence $\gamma \in \{\alpha,\beta\}$. In addition, we have confirmed the fact that $\underline{v}_\gamma = \underline{h}_\gamma$ as pointed out in Important Discussion 2.8F.

Remark: The transition from general function to column vector here is obvious as $\mathcal{G}^-$ relates to a Markov chain on $\{0,1\}$.

To tie things together, note that for $\gamma \in \{\alpha,\beta\}$, application of the Optional-Stopping Theorem at time $T_0 \wedge T_1$ shows that $S_\gamma h_\gamma = h_\gamma$. □

Important Comments. Recall that Definitions 2.3L and 2.3M were entirely dependent upon the validity of Working Hypothesis WH4. Our probabilistic proof of Theorem 2.8I now confirms that WH4 is true and that our consequent definitions are indeed correct.

Proof of Working Hypothesis WH5. Since we have now proved Theorem 2.8I, it is now clear that WH5 is simply a special case of Lemma 2.8H, in that, it corresponds to taking $t = 0$. □

2.8J. Lemma. *We have the following results:*

(a) $h_\alpha(1) \neq h_\beta(1)$,

(b) $\pi(x, y)$, *as defined by (6.2), is given by the formulae (4.1), so that formula must produce non-negative* $\pi(x, y)$,

(c) For $y \in (0, 1)$, $\sum_{x\in\{0,1\}} m_x \pi(x, y) \leq 1$ *and equals* 1 *if and only if* $m_0 + m_1 \geq 1$,

(d) For $x \in \{0, 1\}$, *we have* $\int_{(0,1)} \pi(x, y)\mathrm{d}y \leq 1$ *and equals* 1 *if and only if* $m_0 + m_1 \leq 1$.

Proof of Lemma 2.8J (a). Lemma 2.8H applied at $t = 0$ gives

$$\mathbb{E}^z\left[h_\gamma(Z_0^-)\right] = h_\gamma(z). \tag{8.12}$$

Result (8.12) is true *for all* $\gamma \in \Gamma = \{\alpha, \beta\}$, so that we have

$$\mathbb{E}^z\left[h_\alpha(Z_0^-)\right] = h_\alpha(z), \qquad \text{and} \qquad \mathbb{E}^z\left[h_\beta(Z_0^-)\right] = h_\beta(z). \tag{8.13}$$

We know that $Z_0^- \in \{0, 1\}$ and that $h_\alpha(0) = h_\beta(0) = 1$. Hence, from (8.13), if $h_\alpha(1) = h_\beta(1)$ then $h_\alpha(z) = h_\beta(z)$ for all $z \in [0, 1]$. However, we know that $h_\alpha(z) \neq h_\beta(z)$ for *some* z. Otherwise, $\alpha = \beta$ which we know is false. It follows that $h_\alpha(1) \neq h_\beta(1)$. □

Proof of Lemma 2.8J (b). From the result for α in (8.13) with $y \in (0, 1)$ and the definition of expectation, we have

$$\begin{aligned} \mathbb{E}^y\left[h_\alpha(Z_0^-)\right] &= h_\alpha(0)\mathbb{P}^y(Z_0^- = 0) + h_\alpha(1)\mathbb{P}^y(Z_0^- = 1) = h_\alpha(y) \\ \Leftrightarrow \qquad & \mathbb{P}^y(Z_0^- = 0) + h_\alpha(1)\mathbb{P}^y(Z_0^- = 1) = h_\alpha(y). \end{aligned} \tag{8.14}$$

Similarly, for β we have

$$\mathbb{P}^y(Z_0^- = 0) + h_\beta(1)\mathbb{P}^y(Z_0^- = 1) = h_\beta(y). \tag{8.15}$$

Solving the equations in (8.14) and (8.15) simultaneously, we have

$$\mathbb{P}^y(Z_0^- = 0) = \frac{h_\beta(1)h_\alpha(y) - h_\alpha(1)h_\beta(y)}{h_\beta(1) - h_\alpha(1)}, \qquad \mathbb{P}^y(Z_0^- = 0) = \frac{h_\beta(y) - h_\alpha(y)}{h_\beta(1) - h_\alpha(1)}.$$

Noting that

$$\mathbb{P}^y(Z_0^- = x) := \Pi^{+-}(y, \{x\}) := m_x \pi(x, y), \qquad \text{from (6.2)},$$

we get the desired result as π, defined in terms of a probability, must be non-negative. □

Under Definition 2.6G with $f \equiv 1$, we know the following;

$$(P_0^+ 1)(x) = \mathbb{P}^x(\tau_0^+ < \infty) \quad \text{for } x \in \{0, 1\}, \tag{8.16}$$

$$(P_0^- 1)(y) = \mathbb{P}^y(\tau_0^- < \infty) \quad \text{for } y \in (0, 1). \tag{8.17}$$

Proof of Lemma 2.8J (c). From (8.17), for $y \in (0,1)$, we have

$$(P_0^- 1)(y) = \sum_{x \in \{0,1\}} m_x \pi(x,y) = \mathbb{P}^y \left(\tau_0^- < \infty\right) \leq 1,$$

where Lemma 2.6B (b) gives us equality when $m_0 + m_1 \geq 1$. We have therefore checked that $\Pi^{+-}(y, \cdot)$ is a measure of total mass at most 1 on subsets of $\{0,1\}$. □

Note. From (8.16) we have, for $x \in \{0,1\}$,

$$(P_0^+ 1)(x) = \int_0^1 \pi(x,y)\, \mathrm{d}y = \mathbb{P}^x(\tau_0^+ < \infty).$$

However, we are not able to prove the remaining result of 2.8J by Probability Theory in this way because we do not yet know that $\pi(x,\cdot)$ is the jump-out density from x for Z^+.

Proof of Lemma 2.8J (d). From Lemma 2.3G, we know that $\langle 1, h_{\tilde{\gamma}} \rangle_s = 0$ for $\tilde{\gamma} \in \Gamma \cap (0,\infty)$. Using the definition of $\langle \cdot, \cdot \rangle_s$, this is equivalent to

$$\int_0^1 h_{\tilde{\gamma}}(y) \mathrm{d}y = m_1 h_{\tilde{\gamma}}(1) + m_0, \qquad \text{for } \tilde{\gamma} \in \Gamma \cap (0,\infty). \tag{8.18}$$

Suppose that $m_0 + m_1 \geq 1$. Recall that $\pi(x,y)$ is defined as in (4.1) with $\alpha = 0$. From (8.18) we now get

$$\int_0^1 \pi(0,y)\, \mathrm{d}y = \frac{1}{m_0[h_\beta(1) - 1]} \left\{(1 - m_1) h_\beta(1) - m_0\right\}. \tag{8.19}$$

If $m_0 + m_1 = 1$, then $1 - m_1 = m_0$, and so the RHS of (8.19) clearly reduces to 1. If $m_0 + m_1 > 1$, then $1 - m_1 < m_0$. Therefore

$$\frac{(1 - m_1) h_\beta(1) - m_0}{m_0[h_\beta(1) - 1]} < \frac{m_0[h_\beta(1) - 1]}{m_0[h_\beta(1) - 1]} = 1,$$

giving $\int_0^1 \pi(0,y)\, \mathrm{d}y < 1$. Note that the validity of the above inequality is guaranteed by the fact that $h_\beta(1) - 1 < h_\beta(1) < 0$ as shown in Lemma 2.3O. Similar arguments hold for $\int_0^1 \pi(1,y)\, \mathrm{d}y$.

Suppose that $m_0 + m_1 < 1$. Again, recall the form of $\pi(x,y)$ from (4.1), noting that $\alpha > 0$ here. Let $\Delta h := h_\beta(1) - h_\alpha(1)$. Then, from (8.18), we have

$$\begin{aligned}\int_0^1 \pi(0,y) \mathrm{d}y &= \frac{1}{m_0 \Delta h} \left\{h_\beta(1)[m_1 h_\alpha(1) + m_0] - h_\alpha(1)[m_1 h_\beta(1) + m_0]\right\} \\ &= \frac{1}{m_0 \Delta h} \{m_0 \Delta h\} = 1.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_0^1 \pi(1,y) \mathrm{d}y &= \frac{1}{m_1 \Delta h} \left\{m_1 h_\beta(1) + m_0 - m_1 h_\alpha(1) - m_0\right\} \\ &= \frac{1}{m_1 \Delta h} \{m_1 \Delta h\} = 1,\end{aligned}$$

and so we have the desired result. We have therefore verified that $\Pi^{-+}(x, \cdot)$ is a measure of total mass at most 1 on (Borel subsets of) $(0, 1)$. □

Proof of Working Hypothesis WH2. Most of the desired work has already been done. With our usual definition of H, in the proof of Theorem 2.8I we saw that the necessary PDE holds and $H(\varphi, \cdot) \in \mathcal{D}(\mathcal{H})$. It is also useful to recall Reminder 2.7B at this point. Further details have also been proved in Lemma 2.8J. Due to the martingale property, we may justify the remaining matters probabilistically as well as analytically. In particular, from Definition 2.6G

$$(P_t^- 1)(z) := \mathbb{E}^z\left[1; \tau_t^- < \infty\right] = \mathbb{P}^z(\tau_t^- < \infty) \leq 1,$$

and equals 1 if and only if $m_0 + m_1 \geq 1$ (see Lemma 2.6B). Albeit tedious, it is trivial to prove that

$$\begin{aligned}(P_0^- h)(z) &:= \lim_{t \downarrow 0}(P_t^- h)(z) = (S_\alpha h)(z) + (S_\beta h)(z) \\ &= \begin{cases} \sum_{x \in \{0,1\}} m_x \pi(x, z) h(x) & \text{if } z \in (0,1), \\ h(z) & \text{if } z \in \{0,1\}. \end{cases}\end{aligned}$$

Minimality. With our 'discovered' H, we may simply mimic the arguments given in Section 7 for the τ_t^- situation. □

We have now shown that the P_t^- in Theorem 2.8G is the correct one. We may therefore establish the following result on the long-term behaviour of P_t^-. This will be used to prove WH3.

2.8K. Lemma. *As $t \to \infty$, the following convergence is* **uniform** *in z:*

$$(P_t^- h)(z) \to \begin{cases} c^-(h) & \textit{if } m_0 + m_1 \geq 1, \\ 0 & \textit{if } m_0 + m_1 < 1, \end{cases}$$

where $c^-(h)$ is independent of z and of the form

$$c^-(h) = \frac{1}{h_\beta(1) - 1}\left[h_\beta(1) h(0) - h(1)\right].$$

Proof of Lemma 2.8K. If $m_0 + m_1 < 1$ then the result is obvious due to the fact that $\gamma \in \{\alpha, \beta\}$ is strictly positive. Then $e^{-\frac{1}{2}\gamma^2 t} \to 0$ as $t \to \infty$ for all $\gamma \in \Gamma$. However, if $m_0 + m_1 \geq 1$, then $\alpha = 0$ so that $(P_t^- h)(z) \to (S_0 h)(z)$. The result then follows from the fact that Z_t^- is immortal so that $\mathbb{P}^z(T_0 < T_1) + \mathbb{P}^z(T_1 < T_0) = 1$. The fact that the convergence is uniform is now obvious. □

We pause to clarify that for $h \in C[0, 1]$ and $w \in (0, 1)$,

$$\left(\Pi^{+-} h\right)(w) \ = \ h(0) m_0 \pi(0, w) + h(1) m_1 \pi(1, w),$$

and that $(P_0^- h)(w)$ is given by the right-hand side for all $w \in [0, 1]$.

2.9. Some Identities

It is convenient to collect here some required identities. It should be noted that even the seemingly most simple of results turn out to be quite elusive. The computer package *Mathematica* has been used in some instances.

For $\rho > 0$ and $z \in [0,1]$, we write

$$c_\rho(z) := \cosh \rho z, \qquad s_\rho(z) := \sinh \rho z,$$
$$h^\sharp_\rho(w) := c_\rho(1-w) - m_1 \rho s_\rho(1-w), \qquad h_\rho(w) := c_\rho(w) - m_0 \rho s_\rho(w).$$

The h_ρ notation, which will be used for general $\rho > 0$, is consistent with the h_γ notation used earlier for $\gamma \in \Gamma$. Because of the 'eigenfunction of $\mathcal{H}$' property, we have,

$$\text{for } \gamma \in \Gamma,\ h^\sharp_\gamma(w) = h^\sharp_\gamma(0) h_\gamma(w), \qquad \text{so, in particular, } 1 = h^\sharp_\gamma(1) = h^\sharp_\gamma(0) h_\gamma(1). \tag{9.1}$$

We have, for $\gamma \in \Gamma$ and $\rho > 0$,

$$(\rho - \gamma)(\rho + \gamma)\langle s_\rho, h_\gamma \rangle_s = -\rho + \rho h_\gamma(1) h^\sharp_\rho(0)$$

and

$$(\rho - \gamma)(\rho + \gamma)\langle h_\rho, h_\gamma \rangle_s = \rho h_\gamma(1) \epsilon(\rho), \tag{9.2}$$

where $\epsilon(\rho)$ is the familiar object from (3.1):

$$\epsilon(\rho) := (1 + m_0 m_1 \rho^2) s_\rho(1) - \rho(m_0 + m_1) c_\rho(1).$$

Moreover, if

$$(Qf)(z) := \rho^{-1} \int_0^z s_\rho(z-w) f(w)\, \mathrm{d}w,$$

then, for $\gamma \in \Gamma$,

$$\langle h_\gamma, Qf \rangle_s = \int_{[0,1]} N(w) f(w)\, \mathrm{d}w,$$

where

$$(\rho - \gamma)(\rho + \gamma) N(w) = -h_\gamma(w) + h_\gamma(1) h^\sharp_\rho(w).$$

2.10. Calculation of the Conjectured Resolvent $\{\hat{R}^+_\lambda : \lambda > 0\}$

Heuristics. Believing in duality, we predict that Z^+ has the same laws as $\hat{Z}^+$ where $\hat{Z}^+$ behaves like Brownian motion within $(0,1)$ and, on approaching $x \in \{0,1\}$, jumps into $(0,1)$ according to density $\pi(x,\cdot)$ given by (4.1). We shall construct the resolvent $\{\hat{R}^+_\lambda : \lambda > 0\}$ which $\hat{Z}^+$ would have to possess.

Remark. Think about how to construct an appropriate $(\Omega, \mathcal{F})$ and initial laws for $\hat{Z}^+$. There is always a problem at this point because it is difficult to prove that the constructed $\hat{Z}^+$ is Markov, let alone strong Markov.

Let ${}_{\text{tab}}R_\lambda$ (the notation suggested by Chung's 'taboo' terminology) be the resolvent of the process $\{Z_t : t < T_0 \wedge T_1\}$, and let

$$\psi_\lambda(z,0) := \mathbb{E}^z\left(e^{-\lambda T_0}; T_0 < T_1\right), \qquad \psi_\lambda(z,1) := \mathbb{E}^z\left(e^{-\lambda T_1}; T_1 < T_0\right),$$

the explicit forms of which we know from (8.9). For $z \in (0,1)$ and $f \in C[0,1]$, we surely believe that

$$g_\lambda(z) := (\hat{R}^+_\lambda f)(z) = ({}_{\text{tab}}R_\lambda f)(z) + \sum_{x\in\{0,1\}} \psi_\lambda(z,x)(\hat{R}^+_\lambda f)(x). \tag{10.1}$$

(The analogous result for Brownian motion on $\mathbb{R}$ shows that for $f \in C[0,1]$, ${}_{\text{tab}}g_\lambda := ({}_{\text{tab}}R_\lambda f)$ solves $\lambda({}_{\text{tab}}g_\lambda) - \frac{1}{2}({}_{\text{tab}}g''_\lambda) = f$ on $(0,1)$ with Dirichlet boundary conditions ${}_{\text{tab}}g_\lambda(x) = 0$ for $x \in \{0,1\}$.) Because 0 and 1 are branch points of the Ray process $\hat{Z}^+$, the relevant strong Markov property to which we are making *intuitive* appeal, is really that (due to Meyer and Ray) at Theorem III.41.3 of Rogers and Williams [24] (See Appendix E for further details). Now, (10.1) implies that $g_\lambda \in C[0,1] \cap C^2(0,1)$ and

$$\lambda g_\lambda - \tfrac{1}{2}g''_\lambda = f \quad \text{on } (0,1). \tag{10.2}$$

(Wait for Important Comment 2.10M below for clarification.) Moreover, we also believe that for $x \in \{0,1\}$,

$$g_\lambda(x) = \int_{(0,1)} \pi(x,y)g_\lambda(y)\,dy. \tag{10.3}$$

The 'lateral' condition (10.3) is equivalent to

$$\langle h_\gamma, g_\lambda\rangle_s = 0 \qquad (\gamma \in \Gamma). \tag{10.4}$$

Do remember that equation (10.2) is not the $(\lambda - \hat{\mathcal{G}}^+)^{-1} = \hat{R}^+_\lambda$ equation of Hille-Yosida Theory because f may not belong to the domain of strong convergence to I of $\hat{P}^+_t$ as $t \downarrow 0$.

Definition and calculation of $\{\hat{R}^+_\lambda : \lambda > 0\}$. We now start from scratch with a rigorous analytic definition of the conjectured resolvent. We shall see that for $f \in C[0,1]$, there is a unique solution $g_\lambda \in C[0,1] \cap C^2(0,1)$ of (10.2) with lateral condition (10.3) (equivalently, (10.4)). We shall **define** the linear operator $\hat{R}^+_\lambda$ on $C[0,1]$ via $\hat{R}^+_\lambda f := g_\lambda$. We are going to show that $\{\hat{R}^+_\lambda : \lambda > 0\}$ is a Ray resolvent, and (in Sections 12 and 13) that it is the desired resolvent of Z^+.

2.10A. Analytic verification that (10.1) holds. Recall that we define g_λ via (10.2) and (10.3) Consequently, we expect ${}_{\text{tab}}g_\lambda$ to satisfy the conditions in the comment following equation (10.1). Consider

$${}_{\text{tab}}g_\lambda(z) := g_\lambda(z) - \psi_\lambda(z,0)g_\lambda(0) - \psi_\lambda(z,1)g_\lambda(1). \tag{10.5}$$

Recall that $\delta = \sqrt{2\lambda}$ and we already know that

$$\psi(z,0) = \frac{\sinh[\delta(1-z)]}{\sinh(\delta)}, \qquad \psi(z,1) = \frac{\sinh(\delta z)}{\sinh(\delta)}. \tag{10.6}$$

It follows that

$$\lambda\psi(z,0) - \tfrac{1}{2}\partial_z^2\psi(z,0) = \lambda\psi(z,1) - \tfrac{1}{2}\partial_z^2\psi(z,1) = 0, \quad \text{on } (0,1). \tag{10.7}$$

Also note that

$$\psi(1,0) = \psi(0,1) = 0 \quad \text{and} \quad \psi(0,0) = \psi(1,1) = 1. \tag{10.8}$$

Firstly we check that (10.5) satisfies the PDE for ${}_{\mathrm{tab}}g_\lambda$.

$$\begin{aligned}\lambda({}_{\mathrm{tab}}g_\lambda) - \tfrac{1}{2}({}_{\mathrm{tab}}g_\lambda'') &= \lambda g_\lambda - \tfrac{1}{2}g_\lambda'' - \{\lambda\psi(z,0) - \tfrac{1}{2}\partial_z^2\psi(z,0)\}g(0) \\ &\qquad - \{\lambda\psi(z,1) - \tfrac{1}{2}\partial_z^2\psi(z,1)\}g(1) \\ &= \lambda g_\lambda - \tfrac{1}{2}g_\lambda'' \qquad \text{(by (10.7))} \\ &= f \qquad \text{(by (10.2))}.\end{aligned}$$

This is exactly what we want. Next we deal with the boundary conditions for ${}_{\mathrm{tab}}g_\lambda$. Considering $z = 0$, we have

$$\begin{aligned}{}_{\mathrm{tab}}g_\lambda(0) :&= g_\lambda(0) - \psi(0,0)g_\lambda(0) - \psi(0,1)g_\lambda(1) \\ &= g_\lambda(0) - g_\lambda(0) \qquad \text{(by (10.8))} \\ &= 0.\end{aligned}$$

Of course we get a similar result at $z = 1$, so it follows that ${}_{\mathrm{tab}}g_\lambda(0) = {}_{\mathrm{tab}}g_\lambda(1) = 0$ as desired. □

By elementary calculus, equation (10.2) implies that, for $\lambda > 0$, and with c_ρ and s_ρ as in the previous section,

$$g_\lambda(z) \;=\; A_\lambda(f)c_\rho(z) + B_\lambda(f)s_\rho(z) - 2\rho^{-1}\int_0^z s_\rho(z-w)f(w)\,\mathrm{d}w, \tag{10.9}$$

where, as we shall consistently use,

$$\rho := (2\lambda)^{\frac{1}{2}}, \qquad \lambda = \tfrac{1}{2}\rho^2.$$

Recall that for $\gamma \in \Gamma$, we have $h_\gamma(w) = m_0\pi(0,w)h_\gamma(0) + m_1\pi(1,w)h_\gamma(1)$. Using this fact and the identities of the previous section we find from (10.4) that

$$A_\lambda(f) \;=\; \int_{[0,1]} A_\lambda(0,w)f(w)\,\mathrm{d}w, \qquad B_\lambda(f) \;=\; \int_{[0,1]} B_\lambda(0,w)f(w)\,\mathrm{d}w,$$

where, for $\gamma \in \Gamma$,

$$\begin{aligned}&-\tfrac{1}{2}m_0\rho^2A_\lambda(0,w) - \tfrac{1}{2}\rho B_\lambda(0,w) + m_0\pi(0,w) \\ &= \; h_\gamma(1)\Big[h_\rho^\sharp(w) - \tfrac{1}{2}\rho A_\lambda(0,w)\{-m_1\rho c_\rho(1) + s_\rho(1)\} - \tfrac{1}{2}\rho B_\lambda(0,w)h_\rho^\sharp(0) - m_1\pi(1,w)\Big].\end{aligned}$$

Since $h_\alpha(1) \neq h_\beta(1)$, both sides of the equation just obtained are zero, so

$$\tfrac{1}{2}m_0\rho^2A_\lambda(0,w) + \tfrac{1}{2}\rho B_\lambda(0,w) - m_0\pi(0,w) \;=\; 0 \tag{10.10}$$

and

$$\tfrac{1}{2}\rho\{m_1\rho c_\rho(1) - s_\rho(1)\}A_\lambda(0,w) - \tfrac{1}{2}h_\rho^\sharp(0)\rho B_\lambda(0,w) - m_1\pi(1,w) + h_\rho^\sharp(w) \;=\; 0. \qquad (10.11)$$

As we shall see, equation (10.10) is particularly significant. Multiplying (10.10) by $h_\rho^\sharp(0)$ and adding to (10.11), we find that for $\rho > 0$,

$$\tfrac{1}{2}\rho\epsilon(\rho)A_\lambda(0,w) \;=\; \{(I - P_0^-)h_\rho^\sharp\}(w). \qquad (10.12)$$

For $\rho > 0$ and $\rho \in \Gamma$, both sides of equation (10.12) are zero.

2.10B. Lemma. *We have $\hat{R}_\lambda^+ : C[0,1] \to C[0,1]$, and*

$$(\hat{R}_\lambda^+ f)(z) \;=\; \int_{[0,1]} \hat{r}_\lambda^+(z,w)f(w)\,\mathrm{d}w$$

for a jointly continuous kernel $\hat{r}_\lambda^+$ with

$$\hat{r}_\lambda^+(0,w) \;=\; A_\lambda(0,w), \qquad \hat{r}_\lambda^+(z,x) = 0 \textit{ for } x \in \{0,1\},\ z \in [0,1].$$

Proof of Lemma 2.10B. This is obvious from (10.12) and symmetry. □

Look forward to Corollary B.7 of Appendix B which motivates the following Lemma.

2.10C. Lemma. *As $\lambda \to \infty$, we have the following results*

$$\lambda A_\lambda(0,w) \to \pi(0,w) \quad (w \in (0,1]), \qquad \textit{but} \quad 0 = \lambda A_\lambda(0,0) \not\to \pi(0,0+) = 1/m_0.$$

This result will later emphasize that the lack of uniform convergence, or of 'equi-uniform differentiability' can easily lead one into error in this subject.

Proof of Lemma 2.10C. Multiplying (10.12) by ρ and noting that $\frac{1}{2}\rho^2 = \lambda$, we have

$$\lambda\epsilon(\rho)A_\lambda(0,w) = \rho\, h_\rho^\sharp(w) - \rho m_0 h_\rho^\sharp(0)\pi(0,w) - \rho m_1\pi(1,w). \qquad (10.13)$$

In order to deal with (10.13), we note that $\lambda \to \infty$ is clearly equivalent to $\rho \to \infty$. Observe that

$$\pi(0,0+) = \tfrac{1}{m_0} \qquad \text{and} \qquad \pi(0,1-) = 0.$$

Thus, from (10.13), it then becomes clear that

$$\lambda A_\lambda(0,0) = \lambda A_\lambda(0,1) = 0 \qquad \text{for all } \lambda.$$

In particular, we see that, as $\lambda \to \infty$,

$$\lambda A_\lambda(0,0) \not\to \pi(0,0+), \quad \text{and} \quad \lambda A_\lambda(0,1) \to \pi(0,1).$$

Next recall that

$$\epsilon(\rho) := (1 + m_0 m_1\rho^2)\sinh\rho - (m_0 + m_1)\rho\cosh\rho.$$

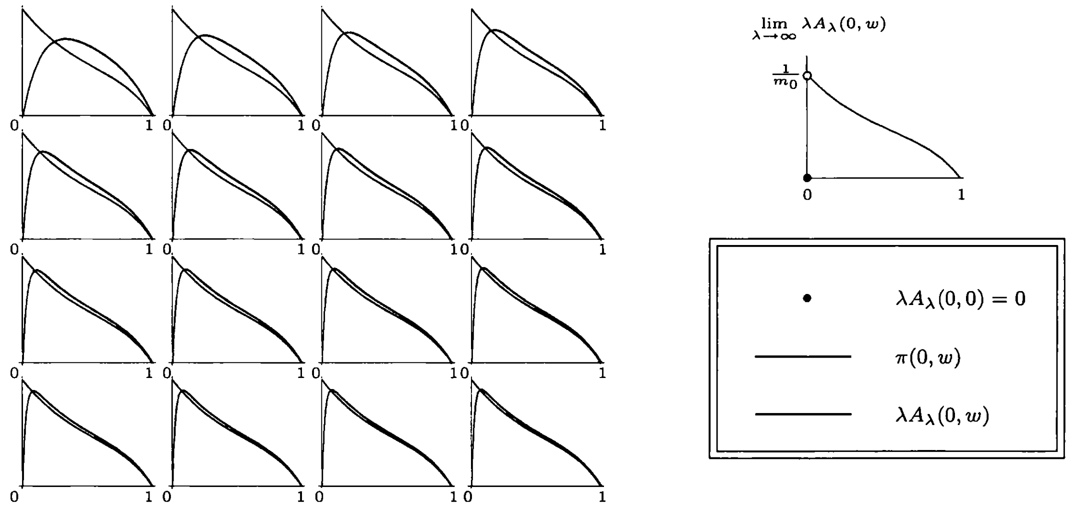

Figure 2.2: $\lambda A_\lambda(0,w)$ against $\pi(0,w)$ for large λ.

From (10.13), it suffices to show that, for $w \in (0,1)$,

$$\lim_{\rho\to\infty} \frac{\rho m_0 h_\rho^\sharp(0)}{\epsilon(\rho)} = -1, \qquad \lim_{\rho\to\infty} \frac{\rho\, h_\rho^\sharp(w) - \rho m_1 \pi(1,w)}{\epsilon(\rho)} = 0. \tag{10.14}$$

For the first limit in (10.14), using the definition of $h_\rho^\sharp$ in Section 9, we have

$$\frac{\rho m_0 h_\rho^\sharp(0)}{\epsilon(\rho)} = \frac{m_0 \rho \cosh \rho - m_0 m_1 \rho^2 \sinh \rho}{(1 + m_0 m_1 \rho^2)\sinh \rho - (m_0 + m_1)\rho \cosh \rho}. \tag{10.15}$$

Now for sufficiently large ρ, it is obvious that $m_0 m_1 \rho^2 \sinh \rho$ is the dominant term in (10.15). Consequently, we now have

$$\frac{\rho m_0 h_\rho^\sharp(0)}{\epsilon(\rho)} \approx \frac{-m_0 m_1 \rho^2 \sinh \rho}{m_0 m_1 \rho^2 \sinh \rho} = -1, \tag{10.16}$$

for sufficiently large ρ. Hence, taking the limit as $\rho \to \infty$ in (10.15), we get the desired result as in (10.14).

For the second limit in (10.14), we have

$$\frac{\rho\, h_\rho^\sharp(w) - \rho m_1 \pi(1,w)}{\epsilon(\rho)} = \frac{\rho \cosh \rho(1-w) - m_1 \rho^2 \sinh \rho(1-w) - \rho m_1 \pi(1,w)}{(1 + m_0 m_1 \rho^2)\sinh \rho - (m_0 + m_1)\rho \cosh \rho}. \tag{10.17}$$

Note that, for any fixed $w \in (0,1)$, we can regard $\pi(1,w)$ as a constant. As in the previous limit, the $\rho^2 \sinh(\cdot)$ term will dominate in both the numerator and denominator of (10.17). Hence, for sufficiently large ρ, we have

$$\begin{aligned} \frac{\rho\, h_\rho^\sharp(w) - \rho m_1 \pi(1,w)}{\epsilon(\rho)} &\approx \frac{-\rho^2 \sinh \rho(1-w)}{\rho^2 \sinh \rho} = \frac{-\sinh \rho(1-w)}{\sinh \rho} \\ &\approx \frac{e^{\rho(1-w)}}{e^\rho} = -e^{-\rho w}. \end{aligned}$$

Since, for $w \in (0,1)$, $e^{-\rho w} \to 0$ as $\rho \to \infty$, we get the desired result. □

Given our proof of Lemma 2.10C, symmetry gives the corresponding result at 1, that is, for $w \in [0,1)$

$$\lambda\, \hat{r}_\lambda^+(1,w) \to \pi(1,w), \qquad \text{as } \lambda \to \infty.$$

$\{\hat{R}_\lambda^+ : \lambda > 0\}$ *as a Ray resolvent.* For the next part of the section, recall the necessary Ray process theory from Appendix B.

2.10D. Theorem. $\{\hat{R}_\lambda^+ : \lambda > 0\}$ *is an honest Feller resolvent on* $C[0,1]$.

In order to prove the above theorem, we firstly need several results. Recall that Lemma 2.10B already gives us part of what we want.

2.10E. Lemma. *We have non-negativity of* $\hat{R}_\lambda^+$, *in that, if* $f \in C[0,1]$ *and* $f \geq 0$, *then* $\hat{R}_\lambda^+ f \geq 0$.

Proof of Lemma 2.10E. Let $f \in C[0,1]$ and $f \geq 0$. Recall that $g(z) := (\hat{R}_\lambda^+ f)(z)$ satisfies (10.2) and (10.3). Note that the subscript λ shall be dropped for typographic convenience. In particular, we need to prove that

$$(\hat{R}_\lambda^+ f)(z) \geq 0 \qquad \text{for all } z \in [0,1].$$

Now for a contradiction suppose that

$$g_* := \inf\{g(z) : z \in [0,1]\} < 0. \tag{10.18}$$

Suppose further that $g(y_0) = g_*$ for some $y_0 \in (0,1)$. Then by (10.3) we have

$$\lambda g_* - \tfrac{1}{2} g_*'' = f(y_0). \tag{10.19}$$

Note that since $\lambda > 0$, $\lambda g_* < 0$ and $f(y_0) \geq 0$. Hence, we need $g_*'' < 0$. However, g_* is defined as a local minimum (as we are working on $(0,1)$) and a necessary condition for a local minimum is that $g_*'' \geq 0$. This contradicts (10.19) and so it follows that

$$g(x) = g_*, \qquad \text{for some } x \in \{0,1\}, \tag{10.20}$$

i.e. g attains its infimum at a boundary point x.

It suffices to show that if (10.20) is true, then we get the desired contradiction to (10.3) in the case when $g < 0$. Note firstly that, for some $x \in \{0,1\}$, (10.20) implies $g(y) > g(x)$ for all $y \in (0,1)$. Next from (10.3), for the x such that (10.19) is true, we have

$$g(x) = (P_0^+ g)(x) = \int_0^1 \pi(x,y) g(y) \mathrm{d}y > \int_0^1 \pi(x,y) g(x) \mathrm{d}y = g(x) \int_0^1 \pi(x,y) \mathrm{d}y.$$

Hence,

$$g(x) > g(x) \int_0^1 \pi(x,y) \mathrm{d}y.$$

Since $g(x) < 0$ by our supposition in (10.18), we have

$$\int_0^1 \pi(x,y)\mathrm{d}y > 1.$$

However, this contradicts Lemma 2.8J(d) so that $g(x) \not< 0$. Therefore $g(x) \geq 0$ as desired. □

2.10F. Lemma. *The resolvent equation holds.*

Proof of Lemma 2.10F. For $f \in C[0,1]$, define $g_\mu = \hat{R}_\mu^+ f$ so that we have (10.2) with g_μ instead of g_λ,

$$\mu g_\mu - \tfrac{1}{2} g_\mu'' = f.$$

Then

$$\lambda g_\mu - \tfrac{1}{2} g_\mu'' = (\lambda - \mu) g_\mu + f,$$

so that g_μ satisfies the same conditions as $\hat{R}_\lambda^+\{(\lambda-\mu)g_\mu + f\}$. Therefore, $g_\mu = \hat{R}_\lambda^+\{(\lambda-\mu)g_\mu + f\}$, which is exactly the required result. □

2.10G. Lemma. *We have*

$$(\lambda \hat{R}_\lambda^+ 1)(z) \leq 1 \qquad \textit{for all } z \in [0,1].$$

Proof of Lemma 2.10G. Once again, for a contradiction suppose that

$$\lambda \bar{g} := \sup\{\lambda g(z) : z \in [0,1]\} > 1. \tag{10.21}$$

Next suppose that $g(z_0) = \bar{g}$ for some $z_0 \in (0,1)$. Then by (10.2) with $f \equiv 1$ combined with (10.21) we have

$$\lambda \bar{g} - \tfrac{1}{2}\bar{g}'' = 1 \qquad \Leftrightarrow \qquad \bar{g}'' > 0. \tag{10.22}$$

Recall that we have defined $\bar{g}$ as a local maximum (as we are working on $(0,1)$) and a necessary condition for this is that $\bar{g}'' \leq 0$. Clearly this contradicts (10.22) so that we must have

$$g(x) = \bar{g}, \qquad \text{for some } x \in \{0,1\}, \tag{10.23}$$

i.e. g attains its maximum at a boundary point x.

We need to show that (10.23) contradicts (10.3) in the case when $g > 1$. Note firstly that, for some $x \in \{0,1\}$, (10.23) implies $g(y) < g(x)$ for all $y \in (0,1)$. Next from (10.3), for the x such that (10.23)is true, as before we have

$$g(x) < g(x) \int_0^1 \pi(x,y)\mathrm{d}y$$

Since $g(x) > 1/\lambda > 0$ by our presumption in (10.21), we have

$$\int_0^1 \pi(x,y)\mathrm{d}y > 1,$$

giving a similar contradiction to before. Hence, we get the desired result. □

2.10H. Corollary. *If $f \in C[0,1]$ and $f \leq 1$, then $\hat{R}_\lambda^+ f \leq 1$.*

Proof of Corollary 2.10H. Now $f \leq 1$ implies $1 - f \geq 0$. Hence, by Lemma 2.10E we have

$$\lambda \hat{R}_\lambda^+(1-f) \geq 0 \qquad \Leftrightarrow \qquad \lambda \hat{R}_\lambda^+ f \leq \underbrace{\lambda \hat{R}_\lambda^+ 1 \leq 1},$$

where the lattermost (highlighted) result follows from Lemma 2.10G. □

Proof of Theorem 2.10D. Referring to Definition B.1 of Appendix B, it can be seen that Lemmas 2.10E, 2.10F, 2.10G and Corollary 2.10H give us almost everything we need. Yet we have only deduced that $\lambda \hat{R}_\lambda^+ \mathbf{1} \leq \mathbf{1}$. The desired 'honesty' property can easily be achieved by adding a coffin state ∂ and extending $\hat{R}_\lambda$ in the obvious way. We will later verify that this augmentation is only necessary when $m_0 + m_1 \geq 1$, that is, when $\hat{Z}^+$ has a finite lifetime. □

2.10I. Theorem. $\{\hat{R}_\lambda^+ : \lambda > 0\}$ *is a Ray resolvent.*

In order to deal with the above theorem, it suffices to firstly establish some necessary results.

2.10J. Lemma. *For $f \in C[0,1]$, as $\lambda \to \infty$, we have*

$$\lambda(\hat{R}_\lambda^+ f)(z) \to (\hat{P}_0^+ f)(z),$$

where

$$\begin{aligned}\left(\hat{P}_0^+ f\right)(y) &:= f(y) \quad (y \in (0,1)),\\ \left(\hat{P}_0^+ f\right)(x) &:= \int_{(0,1)} \pi(x,y) f(y)\,\mathrm{d}y \quad (x \in \{0,1\}).\end{aligned}$$

Proof of Lemma 2.10J. The result at the boundaries is obvious given Lemma 2.10C and symmetry. Away from the boundaries, we recall the decomposition in (10.1). We then follow similar arguments to those given in the proof of Lemma 2.10C to show that, for $y \in (0,1)$ and ψ in (10.6),

$$\lambda \psi_\lambda(y,x) \to 0 \quad \text{as } \lambda \to \infty.$$

Hence, all that it remains to do is to prove that, for $y \in (0,1)$,

$$\lambda({}_{\text{tab}}R_\lambda f)(y) \to f(y) \quad \text{as } \lambda \to \infty.$$

Suppose that ζ_λ is an exponentially distributed random variable with rate λ, independent of the process Z. Then, with T as in Section 8, the 'killed' process may be written as follows

$$\begin{aligned}\lambda({}_{\text{tab}}R_\lambda f)(y) &= \mathbb{E}^y\left[f(Z(\zeta_\lambda)) : \zeta_\lambda < T\right]\\ &= \mathbb{E}^y\left[f\left(Z(\tfrac{\zeta_1}{\lambda})\right) : \zeta_1 < \lambda T\right] \qquad \text{(since } \lambda\zeta_\lambda = \zeta_1.\text{)}\end{aligned}$$

Now $f\left(Z(\frac{\zeta_1}{\lambda})\right)$ is clearly bounded by $||f||_{\text{sup}}$ for $f \in C[0,1]$. Moreover, $\zeta_1 < \lambda T$ for λ sufficiently large. Hence, it is clear that, as $\lambda \to \infty$,

$$\lambda({}_{\text{tab}}R_\lambda f)(y) = \mathbb{E}^y\left[f\left(Z(\tfrac{\zeta_1}{\lambda})\right) : \zeta_1 < \lambda T\right] \to \mathbb{E}^y\left[f(Z_0)\right] = f(y).$$

This completes the proof. □

2.10K. Lemma. *$\mathcal{R}$ separates points of $[0,1]$, where $\mathcal{R}$ is the common range of $\hat{R}_\lambda^+$ on $C[0,1]$.*

Proof of Lemma 2.10K. Suppose $\mathcal{R}$ is dense in $C[0,1]$. For a contradiction, suppose that $\mathcal{R}$ *does not* separate points of $[0,1]$. Then given any two *distinct* points z_1 and z_2 in $[0,1]$, we shall have

$$\lambda(\hat{R}_\lambda^+ f)(z_1) = \lambda(\hat{R}_\lambda^+ f)(z_2) \tag{10.24}$$

for all $f \in C[0,1]$ and all $\lambda > 0$. Hence, by taking the limit as $\lambda \to \infty$ in (10.24), by Lemma 2.10J we therefore have

$$(\hat{P}_0^+ f)(z_1) = (\hat{P}_0^+ f)(z_2) \tag{10.25}$$

for **all** $f \in C[0,1]$. It suffices to consider the following three cases:

1. $z_1,\ z_2 \in (0,1)$,
2. $z_1 \in \{0,1\}$ and $z_2 \in (0,1)$,
3. $z_0 = 0$ and $z_1 = 1$.

Case 1. Since $(\hat{P}_0^+ f)(y) = f(y)$ on $(0,1)$, the function $f(y) = y$ yields the desired contradiction to (10.25) and hence to (10.24).

Case 2. Consider the piecewise linear function

$$f(z) = \begin{cases} z_2 - y, & \text{if } y \in [0, z_2], \\ 0, & \text{if } y \in [z_2, 1]. \end{cases}$$

Clearly $f(z)$ is continuous and $f(z) \geq 0$ on $[0,1]$. In particular, $f(z) > 0$ on $[0, z_2)$ and $f(z_2) = 0$. Now suppose that $z_1 = 0$. Then

$$(\hat{P}_0^+ f)(0) = \int_{(0,1)} \pi(0,y) f(y) \mathrm{d}y > 0. \tag{10.26}$$

Note that we only know that $\pi(0,y)$ is non-negative (see Lemma 2.8J). However, observe that $\pi(0,0) = \frac{1}{m_0} > 0$ so that, due to the fact that $\pi(\cdot, y)$ is continuous, there exists a $\delta > 0$ such that $\pi(0,y) > 0$ on $[0,\delta) \subset [0,1]$. Hence, $\pi(0,y)f(y) > 0$ on $[0, \delta \wedge z_2)$. This allows us to state strictly greater than in (10.26).

Result (10.26) together with the fact that $f(z_2) = 0$ gives the desired contradiction. A similar argument applies to the case in which $z_1 = 1$.

Case 3. If $z_1 = 0$ and $z_2 = 1$ then the result in (10.25) is equivalent to

$$\int_{(0,1)} \Pi^{-+}(0, \mathrm{d}y) f(y) = \int_{(0,1)} \Pi^{-+}(1, \mathrm{d}y) f(y),$$

for all $f \in C[0,1]$. Looking forward to Lemma 2.16F, we know that $C[0,1]$ is measure-determining. As a result, here we have

$$\Pi^{-+}(0, B) = \Pi^{-+}(1, B),$$

for all $B \in \mathcal{B}(0,1)$. It is enough to show that

$$\pi(0,y) = \pi(1,y) \qquad \text{for all } y \in (0,1), \tag{10.27}$$

leads to a contradiction. Recalling the form of $\pi(x,y)$ given in Section 5, following some manipulation of the result in (10.27) we have

$$m_1\{h_\beta(1)h_\alpha(y) - h_\alpha(1)h_\beta(y)\} = m_0\{h_\beta(y) - h_\alpha(y)\}.$$

This clearly implies that $h_\beta(1) = h_\alpha(1)$ which we know is false from Lemma 2.8J.

Remark. Considering the constant function $f(z) = h_\beta(1) - h_\alpha(1)$ would give an equivalent, but less theoretic, result.

We now have the desired contradiction in all three cases. It follows that $\mathcal{R}$ must separate points of $[0,1]$. □

Proof of Theorem 2.10I. From Definition B.4 (of Appendix B), we know that a sufficient condition for our honest Feller resolvent $\{\hat{R}^+_\lambda : \lambda > 0\}$ to be a Ray resolvent is that the common range $\mathcal{R} = \hat{R}^+_\lambda C[0,1]$ of the $\hat{R}^+_\lambda$ operators separates points of $[0,1]$. Hence, given Lemma 2.10K we know that $\hat{R}^+_\lambda$ is a Ray resolvent. □

2.10L. Theorem. *There exists a map* $\hat{P}^+_t$ $(t \geq 0)$ *that maps* $C[0,1]$ *into the space* b$\mathcal{B}[0,1]$ *of bounded Borel functions on* $[0,1]$ *such that for* $f \in C[0,1]$ *and* $z \in [0,1]$, $t \mapsto (\hat{P}^+_t f)(z)$ *is right-continuous, and*

$$\int_0^\infty e^{-\lambda t}(\hat{P}^+_t f)(z)\,dt = (\hat{R}^+_\lambda f)(z),$$

$$\hat{P}^+_{s+t} = \hat{P}^+_s \hat{P}^+_t \text{ for } s,t \geq 0,$$

$$f \geq 0 \text{ implies } \hat{P}^+_t f \geq 0, \qquad \text{and} \qquad \hat{P}^+_t \mathbf{1} \leq \mathbf{1}.$$

Proof of Theorem 2.10L. Given that Theorem 2.10I is now proved, the existence of the $\hat{P}^+_t$ maps is guaranteed by the analytic part of Ray's Theorem of Appendix B. □

[For our example, for $t > 0$ and $f \in C[0,1]$, $\hat{P}^+_t f$ is analytic; and the map $t \mapsto (\hat{P}^+_t f)(z)$ is analytic on $(0,\infty)$; but that's for later.]

