Two classes of stochastic differential equations arising from financial modeling with stochastic volatility.

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Two classes of stochastic differential equations arising from financial modeling with stochastic volatility

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Abstract

This research is mainly concerned with two classes of stochastic differential equations arising from financial modeling with stochastic volatility–reflections and mean-reversion. In Chapter 1, I compare the no free lunch with vanishing risk condition and the no good deal condition for fundamental theorem of asset pricing in a continuous-time market model. I aim to determine the relationship between the conditions. In Chapter 2, I propose to model rating processes arising from rating based models for credit derivatives by SDEs with reflections. Chapter 3 is the infinite dimensional analogue of the mean-reversion type SDEs. The linear term in the drift is linked to the mean term and the nonlinar part can be viewed as the correcting term if the infinite system describes the equilibrium situation. As for the path independence of the Grisanov density, it is exactly corresponding to the equilibrium system.

The motivation of our investigation comes from the mathematical study of economics and finance. In recent years, due to the necessity of stochastic volatility as the measurement of uncertainty in modeling of financial markets, stochastic differential equations have received huge attention from both theoretical and practical aspects [48, 16, 18, 24, 38, 47]. The primary point here is to model the price dynamics or the wealth growth by utilising SDEs, after having established a so-called real world probability space (e.g., the seminal paper [8] by Black and Scholes). To an equilibrium financial market, there must exist a so-called risk neutral probability measure which is absolutely continuous with the given real world probability measure and it is pivotal to determine the path-independence property for the associated density process defined by the Radon-Nikodym derivative.

Chapter 1

Introduction

This research mainly studies two classes of stochastic differential equations arising from financial modeling with stochastic volatility–mean-reversion and reflection. Let us start with a general introduction of the stochastic differential equation (SDE). Stochastic integrals were first introduced by K. Itô [31] to rigorously formulate the SDE. In 1942 this theory was first applied to Kolmogorov's problem of determining Markov processes [30]. Today Itô's theory is applied not only to Markov processes (diffusion processes) but also to a large class of stochastic processes. This framework provides us a powerful tool for describing and analyzing stochastic processes. Since Itô theory may be considered as an integral-differential calculus for stochastic processes, it is often called Itô's stochastic analysis or stochastic calculus. J.L. Doob pointed out the martingale character of stochastic integrals and suggested that a unified theory of stochastic integrals should be establish in the framework of martingale theory. So he plays an important role in the modern theory of stochastic analysis. His program was accomplished by D.L. Fisk, P. Courrège, H. Kunita, S. Watannbe [59] and P. Meyer [66]. The class of stochastic processes to which Itô theory can be applied (usually called Itô processes or locally infinitely divisible processes) is now extended to a class of stochastic processes called semimartingales. Such processes appear to be the most general for which a unified theory of stochastic calculus can be developed. The modern theory of semimartingales and the stochastic calculus on them have been extensively developed in France by Meyer, Dellacherie, Jacod etc, [32]. A somewhat different type of stochastic calculus has been introduced by Stroock and Varadhan under the name of martingale problems [53].

Stochastic processes in finance and economics are developed in concept with the tools of stochastic calculus that are needed to solve practical importance. In 1973, Fischer Black and Myron Scholes used stochastic analysis and an equilibrium argument to compute a famous Black-Scholes formula which represented a triumph for mathematical modeling in finance [4]. It has become an indispensable tool in the trading of options and other financial derivatives. In 1997 Myron Scholes and Robert Merton were awarded the Nobel prize in Economics for their work related to this formula. (Fischer Black died in 1995.)

Virtually all continuous stochastic process of importance in applications satisfy an equation of the form

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t \quad \text{with initiala data} \quad X_0 = x_0.$$

Such SDEs provide an exceptionally effective framework for the construction and analysis of stochastic models. As the coefficients μ and σ of the equation can be interpreted as measures of short-term growth and short-term variability, the modeler has a ready-made pattern for the construction of stochastic processes that reflect real-world behavior. SDEs also provide a link between probability theory and much older but more developed fields of ordinary and partial differential equations. Wonderful consequences flow in both directions. The stochastic modeler benefit from centuries of development of physical sciences, and many classic results of mathematical physics and pure mathematics can be given new intuitive interpretations. In recent years, due to the necessity of stochastic volatility as measurement of uncertainly in modeling of financial markets, SDEs have received huge attention from both theoretical and practical aspect. A growing number of concepts, methods and results from the SDE which can be applied to give a financial model have been studied, [37] [39] [54].

All financial models in this thesis are based on the equilibrium economy. In the economy, an equilibrium state is where the net demand equals to total resources, in other words, that the excess demand is zero, [21]. In this economy, there are no taxes, transaction costs or information asymmetries, that is, any market in this economy is a perfect market. The representative agent is provided a positive initial amount without receiving any intermediate income and only concerned with his terminal wealth. All his consumption takes place at the terminal time T. There are only two kinds of infinitely divisible financial securities available in the market: a bond (risk assets) which pays one unit of consumption at time T and whose net supply is zero, and a stock (risky assets) with an equilibrium price process $(X_t)_{t\geq 0}$. We shall mainly work in an equilibrium market which can be characterized by the utility function of a representative agent. We assume the utility function belongs to the class of increasing, concave and continuously twice differentiable Von Neumann-Morgenstern utility functions. In such an equilibrium market, the representative agent maximizes his expected utility of time $T > 0$ wealth, i.e.,

$$\max E[U(X_T)].$$

It is natural that one would have different utility functions for different terminal date $T > 0$, so we would like to write the utility function U as a function of wealth x and also time t, i.e.,

$$U(x, t).$$

Cox and Leland [10] show that path independence is necessary for expected utility maximization. By path independence, they mean that the value of portfolio will depend only on the asset prices at that point, not on the path followed by the asset in reaching that price. Namely, the utility function U depends on the state price X_t at time t, for $t \geq 0$, that is, the function U is of the form $U(X_t, t)$, for each $t \geq 0$. On the other hand, in an equilibrium market without arbitrage opportunities, there exists a risk neutral probability measure Q which is absolutely continuous with respect to the objective probability P. Under the risk-neutral probability the drift of the stock return is the riskless interest rate $r(t)$. Then the Radon-Nikodym derivative is also a function of the state price X_t at time t, for $t \geq 0$.

Motivated by financial models in an equilibrium market, I have written three chapters in this thesis. Chapter 3 compares the no free lunch with vanishing risk condition and the no good deal condition for the fundamental theorem of asset pricing in a continuous-time market model. Due to the seminal work [13] by Delbaen and Schachermayer, the fundamental theorem of asset pricing became pivotal in mathematical finance, which is a key result in establishing a mathematical framework for pricing and the key condition in the so-called No Free Lunch with Vanishing Risk condition [14]. Since then, many investigations are devoted to generalize this remarkable condition to cover more general situations in the mathematical modelings, cf. eg.[1],[5],[15],[48] and references therein. Most recently, Bion-Nadal and Di Nunno [1] proposed a new condition for pricing in incomplete markets. This condition is named as No Good Deal Condition, which should be thought as an analogy or modified version of the celebrated No Free Lunch with Vanish Risk Condition. In this chapter, I aim to determine the relationship between the conditions. Tools from probability such as martingale, equivalent martingale measure, stochastic integrals, Girsanov transformation are all used in this framework.

In Chapter 4, on modeling credit risk via reflected stochastic differential equations, I propose to model rating processes arising from rating based models for credit derivatives by SDEs with boundary conditions. Rating-based models usually use characteristics such as rating process, yield curve and the recovery rate to compute price of risky assets. Crouhy-Im-Nuelman model and Hull-White model are famous ones in this family. They proposed in [27], [28] defines a rating process X_t which is a pure Brownian motion, but the "default barrier" which is not necessarily a straight line is adapted so as to match the default probability. In order to get a risk-neutral probability, they modify the location of the barrier. We shall follow the rating based framework presented by Douady and Jeanblanc [17] in modeling a defaultable zero coupon bond with a continuous rating process $R = (R_t)_{t\geq 0} \in [0,1]$. This continuous rating process R has an intuitive meaning: it can be seen as an interpolation of rating provided by agencies. More precisely, one can specify the model in such a way that a given agency rating corresponds to some sub-interval $(n_i, n_{i+1}) \subset [0,1]$. Rating migrations correspond to crossing one threshold $n_i \in (0,1)$. In [17], the continuous rating process $R = (R_t)_{t\geq 0} \in [0,1]$ of each bond issuer is determined by the following SDE

$$dR_t = h_t dt + \sigma(R_t, t) dW_t$$

with a given initial value $R_0 \in [0,1]$, where W_t is a Brownian motion, the drift h_t is an integrable function of t and volatility $\sigma(R_t, t)$ is a deterministic function of R_t and t. h_t and

$\sigma(R_t, t)$ are chosen to ensure that for each $R_0 < 1$ implies for all $t > 0$, $R_t < 1$ $a.s.$. $R_0 = 1$ corresponds to a non-defaultable bond and $R_t \equiv 1$, for all $t > 0$. Default happens when $R_t = 0$ which is an absorbing state.

In the basis of Douady and Jeanblanc [17], we propose a natural model of SDE with reflections for the rating process $X(t) \in [0, 1]$. We shall model a "continuous" rating $X(t) \in [0, 1]$, which is incorporated to a bond issuer subject to a possible default, by the following SDE with reflections

$$dX(t) = \theta X(t)dt + \sigma X(t)dB(t) + d\eta(t) - d\bar{\eta}(t)$$

where coefficients θ, σ are positive constants. Here $B(t)$ is a Brownian motion, and $\eta(t)$ is the local time of $X(t)$ at 0. This is a non-decreasing process which only increase when $X(t) = 0$. Similarly $\bar{\eta}(t)$ is the local time of $X(t)$ at 0. It is a non-decreasing process which only increase when $X(t) = 1$. Here we propose a natural model of stochastic differential equation with reflections for the rating process $X(t) \in [0, 1]$. In Chapter 3, we shall use this new diffusion process with reflections feature to model the rating processes and compute risk neutral probability for pricing the defaultable zero-coupon bond.

In Chapter 5, I provide a characterization of the path-independence property in the density process of Girsanov transformation for infinite-dimensional SDEs. From a mathematical viewpoint, as the utility function U is a smooth function, this is equivalent to saying that there exists a function $F : \mathbb{R} \times [0, \infty)$ which is C^2 with respect to the first variable x and C^1 with respect to the second variable t such that

$$F(X_t, t) = \frac{dQ}{dP}.$$

We shall call this property the path independence of the density of the Girsanov transformation. To an equilibrium financial market, there must exist a so-called risk neutral probability measure which is absolutely continuous with the given real world probability measure and it is pivotal to determine the path-independence property for the associated density process defined by the Radon-Nikodym derivative [25, 26]. It is often encountered in the economical and financial market models that one should consider agents in large scale that there are (at least) countably many stocks are treated together so that their pricing dynamics form an infinite-dimensional SDEs. This thesis studies the infinite dimensional analogue of the mean-reversion type SDEs. In Chapter 5, there are some researches which utilize this path independence property to characterize the behaviors of the drift of stock prices and certain ratio between drift coefficient and volatility coefficient in consistence with an equilibrium economy.

Furthermore, from the view point of variational calculus, optimization problems – either in the pattern of maximizing the utility functions (and/or profits) or in the formulation of minimizing the cost functions (and/or risk factors) – are in fact linked with the path-independent property of the pricing trajectories, cf. e.g., [20, 67]. Hence, characterizing the relevant path-independence of the SDEs in terms of (non-linear) PDEs would be interesting and useful.

To our aim, we notice that the methods employed in [63] and in [56] are Itô formula and Girsanov transformation. However, it is not straightforward to have Itô formula in infinite-dimensional so we have to use the finite-dimensional approximation approach here. We will derive a complete link of infinite-dimensional semi-linear SDEs to Burgers-KPZ nonlinear PDEs infinite dimensions. Extensions to more general infinite-dimensional spaces like Banach spaces, multi-Hilbertian spaces as well as locally convex topological vector spaces are interesting and will be considered in the forthcoming works.

Given a real separable Hilbert space $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$. Let $\{W_t\}_{t\geq 0}$ be a cylindrical Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. We consider the following semi-linear stochastic partial differential equation (SPDE) on H

$$\begin{cases} dX_t = \{AX_t + b(t, X_t)\}dt + \sigma(t, X_t)dW_t, & t \geq 0 \\ X_0 = x \in H, \end{cases} \tag{1.1}$$

where $b : [0,\infty) \times H \to H$ and $\sigma : [0,\infty) \times H \to L_A(H)$ are measurable mappings. In this paper, we require the two coefficients fulfill further that $b : [0,\infty) \times H \to H$ and $(t,x) \in [0,\infty) \times H \mapsto e^{tA}\sigma(t,x) \in L_{HS}(H)$ are C^1 with respect to the first variable and C^2 with respect to the second variable respectively. We assume that:

(H1) Assume that $-A$ has discrete spectrum with eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \ldots$$

counting multiplicities such that

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j} < \infty .$$

We let $\{e_j\}_{j\in\mathbb{N}}$ be the corresponding eigen-basis of $-A$ throughout the paper.

(H2) There exist a constant $\epsilon \in (0,1)$ and an increasing function $L : [0,\infty) \to (0,\infty)$ such that

$$\sup_{t\in[0,T]} \left\{ \|b(t,0)\|_H^2 + \int_0^t \|e^{(t-s)A}\sigma(s,0)\|_{HS}^2 s^{-\epsilon} ds \right\} < \infty, \quad \forall T > 0$$

and

$$\|b(t,x) - b(t,y)\|_H + \|e^{tA}\left(\sigma(t,x) - \sigma(t,y)\right)\|_{HS} \leq L(t)\|x-y\|_H, \quad \forall t \geq 0, \quad \forall x, y \in H.$$

It is well known by [12, 6] that (H1) and (H2) imply the existence and uniqueness of the mild solution to (1.1), that is, for any $x \in H$ there exists a unique H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted continuous process $X_t, t \geq 0$, such that P-a.s.

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A} b(s, X_s) ds + \int_0^t e^{(t-s)A}\sigma(s, X_s) dW_s, \quad t \geq 0.$$

Next, I give a brief account of the Girsanov transformation for infinite-dimensional SDEs on H, followed by the main result on the characterization of path-independence of the Girsanov density and its proof.

Theorem 1.0.1. *Assume (H1), (H2), (H3) and let $v : [0,\infty)\times H \to \mathbb{R}$ be in $C_b^{1,2}([0,\infty)\times H)$ such that $[\nabla v(t,\cdot)]x : H \to H \in Dom(A)$ for any $(t,x) \in [0,\infty)\times H$ and $\|A\nabla v(t,\cdot)\|_H$ is bounded locally and uniformly in $t \in [0,\infty)$. If v satisfies*

$$\frac{\partial}{\partial t} v(t,x) = -\frac{1}{2}\{Tr[(\sigma\sigma^*)\nabla^2 v](t,x) + \|\sigma^*\nabla v\|_H^2(t,x)\} - \langle x, A\nabla v(t,x)\rangle_H$$

and

$$b(t,x) = [(\sigma\sigma^*)\nabla v](t,x), \quad \forall (t,x) \in [0,\infty)\times H,$$

then the Girsanov density (5.6) *for* (1.1) *satisfies the following path-indendenpent property*

$$\frac{d\tilde{P}_t}{dP} = \exp\{v(0, X_0) - v(t, X_t)\}, \quad t \geq 0.$$

More specifically, I take stochastic heat equation on a bounded domain as an example demonstrate my work.