Note on the probabilistic part of Ray's Theorem. This part implies that there is a right-continuous strong Markov process $\hat{Z}^+$ with $\{\hat{P}^+_t\}$ as its transition function. Of course, for a boundary point x, the $\mathbb{P}^x$ distribution of $\hat{Z}^+_0$ has density $\pi(x,\cdot)$. We could now prove that the process $\hat{Z}^+$ does behave like Brownian motion inside $(0,1)$ and always jumps in the correct way from the boundary points. Yet, we do not need the probabilistic part of Ray's Theorem. We shall eventually prove the result we really require, namely that $\hat{P}^+_t$ $(t \geq 0)$ is the transition semigroup of $Z^+ = \{Z(\tau^+_t) : t \geq 0\}$. However, we concentrate principally on the resolvent $\{\hat{R}^+_\lambda : \lambda > 0\}$ rather than on the semigroup $\{\hat{P}^+_t\}$.

Connections with the PDE approach. The PDE approach makes us believe that, for $f \in C[0,1]$,

$$\partial_y \left\{(\hat{P}_t^+ f)(y)\right\}\Big|_{y=0} + 2m_0 \partial_t \left\{(\hat{P}_t^+ f)(0)\right\} = 0, \tag{10.28}$$

whence, on taking Laplace transforms, we would have for $g_\lambda := \hat{R}_\lambda^+ f$, equation (10.2) with the different boundary condition

$$g_\lambda'(0) + 2m_0 \left\{\lambda g_\lambda(0) - \int_0^1 \pi(0,z) f(z)\,\mathrm{d}z\right\} = 0, \tag{10.29}$$

with an analogous boundary condition at 1. Referring back to (10.9), one can check that $g_\lambda = \hat{R}_\lambda^+ f$ does have this property in that

$$\rho B_\lambda(f) + 2\lambda m_0 A_\lambda(f) - 2m_0 \int_{[0,1]} \pi(0,z) f(z)\,\mathrm{d}z = 0.$$

Indeed this is exactly what equation (10.10) states. Of course, there is a corresponding result at 1. **The boundary conditions tally,** which proves to be a key result of the thesis.

A direct proof of (10.29). We shall utilize our explicit formulae such as (10.12) later, but it is important to note the crucial point that (10.29) follows from (10.2) and (10.3) may be proved directly without complicated calculations. Because of (10.2), the desired result (10.29) is equivalent to

$$g'(0) + m_0 \int_0^1 \pi(0,y) g''(y)\mathrm{d}y = 0. \tag{10.30}$$

Now we know that $\pi(0,\cdot)$ is a linear combination of $h_\alpha(\cdot)$ and $h_\beta(\cdot)$. Moreover, since, for $\gamma \in \{\alpha,\beta\}$, $\frac{1}{2}\gamma^2 h_\gamma = \mathcal{H} h_\gamma$, we have

$$\gamma^2 h_\gamma = h_\gamma'', \quad m_0 h_\gamma''(0) = -h_\gamma'(0), \quad m_1 h_\gamma''(1) = h_\gamma'(1),$$

whence

$$m_0 \gamma^2 h_\gamma(0) = -h_\gamma'(0), \quad m_1 \gamma^2 h_\gamma(1) = h_\gamma'(1).$$

Using these facts, we may derive (10.30) via integration by parts. Lurking in the background are the facts that $P_t^- h_\gamma = \mathrm{e}^{-\frac{1}{2}\gamma^2 t} h_\gamma$ and the intuitive equation (2.3). □

We may combine equations (10.2) and (10.29) to obtain, for $f \in C[0,1]$ and $\lambda > 0$,

$$(\hat{R}_\lambda^+ f)'(0) + m_0 (\hat{R}_\lambda^+ f)''(0) + 2m_0 \left\{f(0) - \int_0^1 \pi(0,z) f(z)\mathrm{d}z\right\} = 0,$$

whence we see that

$$\text{for } f \in C[0,1] \text{ and } \lambda > 0, \text{ we have } \hat{R}_\lambda^+ f \in \mathcal{D}(\mathcal{H}) \text{ if and only if } \hat{P}_0^+ f = f. \tag{10.31}$$

2.10M. Important comment. The reader may feel that we should have required (10.2) only when $\hat{P}^+_0 f = f$, and then we would have

$$(\hat{R}^+_\lambda f)(0) = A_\lambda(f) \quad \text{whenever } f(x) = \int_0^1 \pi(x,y)f(y)\mathrm{d}y \quad (x \in \{0,1\}).$$

But then (compare Theorem 2.4C) for some $a_0, a_1 \in \mathbb{R}$, we would have for *all* $f \in C[0,1]$,

$$(\hat{R}^+_\lambda f)(0) = A_\lambda(f) + a_0[f(0) - \int_0^1 \pi(0,y)f(y)\mathrm{d}y] + a_1[f(1) - \int_0^1 \pi(1,y)f(y)\mathrm{d}y],$$

and, *since, as is probabilistically obvious, we do not wish* $\hat{R}^+_\lambda$ *to have an atoms at* 0 *or* 1, we want $a_0 = a_1 = 0$.

Eigenfunctions of $\hat{R}^+_\lambda$ *in* L^2. Let $\lambda > 0$. Suppose that $f \in L^2 = L^2((0,1), \text{Leb})$ is not the zero element and that $\hat{R}^+_\lambda f = cf$ for some non-zero constant c. (The possibility that $c = 0$ will be excluded later.) Think of f as a function in $\mathcal{L}^2$ rather than as an equivalence class of such functions. The Dominated-Convergence Theorem shows that $\hat{R}^+_\lambda f \in C[0,1]$, whence, *because* $c \neq 0$, we have $f \in C[0,1]$. It is therefore clear that

$$\hat{P}^+_0 \hat{R}^+_\lambda f = \hat{R}^+_\lambda f. \tag{10.32}$$

This result is obvious away from the boundaries, and at the boundaries, (10.3) gives the desired result. From (10.32) we have

$$\hat{P}^+_0 f = f.$$

By (10.31) it therefore follows that $\hat{R}^+_\lambda f \in \mathcal{D}(\mathcal{H})$ and recalling (10.2) we see that $(\lambda - \mathcal{H})\hat{R}^+_\lambda f = f$. Hence,

$$(\lambda - \mathcal{H})cf = f,$$

so that

$$\mathcal{H}f = (\lambda - c^{-1})f, \tag{10.33}$$

so that f is an eigenfunction of $\mathcal{H}$. Yet, we already know what the eigenfunctions are; namely, h_γ and f_θ. We have deduced that $\hat{P}^+_0 f = f$, which implies that

$$\langle h_\gamma, f\rangle_s = 0 \qquad \text{for all } \gamma \in \Gamma.$$

Since $h_{\tilde{\gamma}}$ is $\langle\text{negative}\rangle_s$, it follows that

$$f \text{ cannot be a multiple of } h_{\tilde{\gamma}} \text{ where } \tilde{\gamma} \in \Gamma \cap (0,\infty). \tag{10.34}$$

It now suffices to show that f **must be a multiple of** f_θ **for some** $\theta \in \Theta$. We achieve this by considering the possible cases for $\sum m_i$.

Case 1: $m_0 + m_1 < 1$. Here α and β are both positive and so the desired result follows from (10.34).

Case 2: $m_0 + m_1 = 1$. In this case $\alpha = 0$ and $\beta > 0$. Once again, f is not a multiple of h_β by (10.34). However, f is a multiple of $h_\alpha = 1$. Observe that

$$\langle h_\alpha, h_\alpha\rangle_s = \langle 1,1\rangle_s = 0 \qquad \text{and} \qquad \langle 1, h_\beta\rangle_s = 0 \quad \text{(see Lemma 2.3G)}.$$

However, $0 \in \Theta$ so that

$$h_\alpha = f_\theta\big|_{\theta=0} = 1,$$

which is the appropriate result.

Case 3: $m_0 + m_1 > 1$. As in the previous case $\alpha = 0$ and $\beta > 0$. Again f is not a multiple of h_β by (10.34). Moreover, it is also clear that f is not a multiple of $h_\alpha = 1$ as

$$\langle h_\alpha, h_\alpha\rangle_s = \langle 1, 1\rangle_s = 1 - (m_0 + m_1) < 0.$$

We have therefore verified that f is a multiple of f_θ for some $\theta \in \Theta$. Given that $\mathcal{H} f_\theta = -\frac{1}{2}\theta^2 f_\theta$, from (10.33) we may deduce that

$$c = (\lambda + \tfrac{1}{2}\theta^2)^{-1}.$$

Thus, for $\theta \in \Theta$, f_θ *is* an eigenfunction of $\hat{R}^+_\lambda$ corresponding to eigenvalue $(\lambda + \frac{1}{2}\theta^2)^{-1}$.

Alternative proof of Proposition 2.3R. Recall that the 'only if' part of the problem is trivial. If f is of the form

$$f(y) = A\cos(\theta y) + B\sin(\theta y),$$

for some constants A and B, then it is obvious that $f \in L^2\big((0,1), \mathrm{Leb}\big)$. We basically 'mimic' the previous discussion. This requires that f is an eigenfunction of $\mathcal{H}$. Thus it is enough to show that

$$\langle h_\gamma, f\rangle_s = 0 \quad \text{for } \gamma \in \Gamma \qquad \Rightarrow \qquad f \text{ is an eigenfunction of } \mathcal{H}.$$

Clearly, we can explicitly show that

$$(\mathcal{H}f)(y) = \tfrac{1}{2}f''(y) = -\tfrac{1}{2}\theta^2 f(y).$$

Hence, all that it remains to do is to show that $f \in \mathcal{D}(\mathcal{H})$, so that the conditions in (1.1) are satisfied. Now,

$$\begin{aligned} &\langle h_\gamma, f\rangle_s = 0 \quad \Leftrightarrow \quad \theta^2\langle h_\gamma, f\rangle_s = 0 \quad \Leftrightarrow \quad \langle h_\gamma, 2\mathcal{H}f\rangle_s = 0 \\ \Leftrightarrow \quad &\int_0^1 h_\gamma(y)2(\mathcal{H}f)(y)\,\mathrm{d}y - 2m_0(\mathcal{H}f)(0) - 2m_0 h_\gamma(1)(\mathcal{H}f)(1) = 0. \end{aligned}$$

We know that the above result is true for all $\gamma \in \Gamma = \{\alpha, \beta\}$. Hence, if we multiply the result for α by $h_\beta(1)$, the result for β by $h_\alpha(1)$, and subtract the corresponding equations we get

$$\begin{aligned} &\int_0^1 \{h_\beta(1)h_\alpha(y) - h_\alpha(1)h_\beta(y)\}(y)2(\mathcal{H}f)(y)\,\mathrm{d}y - 2m_0(\mathcal{H}f)(0)\{h_\beta(1) - h_\alpha(1)\} = 0 \\ \Leftrightarrow \quad &\int_0^1 \pi(0,y)f''(y)\,\mathrm{d}y - f''(0) = 0. \end{aligned}$$

Using (10.30), we may substitute for the integral to yield the boundary condition for 0. Of course, due to the simple form of $\pi(1, y)$, it is even easier to establish the analogous result at 1.

Since $\mathcal{H}f = -\frac{1}{2}\theta^2 f$ and $f \in \mathcal{D}(\mathcal{H})$, it follows that f must be a multiple of f_θ for some $\theta \in \Theta$. □

2.11. $\hat{R}_\lambda^+$ on the Hilbert Space L_+^2 when $m_0 + m_1 \neq 1$

We write L^2 for the standard $L^2(0,1)$ space.

2.11A. Definition. *Conjecture 2.2G and our belief in the form of P_0^+ suggest that we define for real-valued functions f, g on $(0,1)$,*

$$\langle f, g\rangle_+ = \int_{(0,1)} \big((I - W^+)f\big)(y)g(y)\mathrm{d}y, \tag{11.1}$$

where

$$(W^+ f)(y_1) := \int_{(0,1)} W^+(y_1, y_2)f(y_2)\mathrm{d}y_2, \tag{11.2}$$

$W^+(\cdot,\cdot)$ denoting the symmetric kernel

$$W^+(y_1, y_2) := \sum_{x\in\{0,1\}} m_x\pi(x, y_1)\pi(x, y_2). \tag{11.3}$$

We make the 'dual' definition of the $\{0,1\} \times \{0,1\}$-matrix W^- as

$$W^-(x_1, x_2) := \int_{(0,1)} \pi(x_1, y)m_{x_2}\pi(x_2, y)\mathrm{d}y. \tag{11.4}$$

Intuitive interpretation. The operator W^+ should represent the *full-winding operator*

$$(W^+f)(y) = \mathbb{E}^y f(Z_T), \quad \text{where } T := \inf\{t > \tau_0^- : \Phi(t) > 0\}.$$

This is because for $x \in \{0,1\}$ and $y \in (0,1)$, we proved in Lemma 2.8J that $\mathbb{P}^y(Z_0^- = x) = m_x\pi(x,y)$ and we believe that $\mathbb{P}^x(Z_0^+ \in \mathrm{d}y) = \pi(x,y)\mathrm{d}y$. The conjectured dual interpretation of W^- is now clear, and this makes the first part of the following lemma intuitively obvious.

2.11B. Lemma. *W^- is a substochastic 2×2 matrix and is stochastic if and only if $m_0+m_1 = 1$. Furthermore, W^- has distinct real eigenvalues λ_W and μ_W which may be labelled so that $-1 < \lambda_W < \mu_W \leq 1$, and that $\mu_W = 1$ if and only if $m_0 + m_1 = 1$.*

Proof of Lemma 2.11B. Note that W^- is self-adjoint on $L^2(\{0,1\}, m)$. We established in Lemma 2.8J(b) that the π's are non-negative. It follows that $W^-(x_1, x_2) \geq 0$ as desired. Thus, all that it remains to do is to show that

$$S(x_1) := \sum_{x_2\in\{0,1\}} W^-(x_1, x_2) \leq 1 \quad \text{for } x_1 \in \{0,1\}. \tag{11.5}$$

Suppose that $m_0 + m_1 < 1$. Now, from (11.4) we have

$$\begin{aligned} S(x_1) &= W^-(x_1, 0) + W^-(x_1, 1) \\ &= \int_0^1 \pi(x_1, y)m_0\pi(0, y)\mathrm{d}y + \int_0^1 \pi(x_1, y)m_1\pi(1, y)\mathrm{d}y \\ &= \int_0^1 \pi(x_1, y) \sum_{x\in\{0,1\}} m_x\pi(x, y)\ \mathrm{d}y < \int_0^1 \pi(x_1, y)\ \mathrm{d}y = 1, \end{aligned}$$

where the conclusions in the last line follow from Lemma 2.8J(d). Note that we get an equivalent result if $m_0 + m_1 > 1$, except that it suffices to interchange the 'less than' symbol and the final 'equals' sign. Moreover, if $m_0 + m_1 = 1$ then Lemma 2.8J(d) again makes it clear that we get equality throughout so yielding a strictly stochastic matrix. We now know that (11.5) holds in all cases.

Eigenvalues of W^-. We have already remarked that $W^-(\cdot,\cdot) \geq 0$. In fact, $W^-(\cdot,\cdot) > 0$. Referring to the proof of Corollary 2.10K, we have already found that $\exists\ \delta > 0$ such that $\pi(0,y) > 0$ on $[0,\delta)$. We may establish a similar result for $\pi(1,y)$ in that $\pi(1,y) > 0$ on $(\delta, 1]$, where $\delta < 1$. This makes it trivial to show that both $W^-(0,0)$ and $W^-(1,1)$ are strictly positive.

The fact that W^- is self-adjoint on $L^2(\{0,1\},m)$ allows us to deduce the fact that $W^-(1,0) > 0$ is equivalent to $W^-(0,1) > 0$. Thus, it is enough to deal with only one of the terms.

Suppose for a contradiction that $W^-(0,1) = 0$. Then, as the π's are non-negative, we must have

$$\pi(0,y)\pi(1,y) = 0 \qquad \text{for all } y \in (0,1).$$

However, given an earlier remark, this implies that

$$\pi(1,y) = 0 \qquad \text{for all } y \in (0,\delta).$$

Given the form of π in (4.1), this trivially implies $h_\alpha(y) = h_\beta(y)$ on such a neighbourhood. This means $\alpha = \beta$, which gives the desired contradiction.

We now define $a, b, c, d > 0$ in the obvious way so that

$$W^- = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Furthermore, we have established that W^- is substochastic so that we may now deduce that $a, b, c, d < 1$. As usual, we solve

$$\begin{aligned} & \det(\lambda I - W^-) = 0 \\ \Leftrightarrow\quad & \lambda^2 - \lambda(a+d) + ad - bc = 0 \\ \Leftrightarrow\quad & 2\lambda = (a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)} \end{aligned} \tag{11.6}$$

Define

$$R := (a+d)^2 + 4bc - 4ad.$$

Due to the fact that the entries of W^- are strictly positive, observe that

$$R = (a-d)^2 + 4bc > 0.$$

It follows that the underlying eigenvalues of W^- are indeed real and distinct. We define the largest such eigenvalue to be μ_W and the remaining, strictly smaller, eigenvalue to be λ_W.

Firstly suppose that $m_0 + m_1 = 1$. Then, since W^- is strictly stochastic, we know that $b = 1 - a$ and $c = 1 - d$. Substituting these facts into R we find that

$$R = (a + d - 2)^2.$$

Thus, from (11.6) we now have

$$2\lambda = a + d \pm (a + d - 2),$$

so that

$$\mu_W = 1 \qquad \text{and} \qquad \lambda_W = (a + d) - 1 \in (-1, 1).$$

The lattermost result simply follows from the fact that the entries of W^- lie strictly in $(0, 1)$.

Now suppose that $m_0 + m_1 \neq 1$. Then, since W^- is *strictly* sub-stochastic we now have $b < 1 - a$ and $c < 1 - d$. This gives

$$R < (a + d - 2)^2,$$

so that in particular we have

$$-\sqrt{R} > a + d - 2 \qquad \text{and} \qquad +\sqrt{R} < a + d - 2.$$

Using these simple facts in (11.6) this time yields

$$\mu_W < (a + d) - 1 < 1, \qquad \text{and} \qquad \lambda_W > (a + d) - 1 > -1.$$

Hence, we may conclude that $-1 < \lambda_W < \mu_W \leq 1$. □

If some of the following wording seems rather curious, it is because we have to allow for the possibility that $\lambda_W = 0$. For $x \in \{0, 1\}$, let $\ell_x = \pi(x, \cdot) \in L^2$. Then

$$W^+ f = m_0 \langle \ell_0, f \rangle_{L^2} \ell_0 + m_1 \langle \ell_1, f \rangle_{L^2} \ell_1,$$

and

$$W^-(x_1, x_2) = m_{x_2} \langle \ell_{x_1}, \ell_{x_2} \rangle_{L^2}.$$

If a is a row eigenvector of W^- so that $(a_0\ a_1) W^- = (a_0\ a_1)\lambda$, then

$$W^+(a_0 \ell_0 + a_1 \ell_1) = \lambda (a_0 \ell_0 + a_1 \ell_1).$$

So, row eigenvectors of W^- give rise to eigenfunctions of W^+. Via this correspondence, we get a 1-dimensional subspace L_μ^2 [respectively, L_λ^2] of L^2 consisting of eigenfunctions of W^+ associated with eigenvalue μ_W [respectively, λ_W]. We let L_r^2 denote the orthogonal complement of $L_\mu^2 + L_\lambda^2$ in L^2. Clearly,

$$L_r^2 = \{ f : \langle \ell_0, f \rangle_{L^2} = 0 = \langle \ell_1, f \rangle_{L^2} \},$$

and every element f of L_r^2 satisfies $W^+ f = 0$.

Let J_μ, J_λ, J_r be orthogonal projections onto $L_\mu^2, L_\lambda^2, L_r^2$ respectively. Then, for $f \in L^2$, we have, remembering that W^+ is self-adjoint on L^2,

$$f = J_\mu f + J_\lambda f + J_r f \quad \text{(orthogonal decomposition)}$$

and

$$\langle f, f \rangle_+ = (1-\mu_W)\langle J_\mu f, J_\mu f\rangle_{L^2} + (1-\lambda_W)\langle J_\lambda f, J_\lambda f\rangle_{L^2} + \langle J_r f, J_r f\rangle_{L^2}. \tag{11.7}$$

Note that

$$0 \leq \langle f, f\rangle_+ \leq 2\langle f, f\rangle_{L^2} \leq 2K\langle f, f\rangle_+, \tag{11.8}$$

where $K = \max\{1, (1-\lambda_W)^{-1}, (1-\mu_W)^{-1}\} \leq \infty$, a fact that turns out to be extremely useful.

Important note. It is essential to realize that the projections J_μ, J_λ, J_r, arising from the spectral decomposition of W^+ on L^2, have no connection with a spectral decomposition of $\hat{R}_\lambda^+$. Remember that $\hat{R}_\lambda^+$ is *not* self-adjoint on L^2, something reflected in the fact that $\hat{Z}$ can jump out from 0 but cannot jump into 0.

Suppose for the remainder of this section that $m_0+m_1 \neq 1$. Then $\mu_W < 1$ and K in (11.8) is finite. Furthermore, $\langle \cdot, \cdot \rangle_+$ is an inner product on L^2 defining the same topology as the standard topology of L^2. We write L_+^2 for L^2 with the $\langle \cdot, \cdot\rangle_+$ inner product.

2.11C. Proposition. *$\hat{R}_\lambda^+$ is compact on L_+^2.*

Proof of Proposition 2.11C. Let $\lambda > 0$. It is clear from decomposition (10.1) that $\hat{R}_\lambda^+$ is the sum of ${}_{\text{tab}}R_\lambda$, which is well-known to be compact on L^2, and an operator of rank 2. Hence $\hat{R}_\lambda^+$ is compact on L^2 and, due to the "same topology" comment, is therefore compact on L_+^2. Let us examine this point in further detail.

Let f_n be a bounded sequence in L_+^2. By the lattermost inequality in (11.8) it can be seen that f_n is bounded in L^2. Since $\hat{R}_\lambda^+$ is compact on L^2, the sequence $\hat{R}_\lambda^+ f_n$ must have a convergent subsequence. That is, there is a subsequence $\hat{R}_\lambda^+ f_{n_k} \to g$ in L^2. Appealing to the former inequality in (11.8) , it now follows that

$$\left|\left|\hat{R}_\lambda^+ f_{n_k} - g\right|\right|_{L_+^2} \leq 2\left|\left|\hat{R}_\lambda^+ f_{n_k} - g\right|\right|_{L^2} \to 0.$$

□

2.11D. Proposition. *$\hat{R}_\lambda^+$ is self-adjoint relative to $\langle \cdot, \cdot \rangle_+$.*

Proof of Proposition 2.11D. Again let $\lambda > 0$. Since $\hat{R}_\lambda^+$ is a bounded operator on L_+^2 and

$$S_R := \{f \in C[0,1] : \hat{P}_0^+ f = f\} \tag{11.9}$$

is dense in L_+^2 (see Instructive Example 2.16J for justification of a similar result), it is enough to show that

$$\text{for } f, u \in S_R, \text{ we have } \langle \hat{R}_\lambda^+ f, u\rangle_s = \langle f, \hat{R}_\lambda^+ u\rangle_s.$$

(Note that for $f, u \in S_R$, we have $\langle f, u\rangle_s = \langle f, u\rangle_+$.)

So suppose that $f, u \in S_R$, and write $g = \hat{R}^+_\lambda f$, $v = \hat{R}^+_\lambda u$. We know from (10.31) that $g, v \in \mathcal{D}(\mathcal{H})$ and that

$$\lambda g - \mathcal{H}g = f, \qquad \lambda v - \mathcal{H}v = u.$$

Hence, since $\mathcal{H}$ is symmetric relative to $\langle\cdot,\cdot\rangle_s$, we have

$$\lambda\langle g, v\rangle_s - \langle f, v\rangle_s \;=\; \langle \mathcal{H}g, v\rangle_s \;=\; \langle g, \mathcal{H}v\rangle_s \;=\; \lambda\langle g, v\rangle_s - \langle g, u\rangle_s,$$

so that $\langle f, v\rangle_s = \langle g, u\rangle_s$, as required. □

Spectral structure of $\hat{R}^+_\lambda$. We now know that on $\mathcal{L}^2$, $\hat{R}^+_\lambda$ is a compact self-adjoint operator with $\{f_\theta : \theta \in \Theta\}$ as its eigenfunctions normalized to be 1 at 0; moreover,

$$\hat{R}^+_\lambda f_\theta \;=\; (\lambda + \tfrac{1}{2}\theta^2)^{-1} f_\theta \qquad (\theta \in \Theta). \tag{11.10}$$

2.11E. Lemma. $\{f_\theta : \theta \in \Theta\}$ *is a complete orthogonal basis for* L^2_+.

Proof of Lemma 2.11E. The result follows by the Spectral Theorem for compact self-adjoint operators (see Theorem 2.17A in Appendix 2 of this chapter). □

To tie things together, we now require the following Lemma.

2.11F. Lemma. 0 *is not an eigenvalue of* $\hat{R}^+_\lambda$.

Proof of Lemma 2.11F. Recall that this result simply amounts to saying that $c \neq 0$ in the eigenfunctions of $\hat{R}^+_\lambda$ discussion. Because the maximum modulus of the eigenvalues of a compact self-adjoint operator is equal to the norm of that operator (look forward to Theorem 2.17A), we know that for $\mu > 0$, $\mu\hat{R}^+_\mu$ has norm at most 1 on L^2_+. Since $C[0,1]$ is dense in L^2_+ and $\mu\hat{R}^+_\mu f(y) \to f(y)$ for $f \in C[0,1]$ and $y \in (0,1)$ (refer back to Lemma 2.10J), we have $\mu\hat{R}^+_\mu f \to f$ in L^2_+ for $f \in L^2_+$. If now $\hat{R}^+_\lambda f = 0$ for some $f \in L^2_+$, then, by the resolvent equation, $\mu\hat{R}^+_\mu f = 0$ in L^2_+ for all $\mu > 0$, so that $f = 0$ in L^2_+. *Clarification.* The reason we had to treat this differently is that we cannot directly deduce any smoothness property of an element f of L^2_+ from the fact that $\hat{R}^+_\lambda f = 0$ in L^2_+. □

Spectral structure of $\hat{P}^+_t$ $(t > 0)$. For $\theta \in \Theta$ and $z \in [0,1]$, the map $t \mapsto (\hat{P}^+_t f_\theta)(z) = \mathbb{E}^z f_\theta(\hat{Z}_t)$ is right-continuous and

$$\int_0^\infty e^{-\lambda t}(\hat{P}^+_t f_\theta)(z)\,dt \;=\; (\hat{R}^+_\lambda f_\theta)(z) \;=\; (\lambda + \tfrac{1}{2}\theta^2)^{-1} f_\theta(z),$$

whence $(\hat{P}^+_t f_\theta)(z) = e^{-\frac{1}{2}\theta^2 t} f_\theta$. We now arrive at the following extremely useful theorem.

2.11G. Theorem. *For the* $\{\hat{P}^+_t : t > 0\}$ *semigroup we have the following spectral expansion*

$$\hat{P}^+_t f \;=\; \sum_{\theta\in\Theta} e^{-\frac{1}{2}\theta^2 t}\frac{\langle f, f_\theta\rangle_+}{\langle f_\theta, f_\theta\rangle_+} f_\theta \qquad (t > 0). \tag{11.11}$$

Proof of Theorem 2.11G. From Corollary 2.17B (from Section 17, which is an Appendix to this chapter) , for some $\varphi \in \Theta$, we have

$$f = \sum_{\theta \in \Theta} c_\theta f_\theta, \quad \text{so that} \quad \langle f, f_\varphi \rangle_+ = \sum_{\theta \in \Theta} c_\theta \langle f_\theta, f_\varphi \rangle_+ \tag{11.12}$$

for some constants c_θ. Due to orthogonality, the RHS of the second equation in (11.12) is non-zero if and only if $\varphi = \theta$. We therefore have

$$\langle f, f_\theta \rangle_+ = c_\theta \langle f_\theta, f_\theta \rangle_+,$$

so that from the former result in (11.12) we get

$$f = \sum_{\theta \in \Theta} \frac{\langle f, f_\theta \rangle_+}{\langle f_\theta, f_\theta \rangle_+} f_\theta.$$

From linearity we may now deduce that

$$\hat{P}_t^+ f = \sum_{\theta \in \Theta} \frac{\langle f, f_\theta \rangle_+}{\langle f_\theta, f_\theta \rangle_+} (\hat{P}_t^+ f_\theta) = \sum_{\theta \in \Theta} e^{-\frac{1}{2}\theta^2 t} \frac{\langle f, f_\theta \rangle_+}{\langle f_\theta, f_\theta \rangle_+} f_\theta,$$

since $(\hat{P}_t^+ f_\theta) = e^{-\frac{1}{2}\theta^2 t} f_\theta$. □

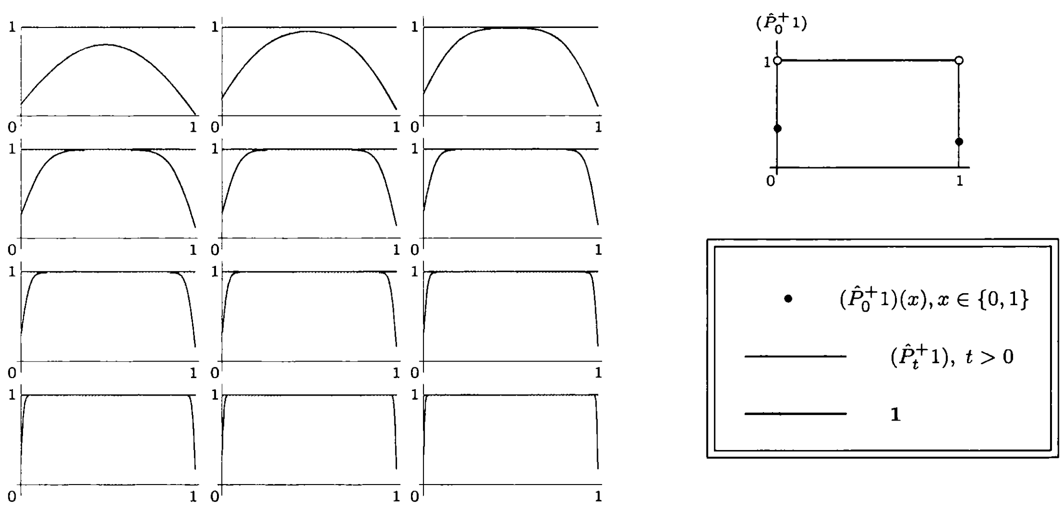

Figure 2.3: $(\hat{P}_t^+ 1)(z)$ for small t and $m_0 + m_1 > 1$.

2.11H. Note. In fact, for $f \in L_+^2$, $t \mapsto \hat{P}_t^+ f$ is strongly continuous over time-interval $[0, \infty)$. The space L_+^2 cannot see the difference between $\hat{P}_0^+$ and the identity map on functions on $[0, 1]$. The space $C[0, 1]$ certainly can. Look forward to Instructive Example 2.16K which demonstrates this point.

2.12. The Probabilistic Semigroup $\{P_t^+ : t \geq 0\}$ when $m_0 + m_1 \neq 1$

Suppose that $m_0 + m_1 \neq 1$. We now confirm that $\hat{P}_t^+ = P_t^+$ for $t \geq 0$. Let $f \in C[0,1]$. It is immediate from (11.11) that $(t,z) \mapsto (\hat{P}_t^+ f)(z)$ is $C^{1,2}$ on $(0,\infty) \times [0,1]$. Moreover, it is an easy exercise to deduce from expansion (11.11) that

$$(\partial_t - \tfrac{1}{2}\partial_z^2)(\hat{P}_t^+ f)(z) = 0.$$

If we put $t = -\varphi$ (so that $\varphi \in (-\infty, 0)$), then the previous result is equivalent to

$$(\partial_\varphi + \tfrac{1}{2}\partial_z^2)(\hat{P}_{-\varphi}^+ f)(z) = 0. \tag{12.1}$$

Hence, in particular, for $z = 0$ we have

$$\partial_\varphi \left\{(\hat{P}_{-\varphi}^+ f)(0)\right\} = -\tfrac{1}{2}\partial_z^2 \left\{(\hat{P}_{-\varphi}^+ f)(z)\right\}\Big|_{z=0}. \tag{12.2}$$

We know that (10.29) holds, and we may now invert the Laplace transform to show that (10.28) holds. Thus, putting $t = -\varphi$ in (10.28), we have

$$\partial_y \left\{(\hat{P}_{-\varphi}^+ f)(z)\right\}\Big|_{z=0} - 2m_0\partial_\varphi \left\{(\hat{P}_{-\varphi}^+ f)(0)\right\} = 0. \tag{12.3}$$

We may now substitute (12.2) into (12.3) in order to eliminate the ∂_φ term and get

$$\partial_z \left\{(\hat{P}_{-\varphi}^+ f)(z)\right\}\Big|_{z=0} + m_0\partial_z^2 \left\{(\hat{P}_{-\varphi}^+ f)(z)\right\}\Big|_{z=0} = 0. \tag{12.4}$$

Next, for $(\varphi, z) \in (-\infty, 0] \times [0,1]$, if we define

$$\hat{F}(\varphi, z) := \left(\hat{P}_{-\varphi}^+ f\right)(z), \tag{12.5}$$

then, for $\varphi < 0$, (12.4) is equivalent to

$$\hat{F}'(\varphi, 0) + m_0 \hat{F}''(\varphi, 0) = 0.$$

Clearly there is a corresponding result at 1. Moreover, the result in (12.1) is exactly the PDE in (1.2) . We therefore know that for every $\varphi < 0$ and $f \in C[0,1]$, we have $\hat{P}_{-\varphi}^+ f \in \mathcal{D}(\mathcal{H})$. Moreover, given Lemma 2.10J , we know that $\hat{F}$ has final condition

$$\hat{F}(0-, y) = f(y) \qquad (y \in (0,1)).$$

Now, by Theorem 2.7A, $\hat{F}$ satisfies the condition that

$$\hat{M}_t := \hat{F}\big(\Phi(t \wedge \tau_0^+), Z(t \wedge \tau_0^+)\big)$$

defines a local martingale under $\mathbb{P}^{\varphi,z}$ (with $z \in [0,1]$, $\varphi < 0$), indeed a bounded martingale.

If $m_0 + m_1 < 1$, then τ_0^+ is almost surely finite. Thus, for $(\varphi, z) \in (-\infty, 0) \times [0, 1]$, by the Optional-Stopping Theorem we have,

$$\begin{aligned}\hat{F}(\varphi, z) &= \mathbb{E}^{\varphi,z}\hat{F}\big(\Phi(0), Z(0)\big) = \mathbb{E}^{\varphi,z}\hat{F}\big(\Phi(\tau_0^+), Z(\tau_0^+)\big)\\ &= \mathbb{E}^{\varphi,z}\hat{F}(0, Z_0^+) = \mathbb{E}^{\varphi,z} f(Z_0^+),\end{aligned}$$

so that, recalling (7.1) , $(\hat{P}_{-\varphi}^+ f)(z) = (P_{-\varphi}^+ f)(z)$. Noting that (a.s.) $Z_0^+ \in (0, 1)$, the lattermost result in the above is simply the final condition for $\hat{F}$.

Now suppose that $m_0 + m_1 > 1$. Then $0 \notin \Theta$, and it is obvious from the spectral expansion that $(\hat{P}_t^+ f)(z) \to 0$ as $t \to \infty$, uniformly in z. Hence, since $\Phi(t) \to -\infty$ (a.s), we have $\hat{M}_t \to 0$ as $t \to \infty$ (a.s.) on $\{\tau_0^+ = \infty\}$. Thus, for $(\varphi, z) \in (-\infty, 0) \times [0, 1]$, this time the OST gives,

$$\begin{aligned}\hat{F}(\varphi, z) &= \mathbb{E}^{\varphi,z}\hat{F}\big(\Phi(0), Z(0)\big) = \mathbb{E}^{\varphi,z}\hat{F}\big(\Phi(\tau_0^+), Z(\tau_0^+)\big)\\ &= \mathbb{E}^{\varphi,z}\hat{M}_\infty \;=\; \mathbb{E}^{\varphi,z}\left\{f(Z_0^+);\ \tau_0^+ < \infty\right\},\end{aligned}$$

so that, again, $(\hat{P}_{-\varphi}^+ f)(z) = (P_{-\varphi}^+ f)(z)$.

2.12A. Remarks. We do not include $\varphi = 0$ in our application of the Optional-Stopping Theorem since $\hat{F}(\varphi, z)$ is only $C^{1,2}$ on $(-\infty, 0) \times [0, 1]$. In addition, our *'almost surely'* references here clearly mean with $\mathbb{P}^{\varphi,z}$ probability 1. It can easily be seen that this is all in order if we recall (7.2) .

One cannot overemphasize the importance of verifying the details of the application of the Optional-Stopping Theorem. The next chapter will confirm this fact.

We can now forget the 'hat' notations, except in the case when $m_0 + m_1 = 1$. *Note that we have proved WH1 when* $m_0 + m_1 \neq 1$. However, for completeness, it now seems appropriate to link our probabilistic description of $P_0^+ f$ for $\Phi(0) = 0$ to the analytical form for $\hat{P}_0^+ f$ as found in Lemma 2.10J.

Clarification of $\Phi(0) = 0$ *situation.* If $y \in (0, 1)$ then, under $\mathbb{P}^y$, $\tau_0^+ = 0$ almost surely. Then, for $f \in C[0, 1]$,

$$(P_0^+ f)(y) = \mathbb{E}^y\left[f(Z_0^+); \tau_0^+ < \infty\right] = \mathbb{E}^y\left[f(Z_0)\right] = f(y) = (\hat{P}_0^+ f)(y).$$

If $x \in \{0, 1\}$ then, under $\mathbb{P}^x$, $\tau_0^+ \neq 0$ almost surely. Here, for $f \in C[0, 1]$, we have

$$\begin{aligned}(P_0^+ f)(x) &= \mathbb{E}^x\left[f(Z_0^+); \tau_0^+ < \infty\right]\\ &= \int_0^1 \mathbb{P}^x(Z_0^+ \in \mathrm{d}y)\ f(y) && \text{(by definition of expectation)}\\ &= \int_0^1 \Pi^{-+}(x, \mathrm{d}y) f(y) && \text{(by Definition 2.6F)}\\ &= \int_0^1 \pi(x, y) f(y)\mathrm{d}y && \text{(by Corollary 2.2C)}\\ &= (\hat{P}_0^+ f)(x).\end{aligned}$$

We may now conclude that our probabilistic situation for $\varphi = 0$ agrees with our final conditions for $\hat{F}$, as established in Lemma 2.10J.

Answering Question 2.4B. We emphasize that for $t > 0$, $P_t^+ : C[0,1] \to C[0,1]$. We denote the norm for $C[0,1]$ as

$$\|f\|_{\text{sup}} := \sup\{|f(z)| : z \in [0,1]\}.$$

An application of Fubini's theorem shows that, for $f \in C[0,1]$,

$$\{R_\lambda^+ f - e^{-\lambda t} P_t^+ R_\lambda^+ f\}(z) = \int_0^t e^{-\lambda s} \left(P_s^+ f\right)(z) ds,$$

whence

$$\left\|R_\lambda^+ f - e^{-\lambda t} P_t^+ R_\lambda^+ f\right\|_{\text{sup}} \leq t\|f\|_{\text{sup}}.$$

Thus, for $f \in \overline{\mathcal{R}}$, where $\mathcal{R}$ is the common range of the R_λ^+ on $C[0,1]$, we have $\|P_t^+ f - f\|_{\text{sup}} \to 0$ as $t \downarrow 0$. Clearly, if $f \in \overline{\mathcal{R}}$, then $P_0^+ f = f$.

Conversely, suppose that $f \in C[0,1]$, $P_0^+ f = f$ and (for the purposes of contradiction) that $f \notin \overline{\mathcal{R}}$. By the Hahn-Banach Theorem, there exists a bounded linear functional $\tilde{\nu}$ on $C[0,1]$ such that $\tilde{\nu}(f) \neq 0$ but $\tilde{\nu}(g) = 0$ for $g \in \overline{\mathcal{R}}$. However, $\lambda \left(R_\lambda^+ f\right)(z) \to \left(P_0^+ f\right)(z) = f(z)$ as $\lambda \to \infty$; and since (by the Riesz Representation Theorem) $\tilde{\nu}$ is a signed measure of finite total variation, the Dominated-Convergence Theorem shows that $0 = \tilde{\nu}(\lambda R_\lambda f) \to \tilde{\nu}(f) \neq 0$, the desired contradiction. Hence,

$$\begin{aligned}\{f \in C[0,1] : \|P_t^+ f - f\|_{\text{sup}} \to 0\} &= \{f \in C[0,1] : P_0^+ f = f\} \\ &= \{f \in C[0,1] : \langle h_\gamma, f\rangle_s = 0,\ \gamma \in \Gamma\}.\end{aligned}$$

However, for $f \in C[0,1]$, for each $t > 0$, the spectral expansion (11.11) converges rapidly in the topology of $C[0,1]$, so that $P_t^+ f$ is the closed linear span in $C[0,1]$ of $\{f_\theta : \theta \in \Theta\}$. Putting the pieces together, we see that for the case when $m_0 + m_1 \neq 1$, we have proved result (4.3) and answered 'Yes' to Question 2.4B.

We now find the invariant density of Z^+ when $m_0 + m_1 < 1$.

2.12B. Corollary to Theorem 2.11G. *Suppose that $m_0 + m_1 < 1$. Then we can rewrite (11.11) as*

$$P_t^+ f = \Lambda(f)\mathbf{1} + \sum_{\theta \in \Theta_+} e^{-\frac{1}{2}\theta^2 t} \frac{\langle f, f_\theta\rangle_+}{\langle f_\theta, f_\theta\rangle_+} f_\theta. \tag{12.6}$$

where

$$\Lambda(f) = \int_{(0,1)} \eta(y) f(y)\, dy, \qquad \eta(y) = \frac{1 - \Pi^{+-}(y, \{0,1\})}{1 - (m_0 + m_1)}. \tag{12.7}$$

This identifies η as being the invariant probability density for $\{P_t^+\}$.

Proof of Corollary 2.12B. This is simply a matter of noticing that $0 \in \Theta$ when $m_0 + m_1 < 1$. □

2.12C. Corollary. *As $t \to \infty$, the following convergence is **uniform** in z:*

$$(P_t^+ f)(z) \to \begin{cases} \Lambda(f) := \int_0^1 \eta(w) f(w) \mathrm{d}w & \text{if } m_0 + m_1 < 1, \\ 0 & \text{if } m_0 + m_1 > 1. \end{cases} \tag{12.8}$$

Proof of Corollary 2.12C. The result is trivial from Corollary 2.12B. □

A weaker version of the above result may be deduced directly from our definition in 2.6G. However, this lacks the benefit of uniform convergence and it is this property that allows us to interchange limit and integral without fuss.

Proof of WH3 when $m_0 + m_1 \neq 1$. Recall Lemma 2.8K in Section 8. If $m_0 + m_1 < 1$, given Corollary 2.12C, elementary properties of limits yields

$$\begin{aligned} \lim_{t\to\infty} \langle P_t^+ f, P_t^- h \rangle_s &= \int_0^1 \underbrace{\lim_{t\to\infty} (P_t^+ f)(y)}_{=\Lambda(f)} \underbrace{\lim_{t\to\infty} (P_t^- h)(y)}_{=0} \mathrm{d}y \\ &\quad - \sum_{x\in\{0,1\}} m_x \underbrace{\lim_{t\to\infty} (P_t^+ f)(x)}_{=\Lambda(f)} \underbrace{\lim_{t\to\infty} (P_t^- h)(x)}_{=0} \\ &= 0 \end{aligned}$$

A similar argument holds when $m_0 + m_1 > 1$. □

Recalling Corollary B.7 of Appendix B, the following result simply verifies the long-term behaviour of P_t^+.

2.12D. Lemma. *If $m_0 + m_1 < 1$, we have $\int_{[0,1]} \lambda A_\lambda(0, w)\ \mathrm{d}w = 1$, as required, and also $\lim_{\lambda \downarrow 0} \lambda A_\lambda(0, w) = \eta(w)$, tallying with the invariant density role of η.*

Proof of Lemma 2.12D. Multiplying both sides of (10.12) by ρ and recalling that $\frac{1}{2}\rho^2 = \lambda$, we have

$$\lambda \epsilon(\rho) A_\lambda(0, w) = \rho h_\rho^\sharp(w) - m_0 \rho h_\rho^\sharp(0) \pi(0, w) - m_1 \rho \pi(1, w). \tag{12.9}$$

Integrating both sides of this equation w.r.t. w, we get

$$\begin{aligned} \epsilon(\rho) \int_{[0,1]} \lambda A_\lambda(0, w) \mathrm{d}w &= \underbrace{\rho \int_{[0,1]} h_\rho^\sharp(w) \mathrm{d}w}_{=\sinh(\rho) - m_1\rho\cosh(\rho) + m_1\rho} \\ &\quad - m_0 \rho h_\rho^\sharp(0) \underbrace{\int_{[0,1]} \pi(0, w) \mathrm{d}w}_{=1} - m_1 \rho \underbrace{\int_{[0,1]} \pi(1, w) \mathrm{d}w}_{=1} \\ &= \sinh(\rho) - m_1 \rho \cosh(\rho) - m_0 \rho h_\rho^\sharp(0) \\ &= \epsilon(\rho), \end{aligned}$$

as desired. Note that we have used Lemma 2.8J and the explicit form of $h_\rho^\sharp$ from Section 10.

We now need to deduce that $\lim_{\lambda \downarrow 0} \lambda A_\lambda(0, w) = \eta(w)$, where η is defined in (12.7). Now $\lambda \downarrow 0$ clearly implies $\rho \downarrow 0$. Recalling (12.9), it is therefore enough to prove the following results

$$\lim_{\rho \downarrow 0} h_\rho^\sharp(w) = \lim_{\rho \downarrow 0} h_\rho^\sharp(0) = 1, \tag{12.10}$$

$$\lim_{\rho \downarrow 0} \frac{\rho}{\epsilon(\rho)} = \frac{1}{1 - (m_0 + m_1)}. \tag{12.11}$$

Result (12.10) is obvious from the definition of $h_\rho^\sharp$. For the result in (12.11), observe that

$$\begin{aligned} \frac{\rho}{\epsilon(\rho)} &= \frac{\rho}{(1 + m_0 m_1 \rho^2)\sinh(\rho) - \rho(m_0 + m_1)\cosh(\rho)} \\ &= \frac{1}{\left(\frac{1}{\rho} + m_0 m_1 \rho\right)\sinh(\rho) - (m_0 + m_1)\cosh(\rho)}. \end{aligned} \tag{12.12}$$

It is also clear that

$$\lim_{\rho \downarrow 0}\{\cosh(\rho)\} = 1 \quad \text{and} \quad \lim_{\rho \downarrow 0}\{\rho \sinh(\rho)\} = 0. \tag{12.13}$$

Furthermore,

$$\lim_{\rho \downarrow 0}\left\{\frac{\sinh(\rho)}{\rho}\right\} = \frac{\mathrm{d}}{\mathrm{d}x}\sinh(x)\bigg|_{x=0} = 1. \tag{12.14}$$

Using results (12.14) and (12.13) in (12.12), we get the desired result. □

2.12E. Lemma. *The probabilistic semigroup $\{P_t^+ : t \geq 0\}$ satisfies $P_t^+ f_\theta = \mathrm{e}^{-\frac{1}{2}\theta^2 t} f_\theta$ for $\theta \in \Theta$.*

Proof of Lemma 2.12E. As a consequence of Theorem 2.7A (a), we already know that

$$t \mapsto \exp(+\tfrac{1}{2}\theta^2 \Phi_t) f_\theta(Z_t) \qquad (\theta \in \Theta) \tag{12.15}$$

defines a local martingale bounded on $[0, \tau_t^+]$ for $t \geq 0$.

Now, if $\tau_t^+ = \infty$ (in which case $m_0 + m_1 > 1$ or if $t = 0$ then $z \neq 0$ or 1), then $\Phi_t \to -\infty$ (a.s.) as $t \to \infty$ by the long-term behaviour of Φ. Thus $\exp\{+\frac{1}{2}\theta^2 \Phi(\tau_t^+)\} = 0$ on $\{\tau_t^+ = \infty\}$.

Next recall that $\Phi(\tau_t^+) = t$ when $\tau_t^+ < \infty$ and $\Phi(0) = 0$. Since τ_t^+ is a valid stopping time, we can legitimately apply the Optional-Stopping Theorem to get

$$\begin{aligned} \mathbb{E}^z\left[\exp\{+\tfrac{1}{2}\theta^2 \Phi(\tau_t^+)\} f_\theta(Z_t^+); \tau_t^+ < \infty\right] &= \mathbb{E}^z\left[\exp\{+\tfrac{1}{2}\theta^2 t\} f_\theta(Z_t^+); \tau_t^+ < \infty\right] \\ &= \mathbb{E}^z\left[\exp\{+\tfrac{1}{2}\theta^2 \Phi(0)\} f_\theta(Z_0)\right]. \\ &= \mathbb{E}^z\left[f_\theta(Z_0)\right] \\ &= f_\theta(z) \end{aligned}$$

In particular, we have

$$\begin{aligned} & \mathbb{E}^z\left[\exp\{+\tfrac{1}{2}\theta^2 t\} f_\theta(Z_t^+); \tau_t^+ < \infty\right] = f_\theta(z) \\ \Leftrightarrow \quad & \mathbb{E}^z\left[f_\theta(Z_t^+); \tau_t^+ < \infty\right] = \exp\{-\tfrac{1}{2}\theta^2 t\} f_\theta(z) \\ \Leftrightarrow \quad & (P_t^+ f_\theta)(z) = \exp\{-\tfrac{1}{2}\theta^2 t\} f_\theta(z). \end{aligned}$$

□

2.12F. Remark. On its own, Lemma 2.12E is not good enough, because we do not know *a priori* that the answer to Question 2.4B is 'Yes'.

2.13. The Case when $m_0 + m_1 = 1$

Suppose now that $m_0 + m_1 = 1$. This 'balanced' case is the most interesting. For example, we might expect our generalized eigenfunction k from (3.16) to feature in an important way in the Probability. This is well illustrated by the fact that if $m_0 = m_1 = \frac{1}{2}$, then the $\{P_t^+\}$ semigroup has invariant density $6(z - z^2)$.

In the notation of Section 12, $\mu_W = 1$ and L^2_μ is the space $[\mathbf{1}]$ spanned by the vector $\mathbf{1}$.

The quotient space $\tilde{L}^2_+$. We explain the idea that $\tilde{L}^2_+$ will denote the quotient space $L^2/[\mathbf{1}]$ associated with the bilinear form $\langle \cdot, \cdot \rangle_+$ defined in Section 11. An element of $\tilde{L}^2_+$ is a coset

$$f + [\mathbf{1}] \;:=\; \{f + c\mathbf{1} : c \in \mathbb{R}\}.$$

Crucial Fact. For this situation, since $1 - \mu_W = 0$, it is important to observe that the term $\langle J_\mu f, J_\mu f \rangle_{L^2} = \langle J_{[1]} f, J_{[1]} f \rangle_{L^2}$ vanishes in (11.7). However, this is not necessarily the case for the corresponding term in the L^2 norm. With this fact in mind, choosing f so that $\langle J_{[1]} f, J_{[1]} f \rangle_{L^2} = 0$ will guarantee the 'norm pinching' result (analogous to (11.8)) for our situation. In particular, for appropriate f we have

$$\langle f, f \rangle_+ \leq 2\langle f, f \rangle_{L^2} \leq 2\tilde{K}\langle f, f \rangle_+, \tag{13.1}$$

where $\tilde{K} = \max\{1, (1 - \lambda_W)^{-1}\} < \infty$.

We define (unambiguously)

$$c_1(f_1 + [\mathbf{1}]) + c_2(f_2 + [\mathbf{1}]) \;:=\; (c_1 f_1 + c_2 f_2) + [\mathbf{1}].$$

Because $\langle \mathbf{1}, g \rangle_+ = \langle (I - W^+)\mathbf{1}, g \rangle_{L^2} = 0$ for all $g \in L^2$, we may also unambiguously define

$$\langle f + [\mathbf{1}], g + [\mathbf{1}] \rangle_+ \;:=\; \langle f, g \rangle_+. \tag{13.2}$$

Suppose that $\langle g, g \rangle_+ = 0$, so $\langle g, g \rangle_{L^2} = \langle W^+ g, g \rangle_{L^2}$. Now,

$$W^+ g \;=\; J_{[1]} g + \lambda_W J_\lambda g + 0 J_r g,$$

and since each J is self-adjoint with $J^2 = J$, we have $\langle g, Jg \rangle_{L^2} = \|Jg\|^2_{L^2}$, whence

$$\begin{aligned} \|g\|^2_{L^2} \;=\; \langle g, g \rangle_{L^2} \;=\; \langle g, W^+ g \rangle_{L^2} \;&=\; \|J_{[1]} g\|^2_{L^2} + |\lambda_W| \|J_\lambda g\|^2_{L^2} \\ &\leq\; \|J_{[1]} g\|^2_{L^2} + \|J_\lambda g\|^2_{L^2} + \|J_r g\|^2_{L^2} \;=\; \|g\|^2_{L^2}, \end{aligned}$$

so that since $|\lambda_W| < 1$, we have $J_\lambda g = 0 = J_r g$, and so $g \in [\mathbf{1}]$. It is clear now that $\langle \cdot, \cdot \rangle_+$ defines an inner product on $\tilde{L}^2_+$. If $(f_n + [\mathbf{1}])$ is a Cauchy sequence for this product, we may choose the representatives f_n so that $J_{[1]} f_n = 0$. Then (f_n) is Cauchy in L^2 and so, for some

$f \in L^2$, we have $f_n \to f$, and now $f_n + [\mathbf{1}] \to f + [\mathbf{1}]$ in $\tilde{L}^2_+$. Hence, $(\tilde{L}^2_+, \langle \cdot, \cdot \rangle_+)$ is a proper Hilbert space.

Since $\hat{P}_t^+ \mathbf{1} = \mathbf{1}$, we may unambiguously define

$$\hat{P}_t^+(f + [\mathbf{1}]) := (\hat{P}_t^+ f) + [\mathbf{1}], \qquad \hat{R}_\lambda^+(f + [\mathbf{1}]) := (\hat{R}_\lambda^+ f) + [\mathbf{1}]. \tag{13.3}$$

We may now transfer all of the arguments of Section 11 to show that in $\tilde{L}^2_+$, we have

$$\hat{P}_t^+(f + [\mathbf{1}]) = \sum_{\theta \in \Theta^+} e^{-\frac{1}{2}\theta^2 t} \frac{\langle f, f_\theta \rangle_+}{\langle f_\theta, f_\theta \rangle_+} f_\theta + [\mathbf{1}] \qquad (t > 0), \tag{13.4}$$

so that

$$\hat{P}_t^+ f = \sum_{\theta \in \Theta^+} e^{-\frac{1}{2}\theta^2 t} \frac{\langle f, f_\theta \rangle_+}{\langle f_\theta, f_\theta \rangle_+} f_\theta + a_t(f)\mathbf{1} \qquad (t > 0), \tag{13.5}$$

for some constant $a_t(f)$. We now settle some points concerning the claim just made.

2.13A. Proposition. *$\hat{R}_\lambda^+$ is compact on $\tilde{L}^2_+$.*

Proof of Proposition 2.13A. As expected, this is similar to the proof of Proposition 2.11C, but with a few key observations. Let $(f_n + [\mathbf{1}])$ be a bounded sequence in the 'blinkered' space $\tilde{L}^2_+$. Then we may shrewdly choose representatives so that $J_{[1]} f_n = 0$ for each n. By the lattermost inequality in (13.1) it can be seen that f_n is bounded in L^2. However, from the proof of Proposition 2.11C , we know that $\hat{R}_\lambda^+$ is compact on L^2, so that the sequence $\hat{R}_\lambda^+ f_n$ must have a convergent subsequence. That is, there is a subsequence $\hat{R}_\lambda^+ f_{n_k} \to g$ in L^2. Appealing to the former inequality in (13.1) , it now follows that

$$\left|\left|\hat{R}_\lambda^+ (f_{n_k} + [\mathbf{1}]) - (g + [\mathbf{1}])\right|\right|_{\tilde{L}^2_+} \le 2 \left|\left|\hat{R}_\lambda^+ f_{n_k} - g\right|\right|_{L^2} \to 0.$$

□

2.13B. Proposition. *$\hat{R}_\lambda^+$ is self-adjoint on $\tilde{L}^2_+$.*

Proof of Proposition 2.13B. The proof follows by identical arguments to those given in Proposition 2.11D. Due to the definitions given in (13.2) and (13.3) , $\hat{R}_\lambda^+$ is a bounded operator on $\tilde{L}^2_+$ and S_R (as defined in (11.9)) is dense in $\tilde{L}^2_+$. □

It is important to observe that our generalized eigenfunction k satisfies

$$\langle k, \mathbf{1} \rangle_s = 2(m_0 m_1 - \tfrac{1}{3}) < 0,$$

so that $\hat{P}_0^+ k \neq k$. *The expression $(m_0 m_1 - \frac{1}{3})$ will keep appearing.*

Assume that $f \in \mathcal{L}^2$ and that

$$\hat{R}_\lambda^+ f = cf + b\mathbf{1} \quad \text{for some real constants } c \neq 0 \text{ and } b.$$

Since $c \neq 0$, we have successively, $f \in C[0,1]$, $\hat{P}_0^+ f = f$, $f \in \mathcal{D}(\mathcal{H})$ and

$$(\lambda - \mathcal{H})(cf + b\mathbf{1}) = f, \text{ whence } c\mathcal{H}f = (c\lambda - 1)f + b\lambda\mathbf{1}.$$

If $b \neq 0$, the assumption that $c\lambda = 1$ leads to a contradiction because $\hat{P}_0^+ k \neq k$. Hence,

$$\mathcal{H}(f - a\mathbf{1}) = (\lambda - c^{-1})(f - a\mathbf{1}) \text{ where } a = b\lambda/(c\lambda - 1)$$

(with $a = 0$ if $b = 0$); and now we are on familiar ground.

2.13C. Lemma. 0 *is not an eigenvalue of* $\hat{R}_\lambda^+$ *on* $\tilde{L}_+^2$.

Proof of Lemma 2.13C. We can do this by an obvious modification of the argument for the corresponding result in Lemma 2.11F. □

The proper spectral expansion. Equation (13.4) misses the key information about the invariant measure of $\{\hat{P}_t^+\}$. For instance, the form

$$\hat{P}_t^+ f \;=\; \sin(t)\mathbf{1} + \sum_{\theta \in \Theta_+} e^{-\frac{1}{2}\theta^2 t} \frac{\langle f, f_\theta \rangle_+}{\langle f_\theta, f_\theta \rangle_+} f_\theta \tag{13.6}$$

would agree with (13.4). However, the desired result reads:

$$\hat{P}_t^+ f \;=\; \Lambda(f)\mathbf{1} + \sum_{\theta \in \Theta_+} e^{-\frac{1}{2}\theta^2 t} \frac{\langle f, f_\theta \rangle_+}{\langle f_\theta, f_\theta \rangle_+} f_\theta \tag{13.7}$$

where

$$\Lambda(f) \;=\; \int_{(0,1)} \eta(y) f(y)\, dy,$$

and

$$\eta(y) \;=\; \frac{k(1)[h_\beta(y) - h_\beta(0)] + [h_\beta(0) - h_\beta(1)]k(y)}{2(\frac{1}{3} - m_0 m_1)[h_\beta(1) - h_\beta(0)]}. \tag{13.8}$$

□

2.13D. Remark. Due to (13.6), (13.7) is merely a *desired* result and so it is not yet proved. We now appeal to probabilistic coupling to resolve the issue.

Let $\{\hat{Z}_t^{(1)+}\}$ and $\{\hat{Z}_t^{(2)+}\}$ be independent strong Markov processes each with the transition semigroup $\{\hat{P}_t^+\}$. Let $\hat{\mathbb{P}}^z$ denote the law of each of $\hat{Z}^{(1)+}$ and $\hat{Z}^{(2)+}$ started at z, and let $\hat{\mathbb{P}}^{(z_1,z_2)}$ denote the (product) law of $\{(\hat{Z}^{(1)+}, \hat{Z}^{(2)+})\}$ started at (z_1, z_2). Let

$$\hat{\sigma} := \inf\left\{t : \hat{Z}_t^{(1)+} = \hat{Z}_t^{(2)+}\right\}.$$

2.13E. Theorem. *Suppose* $m_0 + m_1 = 1$. *For* $f \in C[0,1]$, *there exists a constant* $c^+(f)$, *independent of* z, *such that* $(\hat{P}_t^+ f)(z) \to c^+(f)$, *uniformly in* z, *as* $t \to \infty$.

Recall that we already know the result when $m_0 + m_1 \neq 1$ in Corollary 2.12C. In order to prove the above Theorem, it suffices to prove the following Lemmas.

2.13F. Lemma. *We have* $\hat{\mathbb{P}}^{(z_1,z_2)}(\hat{\sigma} < \infty) = 1$ *for all* $z_1, z_2 \in [0,1]$.

Proof of Lemma 2.13F.

Technical note. For $i \in \{1,2\}$, we know that $\hat{Z}_t^{(i)+}$ is strong Markov relative to the filtration $\mathcal{F}_t^{(i)+}$. Due to independence, we therefore know that $\hat{Z}_t^{(i)+}$ is strong Markov relative to the filtration $\mathcal{F}_t^{(i)+} \times \mathcal{F}_\infty^{(3-i)+}$, and hence relative to $\mathcal{F}_t^{(i)+} \times \mathcal{F}_t^{(3-i)+}$. It follows that the joint process $\{(\hat{Z}^{(1)+}, \hat{Z}^{(2)+})\}$ is strong Markov relative to $\mathcal{G}_t := \mathcal{F}_t^{(1)+} \times \mathcal{F}_t^{(2)+}$. In particular, given our definition of $\hat{\sigma}$, we see that $\hat{\sigma}$ is now a stopping time relative to $\mathcal{G}_t$.

For this section it is convenient to temporarily drop any previous definitions of η and ξ. This is done so that our notation is consistent with that given in III.9 of Rogers & Williams [24].

Let J_1 and J_2 be disjoint compact subintervals of $(0,1)$ each with non-empty interior. The significance of these definitions is clear if one considers one of the boundaries as a starting point. I will denote the familiar indictor function. For $j \in \{1,2\}$ define the events

$$E_j = \left\{ \hat{Z}_{t_2}^{(j)+} \in J_{3-j}; \hat{Z}_r^{(j)+} \in (0,1) \quad \text{for} \quad 0 \le r \le t_2 \right\},$$

E_j :

$$F_j = \left\{ \hat{Z}_{t_1+t_2}^{(j)+} \in J_{3-j}; \hat{Z}_r^{(j)+} \in (0,1) \quad \text{for} \quad t_1 \le r \le t_1 + t_2 \right\}.$$

$$F_j = \left\{ Z_{t_1+t_2}^{(j)+} \in J_{3-j}; Z_r^{(j)+} \in (0,1) \quad \text{for} \quad t_1 \le r \le t_1 + t_2 \right\}.$$

By a similar argument to that given in Appendix C, we may prove that there exist $t_1 > 0$ and $\eta_1 > 0$ such that for $j = 1,2$,

$$\inf_{z \in [0,1]} \hat{\mathbb{P}}^{(z)} \left(\hat{Z}_{t_1}^{(j)+} \in J_j \right) \ge \eta_1. \tag{13.9}$$

(Of course, *any* $t_1 > 0$ will provide a suitable η_1.) In a similar way, we may justify that there exist $t_2 > 0$ and $\eta_2 > 0$ such that for $j = 1,2$,

$$\inf_{z \in I_j} \hat{\mathbb{P}}^{(z)} (E_j) \ge \eta_2. \tag{13.10}$$

(Of course, *any* $t_2 > 0$ will provide a suitable η_2.) Now let

$$\eta_j := I_{E_j} \qquad \text{and} \qquad \xi_j = I_{J_j}(\hat{Z}_{t_1}^{(j)+}).$$

Recalling that $\theta_{t_1}\eta = \theta_{t_1} \circ \eta$ and that θ_{t_1} 'shifts paths' through time t_1, we may now deduce that

$$\theta_{t_1}\eta_j = I_{F_j}, \qquad \text{so that} \qquad \xi_j \theta_{t_1}\eta_j = I_{\{(\hat{Z}_{t_1}^{(j)+} \in J_j);\ F_j\}}.$$

In particular, we now have

$$\hat{\mathbb{E}}^z [\, \xi_j \theta_{t_1} \eta_j] = \hat{\mathbb{P}}^z \left(\hat{Z}_{t_1}^{(j)+} \in J_j;\ F_j \right). \tag{13.11}$$

From Theorem 9.4 of III.9 of Rogers & Williams [24], we have

$$\hat{\mathbb{E}}^z [\, \xi_j \theta_{t_1} \eta_j] = \hat{\mathbb{E}}^z \left[\xi_j \ \hat{\mathbb{E}}^{\hat{Z}_{t_1}^{(j)+}} \eta_j \right]. \tag{13.12}$$

Given (13.12) and (13.11) we now have

$$\begin{aligned}\hat{\mathbb{P}}^z\left(\hat{Z}_{t_1}^{(j)+} \in J_j;\ F_j\right) &= \hat{\mathbb{E}}^z\left[\ \xi_j\ \hat{\mathbb{E}}^{\hat{Z}_{t_1}^{(j)+}}\eta_j\right] = \hat{\mathbb{E}}^z\left[I_{J_j}(\hat{Z}_{t_1}^{(j)+})\hat{\mathbb{E}}^{\hat{Z}_{t_1}^{(j)+}}\left[\ I_{E_j}\right]\right] \\ &= \hat{\mathbb{E}}^z\Big[I_{J_j}(\hat{Z}_{t_1}^{(j)+})\underbrace{\hat{\mathbb{P}}^{\hat{Z}_{t_1}^{(j)+}}(E_j)}_{\geq \eta_2 \text{ by (13.10)}}\Big] \geq \eta_2\hat{\mathbb{E}}^z I_{J_j}(\hat{Z}_{t_1}^{(j)+}) \\ &= \eta_2\ \hat{\mathbb{P}}^z(\hat{Z}_{t_1}^{(j)+} \in J_j) \geq \eta_1\eta_2 \qquad \text{(by (13.9))}.\end{aligned}$$

Thus, we have deduced that there exist $t_j > 0$ and $\eta_j > 0$ such that for $j = 1, 2$,

$$\hat{\mathbb{P}}^z\big(\hat{Z}_{t_1}^{(j)+} \in J_j;\ F_j\big) \geq \eta_1\eta_2. \tag{13.13}$$

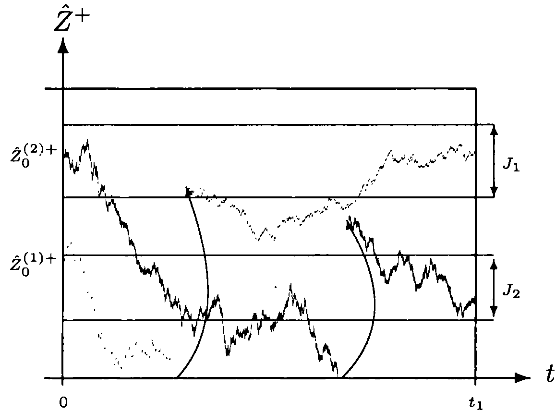

Figure 2.4: Examples of the events $\{\hat{Z}_{t_1}^{(j)+} \in J_j\}$ for $j = 1, 2$.

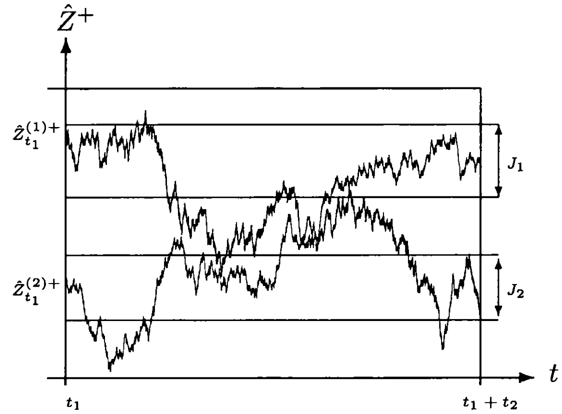

Figure 2.5: Examples of the events $\{\hat{Z}_{t_1}^{(j)+} \in J_j;\ F_j\}$ for $j = 1, 2$.

Due to independence of the underlying processes, we have

$$\hat{\mathbb{P}}^{(z_1,z_2)}(\hat{\sigma} < \infty) \geq \hat{\mathbb{P}}^{z_1}\big(\hat{Z}_{t_1}^{(1)+} \in J_1;\ F_1\big)\hat{\mathbb{P}}^{z_2}\big(\hat{Z}_{t_1}^{(2)+} \in J_2;\ F_2\big), \tag{13.14}$$

for all $z_1, z_2 \in [0, 1]$. As a result of (13.13), this implies that

$$\hat{\mathbb{P}}^{(z_1,z_2)}(\hat{\sigma} < t_1 + t_2)\ \geq\ \delta_0\ :=\ \eta_1^2\eta_2^2 \quad \text{for all } z_1, z_2,$$

which is equivalent to

$$\hat{\mathbb{P}}^{(z_1,z_2)}(\hat{\sigma} \geq t_1 + t_2) \leq 1 - \delta_0 \quad \text{for all } z_1, z_2. \tag{13.15}$$

We are now in a position to prove that

$$\hat{\mathbb{P}}^{(z_1,z_2)}(\hat{\sigma}\ \geq\ n(t_1 + t_2))\ \leq\ (1 - \delta_0)^n \quad \text{for all } z_1, z_2, \text{ and all } n \in \mathbb{N}. \tag{13.16}$$

Proof of (13.16). The proof follows by mathematical induction. From (13.15) we know that (13.16) holds when $n = 1$. Once again we refresh our definitions η and ξ so that, for $k \geq 1$,

$$\xi = I_{\{\hat{\sigma} \geq k(t_1+t_2)\}} \qquad \text{and} \qquad \eta = I_{\{\hat{\sigma} \geq t_1+t_2\}}.$$

This time $\theta_{k(t_1+t_2)}$ represents a time shift of $k(t_1 + t_2)$. It now follows that

$\theta_{k(t_1+t_2)}\eta$ is the event that the processes $\hat{Z}^{(1)+}$ and $\hat{Z}^{(2)+}$ do not collide between $k(t_1+t_2)$ and $(k+1)(t_1+t_2)$.

Now suppose that (13.16) holds for $k \in \mathbb{N}$. It suffices to show that it therefore holds for $k+1$. Now (13.12) clearly holds under the product law $\hat{\mathbb{P}}^{(z_1,z_2)}$. Thus, we may apply (13.12) under our 'new' definitions to get

$$\begin{aligned}
\hat{\mathbb{P}}^{(z_1,z_2)}\big(\hat{\sigma} \geq (k+1)(t_1+t_2)\big) &= \hat{\mathbb{P}}^{(z_1,z_2)}\big(\hat{\sigma} \geq (k+1)(t_1+t_2); \hat{\sigma} \geq k(t_1+t_2)\big) \\
&= \hat{\mathbb{E}}^{(z_1,z_2)}\left[\xi \theta_{k(t_1+t_2)}\eta\right] \\
&= \hat{\mathbb{E}}^{(z_1,z_2)}\left[\xi\ \hat{\mathbb{E}}^{(\hat{Z}^{(1)+}_{k(t_1+t_2)}, \hat{Z}^{(2)+}_{k(t_1+t_2)})}\eta\right] \\
&= \hat{\mathbb{E}}^{(z_1,z_2)}\Big[\xi\ \underbrace{\hat{\mathbb{P}}^{(\hat{Z}^{(1)+}_{k(t_1+t_2)}, \hat{Z}^{(2)+}_{k(t_1+t_2)})}\left(\hat{\sigma} \geq t_1+t_2\right)}_{\leq (1-\delta_0) \text{ by (13.15)}}\Big] \\
&\leq (1-\delta_0)\underbrace{\hat{\mathbb{P}}^{(z_1,z_2)}\big(\hat{\sigma} \geq k(t_1+t_2)\big)}_{\leq (1-\delta_0)^k \text{ as (13.16) holds for } k} \leq (1-\delta_0)^{k+1},
\end{aligned}$$

which is exactly the desired result.

Using (13.16) the (downward) Monotone Convergence Theorem (for sets) yields

$$\hat{\mathbb{P}}^{(z_1,z_2)}\left(\hat{\sigma} = \infty\right) \leq \lim_{n\to\infty}(1-\delta_0)^n = 0 \qquad \Leftrightarrow \qquad \hat{\mathbb{P}}^{(z_1,z_2)}\left(\hat{\sigma} < \infty\right) = 1.$$

□

2.13G. Lemma. *The difference $(\hat{P}_t^+ f)(z_1) - (\hat{P}_t^+ f)(z_2)$ tends to 0 uniformly in $(z_1.z_2)$ as $t \to \infty$. In addition, for $0 \leq t \leq u$,*

$$\sup_z(\hat{P}_t^+ f)(z) \ \geq\ \sup_z(\hat{P}_u^+ f)(z) \ \geq\ \inf_z(\hat{P}_u^+ f)(z) \ \geq\ \inf_z(\hat{P}_t^+ f)(z).$$

Proof of Lemma 2.13G. Given Lemma 2.13F, define

$$W_t := \begin{cases} \hat{Z}_t^{(1)+} & \text{if } t < \hat{\sigma}, \\ \hat{Z}_t^{(2)+} & \text{if } t \geq \hat{\sigma}. \end{cases}$$

We now *assume* that W has the same $\hat{\mathbb{P}}^{(z_1,z_2)}$ law as $\hat{Z}_t^{(1)+}$. We have

$$\begin{aligned}
(\hat{P}_t^+ f)(z_1) - (\hat{P}_t^+ f)(z_2) &= \hat{\mathbb{E}}^{z_1}[f(\hat{Z}_t^{(1)+})] - \hat{\mathbb{E}}^{z_2}[f(\hat{Z}_t^{(2)+})] \\
&= \hat{\mathbb{E}}^{(z_1,z_2)}[f(\hat{Z}_t^{(1)+})] - \hat{\mathbb{E}}^{(z_2,z_2)}[f(\hat{Z}_t^{(2)+})] \quad \text{(by independence)} \\
&= \hat{\mathbb{E}}^{(z_1,z_2)}[f(W_t)] - \hat{\mathbb{E}}^{(z_2,z_2)}[f(\hat{Z}_t^{(2)+})] \quad \text{(by our initial assumption)} \\
&= \hat{\mathbb{E}}^{(z_1,z_2)}[f(W_t) - f(\hat{Z}_t^{(2)+}); t < \hat{\sigma}] \quad \text{(by definition of } W\text{)}
\end{aligned}$$

Thus we may initially deduce that, for all z_1, z_2, we have

$$|(\hat{P}_t^+ f)(z_1) - (\hat{P}_t^+ f)(z_2)| \leq 2||f||\ \hat{\mathbb{P}}^{(z_1,z_2)}(t < \hat{\sigma}).$$

Moreover, given (13.16) in the proof of Lemma 2.13F, we have the much stronger result, that is, for $t > n(t_1 + t_2)$,

$$|(\hat{P}_t^+ f)(z_1) - (\hat{P}_t^+ f)(z_2)| \leq 2||f|| \, \hat{\mathbb{P}}^{(z_1,z_2)}(t < \hat{\sigma}) \leq 2||f||(1 - \delta_0)^n,$$

since $\{\sigma > t\} \subset \{\sigma \geq n(t_1 + t_2)\}$. Such a δ_0 works for all z_1, z_2, so that the difference must tend to zero uniformly. Simply observe that, for all $z \in [0, 1]$, we have

$$(\hat{P}_u^+ f)(z) = (\hat{P}_{u-t}^+ \hat{P}_t^+ f)(z) \leq \sup_z (\hat{P}_t^+ f)(z) \underbrace{(\hat{P}_{u-t}^+ 1)(z)}_{=1} = \sup_z (\hat{P}_t^+ f)(z).$$

Hence, as the above result is true for all z we have,

$$\sup_z (\hat{P}_t^+ f)(z) \ \geq \ \sup_z (\hat{P}_u^+ f)(z) \ \geq \ \inf_z (\hat{P}_u^+ f)(z). \tag{13.17}$$

By a similar argument, we also have

$$(\hat{P}_u^+ f)(z) = (\hat{P}_{u-t}^+ \hat{P}_t^+ f)(z) \geq \inf_z (\hat{P}_t^+ f)(z) \underbrace{(\hat{P}_{u-t}^+ 1)(z)}_{=1} = \inf_z (\hat{P}_t^+ f)(z),$$

which is again true for all $z \in [0, 1]$. We may therefore deduce that

$$\inf_z (\hat{P}_u^+ f)(z) \ \geq \ \inf_z (\hat{P}_t^+ f)(z),$$

so that, together with (13.17) we have the desired result. □

Proof of Theorem 2.13E. Given Lemmas 2.13F and 2.13G the desired result is now obvious. By monotonicity we may deduce that

$$\sup_z (\hat{P}_t^+ f)(z) \downarrow s \qquad \text{and} \qquad \inf_z (\hat{P}_t^+ f)(z) \uparrow i,$$

so that, because of Lemma 2.13G, we must now have, for all z,

$$(\hat{P}_t^+ f)(z) \to s = i, \qquad \text{as } t \to \infty.$$

□

Of course, we could have established a similar result to Theorem 2.13E for the corresponding P_t^- situation. However, there is no need to! In fact, referring to Lemma 2.8K we know exactly what the constant is. In particular, as the underlying convergence is again uniform, we now have

$$\begin{aligned} \lim_{t \to \infty} \langle \hat{P}_t^+ f, P_t^- h \rangle_s &= c^+(f) c^-(h) \langle 1, 1 \rangle_s \\ &= c^+(f) c^-(h)[1 - (m_0 + m_1)] = 0 \end{aligned}$$

We have therefore proved WH3 for the case in which $m_0 + m_1 = 1$.

The following result will allow us to confirm that $c^+(f) \equiv \Lambda(f)$, that is, the explicit form of the constant for the $\hat{P}_t^+ f$ case is $\Lambda(f)$, where η is as in (13.8).

2.13H. Lemma. *If* $m_0 + m_1 = 1$, *we have* $\int_{[0,1]} \lambda A_\lambda(0, w)\ \mathrm{d}w = 1$, *as required, and also* $\lim_{\lambda \downarrow 0} \lambda A_\lambda(0, w)\ \mathrm{d}w = \eta(w)$, *tallying with the invariant density rôle of* η *in* (13.8).

Proof of Lemma 2.13H. The fact that $\int_{[0,1]} \lambda A_\lambda(0, w)\ \mathrm{d}w = 1$ can be deduced an identical argument to that given in the proof of Lemma 2.12D of the previous section. Referring to the remaining arguments given in that Lemma, it suffices to show that

$$\lim_{\lambda \downarrow 0} \lambda A_\lambda(0, w) = \lim_{\rho \downarrow 0} \frac{\rho}{\epsilon(\rho)} \left\{ h^\sharp_\rho(w) - m_0 h^\sharp_\rho(0) \pi(0, w) - m_1 \pi(1, w) \right\} = \eta(w). \tag{13.18}$$

However, since $\lim_{\rho \downarrow 0} \epsilon(\rho) = 0$ when $m_0 + m_1 = 1$, we find we are faced with an indeterminate form. We consequently appeal to L'Hôpitals rule to tackle the limit in (13.18). Define

$$\frac{\rho}{\epsilon(\rho)} \left\{ h^\sharp_\rho(w) - m_0 h^\sharp_\rho(0) \pi(0, w) - m_1 \pi(1, w) \right\} := \frac{\rho L}{\epsilon(\rho)}.$$

Then elementary calculus shows that

$$\frac{\mathrm{d}_\rho\{\rho L\}}{\epsilon'(\rho)} = \frac{h^\sharp_\rho(w) + \rho\ \mathrm{d}_\rho\{h^\sharp_\rho(w)\} - m_0 \pi(0, w) \left\{ h^\sharp_\rho(0) + \rho\ \mathrm{d}_\rho\{h^\sharp_\rho(0)\} \right\} - m_1 \pi(1, w)}{m_0 m_1 \rho^2 \cosh(\rho) + (2 m_0 m_1 - 1) \rho \sinh(\rho)}.$$

If we divide both the numerator and the denominator of the above equation by ρ^2, and examine the consequent limit as $\rho \downarrow 0$, we once again get an indeterminate form. Appealing to L'Hôpitals rule once more, we now have

$$\frac{\mathrm{d}^2_\rho\{\rho L\}}{\epsilon''(\rho)} = \frac{2\mathrm{d}_\rho\{h^\sharp_\rho(w)\} + \rho\ \mathrm{d}^2_\rho\{h^\sharp_\rho(w)\} - m_0 \pi(0, w) \left\{ 2\mathrm{d}_\rho\{h^\sharp_\rho(0)\} + \rho\ \mathrm{d}^2_\rho\{h^\sharp_\rho(0)\} \right\}}{[m_0 m_1 (\rho^2 + 2) - 1] \sinh(\rho) + [(4 m_0 m_1 - 1)\rho] \cosh(\rho)}. \tag{13.19}$$

In order to deal with the appropriate limit, this time it is sufficient to divide both the denominator and numerator of the above by ρ. It is a trivial exercise to show that

$$\lim_{\rho \downarrow 0} \frac{\epsilon''(\rho)}{\rho} = 6 m_0 m_1 - 2 = (-3) 2 \left(\tfrac{1}{3} - m_0 m_1 \right), \tag{13.20}$$

where the significance of the given factorization will soon become clear. Since $m_0 + m_1 = 1$, we know that $\alpha = 0$ so that $\pi(0, w)$ simplifies to the form

$$\pi(0, w) = \frac{h_\beta(1) - h_\beta(y)}{m_0 \{h_\beta(1) - 1\}}.$$

We therefore have

$$\frac{\mathrm{d}^2_\rho\{\rho L\}}{\rho} = \frac{\{h_\beta(1) - 1\}\, \xi(\rho, w) + \{h_\beta(y) - h_\beta(1)\}\, \xi(\rho, 0)}{h_\beta(1) - 1}, \tag{13.21}$$

where

$$\xi(\rho, w) = \frac{2\ \mathrm{d}_\rho\{h^\sharp_\rho(w)\}}{\rho} + \mathrm{d}^2_\rho\{h^\sharp_\rho(w)\}.$$

Recalling the form of k in (3.16) and that $m_1 = 1 - m_0$, one may easily show that

$$\lim_{\rho\downarrow 0} \xi(\rho, w) = (-3)\{k(1) - k(w)\}, \tag{13.22}$$

so that, in particular, we have

$$\lim_{\rho\downarrow 0} \xi(\rho, 0) = (-3)k(1). \tag{13.23}$$

We may now utilize (13.22) and (13.23) in working out the appropriate limit of the terms in (13.21). We accordingly have

$$\begin{aligned}\lim_{\rho\downarrow 0} \frac{\mathrm{d}_\rho^2\{\rho L\}}{\rho} &= \frac{(-3)\left(\{h_\beta(y) - h_\beta(1)\}\, k(1) + \{h_\beta(1) - 1\}\,[k(1) - k(w)]\right)}{h_\beta(1) - 1} \\ &= \frac{(-3)\left(\{h_\beta(y) - 1\}\, k(1) + \{1 - h_\beta(1)\}\, k(w)\right)}{h_\beta(1) - 1}.\end{aligned} \tag{13.24}$$

Noting that $h_\beta(0) = 1$, we may combine results (13.20) and (13.24) to get the desired result in (13.18), where η is of the form given in (13.8). □

Verification of (13.7). Given Corollary B.7, Theorem 2.13E and Lemma 2.13H allow us to confirm that $\Lambda(f) \equiv c^+(f)$. It should be noted that (13.7) may be deduced directly from (13.5) without Theorem 2.13E. Noting that $\hat{P}_u^+ f_\theta = \mathrm{e}^{-\frac{1}{2}\theta^2 u} f_\theta$ and $\hat{P}_u^+ \mathbf{1} = \mathbf{1}$, we may apply $\hat{P}_u^+$ to both sides of (13.5) to get

$$\hat{P}_{(t+u)}^+ f = \sum_{\theta\in\Theta^+} \mathrm{e}^{-\frac{1}{2}\theta^2 (t+u)} \frac{\langle f, f_\theta\rangle_+}{\langle f_\theta, f_\theta\rangle_+} f_\theta + a_t(f)\mathbf{1} \qquad (t > 0), \tag{13.25}$$

for some constant $a_t(f)$. Allowing $u \to \infty$ shows that $\hat{P}_\infty^+ f = a_t(f)$. This result, together with Lemma 2.13H, verifies (13.7).

In light of the above, for $m_0 + m_1 = 1$, we may now accompany (12.8) with the result

$$(\hat{P}_t^+ f)(z) \to \Lambda(f) \equiv \int_0^1 \eta(w) f(w) \mathrm{d}w,$$

uniformly in z and η is given in (13.8).

Proof that $\hat{P}_t^+ = P_t^+$ can now proceed as for the 'unbalanced' case. Of course, we then have result (4.3) and also WH1 for the case $m_0 + m_1 = 1$.

2.14. The Kolmogorov Forward Equation and Riccati Equation

We return to the case of general $m_0, m_1 > 0$.

The Kolmogorov forward equation for the transition density $p_t^+(\cdot,\cdot)$ for $\{P_t^+ : t > 0\}$ relative to Lebesgue measure on $(0,1)$ takes the form:

$$(\partial_t - \tfrac{1}{2}\partial_y^2) p_t^+(z,y) = \tfrac{1}{2}\left\{\partial_w p_t^+(z,w)\right\}\big|_{w=0} \pi(0,y) - \tfrac{1}{2}\left\{\partial_w p_t^+(z,w)\right\}\big|_{w=1} \pi(1,y), \tag{14.1}$$

and we have the interpretation

$$\mathbb{E}^z N_0(t) = \int_0^t \tfrac{1}{2}\left\{\partial_w p_s^+(z,w)\right\}\big|_{w=0} \mathrm{d}s, \tag{14.2}$$

where $N_0(t)$ is the number of jumps out from 0 made by $\{Z_s^+ : 0 < s \le t\}$.

We expect equation (14.1) from integration by parts. Compare, for example, equation (V.38.11) in Volume 2 of Rogers & Williams [24].

Let us understand how intuition suggests (14.2). Let ϵ be a small number. Then the expectation of the time spent in space-interval $(0,\epsilon)$ by a Brownian motion on $\mathbb{R}$ started at ϵ before it hits 0 has expectation exactly ϵ^2. We therefore believe that for Z^+,

$$\begin{aligned}\mathbb{E}^z N_0(t) &\approx \epsilon^{-2}\mathbb{E}^z\{\text{Time spent by } Z^+ \text{ in } (0,\epsilon) \text{ during time-interval } (0,t]\} \\ &= \epsilon^{-2}\int_{s=0}^t \int_{r=0}^{\epsilon} p_s^+(z,r)\mathrm{d}r\mathrm{d}s.\end{aligned}$$

Now, for $s > 0$, $p_s^+(z,0) = 0$, so that

$$p_s^+(z,r) \approx r\left\{\partial_w p_s^+(z,w)\right\}\big|_{w=0},$$

and the meaning of equation (14.2) is clear.

The secret of dealing effectively with the forward equation is to begin by verifying that the following Riccati equation holds for $x \in \{0,1\}$ and $y \in (0,1)$:

$$-\tfrac{1}{2}\partial_y^2 \pi(x,y) = \tfrac{1}{2}\partial_w \pi(x,w)\big|_{w=0}\pi(0,y) - \tfrac{1}{2}\partial_w \pi(x,w)\big|_{w=1}\pi(1,y) \tag{14.3}$$

and that, when $m_0 + m_1 \le 1$, the 'invariant-density' equation:

$$-\tfrac{1}{2}\partial_y^2 \eta(y) = \tfrac{1}{2}\partial_w \eta(w)\big|_{w=0}\pi(0,y) - \tfrac{1}{2}\partial_w \eta(w)\big|_{w=1}\pi(1,y) \tag{14.4}$$

holds.

In fact, π is the minimal non-negative solution of (14.3) subject to conditions

$$\pi(x,\cdot) \in C^2[0,1], \quad \pi(0,0) = m_0^{-1}, \quad \pi(0,1) = 0 = \pi(1,0), \quad \pi(1,1) = m_1^{-1}.$$

In Chapter 4 we comment on the existence of non-minimal non-negative solutions $\tilde{\pi}(\cdot,\cdot)$ and corresponding $\{\tilde{P}_t^+\}$ when $m_0 + m_1 < 1$.

By using the above two equations and linearity, we can show that, with $A_\lambda(0,y)$ as at (10.12), we have

$$\begin{aligned}&\lambda A_\lambda(0,y) - \tfrac{1}{2}\partial_y^2 A_\lambda(0,y) - \pi(0,y) \\ &\quad = \tfrac{1}{2}\left\{\partial_w A_\lambda(0,w)\right\}\big|_{w=0}\pi(0,y) - \tfrac{1}{2}\left\{\partial_w A_\lambda(0,w)\right\}\big|_{w=1}\pi(1,y);\end{aligned} \tag{14.5}$$

and inversion of the Laplace transform (at least formally) yields (14.1) with $z = 0$. One has to remember Lemma 2.10C if one wishes to compare (14.3) with λ times (14.5).