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = \dfrac{\partial v^2}{\partial^2 x}(t,x) + \phi(v(t,x)) + \varphi(v(t,x))\dfrac{\partial^2 W}{\partial t \partial x}(t,x) \quad t \geq 0, x \in (0,1), \\ v(t,0) = v(t,1) = 0, \quad t \geq 0, \\ v(0,x) = v_0(x), \quad x \in (0,1), \end{cases}$$

where $W(t,x)$, $(t,x) \in [0,\infty) \times [0,1]$ is a Brownian sheet $[0,\infty) \times [0,1]$. A Brownian sheet can be regarded as a cylindrical Wiener process on $L^2(0,1)$, see [57].

The organization of this thesis is organized as follows: Chapter 2 prepares some preliminaries on stochastic differential equations, which will be used in later derivations and proofs. First , we introduce Brownian Motions and stochastic integration. Then we show a few well known results on Itô processes and Itô formula. Next we give a brief introduction on SDEs, especially on the existence and uniqueness of the solutions to SDEs and SDEs with reflections. Girsanov theorem and some useful equations are also given in this chapter.

In Chapter 3, we aim to determine the relationship between the no free lunch with vanishing risk condition and the no good deal condition for fundamental theorem of asset pricing in a continuous-time market model. This chapter begins with the basic ideas of the First and Second Fundamental Theorems of asset pricing in the discrete model. Then a general continuous market model is defined and the fundamental theorems of asset pricing are proved in this setting. In the latter scenario we focus on conditions of the model which satisfy no free lunch with vanishing risk. Finally, we present a complete comparison with a thorough derivation. The paper ends with a conclusion to highlight our consideration.

In Chapter 4, under the assumption of equilibrium markets, we propose to model rating processes arising from rating based models for credit derivatives by stochastic differential equations with boundary conditions. Namely, for a rating process $X(t)$ taking value in the unit interval $[0, 1]$, which is assigned to a bond issuer subject to possible default, by stochastic differential equation with two sided reflections.

Chapter 5 is devoted to present links between infinite-dimensional SDEs and nonlinear PDEs of Burgers-KPZ type. we first give a brief account of the Girsanov transformation for SDEs on (infinite-dimensional) a separable Hilbert space H. Then we prove our main result on the characterization of path-independence of the Girsanov density of the SDEs. The final section is devoted to a consideration of parabolic stochastic partial differential equations as an example where we demonstrate an application of our main result.

Chapter 2

Preliminaries

This chapter is intended as an introduction to some elements of mathematical finance and stochastic differential equations (SDEs). We shall present some important analysis tools, Girsanov theorem, Itô formula, Brownian Motion and Stochastic Integration. All the exploration in this chapter is mainly based on Ikeda Watanabe [59], Williams [61], Øksendal [44], and Klebaner [19].

2.1 Concepts of Probability Theory

In this section we give fundamental definitions of probabilistic concepts. Since the theory is more transparent in the discrete case, it is presented first. Then a continuous probability model is defined in this setting.

2.1.1 Discrete Probability Model

A probability model consists of a filtered probability space on which variables of interest are defined. Here we introduce a discrete probability model by using an example of discrete trading in stock.

Filtered Probability Space

A filtered probability space consists of: a sample space of elementary events, a field of events, a probability defined on that field, and a filtration of increasing subfields.

Sample Space

Consider a single stock with price S_t at time $t = 1, 2, \ldots, T$. Denote by Ω the set of all possible values of stock during these times.

$$\Omega = \{\omega : \omega = (S_1, S_2, \ldots, S_T)\} = \mathbb{R}_+^T = (0, +\infty)^T.$$

If we assume that the stock price can go up by a factor u and down by a factor d, then the relevant information reduces to the knowledge of the movements at each time.

$$\Omega = \{\omega : \omega = (a_1, a_2, \ldots, a_T)\} \quad a_t = u \text{ or } d.$$

To model uncertainty about the price in the future, we list all possible future prices, and call it possible states of the world. The unknown future is just one of many possible outcomes, called the true state of the world. As time passes more and more information is revealed about the true state of the world. At time $t = 1$ we know prices S_0 and S_1. Thus the true state of the world lies in a smaller set, subset of Ω, $A \subset \Omega$. After observing S_1 we know which prices did not happen at time 1. Therefore we know that the true state of the world is in A and not in $\Omega \backslash A = \bar{A}$.

Fields of Events

Define by $\mathcal{F}_t$ the information available to investors at time t, which consists of stock prices before and at time t. For example when $T = 2$, at $t = 0$ we have no information about S_1 and S_2, and $\mathcal{F}_0 = \{\emptyset, \Omega\}$, all we know is that a true state of the world is in Ω. Consider the situation at $t = 1$. Suppose at $t = 1$ stock went up by u. Then we know that the true state of the world is in A, and not in its complement $\bar{A}$, where

$$A = \{(u, S_2), S_2 = u \text{ or } d\} = \{(u, u), (u, d)\}.$$

Thus our information at time $t = 1$ is

$$\mathcal{F}_1 = \{\emptyset, \Omega, A, \bar{A}\}.$$

Note that $\mathcal{F}_0 \subset \mathcal{F}_1$, since we do not forget the previous information.

At time t investors know which part of Ω contains the true state of the world. $\mathcal{F}$ is called a field or algebra of sets. $\mathcal{F}$ is a field if

1. $\emptyset, \Omega \in \mathcal{F}$;

2. If $A \in \mathcal{F}$, and $B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$, $A \backslash B \in \mathcal{F}$.

A partition of Ω is a collection of exhaustive and mutually exclusive subsets,

$$\{D_i, \dots, D_k\}, \text{ such that } D_i \cap D_j = \emptyset, \text{ and } \bigcup_i D_i = \Omega.$$

Filtration

A filtration $\mathbb{F}$ is the collection of fields,

$$\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_t, \dots, \mathcal{F}_T\} \quad \mathcal{F}_t \subset \mathcal{F}_{t+1}.$$

$\mathbb{F}$ is used to model a flow of information. As time passes, an observer knows more and more detailed information, that is, finer and finer partitions of Ω. In the example of the price of stock, $\mathbb{F}$ describes how the information about prices is revealed to investors.

Predictable Processes

Suppose that a filtration $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_t, \dots, \mathcal{F}_T\}$ is given. A process H_t is called predictable (with respect to this filtration) if for each t, H_t is $\mathcal{F}_{t-1}$-measurable, that is, the value of the process H at time t is determined by the information up to and including time $t-1$. For example, the number of shares held at time t is determined on the basis of information up to and including time $t-1$. Thus this process is predictable with respect to the filtration generated by the stock prices.

Probability

If Ω is a finite sample space, then we can assign to each outcome ω a probability, $P(\omega)$, that is, the likelihood of it occurring. This assignment can be arbitrary. The only requirement is that $P(\omega) \geq 0$ and $\sum P(\omega) = 1$.

2.1.2 Continuous Probability Model

In this section we define similar probabilistic concepts for a continuous sample space. We start with general definitions.

σ Fields

A σ-field is a field, which is closed with respect to countable unions and countable intersections of its members, that is a collection of subsets of Ω that satisfies

1. $\emptyset, \Omega \in \mathcal{F}$;
2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$;
3. $A_1, A_2, \ldots, A_n, \ldots \in \mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Any subset B of Ω that belongs to $\mathcal{F}$ is called a measurable set.

Borel σ-Fields

The Borel σ-field is the most important example of a σ-field that is used in the theory of functions, Lebesgue integration, and probability. Consider the σ- field $\mathcal{B}$ on R $(\Omega = R)$ generated by the intervals. It is obtained by taking all the intervals first and then all the sets obtained from the intervals by forming countable unions, countable intersections and their complements are included into collection, and countable unions and intersections of these sets are included, etc. It can be shown that we end up with the smallest σ-field which contains all the intervals. One can show that the intersection of σ-fields is again a σ-field. Take the intersection of all σ-fields containing the collection of intervals. It is the smallest σ-field containing the intervals, the Borel σ-field on R. In this model a measurable set is a set from $\mathcal{B}$, a Borel set.

Probability

A probability P on $(\Omega, \mathcal{F})$ is a set function on $\mathcal{F}$, such that

1. $P(\Omega) = 1$;

2. If $A \in \mathcal{F}$, then $P(A^c) = 1 - P(A)$;

3. Countable additivity (σ-additivity): If $A_1, A_2, \ldots, A_n, \ldots \in \mathcal{F}$ are mutually exclusive, then $P(\bigcup_{n=1}^{\infty}) = \sum_{n=1}^{\infty} P(A_n)$.

The σ-additivity property is equivalent to finite additivity plus the continuity property of probability, which states: If $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \ldots A = \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$, then

$$\lim_{n\to\infty} P(A_n) = P(A).$$

A similar property holds for an increasing sequence of events.

Predictable Processes

Recall that in discrete time a process H is predictable if H_n is $\mathcal{F}_{n-1}$ measurable. Predictability in continuous time is harder to define. We recall some general definitions of processes starting with the class of adapted processes.

Definition 2.1. *[19, Def. 8.2, p.212] A process X is called adapted filtration $\mathbb{F} = \{\mathcal{F}_t\}$, if for all t, $X(t)$ is $\mathcal{F}_t$-measurable.*

In construction of the stochastic integral $\int_0^t H(u)dS(u)$, processes H and S are taken to be adapted to $\mathbb{F}$. For a general semimartingale S, the requirement that H is adapted is too weak, it fails to assure measurability of some basic constructions. H must be *predictable*. For our purposes it is enough to describe a subclass of predictable processes which can be defined constructively.

Definition 2.2. *[19, Def. 8.3, p.213] H is predictable if it is one of the following:*

1. *a left-continuous adapted process, in particular, a continuous adapted process;*

2. *a limit (almost sure, in probability) of left-continuous adapted processes.*

3. *a regular right-continuous process such that, for any stopping time τ, H_τ is $\mathcal{F}_\tau$-measurable, the σ-field generated by the sets $A \bigcap \{T < t\}$, where $A \in \mathcal{F}_t$;*

4. *a Borel-measurable function of a predictable process.*

2.2 Brownian Motion

This chapter is mainly about Brownian motion. It is the main process in the calculus of continuous processes.

2.2.1 Introduction

Observations of prices of stocks, positions of a diffusing particle and many other processes observed in time are often modeled by a stochastic process. A stochastic process is an umbrella term for any collection of random variables $X(t)$ depending on time t. Time can be discrete, for example, $t = 0, 1, 2, \ldots,$ or continuous, $t \geq 0$. Calculus is suited more to continuous time processes. At any time t, the observation is described by a random variable which we denote by $X(t)$. A stochastic process $X(t)$ is frequently denoted by X or with a slight abuse of notation also by $X(t)$.

In practice, we typically observe only a single realization of this process, a single path, out of a multitude of possible paths. Any single path is a function of time t, $x_t = x(t)$, $0 \leq t \leq T$; and the process can also be seen as a random function. To describe the distribution and to be able to do probability calculations about the uncertain future, one needs to know the so-called finite-dimensional distributions. Namely, we need to specify how to calculate probabilities of the form $P(X(t) \leq x)$ for any time t, i.e. the probability distribution of the random variable $X(t)$; and probabilities of the form $P(X(t_1) \leq x_1, X(t_2) \leq x_2)$ for any times t_1, t_2, i.e. the joint bivariate distributions of $X(t_1)$ and $X(t_2)$; and probabilities of the form

$$P\Big(X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots X(t_n) \leq x_n\Big), \tag{2.1}$$

for any choice of time points $0 \leq t_1 < t_2 \ldots < tn \leq T$, and any $n \geq 1$ with $x_1, \ldots, x_n \in R$. Often one does not write the formula for (2.1), but merely points out how to compute it.

2.2.2 Properties of Brownian Motion

Botanist R. Brown described the motion of a pollen particle suspended in fluid in 1828. It was observed that a particle moved in an irregular, random fashion. A. Einstein, in 1905, argued that the movement is due to bombardment of the particle by the molecules of the fluid, he obtained the equations for Brownian motion. In 1900, L. Bachelier used the Brownian motion as a model for movement of stock prices in his mathematical theory of speculation. The mathematical foundation for Brownian motion as a stochastic process was done by N. Wiener in 1931, and this process is also called the Wiener process. The Brownian Motion process $B(t)$ serves as a basic model for the cumulative effect of pure noise. If $B(t)$ denotes the position of a particle at time t, then the displacement $B(t) - B(0)$ is the effect of the purely random bombardment by the molecules of the fluid, or the effect of noise over time t.

Definition 2.3. *Brownian motion $\{B(t)\}$ is a stochastic process with the following properties.*

1. *(Independence of increments) $B(t) - B(s)$, for $t > s$, is independent of the past, that is, of $B(u)$, $0 \leq u \leq s$, or of $\mathcal{F}_s$, the σ-field generated by $B(u)$, $u \leq s$.*

2. *(Stationary Normal increments) $B(t) - B(s)$ has Normal distribution with mean 0 and variance $t - s$. This implies (taking $s = 0$) that $B(t) - B(0)$ has $N(0, t)$ distribution.*

3. *(Continuity of paths) $B(\cdot, \omega)$ is continuous for each $\omega \in \Omega$ and $B(t)$, $t \geq 0$ are continuous functions of t.*

The initial position of Brownian motion is not specified in the definition. When $B(0) = x$, a.s. then the process is a Brownian motion started at x. The time interval on which Brownian motion is defined is $[0, T]$ for some $T > 0$, which is allowed to be infinite.

Remark 2.2.1. *A definition of Brownian motion in a more general model (that contains extra information) is given by a pair $\{B(t), \mathcal{F}_t\}$, $t \geq 0$, where $\mathcal{F}_t$ is an increasing sequence of σ-fields (a filtration), $B(t)$ is an adapted process, i.e. $B(t)$ is $\mathcal{F}_t$ measurable, such that Properties 1-3 above hold.*

An important representation used for calculations in processes with independent increments is that for any $s \geq 0$

$$B(t+s) = B(s) + (B(t+s) - B(s)),$$

where the two variables B_s and $(B(t+s) - B(s))$ are independent. An extension of this representation is the process version.

2.2.3 Brownian Motion Paths

An occurrence of Brownian motion observed from time 0 to time T, is a random function of t on the interval $[0, T]$. It is called a realization, a path or trajectory.

Definition 2.4. *[19, p.63] The quadratic variation of Brownian motion $[B, B](t)$ is defined as*

$$[B, B](t) = [B, B]([0, t]) = \lim_{\delta_n \to 0} \sum_{i=1}^{n} |B(t_i^n) - B(t_{i-1}^n)|^2$$

where the limit is taken over all shrinking partitions of $[0, t]$, with $\delta_n = \max_i(t_{i+1}^n - t_i^n) \to 0$ as $n \to \infty$.

It is remarkable that although the sums in the Definition 2.4 are random, their limit is non-random, as the following result shows.