How does the forward equation look from the perspective of Spectral Theory? By the spectral expansion,

$$\begin{aligned} p_t^+(z,y) &= I_\Theta(0)\eta(y) + \textstyle\sum_{\theta\in\Theta_+} \frac{e^{-\frac{1}{2}\theta^2 t} f_\theta(y) f_\theta(z)}{\langle f_\theta, f_\theta\rangle_+} \\ - m_0\pi(0,y) \textstyle\sum_{\theta\in\Theta_+} &\frac{e^{-\frac{1}{2}\theta^2 t} f_\theta(0) f_\theta(z)}{\langle f_\theta, f_\theta\rangle_+} - m_1\pi(1,y)\textstyle\sum_{\theta\in\Theta_+} \frac{e^{-\frac{1}{2}\theta^2 t} f_\theta(1) f_\theta(z)}{\langle f_\theta, f_\theta\rangle_+}. \end{aligned} \tag{14.6}$$

2.14A. Proposition. *The Kolmogorov forward equation in* (14.1) *can be deduced rigorously from this expansion, together with* (14.3) *and* (14.4).

Proof of Proposition 2.14A. From (14.6) we have

$$\begin{aligned} (\partial_t - \tfrac{1}{2}\partial_y^2) p_t^+(z,y) &= -\tfrac{1}{2}\partial_y^2\{\eta(y)\} I_\Theta(0) + \sum_{\theta\in\Theta_+} \{\underbrace{-\tfrac{1}{2}\theta^2 f_\theta(y)}_{=\mathcal{H}f_\theta}\} \frac{e^{-\frac{1}{2}\theta^2 t} f_\theta(z)}{\langle f_\theta, f_\theta\rangle_+} \\ + \sum_{\theta\in\Theta_+} \{\underbrace{-\tfrac{1}{2} f_\theta''(y)}_{=-\mathcal{H}f_\theta}\} \frac{e^{-\frac{1}{2}\theta^2 t} f_\theta(z)}{\langle f_\theta, f_\theta\rangle_+} &- m_0\pi(0,y) \sum_{\theta\in\Theta_+} \{-\tfrac{1}{2}\theta^2 f_\theta(0)\} \frac{e^{-\frac{1}{2}\theta^2 t} f_\theta(z)}{\langle f_\theta, f_\theta\rangle_+} \\ - m_1\pi(1,y) \sum_{\theta\in\Theta_+} \{-\tfrac{1}{2}\theta^2 f_\theta(1)\} \frac{e^{-\frac{1}{2}\theta^2 t} f_\theta(z)}{\langle f_\theta, f_\theta\rangle_+} &+ \tfrac{1}{2} m_0 \partial_y^2\{\pi(0,y)\} \sum_{\theta\in\Theta_+} \frac{e^{-\frac{1}{2}\theta^2 t} f_\theta(0) f_\theta(z)}{\langle f_\theta, f_\theta\rangle_+} \\ + \tfrac{1}{2} m_1 \partial_y^2\{\pi(1,y)\} &\sum_{\theta\in\Theta_+} \frac{e^{-\frac{1}{2}\theta^2 t} f_\theta(1) f_\theta(z)}{\langle f_\theta, f_\theta\rangle_+}. \end{aligned}$$

Next, for $x \in \{0,1\}$, let

$$S_x := \sum_{\theta\in\Theta_+} \frac{e^{-\frac{1}{2}\theta^2 t} f_\theta(x) f_\theta(z)}{\langle f_\theta, f_\theta\rangle_+}.$$

Recall that $f_\theta \in \mathcal{D}(\mathcal{H})$. Thus, using (14.3) and considering the obvious cancellation, we now have

$$\begin{aligned} (\partial_t - \tfrac{1}{2}\partial_y^2) p_t^+(z,y) &= \left\{ \tfrac{1}{2}\partial_y\eta(y)\big|_{y=0}\pi(0,y) + \tfrac{1}{2}\partial_y\eta(y)\big|_{y=1}\pi(1,y) \right\} I_\Theta(0) \\ - \pi(0,y) \sum_{\theta\in\Theta_+} \underbrace{m_0\{-\tfrac{1}{2}\theta^2 f_\theta(0)\}}_{=-f_\theta'(y)|_{y=0}} &\frac{e^{-\frac{1}{2}\theta^2 t} f_\theta(z)}{\langle f_\theta, f_\theta\rangle_+} - \pi(1,y) \sum_{\theta\in\Theta_+} \underbrace{m_1\{-\tfrac{1}{2}\theta^2 f_\theta(1)\}}_{=f_\theta'(y)|_{y=1}} \frac{e^{-\frac{1}{2}\theta^2 t} f_\theta(z)}{\langle f_\theta, f_\theta\rangle_+} \\ &+ m_0\{-\tfrac{1}{2}\partial_w\pi(0,w)\big|_{w=0}\pi(0,y) + \tfrac{1}{2}\partial_w\pi(0,w)\big|_{w=1}\pi(1,y)\} S_0 \\ &+ m_1\{-\tfrac{1}{2}\partial_w\pi(0,w)\big|_{w=0}\pi(0,y) + \tfrac{1}{2}\partial_w\pi(0,1)\big|_{w=0}\pi(1,y)\} S_1. \end{aligned}$$

Rearranging the above we have

$$(\partial_t - \tfrac{1}{2}\partial_y^2) p_t^+(z,y) = C_0\pi(0,y) + C_1\pi(1,y),$$

where

$$C_0 = I_\Theta(0)\tfrac{1}{2}\partial_y\eta(y)\big|_{y=0} + \sum_{\theta\in\Theta_+} \tfrac{1}{2}f'_\theta(y)|_{y=0}\frac{e^{-\frac{1}{2}\theta^2 t}f_\theta(z)}{\langle f_\theta,\, f_\theta\rangle_+}$$
$$- m_0\tfrac{1}{2}\partial_w\pi(0,w)\big|_{w=0}S_0 - m_1\tfrac{1}{2}\partial_w\pi(1,w)\big|_{w=0}S_1,$$
$$C_1 = -I_\Theta(0)\tfrac{1}{2}\partial_y\eta(y)\big|_{y=1} - \sum_{\theta\in\Theta_+} \tfrac{1}{2}f'_\theta(y)|_{y=1}\frac{e^{-\frac{1}{2}\theta^2 t}f_\theta(z)}{\langle f_\theta,\, f_\theta\rangle_+}$$
$$+ m_0\tfrac{1}{2}\partial_w\pi(0,w)\big|_{w=1}S_0 + m_1\tfrac{1}{2}\partial_w\pi(1,w)\big|_{w=1}S_1.$$

By comparison with (14.1), in order for things to tally we need

$$C_0 = \tfrac{1}{2}\left\{\partial_w p_t(z,w)\right\}\big|_{w=0}, \qquad \text{and} \qquad C_1 = -\tfrac{1}{2}\left\{\partial_w p_t(z,w)\right\}\big|_{w=1}.$$

However, by differentiating (14.6) with respect to y, it is clear that we have exactly what we want. □

2.14B. Note. The fact that the left-hand side of (2.3) tends to 0 as $t \to \infty$ has become irrelevant. The result may trivially be deduced from the spectral expansion.

2.15. Traces and Factorizations

2.15A. Lemma. *For $t > 0$,*

$$\int_{(0,1)} p_t^+(w,w)\,\mathrm{d}w = \sum_{\theta\in\Theta} e^{-\frac{1}{2}\theta^2 t}, \tag{15.1}$$

$$\sum_{\{0,1\}} P_t^-(x,\{x\}) = \sum_{\gamma\in\Gamma} e^{-\frac{1}{2}\gamma^2 t}. \tag{15.2}$$

Proof of Lemma 2.15A. From (14.6) with $z = w$, we have

$$\begin{aligned} p_t^+(w,w) &= I_\Theta(0)\eta(w) + \sum_{\theta\in\Theta_+}\frac{e^{-\frac{1}{2}\theta^2 t}f_\theta(w)^2}{\langle f_\theta,\, f_\theta\rangle_+} \\ - m_0\pi(0,w)\sum_{\theta\in\Theta_+}&\frac{e^{-\frac{1}{2}\theta^2 t}f_\theta(0)f_\theta(w)}{\langle f_\theta,\, f_\theta\rangle_+} - m_1\pi(1,w)\sum_{\theta\in\Theta_+}\frac{e^{-\frac{1}{2}\theta^2 t}f_\theta(1)f_\theta(w)}{\langle f_\theta,\, f_\theta\rangle_+}. \end{aligned} \tag{15.3}$$

Integrating both sides of (15.3) with respect to w over $[0, 1]$, using WH5* (which is now proved, of course), we now have

$$\int_0^1 p_t^+(w,w)\,\mathrm{d}w = I_\Theta(0)\underbrace{\int_0^1 \eta(w)\,\mathrm{d}w}_{=1} + \sum_{\theta\in\Theta_+}\frac{\mathrm{e}^{-\frac{1}{2}\theta^2 t}}{\langle f_\theta,\,f_\theta\rangle_+}\int_0^1 f_\theta(w)^2\,\mathrm{d}w$$
$$- \sum_{\theta\in\Theta_+}\frac{\mathrm{e}^{-\frac{1}{2}\theta^2 t}}{\langle f_\theta,\,f_\theta\rangle_+}\, m_0 f_\theta(0)\underbrace{\int_0^1 \pi(0,w) f_\theta(w)\,\mathrm{d}w}_{=f_\theta(0)}$$
$$- \sum_{\theta\in\Theta_+}\frac{\mathrm{e}^{-\frac{1}{2}\theta^2 t}}{\langle f_\theta,\,f_\theta\rangle_+} m_1 f_\theta(1)\underbrace{\int_0^1 \pi(1,w) f_\theta(w)\,\mathrm{d}w}_{=f_\theta(1)}$$
$$= I_\Theta(0) + \sum_{\theta\in\Theta_+}\frac{\mathrm{e}^{-\frac{1}{2}\theta^2 t}}{\langle f_\theta,\,f_\theta\rangle_+}\left\{\int_0^1 f_\theta(w)^2\,\mathrm{d}w - m_0 f_\theta(0)^2 - m_1 f_\theta(1)^2\right\}$$
$$= I_\Theta(0) + \sum_{\theta\in\Theta_+}\frac{\mathrm{e}^{-\frac{1}{2}\theta^2 t}}{\langle f_\theta,\,f_\theta\rangle_+}\langle f_\theta,\,f_\theta\rangle_+$$
$$= \sum_{\theta\in\Theta}\mathrm{e}^{-\frac{1}{2}\theta^2 t}.$$

Remarks: Note that $0 \in \Theta$ if $m_0 + m_1 \leq 1$, and we have already established η as the invariant probability density for this case. However, if $m_0 + m_1 > 1$ then $0 \notin \Theta$ (so $\Theta = \Theta^+$) and so $I_\Theta(0) = 0$. This consequently takes care of the fact that there is *no* invariant probability density when $m_0 + m_1 > 1$.

It also suffices to confirm that, for $t > 0$,

$$\sum_{x\in\{0,1\}} P_t^-(x,\{x\}) = \sum_{\gamma\in\Gamma}\mathrm{e}^{-\frac{1}{2}\gamma^2 t}.$$

Recall from Lemma 2.8H that we have already deduced that

$$\mathbb{E}^z\left[h_\gamma(Z_t^-)\right] = e^{-\frac{1}{2}\gamma^2 t}h_\gamma(z) \qquad \text{for } \gamma\in\Gamma=\{\alpha,\beta\}.$$

From the definition of expectation, we therefore have

$$\mathbb{P}^z(Z_t^-=0)h_\gamma(0) + \mathbb{P}^z(Z_t^-=1)h_\gamma(1) = e^{-\frac{1}{2}\gamma^2 t}h_\gamma(z). \tag{15.4}$$

In particular, taking $z = 0$ and $z = 1$ respectively in (15.4) and noting that $h_\gamma(0) = 1$, we have

$$\mathbb{P}^0(Z_t^-=0) + \mathbb{P}^0(Z_t^-=1)h_\gamma(1) = e^{-\frac{1}{2}\gamma^2 t},$$
$$\mathbb{P}^1(Z_t^-=0) + \mathbb{P}^1(Z_t^-=1)h_\gamma(1) = e^{-\frac{1}{2}\gamma^2 t}h_\gamma(1).$$

Now each of the above results is true *for all* $\gamma \in \{\alpha,\beta\}$. Hence, we may solve the resulting equations simultaneously to get

$$\Delta h\mathbb{P}^0(Z_t^-=0) = h_\beta(1)e^{-\frac{1}{2}\alpha^2 t} - h_\alpha(1)e^{-\frac{1}{2}\beta^2 t}, \tag{15.5}$$
$$\Delta h\mathbb{P}^1(Z_t^-=1) = h_\beta(1)e^{-\frac{1}{2}\beta^2 t} - h_\alpha(1)e^{-\frac{1}{2}\alpha^2 t}, \tag{15.6}$$

where $\Delta h = h_\beta(1) - h_\alpha(1)$. Noting that $\mathbb{P}^x(Z_t^- = x) = P_t^-(x, \{x\})$ and $\Delta h \neq 0$, we may add equations (15.5) and (15.6) to yield the desired result. □

One of Jacobi's theta-function formulae, this one known to Gauss, states that

$$\sum_{n\in\mathbb{Z}} \exp(-\tfrac{1}{2}n^2\pi^2 t) = 2\sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(2n)^2}{2t}\right\}. \tag{15.7}$$

This is the trace formula for Brownian motion on a circle of perimeter 2, the transition density function of which (relative to Lebesgue measure) is symmetric. The left-hand side of (15.7) reflects the fact that the infinitesimal generator $\frac{1}{2}D^2$ has eigenvalues $-\frac{1}{2}n^2\pi^2$ with corresponding eigenfunctions $e^{\pm in\pi w}$. The right-hand side reflects the fact that Brownian motion on $\mathbb{R}$ is shift-invariant and that the circle is $\mathbb{R}/(2\mathbb{Z})$. The sum on the right-hand side is the appropriate $p_t^{\text{circ}}(w, w)$ (obvious notation!) for every w, and the 2 multiplying the sum is the perimeter length. Taking Laplace transforms with parameter $\lambda = \frac{1}{2}\rho^2$ shows that equation (15.7) is equivalent to

$$\sum_{n\in\mathbb{Z}} \frac{2}{\rho^2 + n^2\pi^2} = 2\sum_{n\in\mathbb{Z}} \rho^{-1}e^{-2|n|\rho} = \frac{2\cosh\rho}{\rho\sinh\rho}, \tag{15.8}$$

the second equality being trivial. The equality of the 'extreme' terms is standard Complex Analysis; and Jacobi's formula follows. Euler knew (15.8) in the form

$$\sinh\rho = \rho\prod_{n=1}^{\infty}\left(1 + \frac{\rho^2}{n^2\pi^2}\right). \tag{15.9}$$

(To see the equivalence, simply take ln's and differentiate w.r.t. ρ.)

This makes it plausible that we can crosscheck (15.1) starting from the obvious predictions (with the notation $\Gamma_+ := \Gamma \cap (0, \infty)$):

if $m_0 + m_1 \neq 1$,

$$\tfrac{1}{2}\rho\epsilon(\rho) = \{1 - (m_0 + m_1)\}\tfrac{1}{2}\rho^2 \left\{\prod_{\theta\in\Theta_+}\left(1 + \frac{\frac{1}{2}\rho^2}{\frac{1}{2}\theta^2}\right)\right\}\left\{\prod_{\gamma\in\Gamma_+}\left(1 - \frac{\frac{1}{2}\rho^2}{\frac{1}{2}\gamma^2}\right)\right\}, \tag{15.10}$$

if $m_0 + m_1 = 1$,

$$\tfrac{1}{2}\rho\epsilon(\rho) = 2\left(m_0 m_1 - \tfrac{1}{3}\right)\left(\tfrac{1}{2}\rho^2\right)^2\left(1 - \frac{\frac{1}{2}\rho^2}{\frac{1}{2}\beta^2}\right)\left\{\prod_{\theta\in\Theta_+}\left(1 + \frac{\frac{1}{2}\rho^2}{\frac{1}{2}\theta^2}\right)\right\}. \tag{15.11}$$

The products certainly converge for $\rho \in \mathbb{C}$ and have the correct zeros. To make the Complex Analysis rigorous, more is needed.

2.15B. Verification of the product formulae.

We use the Weierstrass-Hadamard Factorization Theorem (see Titchmarsh [25]) to prove the formulae by Complex Analysis.

Suppose $m_0 + m_1 \neq 1$. Recall that

$$\epsilon(\rho) = (1 + m_0 m_1 \rho^2)\sinh(\rho) - (m_0 + m_1)\rho\cosh(\rho).$$

It follows that $\frac{\epsilon(\rho)}{\rho}$ is an integral function of order 1 with zeros $\pm\gamma$ ($\gamma \in \Gamma_+$) and $\pm i\theta$ ($\theta \in \Theta_+$). The Weierstrass-Hadamard Theorem gives us the following factorization

$$\frac{\epsilon(\rho)}{\rho} = e^{Q(\rho)}P(\rho),$$

where $Q(\rho) = q_0 + q_1\rho$, ($q_i \in \mathbb{C}$) and $P(\rho)$ is a product of terms of the form

$$\left(1 - \frac{\rho}{\eta}\right)e^{\frac{\rho}{\eta}},$$

where η is a general root of $\frac{\epsilon(\rho)}{\rho}$. However, the terms corresponding to η and $-\eta$ combine to give a factor

$$\left(1 - \frac{\rho}{\eta}\right)e^{\frac{\rho}{\eta}}\left(1 + \frac{\rho}{\eta}\right)e^{-\frac{\rho}{\eta}} = \left(1 - \frac{\rho^2}{\eta^2}\right).$$

Hence, we now have

$$\frac{\epsilon(\rho)}{\rho} = e^{Q(\rho)}\prod_{\eta>0}\left(1 - \frac{\rho^2}{\eta^2}\right).$$

Next we consider what happens to $\frac{\epsilon(\rho)}{\rho}$ as we approach zero. This allows us to determine $Q(\rho)$. Considering the Taylor expansions of sinh and cosh about zero, we see that

$$\lim_{\rho\to 0}\frac{\epsilon(\rho)}{\rho} = 1 - (m_0 + m_1).$$

We now have

$$\frac{\epsilon(\rho)}{\rho} = \{1 - (m_0 + m_1)\}e^{q_1\rho}P(\rho).$$

In particular,

$$e^{q_0} = 1 - (m_0 + m_1).$$

Note that this does make sense if $m_0 + m_1 > 1$ as $q_0 \in \mathbb{C}$. Remember that $e^{\pi i} = -1$.

Next we show that $q_1 = 0$. Remember that $\epsilon(\cdot)$ is an odd function. It therefore follows that $\frac{\epsilon(\rho)}{\rho}$ is an even function. Thus, our corresponding factorization must be even. Recall that a function f is even if, and only if, $f(x) = f(-x)$, for all x in its domain. In brief, we simply need

$$e^{q_1\rho}P(\rho) = e^{-q_1\rho}P(-\rho).$$

However, P is clearly even, so we need

$$e^{q_1\rho} = e^{-q_1\rho},$$

which is true for all ρ if and only if $q_1 = 0$. Hence, we now have

$$\begin{aligned} \frac{\epsilon(\rho)}{\rho} &= \{1-(m_0+m_1)\}\prod_{\eta>0}\left(1-\frac{\rho^2}{\eta^2}\right) \\ \Leftrightarrow \quad \epsilon(\rho) &= \{1-(m_0+m_1)\}\rho\prod_{\theta\in\Theta_+}\left(1+\frac{\rho^2}{\theta^2}\right)\prod_{\gamma\in\Gamma_+}\left(1-\frac{\rho^2}{\gamma^2}\right). \end{aligned}$$

If $m_0+m_1=1$, a similar argument applied to $\frac{\epsilon(\rho)}{\rho^3}$, together with

$$\lim_{\rho\to 0}\frac{\epsilon(\rho)}{\rho^3} = m_0m_1 - \tfrac{1}{3},$$

yields

$$\begin{aligned} \frac{\epsilon(\rho)}{\rho^3} &= \{m_0m_1-\tfrac{1}{3}\}\prod_{\eta>0}\left(1-\frac{\rho^2}{\eta^2}\right) \\ \Leftrightarrow \quad \epsilon(\rho) &= \{m_0m_1-\tfrac{1}{3}\}\rho^3\prod_{\theta\in\Theta_+}\left(1+\frac{\rho^2}{\theta^2}\right)\prod_{\gamma\in\Gamma_+}\left(1-\frac{\rho^2}{\gamma^2}\right). \end{aligned}$$

One problem with the Complex Analysis is to exclude the possibility of double roots of $\epsilon(\cdot)$. □

In the light of these product formulae, the trace formula (15.1) becomes equivalent to the identity

$$\int_0^1 r_\lambda^+(w,w)\,\mathrm{d}w = \frac{\epsilon'(\rho)}{\rho\epsilon(\rho)}+\frac{1}{\rho^2}-\frac{2}{\rho^2-\alpha^2}-\frac{2}{\rho^2-\beta^2}. \tag{15.12}$$

2.15C. Verification of (15.12).

Taking Laplace transforms with parameter $\lambda = \frac{1}{2}\rho^2$ in (15.1), we get

$$\int_{(0,1)} r_\lambda^+(w,w)\mathrm{d}w = \sum_{\theta\in\Theta}\frac{2}{\theta^2+\rho^2}. \tag{15.13}$$

Remark. Observe the similarity of (15.13) with the LHS of (15.8) when our roots are of the form $\theta \approx n\pi$.

We are now in a position to crosscheck (15.12) using (15.13) together with (15.10) and (15.11).

Case (i): $m_0+m_1<1$. Here it suffices to use the factorization given in (15.10). If we firstly take natural logarithms of both sides of (15.10) and then differentiate w.r.t. ρ, we have

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\ln\epsilon(\rho) = \frac{\mathrm{d}}{\mathrm{d}\rho}\ln\left\{[1-(m_0+m_1)]\rho\prod_{\theta\in\Theta_+}\left(1+\frac{\rho^2}{\theta^2}\right)\prod_{\gamma\in\Gamma_+}\left(1-\frac{\rho^2}{\gamma^2}\right)\right\}.$$

Using elementary properties of logarithms together with simple differentiation, the above result is equivalent to

$$\frac{\epsilon'(\rho)}{\epsilon(\rho)} = \frac{\mathrm{d}}{\mathrm{d}\rho}\left\{\underbrace{\ln[1-(m_0+m_1)]}_{\text{constant}} + \ln\rho + \sum_{\theta\in\Theta_+}\ln\left(\frac{\theta^2+\rho^2}{\theta^2}\right) + \sum_{\gamma\in\Gamma_+}\ln\left(\frac{\gamma^2-\rho^2}{\gamma^2}\right)\right\}$$
$$= \frac{1}{\rho} + \rho\sum_{\theta\in\Theta_+}\left(\frac{2}{\theta^2+\rho^2}\right) + \rho\sum_{\gamma\in\Gamma_+}\left(\frac{2}{\rho^2-\gamma^2}\right).$$

Dividing both sides by $\rho > 0$ and rearranging, we now have

$$\sum_{\theta\in\Theta_+}\left(\frac{2}{\theta^2+\rho^2}\right) = \frac{\epsilon'(\rho)}{\rho\epsilon(\rho)} - \frac{1}{\rho^2} - \sum_{\gamma\in\Gamma_+}\left(\frac{2}{\rho^2-\gamma^2}\right)$$

$$\Leftrightarrow \quad \sum_{\theta\in\Theta}\left(\frac{2}{\theta^2+\rho^2}\right) = \frac{\epsilon'(\rho)}{\rho\epsilon(\rho)} + \frac{1}{\rho^2} - \sum_{\gamma\in\Gamma_+}\left(\frac{2}{\rho^2-\gamma^2}\right) \qquad \text{(since } 0\in\Theta \text{ for this case.)}$$

$$= \frac{\epsilon'(\rho)}{\rho\epsilon(\rho)} + \frac{1}{\rho^2} - \frac{2}{\rho^2-\alpha^2} - \frac{2}{\rho^2-\beta^2} \qquad \text{(since } \alpha,\beta>0\text{ .)}$$

Recalling (15.13) we get the desired result.

Case (ii): $m_0 + m_1 > 1$. Note that the fact that we are allowing $\rho \in \mathbb{C}$ eradicates the issue of considering the logarithm of a negative number. All we then need to do is to recall that $\Gamma_+ = \{\beta\}$, but $0 \in \Theta$. We may then follow a similar argument to that given in the previous case to deduce that

$$\sum_{\theta\in\Theta}\left(\frac{2}{\theta^2+\rho^2}\right) = \frac{\epsilon'(\rho)}{\rho\epsilon(\rho)} - \frac{1}{\rho^2} - \frac{2}{\rho^2-\beta^2}. \tag{15.14}$$

Case (iii): $m_0 + m_1 = 1$. Again $\Gamma_+ = \{\beta\}$, yet $0 \notin \Theta$, so that $\Theta = \Theta_+$. Observing these facts we may deduce that (15.14) holds. □

2.15D. Direct verification of (15.12).

Direct verification of equation (15.12) is extremely complicated unless one utilizes the indefinite inner product $\langle\cdot,\cdot\rangle_s$. First, we use Section 10 to find a 'symmetric' formula for the 'diagonal' resolvent density:

$$\tfrac{1}{2}\rho\epsilon(\rho)r_\lambda^+(w,w) \;=\; h_\rho^\sharp(w)h_\rho(w) - m_1\pi(1,w)h_\rho(w) - m_0\pi(0,w)h_\rho^\sharp(w). \tag{15.15}$$

(Observe that the right-hand side is 0 when $\rho \in \Gamma$.)

Proof of (15.15). From (10.9) we have

$$\int_0^1 r_\lambda^+(z,w)f(w)\mathrm{d}w \;=\; A_\lambda(f)c_\rho(z) + B_\lambda(f)s_\rho(z) - 2\rho^{-1}\int_0^z s_\rho(z-w)f(w)\,\mathrm{d}w, \tag{15.16}$$

so that in particular

$$\int_0^1 r_\lambda^+(w,w)f(w)\mathrm{d}w \;=\; A_\lambda(f)c_\rho(w) + B_\lambda(f)s_\rho(w).$$

It follows that

$$r_\lambda^+(w,w) = A_\lambda(0,w)c_\rho(w) + B_\lambda(0,w)s_\rho(w).$$

Multiplying both sides by $\frac{1}{2}\rho\epsilon(\rho)$ ($\rho > 0 \notin \Gamma$), we now have

$$\tfrac{1}{2}\rho\epsilon(\rho)r_\lambda^+(w,w) = \tfrac{1}{2}\rho\epsilon(\rho)A_\lambda(0,w)c_\rho(w) + \tfrac{1}{2}\rho\epsilon(\rho)B_\lambda(0,w)s_\rho(w). \tag{15.17}$$

From (10.12), recall that we have

$$\tfrac{1}{2}\rho\epsilon(\rho)A_\lambda(0,w) = h_\rho^\sharp(w) - m_0h_\rho^\sharp(0)\pi(0,w) - m_1\pi(1,w). \tag{15.18}$$

Next, if we take (10.12), multiply by $\epsilon(\rho)$ and rearrange, then we get

$$\tfrac{1}{2}\rho\epsilon(\rho)B_\lambda(0,w) = m_0\pi(0,w)\epsilon(\rho) - \tfrac{1}{2}m_0\rho^2\epsilon(\rho)A_\lambda(0,w).$$

Substitution of (15.18) into the above equation yields

$$\tfrac{1}{2}\rho\epsilon(\rho)B_\lambda(0,w) = m_0m_1\rho\pi(1,w) + \left\{m_0\epsilon(\rho) + m_0^2\rho h_\rho^\sharp(0)\right\}\pi(0,w) - m_0\rho h_\rho^\sharp(w). \tag{15.19}$$

We may now substitute (15.18) and (15.19) into (15.17) to yield

$$\begin{aligned}\tfrac{1}{2}\rho\epsilon(\rho)r_\lambda^+(w,w) = h_\rho^\sharp(w)\underbrace{[c_\rho(w) - m_0\rho s_\rho(w)]}_{=h_\rho(w)} - m_1\pi(1,w)\underbrace{[c_\rho(w) - m_0\rho s_\rho(w)]}_{=h_\rho(w)} \\ - m_0\pi(0,w)\left[h_\rho^\sharp(0)c_\rho(w) - \epsilon(\rho)s_\rho(w) - m_0\rho h_\rho^\sharp(0)s_\rho(w)\right].\end{aligned}$$

Clearly all that it remains to do is to prove that

$$h_\rho^\sharp(0)c_\rho(w) - \epsilon(\rho)s_\rho(w) - m_0\rho h_\rho^\sharp(0)s_\rho(w) = h_\rho^\sharp(w).$$

This follows by recalling the form of $\epsilon(\rho)$ together with the facts that

$$c_\rho(1)c_\rho(w) - s_\rho(1)s_\rho(w) = c_\rho(1-w) \quad \text{and} \quad s_\rho(1)c_\rho(w) - c_\rho(1)s_\rho(w) = s_\rho(1-w).$$

□

If we use the fact that $r_\lambda^+(0,0)$ and $r_\lambda^+(1,1)$ are both zero, integrating both sides of (15.15) w.r.t. the signed measure ν, we now get

$$\tfrac{1}{2}\rho\epsilon(\rho)\int_0^1 r_\lambda^+(w,w)\,\mathrm{d}w \;=\; \langle h_\rho^\sharp, h_\rho\rangle_s - m_1\langle\pi(1,\cdot), h_\rho\rangle_s - m_0\langle\pi(0,\cdot), h_\rho^\sharp\rangle_s.$$

We now verify (15.12) by utilizing (4.1), (9.2) and the result

$$\langle h_\gamma, h_\rho^\sharp\rangle_s \;=\; \frac{\rho\epsilon(\rho) - \gamma\epsilon(\gamma)}{\rho^2 - \gamma^2} \qquad (\gamma \in \mathbb{C},\ \rho \in \mathbb{C},\ \rho \neq \gamma),$$

and its L'Hôpital consequence

$$\langle h_\rho, h_\rho^\sharp\rangle_s \;=\; \tfrac{1}{2}[\epsilon'(\rho) + \rho^{-1}\epsilon(\rho)].$$

2.15E. Remark. The product formulae may be deduced trivially from (15.12) and (15.1) by simply reversing the argument given in 2.15C. Furthermore, we now have a perfectly legitimate *direct* proof of (15.12), so that the forms given in (15.10) and (15.11) are certainly correct. This therefore eradicates the possibility of repeated roots as we would otherwise expect terms of the form $\left(1 + \frac{\rho^2}{\theta^2}\right)^k$ or $\left(1 - \frac{\rho^2}{\gamma^2}\right)^k$, for $k \geq 2$.

2.16. Appendix 1: Some Functional Analysis and Required Measure Theory

We begin by stating an advanced version of the Monotone-Class Theorem.

2.16A. Theorem (Monotone-Class). *Let $\mathcal{H}$ be a vector space of bounded real-valued functions on a set S. Suppose that $\mathcal{H}$ contains constant functions, is closed under uniform convergence, and has the following property: for a uniformly bounded sequence (f_n) of non-negative functions in $\mathcal{H}$ such that $f_n(s) \uparrow f(s)(\forall s)$, we must have $f \in \mathcal{H}$. If $\mathcal{H}$ contains a subset $\mathcal{C}$ that is closed under multiplication, then $\mathcal{H}$ contains every bounded $\sigma(\mathcal{C})$- measurable function from S to $\mathbb{R}$.*

Notes: If $\mathcal{H}$ is closed under uniform convergence then the following holds:

$$f_n \in \mathcal{H} \quad \text{and} \quad f_n \mapsto f \text{ uniformly on } S \quad \Rightarrow \quad f \in \mathcal{H}.$$

A sequence (f_n) is *uniformly bounded* if, for some constant $M > 0$,

$$|f_n(s)| \leq M \qquad \forall s \in S \quad \text{and} \quad \forall n.$$

Hereafter, for typographic neatness, Theorem 2.16A will be referred to as the MCT.

Proof of Theorem 2.16A. See Volume 1 of Rogers & Williams [24]. There, a more elementary version of the MCT is also given. Although this alternative version is easier to prove, it introduces the difficulty of dealing with indicator functions. □

For the purpose of the following Lemma, we let (S, Σ, μ) be a measure space.

2.16B. Lemma (Monotone-Convergence).

(a) *If $F_n \in \Sigma$ ($n \in \mathbb{N}$) and $F_n \uparrow F$, then $\mu(F_n) \uparrow \mu(F)$.*

(b) *If $G_n \in \Sigma, G_n \downarrow G$ and $\mu(G_k) < \infty$ for some k, then $\mu(G_n) \downarrow \mu(G)$.*

From now on the above lemma will simply be referred to as MON.

Proof of Lemma 2.16B. See Williams [29] for a proof. □

2.16C. Some important spaces. We now introduce the following spaces:

- $C[0,1]$ where, for $f \in C[0,1]$, we define the supremum norm $||f|| := \sup_{z\in[0,1]} |f(z)|$;
- $L^2[0,1]$ which is an abbreviation for $L^2\big([0,1], \mathcal{B}([0,1]), \text{Leb}\big)$ with

$$||f||_{L^2} := \left\{ \int_0^1 |f(z)^2| dz \right\}^{\frac{1}{2}}.$$

We next establish some helpful results concerning the two spaces. It is well-known that the spaces are examples of Banach spaces.

2.16D. Lemma. $C[0,1]$ *is dense in* $L^2[0,1]$.

Proof of Lemma 2.16D. Suppose for a contradiction that $C[0,1]$ is not dense in L^2. Then there exists a non-zero bounded linear functional λ on L^2 such that $\lambda(h) = 0$ for all $h \in C[0,1]$. But for the Hilbert space L^2, the bounded linear functional λ must take the form

$$\lambda(f) = \langle f, g \rangle = \int_0^1 f(y)g(y)\mathrm{d}y,$$

where $g \in L^2$. It is a trivial exercise to show that this implies $g \in L^1$, in that,

$$\int_0^1 |g(y)|\mathrm{d}y < \infty.$$

Now let $\mathcal{H}$ be the class of bounded measurable functions h on $[0,1]$ such that

$$\int_0^1 h(y)g(y)\mathrm{d}y = 0 \quad (= \lambda(h)).$$

We now appeal to the MCT. Some fundamental theorems of Measure Theory will be used to prove that $\mathcal{H}$ satisfies the necessary assumptions in the MCT.

$\mathcal{H}$ clearly is a vector space. We simply use linearity. Moreover, $\mathcal{H}$ contains constant functions as these are continuous.

$\mathcal{H}$ is closed under uniform convergence if, whenever $(h_n) \in \mathcal{H}$ and $|h(y) - h_n(y)| \leq \epsilon$ for all $y \in [0,1]$ (i.e. we know that for all $\epsilon > 0$ there always exists a suitable n), then $h \in \mathcal{H}$. Consider the following,

$$\begin{aligned}
\left| \int_0^1 h(y)g(y)\mathrm{d}y - \int_0^1 h_n(y)g(y)\mathrm{d}y \right| &= \left| \int_0^1 \{h(y) - h_n(y)\}\, g(y)\mathrm{d}y \right| \\
&\leq \int_0^1 \left| \{h(y) - h_n(y)\}\, g(y) \right| \mathrm{d}y \\
&\leq \int_0^1 |h(y) - h_n(y)|\ |g(y)|\mathrm{d}y \\
&\leq \epsilon \int_0^1 |g(y)|\mathrm{d}y = \epsilon K,
\end{aligned}$$

where K is simply a constant since $g \in L^1$. Hence, we have

$$\left| \int_0^1 h(y)g(y)\mathrm{d}y - \int_0^1 h_n(y)g(y)\mathrm{d}y \right| \leq \epsilon K. \tag{16.1}$$

By allowing $\epsilon \to 0$ in (16.1), we see that

$$0 = \int_0^1 h_n(y)g(y)\mathrm{d}y = \int_0^1 h(y)g(y)\mathrm{d}y,$$

so that $h \in \mathcal{H}$ as desired.

The final condition is also quite straightforward to satisfy. If $h_n \in \mathcal{H}$ and $h_n(y) \uparrow h(y)$ for all $y \in [0,1]$, then, by the MON we can easily deduce that

$$0 = \int_0^1 h_n(y)g(y)\mathrm{d}y \uparrow \int_0^1 h(y)g(y)\mathrm{d}y,$$

so that $\int_0^1 h(y)g(y)\mathrm{d}y = 0$ and so it follows that $h \in \mathcal{H}$.

Next take $\mathcal{C} = C[0,1] \subseteq \mathcal{H}$. Clearly $\mathcal{C}$ is closed under multiplication. Then, by the MCT, we have

$\mathcal{H}$ contains every bounded $\sigma(\mathcal{C})$-measurable function from $[0,1]$ to $\mathbb{R}$.

In other words,

$$\int_0^1 h(y)g(y)\mathrm{d}y = 0 \tag{16.2}$$

for *every* bounded measurable function h on $[0,1]$. Recall that $g \in L^1$. However, this *does not* imply that g is bounded on $[0,1]$. This fact motivates the following definition. Let

$$g_n(y) := \begin{cases} g(y) & \text{if } |g(y)| \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Then g_n is certainly bounded on $[0,1]$ for each n. Thus by (16.2), we have

$$\int_0^1 g_n(y)g(y)\mathrm{d}y = 0, \qquad \text{so that} \qquad \int_0^1 g(y)^2\mathrm{d}y = 0.$$

It follows that $g = 0$ almost everywhere and so λ *is* the zero functional. This gives us the desired contradiction. □

2.16E. Definition. *Let S be a class of functions on a space U. If μ_1 and μ_2 are finite measures on (Borel subsets of) U, then S is said to be measure-determining if $\mu_1(f) := \int f\mathrm{d}\mu_1 = \mu_2(f)$ for all f in S implies that $\mu_1 = \mu_2$.*

In accordance with the usual convention, 'measure' will always mean 'non-negative measure' (as opposed to 'signed measure').

2.16F. Lemma. *Suppose that μ and ν are finite measures on $\mathcal{B}[0,1]$. Then the space $C[0,1]$ is measure-determining.*

Proof of Lemma 2.16F. From Definition 2.16E, we suppose that

$$\int_{[0,1]} f(s)\mu(\mathrm{d}s) = \int_{[0,1]} f(s)\nu(\mathrm{d}s) \tag{16.3}$$

for every $f \in C[0,1]$. Then it suffices to show that $\mu = \nu$. That is, $\mu(B) = \nu(B)$ for every $B \in \mathcal{B}[0,1]$. We shall show that we can extend (16.3) from functions in $C[0,1]$ to functions in $\mathrm{b}\mathcal{B}[0,1]$, the space of bounded Borel functions on $[0,1]$. Taking $f = I_B$ will then clinch things. We again appeal to the MCT. Let

$\mathcal{H}$ = {class of bounded Borel functions f such that (16.3) holds}.

We now basically 'mimic' arguments presented in the proof of Lemma 2.16D.

Again, by linearity, $\mathcal{H}$ is a vector space which contains constant functions as constant functions are continuous.

$\mathcal{H}$ is closed under uniform convergence if, whenever $(f_n) \in \mathcal{H}$ and $|f(s)-f_n(s)| \leq \epsilon \quad \forall s \in S$ (i.e. we know that for all $\epsilon > 0$ there always exists a suitable n), then $f \in \mathcal{H}$. It is clear that

$$\begin{aligned}\left|\int_{[0,1]} f(s)\mu(\mathrm{d}s) - \int_{[0,1]} f_n(s)\mu(\mathrm{d}s)\right| &= \left|\int_{[0,1]} (f(s) - f_n(s))\mu(\mathrm{d}s)\right| \\ &\leq \int_{[0,1]} |f(s) - f_n(s)|\mu(\mathrm{d}s) \\ &\leq \epsilon \int_{[0,1]} \mu(\mathrm{d}s).\end{aligned}$$

Hence, we have

$$\left|\int_{[0,1]} f(s)\mu(\mathrm{d}s) - \int_{[0,1]} f_n(s)\mu(\mathrm{d}s)\right| \leq \epsilon K, \tag{16.4}$$

where K is a constant. A similar argument shows that

$$\left|\int_{[0,1]} f_n(s)\nu(\mathrm{d}s) - \int_{[0,1]} f(s)\nu(\mathrm{d}s)\right| \leq \epsilon K', \tag{16.5}$$

where K' is also a constant. Adding (16.4) and (16.5), using the triangle inequality, and noting that (16.3) holds for f_n, we obtain

$$\left|\int_{[0,1]} f(s)\mu(\mathrm{d}s) - \int_{[0,1]} f(s)\nu(\mathrm{d}s)\right| \leq \epsilon(K + K'), \tag{16.6}$$

It is clear that, by allowing $\epsilon \to 0$ in (16.6), $f \in \mathcal{H}$.

If $f_n(s) \uparrow f(s) \quad \forall s \in S$, then, by the MON, we can deduce that

$$\int_{[0,1]} f_n(s)\mu(\mathrm{d}s) \uparrow \int_{[0,1]} f(s)\mu(\mathrm{d}s), \qquad \int_{[0,1]} f_n(s)\nu(\mathrm{d}s) \uparrow \int_{[0,1]} f(s)\nu(\mathrm{d}s).$$

Moreover, if $f_n(s) \in \mathcal{H}$, then (16.3) holds for f_n and so it follows that $f \in \mathcal{H}$, as desired.

Note. The fact that the sequence is uniformly bounded was not needed. However, if we alternatively decided to use the Dominated-Convergence Theorem (see Williams [29]), then we would have made use of the fact.

Take $\mathcal{C} = C[0,1] \subseteq \mathcal{H}$. Clearly $\mathcal{C}$ is closed under multiplication. Then, by the MCT, we have

$\mathcal{H}$ contains every bounded $\sigma(\mathcal{C})$-measurable function from S to $\mathbb{R}$.

It now suffices to show that $\sigma(\mathcal{C}) = \mathcal{B}[0,1]$. To do this, it is enough to prove by bare hands that we can express the indicator function of an arbitrary closed sub-interval of $[0,1]$ as the pointwise limit of a sequence of continuous functions.

$$f_n \text{ is a continuous sequence of functions} \quad \Rightarrow \quad f_n \in \mathrm{m}\sigma(\mathcal{C}).$$

We can also say that

$$f_n(s) \mapsto f(s) \quad \forall s \quad \Rightarrow \quad f \in \mathrm{m}\sigma(\mathcal{C}).$$

In our particular case, the following sequence of functions suffices. For $0 < a < b < 1$, define

$$f_n(s) = \begin{cases} 0, & \text{for } s \in [0, a(1-\frac{1}{n})), \\ \frac{ns}{a} - (n-1), & \text{for } s \in [a(1-\frac{1}{n}), a), \\ 1, & \text{for } s \in [a,b], \\ \frac{n}{(1-b)}(b-s)+1, & \text{for } s \in (b, b+\frac{(1-b)}{n}], \\ 0, & \text{for } s \in (b+\frac{(1-b)}{n}, 1]. \end{cases}$$

Clearly, $f_n(s) \mapsto f(s) = I_{[a,b]}(s) \quad \forall s \in [0,1]$. Hence, as $\mathcal{C} \subseteq \sigma(\mathcal{C})$,

$[a,b] = \{s : f(s) = 1\} \in \sigma(\mathcal{C}), \quad \therefore \quad \sigma(\mathcal{C})$ contains **all** closed sub-intervals of [0,1].

We know that the smallest σ-algebra on $[0,1]$ containing the closed sub-intervals of $[0,1]$ is $\mathcal{B}[0,1]$. This means that $\sigma(\mathcal{C}) = \mathcal{B}[0,1]$ (See, for example, Chapter 1 of Williams [29].)

Remark: In Williams [29] we see that $\mathcal{B}(S)$ is the smallest σ-algebra generated by the *open* intervals of S. The fact that we are dealing with closed intervals in the problem makes no difference as we can simply consider the complements of these intervals which are indeed open.

We can now deduce that $\mathcal{H} \supseteq \mathrm{b}\mathcal{B}[0,1]$. Hence,

$I_B \in \mathcal{H}$ for every $B \in \mathcal{B}[0,1]$, i.e. (16.3) holds for $f = I_B$.

Thus, taking $f = I_B$ in (16.3), we have

$$\int_{[0,1]} I_B(s)\mu(\mathrm{d}s) = \int_{[0,1]} I_B(s)\nu(\mathrm{d}s),$$

for every $B \in \mathcal{B}[0,1]$, and so it follows that $\mu(B) = \nu(B)$ for every $B \in \mathcal{B}[0,1]$. We have shown that $C[0,1]$ is measure-determining. □

2.16G. Some additional spaces. We now introduce the dual spaces of the Banach spaces described in 2.16C.

- Considering $C[0,1]$, then its dual space $C[0,1]^*$ is the space of bounded signed measures μ on $\big([0,1], \mathcal{B}[0,1]\big)$, so that μ may be written as the difference of two finite (positive) measures. Of course, $\mu(f)$ is simply the integral $f\mathrm{d}\mu$. If μ is a (positive) measure, then $||f||$ in $C[0,1]^*$ is just $\mu([0,1])$.

- For $L^2[0,1]$, then $L^2[0,1]^*$ may be identified with $L^2[0,1]$ in that if $\ell \in L^2[0,1]^*$, then there exists $g \in L^2[0,1]$ such that

$$\ell(f) = \langle g, f\rangle_{L^2} = \int_{[0,1]} g(y)f(y)\mathrm{d}y.$$

(See Dunford & Schwarz [6] for additional information).

2.16H. Lemma. *Let S be a linear subspace of the Banach space $C[0,1]$. Suppose that whenever $\ell \in C[0,1]^*$ is such that $\ell(f) = 0$ for all $f \in S$, then $\ell = 0$. Then S is a dense subspace of $C[0,1]$.*

Proof of Lemma 2.16H. This is a special case of a result given in III 3.13 of Dunford & Schwartz [6]. It is a consequence of the Hanh-Banach Theorem. □

2.16I. Lemma. *Let $U = [0,1]$. A linear subspace S of $C[0,1]$ is measure-determining if and only if S is dense in $C[0,1]$.*

Proof of Lemma 2.16I. ($\Rightarrow$) Suppose that $S \subset C[0,1]$ is measure-determining, that is, for μ, ν finite measures on $\mathcal{B}[0,1]$,

$$\mu(f) = \nu(f) \quad \textit{for all } f \in S \qquad \Rightarrow \qquad \mu(B) = \nu(B) \quad \textit{for all } B \in \mathcal{B}[0,1].$$

Now

$$\mu(f) = \nu(f) \qquad \Leftrightarrow \qquad (\mu - \nu)(f) = 0 \qquad \Leftrightarrow \qquad l(f) = 0 \qquad \textit{for all } f \in S,$$

where $l \in C[0,1]^*$. Hence the fact that S is measure-determining here is equivalent to

$$l(B) = 0 \qquad \textit{for all } B \in \mathcal{B}[0,1].$$

From Lemma 2.16H, it follows that S is dense in $C[0,1]$.

($\Leftarrow$) Next suppose that S is dense in $C[0,1]$. In Lemma 2.16F we established that $C[0,1]$ is measure determining, so that S is certainly measure-determining. □

2.16J. Instructive Example. We will show that the linear space of functions f in $C[0,1]$ such that $5f(0) = \int_{[0,1]} f(y)\mathrm{d}y$ is not measure-determining but is dense in $L^2([0,1], \mathrm{Leb})$.

Recall that $U = [0,1]$. Define

$$S := \left\{ f \in C[0,1] : \; 5f(0) = \int_{[0,1]} f(y)\mathrm{d}y \right\} \subset C[0,1].$$

Clearly

$$5\int_{[0,1]} f(y)\, \mathrm{d}\delta_0(y) = \int_{[0,1]} f(y)\, \mathrm{dLeb}(y) \qquad \textit{for all } f \in S,$$

where the measures $(5\delta_0)(\cdot)$ and $(\text{Leb})(\cdot)$ are measures on $(U, \mathcal{B}(U))$. However, it is obvious that

$$5\delta_0 \neq \text{Leb} \qquad \text{on } (U, \mathcal{B}(U)),$$

and so S is not measure-determining ($\Rightarrow$ *S is not dense in $C[0,1]$ $\not\Rightarrow$ S is not dense in L^2*). For example, consider the set $B = (0, \frac{1}{2}) \in \mathcal{B}[0,1]$. Then $(5\delta_0)(B) = 0$, but $\text{Leb}(B) = \frac{1}{2}$.

Since $C[0,1]$ is dense in $L^2[0,1]$ (from Lemma 2.16D), we need only show that if $h \in C[0,1]$, then h may be approximated in L^2 by some h_n in S. For some α_n, $h_n = h - \alpha_n f_n \in S$ where f_n is as in the next Instructive Example. We have $\alpha_n \to h(0) - \frac{1}{5}\int_0^1 h(z)\mathrm{d}z$, so $||h_n - h||_{L^2} = |\alpha_n|.||f_n||_{L^2} \to 0$. Hence, it follows that S is dense in $L^2[0,1]$.

2.16K. Instructive Example. This example highlights a distinction between the spaces $L^2[0,1]$ and $C[0,1]$. The distinction remains a key point in Chapter 2, especially when it comes to the P_t^+ semigroup. Consider the function

$$f_n(x) = \begin{cases} 1 - nx & \text{on } [0, \frac{1}{n}], \\ 0 & \text{on } [\frac{1}{n}, 1]. \end{cases}$$

Then, we have

$$||f_n||_{\text{sup}} := \sup_{x\in[0,1]} |f_n(x)| = 1, \qquad ||f_n||_{L^2[0,1]} := \left(\int_0^1 f_n(x)^2 \mathrm{d}x\right)^{\frac{1}{2}} = \frac{1}{\sqrt{3n}}.$$

In particular, we see that

$$f_\infty(x) := \lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \in (0,1], \end{cases} \qquad \text{so that} \quad f_\infty(0) = \begin{cases} 1 & \text{in } C[0,1], \\ 0 & \text{in } L^2. \end{cases}$$

This example demonstrates the fact that the L^2 space cannot see the difference between $f_\infty(x)$ and the function identically equal to zero on $[0,1]$. However, the space $C[0,1]$ does not have this difficulty.

2.16L. Definition (Hilbert-Adjoint maps). *Let $H_1 = L^2(S_1, \Sigma_1, \mu_1)$ and $H_2 = L^2(S_2, \Sigma_2, \mu_2)$. Suppose that $T : H_1 \to H_2$ is a bounded linear map. Then we define the adjoint map $T^* : H_2 \to H_1$ via the fact that, for $f_1 \in H_1$ and $g_2 \in H_2$,*

$$\langle Tf_1, g_2\rangle = \langle f_1, T^* g_2\rangle.$$

The T^* is a bounded linear map with the same norm as T.

(*Justification:* The map $f_1 \mapsto \langle Tf_1, g_2\rangle$ lies in the dual space of H_1, so defines an element $T^* g_2$ of H_1, etc.)

2.16M. Definition (Self-adjoint). *Suppose that $H = L^2(S, \Sigma, \mu)$ and that T is a Bounded Linear Operator from H to H. We call T self-adjoint if $T^* = T$.*

2.16N. Definition. *The space* $L^2(\{0,1\},m)$. *Suppose that* $\underline{u}, \underline{v}$ *are column vectors of length 2 in* $L^2(\{0,1\},m)$, *then the associated inner product is given by*

$$\langle \underline{u}, \underline{v} \rangle = \sum_{x \in \{0,1\}} m_x u(x) v(x).$$

2.16O. Proposition. *The* 2×2 *matrix* W^- *as defined in* (11.4) *is self-adjoint relative to* $L^2(\{0,1\},m)$.

Proof of Proposition 2.16O. The proof is trivial. □

2.16P. Proposition. W^+ *is self-adjoint on* $L^2[0,1]$, *where* W^+ *is defined in* (11.2) *and* (11.3).

Proof of Proposition 2.16P. This follows by Fubini's Theorem. □

2.17. Appendix 2: Additional Results for Operators on L^2

Let T be a self-adjoint compact operator on a (non-zero) Hilbert space X. For complex λ, let X_λ be the λ-eigenspace

$$X_\lambda = \{x \in X : Tx = \lambda x\}$$

of T on X. We arrive at the following theorem.

2.17A. Theorem.

- *T has only countably many eigenvalues, and there is a complete orthonormal basis consisting of eigenvectors.*
- *One or the other of* $\pm||T||_X$ *is a eigenvalue of* T.

Proof of Theorem 2.17A. A proof can be found in Garrett [11]. □

2.17B. Corollary. *Given any* $x \in X$, *we have*

$$x = \sum_\lambda c_\lambda x_\lambda$$

where $c_\lambda \in \mathbb{R}$ *and* $x_\lambda \in X_\lambda$, *the series converging in the topology of* X.

Chapter 3

One-Boundary with Drift

Summary

For this problem the range of the underlying stochastic process is not compact. As we shall see, this almost immediately causes problems with the functional analysis. In particular, we have difficulties with domains of operators and with the description of symmetry. Furthermore, there is no series spectral expansion. This emphasizes the importance, and indeed the great benefit, of such an expansion in the previous chapter. The main loss is the ability to easily deduce that the $P_t^+ f$ semigroup is $C^{1,2}$. This alone makes the route through this chapter much more arduous. Thus, the detailed study of more general continuity issues is deferred to the final chapter.

As demonstrated in the previous chapter, many analytic statements either asserted, or were consequences of, martingale properties. It is therefore unsurprising that a lack of bounded martingales in certain cases for this problem further restrict us.

Due to the similarity of the setup to this problem with the previous two-boundary problem, many definitions will be re-stated.

Notation: within the bounds of reason, we use y to denote a point of the open interval $(0, \infty)$, and z for a point of the interval $[0, \infty)$.

3.1. The Operator $\mathcal{H}$ and Indefinite Inner Product $\langle \cdot, \cdot \rangle_s$

We begin by defining the following spaces;

$$\begin{aligned} L &:= C^2[0,\infty) \cap C[0,\infty], \\ M &:= C^{1,2}\big((-\infty,0)\times[0,\infty)\big) \cap C\big((-\infty,0)\times[0,\infty]\big), \\ N &:= C^{1,2}\big((0,\infty)\times[0,\infty)\big) \cap C\big((0,\infty)\times[0,\infty]\big). \end{aligned}$$

3.1A. Definition. *We define the operator $\mathcal{H}$ with domain $\mathcal{D}(\mathcal{H})$ to consist of those $\mathbb{C}$-valued functions in L which satisfy the ('reverse Feller') boundary condition*

$$\tfrac{1}{2}f''(0) + (\mu+1)f'(0) \;=\; 0, \tag{1.1}$$

and, for $f \in \mathcal{D}(\mathcal{H})$, $\mathcal{H}f = \frac{1}{2}f'' + \mu f'$.

Nowhere do we need to extend further the domain of $\mathcal{H}$. For $f \in C[0,\infty]$, we shall be interested in the equation

$$\partial_\varphi F + \mathcal{H}F \;=\; 0, \qquad \big((\varphi,z) \in (-\infty,0)\times[0,\infty)\big) \tag{1.2}$$

with final condition

$$F(0-,y) \;=\; f(y) \qquad (y \in (0,\infty)), \tag{1.3}$$

the final condition *not* being imposed at the boundary point 0. The solution F must belong to M and must satisfy $F(\varphi,\cdot) \in \mathcal{D}(\mathcal{H})$ and $F(\varphi,\infty) = f(\infty)$ for $\varphi < 0$.

For the remainder of the section, it will shorty become clear that we need to consider functions on $\mathbb{R}^+$ with compact support, hereafter denoted by $C_K[0,\infty)$. In some instances, simply stating that $f,g \in L$ is insufficient (particularly if $\mu \geq 0$) as it may lead to problems concerning existence of integrals and certain limits. It turns out that this does not restrict us too much. It simply means that we are only carrying out the analysis heuristically in order to point us in the right direction.

3.1B. Definition. *For $\mathbb{C}$-valued $f,g \in L \cap C_K[0,\infty)$, and with $\overline{g}$ denoting the complex conjugate of g, define the 'indefinite inner product' $\langle\cdot,\cdot\rangle_s$ (subscript 's' for 'signed') via*

$$\begin{aligned}\langle f,g\rangle_s \;&=\; \int_{(0,\infty]} e^{2\mu y} f(y)\overline{g}(y)\mathrm{d}y - \tfrac{1}{2}f(0)\overline{g}(0) \\ &=\; \int_{[0,\infty]} f(y)\overline{g}(y)\nu(\mathrm{d}y),\end{aligned} \tag{1.4}$$

where ν is the signed measure with $\nu(\mathrm{d}y) = e^{2\mu y}\mathrm{d}y$ $(y>0)$ and $\nu\{0\} = -\frac{1}{2}$. An element f of $L \cap C_K[0,\infty)$ will be called $\langle\text{positive}\rangle_s$ if $\langle f,f\rangle_s > 0$, $\langle\text{negative}\rangle_s$ if $\langle f,f\rangle_s < 0$, $\langle\text{neutral}\rangle_s$ if $\langle f,f\rangle_s = 0$.

Note that the integral in (1.4) is well defined given our added restraint of compact support.

3.1C. Lemma. *For $f,g \in \mathcal{D}(\mathcal{H}) \cap C_K[0,\infty)$, we have $\langle \mathcal{H}f,g\rangle_s = \langle f,\mathcal{H}g\rangle_s$.*

Proof of Lemma 3.1C. Trivial calculations show that, for $f,g \in \mathcal{D}(\mathcal{H}) \cap C_K[0,\infty)$,

$$\begin{aligned}\langle \mathcal{H}f,g\rangle_s \;&=\; I + \tfrac{1}{2}\lim_{y\to\infty} e^{2\mu y} f(y)g'(y) = I, \\ \langle f,\mathcal{H}g\rangle_s \;&=\; I + \tfrac{1}{2}\lim_{y\to\infty} e^{2\mu y} f'(y)g(y) = I,\end{aligned}$$

where

$$I = -\tfrac{1}{2}\int_0^{\infty} e^{2\mu y} f'(y)(\overline{g})'(y)\mathrm{d}y.$$

To understand why the above limits vanish, simply observe that $e^{2\mu y}f(y)g'(y) = e^{2\mu y}f'(y)g(y) = 0$ for all y outside the compact support. □

3.1D. Remark. The compact support assumption *suggests* that $\mathcal{H}$ is symmetric relative to $\langle\cdot,\cdot\rangle_s$.

Under the assumption that $\mathcal{H}$ is symmetric relative to $\langle\cdot,\cdot\rangle_s$, we may follow an analogous argument to that given in the previous chapter to deduce that all eigenvalues are real and so are the corresponding eigenfunctions. *Thus, from now on, we consider only real-valued functions.* It will later be confirmed that the semigroups $\{P_t^{\pm}\}$ are Ray semigroups.

Working Hypothesis WH1. *For $f \in C[0,\infty]$ with $f \geq 0$, there exists a* minimal *non-negative solution F of equation (1.2) with final condition (1.3) in that any other such solution $\tilde{F}$ satisfies $\tilde{F}(\varphi,z) \geq F(\varphi,z)$ for all $(\varphi,z) \in (-\infty,0]\times[0,\infty)$. Define $(P_t^+f)(z) = F(-t,z)$ for $t > 0$ and extend P_t^+ (as we may) to $C[0,\infty]$ by linearity. Then $\{P_t^+ : t > 0\}$ defines a one-parameter semigroup of non-negative operators on $C[0,\infty]$, so $P_{s+t}^+ = P_s^+P_t^+$. We will of course have $P_t^+\mathbf{1} \leq \mathbf{1}$, where $\mathbf{1}$ is the constant function equal to 1 on $[0,\infty]$. For $f \in C[0,\infty]$ and $z \in [0,\infty)$, the limit*

$$\left(P_0^+f\right)(z) := \lim_{t\downarrow 0}(P_t^+f)(z)$$

exists and

$$\left(P_0^+f\right)(y) = f(y) \quad (y \in (0,\infty)),$$

$$\left(P_0^+f\right)(0) = \int_{(0,\infty)} \Pi^{-+}(0,\mathrm{d}y)f(y),$$

where $\Pi^{-+}(0,\cdot)$ is a measure of total mass at most 1 on (Borel subsets of) the open *interval $(0,\infty)$. Note that P_0^+ does* not *map $C[0,\infty]$ into $C[0,\infty]$. We have $P_0^+P_t^+ = P_t^+P_0^+ = P_t^+$.*

3.1E. Important Remark. In Chapter 4, we give additional examples of 'whole solution semigroups' which are non-minimal for the case when $\mu < -1$. For $\mu \geq -1$, we see that there is only one semigroup solution of the desired form.

The second Working Hypothesis of the section will be trivial to prove for this example as we only have one-boundary. For $h \in C[0,\infty]$, we consider the PDE

$$\partial_\varphi H + \mathcal{H}H = 0, \quad \left((\varphi,z) \in (0,\infty)\times[0,\infty)\right) \tag{1.5}$$

with initial condition

$$H(0+,0) \;=\; h(0). \tag{1.6}$$

Note that $H(0+,\cdot)$ *is only* specified at the boundary . The solution H must belong to N and must satisfy $H(\varphi,\cdot) \in \mathcal{D}(\mathcal{H})$ for $\varphi > 0$.

Working Hypothesis WH2. *For $h \in C[0,\infty]$ with $h \geq 0$, there exists a* minimal *non-negative solution H of equation (1.5) with initial condition (1.6) in that any other such solution $\tilde{H}$ satisfies $\tilde{H}(\varphi,z) \geq H(\varphi,z)$ for all $(\varphi,z) \in (0,\infty)\times[0,\infty)$. Define $(P_t^- h)(z) = H(t,z)$ for $t > 0$ and extend P_t^- (as we may) to $C[0,\infty]$ by linearity. Then $\{P_t^- : t > 0\}$ defines a one-parameter semigroup of non-negative operators on $C[0,\infty]$, so $P_{s+t}^- = P_s^- P_t^-$. We have $P_t^- \mathbf{1} \leq \mathbf{1}$. For $h \in C[0,\infty]$ and $z \in [0,\infty)$, the limit*

$$\left(P_0^- h\right)(z) \;:=\; \lim_{t \downarrow 0}(P_t^- h)(z)$$

exists and

$$\begin{aligned} \left(P_0^- h\right)(0) &= h(0), \\ \left(P_0^- h\right)(y) &= \Pi^{+-}(y,\{0\})h(0) \quad (y \in (0,\infty)). \end{aligned}$$

where $\Pi^{+-}(y,\{0\})$ is a measure of total mass at most 1 *. This time, P_0^-* does *map $C[0,\infty]$ into $C[0,\infty]$. We have $P_0^- P_t^- = P_t^- P_0^- = P_t^-$. For $h \in C[0,\infty]$, $(P_t^- h)(z)$ depends only on the values of h at* 0.

3.1F. Definition. For $y \in (0,\infty)$, let

$$\pi(0,y) \;:=\; 2\Pi^{+-}(y,\{0\}).$$

3.2. Duality

Drawing an analogy with the two-boundary problem of the previous chapter, this time duality arguments suggest that

$$\Pi^{-+}(0,\mathrm{d}y) \;=\; \pi(0,y)e^{2\mu y}\,\mathrm{d}y \qquad \text{on } (0,\infty) \tag{2.1}$$

in the Radon-Nikodým sense.

3.2A. Remark. At this stage it must be emphasized that this is only a suggested form of Π^{-+}. This is because the duality argument here relies on the fact that $\mathcal{H}$ is symmetric relative to $\langle\cdot,\cdot\rangle_s$, a fact that is again only suggested by the compact support assumption. Later arguments will then confirm matters, in that, the given Π^{-+} is indeed correct.

3.3. The Processes Z and Φ

Let $Z = \{Z(t) : t \geq 0\}$ be a reflected Brownian motion, with drift μ, on $[0, \infty)$ reflected at the boundary 0. We therefore have

$$dZ(t) = dB(t) + \mu dt + dL_0(t),$$

for some Brownian motion B on $\mathbb{R}$ and continuous non-decreasing processes L_0 with

$$\int_0^t I_{\{0\}}(Z(s))dL_0(s) = L_0(t),$$

so that L_0 grows only when Z is at 0. The process L_0 is the familiar local-time process at 0.

The fluctuating additive functional Φ. We define Φ via the equation

$$d\Phi(t) = dt - dL_0(t).$$

For the moment, we concentrate on the situation when $\Phi(0) = 0$.

For $z \in [0, \infty)$, we write $\mathbb{P}^z$ for the law of the (Markov) process (Φ, Z) when $\Phi(0) = 0$ and $Z(0) = z$. As usual, $\mathbb{E}^z$ denotes the expectation associated with $\mathbb{P}^z$. A statement about Z will be said to hold almost surely (a.s.) if it has $\mathbb{P}^z$ probability 1 for every z.

3.4. The Behaviour of Φ

Before we deduce some crucial results on the long-term behaviour of our functional Φ, we begin by introducing some additional important results. The following results on Brownian motion are well-known.

3.4A. Definition. *After throwing away a null set of* ω *the following statement about* $BM(\mathbb{R})$ B *is true. Let* f *be a continuous function on* $[0, 1]$. *We call* f *approximable if there is a (random) sequence* (n_k) *of positive integers with* $n_k \to \infty$ *such that given* $\epsilon > 0$, *there exists (a random)* k_0 *such that for* $k > k_0$,

$$\left| \frac{B(n_k u)}{\sqrt{n_k \log \log n_k}} - f(u) \right| < \epsilon \qquad \text{for all } u \in [0, 1]. \tag{4.1}$$

3.4B. Theorem (Part of Strassen's Law of the Iterated Logarithm).
f is approximable if and only if there exists a $g \in L^1[0, 1]$ *such that*

$$\int_0^t g(s)ds = f(t) \quad (t \in [0, 1]), \tag{4.2}$$

and

$$\tfrac{1}{2} \int_0^1 g(s)^2 ds \leq 1. \tag{4.3}$$

Proof of Theorem 3.4B. For the full result and its proof, see Freedman [10]. □

3.4C. Theorem (Law of the Iterated Logarithm). *We have the following result:*

$$\mathbb{P}\left\{\limsup_{t\uparrow\infty} \frac{|B_t|}{\sqrt{2t\log\log t}} = 1\right\} = 1. \tag{4.4}$$

Proof of Theorem 3.4C. For a direct proof, consult I.16 of Rogers & Williams [24]. The $\mathbb{P}(\limsup \geq 1) = 1$, the deeper part of the LIL Theorem, follows from the part of Strassen's Law given above. The other half follows from the full Strassen Law. Again, see Freedman [10] for further details. □

3.4D. Corollary. *B_t grows much more slowly than t for large t.*

Proof of Corollary 3.4D. This is quite clear in light of Theorem 3.4C. □

3.4E. Theorem. *We have the following situation:*

- *if $\mu > -1$, then (a.s.) $\Phi(t) \to +\infty$ as $t \to \infty$,*
- *if $\mu < -1$, then (a.s.) $\Phi(t) \to -\infty$ as $t \to \infty$.*

Additionally, if $\mu = -1$, then (a.s.) Φ fluctuates infinitely, in that,

$$\limsup \Phi(t) = +\infty, \qquad \liminf \Phi(t) = -\infty. \tag{4.5}$$

3.4F. Remarks. For $\mu > 0$, it is obvious that (a.s.) $\Phi(t) \to +\infty$ as $t \to \infty$. Furthermore, for $\mu < 0$, it is well known that (a.s.) $t^{-1}L_0(t) \to |\mu|$, so that the result when $\mu \neq -1$ is clear. This well known result is a consequence of a general account given in Itô and McKean [14]. It should be noted that the Ergodic Theorem is used there. This causes the main difficulty. However, such results can easily be deduced via Lévy's presentation of drifting Brownian motion which gives us a convenient normalization (in law) for the local time process.

By comparison with the two-boundary case, the result in (4.5) is difficult to justify as the generalized eigenfunction ξ is unbounded (see Section 9). This fact does indeed hinder us, in that we cannot deduce that Φ fluctuates infinitely via a quadratic variation result as demonstrated in the previous chapter. This is one of the main reasons for considering Lévy's presentation. However, only the $\liminf$ can be deduced directly from the Lévy presentation. The $\limsup$ result can be proved, following some work, by Theorem's 3.4B and 3.4C above.

Proof of Theorem 3.4E. Suppose B_t is a BM($\mathbb{R}$) with starting state zero. Then $B_t + \mu t$ is a $\mathrm{BM}_\mu(\mathbb{R})$ (obvious notation!). Thus, by Lévy's presentation (see Theorem C.1 of Appendix C), it follows that

$$D_t := B_t + \mu t + \mathcal{L}_t^D,$$

is a reflecting $\mathrm{BM}_\mu(\mathbb{R})$ on $[0, \infty)$ where $\mathcal{L}_t^D := -\min\{B_s + \mu s : s \leq t\}$ is the local time (at zero) of D_t. Hence, we now have

$$\Phi_t = t - \mathcal{L}_t^D = t + \min\{B_s + \mu s : s \leq t\}. \tag{4.6}$$

As expected, we consider the following cases separately:

1. $\mu > -1$,
2. $\mu < -1$,
3. $\mu = -1$.

Note again that we know exactly what to expect in cases 1 and 2 from Itô and McKean [14].

Case 1: $\mu > -1$. Here,

$$\begin{aligned}
\Phi_t &= t + \min\{B_s + \mu s : s \leq t\} \\
&\geq t + \min\{B_s : s \leq t\} + \min\{\mu s : s \leq t\} \\
&= \hat{\Phi}_t = \begin{cases} t + \min\{B_s : s \leq t\} & \text{if } \mu \geq 0, \\ (\mu + 1)t + \min\{B_s : s \leq t\} & \text{if } \mu \in (-1, 0]. \end{cases}
\end{aligned}$$

We know from Corollary 3.4D that B_t grows much more slowly than t. It follows that

$$\lim_{t \to \infty} \hat{\Phi}_t = \infty \qquad \Rightarrow \qquad \lim_{t \to \infty} \Phi_t = \infty.$$

Case 2: $\mu < -1$. In this case, we have

$$\begin{aligned}
\Phi_t &= t + \min\{B_s + \mu s : s \leq t\} \\
&\leq t + B_t + \mu t \\
&= (\mu + 1)t + B_t \\
&= -Kt + B_t \qquad \text{(for some constant } K > 0).
\end{aligned}$$

Once again by Corollary 3.4D we know that B_t grows more slowly than t, hence we have

$$\lim_{t \to \infty} \Phi_t = -\infty.$$

Case 3: $\mu = -1$. We hope to deduce that we have infinite fluctuation in this case, in that, the conditions in (4.5) hold. The lattermost result in (4.5) follows from the previous case. In particular, we saw that

$$\Phi_t \leq B_t.$$

However, (a.s.) we know that

$$\liminf B_t = -\infty,$$

so we must have (a.s.)

$$\liminf \Phi_t = -\infty.$$

The former result in (4.5) is a little more difficult to justify. By Theorem 3.4B since $g(s) = \sqrt{2} \in L^1[0,1]$, it is clear that $f(t) = t\sqrt{2}$ is approximable. This $f(t)$ is the maximal one (simply maximize $f(1)$ (given by (4.2)), subject to (4.3), so that $g(s) = \sqrt{2}$ $(\in L^2[0,1])$), yet it need not be. Given any ϵ, we therefore know that the inequality in (4.1) is true with $f(u) = u\sqrt{2}$ for sufficiently large n_k. Thus, in particular, we may take $\epsilon = \frac{\sqrt{2}}{4}$, so that for sufficiently large n_k we have

$$\left| \frac{B(n_k u)}{\sqrt{n_k \log\log n_k}} - u\sqrt{2} \right| < \frac{\sqrt{2}}{4} \qquad \text{for all } u \in [0,1].$$

Consequently, we have

$$B(n_k u) > \left(u - \tfrac{1}{4}\right) \sqrt{2 n_k \log\log n_k}. \tag{4.7}$$

Next by simply amending the form for Φ_t in (4.6) to account for n_k, we now get

$$\begin{aligned}
\Phi_{n_k} &= n_k + \inf\left\{B(n_k u) - n_k u : u \in [0,1]\right\} \\
&> n_k + \inf\left\{\left(u - \tfrac{1}{4}\right)\sqrt{2n_k \log\log n_k} - n_k u : u \in [0,1]\right\} \qquad \text{(by (4.7))} \\
&= \inf\left\{n_k \underbrace{(1-u)}_{\geq 0} + \left(u - \tfrac{1}{4}\right)\sqrt{2n_k \log\log n_k} : u \in [0,1]\right\}.
\end{aligned}$$

If $u \in [0, \frac{1}{2})$, then for n_k sufficiently large

$$\begin{aligned}
\Phi_{n_k} &> n_k(1-u)\Big|_{u=\frac{1}{2}} + \left(u - \tfrac{1}{4}\right)\sqrt{2n_k \log\log n_k}\ \Big|_{u=0} \\
&= \tfrac{1}{2} n_k - \tfrac{1}{4}\sqrt{2n_k \log\log n_k} \quad \longrightarrow \infty \qquad \text{as} \quad n_k \longrightarrow \infty.
\end{aligned}$$

If $u \in [\frac{1}{2}, 1]$, then for n_k sufficiently large

$$\begin{aligned}
\Phi_{n_k} &> n_k(1-u)\Big|_{u=1} + \left(u - \tfrac{1}{4}\right)\sqrt{2n_k \log\log n_k}\ \Big|_{u=\frac{1}{2}} \\
&= \tfrac{1}{4}\sqrt{2n_k \log\log n_k} \quad \longrightarrow \infty \qquad \text{as} \quad n_k \longrightarrow \infty.
\end{aligned}$$

It now follows that

$$\lim_{n_k \to \infty} \Phi_{n_k} = +\infty,$$

so that, as desired, we get

$$\limsup \Phi_t = +\infty.$$

□

Short-term behaviour of Φ. If $Z_0 = 0$, then initially $L_0(t)$ will grow faster than t so that there will be a (random) non-empty time-interval $(0, \delta)$ on which $\Phi < 0$. See Appendix C for further details.

3.5. The Processes Z^+ and Z^-

3.5A. Definition (The time-substitutions $\tau^{\pm}$). *For $t \geq 0$, we define (with the strict '>' conditions again being important)*

$$\tau_t^+ := \inf\{u : \Phi(u) > t\}, \qquad \tau_t^- := \inf\{u : -\Phi(u) > t\},$$

with the usual convention that $\inf(\emptyset) = \infty$.

3.5B. Lemma. *The following results hold.*

(a) $\mathbb{P}^z(\tau_t^+ < \infty) = 1$ *if and only if either* $\mu \geq -1$ *or both* $t = 0$ *and* $z \in (0, \infty)$.

(b) $\mathbb{P}^z(\tau_t^- < \infty) = 1$ *if and only if either* $\mu \leq -1$ *or both* $t = 0$ *and* $z = 0$.

Proof of Lemma 3.5B. This is trivial given Theorem 3.4E and Lemma 2.16B. □

3.5C. Definition (The processes $Z^{\pm}$). *For $t \geq 0$, we define*

$$Z^+(t) := Z(\tau_t^+), \qquad Z^-(t) := Z(\tau_t^-),$$

with the usual convention that $Z^{\pm}(t) = \partial$ if $\tau_t^{\pm} = \infty$, where ∂ is a 'coffin state'. For instance, we can only have $Z^+(t) = \partial$ here if $\mu < -1$.

3.5D. Hypothesis. *For the process Z^+, we have the following situation*

(a) if $\mu > 0$, then Z^+ is transient: $Z_t^+ \to \infty$, a.s.;

(b) if $\mu = 0$ then Z^+ is null-recurrent: for any t_0 and any $z \in [0, \infty)$, there will (a.s.) exist a random $t > t_0$ such that $Z_t^+ = z$, but for any interval I, $\mathbb{P}(Z_t^+ \in I) \to 0$ as $t \to \infty$;

(c) if $-1 \leq \mu < 0$, then Z^+ is positive recurrent, and (a.s.) for any interval I, $\mathbb{P}(Z_t^+ \in I) \to \int_I \eta(y)\mathrm{d}y$ as $t \to \infty$, where η is the invariant density for Z^+;

(d) if $\mu < -1$, then (a.s.) Z^+ has finite lifetime.

Comments on Hypothesis 3.5D. This is given for the same reasons as in the corresponding point in the previous chapter.

3.5E. Lemma. *Z^+ and Z^- are strong Markov processes.*

Proof of Lemma 3.5E. The proof is similar to the corresponding result in Chapter 2. □

It is clear that Z^- is a Markov chain on $\{0\} \cup \{\partial\}$. Under $\mathbb{P}^0$ the value τ_0^+ will (a.s.) be strictly positive and Z_0^+ will belong to $(0, \infty)$. We see that Z^+ is therefore a process which behaves like Brownian motion inside $(0, \infty)$ but which, on approaching a point 0, jumps into $(0, \infty)$ according to some measure $\Pi^{-+}(0, \cdot)$ (of total mass at most 1) on (Borel subsets of) $(0, \infty)$ (and jumps to ∂ with probability $1 - \Pi^{-+}(0, (0, \infty))$.

3.5F. Definition. To be specific, we let

$$\begin{aligned}\Pi^{-+}(0, J) &:= \mathbb{P}^0(Z_0^+ \in J) \qquad (J \in \mathcal{B}(0,\infty)),\\ \Pi^{+-}(y, \{0\}) &:= \mathbb{P}^y(\tau_0^- < \infty) \qquad (y \in (0,\infty)),\end{aligned}$$

to accompany Definition 3.1F.

3.5G. Definition (The transition semigroups $P_t^\pm$). *For $t \geq 0$, we now* define *the map $P_t^\pm$ on $C[0,\infty]$ via*

$$\left(P_t^\pm f\right)(z) := \mathbb{E}^z\left(f(Z_t^\pm);\ \tau_t^\pm < \infty\right) \quad (f \in C[0,\infty],\ z \in [0,\infty)).$$

3.6. The Probabilistic Significance of the PDE for F

Let $\Phi(0)$ take an initial value $\varphi < 0$ and let $\mathbb{P}^{\varphi,z}$ denote the law of (Φ, Z) for this new situation: it is the $\mathbb{P}^z$ law of $(\Phi + \varphi, Z)$. In the following theorem, this allows us to include $z = 0$ in the general starting point.

3.6A. Theorem.

(a) *Suppose that $F \in M$ with continuous extension to $\{0\} \times (0,\infty)$, and define*

$$U_t := F\left(\Phi(t \wedge \tau_0^+), Z(t \wedge \tau_0^+)\right).$$

Then our PDE

$$\partial_\varphi F + \mathcal{H}F = 0$$

holds on $(-\infty, 0) \times [0,\infty)$ if and only if

U is a local martingale under each $\mathbb{P}^{\varphi,z}$ with $z \in [0,\infty)$, $\varphi < 0$.

(b) *Suppose that $H \in N$ with continuous extension to $\{0\} \times \{0\}$, and define*

$$V_t := H\left(\Phi(t \wedge \tau_0^-), Z(t \wedge \tau_0^-)\right).$$

Then our PDE

$$\partial_\varphi H + \mathcal{H}H = 0$$

holds on $(0,\infty) \times [0,\infty)$ if and only if

V is a local martingale under each $\mathbb{P}^{\varphi,z}$ with $z \in [0,\infty)$, $\varphi > 0$.

Proof of Theorem 3.6A. Again, the required proof follows by similar arguments to those given in the proof of 2.7A of Chapter 2. □

3.7. Finding Π^{+-} Rigorously

Let $c > 0$. We now amend the setup as given in Section 1 of this chapter, in that the following situation corresponds to considering the operator $\mathcal{H}_c$ with domain $\mathcal{D}(\mathcal{H}_c)$ consisting of real valued functions in L which satisfy the ('reverse Feller') boundary condition

$$\tfrac{1}{2}h''(0) + (\mu + 1)h'(0) - ch(0) = 0. \tag{7.1}$$

Additionally for $h \in \mathcal{D}(\mathcal{H}_c)$, we have $\mathcal{H}_c h = \frac{1}{2}h'' + \mu h' - ch$. This relates to killing the underlying process at the random time ζ. One of the main benefits of this is that we can obtain Π^{+-}, for all μ, from one calculation.

3.7A. Lemma. *For $c > 0$, the process*

$$M_t := H(t, \Phi_t, Z_t) = \exp(-ct - \tfrac{1}{2}\gamma_c^2\Phi_t)h_c(Z_t) \tag{7.2}$$

is a local martingale bounded on $[0, \tau_u^-]$ for $u \geq 0$, where $h_c \in C[0, \infty]$ is given by

$$h_c(z) = \exp(-\tfrac{1}{2}\gamma_c^2 z) \qquad \textit{for } z \in [0, \infty], \tag{7.3}$$

and $\gamma_c^2 = 2(\mu + 1) + \sqrt{4(\mu + 1)^2 + 8c} \geq 0$.

Proof of Lemma 3.7A. Applying Itô's formula to M_t in (7.2), we have

$$\begin{aligned}
\mathrm{d}M_t &= (\partial_t H)(t, \Phi_t, Z_t)\mathrm{d}t + (\partial_\varphi H)(t, \Phi_t, Z_t)\mathrm{d}\Phi_t \\
&\qquad + (\partial_z H)(t, \Phi_t, Z_t)\mathrm{d}Z_t + \tfrac{1}{2}(\partial_z^2 H)(t, \Phi_t, Z_t)\mathrm{d}t \\
&= \exp(-ct - \tfrac{1}{2}\gamma_c^2\Phi_t)\left[-ch_c(Z_t)\mathrm{d}t - \gamma_c^2 h_c(Z_t)\mathrm{d}\Phi_t + h_c'(Z_t)\mathrm{d}Z_t + \tfrac{1}{2}h_c''(Z_t)\mathrm{d}t\right] \\
&= \exp(-ct - \tfrac{1}{2}\gamma_c^2\Phi_t)\Big[- ch_c(Z_t)\mathrm{d}t - \gamma_c^2 h_c(Z_t)\left\{\mathrm{d}t - \mathrm{d}L_0(t)\right\} \\
&\qquad + h_c'(Z_t)\big\{\mathrm{d}B_t + \mu\mathrm{d}t + \mathrm{d}L_0(t)\big\} + \tfrac{1}{2}h_c''(Z_t)\mathrm{d}t\Big].
\end{aligned}$$

Then, by the martingale preservation property, it follows that we need

$$\begin{aligned}
\tfrac{1}{2}h_c'' + \mu h_c' - \tfrac{1}{2}\gamma_c^2 h_c - ch_c &= 0 \qquad \text{on } (0, \infty), \\
h_c' + \tfrac{1}{2}\gamma_c^2 h_c &= 0 \qquad \text{at zero,}
\end{aligned}$$

with $|h_c(y)| < \infty$. This implies $h_c \in C[0, \infty]$, and then elementary calculus gives the desired result. □

3.7B. Remark. Clearly $h_c(\infty) = 0$. Note that, as expected, h_c satisfies (7.1). Therefore, H would now satisfy the PDE in Theorem 3.6A(b) with $\mathcal{H} = \mathcal{H}_c$, but with an additional $+\partial_t H$. One could have easily further generalized the theorem to account for this case. However, the form given there is what is later required.

3.7C. Theorem. *We have the following explicit form for* Π^{+-}*;*

$$\Pi^{+-}(z,\{0\}) := \mathbb{P}^z(\tau_0^- < \infty) = \begin{cases} e^{-2(\mu+1)z} & \text{if } \mu \geq -1, \\ 1 & \text{if } \mu \leq -1. \end{cases}$$

Proof of Theorem 3.7C. Given Lemma 3.7A, we now have an appropriate h_c such that H in (7.2) is a local martingale. Clearly, $\exp(-c\tau_0^-) = 0$ on the set $\{\tau_0^- = \infty\}$. Moreover, if $\tau_0^- = \infty$, then $\Phi(t) \geq 0$ for all $t \geq 0$, simply by definition of τ_0^-. This makes it clear that

$$\exp\{-c\tau_0^- - \tfrac{1}{2}\gamma_c^2\Phi(\tau_0^-)\} = 0 \qquad \text{on the set } \{\tau_0^- = \infty\}. \tag{7.4}$$

Note that Theorem 3.4E enables us to strengthen some of the above. If $\tau_0^- = \infty$ (in which case we must have $\mu < -1$), then $\Phi_t \to \infty$ as $t \to \infty$, so we additionally have $\Phi(\tau_0^-) = \infty$ on the set $\{\tau_0^- = \infty\}$, so that $\exp\{-\frac{1}{2}\gamma_c^2\Phi(\tau_0^-)\} = 0$. This further verifies the result in (7.4). Hence, such a case gives no contribution to the underlying expectation, in that

$$\begin{aligned} &\mathbb{E}^z\left[\exp\{-c\tau_0^- - \tfrac{1}{2}\gamma_c^2\Phi(\tau_0^-)\}h_c(Z_0^-); \tau_0^- < \infty\right] \\ &\qquad = \mathbb{E}^z\left[\exp\{-c\tau_0^- - \tfrac{1}{2}\gamma_c^2\Phi(\tau_0^-)\}h_c(Z_0^-)\right]. \end{aligned} \tag{7.5}$$

Now observe that $\Phi(\tau_0^-) = 0$ when $\tau_0^- < \infty$, $\Phi(0) = 0$ and $h_c(Z_0^-) = h_c(0) = 1$. Since τ_0^- is a valid stopping time, we can legitimately apply the Optional-Stopping Theorem to get

$$\begin{aligned} \mathbb{E}^z\left[\exp\{-c\tau_0^- - \tfrac{1}{2}\gamma_c^2\Phi(\tau_0^-)\}h_c(Z_0^-); \tau_0^- < \infty\right] &= \mathbb{E}^z\left[\exp\{-c\tau_0^-\}; \tau_0^- < \infty\right] \\ &= \mathbb{E}^z\left[h_c(Z_0)\right] \\ &= h_c(z). \end{aligned}$$

Thus, in particular, we have

$$\mathbb{E}^z\left[\exp\{-c\tau_0^-\}; \tau_0^- < \infty\right] = h_c(z). \tag{7.6}$$

Letting $c \downarrow 0$ in (7.6), by Lemma 2.16B, we find that

$$\mathbb{E}^z[\,1\,; \tau_0^- < \infty] = \lim_{c\downarrow 0} h_c(z).$$

The result now follows by elementary Probability Theory and considering the appropriate limits. □

3.7D. Corollary. *We now have the following explicit form for* Π^{-+}*;*

$$\Pi^{-+}(0, dy) = 2e^{-k(\mu)y}dy, \text{ where } k(\mu) = \begin{cases} 2 & \text{if } \mu \geq -1, \\ -2\mu & \text{if } \mu \leq -1. \end{cases} \tag{7.7}$$

Proof of Corollary 3.7D. Assuming the duality result in (2.1) and recalling the definition given in 3.1F, the given result follows from Theorem 3.7C. □

3.7E. Proposition. *It is now clear that* $\Pi^{+-}(y, \{0\})$ *is a measure of total mass at most* 1. *The same can be said about the measure* $\Pi^{-+}(0, \cdot)$ *on* (Borel subsets of) *the interval* $(0, \infty)$.

Proof of Proposition 3.7E. This is trivial. □

3.7F. Important Remark. Observe that, in the case when $\mu \geq -1$, the probability density of 'half-winding' is *independent* of the drift component. This means that the density of half-winding remains unchanged irrespective of the magnitude of μ, which initially seems unreasonable. This fact motivates the following section.

3.8. Heuristic Explanation of Why Π^{-+} is Independent of μ when $\mu \geq -1$

For an explanation of the independence result we appeal to the Cameron-Martin-Girsanov Theorem below. However, there is a problem in justifying the *uniform integrability* property necessary for its use. This accounts for the use of 'heuristic explanation'in the section title.