Theorem 2.2.1. *[19, Theorem. 3.4, p.63] The Quadratic variation of a Brownian motion over $[0, t]$ is t.*

Properties of Brownian paths

$B(t)$ is as functions of t have the following properties. Almost every sample path $B(t)$, $0 \leq t \leq T$

1. is a continuous function of t;

2. is not monotone in any interval, no matter how small the interval is;

3. is not differentiable at any point;

4. has infinite variation on any interval, no matter how small it is;

Properties 1 and 3 of Brownian motion paths state that although any realization $B(t)$ is a continuous function of t, it has increments $\Delta B(t)$ over an interval of length Δt much larger than Δt as $\Delta t \to 0$. Since $E(B(t+\Delta t) - B(t))^2 = \Delta t$, it suggests that the increment is roughly like $\sqrt{t}$. This is made precise by the quadratic variation Property 5.

Theorem 2.2.2. *[19, Theorem. 3.5, p.64] For any $t \in [0, +\infty)$ almost all trajectories of Brownian motion are not differentiable at t.*

2.3 Stochastic Calculus

In this chapter stochastic integrals with respect to Brownian motion are introduced and their properties are given. They are also called Itô integrals, and the corresponding calculus Itô calculus. For more details and further background we refer to reader to Klebaner [19].

2.3.1 Definition of the Itô Integral

Our goal is to define the stochastic integral $\int_0^T X(t)dB(t)$, also denoted $X \cdot B$. This integral should have property that if $X(t) = 1$ then $\int_0^T dB(t) = B(T) - B(0)$. Similarly, if $X(t)$ is a constant c, then the integral should be $c(B(T) - B(0))$. In this way we can integrate constant processes with respect to B. The integral over $(0, T]$ should be the sum of integrals over two subintervals $(0, a]$ and $(a, T]$. Thus if $X(t)$ takes two values c_1 on $(0, a]$, and c_2 on $(a, T]$, then the integral of X with respect to B is easily defined. In this way the integral is defined for simple processes, that is, processes which are constant on finitely many intervals. By the limiting procedure the integral is then defined for more general processes.

Itô Integral of Simple Processes

We call $X(t)$ is a simple non-random process if there exist times $0 = t_0 < t_1 < \ldots < t_n =$

T and constants $c_0, c_1, \ldots, c_{n-1}$ such that

$$X(t) = c_0 I_0(t) + \sum_{i=1}^{n-1} c_i I_{(t_i, t_{i+1}]}(t).$$

Then the Itô integral $\int_0^T X(t)dB(t)$ is defined by

$$\int_0^T X(t)dB(t) := \sum_{i=0}^{n-1} c_i(B(t_{i+1}) - B(t_i)) \tag{2.2}$$

Remark 2.3.1. According to the independence property of Brownian increments, the integral defined in (2.2) is a Gaussian random variable with mean zero and variance

$X(t)dB(t)$

$$Var\left(\int_0^T X(t)dB(t)\right) = Var\left(\sum_{i=0}^{n-1} c_i(B(t_{i+1}) - B(t_i))\right) = \sum_{i=0}^{n-1} Var\Big(c_i(B(t_{i+1}) - B(t_i))\Big)$$
$$= \sum_{i=0}^{n-1} c_i^2(t_{i+1} - t_i) = \int_0^t E(X^2(s))ds.$$

To integrate random processes, it is important to allow for constants c_i in (2.2) to be random. If c_i's are replaced by random variables ξ_i's, then, in order to have convenient properties of the integral, the random variable ξ_i's are allowed to depend on the values of $B(t)$ for $t \leq t_i$, but not on future values of $B(t)$ for $t > t_i$. If $\mathcal{F}_t$ is the σ-field generated by Brownian motion up to time t, then ξ_i is $\mathcal{F}_{t_i}$-measurable. The approach of defining the integral by approximation can be carried out for the class of adapted processes $X(t)$, $0 \leq t \leq T$.

Definition 2.5. *[19, Def. 4.1, p.92] A process X is called adapted to the filtration $\mathbb{F} = \{\mathcal{F}_t\}$ if for all t, $X(t)$ is $\mathcal{F}_t$-measurable.*

Remark 2.3.2. In order that the integral has desirable properties, in particular that the expectation and the integral can be interchanged (by Fubini's Theorem), the requirement that X is adapted is too weak, and a stronger condition, a progressive (progressively measurable) process, is needed.

Definition 2.6. X is progressive if it is a measurable function in the pair of variables (t, ω), i.e., $\mathcal{B}([0,t]) \times \mathcal{F}_t$ measurable as a map from $[0,t] \times \Omega$ into R.

Remark 2.3.3. It can be seen that every adapted right-continuous with left limits or left-continuous with right limits (i.e., cádlág) process is progressive.

Definition 2.7. [19, Def. 4.2, p.93] A process $X = \{X(t), 0 \le t \le T\}$ is called a simple adapted process if there exist times $0 = t_0 < t_1 < \ldots < t_n = T$ and random variables $\xi_0, \xi_1, \ldots, \xi_{n-1}$, such that ξ_0 is a constant, ξ_i is $\mathcal{F}_{t_i}$-measurable (depends on the values of $B(t)$ for $t \le t_i$, but not on values of $B(t)$ for $t > t_i$), and $E(\xi_i^2) < \infty$, $i = 0, 1, \ldots, n-1$; such that

$$X(t) = \xi_0 I_0(t) + \sum_{i=0}^{n-1} \xi_i I_{(t_i, t_{i+1}]}(t).$$

For simple adapted processes Itô integral $\int_0^T X dB$ is defined as a sum

$$\int_0^T X(t)dB(t) = \sum_{i=0}^{n-1} \xi_i (B(t_{i+1}) - B(t_i)).$$

Remark 2.3.4. Note that when ξ_i's are random, the integral need not have a normal distribution.

Remark 2.3.5. Simp $\int_0^T I_{(a,b]}(t)dB(t) = B(b) - B(a), \quad \int_0^T I_{(a,b]}(t)X(t)dB(t) = \int_a^b X(t)dB(t),$
One can take right-con with respect to general martingales, other than the Brownian motion, only left-continuous functions are taken.

Properties of the Itô Integral of Simple Adapted Processes

In what follows we recall some basic properties of the Itô integral of simple processes.

(P1) **Linearity:**. If $X(t)$ and $Y(t)$ are simple processes and α and β are some constants then $\int_0^T (\alpha X(t) + \beta Y(t))dB(t) = \alpha \int_0^T X(t)dB(t) + \beta \int_0^T Y(t)dB(t)$;

$$\int_0 (\alpha X(t) + \beta Y(t))dB(t) = \alpha \int_0 X(t)dB(t) + \beta \int_0 Y(t)dB(t);$$

(P2)

$$\int_0^T I_{(a,b]}(t)dB(t) = B(b) - B(a), \quad \int_0^T I_{(a,b]}(t)X(t)dB(t) = \int_a^b X(t)dB(t),$$

where $I_{(a,b]}(t) = 1$ for $t \in (a, b]$, and zero otherwise.

(P3) **Zero Mean:** $E\int_0^T X(t)dB(t) = 0$;

(P4) **The Itô Isometry:**

$$E\left(\int_0^T X(t)dB(t)\right)^2 = \int_0^T E\left(X^2(t)\right) dt.$$

Itô Integral of Adapted Processes

Let $X(t)$ be an $\mathcal{F}_t$-adapted process and assume that $\{X^n(t)\}_{n\in\mathbb{N}}$ is a sequence of simple processes such that

$$E\int_0^T |X^n(t) - X(t)|^2 dt \to 0.$$

Any $\mathcal{F}_t$-adapted processes can be approximated by a sequence of simple processes in $L^2(P)$.

Then we define

$$\int_0^T X(t)dB(t) := \lim_{n\to\infty} \int_0^T X^n(t)dB(t) \quad \text{in } L^2(P).$$

It is clear that the limits of $\int_0^T X^n(t)dB(t)$ does not depend on the choice of the approximation sequence $\{X^n(t)\}_{n\in N}$.

Theorem 2.3.1. ([19, Theorem 4.3, p.96]) Let $X(t)$ be a regular adapted process such that $\int_0^T |X(t)|^2 dt < \infty$ with probability one. Then Itô integral $\int_0^T X(t)dB(t)$ is well-defined and enjoys the properties (P1)-(P4).

Let X be a regular adapted process, such that $\int_0^T X^2(s)ds < \infty$ with probability one, so that $\int_0^t X(s)dB(s)$ is defined for any $t \leq T$.

The Itô integral also possesses the following properties (see, e.g., [41, Theorem 5.9, p.22], [19, Theorem 4.7, p.101]): for and $t > s$,

- $\int_a^b X(s)dB(s)$ is $\mathcal{F}_b$-measurable;
- $E\left(\int_s^t X(u)dB(u)|\mathcal{F}_s\right) = 0$;
- $E\left(\left|\int_s^t X(u)dB(u)\right|^2 |\mathcal{F}_s\right) = \int_s^t E(|X(u)|^2|\mathcal{F}_s)du$;
- Martingale Property: $Y(t) := \int_0^t X(s)dB(s), 0 \leq t \leq T$, is a square integrable martingale if $E\int_0^T |X(s)|^2 ds < \infty$, i.e., $E\left(Y(t)|\mathcal{F}_s\right) = Y(s)$.

2.3.2 Itô Integral Processes

Let X be a regular adapted process, such that $E(\int_0^T X^2(s)ds) < \infty$ with probability one, so that $\int_0^t X(s)dB(s)$ is defined for any $t \leq T$. Since it is a random variable for any fixed t, $\int_0^t X(s)dB(s)$ as a function of the upper limit t defines a stochastic process

$$Y(t) = \int_0^t X(s)dB(s).$$

It is possible to show that there is a version of the Itô integral $Y(t)$ with continuous sample paths. It is always assumed that the continuous version of the Itô integral is taken. It will be seen later in this section that the Itô integral has a positive quadratic variation and infinite variation.

Martingale Property of the Itô Integral

It is intuitively clear from the construction of Itô integrals that they are adapted. To see this more formally, Itô integrals of simple processes are clearly adapted, and also continuous. Since $Y(t)$ is a limit of integrals of simple processes, it is itself adapted.

Suppose that in addition to the condition $\int_0^T X^2(s)ds < \infty$, condition $\int_0^T E(X^2(t))dt < \infty$ holds. (The latter implies the former by Fubinis theorem.) Then $Y(t) = \int_0^t X(s)dB(s), 0 \leq t \leq T$, is defined and possesses first two moments. It can be shown, first for simple processes and then in general, that for $s < t$,

$$E\left(\int_s^t X(u)dB(u)|\mathcal{F}_s\right) = 0.$$

Thus

$$\begin{aligned}
E\left(Y(t)|\mathcal{F}_s\right) &= E\left(\int_0^t X(u)dB(u)|\mathcal{F}_s\right) \\
&= \int_0^s X(u)dB(u) + E\left(\int_s^t X(u)dB(u)|\mathcal{F}_s\right) \\
&= \int_0^s X(u)dB(u) \\
&= Y(s).
\end{aligned}$$

Therefore $Y(t)$ is a martingale. The second moments of $Y(t)$ are given by the isometry property,

$$E\left(\int_0^t X(s)dB(s)\right)^2 = \int_0^t E\left(X^2(s)\right)ds.$$

This shows that $\sup_{t\le T} E\left(Y^2(t)\right) = \int_0^T EX^2(s)ds < \infty$.

Definition 2.8. *[19, Def. 4.6, p.101] A martingale is called square integrable on $[0,T]$ if its second moments are bounded.*

Thus we have

Theorem 2.3.2. *[19, theorem 4.7, p.101] Let $X(t)$ be an adapted process such that $\int_0^T EX^2(s)ds < \infty$. Then $Y(t) = \int_0^t X(s)dB(s), 0 \le t \le T$, is a continuous zero mean square integrable martingale.*

Theorem 2.3.2 above provides a way of constructing martingales.

Corollary 2.1. *[19, Corollary 4.8, p.101] For any bounded function f on R, $\int_0^t f(B(s))dB(s)$ is a square integrable martingale.*

Quadratic Variation and Covariation of Itô Integrals

The Itô integral $Y(t) = \int_0^t X(s)dB(s), 0 \le t \le T$, is a random function of t. It is continuous and adapted. The quadratic variation of Y is defined by

$$[Y,Y](t) = \lim \sum_{i=0}^{n-1} \left(Y(t_{i+1}^n) - Y(t_i^n)\right)^2,$$

where for each n, $\{t_i^n\}_{i=0}^n$, is a partition of $[0,T]$, and the limit is in probability, taken over all partitions with $\delta_n = \max(t_{i+1}^n - t_i^n) \to 0$ as $n \to \infty$.

Theorem 2.3.3. *[19, Theorem 4.9, p.101] The quadratic variation of the Itô integral $\int_0^t X(s)dB(s), 0 \le t \le T$ is given by*

$$\left[\int_0^t X(s)dB(s), \int_0^t X(s)dB(s)\right](t) = \int_0^t X^2(s)ds. \tag{2.3}$$

Let now $Y_1(t)$ and $Y_2(t)$ be Itô integrals of $X_1(t)$ and $X_2(t)$ with respect to the same Brownian motion $B(t)$. Then, clearly, the process $Y_1(t) + Y_2(t)$ is also an Itô integral of $X_1(t) + X_2(t)$ with respect to $B(t)$.

Quadratic covariation of Y_1 and Y_2 on $[0, t]$ is defined by

$$[Y_1, Y_2](t) = \frac{1}{2}\Big([Y_1 + Y_2, Y_1 + Y_2](t) - [Y_1, Y_1](t) - [Y_2, Y_2](t)\Big).$$

By (2.3) it follows that

$$[Y_1, Y_2](t) = \int_0^t X_1(s)X_2(s)ds.$$

It is clear that $[Y_1, Y_2](t) = [Y_2, Y_1](t)$, and it can be seen that quadratic covariation is given by the limit in probability of products of increments of the processes Y_1 and Y_2 when partitions $\{t_i^n\}$ of $[0, t]$ shrink,

$$[Y_1, Y_2](t) = \lim \sum_{i=0}^{n-1} \left(Y_1(t_{i+1}^n) - Y_1(t_i^n)\right)\left(Y_2(t_{i+1}^n) - Y_2(t_i^n)\right).$$

2.3.3 Itô Formula

Let X_t be a d-dimensional Itô process on $t \geq 0$ with the stochastic differential

$$dX_t = R_t dt + Z_t dB_t, \quad X_0 = x, \tag{2.4}$$

where $\{B_t\}_{t\geq 0}$ is an n-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ satisfying the usual conditions, and $R : [0, \infty) \to R^d$, $Z : [0, \infty) \to R^d \times R^n$ are progressively measurable. Let $f \in C^{1,2}(R_+ \times R^d; R)$, the family of all real-valued functions $f(t, x)$ defined on $R_+ \times R^d$ such that they are continuously once differentiable in t and twice in x. Then the following Itô formula (see, e.g., [29, Theorem 5.1, p.66])

$$\begin{aligned} f(t, X_t) =& f(0, X_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s)ds + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}(s, X_s)R_s^i ds \\ &+ \sum_{i=1}^{d}\sum_{j=1}^{n} \int_0^t \frac{\partial f}{\partial x_i}(s, X_s)Z_s^{ij} dB_s^j \\ &+ \frac{1}{2}\sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 f}{\partial x_i x_j}(s, X_s)(ZZ')_s^{ij} ds. \end{aligned} \tag{2.5}$$

The Itô formula (2.5) can also be rewritten in a compact form

$$
\begin{aligned}
f(t, X_t) =& f(0, X_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s)ds + \int_0^t \langle \nabla f(s, X_s), dX_s \rangle \\
&+ \frac{1}{2}\int_0^t \text{trace}(Z_s^T \nabla^2 f(s, X_s) Z_s)ds,
\end{aligned}
$$

where ∇ and ∇^2 stand for the gradient and Hessian operators with respect to the second variable respectively, i.e.,

$$
\nabla f(t, x) = \Big(\frac{\partial f(t,x)}{\partial x_1}, \cdots, \frac{\partial f(t,x)}{\partial x_d}\Big)
$$

and

$$
\nabla^2 f(t, x) = \begin{pmatrix} \frac{\partial^2 f(t,x)}{\partial x_1^2} & \cdots & \frac{\partial^2 f(t,x)}{\partial x_1 \partial x_d} \\ \cdots & \cdots & \cdots \\ \frac{\partial^2 f(t,x)}{\partial x_d \partial x_1} & \cdots & \frac{\partial^2 f(t,x)}{\partial x_d^2} \end{pmatrix}
$$

2.4 Martingale Representation Theorem

The following result is used several times in the Chapters 3 on continuous market models as the Martingale Representation Theorem can be used to establish the existence of a hedging strategy. Suppose that $B = \{B_t, t \in [0,T]\}$ defined on the complete probability space $(\Omega, \mathcal{F}, P)$ is standard n-dimensional Brownian motion (for some $n \geq 1$) and let $\{\mathcal{F}_t, t \in [0,T]\}$ be its standard filtration. Without loss of generality we assume that $\mathcal{F} = \mathcal{F}_T$. It is important for this result that the filtration is the standard one generated by the Brownian motion. For a proof of this theorem, see for example Revuz and Yor [47], Theorem V.3.4.

Theorem 2.4.1. *(Martingale Representation Theorem) Suppose that $\{B_t, \mathcal{F}_t, t \in [0,T]\}$ is a right continuous local martingale. There is an adapted, n-dimensional process $\eta = \{\eta_t, t \in [0,T]\}$ satisfying*

1. *$\eta : [0,T] \times \Omega \to R^n$ is ($\mathcal{B}_T \times \mathcal{F}_T$-measurable where $\eta(t, \omega) = \eta_t(\omega)$ for $t \in [0,T]$ and $\omega \in \Omega$;*

2. *$\int_0^T |\eta_s|^2 ds < \infty \quad$ P-a.s.,*

such that P-a.s.,

$$M_t = M_0 + \int_0^t \eta_s dB_s \qquad \textit{for } t \in [0, T].$$

2.5 Introduction to stochastic differential equations

Consider a Stochastic Differential Equation (SDE) in the framework

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad t \geq 0 \tag{2.6}$$

with the initial value $X_0 = x \in R^n$. Here $b : [0, \infty) \times R^n \to R^n$, $\sigma : [0, \infty) \times R^n \to R^{n \times m}$ be measurable, and $\{W_t\}_{t \geq 0}$ is an m-dimensional Brownian motion defined on the stochastic basis $(\Omega, \mathcal{F}, P)$ equipped with the reference family $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions, i.e., $\mathcal{F}_{t_+} := \cap_{s>t} \mathcal{F}_s = \mathcal{F}_t$, $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$, and $\mathcal{F}_0$ contains all P-null sets.

Next we recall two kinds of notions of solutions to (2.6).

Definition 2.9. (Strong Solution) A process $X(t)$ is called a strong solution of (2.6) if for all $t > 0$ the integrals $\int_0^t b(s, X(s))ds$ and $\int_0^t \sigma(s, X(s))dW(s)$ exist a.s., and

$$X(t) = X(0) + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s), \qquad a.s.$$

Definition 2.10. We shall say that the pathwise uniqueness holds for (2.6), if, for any two solutions $\{X_t(x)\}_{t \geq 0}, \{Y_t(y)\}_{t \geq 0}$ that $X_t(x)$ is the solution with $X_0(x) = x$ and $Y_t(y)$ is the solution with $Y_0(y) = y$ defined on the same quadruplet $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, $x = y$ implies $X_t(x) = Y_t(y)$ a.s.

Definition 2.11. (Weak Solution) If there exist a probability space with a filtration, a Brownian motion $\hat{W}(t)$ and a process $\hat{X}(t)$ adapted to that filtration, such that: $\hat{X}(0)$ has the given distribution, for all t the integral below are defined, and $\hat{X}(t)$ satisfies

$$\hat{X}(t) = \hat{X}(0) + \int_0^t \mu(u, \hat{X}(u))ds + \int_0^t \sigma(u, \hat{X}(u))d\hat{W}(u),$$

then $\hat{X}(t)$ is called a weak solution to (2.6).

Definition 2.12. A weak solution is called unique if whenever $X(t)$ and $X'(t)$ are two solutions (perhaps on different probability spaces) such that the distributions of $X(0)$ and $X'(0)$ are the same, then all finite-dimensional distributions of $X(t)$ and $X'(t)$ are the same.

Remark 2.5.1. The concept of weak solution allows us to give a meaning to an SDEs when strong solutions do not exist. Weak solution are solutions in distribution, they can be realized (defined) on some other probability space and exist under less stringent conditions on the coefficients of the SDE.

For the strong solution, we need to give a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t>0}, P)$ and the Brownian motion W_t in advance. For the weak solution, we need to construct a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t>0}, P)$ and the Brownian motion, and then ask for X_t such that the equation considered.

To guarantee the existence and uniqueness of strong solutions of (2.6), we impose the following conditions.

Theorem 2.5.1. (Existence and Uniqueness of Strong Solutions: Global Case) Let $T > 0$ be fixed and assume that there exist $L_1, L_2 > 0$ such that, for any $x, y \in R^n$ and $t \in [0, T]$,

$$|b(t,x)| + \|\sigma(t,x)\| \leq L_1(1+|x|) \qquad \text{(Linear Growth Condition)} \tag{2.7}$$

and

$$|b(t,x) - b(t,y)| + \|\sigma(t,x) - \sigma(t,y)\| \leq L_2|x-y|. \qquad \text{(Lipschitz Condition)} \tag{2.8}$$

Then (2.6) has a unique strong solution $\{X_t(x)\}_{t\geq 0}$ with the starting point $x \in \mathbb{R}^n$ such that $\sup_{0\leq t\leq T} E|X_t(x)|^p < \infty$ for any $p > 0$.

Theorem 2.5.2. (Existence and Uniqueness of Strong Solutions: Local Case) Replace the global Lipschitz condition (2.8) by the following **local Lipschitz condition**: for $N \geq 1$, there exists $K_N > 0$ such that

$$|b(t,x) - b(t,y)| + \|\sigma(t,x) - \sigma(t,y)\| \leq K_N|x-y|, \quad |x| \vee |y| \leq N.$$

Then (2.6) has a unique strong solution $\{X_t(x)\}_{t\geq 0}$ with the starting point $x \in R^n$ such that $\sup_{0\leq t\leq T} E|X_t(x)|^p < \infty$ for any $p > 0$.

Remark 2.5.2. Let $b(x) := x \sin x, x \in R$. It is easy to see that b is of linear growth, but satisfies a local Lipschitz condition, not a global one.

Remark 2.5.3. Clearly, by definition, a strong solution is also a weak solution. Uniqueness of the strong solution (pathwise uniqueness) implies uniqueness of the weak solution, (a result of Yamada and Watanabe (1971)) [65].

2.6 Girsanov Theorem

Girsanov Theorem is another powerful probabilistic tool to solves SDEs by changing the underlying probability measure, so that the process which was the driving Brownian motion becomes, under the new probability measure, the solution to the differential equation.

Lemma 2.6.1. (Girsanov Transformation) Let $\varphi(\cdot)$ be an $\mathcal{F}_t$-predictable process such that

$$E \exp\Big(\int_0^T \varphi(s)dW_s - \frac{1}{2}\int_0^T |\varphi(s)|^2 ds\Big) = 1. \tag{2.9}$$

Then the process

$$\hat{W}(t) := W(t) - \int_0^t \varphi(s)ds, \quad t \in [0,T],$$

is Brownian motion with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ on the probability space $(\Omega, \mathcal{F}, \hat{P})$, where

$$d\hat{P}(\omega) := \exp\Big(\int_0^T \varphi(s)dW_s - \frac{1}{2}\int_0^T |\varphi(s)|^2 ds\Big)dP(\omega).$$

Remark 2.6.1. The following Novikov condition:

$$E \exp\Big(\frac{1}{2}\int_0^T |\varphi(s)|^2 ds\Big) < \infty$$

is one of the sufficient condition such that (2.9) holds.

2.7 Some Fundamental Inequalities

For later use, we recall some fundamental inequalities.

Lemma 2.7.1. (The Gronwall Inequality [19, Theorem 1.20, p.18]) Let $f(t)$, $g(t)$ and $h(t)$ be continuous non-negative functions on interval $[a,b]$, and

$$f(t) \leq g(t) + \int_a^t h(s)f(s)ds, \quad \text{for } t \in [a,b].$$

Then

$$f(t) \leq g(t) + \int_a^t g(s)h(s)\exp\Big(\int_a^t h(\alpha)d\alpha\Big)ds \quad \alpha \in [a,b].$$

In particular, if g is non-decreasing, then

$$f(t) \leq g(t)\exp\Big(\int_a^t h(\alpha)d\alpha\Big) \quad \alpha \in [a,b].$$

Lemma 2.7.2. (The Chebyshev Inequality [41, Theorem 1.20, p.18]) For each constant $c > 0$ and any random variable Y such that $E|Y|^p < \infty$ for some $p > 0$,

$$P(Y \geq c) \leq \frac{E|Y|^p}{c^p}.$$

Lemma 2.7.3. (The Hölder Inequality [35, Theorem 7.3, p.40]) If S is a measurable subset of $\mathbb{R}^n$ with the Lebesgue measure, and f and g are measurable real-or complex-valued function on S, then

$$\int_S |f(x)g(x)|dx \leq \Big(\int_S |f(x)|^p dx\Big)^{\frac{1}{p}}\Big(\int_S |g(x)|^q dx\Big)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q > 1$.

Lemma 2.7.4. (The Burkholder-Davis-Gundy inequality) [19, Theorem 7.34, p.201]) Let $g \in L^2(R_+; R^{d\times m})$. Define, for $t \geq 0$,

$$x(t) = \int_0^t g(s)dB(s) \quad \text{and} \quad A(t) = \int_0^t |g(s)|^2 ds.$$

Then for every $p > 0$, there exist universal positive constants c_p, C_p (depending only on p), such that

$$c_p E|A(t)|^{\frac{p}{2}} \leq E\Big(\sup_{0\leq s\leq t}|x(s)|^p\Big) \leq C_p E|A(t)|^{\frac{p}{2}}$$

for all $t \geq 0$. In particular, one may take

$$c_p = \Big(\frac{p}{2}\Big)^p, \quad C_p = \Big(\frac{32}{p}\Big)^{\frac{p}{2}} \quad \text{if } 0 < p < 2;$$

$$c_p = 1, \quad C_p = 4 \quad \text{if } p = 2;$$

$$c_p = \Big(2p\Big)^{\frac{p}{2}}, \quad C_p = [p^{p+1}/2(p-1)^{p-1}]^{\frac{p}{2}} \quad \text{if } p > 2.$$

Chapter 3

A Comparison of No Free Lunch With Vanishing Risk Condition And No Good Deal Condition

3.1 Introduction

Due to the seminal work [13] by Delbaen and Schachermayer, the fundamental theorem of asset pricing became pivotal in mathematical finance, which is a key result in establishing a mathematical framework for pricing and the key condition in the so-called No Free Lunch with Vanishing Risk condition [14]. Since then, many investigations are devoted to generalize this remarkable condition to cover more general situations in the mathematical modelings, cf. eg.[1],[5],[15],[48] and references therein. Most recently, Bion-Nadal and Di Nunno [1] proposed a new condition for pricing in incomplete markets. This condition is named as No Good Deal Condition, which should be thought as an analogy or modified version of the celebrated No Free Lunch with Vanish Risk Condition.

This chapter begins with the basic ideas of First and Second Fundamental Theorems of asset pricing in the discrete model. Then a general continuous market model is defined and the fundamental theorems of asset pricing are proved in this setting. The objective is to compare these two conditions in some simplified models. We aim to seek certain links between the No Free Lunch with Vanishing Risk condition and the No Good Deal condition by explicating them into several simple models so that one can compare them more concretely. Our discussions reveal the essential properties of these models.

The rest of the chapter is organized as follows. In the next section, we start with the discrete time market model. Then introduce the basic concepts of the First and Second Fundamental Theorems of asset pricing in the continuous model. Then a general continuous market model is defined and the fundamental theorems of asset pricing are proved in this setting. In the latter scenario we focus on conditions of the model which satisfy no free lunch with vanishing risk. Tools from probability such as martingale, equivalent martingale measure, stochastic integrals, Girsanov transformation are all used in this framework. In Section 3, we present a complete comparison with a thorough derivation. The paper ends with a conclusion to highlight our consideration.

3.2 Discrete Time Market Model

Let us first consider the discrete-time market. We consider a market model in which $d+1$ assets are priced at time $t = 0, 1, \dots, T$. Let the random vector $\bar{S}_t = (S_t^0, S_t) = (S_t^0, S_t^1, \dots, S_t^d)_{t=1,\dots,T}$ be an adapted process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, $t = 0, 1, \dots, T$. Note that if $\bar{S}$ is not a semi-martingale, then the space of S-integrable process cannot include all the local bounded process. The price of the i^{th} asset at time t is modeled as non-negative random variable S_t^i. $\bar{S}_t$ is assumed to be measurable with respect to a σ-algebra $\mathcal{F}_t \subset \mathcal{F}$. Here S_t^0 is a riskless bond which will pay a sure amount at time T. S_t is a risky stock price process.

A trading strategy is a predictable R^{d+1}-valued process $\bar{\xi}_t = (\xi_t^0, \xi_t) = (\xi_t^0, \xi_t^1, \dots, \xi_t^d)_{t=1,\dots,T}$. The value ξ_t^i of a trading strategy $\bar{\xi}_t$ corresponds to the quantity of shares of the i^{th} asset held during the t^{th} trading period between $t-1$ and t. Thus, $\xi_t^i S_{t-1}^i$ is the amount invested into i^{th} asset at time $t-1$, while $\xi_t^i S_t^i$ is the resulting value at time t [23, p210].

The total value of the portfolio $\bar{\xi}_t$ at time $t-1$ is

$$V_{t-1}(\bar{\xi}_t) = \bar{\xi}_t \cdot \bar{S}_{t-1} = \sum_{i=0}^{d} \xi_t^i S_{t-1}^i.$$

By time t, the value of the portfolio $\bar{\xi}_t$ has changed to

$$V_t(\bar{\xi}_t) = \bar{\xi}_t \cdot \bar{S}_t = \sum_{i=0}^{d} \xi_t^i S_t^i.$$

Definition 3.1. *A trading strategy $\bar{\xi}_t$ is called self-financing if*

$$\bar{\xi}_t \cdot \bar{S}_t = \bar{\xi}_{t+1} \cdot \bar{S}_t. \tag{3.1}$$

(3.1) *means that the portfolio is always rearranged in such a way that its value is preserved. It follows that*

$$\bar{\xi}_{t+1} \cdot \bar{S}_{t+1} - \bar{\xi}_t \cdot \bar{S}_t = \bar{\xi}_{t+1} \cdot (\bar{S}_{t+1} - \bar{S}_t) \tag{3.2}$$

In fact, a trading strategy is self-financing if and only if (3.2) holds for $t = 1, \ldots, T-1$. It follows that

$$\bar{\xi}_t \cdot \bar{S}_t = \bar{\xi}_1 \cdot \bar{S}_0 + \sum_{i=1}^{t} \bar{\xi}_i \cdot (\bar{S}_i - \bar{S}_{i-1}) \qquad t = 1, \ldots .T.$$

Here, the constant $\bar{\xi}_1 \cdot \bar{S}_0$ can be interpreted as the initial investment for the purchase of the portfolio $\bar{\xi}_1$.