3.8A. Theorem (CMG change of measure). *Let* $(\Omega, \mathcal{F}^\circ, \{\mathcal{F}_t^\circ\}, \mathbb{P})$ *relate to a Brownian motion on* $\mathbb{R}$. *Suppose that* γ *is an* $\{\mathcal{F}_{t+}^\circ\}$ *previsible* $\mathbb{R}$*-valued process such that:*

$$\zeta_t \equiv \exp\left(\int_0^t \gamma_s \mathrm{d}X_s - \tfrac{1}{2}\int_0^t |\gamma_s|^2 \mathrm{d}s\right) \tag{8.1}$$

defines a uniformly integrable martingale ζ *on each finite interval* $[0, u]$. *Corollary IV.37.11 of Rogers & Williams [24] shows that this will be true when* γ *is a bounded process. Then there exists a measure* $\mathbb{Q}$ *on* $(\Omega, \mathcal{F}^\circ)$ *such that:*

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_{t+}^\circ} = \zeta_t, \quad \forall t,$$

and, under $\mathbb{Q}$,

$$X_t - \int_0^t \gamma_s \mathrm{d}s$$

defines a Brownian motion relative to $\mathcal{F}_{t+}^\circ$. If T is an $\{\mathcal{F}_{t+}^\circ\}$ stopping time, and $\zeta_{t\wedge T}$ is uniformly integrable, then

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_{T+}^\circ} = \zeta_T, \tag{8.2}$$

It follows from equation (7.6) when $\mu = 0$, we have

$$\mathbb{E}_{\mu=0}^z\left[e^{-c\tau_0^-}\right] = e^{-\alpha z},$$

where $\alpha = 1 + \sqrt{1+2c}$. Next define

$$\Pi_c^{-+}(0, \mathrm{d}y) := \mathbb{E}^0_{\mu=0}\left[e^{-c\tau_0^+}; Z_0^+ \in \mathrm{d}y\right],$$

$$\Pi_c^{+-}(y, \{0\}) := \mathbb{E}^y_{\mu=0}\left[e^{-c\tau_0^-}; \tau_0^- < \infty\right] = \mathbb{E}^y_{\mu=0}\left[e^{-c\tau_0^-}\right].$$

It is clear that our previous duality argument will hold for the 'killed' situation. This therefore suggests that

$$\Pi_c^{-+}(0, \mathrm{d}y) = 2\Pi_c^{+-}(y, \{0\}),$$

which is equivalent to

$$\mathbb{E}^0_{\mu=0}\{e^{-c\tau_0^+}; Z(\tau_0^+) \in \mathrm{d}y\} \;=\; 2e^{-\alpha z}\mathrm{d}y. \tag{8.3}$$

As a consequence we now have

$$\mathbb{E}^0_{\mu=0}e^{-\gamma Z(\tau_0^+) - c\tau_0^+} = \frac{2}{\gamma+\alpha}. \tag{8.4}$$

Justification that (8.4) **can be obtained from** (8.3)**.** Multiplying the RHS of (8.3) by $e^{-\gamma y}$ and integrating over $[0, \infty)$, it is clear that

$$\int_0^\infty e^{-\gamma y} 2e^{-\alpha y}\mathrm{d}y = \frac{2}{\gamma+\alpha},$$

the RHS being exactly the RHS in (8.4). This tells us how to proceed. All that it remains to do is to deal with the LHS in (8.3). Using elementary results from conditional expectation, we have

$$\begin{aligned}
\text{LHS in (8.3)} &= \mathbb{E}^0_{\mu=0}\{e^{-c\tau_0^+}; Z(\tau_0^+) \in \mathrm{d}y\} \\
&= \mathbb{E}^0_{\mu=0}\left[e^{-c\tau_0^+} \middle| Z(\tau_0^+) \in \mathrm{d}y\right] \mathbb{P}^0_{\mu=0}(Z_0^+ \in \mathrm{d}y) \\
&= \int_0^\infty e^{-ct}\, \mathbb{P}^0_{\mu=0}(\tau_0^+ \in \mathrm{d}t \big| Z_0^+ \in \mathrm{d}y)\mathbb{P}^0_{\mu=0}(Z_0^+ \in \mathrm{d}y).
\end{aligned}$$

Next, if we multiply the lattermost result by $e^{-\gamma y}$ and integrate over $[0, \infty)$, we have

$$\begin{aligned}
& \int_0^\infty\int_0^\infty e^{-\gamma y - ct}\, \underbrace{\mathbb{P}^0_{\mu=0}(\tau_0^+ \in \mathrm{d}t \big| Z_0^+ \in \mathrm{d}y)\mathbb{P}^0_{\mu=0}(Z_0^+ \in \mathrm{d}y)}_{*} \\
\Leftrightarrow \quad & \int_0^\infty\int_0^\infty e^{-\gamma y - ct}\, \mathbb{P}^0_{\mu=0}(\tau_0^+ \in \mathrm{d}t, Z_0^+ \in \mathrm{d}y) \\
\Leftrightarrow \quad & \mathbb{E}^0_{\mu=0}e^{-\gamma Z(\tau_0^+) - c\tau_0^+}.
\end{aligned}$$

Note that we have used the definition of the conditional density of the law of $\tau_0^+ \big| Z_0^+$ in order to express the desired result in terms of the joint law of Z_0^+ and τ_0^+. □

Let W_t be the canonical 'coordinate process', so Ω is the space of continuous functions and $W_t(\omega) = \omega(t)$. Let $\mathbb{P}^0_{\mu=0}$ be the Wiener measure, so W is a $\mathbb{P}^0_{\mu=0}$ Brownian motion.

$\mu \in \mathbb{R}$ is clearly $\{\mathcal{F}^{\circ}_{t+}\}$ previsible as it is simply a constant. Moreover, it is well known that $\zeta_t := \exp(\mu W_t - \frac{1}{2}\mu^2 t)$ defines a $\mathbb{P}^0_{\mu=0}$-martingale ζ with initial value 1. Hence, by the Cameron-Martin-Girsanov Theorem we can consistently define a measure $\mathbb{P}^0_{\mu=\mu}$ on $(\Omega, \mathcal{F}^{\circ}_{\infty})$ via

$$\frac{d\mathbb{P}^0_{\mu=\mu}}{d\mathbb{P}^0_{\mu=0}} = \zeta_t \quad \text{on } \mathcal{F}^{\circ}_{t+} \quad \text{for finite } t. \tag{8.5}$$

Here the measure $\mathbb{P}^0_{\mu=\mu}$ is said to be equivalent to $\mathbb{P}^0_{\mu=0}$ on each $\{\mathcal{F}^{\circ}_{t+}\}$. However, the equation just given is of course false when $t = \infty$, so that the measures are not equivalent on $\{\mathcal{F}^{\circ}\}$ which is the completion of $\{\mathcal{F}^{\circ}_{t+}\}$. In order to fully justify the use of the CMG Theorem, we need to show that:

1. $W_t - \mu t$ is a $\mathbb{P}^0_{\mu=\mu}$-Brownian motion,
2. $\{\zeta_{t\wedge\tau_0^+}\}$ is a UI martingale.

For reasons already given, the latter point in the above is ignored. We therefore concentrate solely on point 1. By Lévy's characterization of BM, in order to show that $W_t - \mu t$ is a $\mathbb{P}^0_{\mu=\mu}$-BM we need to show that
(i) $W_t - \mu t$ is a $\mathbb{P}^0_{\mu=\mu}$-martingale,
(ii) $W_t - \mu t$ has $\mathbb{P}^0_{\mu=\mu}$ quadratic variation t.

However, (i) is the same as saying that

$$X_t := G(t, W_t) = (W_t - \mu t)\zeta_t \quad \text{defines a } \mathbb{P}^0_{\mu=0}\text{-martingale.}$$

This can be proved directly as follows.

It is clear that $G \in C^{1,2}\big([0,\infty) \times \mathbb{R}\big)$. We can therefore apply Itô's formula to get

$$\begin{aligned} \mathrm{d}X_t &= \partial_t G(t, W_t)\mathrm{d}t + \partial_w G(t, W_t)\mathrm{d}W_t + \tfrac{1}{2}\partial^2_w G(t, W_t)(\mathrm{d}W_t)^2 \\ &= -\zeta_t\left\{\tfrac{1}{2}\mu^2(W_t - \mu t) + \mu\right\}\mathrm{d}t + \zeta_t\left\{\mu(W_t - \mu t) + 1\right\}\mathrm{d}W_t \\ &\qquad\qquad + \zeta_t\left\{\tfrac{1}{2}\mu^2(W_t - \mu t) + \mu\right\}\mathrm{d}t \\ &= \zeta_t\left\{\mu(W_t - \mu t) + 1\right\}\mathrm{d}W_t \\ &= \left\{\mu X_t + \zeta_t\right\}\mathrm{d}W_t. \end{aligned}$$

The martingale preservation property now yields the desired result.

Similarly point (ii) is equivalent to $(W_t - \mu t)^2 - t$ defines a $\mathbb{P}^0_{\mu=\mu}$-martingale, which is the same as saying that

$$Y_t := H(t, W_t) = \{(W_t - \mu t)^2 - t\}\zeta_t \qquad \text{defines a } \mathbb{P}^0_{\mu=0}\text{-martingale.}$$

This can be proved in a similar way to (i) as $H \in C^{1,2}\big([0,\infty) \times \mathbb{R}\big)$. However, this time Itô's formula gives

$$\mathrm{d}Y_t := \{\mu Y_t + 2(W_t - \mu t)\}\mathrm{d}W_t,$$

so that once again the martingale preservation property prevails.

Recall that W_t is a $\mathbb{P}^0_{\mu=0}$ BM. We now know that, under $\mathbb{P}^0_{\mu=\mu}$, $W_t - \mu t$ is a BM. Then,

$$\mathrm{d}W_t = \mu \mathrm{d}t + [\mathrm{d}W_t - \mu \mathrm{d}t].$$

Hence, we have changed W_t to a BM with drift μ under $\mathbb{P}^0_{\mu=\mu}$.

We now introduce Lévy's presentation in order to get the desired reflecting BM on $[0,\infty)$. We take

$$\widetilde{Z}_t = W_t - \min_{s\le t} W_s, \qquad \widetilde{L}_t = -\min_{s\le t} W_s,$$

so that $\widetilde{Z}_t$ defines a reflecting BM with zero drift under $\mathbb{P}^0_{\mu=0}$ and drift μ under $\mathbb{P}^0_{\mu=\mu}$. Recall that such a presentation only gives us an equivalence in law to the underlying processes Z_t and L_t. This accounts for the $\widetilde{Z}$ notation. We write τ for τ_0^+. Then $\tau - \widetilde{L}_\tau = 0$. Next we can use the characterization in (8.5) in order to relate the two measures.

$$\begin{aligned}\mathbb{E}^0_{\mu=\mu} e^{-\theta \widetilde{Z}_0^+} &= \mathbb{E}^0_{\mu=0} e^{-\theta \widetilde{Z}_0^+} \left.\frac{\mathrm{d}\mathbb{P}^0_{\mu=\mu}}{\mathrm{d}\mathbb{P}^0_{\mu=0}}\right|_{t=\tau} = \mathbb{E}^0_{\mu=0} e^{-\theta \widetilde{Z}_0^+} \zeta_{t=\tau} \\ &= \mathbb{E}^0_{\mu=0} e^{-\theta[W(\tau)+\widetilde{L}(\tau)]+\mu W(\tau)-\frac{1}{2}\mu^2\tau}.\end{aligned}$$

Now define

$$\begin{aligned} I :&= -\theta[W(\tau)+\widetilde{L}(\tau)] + \mu W(\tau) - \tfrac{1}{2}\mu^2\tau \\ &= -(\theta-\mu)[W(\tau)+\widetilde{L}(\tau)] - \mu\widetilde{L}(\tau) - \tfrac{1}{2}\mu^2\tau \\ &= -(\theta-\mu)\widetilde{Z}_0^+ - \tau\left(\mu + \tfrac{1}{2}\mu^2\right) \qquad \text{(since } \widetilde{L}(\tau)=\tau.\text{)}\end{aligned}$$

It follows from (8.4) that

$$\mathbb{E}^0_{\mu=0} e^I = \frac{2}{\theta-\mu+\alpha},$$

where $\alpha = 1 + |\mu+1|$. Clearly there are two cases to consider.

Case (i): $\mu \ge -1 \;\Rightarrow\; |\mu+1| = \mu+1$ so that $\alpha = 2+\mu$. Here we have

$$\mathbb{E}^0_{\mu=\mu} e^{-\theta Z_0^+} = \frac{2}{\theta+2},$$

and so we expect that

$$\mathbb{E}^0_{\mu=\mu}[1;\ Z_0^+ \in \mathrm{d}y] = \Pi^{-+}(0,\mathrm{d}y) = 2e^{-2y}\mathrm{d}y,$$

which is the desired independence result.

Case (ii): $\mu \le -1 \;\Rightarrow\; |\mu+1| = -\mu-1$ so that $\alpha = -\mu$. This case yields

$$\mathbb{E}^0_{\mu=\mu} e^{-\theta Z_0^+} = \frac{2}{\theta-2\mu},$$

and we therefore expect that

$$\mathbb{E}^0_{\mu=\mu}[1;\ Z_0^+ \in \mathrm{d}y] = \Pi^{-+}(0,\mathrm{d}y) = 2e^{2\mu y}\mathrm{d}y,$$

by using a similar to that used to show that (8.4) may be obtained from (8.3).

3.9. Eigenfunctions of $\mathcal{H}$

We now define

$$h_\mu(y) := \Pi^{+-}(y, \{0\}). \tag{9.1}$$

Trivially, if $\mu > -1$ then $2(\mu+1)$ is an eigenvalue of $\mathcal{H}$ corresponding to the eigenfunction $h_\mu(y)$. Alternatively, if $\mu \le -1$ then 0 is the corresponding eigenvalue as h_μ is the constant function 1. It is not difficult to show that each such eigenfunction h_μ is $\langle\text{negative}\rangle_s$, except for the case when $\mu = -1$ in which h_μ is $\langle\text{neutral}\rangle_s$. In light of the above, let

$$e(\mu) := \begin{cases} 2(\mu+1) & \text{if } \mu \ge -1, \\ 0 & \text{if } \mu \le -1. \end{cases} \tag{9.2}$$

At first, it seems that $f_\theta(y) = \cos(\theta y) + \frac{1}{2}\theta\sin(\theta y)$ is an eigenfunction of $\mathcal{H}$ corresponding to the negative eigenvalue $-\frac{1}{2}\theta^2$. However, even though $f_\theta(y)$ is bounded on $[0,\infty)$, $\langle f_\theta, f_\theta\rangle_s$ is *not defined* in this case as the corresponding integral turns out to be infinite. Hence, f_θ *is not* an eigenfunction of $\mathcal{H}$: it does not belong to the right space. This is one of the main reasons why we had to proceed differently in this case.

3.9A. Generalized eigenfunctions. Here, problems concerning the underlying domain of $\mathcal{H}$ are further emphasized. In the case when $\mu = -1$, if our underlying functions have to be bounded, then a generalized eigenfunction of $\mathcal{H}$ corresponding to the eigenvalue 0 does not exist. Recall that such a generalized eigenfunction, ξ say, would have to satisfy $\xi \in \mathcal{D}(\mathcal{H})$ (so that ξ must be bounded), $\mathcal{H}\xi = 1$ and $\mathcal{H}^2\xi = 0$. One should then observe that, for this case, (1.1) simply reduces to $\xi''(0) = 0$. Ignoring the need for 'boundedness', we have $\xi(y) = -y$.

3.10. The $\{P_t^-\}$ Semigroup

3.10A. Theorem. *For $z \in [0,\infty)$ and $h \in C[0,\infty]$, we have*

$$(P_t^- h)(z) = e^{-e(\mu)(z+t)} h(0),$$

where $e(\mu)$ corresponds to the appropriate eigenvalue of $\mathcal{H}$ as given in (9.2).

Proof of Theorem 3.10A. It suffices to appeal to arguments given in the proof of Theorem 3.7C. More specifically, we modify matters in that we apply the Optional-Stopping Theorem at τ_t^- and we observe the facts that $\Phi(\tau_t^-) = -t$ and $Z_t^- = 0$. This time the Monotone Convergence Theorem gives

$$\mathbb{P}^z(\tau_t^- < \infty) = e^{-e(\mu)(z+t)}.$$

This result together with Definition 3.5G gives us exactly what we need. □

The following Corollary to the above Theorem is crucial in regard to the legitimate application of Theorem 3.6A.

3.10B. Corollary. *We now have* $(t, z) \mapsto (P^-_t h)(z)$ *is* $C^{1,2}$.

Proof of Corollary 3.10B. This is obvious. □

3.10C. Remarks. If we draw an analogy with the calculation of $P^-_t h$ in Chapter 2, it is immediately clear that affairs are much simpler. This is as we only have a single boundary so that, in particular, $Z^-_t = 0$ if $\tau^-_t < \infty$. Viewing Z^- as a Markov chain on the solitary state 0, we see the 'Q-matrix' corresponding to this situation is simply the value $-e(\mu)$.

Proof of WH2. Given Theorem 3.10A, we define $H(t, z) := (P^-_t h)(z)$ for $t > 0$. By arguments given in Section 7 of Chapter 2 it follows that $H\left(\Phi(t \wedge \tau^-_0), Z(t \wedge \tau^-_0)\right)$ is a martingale. We may therefore legitimately use the 'if' part of Theorem 3.6A (b) to show that the necessary PDE holds and $H(\varphi, \cdot) \in \mathcal{D}(\mathcal{H})$. The martingale property allows us to legitimately justify the remaining matters by probability as well as by analysis. Of course, due to the simple form of $P^-_t h$, everything here can easily be confirmed directly. □

3.11. Calculation of the Conjectured Resolvent $\{\hat{R}^+_\lambda : \lambda > 0\}$

If $Z(0) = 0$, there is no good way of applying the Strong Markov Theorem at a stopping time which is less than τ^+_0. (The only thing that comes to mind is $\sup\{u : \Phi(u) < 0\}$. However, this is clearly not a stopping time!) This is one of the features which makes it necessary to begin by relying on guesswork.

Heuristics. Believing in duality, we guess that Z^+ is a Ray process and has the same laws as $\hat{Z}^+$ where $\hat{Z}^+$ behaves like a drifting Brownian motion on $(0, \infty)$ and, on approaching 0, jumps into $(0, \infty)$ according to the conjectured density $\pi(0, \cdot)e^{2\mu(\cdot)} = 2e^{-k(\mu)(\cdot)}$, where k is given in (7.7). We shall construct the resolvent $\{\hat{R}^+_\lambda : \lambda > 0\}$ which $\hat{Z}^+$ would have to possess.

For typographic neatness we shall use the following notation:

$$\beta := (\mu^2 + 2\lambda)^{\frac{1}{2}}, \quad \text{so that} \quad \alpha_* := (\mu^2 + 2\lambda)^{\frac{1}{2}} + \mu = \beta + \mu,$$
$$\alpha^* := (\mu^2 + 2\lambda)^{\frac{1}{2}} - \mu = \beta - \mu.$$

Since $\lambda > 0$, note that both α_* and α^* are strictly positive. Furthermore, $\alpha_* + \alpha^* = 2\beta$ and $\alpha_*\alpha^* = 2\lambda$.

Suppose that $T_0 := \inf\{t : Z_t = 0\}$ as usual. Let ${}_{\text{tab}}R_\lambda$ be the resolvent of the process $\{Z_t : t < T_0\}$, which is therefore the resolvent of *drifting* BM killed at 0. Note further that

$$\mathbb{E}^z\left(e^{-\lambda T_0}\right) = e^{-\alpha_* z}.$$

For $f \in C[0, \infty]$, we expect to have

$$\lambda(\hat{R}^+_\lambda f)(\infty) = \lambda({}_{\text{tab}}R_\lambda f)(\infty) = f(\infty). \tag{11.1}$$

Now for $z \in (0, \infty]$, we surely believe that

$$g_\lambda(z) := (\hat{R}^+_\lambda f)(z) = ({}_{\text{tab}}R_\lambda f)(z) + e^{-\alpha_* z}(\hat{R}^+_\lambda f)(0). \tag{11.2}$$

From the analogous result for drifting Brownian motion on $\mathbb{R}$, it is a trivial exercise to show that for $f \in C[0,\infty]$,

$$({}_{\mathrm{tab}}R_\lambda f)(z) = \int_0^\infty {}_{\mathrm{tab}}r_\lambda(z,w)f(w)\mathrm{d}w, \tag{11.3}$$

where

$$ {}_{\mathrm{tab}}r_\lambda(z,w) = \begin{cases} \frac{1}{\beta}\left\{e^{-\alpha^*(w-z)} - e^{-\alpha_* z - \alpha^* w}\right\} & \text{if } z < w, \\ \\ \frac{1}{\beta}\left\{e^{-\alpha_*(z-w)} - e^{-\alpha_* z - \alpha^* w}\right\} & \text{if } z > w. \end{cases} \tag{11.4}$$

Hence it is clear that ${}_{\mathrm{tab}}g_\lambda := ({}_{\mathrm{tab}}R_\lambda f)$ solves $\lambda({}_{\mathrm{tab}}g_\lambda) - \frac{1}{2}({}_{\mathrm{tab}}g''_\lambda) - \mu({}_{\mathrm{tab}}g'_\lambda) = f$ on $(0,\infty)$ with Dirichlet boundary condition ${}_{\mathrm{tab}}g_\lambda(0) = 0$ and that it converges to the limit as given in (11.1). Additionally, note that ${}_{\mathrm{tab}}r_\lambda(z,\{0\}) = 0$ and that $\lambda_{\mathrm{tab}}r_\lambda(\infty,\{\infty\}) = 1$ for all λ. Of course, as we shall soon see, the corresponding result is true for the kernel $\hat{r}^+$. This reflects the fact that the 'point' ∞ is absorbing.

Because 0 is a branch point of the Ray process $\hat{Z}^+$, the relevant strong Markov property to which we are making *intuitive* appeal, is really that (due to Meyer and Ray) at Theorem III.41.3 of Rogers and Williams [24]. Now, (11.2) implies that $g_\lambda \in C[0,\infty] \cap C^2(0,\infty)$ and

$$\lambda g_\lambda - \tfrac{1}{2}g''_\lambda - \mu g'_\lambda = f \quad \text{on } (0,\infty), \tag{11.5}$$

with (11.1) at ∞. Moreover, we also believe that,

$$g_\lambda(0) = \int_{(0,\infty]} \Pi^{-+}(0,\mathrm{d}y)g_\lambda(y), \tag{11.6}$$

where the *guessed* value of Π^{-+} is given in (7.7).

Trivially, the 'lateral' condition (11.6) is equivalent to

$$\langle h_\mu, g_\lambda\rangle_s = 0, \tag{11.7}$$

where $h_\mu(z)$ is defined in (9.1).

Do remember that equation (11.5) is not the $(\lambda - \hat{\mathcal{G}}^+)^{-1} = \hat{R}_\lambda^+$ equation of Hille-Yosida theory because f may not belong to the domain of strong convergence to I of $\hat{P}_t^+$ as $t \downarrow 0$.

Definition and calculation of $\{\hat{R}_\lambda^+ : \lambda > 0\}$. We now begin the rigorous study of the conjectured resolvent. We shall see that for $f \in C[0,\infty]$, there is a unique solution $g_\lambda \in C[0,\infty] \cap C^2(0,\infty)$ of (11.5) with lateral condition (11.6) (equivalently, (11.7)). We shall define the linear operator $\hat{R}_\lambda^+$ on $C[0,\infty]$ via $\hat{R}_\lambda^+ f := g_\lambda$. We are going to show that $\{\hat{R}_\lambda^+ : \lambda > 0\}$ is a Ray resolvent, and (eventually) that it is the desired resolvent of Z^+.

3.11A. Analytic verification that (11.2) holds. With the definition of $\hat{R}_\lambda^+$ just given, we simply verify the 'Dirichlet' description of ${}_{\mathrm{tab}}R_\lambda f$ as defined via (11.2) from $\hat{R}_\lambda^+$ as in Section 10 of the previous chapter. □

By elementary calculus, equation (11.5) implies that

$$\begin{aligned} g_\lambda(z) \;=\; A_\lambda(f)e^{-\alpha_* z} + \frac{1}{\beta}\int_z^\infty \left\{e^{-\alpha^*(w-z)} - e^{-\alpha_* z-\alpha^* w}\right\} f(w)\,\mathrm{d}w \\ + \frac{1}{\beta}\int_0^z \left\{e^{-\alpha_*(z-w)} - e^{-\alpha_* z-\alpha^* w}\right\} f(w)\,\mathrm{d}w. \end{aligned} \tag{11.8}$$

Using (11.7) we find that

$$A_\lambda(f) = \int_{[0,\infty]} A_\lambda(0,w)f(w)\,\mathrm{d}w,$$

where

$$A_\lambda(0,w) \;=\; \frac{4}{(k(\mu)-\alpha^*)((k(\mu)-2)+\alpha_*)}\left\{e^{-\alpha^* w} - e^{-k(\mu)w}\right\}. \tag{11.9}$$

From (11.8) we can easily check that g_λ satisfies (11.5) and (11.1), in that,

$$\lim_{z\to\infty} g''_\lambda(z) = \lim_{z\to\infty} g'_\lambda(z) = 0 \qquad \text{and} \qquad \lim_{z\to\infty} \lambda g_\lambda(z) = f(\infty).$$

All that one needs to do is to observe that

$$\lim_{z\to\infty} e^{\alpha^* z}\int_z^\infty e^{-\alpha^* w} f(w)\mathrm{d}w = \frac{1}{\alpha^*}f(\infty),$$
$$\lim_{z\to\infty} e^{-\alpha_* z}\int_0^z e^{\alpha_* w} f(w)\mathrm{d}w = \frac{1}{\alpha_*}f(\infty),$$

together with

$$\lim_{z\to\infty} e^{-\alpha_* z}\int_z^\infty e^{-\alpha^* w} f(w)\mathrm{d}w = \lim_{z\to\infty} e^{-\alpha_* z}\int_0^z e^{-\alpha^* w} f(w)\mathrm{d}w = 0.$$

3.11B. Lemma. *We have* $\hat{R}^+_\lambda : C[0,\infty] \to C[0,\infty]$, *and*

$$(\hat{R}^+_\lambda f)(z) \;=\; \int_{[0,\infty]} \hat{r}^+_\lambda(z,w)f(w)\,\mathrm{d}w$$

for a jointly continuous kernel $\hat{r}^+_\lambda$ *with*

$$\hat{r}^+_\lambda(0,w) \;=\; A_\lambda(0,w), \qquad \hat{r}^+_\lambda(z,0) = 0 \textit{ for all } z \in [0,\infty], \qquad \lambda\,\hat{r}^+_\lambda(\infty,\infty) = 1 \textit{ for all } \lambda.$$

Proof of Lemma 3.11B. Given (11.9), matters are obvious here. □

As in the previous chapter, look forward to Corollary B.7 of Appendix B which motivates the following result.

3.11C. Lemma. *We have*

$$\lambda A_\lambda(0,w) \to e^{2\mu w}\pi(0,w) \quad (w\in(0,\infty]), \quad \text{but} \quad 0 = \lambda A_\lambda(0,0) \nrightarrow \lim_{w\to 0+}\left\{e^{2\mu w}\pi(0,w)\right\} = 2.$$

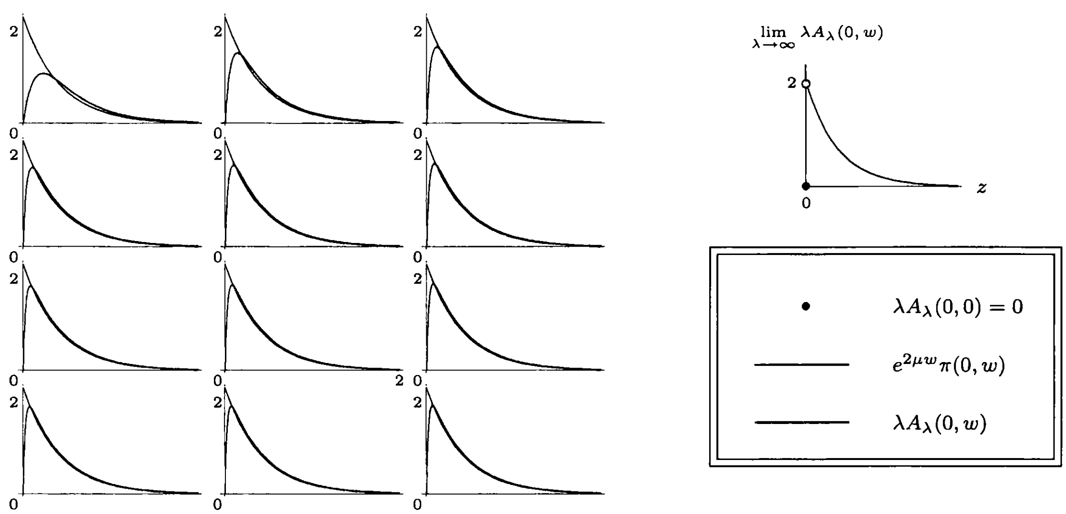

Figure 3.1: $\lambda A_\lambda(0,w)$ against $e^{2\mu w}\pi(0,w)$ for large λ.

Proof of Lemma 3.11C. All that we need to do is to observe that, for $w > 0$,

$$\lim_{\lambda\to\infty} e^{-\alpha^* w} = 0 \quad \text{and} \quad \lim_{\lambda\to\infty} \frac{4}{(k(\mu)-\alpha^*)((k(\mu)-2)+\alpha_*)} = -2,$$

for both cases for $k(\mu)$. The result at 0 is obvious. □

Noting that $g_\lambda(0) = A_\lambda(f)$, we can substitute (11.8) into (11.6) to get

$$g_\lambda(0) = \left\{1 + \frac{2}{(\kappa(\mu)-2)+\alpha_*}\right\} \int_0^\infty \Pi^{-+}(0, dy)({}_{\text{tab}}R_\lambda f)(y). \qquad (11.10)$$

One should simply observe that the particular solution in (11.8) is exactly what recognize as $({}_{\text{tab}}R_\lambda f)$ in (11.4). Given the numerous ways of expressing the solution to (11.5), this is the main reason why the form in (11.8) was chosen. Formula (11.10) is the 'Reuter' formula (actually a Feller-Reuter-Neveu formula) which has a useful interpretation. This will help us to deduce the desired $C^{1,2}$ properties of the underlying semigroup, which we have not yet shown exists.

3.11D. Remark. Note that, as expected, (11.10) follows by substituting the resolvent decomposition (11.2) directly into (11.6). However, this was not the correct route to take given our preliminary guesswork!

$\{\hat{R}^+_\lambda : \lambda > 0\}$ *as a Ray resolvent.* Firstly recall the necessary Ray process theory from Appendix B.

3.11E. Theorem. $\{\hat{R}^+_\lambda : \lambda > 0\}$ *is an honest Feller resolvent on* $C[0,\infty]$.

In order to prove the above theorem, we firstly need several results. We have already established that $\hat{R}_\lambda^+ : C[0,\infty] \to C[0,\infty]$.

3.11F. Lemma. *We have non-negativity of $\hat{R}_\lambda^+$, in that, if $f \in C[0,\infty]$ and $f \geq 0$, then $\hat{R}_\lambda^+ f \geq 0$.*

Proof of Lemma 3.11F. Let $f \in C[0,\infty]$ and $f \geq 0$. Recall that $g(z) := (\hat{R}_\lambda^+ f)(z)$ satisfies the conditions in (11.5) and (11.6). Note that the subscript λ will be dropped for typographic convenience. It suffices to prove the following

$$(\hat{R}_\lambda^+ f)(z) \geq 0 \qquad \text{for all } z \in [0,\infty].$$

Now for a contradiction suppose that

$$g_* := \inf\{g(z) : z \in [0,\infty]\} < 0. \tag{11.11}$$

Suppose further that $g(y_0) = g_*$ for some $y_0 \in (0,\infty)$. Then by (11.5) we have

$$\lambda g_* - \tfrac{1}{2} g_*'' - \mu g_*' = f(y_0). \tag{11.12}$$

Recall that g_* is defined as a local minimum so that we must have $g_*' = 0$ and $g_*'' \geq 0$. However, $\lambda g_* < 0$ and we have $f(y_0) \geq 0$, which clearly gives the desired contradiction to our supposition in (11.12). It therefore follows that

$$g(x) = g_*, \qquad \text{for } x \in \{0,\infty\}, \tag{11.13}$$

i.e. g attains its infimum at the boundary point 0 or at ∞. The result at $z = \infty$ follows immediately from (11.1) since $\lambda, f > 0$. We can therefore restrict out attention to 0.

It suffices to show that if (11.13) is true, then we get the desired contradiction to (11.6) in the case when $g < 0$. Note firstly that (11.13) implies $g(y) > g(0)$ for all $y \in (0,\infty)$. Using this fact in (11.6) we have

$$g(0) = \int_0^\infty 2e^{-k(\mu)y} g(y)\mathrm{d}y > g(0)\int_0^\infty 2e^{-k(\mu)y}\mathrm{d}y = g(0)\frac{2}{k}.$$

Since $g(0) < 0$ by our supposition in (11.11), we have

$$k(\mu) < 2,$$

which clearly contradicts what we recognize as $k(\mu)$ in (7.7). □

3.11G. Lemma. *The resolvent equation holds.*

Proof of Lemma 3.11G. The desired proof is identical to that given in Lemma 2.10F. □

3.11H. Lemma. *For $f \in C[0,1]$, we have the following results:*

(a) $\lambda({}_{\text{tab}}R_\lambda 1)(z) = 1 - e^{-\alpha_* z}$,

(b) $\lambda(\hat{R}_\lambda^+ 1)(0) \leq 1$.

Proof of Lemma 3.11H (a). We take $f \equiv 1$ in (11.3) and then use elementary integration. Observe that $\lambda({}_{\text{tab}}R_\lambda f)(z)$ is simply the probability that Brownian motion killed at 0 is alive at an exponential time. □

Proof of Lemma 3.11H (b). From (11.9), we can easily deduce that

$$\lambda(\hat{R}^+_\lambda 1)(0) = \frac{4\lambda}{(k - 2 + \alpha_*)\alpha^* k(\mu)},$$

which equals 1 if $k(\mu) = 2$. If $k(\mu) = -2\mu$ (in which case $\mu \le -1$) we have

$$\lambda(\hat{R}^+_\lambda 1)(0) = \frac{\lambda}{\mu^2\alpha_* - \mu\alpha^* - \mu\lambda}.$$

However,

$$\mu^2\alpha_* - \mu\alpha^* - \mu\lambda \ge \alpha_* + \alpha^* + \lambda > \lambda > 0,$$

and so we have the desired result. □

3.11I. Corollary. $\lambda\hat{R}^+_\lambda \mathbf{1} \le 1$.

Proof of Corollary 3.11I. From the decomposition in (11.2) we have

$$\begin{aligned}
\lambda(\hat{R}^+_\lambda 1)(z) &= \lambda({}_{\text{tab}}R_\lambda 1)(z) + e^{-\alpha_* z}\lambda(\hat{R}^+_\lambda 1)(0) \\
&= 1 - e^{-\alpha_* z} + e^{-\alpha_* z}\lambda(\hat{R}^+_\lambda 1)(0) && \text{(by Lemma 3.11H (a))} \\
&\le 1 - e^{-\alpha_* z} + e^{-\alpha_* z} && \text{(by Lemma 3.11H (b))} \\
&= 1.
\end{aligned}$$

Finally from (11.1) we have $\lambda(\hat{R}^+_\lambda 1)(\infty) = 1$, which yields the desired result. □

3.11J. Corollary. *If $f \in C[0,1]$ and $f \le 1$, then $\hat{R}^+_\lambda f \le 1$.*

Proof of Corollary 3.11J. The result is a trivial consequence of Lemma 3.11F and Corollary 3.11I. □

Proof of Theorem 3.11E. Recalling Definition B.1 (of Appendix B), to get the desired result, it suffices to combine the results in Lemmas 3.11G and 3.11F and Corollary's 3.11I and 3.11J. However, as in the proof Theorem 2.10D, the desired 'honesty' property may require adding a coffin state ∂ and extending $\hat{R}_\lambda$ in the obvious way. This only applies to the case when $\mu < -1$. □

3.11K. Theorem. $\{\hat{R}^+_\lambda : \lambda > 0\}$ *is a Ray resolvent.*

Before we are able to prove the above theorem, some important results need to be established.

3.11L. Lemma. *For $f \in C[0,\infty]$, as $\lambda \to \infty$, we have*

$$\lambda(\hat{R}^+_\lambda f)(z) \to (\hat{P}^+_0 f)(z),$$

where, as expected,

$$\left(\hat{P}^+_0 f\right)(y) := f(y) \quad (y \in (0,\infty]),$$
$$\left(\hat{P}^+_0 f\right)(0) := \int_{(0,\infty]} e^{2\mu y}\pi(0,y)f(y)\,dy.$$

Proof of Lemma 3.11L. The result for $z = 0$ follows directly from Lemma 3.11C. We also know that $\lambda(\hat{R}^+_\lambda f)(\infty) = f(\infty)$, so we only need to deal with the points in $(0,\infty)$. We do this probabilistically.

Suppose that ζ_λ is exponential with rate λ, independent of the process Z. Then, with T_0 as at the beginning of the section, we have

$$\begin{aligned}\lambda({}_{\text{tab}}R_\lambda f)(z) &= \mathbb{E}^z\left[f(Z(\zeta_\lambda)) : \zeta_\lambda < T_0\right] \\ &= \mathbb{E}^z\left[f\left(Z(\tfrac{\zeta_1}{\lambda})\right) : \zeta_1 < \lambda T_0\right] \qquad (\text{since } \lambda\zeta_\lambda = \zeta_1).\end{aligned}$$

Now $f\left(Z(\frac{\zeta_1}{\lambda})\right)$ is clearly bounded by $||f||_{\text{sup}}$ for $f \in C[0,\infty]$. In addition, $\zeta_1 < \lambda T_0$ for λ sufficiently large. Hence, as $\lambda \to \infty$,

$$\lambda({}_{\text{tab}}R_\lambda f)(z) = \mathbb{E}^z\left[f\left(Z(\tfrac{\zeta_1}{\lambda})\right) : \zeta_1 < \lambda H_0\right] \to \mathbb{E}^z\left[f(Z_0)\right] = f(z).$$

□

3.11M. Lemma. *$\mathcal{R}$ separates points of $[0,\infty]$, where $\mathcal{R}$ is the common range of $\hat{R}^+_\lambda$ on $C[0,\infty]$.*

Proof of Lemma 3.11M. Suppose $\mathcal{R}$ is dense in $C[0,\infty]$. Then if $\mathcal{R}$ does not separate two distinct points z_1 and z_2 of $[0,\infty]$ so $g(z_1) = g(z_2)$ for all $g \in \mathcal{R}$, then for all λ,

$$\underbrace{(\lambda\hat{R}^+_\lambda f)}_{\in\mathcal{R}}(z_1) = (\lambda\hat{R}^+_\lambda f)(z_2), \quad \text{for all } f \in C[0,\infty].$$

Taking the limit as $\lambda \to \infty$ in the above we have

$$(\hat{P}^+_0 f)(z_1) = (\hat{P}^+_0 f)(z_2), \quad \text{for all } f \in C[0,\infty].$$

Hence, in order to get the desired contradiction, it suffices to show that if z_1 and z_2 are distinct points of $[0,\infty]$, then for *some* continuous f on $[0,\infty]$ we have

$$(\hat{P}^+_0 f)(z_1) \neq (\hat{P}^+_0 f)(z_2).$$

It is now trivial to prove that $\mathcal{R}$ separates points. We just take $f = 1$ and $f = e^{-ky}$. □

Proof of Theorem 3.11K. From Definition B.4, we know that a sufficient condition for our honest Feller resolvent $\{\hat{R}^+_\lambda : \lambda > 0\}$ to be a Ray resolvent is that the common range

$\mathcal{R} = \hat{R}^+_\lambda C[0,\infty]$ of the $\hat{R}^+_\lambda$ operators separates points of $[0,\infty]$. Hence, given Lemma 3.11M we know that $\hat{R}^+_\lambda$ is a Ray resolvent. □

3.11N. Theorem. *There exists a map* $\hat{P}^+_t$ $(t \geq 0)$ *mapping* $C[0,\infty]$ *into the space* $\mathrm{b}\mathcal{B}[0,\infty]$ *of bounded Borel functions on* $[0,\infty]$ *such that for* $f \in C[0,\infty]$ *and* $z \in [0,\infty]$, $t \mapsto (\hat{P}^+_t f)(z)$ *is right-continuous, and*

$$\int_0^\infty \mathrm{e}^{-\lambda t}(\hat{P}^+_t f)(z)\,\mathrm{d}t \;=\; \lambda(\hat{R}^+_\lambda f)(z),$$

$$\hat{P}^+_{s+t} = \hat{P}^+_s \hat{P}^+_t \textit{ for } s,t \geq 0,$$

$$f \geq 0 \textit{ implies } \hat{P}^+_t f \geq 0, \qquad \textit{and} \qquad \hat{P}^+_t \mathbf{1} \leq \mathbf{1}.$$

Proof of Theorem 3.11N. We have now proved Theorem 3.11K so the existence of the $\hat{P}^+_t$ maps is guaranteed by Theorem B.5 of Appendix B. □

3.11O. Remark. For comments on the probabilistic part of Ray's Theorem see the corresponding section of Chapter 2.

Connections with the PDE approach. The PDE approach makes us believe that, for $f \in C[0,\infty]$,

$$\partial_y \left\{(\hat{P}^+_t f)(y)\right\}\Big|_{y=0} + \partial_t \left\{(\hat{P}^+_t f)(0)\right\} \;=\; 0. \tag{11.14}$$

Note the use of 'believe' here, due to the fact that we do not yet know that $(\hat{P}^+_t f)(z)$ is $C^{1,2}$. On taking Laplace transforms in (11.14), we would have for $g_\lambda := \hat{R}^+_\lambda f$, equation (11.5) with the different boundary condition

$$g'_\lambda(0) + \lambda g_\lambda(0) - \int_0^\infty \Pi^{-+}(0,\mathrm{d}w) f(w) \;=\; 0. \tag{11.15}$$

One can easily verify that $g_\lambda = \hat{R}^+_\lambda f$ does have this property. From (11.8), it is simply enough to prove that

$$(\alpha_* - \lambda)A_\lambda(0,w) = 2e^{-\alpha^* w} - 2e^{-k(\mu)w}. \tag{11.16}$$

Noting that $\alpha_* = \alpha^* + 2\mu$, this is exactly equation (11.9). *The boundary condition tallies, in that* $(\hat{P}^+_t f)(\cdot) \in \mathcal{D}(\mathcal{H})$.

A direct proof of (11.15). For this case, the fact that (11.15) follows from (11.5) and (11.6) may *not* be proved directly, unless we make some vital assumptions. This time, as a consequence of (11.5), the desired result in (11.15) is equivalent to

$$g'_\lambda(0) + \mu\int_0^\infty \Pi^{-+}(0,\mathrm{d}y) g'_\lambda(y) + \tfrac{1}{2}\int_0^\infty \Pi^{-+}(0,\mathrm{d}y) g''_\lambda(y) \;=\; 0. \tag{11.17}$$

However, we only know that $g_\lambda \in C[0,\infty] \cap C^2(0,\infty)$. This certainly *does not* imply that both g'_λ and g''_λ are bounded in the limit as $y \to \infty$. Given the presence of g'_λ and g''_λ in (11.17), this clearly causes a problem!

Recall that $h_\mu(y) = e^{-y(k(\mu)+2\mu)}$ and that $\mathcal{H}h_\mu = 2(\mu+1)h_\mu$ in the case when $\mu \geq -1$. Furthermore, $h'_\mu(y) = -2(\mu+1)h_\mu(y)$, so that in particular we have $h_\mu(0) = 1$ and $h'_\mu(0) = -2(\mu+1)$. Assuming that both g'_λ and g''_λ are bounded in the appropriate limit, elementary integration by parts combined with these facts then allows us to simplify matters considerably and therefore prove (11.17).

If $\mu \leq -1$, then $h_\mu = 1$ and so zero is the corresponding e-value of $\mathcal{H}$. As a result, (11.7) now becomes redundant, in that (11.17) may be deduced solely from (11.5) via integration by parts.

The lack of rigour here in our assumptions about g'_λ and g''_λ further emphasize the importance of finding an explicit solution to (11.5) as in (11.8).

3.12. Deducing that $\hat{P}_t^+$ is $C^{1,2}$

Due to the absence of eigenfunctions corresponding to negative eigenvalues here, it is almost impossible to achieve a precise spectral expansion for $\hat{P}_t^+ f$ as we did for the two-boundary problem. Consequently, deducing that $\hat{P}_t^+$ is $C^{1,2}$ immediately becomes a more strenuous issue. The continuity problem is considered in complete generality for the driftless case in the final chapter. Therefore, we shall merely comment on calculations specific to this case. In fact, it turns out that the calculations for this particular case are more 'awkward' than those for the general case.

Clearly we need a convenient explicit form for the $\{P_t^+\}$ semigroup. Due to the connection between Laplace transforms and convolutions, it is unsurprising that such an explicit form can be written in terms of the convolution of two functions. We therefore begin with the following definition.

3.12A. Definition. *The convolution of two functions $f(t)$ and $g(t)$ is defined as*

$$(f * g)(t) := \int_0^t f(s)g(t-s)\mathrm{d}s.$$

It is trivial that $(f * g)(t) = (g * f)(t)$, when the underlying integrals exist. Next we consider the following useful theorem concerning convolutions. $\mathcal{L}$ denotes the familiar laplace transform.

3.12B. Theorem. *If $\mathcal{L}\{f(t)\} = F(\lambda)$ and $\mathcal{L}\{g(t)\} = G(\lambda)$, then*

$$\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(\lambda)G(\lambda).$$

We now have the following useful corollary.

3.12C. Corollary.

$$\mathcal{L}^{-1}\{F(\lambda)G(\lambda)\} = (f * g)(t) = (g * f)(t).$$

Proof of Theorem 3.12B and its Corollary. For a proof, see Finney & Ostberg [9]. □

3.12D. Working Hypothesis. *For all* $\mu \in \mathbb{R}$, *we have*

$$(\hat{P}_t^+ f)(z) \in N := C^{1,2}\big((0,\infty) \times [0,\infty)\big) \cap C\big((0,\infty) \times [0,\infty]\big).$$

Comments on proof of Working Hypothesis 3.12D. For a proof in the 'driftless' case, see the final section of Chapter 4. It will become quite clear that the argument given there may be modified to account for this case. However, we shall now comment on issues specific to this case. The following function plays a key role;

$$u(t) := \int_0^\infty \Pi^{-+}(0, \mathrm{d}y)({}_{\mathrm{tab}}P_t^+ f)(y), \tag{12.1}$$

and referring to VI.55 of Rogers & Williams [24], we find that for $f \in C[0,\infty]$

$$({}_{\mathrm{tab}}P_t^+ f)(z) = \int_0^\infty {}_{\mathrm{tab}}p_t^+(z,w) f(w)\mathrm{d}w,$$

where

$${}_{\mathrm{tab}}p_t^+(z,w) = \frac{1}{\sqrt{2\pi t}} \exp\{\mu(w-z) - \tfrac{1}{2}\mu^2 t\} \left[\exp\left\{-\tfrac{(w-z)^2}{2t}\right\} - \exp\left\{-\tfrac{(w+z)^2}{2t}\right\}\right]. \tag{12.2}$$

The explicit form of $u(t)$ is now clear. Corollary 3.12C may be used with the 'Reuter' formula in (11.10) to deduce that

$$a(t) := (\hat{P}_t f)(0) = u(t) + (u * \eta)(t). \tag{12.3}$$

where

$$\eta(t) = \exp(-\tfrac{1}{2}\mu^2 t)\left\{\frac{\sqrt{2}}{\sqrt{\pi t}} - \exp\left\{\tfrac{1}{2}(k(\mu)+\mu-2)^2 t\right\}(k(\mu)+\mu-2)\Upsilon_\mu(t)\right\}, \tag{12.4}$$

where $\Upsilon_\mu(t) = 2 - 2\Phi\Big((\mu + k(\mu) - 2)\sqrt{t}\Big)$ and $\Phi(\cdot)$ is the distribution of the standard normal variable.

Initially, the key step is to prove that $a(\cdot)$ is continuously differentiable on $(0,\infty)$. Though both u and η are differentiable on $(0,\infty)$, the problem is that they do not behave well at $t = 0$. The way to proceed by bare hands is explained in Chapter 4.

It is possible to further generalize matters in order to get an expression for $(\hat{P}_t^+ f)(z)$ for general $z \in (0, \infty)$. In particular, we may now deduce that

$$(\hat{P}_t^+ f)(z) = ({}_{\text{tab}}P_t^+ f)(z) + \int_0^t b(s)a(t-s)\mathrm{d}s, \tag{12.5}$$

where

$$b(s) = \frac{y}{\sqrt{2\pi s^3}} \exp\{\tfrac{-(y+\mu s)^2}{2s}\}.$$

Notice that $b(t)$ represents the probability density for the first-passage time of Brownian motion with negative drift, to hit level y starting from 0. In particular, we have

$$\int_0^\infty \frac{y}{\sqrt{2\pi s^3}} \exp\{\tfrac{-(y+\mu s)^2}{2s}\}\mathrm{d}s = \begin{cases} 1 & \text{if } \mu \leq 0, \\ e^{-2\mu y} & \text{if } \mu > 0. \end{cases} \tag{12.6}$$

(Referring to I.8 of Rogers & Williams [24], it can be seen that the case given there corresponds to taking α^* rather than α_* in the above. There it can also be seen that $\mathcal{L}\{b(t)\} = e^{-\alpha_* y}$, a fact used to deduce (12.5).) □

3.13. The Probabilistic Semigroup $\{P_t^+ : t \geq 0\}$

As promised, we now confirm that $\hat{P}_t^+ = P_t^+$ for $t \geq 0$. Recall that for the two boundary case, we had to consider the 'problem' case separately as the spectral expansion was only valid for $m_0 + m_1 \neq 1$. However, for this example there are no such worries, in that we do not need to consider quotient spaces in order to deal with the case when $\mu = -1$.

Let $f \in C[0, \infty]$. In the previous section we have deduced that $(\hat{P}_t^+ f)(z) \in N$ *for all* μ. Furthermore, we may also deduce that $(\partial_t - \frac{1}{2}\partial_y^2)(\hat{P}_t^+ f)(y) = 0$. We know that (11.15) holds, and we may now legitimately invert the Laplace transform to show that (11.14) holds. We therefore know that for every $t > 0$ and $f \in C[0, \infty]$, we have $\hat{P}_t^+ f \in \mathcal{D}(\mathcal{H})$. Moreover, given Lemma 3.11L, we know that $\hat{F}$ has final condition

$$\hat{F}(0-, y) = f(y) \qquad (y \in (0, \infty)).$$

Hence, if for $(\varphi, z) \in (-\infty, 0] \times [0, \infty]$ we define

$$\hat{F}(\varphi, z) := \left(\hat{P}_{-\varphi}^+ f\right)(z), \tag{13.1}$$

then $\hat{F}$ satisfies the condition (see Theorem 3.6A) that

$$\hat{M}_t := \hat{F}\big(\Phi(t \wedge \tau_0^+), Z(t \wedge \tau_0^+)\big)$$

defines a local martingale, indeed a bounded martingale. We are now in a position to utilize the Optional-Stopping Theorem.

If $\mu \geq -1$, then τ_0^+ is almost surely finite, so we have for $(\varphi, z) \in (-\infty, 0) \times [0, \infty]$,

$$\begin{aligned}\hat{F}(\varphi, z) &= \mathbb{E}^{\varphi,z}\hat{F}(\Phi(0), Z(0)) = \mathbb{E}^{\varphi,z}\hat{F}(\Phi(\tau_0^+), Z(\tau_0^+)) \\ &= \mathbb{E}^{\varphi,z}\hat{F}(0, Z_0^+) = \mathbb{E}^{\varphi,z}f(Z_0^+),\end{aligned}$$

so that $(\hat{P}_{-\varphi}^+ f)(z) = (P_{-\varphi}^+ f)(z)$. The lattermost result is simply our final condition for $\hat{F}$.

Now suppose that $\mu < -1$. First, we pause to make some important remarks.

3.13A. Remarks. In the corresponding section of the previous chapter (i.e. when $m_0 + m_1 > 1$), we were easily able to deduce that $(\hat{P}_t^+ f)(z) \to 0$ as $t \to \infty$, uniformly in z, via the spectral expansion. Here, however, we are not so fortunate, as we do not have an explicit form for $(\hat{P}_t^+ f)(z)$ to exploit.

Note too that we cannot yet talk about Φ_t in relation to $\hat{P}_t^+ f$ (as in Definition 3.5G) as this is precisely what we are striving to prove.

Even with the previous remarks in mind, all is not lost, as we have enough material in our 'hatted' scene to deduce the following lemma.

3.13B. Lemma. *For $f \in C_K[0, \infty)$, we have $(\hat{P}_t^+ f)(0) \to 0$ as $t \to \infty$.*

Proof of Lemma 3.13B. Suppose that $\hat{Z}^+$ has transition semigroup $\{\hat{P}_t^+\}$. Suppose that $\hat{Z}^+$ 'starts at 0': in fact it jumps away from 0 immediately. But $\hat{Z}^+$ must keep approaching 0 (since $\mu < -1$, but $\mu \leq 0$ is enough) and every time it approaches 0, it has chance $1 - 1/|\mu|$ of dying. So, it must die at some random time $\hat{\zeta}$. Now for $f \in C_K[0, \infty)$, we have

$$(\hat{P}_t^+ f)(0) = \hat{\mathbb{E}}^0\left(f(\hat{Z}_t^+)\,; t < \hat{\zeta}\right) \leq \|f\|_{\sup}\,\hat{\mathbb{P}}^0(t < \hat{\zeta}) \to 0 \tag{13.2}$$

as $t \to \infty$. □

3.13C. Remarks. The result in (13.2) may clearly be generalized for $(\hat{P}_t^+ f)(z)$ with the $\hat{\mathbb{P}}^z$ measure. Now suppose $f \equiv 1$, so that $(\hat{P}_t^+ 1)(z) = \hat{\mathbb{P}}^z(t < \hat{\zeta})$. Then, whatever the value of t, for very large z, $(\hat{P}_t^+ f)(z)$ will be very close to 1. Consequently, even though $(\hat{P}_t^+ f)(z) \to 0$ pointwise, no uniformity can be claimed! This explains why we choose to focus solely on the case in which $z = 0$.

Suppose that f is non-negative and in $C_K[0, \infty)$. Then $\hat{M}$ is a non-negative local martingale, hence a non-negative supermartingale, and so (a.s.) $\hat{M}_\infty$ exists by Doob's Convergence Theorem (see (11.5) of Williams [29]). We now show that if $\tau_0^+(\omega) = \infty$, then $\hat{M}_\infty(\omega) = 0$ (a.s.) for the $\hat{\mathbb{P}}^{\varphi,z}$ measure. We now appeal to the underlying Z process which we *know* is a reflecting Brownian motion with drift $\mu < -1$. Then there must be a sequence of times $T_1(\omega) \leq T_2(\omega) \leq \cdots$ with $T_n(\omega) \to \infty$ at which $Z(\cdot, \omega) = 0$ (because $\mu < -1$, but again $\mu \leq 0$ is enough!). For example, let $T_n(\omega)$ be the first time, after time n, when $Z(\cdot, \omega)$ is at 0. Next assume that $\tau_0^+(\omega) = \infty$. Now, using Lemma 3.13B. and the fact that $\Phi_t(\omega) \to -\infty$, we have $\hat{M}(T_n(\omega)) \to 0$, so $\hat{M}_\infty = 0$ and $\hat{M}_t(\omega) \to 0$ as $t \to \infty$. Thus, again, $(\hat{P}_{-\varphi}^+ f)(z) = (P_{-\varphi}^+ f)(z)$.

The fact that our initial analysis was only heuristic is now unimportant. Also, any remaining issues for the $\varphi = 0$ situation may be resolved in a similar fashion to the argument presented in Section 12 of the previous chapter.

3.13D. Extremely Important Discussion. *Suppose that* $\mu < -1$. In light of (7.7) we now generalize our definition of $\hat{F}$ in (13.1) so that, for $\kappa \geq 2$, we let $\{P_t^\kappa\}$ be the Ray semigroup for "Brownian motion with drift μ and with jump-out measure J from 0 satisfying $J(0, \mathrm{d}y) = 2\mathrm{e}^{-\kappa y}\mathrm{d}y$". The notation F^κ and M^κ is therefore obvious! A rather complicated calculation shows that both for $\kappa = 2$ and $\kappa = -2\mu$, and for no other $\kappa \geq 2$, we have

$$(\partial_\varphi + \mu\partial_z + \tfrac{1}{2}\partial_z^2)F^\kappa \;=\; 0 \quad \text{on } (-\infty, 0) \times [0, \infty),$$

with

$$(\mu\partial_z + \tfrac{1}{2}\partial_z^2)F^\kappa + \partial_z F^\kappa \;=\; 0 \quad \text{on } (-\infty, 0) \times \{0\},$$

so that $(\hat{P}^\kappa_{-\varphi} f) \in \mathcal{D}(\mathcal{H})$ for both values of κ. Equivalently, both for $\kappa = 2$ and $\kappa = -2\mu$,

$$M_t^\kappa \;:=\; F^\kappa\big(\Phi(t \wedge \tau_0^+), Z(t \wedge \tau_0^+)\big)$$

defines a bounded martingale. However, when $\kappa = 2$, M_t^κ does not tend to 0 on the set where $\tau_0^+ = \infty$; and this explains why $\{P_t^\kappa\}$ is not the correct probabilistic ('minimal non-negative') semigroup.

3.14. The Kolmogorov Forward Equation and Riccati Equation

The Kolmogorov forward equation for the transition density $p_t^+(\cdot,\cdot)$ for $\{P_t^+ : t > 0\}$ relative to Lebesgue measure on $(0, \infty)$ takes the form:

$$(\partial_t - \tfrac{1}{2}\partial_y^2 + \mu\tfrac{1}{2}\partial_y)p_t^+(z, y) \;=\; \tfrac{1}{2}\left\{\partial_w p_t^+(z, w)\right\}\big|_{w=0} e^{2\mu y}\pi(0, y), \tag{14.1}$$

with the interpretation and explanation given in Chapter 2. As before, it is extremely useful to show that the following Riccati equation holds for $y \in (0, \infty)$:

$$\begin{aligned} -\tfrac{1}{2}\partial_y^2\{e^{2\mu w}\pi(0, y)\} + \mu\partial_y\{e^{2\mu y}\pi(0, y)\} \\ = \tfrac{1}{2}\partial_w\{e^{2\mu w}\pi(0, w)\}\big|_{w=0} e^{2\mu y}\pi(0, y) - 2\mu e^{2\mu y}\pi(0, y) \end{aligned} \tag{14.2}$$

and that, when $\mu \in [-1, 0)$, the 'invariant-density' equation:

$$-\tfrac{1}{2}\partial_y^2\eta(y) + \mu\partial_y\eta(y) \;=\; \tfrac{1}{2}\partial_w\eta(w)\big|_{w=0} e^{2\mu y}\pi(0, y). \tag{14.3}$$

holds, where η is the invariant density of P_t^+. Here, we initially 'cheat' as we shall soon use (14.3) to find η. However, once we have η we can independently crosscheck that it is indeed the correct one, which is equivalent to showing that (14.3) holds.

By using the above two equations and linearity, we can show that, with the alternative form for $A_\lambda(0, y)$ as at (11.16), we have

$$\begin{aligned} \lambda A_\lambda(0, y) - \tfrac{1}{2}\partial_y^2 A_\lambda(0, y) + \mu\partial_y A_\lambda(0, y) - e^{2\mu y}\pi(0, y) \\ = \tfrac{1}{2}\left\{\partial_w A_\lambda(0, w)\right\}\big|_{w=0} e^{2\mu y}\pi(0, y); \end{aligned} \tag{14.4}$$

and inversion of the Laplace transform (at least formally) yields (14.1) with $z = 0$. One has to remember Lemma 3.11C if one wishes to compare (14.2) with λ times (14.4).

Once again, recall that we do not have a spectral expansion for P_t^+. Hence, even with the aid of (14.2) and (14.3), deducing (14.1) rigorously is demanding.

Invariant probability density for $\{P_t^+\}$ when $\mu \in [-1, 0)$. Here we give the invariant measure remarked on in Hypothesis 3.5D. For $\mu \in (-1, 0)$, from (14.3) together with elementary calculus, we may show that

$$\eta(w) = \frac{2|\mu|}{1 - |\mu|}\left(e^{-2|\mu|w} - e^{-2w}\right),$$

and, in the case when $\mu = -1$, we have

$$\eta(w) = 4we^{-2w}. \tag{14.5}$$

As expected, we may easily check that $\int \eta(w)\mathrm{d}w = 1$.

In the following Lemma, we may now crosscheck matters via the expected properties of $A_\lambda(0, y)$ (see Corollary B.7 of Appendix B).

3.14A. Lemma. *If $\mu \geq -1$, we have $\int_0^\infty \lambda A_\lambda(0, y)\mathrm{d}y = 1$, as required, and also $\lim_{\lambda \downarrow 0} \lambda A_\lambda(0, w) = \eta(w)$, so that everything tallies.*

Proof of Lemma 3.14A. Using (11.16), the first part of the Lemma follows by elementary integration. L'Hôpitals rule deals with the limit. □

Chapter 4

Non-Minimal Solutions and Regularity

Summary

The Riccati equation in (14.2) of Chapter 3 is used to reveal some rather surprising explicit non-minimal non-negative solutions to a corresponding PDE. Moreover, as promised in the previous chapter, smoothness issues for the general semigroup case are resolved here. Further details and different methods may be found in the paper with Stroock and Williams which presents the satisfying general existence and uniqueness theorem for the type of PDEs we have been studying.

4.1. Further Solutions to the PDE in Chapter 3

Before beginning this section, one should recall the necessary PDE in (1.2) of Chapter 3 and the conditions that follow it. Hence, in the material that follows, when we say 'our PDE' we clearly mean the PDE in (1.2) equipped with the appropriate boundary conditions.

Here we will comment on 'non-negative semigroup solutions' of our PDE in (1.2). Repeating some of what has been said previously, we mean that we have a one-parameter semigroup $\{P_t : t > 0\}$ of bounded non-negative operators on $C[0,\infty]$ such that $F(\varphi, z) := (P_{-\varphi}f)(z)$ for $(\varphi, z) \in (-\infty, 0) \times [0,\infty]$ defines a solution of (1.2). Our semigroups on $C[0,\infty]$ will be strongly continuous on $(0,\infty)$ but not at 0; however, for every z in $[0,\infty]$, $(P_0 f)(z) := \lim_{t\downarrow 0}(P_t f)(z)$ will exist (though not uniformly in z). Assuming Working Hypothesis 3.12D, we insist that

$$(P_t^+ f)(z) \in C^{1,2}\big((0,\infty)\times[0,\infty)\big) \cap C\big((0,\infty)\times[0,\infty]\big). \tag{1.1}$$

Lateral conditions. As seen in Sections 10 and 11 of Chapters 2 and 3 respectively, the natural thing to do is to focus on non-negative semigroup solutions where the 'infinitesimal generator' of $\{P_t\}$ has certain lateral conditions determining its domain. Hence, by analogy with Chapter 3 for this case, we begin with the desired resolvent conditions, but with a generalized version of the lateral condition given in (11.6).

To be precise, we let $J \in C^2[0,\infty)$ with $J(\cdot) \geq 0$ and $\int_0^\infty J(y)\mathrm{d}y \leq 1$. Let $f \in C[0,\infty]$. Then for $\lambda > 0$, we may define $g_\lambda := R_\lambda f$ to be the unique solution in $C[0,\infty] \cap C^2(0,\infty)$ of

$$\lambda g_\lambda - \tfrac{1}{2} g_\lambda'' - \mu g_\lambda' = f \quad \text{on } (0,\infty), \tag{1.2}$$

subject to conditions

$$g_\lambda(0) = \int_{(0,\infty)} J(y) g_\lambda(y)\mathrm{d}y, \quad \lambda g_\lambda(\infty) = f(\infty). \tag{1.3}$$

Then, for $z \in [0,\infty]$, the function $t \mapsto (P_t f)(z)$ on $(0,\infty)$ is the unique right-continuous (in our context, continuous) function such that for $\lambda > 0$,

$$\int_0^\infty \mathrm{e}^{-\lambda t}(P_t f)(z)\,\mathrm{d}t = (R_\lambda f)(z). \tag{1.4}$$

The semigroup property $P_t P_u = P_{t+u}$ holds, and $f \geq 0$ implies that $P_t f \geq 0$. The limit

$$(P_0 f)(z) := \lim_{t \downarrow 0} (P_t f)(z)$$

exists, and

$$(P_0 f)(y) = f(y) \quad \text{for } y \in (0,1), \text{ while}$$
$$(P_0 f)(0) = \int_0^\infty J(y) f(y)\mathrm{d}y.$$

4.1A. Remark. Existence of a semigroup with the above properties may be established by the 'Ray'methods of previous chapters.

4.1B. Theorem (Comparison). *If $\{P_t\}$ is associated with the non-negative function J and $\{\tilde{P}_t\}$ is associated with the non-negative function $\tilde{J}$ then*

$$\begin{aligned} (\tilde{P}_t f)(z) \geq (P_t f)(z) & \quad \textit{for all } (t,z) \textit{ if and only if} \\ \tilde{J}(y) \geq J(y) & \quad \textit{for all } y \in (0,\infty). \end{aligned} \tag{1.5}$$

In (1.5), $t \geq 0$, $z \in [0,\infty]$.

Proof of Theorem 4.1B. The 'only if' part is obvious on letting $t \downarrow 0$. The 'if' part is obvious from the probabilistic interpretation of Ray processes. □

4.1C. Theorem. *The function J in $C[0,\infty]$ yields a non-negative semigroup solution of our PDE if and only if the following 'Riccati' equation holds:*

$$\tfrac{1}{2}J''(y) - \mu J'(y) + \{\tfrac{1}{2}J'(0) - \mu J(0)\}J(y) = 0, \tag{1.6}$$

$$J(0) = 2, \quad J(\cdot) \geq 0, \quad \int_0^\infty J(y)\mathrm{d}y < \infty. \tag{1.7}$$

Proof of Theorem 4.1C. We begin with the 'only if' part of the result. Firstly observe that if $g_\lambda \in C_K^\infty[0,\infty)$ (the space of smooth functions on $[0,\infty)$ with compact support) and one *defines* f via (1.2), then $f \in C_K^\infty[0,\infty)$ and $R_\lambda f = g_\lambda$. Assuming the continuity result in (1.1), we have already confirmed that (see Section 13 of Chapter 3) the fact our PDE holds is equivalent to, for $f \in C[0,\infty]$,

$$\partial_y\Big\{(P_tf)(y)\Big\}\Big|_{y=0} + \partial_t\Big\{(P_tf)(0)\Big\} = 0, \tag{1.8}$$

or, in Laplace-transformed version for $\lambda > 0$,

$$g'_\lambda(0) + \lambda g_\lambda(0) - \int_0^\infty J(y)f(y)\mathrm{d}y = 0.' = 0. \tag{1.9}$$

Of course, strictly speaking, what we did previously was for the 'minimal' J, but we are now considering a general non-minimal J. Drawing further analogies with Chapter 3, we see that (1.9) is equivalent to

$$g'_\lambda(0) + \mu\int_0^\infty J(y)g'_\lambda(y)\mathrm{d}y + \tfrac{1}{2}\int_0^\infty J(y)g''_\lambda(y)\mathrm{d}y = 0,$$

and now, integration by parts reduces this to the form

$$\mathcal{L}(g_\lambda) = g'_\lambda(0)[1 - \tfrac{1}{2}J(0)] + \int_{(0,\infty)} R(y)g_\lambda(y)\mathrm{d}y = 0. \tag{1.10}$$

where

$$R(y) := \tfrac{1}{2}J''(y) - \mu J'(y) + \{\tfrac{1}{2}J'(0) - \mu J(0))\}J(y). \tag{1.11}$$

We know that (1.10) holds for any function $g_\lambda \in C_K^\infty[0,\infty)$ such that $\Lambda(g_\lambda) = 0$, where Λ is the linear functional on $C_K^\infty[0,\infty)$ defined by

$$\Lambda(g_\lambda) = g_\lambda(0) - \int_0^\infty J(y)g_\lambda(y)\mathrm{d}y.$$

Hence, by an argument already seen in the proof of Theorem 2.4C of Chapter 2, we may uniquely define a linear map $\ell : \mathbb{R} \to \mathbb{R}$ via

$$\ell\big(\Lambda(g_\lambda)\big) := \mathcal{L}(g_\lambda) \quad \text{for } C_K^\infty[0,\infty),$$

so that $\mathcal{L}(g_\lambda) = c\Lambda(g_\lambda)$ for some $c \in \mathbb{R}$. Hence, we have

$$g'_\lambda(0)[1 - \tfrac{1}{2}J(0)] + \int_0^\infty R(y)g_\lambda(y)\mathrm{d}y = cg_\lambda(0) - c\int_0^\infty J(y)g_\lambda(y)\mathrm{d}y$$

for *all* g in $C_K^\infty[0,\infty)$.

We can avoid direct appeal to distribution theory via the following elementary argument. Choose a uniformly bounded sequence g_n of functions in with compact support in $[0,\infty]$ such that $g'_n(0) = 1$ for every n and $g_n(z) \to 0$ for $z \in [0,\infty]$. By the Dominated-Convergence Theorem, we must have $1 - \frac{1}{2}J_0(0) = 0$. Now choose a uniformly bounded sequence g_n of functions in $C_K^\infty[0,\infty)$ with $g_n(0) = 1$ and $g_n(z) \to 0$ for $z \in (0,\infty]$. We see that $c = 0$ and our Riccati equation follows immediately.

The 'if' part here is obtained by reversing the argument.

Solving the Riccati equation (1.6) subject to (1.7). If we put $J'(0) = b$ and $c = \sqrt{\mu^2 + 4\mu - b}$, we find that (except for the case when $c = 0$ considered separately below) the solution of equation (1.6) is

$$J(y) = J^{\mu,c}(y) := e^{\mu y}\left[2\cosh cy + c^{-1}(\mu^2 + 2\mu - c^2)\sinh cy\right].$$

Now for J to be non-negative, c must be real because if $c = i\omega$, then $J(y)$ would have the form $Re^{\mu y}\cos(\omega(y + \varphi))$, and so would oscillate in sign. So c is real and we may and do assume that $c > 0$. Since $\cosh cy/\sinh cy \to 1$ as $y \to \infty$, for J to be non-negative we must have $c^{-1}(\mu^2 + 2\mu - c^2) \geq -2$. Thus,

$$J(y) \geq e^{\mu y}2(\cosh cy - \sinh cy) = 2e^{(\mu - c)y}.$$

But $\mu^2 + 2\mu - c^2 \geq -2c$, so that $(\mu + 1)^2 \geq (c - 1)^2$, and

$$\mu - c \geq \mu - 1 - |\mu + 1| = \begin{cases} -2 & \text{if } \mu \geq -1, \\ 2\mu & \text{if } \mu \leq -1. \end{cases}$$

If $c = 0$, we have the solution

$$S(y) = e^{\mu y}\left[2 + (\mu^2 + 2\mu)y\right].$$

We see that S can be non-negative only if $\mu^2 + 2\mu \geq 0$ so that $\mu \geq 0$ or $\mu \leq -2$. If $\mu \geq 0$ then $S(y) \geq 2 \geq 2e^{-2y}$; and if $\mu \leq -2$, then $S(y) \geq 2e^{\mu y} \geq 2e^{2\mu y}$.

So, we have found that the minimal non-negative solution $J^{\min}$ of equation (1.6) is given by

$$J^{\min}(y) = \begin{cases} 2e^{-2y} & \text{if } \mu \geq -1, \\ 2e^{2\mu y} & \text{if } \mu \leq -1, \end{cases} \tag{1.12}$$

thus confirming what we already know. □

4.1D. A uniqueness result when $\mu \geq -1$. When $\mu \geq -1$, the solution with $c = \mu + 2$ is the only one with $\int J^{\mu,c}(y)dy \leq 1$ (yet, not the only one $< \infty$), and then $\int J^{\mu,c}(y)dy = 1$. Thus there is only one semigroup solution of the type we are considering.

4.1E. A strange aspect of the case when $\mu < -1$. Consider the case when $\mu < -1$. Then $\max(2 + \mu, 0) \leq c \leq -\mu$. For the minimal non-negative semigroup $P^{\min}$ (corresponding to $c = -\mu$), we have $\int J^{\min}(y)dy < 1$, so that $P_t^{\min}\mathbf{1} < \mathbf{1}$ for all $t > 0$. However, all the other (μ, c) semigroups have $\int J^{\mu,c}(y)dy = 1$, and so are 'honest' Markov semigroups satisfying $P_t^{\mu,c}\mathbf{1} = 1$ for all $t > 0$. When $c \neq -\mu$ we have for $f \in C[0,\infty]$,

$$\text{as } t \to \infty, \quad (P_t^{\mu,c}f)(z) \to \eta(f) = \int f(y)\eta(y)dy, \tag{1.13}$$

where $\eta = \eta^{\mu,c}$ satisfies

$$\tfrac{1}{2}\eta''(y) - \mu\eta'(y) + \tfrac{1}{2}\eta'(0)J^{\mu,c}(y) \;=\; 0, \tag{1.13a}$$
$$\eta(0) = 0, \quad \int_0^\infty \eta(y)\mathrm{d}y = 1. \tag{1.13b}$$

Bounded solutions. If $\mathbb{P}^{\varphi,z}(\tau_0^+ < \infty) = 1$ for all $\varphi \le 0$ and all $z \in [0,\infty)$, something which happens if and only if $\mu \ge -1$, then F^{prob} is the unique bounded solution of (1.2). We have discovered other bounded solutions $F^{\mu,c}$ when $\mu < -1$. For a solution $F^{\mu,c}$, we have

$$F^{\mu,c}(\varphi, z) \;=\; \mathbb{E}^{\varphi,z}[f(Z_0^+) : \tau_0^+ < \infty] + \eta^{\mu,c}(f)\mathbb{P}^{\varphi,z}[\tau_0^+ = \infty] \tag{1.14}$$

where $\eta^{\mu,c}$ is given by (1.13). Equation (1.14) explains a great deal.

4.2. Further Solutions to the PDE in Chapter 2

Again, one should recall the necessary PDE in (1.2) of Chapter 2 and the conditions that follow it. It is also necessary to recall the basics of that chapter concerning the eigenvalues and eigenvectors of $\mathcal{H}$.

Here we will comment on 'non-negative semigroup solutions' of our PDE (1.2). Again repeating some of what has been said previously, we mean that we have a one-parameter semigroup $\{P_t : t > 0\}$ of bounded non-negative operators on $C[0,1]$ such that $F(\varphi, z) := (P_{-\varphi}f)(z)$ for $(\varphi, z) \in (-\infty, 0) \times [0,1]$ defines a solution of (1.2). Our semigroups on $C[0,1]$ will be strongly continuous on $(0,\infty)$ but not at 0; however, for every z in $[0,1]$, $(P_0 f)(z) := \lim_{t\downarrow 0}(P_t f)(z)$ will exist (though not uniformly in z).

Semigroup solutions link together in a coherent way solutions for individual f. But, of course, there is no reason why a solution for an individual f need be part of a semigroup solution. Indeed, as already mentioned in Example 2.1D, it seems that when $m_0 + m_1 > 1$, and $f \equiv 1$ on $[0,1]$, then the obvious, but non-minimal, solution $F \equiv 1$ does not derive from a semigroup solution.

Lateral conditions. As seen in Section 11 of Chapter 2, the natural thing to do is to focus on non-negative semigroup solutions where the 'infinitesimal generator' of $\{P_t\}$ has certain lateral conditions determining its domain. Hence, by analogy with Chapter 2, we begin with the desired resolvent conditions, but with a generalized version of the lateral condition given in (10.3).

To be precise, we consider two non-negative functions $J_0(\cdot)$ and $J_1(\cdot)$ in $C^2[0,1]$. Let $f \in C[0,1]$. Then there exists $\lambda_0 \ge 0$ (independent of f) such that for $\lambda > \lambda_0$, we may define $g_\lambda := R_\lambda f$ to be the unique solution in $C[0,1] \cap C^2(0,1)$ of

$$\lambda g_\lambda - \tfrac{1}{2}g_\lambda'' \;=\; f \quad \text{on } (0,1), \tag{2.1}$$

subject to the lateral conditions

$$g_\lambda(b) \;=\; J_b(g_\lambda) \;:=\; \int_{(0,1)} J_b(y)g_\lambda(y)\mathrm{d}y, \quad (b \in \{0,1\}). \tag{2.2}$$

4.2A. Remark. The key difference to what we have seen previously is that we must allow $\int_0^1 J_b(y) > 1$. As a consequence, the resolvent corresponding to the above setup need not be a Ray resolvent. In fact, it turns out that only the minimal J_b satisfies the condition that $\int_0^1 J_b(y) \leq 1$. Referring to Lemmas 2.10E and 2.10G (of Chapter 2) we see that this immediately causes problems. In particular, we may not use Ray's Theorem to justify the existence of the appropriate semigroup. The only way to get around this would be to introduce processes with births and deaths, a study of which would substantially increase the length of this chapter. Clearly, we may establish analogues to Theorem's 4.1B and 4.1C. See the paper by Andrews, Stroock and Williams.

A non-minimal semigroup. If $m_0 + m_1 < 1$, so that $0 < \alpha < \beta$, we may obtain a non-negative solution semigroup $\{\tilde{P}_t\}$ by taking $\alpha = 0$ with the usual β in (4.1) of Chapter 2. The question of whether or not the resulting $\tilde{J}_i$'s are non-negative is resolved by Lemma 2.3O, which says that $h_\beta(z)$ is monotonic in z.

In this instance $\tilde{P}_t f_\theta = e^{-\frac{1}{2}\theta^2 t} f_\theta$ for $\theta \in \Theta^+$ but the condition that $\tilde{P}_t f_0 = f_0$ fails, but can be replaced by $\tilde{P}_t h_\alpha = e^{\frac{1}{2}\alpha^2 t} h_\alpha$. From the point of view of Wiener-Hopf Theory which generally 'splits things' according to the complete set of eigenvalues in some half-plane, this is a rather strange switch. For example, $\langle \tilde{P}_0 h_\alpha, \tilde{P}_0 h_\alpha \rangle_s = \langle h_\alpha, h_\alpha \rangle_s < 0$, so we do not have a proper inner product in analogy with $\langle \cdot, \cdot \rangle^{\min} := \langle P_0^{\min} u, P_0^{\min} v \rangle_s$.

4.3. Regularity at the Boundary

The continuity hypothesis in the latter part of the previous chapter is now dealt with. Here the 'driftless' case is considered in order to simplify matters. We have already commented on the nature of the underlying functions in the general drift case.

4.3A. Theorem. *Let J be a bounded non-negative continuous function on $(0,\infty)$ with $\int_0^\infty J(y)\mathrm{d}y \leq 1$. Then there exists a semigroup $\{P_t\}$ of bounded non-negative operators on $C[0,\infty]$, strongly continuous on $(0,\infty)$ but not at 0, uniquely determined by the following property:*
for $\lambda > 0$ and $f \in C[0,\infty]$, the function g_λ with

$$g_\lambda(z) \;:=\; (R_\lambda f)(z) \;:=\; \int_0^\infty \mathrm{e}^{-\lambda t}(P_t f)(z)\mathrm{d}t$$

is the unique solution in $C^2[0,\infty) \cap C[0,\infty]$ of

$$\lambda g_\lambda - \tfrac{1}{2}g_\lambda'' = f, \quad g_\lambda(\infty) = \lambda^{-1}f(\infty), \quad g_\lambda(0) \;=\; \int_0^\infty J(y)g(y)\mathrm{d}y.$$

For $f \in C[0,\infty]$, the limit $(P_0 f)(z) := \lim_{t\downarrow 0}(P_t f)(z)$ exists, and $(P_0 f)(y) = f(y)$ for $y \in (0,\infty]$, while $(P_0 f)(0) = \int_0^\infty J(y)f(y)\mathrm{d}y$.
For $f \in C[0,\infty]$,

$$(t,z) \mapsto (P_t f)(z) \text{ is in } C^{1,2}((0,\infty)\times[0,\infty))$$

and

$$(\partial_t - \tfrac{1}{2}\partial_z^2)(P_t f)(z) = 0 \text{ on } (0,\infty)\times[0,\infty).$$

Moreover, for $f \in C[0,\infty]$,

$$g_\lambda'(0) \;=\; \int_0^\infty \mathrm{e}^{-\lambda t}\partial_z(P_t f)(z)\big|_{z=0}\mathrm{d}t.$$

Note. The crucial point is the extension of regularity to the boundary $(0,\infty)\times\{0\}$.

Proof of Theorem 4.3A. The existence results follow immediately from similar arguments to those used in Section 11 of Chapter 3, but with our general 'jump out' measure J.

Let $\{{}_{\mathrm{tab}}P_t : t > 0\}$ be the Dirichlet heat kernel for $(\partial_t - \frac{1}{2}\partial_x^2)$ on $(0,\infty)$, so that

$$({}_{\mathrm{tab}}P_t f)(z) \;=\; \int_0^\infty {}_{\mathrm{tab}}p_t(z,w)f(w)\mathrm{d}w,$$

where

$${}_{\mathrm{tab}}p_t(z,w) \;=\; (2\pi t)^{-\frac{1}{2}}\left\{\exp\left(-\frac{(w-z)^2}{2t}\right) - \exp\left(-\frac{(w+z)^2}{2t}\right)\right\}.$$

Let $\{{}_{\text{tab}}R_\lambda : \lambda > 0\}$ be the resolvent kernel for $\{{}_{\text{tab}}P_t\}$, so that

$$({}_{\text{tab}}R_\lambda f)(z) = \int_0^\infty \mathrm{e}^{-\lambda t}({}_{\text{tab}}P_t f)(z)\mathrm{d}t = \int_0^\infty {}_{\text{tab}}p_t(z,w)f(w)\mathrm{d}w,$$

where

$${}_{\text{tab}}p_t(z,w) = \gamma^{-1}\left(\mathrm{e}^{-\gamma|w-z|} - \mathrm{e}^{-\gamma|z+w|}\right),$$

γ again denoting

$$\gamma = \sqrt{2\lambda}.$$

We now have

$$g_\lambda(y) = ({}_{\text{tab}}R_\lambda f)(y) + \widehat{K_\lambda}(f)\mathrm{e}^{-\gamma y}, \tag{3.1}$$

where

$$\widehat{K_\lambda}(f) = \left(1 - \widehat{J_\gamma}\right)^{-1}\int_0^\infty ({}_{\text{tab}}R_\lambda f)(y)J(y)\mathrm{d}y. \tag{3.2}$$

This is exactly the Reuter formula for our situation. In contrast to (11.10) of Chapter 3, we take $\mu = 0$ and $\widehat{J_\gamma} = \int_0^\infty e^{-\gamma y}J(y)\mathrm{d}y$. An alternative form for $\widehat{J_\gamma}$ will be given shortly.

The map $(t,x) \mapsto ({}_{\text{tab}}P_t f)(x)$ is $C^{1,2}$ and satisfies $(\partial_t - \frac{1}{2}\partial_x^2)({}_{\text{tab}}P_t f)(x) = 0$ on $(0,\infty)\times[0,\infty)$. This may be proved by applying Fubini's theorem to the expressions obtained by formally differentiating through integrals to show that they 'integrate up correctly' (see Appendix A16 of Williams [29]).

If

$$\psi(t) := \int_0^\infty J(y)\frac{y\exp(-y^2/2t)}{\sqrt{2\pi t^3}}\,\mathrm{d}y,$$

then $\widehat{J_\gamma} = \int_0^\infty \mathrm{e}^{-\lambda t}\psi(t)\mathrm{d}t$. We have, with $\|J\| := \sup J(y)$,

$$\psi(t) \le \frac{\|J\|}{\sqrt{2\pi t}}, \quad \psi'(t) \le \frac{C_1\|J\|}{\sqrt{\pi t^3}} \quad (t > 0).$$

Now, since we have the arc-sine/beta-function formula

$$\int_0^t \frac{1}{\sqrt{\pi s}}\frac{1}{\sqrt{\pi(t-s)}}\,\mathrm{d}s = 1,$$

we have (with '$*$' denoting convolution) $\psi * \psi(t) \le \frac{1}{2}\|J\|^2$ for all $t > 0$. If

$$\psi_{k*} := \psi * \psi * \cdots * \psi \quad (k \text{ factors}),$$

then, by induction using the beta-function formula,

$$\psi_{n*}(t) \le C_0^n\frac{t^{\frac{1}{2}n-1}}{\Gamma(\frac{1}{2}n)}, \qquad C_0 := 2^{-\frac{1}{2}}\|J\|.$$

Differentiating convolutions. If $a(\cdot), b(\cdot) \in C^1[0,\infty)$ and $b(0) = 0$, then

$$(a*b)' = a*(b').$$

In general, for $a(\cdot), b(\cdot) \in C^1(0,\infty)$ with suitable growth at 0, we have to write

$$(a * b)(t) = t\int_0^1 a(tu)b(t(1-u))\mathrm{d}u,$$

and differentiate to get

$$\begin{aligned}(a * b)'(t) &= t^{-1}(a*b)(t) + t\int_0^1 ua'(tu)b(t(1-u))\mathrm{d}u \\ &\quad + t\int_0^1 a(tu)(1-u)b'(t(1-u))\mathrm{d}u.\end{aligned}$$

So,

$$\begin{aligned}|(\psi * \psi)'(t)| &\le \frac{(\psi*\psi)(t)}{t} + 2t\int_0^1 \frac{uC_1}{\sqrt{\pi u^3 t^3}}\frac{C_0}{\sqrt{\pi(1-u)t}}\,\mathrm{d}u \\ &\le \frac{C_0^2}{t} + \frac{2C_0C_1}{t}.\end{aligned}$$

Now, we find that $|(\psi_{4*})'(t)| \le M_4$, a constant, and by the easy formula for differentiating convolutions, we have, for $n \ge 3$,

$$|(\psi_{(n+4)*})'(t)| \le \frac{M_4 C_0^n t^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n+1)}.$$

Let T be an arbitrary, but now fixed, number in $(0,\infty)$. We shall henceforth restrict t to lie in $(0,T]$, and we shall work with a fixed f in $C[0,\infty]$.

Now, for $\lambda > 0$,

$$\widehat{K_\lambda}(f) = \int_0^\infty \mathrm{e}^{-\lambda t}K_f(t)\mathrm{d}t = \mathcal{L}\{K_f(t)\},$$

where $K_f(t)$ is our $(P_t f)(0)$. Equation (3.2) may alternatively be written

$$\widehat{K_\lambda}(f) = \left(1 + \sum_{k=1}^\infty \widehat{J_\gamma}^k\right)\int_0^\infty ({}_{\mathrm{tab}}R_\lambda f)(y)J(y)\mathrm{d}y. \tag{3.3}$$

Hence, by inverting the Laplace transform and using Corollary 3.12C, we have

$$K_f(t) = W(t) + (\varphi * W)(t),$$

where

$$W(t) := \int_0^\infty ({}_{\mathrm{tab}}P_t f)(y)J(y)\mathrm{d}y \quad \text{and} \quad \mathcal{L}\{\varphi(t)\} = \sum_{k=1}^\infty \widehat{J_\gamma}^k.$$

Since $\mathcal{L}\{\psi(t)\} = \widehat{J_\gamma}$, we may use Theorem 3.12B to deduce that

$$\varphi(t) = \psi(t) + (\psi*\psi)(t) + \cdots = \sum_{k=1}^\infty \psi_{k*}(t).$$

Our estimates show that φ is in $C^1(0,\infty)$ and $\varphi(t) \le A_0 t^{-\frac{1}{2}}$, $|\varphi'(t)| \le A_1 t^{-\frac{3}{2}}$ (for $t \in (0,T]$). Since $\int J \le 1$, we find that for some constant $C(J)$ determined by J,

$$W(t) \le C(J), \qquad W'(t) \le C(J)/t.$$

We therefore find that $K(t) = K_f(t)$ satisfies

$$K(t) \le A(J), \quad K'(t) \le A(J)t^{-1}, \qquad (t \in (0,T]).$$

We assume that the reader will trust that similar arguments to those above establish that K is in fact in $C^2(0,\infty)$. We do not need estimates on K''.

Regularity at the boundary. From (3.1), we have

$$(P_t f)(y) = ({}_{\text{tab}}P_t f)(y) + U(t,y),$$

where

$$U(t,y) \ := \int_0^t \frac{y e^{-\frac{1}{2}y^2/s}}{\sqrt{2\pi s^3}} K(t-s)\, ds. \tag{3.4}$$

We wish to show that

$$t \mapsto U(t,y) \in C^{1,2}((0,\infty) \times [0,\infty))$$

and that $U(t,y)$ satisfies the heat equation. Of course,

$$U(t,0) = K(t).$$

We base everything on the formula

$$\begin{aligned} U(t,y) \ &= \int_0^t \frac{y e^{-\frac{1}{2}y^2/s}}{\sqrt{2\pi s^3}} [K(t-s)] - K(t)]\, ds \\ &\qquad + K(t) - K(t) \int_t^\infty \frac{y e^{-\frac{1}{2}y^2/s}}{\sqrt{2\pi s^3}}\, ds. \end{aligned} \tag{3.5}$$

It is now easy to check that, as $y \to 0$,

$$\partial_y U(t,y) \ \to D_y U(t,0) \ := \ \partial_z U(t,z)\big|_{z=0}.$$

From equation (3.5), we obtain for $y > 0$,

$$\begin{aligned} \partial_t U(t,y) \ &= \int_0^t \frac{y e^{-\frac{1}{2}y^2/s}}{\sqrt{2\pi s^3}} [K'(t-s) - K'(t)]\, ds \ + \ K'(t) \\ &\quad - K'(t) \int_t^\infty \frac{y e^{-\frac{1}{2}y^2/s}}{\sqrt{2\pi s^3}}\, ds \ - \ \frac{y e^{-\frac{1}{2}y^2/t}}{\sqrt{2\pi t^3}} K(t). \end{aligned}$$

As $y \to 0$, this does tend to $K'(t)$, as we wish. It was in the above formula that we needed the fact that $K''(t)$ exists.