It will simplify computation to use discounted asset price processes. For $i = 0, 1, \ldots, d$, we define

$$S_t^{*,i} = \frac{S_t^i}{S_t^0}, \qquad t = 1, \ldots .T.$$

Then $S_t^* = (S^{*,0}, S_t^{*,1}, \ldots, S_t^{*,d})$ is the value of the vector of discounted assets prices at time t.

Definition 3.2. *The discounted value process for a trading strategy $\bar{\xi}_t$ is defined by*

$$V_t^*(\bar{\xi}_t) = \frac{V_t(\bar{\xi})}{S_t^0}$$

It is also given by $V_t^* = \bar{\xi}_t \cdot \bar{S}_t^*$ for $t = 1, \ldots .T$.

Definition 3.3. *A self-financing trading strategy $\bar{\xi}_t$ is called an arbitrage opportunity if its value process V^* satisfies*

1. $V_0^* = 0,$
2. $V_T^* \geq 0 \qquad P\text{--}a.s.,$
3. $P[V_T^* > 0] > 0.$

A model satisfies the no-arbitrage condition if such a strategy does not exist. The market model is said to be viable if it has no arbitrage opportunities.

Definition 3.4. *The discounted gain process for ξ is defined by*

$$G_t^*(\xi) = \sum_{s=1}^{t} \xi_s \cdot (\Delta S_s^*), \qquad t = 1, \dots . T. \tag{3.3}$$

Where $\Delta S_s^* = S_s^* - S_{s-1}^*$. Clearly, We set $G_0^* = 0$, It involves only the risky assets, since $\Delta S_s^{*,0} = 0$ for $s = 1 . \dots , T$.

3.2.1 First Fundamental Theorem of Asset Pricing in the discrete time

An equivalent martingale measure is a probability measure Q on $(\Omega, \mathcal{F})$, such that Q is equivalent to P and S is a martingale equivalent under Q, ie, for each $t \in \{1, 2, \dots, T\}$,

$$E_Q\left[S_{n+1}|\mathcal{F}_n\right] = S_n.$$

Theorem 3.2.1. *([51, Theorem 1.2, p4])* **(First Fundament Theorem of Asset Pricing in the discrete time model)** *A model is arbitrage-free if and only if exists an equivalent martingale measure. i.e., $M(P) \neq \emptyset$, Let $M(P)$ denote the totality of equivalent martingale measure on $(\Omega, \mathcal{F}, P)$, where Q is equivalent to P and S is a $(\mathcal{F}_n, Q)$-martingale.*

Proof. The argument is motivated by[61, p35]. We first prove the "if" part. Suppose there is an equivalent martingale measure Q. For a proof by contradiction, suppose ξ is an arbitrage opportunity, that is ξ is a trading strategy with initial value $V_0(\xi) = 0$, and the final value $V_T(\xi) \geq 0$, It follows that the discounted values satisfy $V_0^*(\xi) = 0$, $V_T^*(\xi) > 0$, and since Q is equivalent to P, $E_Q[V_T^*] > 0$. As $\{V_t^*(\xi), \mathcal{F}_t\}_{t=0,\ldots,T}$ is a martingale under Q, and so

$$E_Q[V_T^*(\xi)] = E_Q[V_0^*(\xi)].$$

However, the left side object above is strictly positive whereas the right member is zero, which yields the contradiction. Thus there cannot be an arbitrage opportunity in the finite market model.

We now turn to prove "only if" part. Suppose that the finite market is viable. Since Ω is a finite set, for any random variable Y defined on $(\Omega, \mathcal{F})$, by enumerating Ω as $\{\omega_1, \ldots, \omega_n\}$, we may view Y as $(Y(\omega_1), \ldots, Y(\omega_n)) \in R^n$. Since $\mathcal{F}$ consists of all subsets of all subsets of Ω, any point in R can be thought of a real-valued random variable defined on Ω. Thus, there is a one-to-one correspondence between points in R^n and random variables defined on Ω. Adopting this point of view for the terminal discounted gain random variable $G_T^*(\xi)$, we define

$$L = \{G_T^*(\xi) : \xi \text{ is a trading strategy such that } V_0(\xi) = 0\}.$$

Note that L is a linear space, since $G_T^*(\xi)$ is linear in ξ and any linear combination of trading strategies with initial values of zero is again a trading strategy with the same initial value. Also, L is non-empty because the origin is contained in L. Let

$$D = \{Y \in R^n : Y_i \geq 0 \text{for} i = 1, \ldots, n \text{ and } Y_j > 0 \text{for some } j\}.$$

Thus, D is the positive orthant in R^n with the origin removed.

$$C = \left\{ Y \in D : \sum_{i=1}^{n} Y_i = 1 \right\}.$$

Then C is a convex, compact, non-empty subset of R^n and $L \cap C = \emptyset$. By applying the Separating Hyperplane Theorem. We see that there is a vector $Z \in \mathbb{R}^n$ $\{0\}$ such that the hyperplane. $H = \{Y \in \mathbb{R}^n : Y \cdot Z = 0\}$ contains L and $Z \cdot Y > 0$ for all $Y \in F$. By setting

$Y_i = 1$ if $i = j$ and $Y_i = 0$ if $i \neq j$. We see that $Z_j > 0$ for each $j \in \{1, \ldots, n\}$. Define

$$Q(\{\omega_i\}) = \frac{Z_i}{\sum_{j=1}^{n} Z_j}, \qquad i = 1, \ldots .n.$$

Then Q is a probability measure on $(\Omega, \mathcal{F})$ and it is equivalent to P, Moreover, for any trading strategy ξ such that $V_0(\xi) = 0$, we have

$$\begin{aligned} E_Q[G_T(\xi)] &= \sum_{i=1}^{n} G_T(\xi)(\omega_i) \frac{Z_i}{\sum_{j=1}^{n} Z_j} \\ &= \frac{G_T(\xi) \cdot Z}{\sum_{j=1}^{n} Z_j} \\ &= 0. \end{aligned} \tag{3.4}$$

where the last line follows from the fact that Z is perpendicular to H, which contains L.

Note that $G_T^*(\xi)$ involves only $(\xi_t^1, \ldots, \xi_t^d)$. Given $\xi^1, \ldots, \xi^d$, where for $i = 1, \ldots, d$, $\xi^i = \{\xi_t^i\}_{t=1\ldots,T}$ and ξ_t^i is a real valued, $\mathcal{F}_{t-1}$ measurable random variable for each t, there is a unique time-ordered set of T real-valued random variable $\xi^0 = \{\xi_t^0\}_{t=1,\ldots,T}$ such that $\bar{\xi} = \{\xi_t^0, \xi_t^1, \ldots, \xi_t^d\}_{t=1,\ldots,T}$ is a trading strategy with an initial value of zero. Upon substituting this in (3.4), we see that

$$\begin{aligned} 0 = E_Q[G_T^*(\xi)] &= E_Q\left[\sum_{t=1}^{T} \xi_t \cdot \Delta S_t^*\right] \\ &= E_Q\left[\sum_{t=1}^{T}\sum_{i=1}^{d} \xi_t^i \Delta S_t^{*,i}\right]. \end{aligned}$$

For each fixed $i \in \{1, \ldots, d\}$. If we set $\xi_t^j = 0$ for all t and $j \neq i$, we obtain

$$0 = E_Q\left[\sum_{t=1}^{T} \xi_t^i \Delta S_t^{*,i}\right]$$

for each $\xi^i = \{\xi_t^i\}_{t=1\ldots,T}$ such that ξ_t^i is a real-valued $\mathcal{F}_{t-1}$-measurable random variable for each t. It then follows from lemma 3.2.1, proved below, that for $i \in \{1, \ldots, d\}$, $S^{*,i}$ is a martingale under Q, Hence Q is an equivalent martingale measure. □

3.2.2 Second Fundamental Theorem of Asset Pricing in the discrete time

We will frequently refer to the random variable X, which represents a European contingent claim. For a European contingent claim X, we let $X^* = X/S_T^0$, the discounted value of X. A replicating (or hedging) strategy for a European contingent claim X is a trading strategy $\bar{\xi}$ such that $V_T(\bar{\xi}) = X$. If there exists such a replicating strategy,the European contingent claim is attainable. The finite market model is said to be complete if all European contingent claims are attainable.

Theorem 3.2.2. ([61, Theorem 3.3.1, p40]) Suppose that the finite market model is variable and X is a replicable European contingent claim, Then the value process $\{V_t(\bar{\xi}), t = 1, \dots, T\}$ is the same for trading strategies $\bar{\xi}$ for X. Indeed, for any trading strategy $\bar{\xi}$ and any equivalent martingale measure Q, we have

$$V_t^*(\bar{\xi}) = E_Q[X^*|\mathcal{F}_t], t = 1, \dots, T.$$

Before proceeding to the next result, we need to use the following Lemma 3.2.1.

Lemma 3.2.1. *([61, Lemma 3.2.6, p39]) Let $M = \{M_t, t = 0, 1, \dots, T\}$ be a real-valued process such that $M_t \in \mathcal{F}_t$. Then, M is a martingale if and only if*

$$E\left[\sum_{t=1}^{T} \eta_t \Delta M_t\right] = 0$$

for all $\eta = \{\eta_t, t = 1, \dots, T\}$ such that η_t is a real-valued $\mathcal{F}_{t-1}$-measurable random variable for $t = 1, \dots, T$. Here $\Delta M_t = M_t - M_{t-1}$.

Theorem 3.2.3. (Second Fundamental Theorem of Asset Pricing in the discrete time) *([61, Theorem 3.3.2, p41])A variable finite market model is complete if and only in the admits a unique equivalent martingale measure.*

Proof. The argument follows the lines of [61, p41]. If the finite market model X is viable and replicable, there is an equivalent martingale measure Q due to the first fundamental theorem of asset pricing. Let $\tilde{Q}$ be another equivalent martingale measure. For any $A \in \mathcal{F}_T$ and $X = \chi A$, according to theorem 3.2.2, one has

$$E_Q[X^*] = E_{\tilde{Q}}[X^*].$$

Multiplying both sides by the deterministic quantity S_T^0 yields

$$E_Q[X] = E_{\tilde{Q}}[X].$$

This of course implies from $X = \chi A$ that

$$Q(A) = \tilde{Q}(A).$$

Then we conclude that $Q = \tilde{Q}$ by the arbitrariness of $A \in \mathcal{F}_T = \mathcal{F}$.

If the market is not complete, then there exists a European contingent claim X unattainable. Set

$$\mathcal{P} := \{\xi = (\xi^1, \ldots, \xi^d) : \xi^i = \{\xi_t^i\}_{t=1,\ldots,T} \text{ and } \xi_t^i \in \mathcal{F}_{t-1}\}.$$

Then there is no pair (c, ξ) with $\xi \in \mathcal{P}$ and $c \in \mathbb{R}$ such that

$$c + \sum_{t=1}^{T} \xi_t \cdot \Delta S_t^* = X^*,$$

in which $S^* = (S^{*,1}, \ldots, S^{*,d})$.

Let

$$L = \left\{ c + \sum_{t=1}^{T} \xi_t \cdot \Delta S_t^* : \xi \in \mathcal{P}, c \in \mathbb{R} \right\}.$$

Observe that L is a strict subspace of $\mathbb{R}^n$. So there exists $Z \in L^\perp$, the orthogonal complement of L in $\mathbb{R}^n$. Then it follows that

$$\sum_{\omega \in \Omega} Z(\omega) Y(\omega) = 0 \qquad \text{for all } Y \in L \tag{3.5}$$

Since the finite market model is viable, there is at least one equivalent martingale measure Q, then $Q(\{\omega\}) > 0$ for all $\omega \in \Omega$. Set

$$\tilde{Z}(\omega) := \frac{Z(\omega)}{Q(\omega)} \qquad \text{and} \qquad Q^*(\{\omega\}) := \left(1 + \frac{\tilde{Z}(\omega)}{2\|\tilde{Z}\|_\infty}\right) Q(\{\omega\}) \qquad \text{for all } \omega \in \Omega,$$

in which $\|\tilde{Z}\|_\infty := \max_{\omega\in\Omega} |\tilde{Z}(\omega)|$. Thus (3.5) can be rewritten as

$$E_Q[\tilde{Z}Y] = 0 \qquad \text{for all } Y \in L. \tag{3.6}$$

Since $\tilde{Z} \neq 0$, $Q^* = Q$. Moreover, $Q^*(\{\omega\}) > 0$ for each $\omega \in \Omega$. By the definition of Q^*, note from (3.6) with $Y = 1 \in L$ that

$$\begin{aligned} Q^*(\Omega) &= Q(\Omega) + \frac{1}{2\|\tilde{Z}\|_\infty} \sum_{\omega\in\Omega} \tilde{Z}(\omega) Q(\{\omega\}) \\ &= 1 + \frac{1}{2\|\tilde{Z}\|_\infty} E_Q\left[\tilde{Z}\right] \\ &= 1. \end{aligned}$$

Thus, Q^* is a probability measure that is equivalent to Q.

We finally need to check that S^* is a martingale under Q^*. For any $\xi \in \mathcal{P}$, by the definition of Q^*, it is easy to see that

$$E_{Q^*}\left[\sum_{t=1}^{T} \xi_t \cdot \Delta S_t^*\right] = E_Q\left[\sum_{t=1}^{T} \xi_t \cdot \Delta S_t^*\right] + \frac{1}{2\|\tilde{Z}\|_\infty} E_Q\left[\tilde{Z} \sum_{t=1}^{T} \xi_t \cdot \Delta S_t^*\right]. \tag{3.7}$$

The first term on the first right side of the equality above is zero, by lemma 3.2.1, since S^* is a martingale under Q. The second term is zero by (3.6), since $Y = \sum_{t=1}^{T} \xi_t \cdot \Delta S_t^*$. On setting $\xi_t^j = 0$ for all $j \neq i$, and $t = 1, 2, \ldots, T$, and applying lemma3.2.1 again, it follows that $S^{*,i}$ is a Q^*-martingale and for $i = 1, \ldots, d$, and since this is trivially so for $i = 0$, it follows that S^* is a Q^* martingale and hence Q^* is an equivalent martingale measure that is different from Q. □

3.3 Continuous Market Model

We consider a market model in which $d+1$ assets are priced at time $t \in [0, T]$. Our model has two assets, a risky stock and a riskless bond. We use $S_t^i = (S_t^1, \ldots, S_t^d)_{t\in[0,T]}$ to denote d risky stock price processes. We use S_t^0 as the riskless bound with growing $S_t^0 = 1 + r$, r is a given interest rate.

Let $\bar{S}_t = (S_t^0, S_t^1, \ldots, S_t^d)_{t\in[0,T]}$ denote the corresponding price processes for this multi asset, which can be viewed as a vector valued stochastic process. In general, we take $\bar{S}_t$

to be a semi-martingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$. Here as usual, the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is assumed to be right-continuous. The price of the i^{th} asset at time t is modeled as non-negative random variable S_t^i. We assume that $(S_t^1, \ldots, S_t^d)_{t\in[0,T]}$ is $\{\mathcal{F}_t\}$-adapted.

Recall a trading strategy is an $\{\mathcal{F}_t\}$-predictable R^{d+1}-valued process $\bar{\xi}_t = (\xi_t^0, \xi_t) = (\xi_t^0, \xi_t^1, \ldots, \xi_t^d)_{t\in[0,T]}$. The value ξ_t^i of a trading strategy $\bar{\xi}_t$ corresponds to the quantity of assets of the i^{th} asset held at time t. We simplify computation to use discounted asset price processes. For $i = 0, 1, \ldots, d$, we define

$$S_t^{*,i} = \frac{S_t^i}{S_t^0}, \qquad t \in [0,T].$$

Then $\bar{S}_t^* = (S_t^{*,0}, S_t^{*,1}, \ldots, S_t^{*,d})$ is the value of the vector of discounted assets prices at time t. Next we will introduce some definitions.

Definition 3.5. *([61, Lemma 4.2.1, p59]) A trading strategy $\bar{\xi}_t = (\xi_t^0, \xi_t^1, \ldots, \xi_t^d)_{t\geq 0}$ is called self-financing if and only if V_t^* is a continuous, adapted process such that P-a.s., for each $t \in [0,T]$,*

$$V_t^* = \bar{\xi}_t \cdot \bar{S}_t^* = \sum_{i=0}^{d} \int_0^t \xi_s^i dS_s^{*,i}, \qquad t \in [0,T], i = 0, 1, \ldots, d. \tag{3.8}$$

Definition 3.6. *A self-financing trading strategy $\bar{\xi}_t$ is called an arbitrage opportunity if the discounted value process V_t^* satisfies the following*

1. $V_0^* = 0$;

2. *there exists a constant a such that* $P(\{\omega \in \Omega : V_t^*(\omega) \geq a, \text{ for all } t \in [0,T]\}) = 1$;

3. $V_T^* \geq 0 \qquad P\text{−}a.s.$;

4. $P[V_T^* > 0] > 0$.

A model satisfies the No-Arbitrage condition if such a strategy does not exist.

It turns out that in the continuous-time setting, the No-Arbitrage condition does not guarantee the existence of an equivalent local martingale measure (see Example 7.7 in F.

Debaen and W. Schachermayer in [13]). A stronger condition is needed. The following modification of the no-arbitrage property was introduced by A. N. Shiryaev, A. S. Cherny in [51].

Definition 3.7. *([51, Definition 1.6, p6])A trading strategy $\bar{\xi}_t = (\xi_t^0, \xi_t^1, \dots, \xi_t^d)_{t\in[0,T]}$ fulfills free lunch with vanishing risk condition, if*

1. $V_0^* = 0$;

2. *for each $k = 0, 1 \dots, d$, there exists a constant a_k such that*

$$P(\{\omega \in \Omega : V_t^{*,k}(\omega) \geq a_k, \text{ for all } t \in [0,T]\}) = 1,$$

   where $V_t^{*,k} := \xi_t^k S_t^{*,k}$;

3. *for each k, $V_T^{*,k} \geq -\frac{1}{k}$, $P - a, s,$;*

4. *there exist constants a such that $P(\{\omega \in \Omega : V_t^*(\omega) \geq a, t \in [0,T]\}) = 1$;*

5. *there exist constant $\delta_1 > 0$, $\delta_2 > 0$ such that, for each k, $P\{V_T^{*,k} > \delta_1\} > \delta_2$.*

A model satisfies the No Free Lunch with Vanishing Risk condition if such a sequence of strategies does not exist.

Definition 3.8. *A trading strategy $\bar{\xi}_t = (\xi_t^0, \xi_t^1, \dots, \xi_t^d)_{t\in[0,T]}$ realizes free lunch with bounded risk if it satisfies condition 1, 2 of Definition ?? as well as the following conditions:*

1. *there exists a constant a such that, for all $k = 0, 1, \dots, d$,*

$$P(\{\omega \in \Omega : V_t^{*,k}(\omega) \geq a, \text{ for all } t \in [0,T]\}) = 1;$$

2. *there exist constants $\delta_1 > 0$, $\delta_2 > 0$ such that, for each k, $P\{V_T^{*,k} > \delta_1\} > \delta_2$. and, for each $\delta > 0$, $P\{V_T^{*,k} < -\delta\} \to 0$.*

A model satisfies No Free Lunch with Bounded Risk condition if such a sequence of strategies does not exist.

Theorem 3.3.1. *([13, Theorem 1.1, p479])* **(Fundamental Theorem of Asset Pricing)** *Let $\bar{S}_t$ be a locally bounded, $(d+1)$-dimensional vector valued semi-martingale. An equivalent local martingale measure exists for $\bar{S}_t$ if and only if the No Free Lunch with Vanishing Risk condition holds.*

3.4 No good deal condition

Here we will focus on the no-good deal condition. Following J. Biog-Nasal and G. Dvi Nunno [3] we assume that the given $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfies that $\mathcal{F}_T = \mathcal{F}$. We work in an L_∞-framework and consider claims as elements of the space $L_\infty(\mathcal{F}_t) := L_\infty(\Omega, \mathcal{F}_t, P)$ of random variables with finite norm $\|X\|_\infty := \text{esssup}|X|$, $X \in L_\infty(\mathcal{F}_t)$. For any time $t \in [0,T]$, let $L_t \subseteq L_\infty(\mathcal{F}_t)$ denote the linear sub-space representing all market claims that are payable at time t. Note that on a complete market $L_t = L_\infty(\mathcal{F}_t)$. For a given asset $X \in L_t$ we denote the systems of prices by x_{st}, $0 \leq s \leq t \leq T$. We assume that price $x_{st}(X)$ $0 \leq s \leq t \leq T$, for marked assets $X \in L_t$ are given and we describe them in axiomatic form where $x_{st}(X)$ denote the price of asset X from s to t . Here we set the bounds on prices $m_{st}(X) \leq x_{st}(X) \leq M_{st}(X)$ and we study the existence of a pricing measures P_0 that allows a linear representation $x_{st}(X) = E_{P_0}[X|\mathcal{F}_s], X \in L_t$, fulfilling the given bounds. The pricing measure P_0 will reflect the choices of bounds. When we use + in the notation of space, we refer to the corresponding cone of the non-negative elements.

Next we consider no-good-deal pricing measures. The good-deal bound is a way to restrict the choice of equivalent martingale measures in incomplete markets. The idea is to consider martingale measures that not only rule out arbitrage possibilities, but also deals with "too good to be true". As usual we work with general price systems and not with specific price dynamics.

Following Chicharee and Sa Requejo [9], a good-deal of level $\delta > 0$ is a non-negative $\mathcal{F}_T$-measurable payoff X such that

$$\frac{E(X) - E_Q(X)}{\sqrt{Var(X)}} \geq \delta.$$

Accordingly, a probability measure Q is equivalent to P is a no good-deal pricing measure if there are no good-deals of level δ under Q, i.e.,

$$E_Q[X] \geq E[X] - \delta\sqrt{Var(X)}, \qquad X \geq 0. \tag{3.9}$$

Note that (3.9) holds for all $X \in L_\infty(\mathcal{F}_T)$ as we have that $X + \|X\|_\infty \geq 0$. Hence also the relation

$$E_Q[X] \leq E[X] + \delta\sqrt{Var(X)}, \qquad X \geq 0$$

holds true for all $X \in L_\infty(\mathcal{F}_T)$. This motivates the following extended general definition of no good-deals pricing measure.

Definition 3.9. *([3, Theorem 6.1, p24]) A probability measure Q equivalent to P is a no good-deal pricing measure if there are no good-deals of level $\delta > 0$ under Q, i.e.,*

$$-\delta \leq \frac{E(X) - E_Q(X)}{\sqrt{Var(X)}} \leq \delta.$$

for all $X \in L_2(\mathcal{F}_T, P) \cap L_1(\mathcal{F}_T, P)$.

3.5 Comparison and further discussions

In general, for fixed $T > 0$, consider a stochastic differential equation on $[0, T] \times R^n$,

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{3.10}$$

where $\mu : [0, T] \times R^n \to R^n$, $\sigma : [0, T] \times R^n \to R^{n \times m}$*and* W_t is an m-dimensional Brownian motion. Under linear growth condition and Lipschitz condition

$$|\mu(t, x)| + \|\sigma(t, x)\| \leq C_t(1 + |x|) \quad x \in R^n, t \in [0, T];$$

$$|\mu(t, x) - \mu(t, y)| + \|\sigma(t, x) - \sigma(t, y)\| \leq C_t|x - y| \quad x, y \in \mathbb{R}^n, t \in [0, T].$$

for some constant $C_t > 0$, for every $t \in [0, T]$, (3.10) admits a unique strong solution $(X_t)_{t \geq 0}$ for a given initial X_0.

In the special case, let $(S_t)_{t\in[0,T]}$ be the price process satisfying the following Black-Scholes pricing dynamics

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

along with a bank account $dS_t^0 = rdt$, where r, μ, σ are positive constants. Given initial data S_0, S_t is determined uniquely by the above equation, and S_t is given explicitly by $S_t = S_0\left[\exp(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\right]$. The discounted price process is given by $S_t^* = \frac{S_t}{S_t^0}$. We then have discounted price:

$$S_t^* = S_0 \exp\left[(\mu - r - \frac{1}{2}\sigma^2)t + \sigma W_t\right]. \tag{3.11}$$

Now applying Itô formula for (3.11), we have

$$dS_t^* = (\mu - r)S_t^* dt + \sigma S_t^* dW_t. \tag{3.12}$$

According to (3.12) and Definition 3.5 with $d = 1$ and $V_0^* = e^{rT}$, the value process is then

$$\begin{aligned}
V_T^* &= V_0^* + \int_0^T \xi_t dS_t^* \\
&= V_0^* + \int_0^T \xi_t S_t^* \{(\mu - r)dt + \sigma dW_t\} \\
&= e^{rT} + (\mu - r)\int_0^T \xi_t S_t^* dt + \sigma \int_0^T \xi_t S_t^* dW_t \\
&= e^{rT} + (\mu - r)S_0 \int_0^T \xi_t e^{(\mu - r - \frac{1}{2}\sigma^2)t + \sigma W_t} dt + \sigma S_0 \int_0^T \xi_t e^{(\mu - r - \frac{1}{2}\sigma^2)t + \sigma W_t} dW_t.
\end{aligned}$$

Let

$$f(t, x) = \frac{1}{\sigma}\xi_t e^{(\mu - r - \frac{1}{2}\sigma^2)t} e^{\sigma x},$$

then we have

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \xi_t e^{(\mu - r - \frac{1}{2}\sigma^2)t} e^{\sigma x}, \\
\frac{\partial^2 f}{\partial x^2} &= \sigma \xi_t e^{(\mu - r - \frac{1}{2}\sigma^2)t} e^{\sigma x},
\end{aligned}$$

and

$$\frac{\partial f}{\partial t} = \frac{1}{\sigma}(\mu - r - \frac{1}{2}\sigma^2)\xi_t e^{(\mu - r - \frac{1}{2}\sigma^2)t} e^{\sigma x} + \frac{1}{\sigma} e^{(\mu - r - \frac{1}{2}\sigma^2)t} e^{\sigma x} \frac{\partial \xi_t}{\partial t}$$

where we assume ξ_t is differentiable w.r.t t.

Clearly, in terms of the Itô formula [44, Theorem 4.2.1, p48]

$$\begin{aligned}
\int_0^T \frac{\partial f}{\partial x}(t, W_t)dW_t &= f(T, W_T) - f(0, W_0) - \int_0^T \left[\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, W_t) + \frac{\partial f}{\partial t}(t, W_t)\right] dt \\
&= \frac{1}{\sigma}\xi_T e^{(\mu-r-\frac{1}{2}\sigma^2)T} e^{\sigma W_T} - \frac{1}{\sigma}\xi_0 - \int_0^T \left[\frac{1}{2}\sigma\xi_t e^{(\mu-r-\frac{1}{2}\sigma^2)t} e^{\sigma W_t}\right. \\
&\quad \left. + \frac{1}{\sigma}(\mu - r - \frac{1}{2}\sigma^2)\xi_t e^{(\mu-r-\frac{1}{2}\sigma^2)t} e^{\sigma W_t} + \frac{1}{\sigma} e^{(\mu-r-\frac{1}{2}\sigma^2)t} e^{\sigma W_t} \frac{\partial \xi_t}{\partial t}\right] dt \\
&= \frac{1}{\sigma}\xi_T e^{(\mu-r-\frac{1}{2}\sigma^2)T} e^{\sigma W_T} - \frac{1}{\sigma}\xi_0 - \int_0^T \left[\frac{1}{\sigma}(\mu - r)\xi_t + \frac{1}{\sigma}\frac{\partial \xi_t}{\partial t}\right] e^{(\mu-r-\frac{1}{2}\sigma^2)t} e^{\sigma W_t} dt.
\end{aligned}$$

Thus,

$$\begin{aligned}
V_T^* &= e^{rT} + (\mu - r)S_0 \int_0^T \xi_t e^{(\mu-r-\frac{1}{2}\sigma^2)t+\sigma W_t} dt + \sigma S_0 \left\{\frac{1}{\sigma}\xi_T e^{(\mu-r-\frac{1}{2}\sigma^2)T} e^{\sigma W_T} - \frac{1}{\sigma}\xi_0\right. \\
&\quad \left. - \int_0^T \left[\frac{1}{\sigma}(\mu - r)\xi_t + \frac{1}{\sigma}\frac{\partial \xi_t}{\partial t}\right] e^{(\mu-r-\frac{1}{2}\sigma^2)t} e^{\sigma W_t} dt\right\} \\
&= e^{rT} + S_0\xi_T e^{(\mu-\frac{1}{2}\sigma^2)T} e^{\sigma W_T} - S_0 \int_0^T \frac{\partial \xi_t}{\partial t} e^{(\mu-r-\frac{1}{2}\sigma^2)t} e^{\sigma W_t} dt \\
&= \int_0^T \left(\frac{1}{T}e^{rT} + \frac{1}{T}S_0\xi_T e^{(\mu-r-\frac{1}{2}\sigma^2)T} e^{\sigma W_T} - S_0 \frac{\partial \xi_t}{\partial t} e^{(\mu-r-\frac{1}{2}\sigma^2)t} e^{\sigma W_t}\right) dt.
\end{aligned}$$

By the item 5 of definition (3.7), if there are constants $\delta_1 > 0$, $\delta_2 > 0$, then free lunch with vanishing risk exists,

$$P\left\{\int_0^T \left(\frac{1}{T}e^{rT} + \frac{1}{T}S_0\xi_T e^{(\mu-r-\frac{1}{2}\sigma^2)T} e^{\sigma W_T} - S_0 \frac{\partial \xi_t}{\partial t} e^{(\mu-r-\frac{1}{2}\sigma^2)t} e^{\sigma W_t}\right) dt > \delta_1\right\} > \delta_2. \quad (3.13)$$