We need to prove that $\partial_t U(t,y) = \frac{1}{2}\partial_y^2 U(t,y)$ at $y = 0$. As $y \to 0$, the expression

$$y^{-1}[\partial_y U(t,y) - D_y U(t,0)]$$

needs to converge to $2K'(t)$. It is easy to check that this amounts to saying that

$$\int_0^t \frac{e^{-\frac{1}{2}y^2/s} - 1 - s^{-1}y^2 e^{-\frac{1}{2}y^2/s}}{y} \frac{1}{\sqrt{2\pi s^3}}[K(t-s) - K(t)]\, ds \;\to 2K'(t).$$

But because of a well-known 'hitting-time' convergence to the delta function,

$$\int_0^t -s^{-1}\frac{y e^{-\frac{1}{2}y^2/s}}{\sqrt{2\pi s^3}}[K(t-s) - K(t)]\, ds \;\to K'(t).$$

So, we need to prove that

$$\int_0^t \frac{1 - e^{-\frac{1}{2}y^2/s}}{y} \frac{1}{\sqrt{2\pi s^3}}[K(t) - K(t-s)]\, ds \to K'(t);$$

and it is enough to show that as $y \to 0$,

$$\delta(y,s) \;:=\; \frac{1 - e^{-\frac{1}{2}y^2/s}}{y\sqrt{2\pi s}}$$

converges to the δ-function $\delta_0(s)$. But it is obvious that for $\eta > 0$,

$$\int_\eta^\infty \delta(y,s) ds \;\le\; \int_\eta^\infty \frac{y^2}{y\sqrt{2\pi s^3}}\, ds \to 0 \qquad (y \to 0).$$

Moreover, as $\gamma \downarrow 0$,

$$\int_0^\infty e^{-\frac{1}{2}\gamma^2 s}\delta(y,s)\, ds \;=\; \frac{1 - e^{-\gamma y}}{\gamma y} \to 1,$$

and we have the result we need.

To be sure, there are some other things to check; but we have done all that matters. □

Appendix A

Additive Functionals

In Wiener-Hopf theory, continuous additive functionals play a vital role. They are responsible for the consequent 'windings' of the (Wiener-Hopf) process when the stochastic process is viewed with respect to the additive functional rather than simply time. We therefore establish some useful results.

Here we assume that we are working with the canonical model in which the sample space Ω is the space of paths ω.

A.1. Definition. *For $0 \leq t \leq \infty$, we define the 'time-shift' map θ_t as*

$$\theta_t : \Omega \to \Omega \quad \textit{such that} \quad \theta_t \omega(s) = \omega(t+s), \quad \textit{for all } s,$$

with the usual convention that $\infty + s = s + \infty = \infty$, for all s. Furthermore, if T is a map from Ω to $[0, \infty]$, define

$$\theta_T \omega = \theta_{T(\omega)} \omega.$$

A.2. Definition. *A is a fluctuating perfectly continuous additive functional of some process X if the following properties hold:*

- *$t \mapsto A_t$ is continuous,*
- *A_t is $\{\mathcal{F}_t\}$ adapted,*
- *$A_{s+t} = A_s + A_t \circ \theta_s$, $\forall s$, $\forall t$.*

In addition, we also define the inverses of A as follows

$$\tau_t^+ := \inf\{u : A(u) > t\} \quad \textit{and} \quad \tau_t^- := \inf\{u : A(u) < -t\}. \tag{A.1}$$

In the above definitions we make the usual convention that $\inf \emptyset = \infty$ and we allow the usual notational switches $A(t) \equiv A_t$.

We now establish some useful results concerning both τ_t^+ and τ_t^-. However, due to the similarity of the proofs, we concentrate solely on proving the 'plus' case for each result.

A.3. Proposition. *For* $t \geq 0$, $\tau_t^{\pm} < \infty$ *if and only if there exists some* u *such that* $\pm A(u) > t$*; then we have* $A(\tau_t^{\pm}) = \pm t$.

Proof of Proposition A.3. The former part of the proposition is simply a consequence of our convention that $\inf \emptyset = \infty$.

Suppose $\tau_t^+ < \infty$. Given the definition of τ_t^+, note that $A(s) \leq t$ for $s < \tau_t^+$. Given any $\delta > 0$, $\tau_t^+ + \delta$ is not a lower bound for the set $\{u : A(u) > t\}$. Hence, for some u with $u < \tau_t^+ + \delta$, we have $A(u) > t$. Furthermore, $A(u)$ is continuous, and

$$\lim_{v \downarrow \tau_t^+} A(v) \geq t, \qquad \lim_{s \uparrow \tau_t^+} A(s) \leq t.$$

It follows that $A(\tau_t^+) = +t$. □

Next define

$$A^*(u) := \sup_{r \leq u} A(r), \qquad \text{and} \qquad A_*(u) := \inf_{r \leq u} A(r), \tag{A.2}$$

which naturally lead us to the following definitions

$$\hat{\tau}_t^+ := \inf\{u : A^*(u) > t\} \qquad \text{and} \qquad \hat{\tau}_t^- := \inf\{u : -A_*(u) > t\}.$$

In order to simplify matters we arrive at the following proposition.

A.4. Proposition. *We have* $\hat{\tau}_t^{\pm} = \tau_t^{\pm}$ *for all* $t \geq 0$.

Proof of Proposition A.4. We know that

$$A^*(u) \geq A(u) \qquad \text{for all } u. \tag{A.3}$$

Hence, from (A.3) and the definitions in (A.1), it is clear that

$$\hat{\tau}_t^+ \leq \tau_t^+.$$

For a contradiction suppose that $\hat{\tau}_t^+ < \tau_t^+$. Once again, given any $\delta > 0$, we are familiar with the fact that $\hat{\tau}_t^+ + \delta$ is not a lower bound for the set $\{u : A^*(u) > t\}$. Hence, there must exist a $u < \hat{\tau}_t^+ + \delta$ such that $A^*(u) > t$ for all $\delta > 0$. However, $\hat{\tau}_t^+ < \tau_t^+$ and since the result is true for all $\delta > 0$ (*in which case it is true for* $\delta = \tau_t^+ - \hat{\tau}_t^+$), it follows that there exists a $u < \tau_t^+$ such that

$$A^*(u) > t.$$

As A^* is defined as the supremum of A as in (A.2), trivially there must exist a $v \leq u < \tau_t$ such that

$$A(v) > t.$$

However, this contradicts the definition of τ_t, in that, we must have

$$A(u) \leq t \qquad \text{for all } u < \tau_t^+.$$

We now have the desired contradiction so that $\hat{\tau}_t^+ = \tau_t^+$. □

Given Proposition A.4, we are now able to make the following alternative definition of τ_t^+;

$$\tau_t^+ := \inf\{u : A_u^* > t\}, \qquad \text{for every } t \geq 0. \tag{A.4}$$

Clearly, we can make a similar definition for τ_t^-. We can now work with A^* (or A_*), so that we have the benefit of the monotonicity at our disposal.

A.5. Proposition. *For $t \geq 0$, $\tau_t^+ < \infty$ [respectively, $\tau_t^- < \infty$] if and only if there exists some u such that $A^*(u) > t$ [respectively, $-A_*(u) > t$]; then we have $A^*(\tau_t^+) = t$ [respectively, $A_*(\tau_t^-) = -t$].*

Proof of Proposition A.5. Given the definition in (A.4), this is now simply an alternative version of Proposition A.3. □

A.6. Proposition. *We have the following characterizations of τ_t^+ and τ_t^-:*

$\tau_t^+ = r < \infty$ if and only if $A^*(r) = t$ and $A^*(v) > t$ whenever $v > r$.
$\tau_t^- = r < \infty$ if and only if $A_*(r) = -t$ and $A_*(v) < -t$ whenever $v > r$.

Proof of Proposition A.6. ($\Rightarrow$) Suppose that $\tau_t^+ = r < \infty$. Then $A^*(r) = t$ by Proposition A.5. Again, given any $\delta > 0$, $\tau_t^+ + \delta$ is not a lower bound for the set $\{u : A_u^* > t\}$. Hence, by monotonicity of A^*, $A^*(v) > t$ for all $v > r$.
($\Leftarrow$) Conversely suppose that $A^*(r) = t$ and $A^*(v) > t$ whenever $v > r$. Then the set $\{u : A_u^* > t\}$ is not null so that $\tau_t^+ = r < \infty$. Note that we cannot have $\tau_t^+ < r$ in the previous statement due to monotonicity of A^* again. □

We now have enough ammunition to state and prove the following three Lemmas.

A.7. Lemma. *τ_t^+ and τ_t^- are right and left continuous respectively.*

Proof of Lemma A.7. Again, similarity of the two proofs makes it enough to only prove the 'plus' result. In particular, we need to show that

$$\lim_{v \downarrow t} \tau_v^+ = \tau_t^+.$$

Define $L := \lim_{v\downarrow t} \tau_v^+$, which exists by monotonicity. Then it suffices to prove that $L = \tau_t^+$. Using our characterization of τ_t^+, we need to prove that $A^*(L) = t$ and $A^*(v) > t$ whenever $v > L$. Now, as A^* is continuous,

$$A^*(L) = A^*\left(\lim_{v\downarrow t} \tau_v^+\right) = \lim_{v\downarrow t} A^*(\tau_v^+) = \lim_{v\downarrow t} v = t.$$

We can certainly say that $L \geq \tau_t^+$, simply by monotonicity of τ_t^+. Therefore, $v > L$ implies $v > \tau_t$. Recall that A^* is also monotone. Let $v = \tau_t^+ + \delta$, for some $\delta > 0$. Then, $\tau_t^+ + \delta$ is not a lower bound for the set $\{u : A^*(u) > t\}$, and so $A^*(\tau_t^+ + \delta) > t$. It follows that

$$\lim_{v\downarrow t} \tau_v = \tau_t^+.$$

□

A.8. Lemma. *The function $(u,\omega) \mapsto \tau_u^+(\omega)$ on $[0,\infty) \times \Omega$ is the unique function such that $A(\tau_u^+(\omega),\omega)) = u$ and $u \mapsto \tau_u^+(\omega)$ is right-continuous.*

Proof of Lemma A.8. Suppose that $(u,\omega) \mapsto \tau_u^+(\omega)$ has the described properties and that $v > \tau_u^+(\omega)$. Right-continuity shows that for some $\delta > 0$, $v > \tau_{u+\delta}^+(\omega)$, whence $A_v^*(\omega) > u + \delta$. □

A.9. Lemma. *For all u and all t, we have*

$$\tau_{t+u}^+ = \tau_t^+ + \tau_u^+ \circ \theta_{\tau_t^+}. \tag{A.5}$$

Proof of Lemma A.9. It is clear that

$$A\left(\tau_{u+t}^+\right) - A\left(\tau_t^+\right) = u + t - t = u.$$

Thus, from the 'time-shift' condition in Definition A.2 we may deduce that

$$A_{\tau_{t+u}^+(\omega) - \tau_t^+(\omega)} \circ \theta_{\tau_t^+} = u.$$

Because in addition $u \mapsto (\tau_{t+u}^+ - \tau_t^+)(\theta_{\tau_t^+}(\omega))$ is right-continuous, we have

$$\tau_{t+u}^+(\omega) - \tau_t^+(\omega) = \tau_u^+ \circ \theta_{\tau_t^+}(\omega),$$

as desired. □

Remark. The above material is intended to apply to our Φ's in Sections 5 and 3 of Chapters 2 and 3 respectively. The given Φ's are continuous additive functionals as the corresponding local times are additive functionals, proof of which can be found in Rogers & Williams [24].

Appendix B

Some Ray Process Theory

Notation: Let E be a compact real interval. We write $C(E)$ for the space of all ($\mathbb{R}$-valued) continuous functions on E. For reasons that have already been pointed out, consider ∂ as an isolated point from E.

We now present some necessary results that form part of the rich theory of Ray processes. For further details, see Chapter 3 of Rogers & Williams [24].

B.1. Definition. $\{R^+_\lambda : \lambda > 0\}$ *is said to be an honest Feller resolvent on* $C(E)$ *if the following conditions hold:*

1. $R^+_\lambda : C(E) \to C(E)$,
2. $0 \le f \le 1 \quad \Rightarrow \quad 0 \le \lambda R^+_\lambda f \le 1$,
3. $\lambda R^+_\lambda 1 = 1$,
4. *The resolvent equation holds;*

$$R^+_\lambda - R^+_\mu + (\lambda - \mu) R^+_\lambda R^+_\mu \ = \ 0.$$

B.2. Definition (CSM$^\alpha$). *For* $\alpha \ge 0$*, a function* $f \in C(E)$ *is called a (continuous)* α*-supermedian function relative to* $\{R_\lambda\}$*, hereafter written* $f \in$*CSM*$^\alpha$*, if*

$$0 \le \lambda R_{\lambda+\alpha} f \le f \quad (\forall \lambda > 0).$$

B.3. Lemma. *Let* $\alpha, f \ge 0$ *and* $f \in C[0,1]$*. If* $\{R_\lambda\}$ *is an honest Feller resolvent, then* $R^+_\alpha f \in$*CSM*$^\alpha$.

Proof of Lemma B.3. For $f \ge 0$ in $C[0,1]$, we may use the resolvent equation to deduce that

$$0 \le \lambda R^+_{\lambda+\alpha} R^+_\alpha f = R^+_\alpha f - R^+_{\lambda+\alpha} f \le R^+_\alpha f,$$

which is exactly the desired result. □

B.4. Definition (Ray resolvent). *An honest Feller resolvent on E is called a Ray resolvent if*

$$\bigcup_{\alpha \geq 0} CSM^{\alpha} \text{ separates points of } E.$$

B.5. Theorem (Ray's Theorem: analytic part). *Let $\{R_\lambda : \lambda > 0\}$ be an honest Ray resolvent on a compact metric space. Then there exists a unique honest measurable transition function $\{P_t\}$ on $(E, \mathcal{B}(E))$ such that*

(i) $t \mapsto (P_t f)(z)$ *is right-continuous on* $[0, \infty)$ *for* $z \in E$ *and* $f \in C(E)$;

(ii) $(R_\lambda f)(z) = \int_0^\infty e^{-\lambda t}(P_t f)(z)\mathrm{d}t \quad (z \in E,\ f \in C(E),\ \lambda > 0)$.

Remark: The above theorem not only provides us with the existence of a semigroup, but it tells us that the semigroup inherits honesty properties from the honest resolvent.

Proof of Theorem B.5. Again see Chapter 3 of Rogers & Williams [24]. □

For the purpose of the following Lemma, assume that $\{R_\lambda : \lambda > 0\}$ is an honest Ray resolvent on a compact metric space E with density r_λ. Then Ray's Theorem guarantees the existence of an honest semigroup. Suppose that such a semigroup has transition density p_t.

B.6. Lemma. *For $z \in E$, we have*

$$\lambda r_\lambda(z, w) = \int_0^\infty e^{-s} p_{s/\lambda}(z, w)\mathrm{d}s. \tag{B.1}$$

Proof of Lemma B.6. Now, for $z \in E$,

$$\begin{aligned}(R_\lambda f)(z) &= \int_E r_\lambda(z, w) f(w)\mathrm{d}w \\ &= \int_0^\infty e^{-\lambda t}(P_t f)(z)\mathrm{d}t = \int_0^\infty e^{-\lambda t} \int_E p_t(z, w) f(w)\mathrm{d}w\ \mathrm{d}t \\ &= \int_E \int_0^\infty e^{-\lambda t} p_t(z, w)\mathrm{d}t\ f(w)\mathrm{d}w \quad \text{(by Fubini's Theorem)}\end{aligned}$$

From the above, it follows that

$$\lambda r_\lambda(z, w) = \int_0^\infty \lambda e^{-\lambda t} p_t^+(z, w)\mathrm{d}t.$$

The obvious substitution gives us the desired result. □

B.7. Corollary. *We have the following limits;*

$$\lim_{\lambda \to \infty} \lambda r_\lambda(z, w) = p_0^+(z, w), \qquad \lim_{\lambda \downarrow 0} \lambda r_\lambda(z, w) = p_\infty^+(z, w).$$

Proof of Corollary B.7. This is trivial. □

Appendix C

Applications of Lévy's Presentation of Brownian Motion

C.1. Remaining Points in the Proof of Theorem 2.7A

The following result turns out to be extremely useful, mainly as it provides us with a simple normalization (in law) to our local time processes. Script notation is used here to make the important distinction between 'pathwise' and 'law-wise' equivalence.

C.1. Theorem (Lévy's presentation). *Suppose that $\mathcal{W}$ is a Brownian motion, with drift $\mu \in \mathbb{R}$, on $\mathbb{R}$ started at zero. Let $\mathcal{L}_u := -\min\{\mathcal{W}_s : s \leq u\}$, then $\{\mathcal{W}_u + \mathcal{L}_u\}$ is a reflecting Brownian motion with drift μ on $[0, \infty)$ with local time $\mathcal{L}$ at* 0.

Proof of C.1. Refer to Volume 2 of Rogers & Williams [24]. □

We now examine results 1° and 2° in the proof of Theorem 2.7A (a). The results are re-stated below. The symbol ϵ now reverts to its normal use in Mathematics.

1°. If N is any neighbourhood in $(-\infty, 0) \times (0, 1)$, then we have

$$\mathbb{P}^{\varphi,z}\big((\Phi_t, Z_t) \in N;\ \text{for some } t < \tau_0^+\big) > 0.$$

2°. Let $x \in \{0, 1\}$. If N^* is a relatively open subset of $(-\infty, 0) \times [0, 1]$ such that $N^* \cap \{(-\infty, 0) \times \{x\}\}$ is non-empty, then with positive $\mathbb{P}^{\varphi,z}$ probability, we have

$$\int_0^{\tau_0^+} I_{N^*}(\Phi_t, Z_t) \mathrm{d}L_x(t) > 0.$$

C.2. Proof of Result 1°. We consider the case when $Z(0) = 0$ and $\Phi(0) = \varphi < 0$. Choose (φ^*, z^*) in N and $\epsilon \in (0, -\varphi/4)$ such that if $|\varphi_1 - \varphi^*| < 4\epsilon$ and $|z_1 - z^*| < 4\epsilon$, then $(\varphi_1, z_1) \in N$. Note that $\max(\varphi, \varphi^*) < -4\epsilon$ and $z^* < 1 - 4\epsilon$. Let $t > \max(\varphi^* - \varphi, 0)$. We wish to show that the event that the following statements hold simultaneously:

$$
\begin{aligned}
&|Z_t - z^*| < 2\epsilon, \qquad |\Phi_t - \varphi^*| < 2\epsilon,\\
&\sup_{u \le t} \Phi_u \le -\epsilon \quad \text{so that } \tau_0^+ > t,\\
&\sup_{u \le t} Z_t \le 1 - \epsilon \quad \text{so that } \Phi_u = \varphi + u - 2m_0 L_0(u) \text{ for } u \le t,
\end{aligned}
$$

has positive $\mathbb{P}^{\varphi,z}$ probability.

Because it refers to a situation in which Z stays away from 1 during time-interval $[0, t]$, the event just described has the same probability as it would have for a Brownian motion on $[0, \infty)$ reflected at 0. If $\mathcal{W}$ is a Brownian motion on $\mathbb{R}$ started at 0 and $\mathcal{L}_u := -\min\{\mathcal{W}_s : s \le u\}$, then by Lévy's Theorem above applied at $\mu = 0$, $\{\mathcal{W}_u + \mathcal{L}_u\}$ is a reflecting Brownian motion on $[0, \infty)$ with local time $\mathcal{L}$ at 0.

We can easily find a piecewise-linear continuous function $w : [0, t] \to \mathbb{R}$ (consisting of just two linear pieces) such that if $\ell_t := -\min\{w_s : s \le t\}$, then

$$
\begin{aligned}
&w_t + \ell_t = z^*, \qquad \varphi + t - 2m_0\ell_t = \varphi^*,\\
&\sup_{u \le t}(\varphi + u - 2m_0\ell_u) \le -3\epsilon,\\
&\sup_{u \le t}(w_t + \ell_t) \le 1 - 3\epsilon.
\end{aligned}
\tag{C.1}
$$

By the Cameron-Martin formula, the event that $|\mathcal{W}_u - w_u| < \epsilon/(1 + 2m_0)$ for all u in $[0, t]$ has positive probability. But on this event,

$$
\begin{aligned}
&|\mathcal{W}_t + \mathcal{L}_t - z^*| < 2\epsilon, \qquad |\varphi + t - 2m_0\mathcal{L}_t - \varphi^*| < 2\epsilon,\\
&\sup_{u \le t}(\varphi + u - 2m_0\mathcal{L}_u) \le -\epsilon,\\
&\sup_{u \le t}(\mathcal{W}_t + \mathcal{L}_t) \le 1 - \epsilon,
\end{aligned}
$$

so that Result 1° is proved when $z = 0$. If $z \in (0, \frac{1}{2}]$, then Z can hit 0 almost immediately, an idea that can again be dealt with using a straight line. Hence, all that remains is the case when the 'new' Φ_0 is very close to φ and the 'new' $Z_0 = 0$. This is exactly what we have above! Therefore, Lévy's idea may be started afresh with respect to the new starting situation. The remaining cases follow by symmetry. □

C.3. Proof of result 2°. Result 2° follows by similar, but surprisingly much simpler, arguments to those used to prove 1°. It turns out that a single straight line suffices, provided the fact that $t < \tau_0^+$ is obeyed.

The following figures consider the case when $x = 1$ and $z_0 \in (0, 1)$ in result 2°. Notice that Lévy's idea is started afresh once w_u hits 1. □

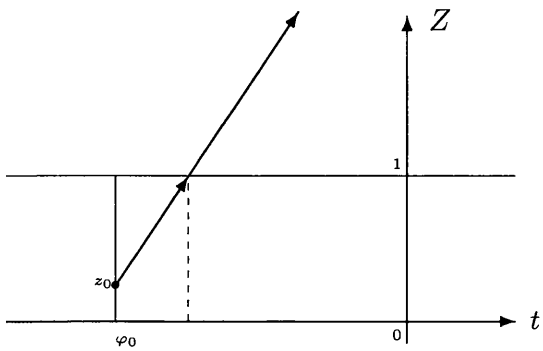

Figure C.1: An appropriate w_t.

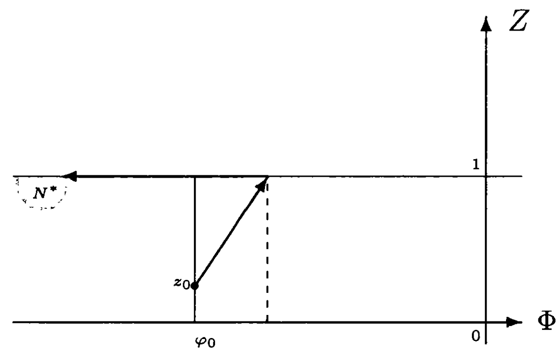

Figure C.2: An appropriate w_t against Φ_t.

C.4. Further comments on the proof of Proposition 2.8A. Recall that we let $\mathcal{G}^-$ denote the Q-matrix of Z^- when Z^- is considered as a Markov chain on $\{0, 1\}$. Proposition 2.8A states that the off-diagonal elements of $\mathcal{G}^-$ are strictly positive and it was remarked that such a fact could be proved by the arguments presented above. Further details will now be given. It is certainly enough to prove that

$$\mathbb{P}^0\left(Z_t^- = 1\right) > 0, \quad \text{for some } t.$$

Of course, symmetry gives the corresponding result at 1. It is now clear that the proof of the result in 2° covers this case. □

C.2. Growth of Local Time: Instructive Example

The object of this example on Lévy's $\mathcal{L}$ and $\mathcal{W}$ (started at 0) is to show that $\mathcal{L}$ initially grows faster than any positive multiple at of t. We have $\mathcal{L}_t \cong \max_{s\le t} \mathcal{W}_s$ (identity in law) so that for $c > 0$, $\mathcal{L}_{c^2t} \cong c\mathcal{L}_t$, and hence, for $t < 1$, we have (taking $c = t^{-\frac{1}{2}}$)

$$\mathcal{L}_{c^2t} = \mathcal{L}_1 \cong \tfrac{1}{\sqrt{t}}\mathcal{L}_t \quad \text{so that} \quad \sqrt{t}\mathcal{L}_1 \cong \mathcal{L}_t.$$

Thus, we have

$$\mathbb{P}[\mathcal{L}_t < 3at] \;=\; \mathbb{P}[\mathcal{L}_1 < 3a\sqrt{t}].$$

By considering a simple diagram, it is clear that

$$\mathbb{P}[\mathcal{L}_t < u] \;= \mathbb{P}[\rho_u > t],$$

where $\rho_u := \inf\{v : \mathcal{W}_v = u\}$, so that

$$\mathbb{P}[\mathcal{L}_1 < u] \;= \mathbb{P}[\rho_u > 1].$$

Let $u = 3a\sqrt{t}$. Using the probability density function for ρ_u we have

$$\mathbb{P}[\rho_u > 1] = \int_1^\infty \frac{|u|}{\sqrt{2\pi v^3}} \exp\left(-\frac{u^2}{2v}\right) dv.$$

Considering the substitution $v = \frac{u^2}{w^2}$, we equivalently have

$$\mathbb{P}[\rho_u > 1] = \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{1}{2}} \int_0^u \exp\left(-\frac{w^2}{2}\right) dw.$$

Next note that $e^{-\frac{w^2}{2}} \leq 1$ so that

$$\mathbb{P}[\rho_u > 1] \leq \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{1}{2}} \int_0^{3a\sqrt{t}} dw = \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{1}{2}} 3a\sqrt{t} = K\sqrt{t},$$

where K is clearly a constant. We now have

$$\mathbb{P}[\mathcal{L}_t < 3at] \leq K\sqrt{t}.$$

Next let $t = 2^{-n}$ (< 1) for $n \in \mathbb{N}$, so that

$$\mathbb{P}[\mathcal{L}(2^{-n}) < 3a2^{-n}] \leq K2^{-\frac{n}{2}}.$$

Summing both sides over $n \in \mathbb{N}$, we find that

$$\sum_n \mathbb{P}[\mathcal{L}(2^{-n}) < 3a2^{-n}] = \frac{K}{\sqrt{2}-1} < \infty.$$

Thus, by the *First Borel-Cantelli Lemma*, it follows that

$$\mathbb{P}[\limsup E_n] = 0 \qquad \Leftrightarrow \qquad \mathbb{P}[(\limsup E_n)^c] = 1,$$

where we define $E_n := \{\mathcal{L}(2^{-n}) < 3a2^{-n}\}$. Now note that

$$(\limsup E_n)^c = \liminf E_n^c,$$

and so we have

$$\mathbb{P}[(\limsup E_n)^c] = 1 \qquad \Leftrightarrow \qquad \mathbb{P}[\liminf E_n^c] = 1.$$

By using the definition of lim inf we have

$$\begin{aligned} \liminf E_n^c &= \{\omega : \omega \in E_n^c \text{ for all large } n\} \\ &= \{\omega : \text{ for some } m(\omega) > 0, \omega \in E_n^c, \text{ for all } n \geq m(\omega)\} \\ &= \{\exists m(\omega) > 0 \text{ such that } \mathcal{L}(2^{-n}) \geq 3a2^{-n} \text{ for all } n \geq m(\omega)\}. \end{aligned}$$

Clearly we are dealing with a discrete time event, and we need a real time statement.

Let $\delta(\omega) = 2^{-m(\omega)}$. If $t < \delta(\omega)$ then $2^{-(n+1)} \leq t < 2^{-n}$ for some $n \geq m(\omega)$. Then since $\mathcal{L}$ is a non-decreasing process, we have

$$\mathcal{L}(t) \geq \mathcal{L}\left(2^{-(n+1)}\right) \geq 3a2^{-(n+1)} > a2^{-n} > at.$$

Hence on the set $\liminf E_n^c$ of probability 1, there exists $\delta(\omega) > 0$ such that $\mathcal{L} > at$ for $t < \delta(\omega)$.

□

Appendix D

Further Probabilistic Aspects

Classical Probabilistic Wiener-Hopf Theory is about *fluctuations* of functionals; and in the case of Chapters 2 and 3, the relevant functional is Φ. A typical problem of the classical theory is the following: what is the distribution of the supremum of the Φ process? (The supremum will, of course, a.s. be ∞ if $m_0 + m_1 \leq 1$ or $\mu \geq -1$.) In the two-boundary problem for instance, we have solved this particular question because

$$\int_0^\infty e^{-\lambda t}\mathbb{P}^z[\sup_u \Phi_u > t]dt = \int_0^1 r_\lambda^+(z,w)dw,$$

and we know $r_\lambda^+(z,w)$. The classical theory would utilize such expressions as $\mathbb{E}\exp(i\theta\Phi_\zeta)$ where ζ is an exponentially distributed random variable of rate c, whence mean c^{-1}, ζ being independent of the Z-process. A limiting case of this reads: for $m_0 + m_1 < 1$ and $0 < \rho < \alpha$ (the smaller positive root of ϵ), we have

$$\tfrac{1}{2}\rho\epsilon(\rho)\mathbb{E}^0\exp(-\tfrac{1}{2}\rho^2\Phi_t)dt = \int_0^1 h_\rho^\sharp(w)dw,$$

as follows (after some calculation) from the Feynman-Kac formula.

Analogues of the splendid results in Bertoin [2], Bingham [3], Greenwood and Pitman [12], are, of course, well worth pursuing here. A familiar difficulty is that killing our process at the random time ζ greatly complicates calculations. The effect of the killing is to replace $\mathcal{H}$ by $\mathcal{H}_c$ where

$$\mathcal{H}_c f = \tfrac{1}{2}f'' - cf$$

with boundary condition $2m_0[\frac{1}{2}f''(0) - cf(0)] + f'(0) = 0$ at 0 and the analogous condition at 1. Refer back to (7.1) of Chapter 3 for the 'killed' setup for the drift case.

Taking up such matters here would substantially lengthen this thesis and would introduce material of a completely different flavour from the 'indefinite inner product' approach which we have been trying to advertise.

Let us now answer one final question.

How did we know what boundary conditions to impose on $\mathcal{H}$ at (1.1) *of Chapter 2?*

One answer is to take a Markov-chain analogue. Let $E_n = \{0, 1/n, 2/n, \ldots, n/n\}$ with $n+1$ points, and let Q_n (the Q-matrix of an approximating Markov chain Z_n) be the $E_n \times E_n$ matrix

$$Q_n = n^2 \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{2} & -1 & \frac{1}{2} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & -1 & \frac{1}{2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Let v_n be the function on E_n with

$$v_n = \begin{cases} 1 & \text{on } \{1/n, 2/n, \ldots, (n-1)/n\}, \\ -m_0(n-1) & \text{at } 0, \\ -m_1(n-1) & \text{at } 1, \end{cases}$$

and let V_n be the operator of multiplication by v_n. For Φ_n, we take

$$\Phi_n(t) := \int_0^t v_n(Z_n(s))\mathrm{d}s.$$

Wiener-Hopf theory for Markov chains tells us that the analogue of $\mathcal{H}$ is $\mathcal{H}_n = V_n^{-1} Q_n$. For further details see, for example, Barlow, Rogers and Williams [1], Williams [26] and references therein. Clearly V_n^{-1} is the $E_n \times E_n$ matrix

$$V_n^{-1} = \begin{pmatrix} \frac{-1}{m_0(n-1)} & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \ddots & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \frac{-1}{m_1(n-1)} \end{pmatrix}.$$

Suppose the function f on $[0,1]$ restricts to E_n, so that it can be viewed as a column vector of length $n+1$. Then

$$V_n^{-1} Q_n f = n^2 \begin{pmatrix} \frac{1}{2m_0(n-1)} & \frac{-1}{2m_0(n-1)} & & & \\ \frac{1}{2} & -1 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{2} & -1 & \frac{1}{2} \\ & & & \frac{-1}{2m_1(n-1)} & \frac{1}{2m_1(n-1)} \end{pmatrix} \begin{pmatrix} f(0) \\ f(\frac{1}{n}) \\ \vdots \\ f(\frac{n-1}{n}) \\ f(1) \end{pmatrix}. \tag{D.1}$$

In particular, we see that

$$(V_n^{-1} Q_n f)(0) = \frac{n^2}{2m_0(n-1)} \left\{ f(0) - f(\tfrac{1}{n}) \right\}.$$

Using the Taylor series expansion for $f(\frac{1}{n})$ about 0 and considering the obvious cancellation we alternatively have

$$(V_n^{-1} Q_n f)(0) = \frac{-n^2}{2m_0(n-1)} \left\{ \frac{f'(0)}{n} + \frac{f''(0)}{2n^2} + O\left(\tfrac{1}{n^3}\right) \right\}. \tag{D.2}$$

From (D.1) we also have

$$(V_n^{-1}Q_nf)(\tfrac{1}{n}) = \tfrac{n^2}{2}\left\{f(0) - 2f(\tfrac{1}{n}) + f(\tfrac{2}{n})\right\}.$$

This time we consider the Taylor expansions of $f(0)$ and $f(\frac{2}{n})$ both about $\frac{1}{n}$ to get

$$(V_n^{-1}Q_nf)(\tfrac{1}{n}) = \tfrac{1}{2}f''(\tfrac{1}{n}) + O\left(\tfrac{1}{n}\right). \tag{D.3}$$

As $n \to \infty$ we expect

$$(V_n^{-1}Q_nf)(\tfrac{1}{n}) - (V_n^{-1}Q_nf)(0) \to 0. \tag{D.4}$$

Now from (D.2) and (D.3) we have

$$(V_n^{-1}Q_nf)(\tfrac{1}{n}) - (V_n^{-1}Q_nf)(0) = \left\{-\tfrac{1}{2}f''(\tfrac{1}{n}) - \frac{n}{2m_0(n-1)}f'(0)\right\} - \frac{1}{4m_0(n-1)}f''(0) + O\left(\tfrac{1}{n}\right).$$

In order to tally with the result in (D.4), we clearly need

$$\lim_{n\to\infty}\left\{f''(\tfrac{1}{n}) + \frac{f'(0)}{m_0(1-1/n)}\right\} = 0 \quad \Rightarrow \quad m_0f''(0) + f'(0) = 0.$$

Similar calculations for $(V_n^{-1}Q_nf)(1)$ and $(V_n^{-1}Q_nf)(\frac{n-1}{n})$ give the corresponding result at 1. □

Further Calculations. One could further pursue this discrete problem. However, the calculations are horrendous! One *should* find that the matrix $V_n^{-1}Q_n$ has two strictly positive distinct eigenvalues if and only if $m_0 + m_1 < 1$, and so on.

Appendix E

Derivation of the Resolvent Decomposition in Chapter 2

Here we derive the resolvent decomposition given in (10.1) of Chapter 2. The argument can clearly be modified to give the corresponding decomposition in (11.2) of Chapter 3.

It is well-known that the resolvent may be defined as follows

$$\begin{aligned} g_\lambda(z) := (\hat{R}_\lambda^+ f)(z) : &= \mathbb{E}^z \int_0^\infty e^{-\lambda t} f(\hat{Z}_t^+)\, \mathrm{d}t \\ &= \mathbb{E}^z \int_0^T e^{-\lambda t} f(\hat{Z}_t^+)\, \mathrm{d}t \;+\; \mathbb{E}^z \int_T^\infty e^{-\lambda t} f(\hat{Z}_t^+)\, \mathrm{d}t \\ &= ({}_{\mathrm{tab}}\hat{R}_\lambda^+ f)(z) \;+\; \mathbb{E}^z \int_T^\infty e^{-\lambda t} f(\hat{Z}_t^+)\, \mathrm{d}t. \end{aligned} \tag{E.1}$$

It suffices to examine the last term in (E.1). Recall that $T = T_0 \wedge T_1$, so that

$$\begin{aligned} \mathbb{E}^z \int_T^\infty e^{-\lambda t} f(\hat{Z}_t^+)\, \mathrm{d}t = \mathbb{E}^z &\left[\int_T^\infty e^{-\lambda t} f(\hat{Z}_t^+)\, \mathrm{d}t \;;\; T_0 < T_1 \right] \\ &+ \mathbb{E}^z \left[\int_T^\infty e^{-\lambda t} f(\hat{Z}_t^+)\, \mathrm{d}t \;;\; T_1 < T_0 \right]. \end{aligned} \tag{E.2}$$

Next define $Y_t = \hat{Z}_{T+t}^+$ and note that

$$\int_T^\infty e^{-\lambda t} f(\hat{Z}_t^+)\, \mathrm{d}t = e^{-\lambda T} \int_0^\infty e^{-\lambda t} f(Y_t)\, \mathrm{d}t. \tag{E.3}$$

For convenience define

$$F(\hat{Z}_t^+) := \int_0^\infty e^{-\lambda t} f(\hat{Z}_t^+)\, \mathrm{d}t, \tag{E.4}$$

noting that

$$\mathbb{E}^z \left[F(\hat{Z}_t^+) \right] = g_\lambda(z).$$

Because 0 and 1 are branch points of the Ray process $\hat{Z}^+$, the relevant strong Markov property to which we are making *intuitive* appeal, is really that (due to Meyer and Ray) at Theorem III.41.3 of Rogers and Williams. This allows us to avoid any confusion as we know that the $\hat{Z}^+$ process only 'lives' on $(0,1)$. Next we consider an application of the strong Markov property (SMP).

$$\begin{aligned} \mathbb{E}^z \left[F(Y) \mid \mathcal{F}_T \right] &= g_\lambda(Y_0) \\ &= \mathbb{E}^{\hat{Z}_t^+} \left[F(\hat{Z}_t^+) \right] \\ &= g_\lambda(\hat{Z}_T^+) \\ &= \begin{cases} g_\lambda(0) & \text{if } T = T_0, \\ g_\lambda(1) & \text{if } T = T_1. \end{cases} \end{aligned} \tag{E.5}$$

We are now in a position to simplify the RHS of (E.2)

$$\begin{aligned} &\mathbb{E}^z \left[\int_T^\infty e^{-\lambda t} f(\hat{Z}_t^+)\, \mathrm{d}t \ ; \ T_0 < T_1 \right] \\ &= \mathbb{E}^z \left[e^{-\lambda T} \int_0^\infty e^{-\lambda t} f(Y_t)\, \mathrm{d}t \ ; \ T_0 < T_1 \right] && \text{(by (E.3))} \\ &= \int_{\{T_0 < T_1\}} \left[e^{-\lambda T_0} \int_0^\infty e^{-\lambda t} f(Y_t)\, \mathrm{d}t \right] \mathrm{d}\mathbb{P}_z \\ &= \int_{\{T_0 < T_1\}} \mathbb{E}^z \left[e^{-\lambda T_0} \int_0^\infty e^{-\lambda t} f(Y_t)\, \mathrm{d}t \middle| \ \mathcal{F}_{T_0} \right] \mathrm{d}\mathbb{P}_z && \text{(by Tower Property)} \\ &= \int_{\{T_0 < T_1\}} e^{-\lambda T_0} \mathbb{E}^z \left[\int_0^\infty e^{-\lambda t} f(Y_t)\, \mathrm{d}t \middle| \ \mathcal{F}_{T_0} \right] \mathrm{d}\mathbb{P}_z \\ &= \int_{\{T_0 < T_1\}} e^{-\lambda T_0}\, \mathbb{E}^{\hat{Z}_{T_0}^+} \left[F(\hat{Z}^+) \right] \mathrm{d}\mathbb{P}_z && \text{(by the SMT in (E.5))} \\ &= \int_{\{T_0 < T_1\}} e^{-\lambda T_0} g_\lambda(0)\, \mathrm{d}\mathbb{P}_z = g_\lambda(0)\, \mathbb{E}^z \left[e^{-\lambda T_0};\ T_0 < T_1 \right] \\ &= (\hat{R}_\lambda^+ f)(0) \psi_\lambda(z, 0). \end{aligned} \tag{E.6}$$

A similar argument yields the corresponding condition at one.

$$\mathbb{E}^z \left[\int_T^\infty e^{-\lambda t} f(\hat{Z}_t^+)\, \mathrm{d}t \ ; \ T_1 < T_0 \right] = (\hat{R}_\lambda^+ f)(1) \psi_\lambda(z, 1). \tag{E.7}$$

Substitution of (E.6) and (E.7) into (E.1) yields

$$g_\lambda(z) := (\hat{R}_\lambda^+ f)(z) = ({}_{\text{tab}}\hat{R}_\lambda^+ f)(z) + \sum_{x \in \{0,1\}} (\hat{R}_\lambda^+ f)(x) \psi_\lambda(z, x).$$

Note that we rely on the following points:

$\{T_0 < T_1\} \in \mathcal{F}_{T_0}$ and $e^{-\lambda T_0}$ is measurable w.r.t. $\mathcal{F}_{T_0}$.

We need to prove these points, so we must show

$$\{T_0 < T_1\} \wedge \{T_0 < t\} \in \mathcal{F}_t.$$

We have $T_0(\omega) < T_1(\omega)$ and $T_0(\omega) < t$ if and only if for some rational $r < 1$ it is true for every n in $\mathbb{N}$ that for some rational $s < t$ we have

$$\sup\{Z_q(\omega) : q \in \mathbb{Q}, 0 < q < s\} < r \qquad \text{and} \qquad \inf\{Z_q(\omega) : q \in \mathbb{Q}, 0 < q < s\} < \frac{1}{n}.$$

As matters are only intuitive here, proof of these points will be ignored.

List of Figures

Bibliography

[1] Barlow, M.T., Rogers, L.C.G.R. and Williams, D., Wiener-Hopf factorization for matrices, *Séminaire de probabilités, XIV* (ed. J. Azéma and M. Yor), Lecture Notes in Math. 784, Springer, Berlin, Heidelberg, New York, pp. 324–331, 1980.

[2] Bertoin, J., *Lévy Processes*, Cambridge University Press, 1996.

[3] Bingham, N.H., Fluctuation theory in continuous time, *Adv. Appl. Prob.* **7**, pp. 705–766, 1975.

[4] Bollobás, B., *Linear Analysis*, Cambridge University Press, 1990.

[5] A. Dijksma, I. Gohberg, MA Kaashoek and R. Mennicken, (editors), *Contributions to Operator Theory in Spaces with an Indefinite Metric : The Heinz Langer Anniversary Volume*, Birkhauser, Boston, 1998.

[6] Dunford, N. and Schwarz, J.T., *Linear Operators, Part 1*, Wiley, New York, 1957.

[7] Durrett, R., *Stochastic calculus - A practical introduction*, CRC Press, Boca Raton, 1996.

[8] Egorov, I.E., Pyatkov, S.G., and Popov, S.V., *Non-classical operator-differential equations* (in Russian), Nauka, Novosibirsk, 2000.

[9] Finney, R.L. and Ostberg, D.R., *Elementary Differential Equations with Linear Algebra*, Addison-Wesley, 1976.

[10] Freedman, D., *Brownian motion and Diffusion*, Holden-Day, 1971.

[11] Garrett, P., Compact operators on Hilbert space, `www.math.umn.edu/~garrett/m/fun/cpt_ops_hsp.ps`

[12] Greenwood, P.E. and Pitman, J.W., Fluctuation identities for Lévy processes and splitting at the maximum, *Adv. Appl. Prob.* **12**, pp. 893–902, 1980.

[13] Harrison, J.M., *Brownian Motion and Stochastic Flow Systems*, Wiley, 1985.

[14] Itô, K. and McKean, H.P., *Diffusion processes and their sample paths*, Springer, Berlin, Heidelberg, New York, 1965.

[15] Karatzas,I. and Schreve, S.E., *Brownian Motion and Stochastic Calculus*, Springer, Berlin, Heidelberg, New York, 1991.

[16] Kennedy, J.E., A probabilistic view of some algebraic results in Wiener-Hopf theory for symmetrizable Markov chains. In *Stochastics and Quantum Mechanics* (edited by A. Truman and I.M.Davies), World Scientific, 1992, pp. 165–177, 1990.

[17] Lindvall, T., *Lectures on the Coupling Method*, Wiley, Chichester, New York, 1992.

[18] Maisonneuve, B., *Systèmes Régénératifs*, Astérisque **15**, Societe Mathématique de France, 1974.

[19] Noble, B., *Methods based on the Wiener-Hopf technique*, 2nd edition, Chelsea, New York, 1988.

[20] Norris, J.R, *Markov chains*, Cambridge University Press, 1997.

[21] Pitman, J.W., On coupling of Markov chains, *Z. Wahrsch. verw. Gebiete* **35**, pp. 315–322, 1976.

[22] Revuz, D. and Yor, M., *Continuous martingales and Brownian motion*, Springer, Berlin, Heidelberg, New York, 1991.

[23] Rogers, L.C.G., Time-reversal of the noisy Wiener-Hopf factorisation, *Proc. Symposia Pure Math.*, **57**, (edited by M.C. Cranston and M. Pinsky) 129–135, Amer. Math. Soc., Providence RI, 1995.

[24] Rogers, L.C.G. and Williams, D., *Diffusions, Markov prcesses and martingales, Volume I: Foundations, Volume II: Itô calculus*, Cambridge University Press, 2000 (originally published by Wiley).

[25] Titchmarsh, E.C., *The theory of functions*, Oxford University Press, 1952.

[26] Williams, D., Some aspects of Wiener-Hopf factorization, *Phil. Trans. Royal Soc. Lond. A* **335**, pp. 593–608, 1991.

[27] Williams, D., On Rogers' time-reversal theorem in Wiener-Hopf theory with noise (submitted).

[28] Williams, D., *Weighing the Odds*, Cambridge University Press, 2001.

[29] Williams, D., *Probability with Martingales*, Cambridge University Press, 1991.