Set $X(t) := \frac{1}{T}e^{rT} + \frac{1}{T}S_0\xi_T e^{(\mu-r-\frac{1}{2}\sigma^2)T} e^{\sigma W_T} - S_0 \frac{\partial \xi_t}{\partial t} e^{(\mu-r-\frac{1}{2}\sigma^2)t} e^{\sigma W_t}$. Then (3.13) reduce to

$$P\left\{\int_0^T X(t)dt > \delta_1\right\} > \delta_2.$$

For any $p > 0$, by Chebyshev's inequality,

$$\delta_2 \leq P\left\{|\int_0^T X(t)dt| > \delta_1\right\} \leq \frac{E|\int_0^T X(t)dt|^p}{\delta_1^p}.$$

For any $p, q > 1$ and $\frac{1}{p} + \frac{1}{p} = 1$,

$$E\left(|\int_0^T X(t)dt|^p\right) \leq T^{\frac{p}{q}} \int_0^T E\left(X(t)\right)^p dt.$$

More generally, if $p = q = 2$, in that case, we can get

$$\frac{\delta_2\delta_1^2}{T} \leq \int_0^T E\left(X(t)\right)^2 dt.$$

We calculate $\int_0^T E\left(X(t)\right)^2 dt$. By $E(e^{\sigma W_t - \frac{1}{2}\sigma^2 t}) = 1$, assume $\xi_t = Ct$, we can compute

$$\int_0^T E\left(X(t)\right)^2 dt = \int_0^T E\left(\frac{1}{T}e^{rT} + (\frac{1}{T}S_0\xi_T e^{(\mu-r-\frac{1}{2}\sigma^2)T}e^{\sigma W_T} - S_0Ce^{(\mu-r-\frac{1}{2}\sigma^2)t}e^{\sigma W_t})\right)^2 dt$$
$$= \frac{1}{T}e^{2rT} + S_0^2C^2e^{(2(\mu-r)+\sigma^2)T}\left(T - \frac{2}{\sigma^2+(\mu-r)} + \frac{1}{\sigma^2+2(\mu-r)}\right)$$
$$+ 2S_0^2C^2\frac{e^{(\mu-r)T}}{\sigma^2+(\mu-r)} - \frac{C^2S_0^2}{\sigma^2+2(\mu-r)}$$
$$+ 2S_0Ce^{\mu T} - \frac{2S_0C}{T(\mu-r)}e^{\mu T} + \frac{2S_0C}{T(\mu-r)}e^{rT} \geq \frac{\delta_1^2\delta_2}{T}. \tag{3.14}$$

Let

$$I = \frac{1}{T}e^{2rT} + S_0^2C^2e^{2(\mu-r)-\sigma^2)T}\left(T - \frac{2}{\sigma^2+(\mu-r)} + \frac{1}{\sigma^2+2(\mu-r)}\right)$$
$$+ 2S_0^2C^2\frac{e^{(\mu-r)T}}{\sigma^2+(\mu-r)} - \frac{C^2S_0^2}{\sigma^2+2(\mu-r)}$$
$$+ 2S_0Ce^{\mu T} - \frac{2S_0C}{T(\mu-r)}e^{\mu T} + \frac{2S_0C}{T(\mu-r)}e^{rT}$$

$$J = e^{(2(\mu-r)+\sigma^2)T}\left(T - \frac{3}{2(\sigma^2+(\mu-r))}\right) + \frac{2(\mu-r)T+1}{\sigma^2+2(\mu-r)} + \frac{e^{2rT}}{TS_0^2C^2} + \frac{2e^{\mu T}}{S_0C}.$$

Obviously $I \geq J$. Thus, if

$$e^{(2(\mu-r)+\sigma^2)T}\left(T - \frac{3}{2(\sigma^2+(\mu-r))}\right) + \frac{2(\mu-r)T+1}{\sigma^2+2(\mu-r)} + \frac{e^{2rT}}{TS_0^2C^2} + \frac{2e^{\mu T}}{S_0C} \geq \frac{\delta_1^2\delta_2}{TC^2S_0^2}. \tag{3.15}$$

Then

$$\int_0^T E\left(X(t)\right)^2 dt \geq \frac{\delta_2\delta_1^2}{T}.$$

Clearly, (3.15) is stronger than(3.14).

On the other hand, under no good deal conditions, the price is given by $x_{s,t}$. $0 \leq s < t < T$. We define $\mu : (0,T) \times R \to R$, $\sigma : (0,T) \times R \to R$,

$$x_{s,t}(X) := \int_s^t \mu(r, X_r)dr + \int_s^t \sigma(r, X_r)dW_r. \tag{3.16}$$

By Girsanov transformation, we have

$$X_t = X_0 + \int_0^t \sigma(s, X_s)d\tilde{W}_s,$$

here $\tilde{W}_t = W_t + \int_0^t \sigma^{-1}(s, X_s)\mu(s, X_s)ds$ and $\tilde{W}_t$ is a Q-Brownian motion, where Q is defined as $dQ = e^{\int_0^t \sigma^{-1}(s,X_s)\mu(s,X_s)dB_s - \frac{1}{2}\int_0^t [\sigma^{-1}(s,X_s)\mu(s,X_s)]^2 ds} dP$. Taking expectation on both sides yields that,

$$\begin{aligned} E_Q(X_t) &= E(X_0) + E_Q \int_0^t \sigma(s, X_s) d\tilde{W}_s \\ &= E(X_0) + 0 \\ &= E(X_0) \end{aligned}$$

and

$$E(X_t) = E \int_0^t \mu(r, X_r)dr + E(X_0).$$

Therefore,

$$\begin{aligned} E(X_t) - E_Q(X_t) &= E \int_0^t \mu(r, X_r)dr + E(X_0) - E(X_0) \\ &= E \int_0^t \mu(r, X_r)dr. \end{aligned} \tag{3.17}$$

By the definition of Variance,

$$\begin{aligned} \mathrm{Var}(X_t) &= E_Q(X_t - E_Q(X_t))^2 \\ &= E_Q(X_0 + \int_0^t \sigma(s, X_s)d\tilde{W}_s - X_0)^2 \\ &= E_Q(\int_0^t \sigma(s, X_s)d\tilde{W}_s)^2 \\ &= E \int_0^t \sigma^2(s, X_s)ds. \end{aligned} \tag{3.18}$$

Considering a special case of above equation with the riskless bond S_u^0 and interest rate ι, we have

$$x_{s,t}(S_t) := \iota \int_s^t S_u^0 du + \int_s^t \mu S_r^* dr + \int_s^t \sigma S_u^* dW_u.$$

According to the definition (3.9), and identities (3.17) and (3.18), the no good-deal condition can be satisfied if

$$-\delta \leq \frac{E(\int_0^T r S_t^0 dt + \int_0^T \mu S_t^* dt)}{\sqrt{E(\int_0^T r^2 (S_t^0)^2 + \int_0^T \sigma^2 S_t^{*2} dt)}} \leq \delta. \tag{3.19}$$

Compute (3.19),

$$E(\int_0^T rS_t^0 dt + \int_0^T \mu S_t^* dt) = E(\int_0^T re^{rt} dt + \mu \int_0^T S_0 e^{(\mu - r - \frac{1}{2}\sigma^2)t + \sigma W_t} dt)$$
$$= (e^{rT} - 1) + \frac{\mu S_0}{\mu - r}\left(e^{(\mu - r)T} - 1\right).$$

and

$$\sqrt{E(\int_0^T r^2 (S_t^0)^2 dt + \int_0^T \sigma^2 S_t^{*2} dt)} = \sqrt{\frac{r}{2}(e^{2rT} - 1) + \sigma^2 S_0^2 \frac{e^{(2(\mu - r) + \sigma^2)T} - 1}{2(\mu - r) + \sigma^2}}.$$

Then we can get

$$\frac{E(\int_0^T rS_t^0 dt + \int_0^T \mu S_t^* dt)}{\sqrt{E(\int_0^T r^2 (S_t^0)^2 + \int_0^T \sigma^2 S_t^{*2} dt)}}$$
$$= \frac{(e^{rT} - 1) + \frac{\mu S_0}{\mu - r}\left(e^{(\mu - r)T} - 1\right)}{\sqrt{\frac{r}{2}(e^{2rT} - 1) + \sigma^2 S_0^2 \frac{e^{(2(\mu - r) + \sigma^2)T} - 1}{2(\mu - r) + \sigma^2}}}.$$

Now substituting the result above into (3.19), we derive the following

$$-\delta \leq \frac{(e^{rT} - 1) + \frac{\mu S_0}{\mu - r}\left(e^{(\mu - r)T} - 1\right)}{\sqrt{\frac{r}{2}(e^{2rT} - 1) + \sigma^2 S_0^2 \frac{e^{(2(\mu - r) + \sigma^2)T} - 1}{2(\mu - r) + \sigma^2}}} \leq \delta. \tag{3.20}$$

Now we summarize the above derivation as the following main result

Theorem 3.5.1. *Under conditions* $T \geq \frac{3}{2(\sigma^2 + (\mu - r))}$, $\delta \leq \sqrt{\frac{2}{r}}$ *and* $r \geq (\mu - \mu S_0) \vee 0$. *Condition* (3.20) *can imply Condition* (3.14) *which means in financial market, the no-good deal condition for fundamental theorem is more general than the No Free Lunch with Vanishing Risk condition.*

Proof. (3.20) can be reduced to

$$e^{(2(\mu - r) + \sigma^2)T} \geq \Big((1 - \frac{r\delta^2}{2})e^{2rT} + 2(\mu S_0 + 1)e^{rT} + \left[\frac{\mu S_0}{\mu - r}e^{(\mu - r)T} - \frac{\mu S_0(\mu - r) + \mu^2 S_0^2}{\mu^2 S_0^2}\right]^2$$
$$- \frac{(\mu - r)^2}{\mu^2 S_0^2} - \frac{2(\mu - r)}{\mu S_0} + \frac{2\mu S_0}{(\mu - r)} + \frac{\mu^2 S_0^2}{(\mu - r)^2}$$
$$+ \frac{r\delta^2}{2} + \frac{\delta^2 \sigma^2 S_0^2}{2(\mu - r) + \sigma^2} + 1\Big)\frac{2(\mu - r) + \sigma^2}{\delta^2 \sigma^2 S_0^2}. \tag{3.21}$$

If (3.20) is true, putting (3.21) into LHS of (3.15) then yields that

$$
\begin{aligned}
LHS\,\text{of}\,(3.15) \geq \Bigg[\Big((1-\frac{r\delta^2}{2})e^{2rT} &+ 2(\mu S_0+1)e^{rT} + \left[\frac{\mu S_0}{\mu-r}e^{(\mu-r)T} - \frac{\mu S_0(\mu-r)+\mu^2S_0^2}{\mu^2S_0^2}\right]^2 \\
&- \frac{(\mu-r)^2}{\mu^2S_0^2} - \frac{2(\mu-r)}{\mu S_0} + \frac{2\mu S_0}{(\mu-r)} + \frac{\mu^2S_0^2}{(\mu-r)^2} \\
&+ \frac{r\delta^2}{2} + \frac{\delta^2\sigma^2S_0^2}{2(\mu-r)+\sigma^2} + 1\Big)\frac{2(\mu-r)+\sigma^2}{\delta^2\sigma^2S_0^2}\Bigg] \\
&\left(T - \frac{3}{2(\sigma^2+(\mu-r))}\right) + \frac{2(\mu-r)T+1}{\sigma^2+2(\mu-r)} + \frac{e^{2rT}}{TS_0^2C^2} + \frac{2e^{\mu T}}{S_0C}.
\end{aligned}
\tag{3.22}
$$

Let us assume (3.22) ≥ 0, then it is easy to find

$$
(1-\frac{r\delta^2}{2}) \geq 0
$$

$$
T - \frac{3}{2(\sigma^2+(\mu-r))} \geq 0
$$

$$
-\frac{(\mu-r)^2}{\mu^2S_0^2} - \frac{2(\mu-r)}{\mu S_0} + \frac{2\mu S_0}{(\mu-r)} + \frac{\mu^2S_0^2}{(\mu-r)^2} \geq 0.
$$

Therefore, we have under conditions $T \geq \frac{3}{2(\sigma^2+(\mu-r))}$, $\delta \leq \sqrt{\frac{2}{r}}$ and $r \geq (\mu-\mu S_0) \vee 0$, (3.20) can imply (3.15). As (3.15) is stronger than (3.14), we then prove that (3.20) implies (3.14).

□

Remark 3.5.1. *If* $T - \frac{3}{2(\sigma^2+(\mu-r))} \leq 0$, *it might not feasible to compare. We can not compare them in a short time.*

On the other hand, we would like to examine whether (3.14) implies (3.20). Assume (3.14) is true, then we have

$$
\begin{aligned}
&e^{2(\mu-r)-\sigma^2)T}\left(T - \frac{2}{\sigma^2+(\mu-r)} + \frac{1}{\sigma^2+2(\mu-r)}\right) \geq \\
&\frac{\delta_1^2\delta_2}{TS_0^2C^2} - \frac{1}{TS_0^2C^2}e^{2rT} - \frac{2e^{(\mu-r)T}}{\sigma^2+(\mu-r)} + \frac{1}{\sigma^2+2(\mu-r)} - \frac{2e^{\mu}T}{S_0C} \\
&+ \frac{2e^{\mu T}}{TS_0C(\mu-r)} - \frac{2e^{rT}}{TS_0C(\mu-r)}.
\end{aligned}
\tag{3.23}
$$

Letting RHS of (3.23) ≥ 0, we need

$$
T - \frac{2}{\sigma^2+(\mu-r)} + \frac{1}{\sigma^2+2(\mu-r)} \geq 0
$$

and

$$1-\frac{1}{T(\mu-r)}\geq 0.$$

We can show that:

$$T\geq\frac{\sigma^2+3(\mu-r)}{[\sigma^2+(\mu-r)][\sigma^2+2(\mu-r)]}$$

and

$$T\geq\frac{1}{\mu-r}.$$

We conclude that

$$T\geq\left\{\frac{\sigma^2+3(\mu-r)}{[\sigma^2+(\mu-r)][\sigma^2+2(\mu-r)]}\vee\frac{1}{\mu-r}\right\}. \tag{3.24}$$

Consequently, our conclusion is that under the condition (3.24), (3.14) implies (3.21). From (3.21), we work backwards

$$\left[(e^{rT}-1)+\frac{\mu S_0}{\mu-r}\left(e^{(\mu-r)T}-1\right)\right]^2\leq\delta^2(\frac{r}{2}(e^{2rT}-1)+\sigma^2S_0^2\frac{e^{(2(\mu-r)+\sigma^2)T}-1}{2(\mu-r)+\sigma^2}),$$

then we obtain

$$0\leq(e^{rT}-1)+\frac{\mu S_0}{\mu-r}\left(e^{(\mu-r)T}-1\right)\leq\delta\sqrt{\frac{r}{2}(e^{2rT}-1)+\sigma^2S_0^2\frac{e^{(2(\mu-r)+\sigma^2)T}-1}{2(\mu-r)+\sigma^2}},$$

The above is equivalent to (3.20). We summarize our discussion by the follows

Theorem 3.5.2. *Under conditions* $T\geq\left\{\frac{\sigma^2+3(\mu-r)}{[\sigma^2+(\mu-r)][\sigma^2+2(\mu-r)]}\vee\frac{1}{\mu-r}\right\}$, (3.14) *implies* (3.20)*,which means no-good deal condition for fundamental theorem is weaker than No Free Lunch with Vanishing Risk condition.*

We now turn to the situation in higher dimensions. We consider a finite time interval $[0,T]$ as the interval during which trading may take place. We assume as given a complete probability space $(\Omega,\mathcal{F},P;\mathcal{F}_t)$ on which is defined a stranded m-dimensional Browning motion $W=\{W_t,t\in[0,T]\}$. In particular, $W=(W^1,W^2,\ldots,W^m)$ is an m-dimensional process defined on the time interval $[0,T]$. Our multi-dimensional model has $d+1$ assets where d is a strictly positive integer.

Sometimes, the money market asset is referred to as a riskless asset, denoted by S_t^0 which is given by

$$\frac{dS_t^0}{S_t^0}=rdt.$$

We assume that there are d stock with continuous, adapted price process $S_t^i = (S_t^0, S_t^1, \dots, S_t^d)$ which satisfying the following high-dimensional Black-Scholes pricing dynamics and

$$\frac{dS_t^i}{S_t^i} = \mu^i d_t + \sum_{j=1}^{m} \sigma^{ij} dW_t^j.$$

Here the solution can be explicitly given as follows

$$S_t^i = S_0^i \exp\left((\mu^i - \frac{1}{2}\sum_{j=1}^{m}(\sigma\sigma')^{ij} t + \sum_{j=1}^{m} \sigma^{ij} W_t^j \right) \tag{3.25}$$

where S_0^i is a strictly positive constant and μ^i is the i^{th} component of a d-dimensional drift vector, and $\sigma = (\sigma^{ij})_{1 \le i \le d, 1 \le j \le m}$ is a $(d \times m)$-matrix.

Now applying Itô formula then yields that:

$$dS_t^{*i} = S_t^{*i}(\mu^i - r)dt + S_t^{*i} \sum_{j=1}^{m} \sigma^{ij} dW_t^j. \tag{3.26}$$

A trading strategy in this case is a $(d+1)$-dimensional process $\xi = \{\xi_t^0, \xi_t^1, \dots, \xi_t^d, t \in [0,T]\}$. The value at t of the portfolio associated with ξ is given by

$$\begin{aligned} V_t^* &= \bar{\xi}_t \cdot \bar{S}_t^* \\ &= \sum_{i=0}^{d} \int_0^t \xi_s^i dS_s^{*,i} \qquad t \in [0,T], i = 0, 1, \dots, d. \end{aligned}$$

Putting (3.26) into above, we get

$$\begin{aligned} V_T^* - V_0^* &= \sum_{i=0}^{d} \int_0^T \xi_t^i dS_t^{*i} \\ &= \sum_{i=0}^{d} \int_0^T \xi_t^i \left(S_t^{*i}(\mu^i - r)dt + S_t^{*i} \sum_{j=1}^{m} \sigma^{ij} dW_t^j \right) \\ &= \sum_{i=1}^{d} \int_0^T \xi_t^i S_t^{*i}(\mu^i - r)dt \\ &+ \sum_{i=1}^{d} \sum_{j=1}^{m} \int_0^T \xi_t^i S_t^{*i} \sigma^{ij} dW_t^j \end{aligned}$$

here let $\xi_T^0 = 1$, $S_T^0 = e^{rT}$,

$$V_T^* = e^{rT} + \sum_{i=1}^{d} \int_0^T (\mu^i - r)\xi_t^i S_0^i \exp\left((\mu^i - r - \frac{1}{2}\sum_{j=1}^{m}(\sigma\sigma')^{ij}t + \sum_{j=1}^{m}\sigma^{ij}W_t^j\right) dt$$
$$+ \sum_{i=1}^{d}\sum_{j=1}^{m}\sigma^{ij}\int_0^T \exp\left((\mu^i - r - \frac{1}{2}\sum_{j=1}^{m}(\sigma\sigma')^{ij}t + \sum_{j=1}^{m}\sigma^{ij}W_t^j\right) dW_t^j.$$

Remark 3.5.2. *For fixed i, j, $(\sigma\sigma')^{ij}$ is given by*

$$(\sigma\sigma')^{ij} = \sum_{k=1}^{m}\sigma^{ik}\sigma^{jk},$$

where $'$ denotes transpose.

Proposition 3.5.1. *Each price process S_t^i can be represented as $S_0^i e^{(\mu^i - r - \frac{1}{2}(\sigma^i)^2)t + \sigma^i W_t}$, where*

$$(\sigma^i)^2 = \sum_{j=1}^{m}(\sigma^{ij})^2 = (\sigma\sigma^*)^{ii} > 0$$

and

$$\sum_{j=1}^{m}\sigma^{ij}dW_t^j = \sigma^i dW_t. \tag{3.27}$$

Proof. From above equation (3.27), we have

$$W_t = \frac{\sum_{j=1}^{m}\sigma^{ij}dW_t^j}{\sigma^i}.$$

According to definition of quadratic variation

$$\begin{aligned} d\langle W, W\rangle(t) &= \frac{1}{(\sigma^i)^2}\langle\sum_{j=1}^{m}\sigma^{ij}dW_t^j, \sum_{j=1}^{m}\sigma^{ij}dW_t^j\rangle \\ &= \frac{1}{(\sigma^i)^2}\sum_{j=1}^{m}(\sigma^{ij})^2 dt \\ &= dt. \end{aligned}$$

Therefor W_t is a Browning motion and $S_t^i = S_0^i e^{(\mu^i - r - \frac{1}{2}(\sigma^i)^2)t + \sigma^i W_t}$. We are done. □

This proposition shows that the result from Multi-dimensional model is essentially similar to the one-dimensional case. In other words, one-dimensional model is a special case in high dimensional model. By item 5 of Definition (3.7), and Proposition(3.5.1) if there is constant $\delta_1 > 0,\ \delta_2 > 0$, then free lunch with vanishing risk exists,

$$P\Big\{\int_0^T \big(\frac{1}{T}e^{rT} + \frac{1}{T}S_0^i\xi_T^i e^{(\mu^i - r - \frac{1}{2}\sum_{j=1}^m(\sigma\sigma')^{ij})T}e^{\sum_{j=1}^m \sigma^{ij}W_T^j} - S_0^i\frac{\partial \xi_t}{\partial t}e^{(\mu^i - r - \frac{1}{2}\sum_{j=1}^m(\sigma\sigma')^{ij})t}e^{\sum_{j=1}^m \sigma^{ij}W_t^j}\big)dt > \delta_1\Big\} > \delta_2. \tag{3.28}$$

Set $X(t) := \frac{1}{T}e^{rT} + \frac{1}{T}S_0^i\xi_T^i e^{(\mu^i - r - \frac{1}{2}\sum_{j=1}^m(\sigma\sigma')^{ij})T}e^{\sum_{j=1}^m \sigma^{ij}W_T^j} - S_0^i\frac{\partial \xi_t}{\partial t}e^{(\mu^i - r - \frac{1}{2}\sum_{j=1}^m(\sigma\sigma')^{ij})t}e^{\sum_{j=1}^m \sigma^{ij}W_t^j}$
Then (3.28) reduce to

$$P\Big\{\int_0^T X(t)dt > \delta_1\Big\} > \delta_2.$$

For any $p > 0$, and $\frac{1}{p}+\frac{1}{p} = 1$, by Tchebychev's inequality, we can get $\delta_2 \le P\Big\{|\int_0^T X(t)dt| > \delta_1\Big\} \le \frac{E|\int_0^T X(t)dt|^p}{\delta_1^p}$. More generally, if $p = q = 2$, in that case, we have that

$$\frac{\delta_2\delta_1^2}{T} \le \int_0^T E\left(X(t)\right)^2 dt.$$

We calculate $\int_0^T E\left(X(t)\right)^2 dt$. By $E(e^{-\frac{1}{2}\sum_{j=1}^m(\sigma\sigma')^{ij}t + \sum_{j=1}^m \sigma^{ij}W_t^j}) = 1$, and $\xi_t^i = C^i t$, where $i = 1, \ldots, d,\ j = 1, \ldots, m$. We can get the following inequality

$$\frac{1}{T}e^{2rT} + S_0^{i\,2}C^{i\,2}e^{2((\mu^i - r) + \sum_{j=1}^m(\sigma\sigma')^{ij})T}\left(T - \frac{2}{\sum_{j=1}^m(\sigma\sigma')^{ij} + (\mu - r)} + \frac{1}{\sum_{j=1}^m(\sigma\sigma')^{ij} + 2(\mu^i - r)}\right)$$
$$+ 2S_0^{i\,2}C^{i\,2}\frac{e^{(\mu^i - r)T}}{\sum_{j=1}^m(\sigma\sigma')^{ij} + (\mu^i - r)} - \frac{C^{i\,2}S_0^{i\,2}}{\sum_{j=1}^m(\sigma\sigma')^{ij} + 2(\mu^i - r)}$$
$$+ 2S_0^iC^ie^{\mu^iT} - \frac{2S_0^iC^i}{T(\mu^i - r)}e^{\mu^iT} + \frac{2S_0^iC^i}{T(\mu^i - r)}e^{rT} \ge \frac{\delta_1^2\delta_2}{T}. \tag{3.29}$$

In no good deal conditions, the price is $x_{s,t}^i$. $0 \le s < t < T$. We define $\mu : (0,T) \times R \to R^d$, $\sigma : (0,T) \times R \to R^{d\times m}$,

$$x_{s,t}^i(X) := \int_s^t \mu^i(u, X_u)du + \sum_{j=1}^m \int_s^t \sigma^{ij}(u, X_u)dW_u^j. \tag{3.30}$$

Considering a special case of (3.30) with the riskless bond, we have

$$x_{s,t}^i(S_t) := \int_s^t rS_u^0du + \int_s^t \mu^iS_u^{i,*}du + \sum_{j=1}^m \int_s^t \sigma^{ij}S_u^{i,*}dW_u^j. \tag{3.31}$$

By the definition, the no good-deal condition can be satisfied if

$$-\delta \leq \frac{E(\int_0^T rS_t^0 dt + \int_0^T \mu^i S_t^{*i} dt)}{\sqrt{E(\int_0^T r^2(S_t^0)^2 + \sum_{j=1}^m \int_0^T (\sigma\sigma')^{ij}(S_t^{*i})^2 dt)}} \leq \delta. \tag{3.32}$$

We can find the explicit solutions

$$-\delta \leq \frac{(e^{rT}-1) + \frac{\mu^i S_0}{\mu^i - r}\left(e^{(\mu^i - r)T} - 1\right)}{\sqrt{\frac{r}{2}(e^{2rT}-1) + \sum_{j=1}^m (\sigma\sigma')^{ij} {S_0^i}^2 \frac{e^{(2(\mu^i-r)+\sum_{j=1}^m(\sigma\sigma')^{ij})T}-1}{2(\mu^i-r)+\sum_{j=1}^m(\sigma\sigma')^{ij}}}} \leq \delta. \tag{3.33}$$

We summarize our above discussion as the following results

Theorem 3.5.3. *Under conditions* $T \geq \frac{3}{2(\sum_{j=1}^m (\sigma\sigma')^{ij} + (\mu^i - r))}$, $\delta \leq \sqrt{\frac{2}{r}}$ *and* $r \geq (\mu^i - \mu^i S_0^i) \vee 0$. (3.33) *can imply* (3.29) *which means in financial market, the no-good deal condition for fundamental theorem is more general than the no free lunch with vanishing risk condition.*

Theorem 3.5.4. *Under conditions* $T \geq \left\{ \frac{\sum_{j=1}^m (\sigma\sigma^*)^{ij} + 3(\mu^i - r)}{[\sum_{j=1}^m (\sigma^{ij})^2 + (\mu^i - r)][\sum_{j=1}^m (\sigma\sigma')^{ij} + 2(\mu^i - r)]} \vee \frac{1}{\mu - r} \right\}$, (3.29) *implies* (3.33)*,which means the no-good deal condition for fundamental theorem is weaker than the no free lunch with vanishing risk condition.*

3.6 Conclusion

In this paper, we gave a detailed discussion on comparison of the no free lunch with the vanishing risk condition and no good deal condition for the fundamental theorems on option pricing. We used concrete examples to explore our comparison. Our examples are simple but intrinsic in mathematical modeling.

Chapter 4

Modelling Credit Ratings via Reflected Stochastic Differential Equations

4.1 Introduction

The recent turmoil in the international financial markets has drawn attention to the significance of correctly assessing and pricing credit risk. The credit rating provided by rating agencies has been criticized for their inability to predict major corporate defaults. Two basic model categories are: *reduced models* and *optional pricing models.* Structure models, default-intensity-based models and rating-based models belong to the reduced models category. A well utilized structure model is proposed in Merton's article [42] which describes that the firm value, which depends on the investor's risk aversion, is a deterministic function of three variables: the yield curve, the probability of default, and maturity. The seminal articles concerning the family of default-intensity-based models are Jarrow-Turnbull [33] and Lando [38]. In those articles, only the possible default of bond issues is observed but their rating is ignored. Clearly, the rating in those formulation takes only two possible values: either a prescribed value $\bar{X}$ or 0.

Rating-based models usually use characteristics such as rating process, yield curve and the recovery rate to compute price of risky assets. Crouhy-Im-Nuelman model and Hull-White model are famous ones in this family. M. Crouhy, J. Im and G. Nuelman in [11] model the rating as Markov chain with finitely many states, in order to mimic agency ratings. They build a risk-neutral probability that is inconsistent with interpolation of discontinuous rating by continuous ones. While Hull-White model proposed in [27], [28] defines a rating process $X(t)$ which is a pure Brownian motion, but the "default barrier" which is not necessarily a straight line is adapted so as to match the default probability. In order to get a risk-neutral probability, they modify the location of the barrier. In addition, M. Avellaneda and J.-Y. Zhu in [2] introduce the idea of "risk-neutral-distance-to-default process" of a firm. They characterize risk-neutrality by the fact that the default index satisfies a Fokker-Planck type parabolic PDE and show the easiness of calibration and the "square-root" shape of barriers.

This chapter is motivated by Douady and Jeanblanc [17], which presents a rating-based credit risk model that is both tractable (in terms of statistics), considers the effects of credit derivative pricing and hedging, and flexible enough to reproduce the real features of credit events in financial markets. In this paper, we propose to use stochastic differential equations with reflections to model rating processes, such that the object automatically falls into the values between 0 and 1. Under the framework of a rating-based model in pricing "zero-coupon-bond" presented in [18], we shall model a "continuous" rating $X(t) \in [0,1]$, which is incorporated to a bond issuer subject to a possible default, by the following stochastic differential equation with reflections

$$dX(t) = \theta X(t)dt + \sigma X(t)dB(t) + d\eta(t) - d\bar{\eta}(t).$$

where coefficients θ, σ are positive constants. Here $B(t)$ is a Brownian motion, and $\eta(t)$ is the local time of $X(t)$ at 0. This is a non-decreasing process which only increase when $X(t) = 0$. Similarly $\bar{\eta}(t)$ is the local time of $X(t)$ at 0. It is a non-decreasing process which only increase when $X(t) = 1$. Here we propose a natural model of stochastic differential equation with reflections for the rating process $X(t) \in [0,1]$.

The rest of the chapter is organized as follows: in Section 4.2, we first introduce the model studies in [17] and then we present a rating-based model in pricing "zero-coupon-bond" in which a bond issuer subject to possible default is assigned a "continuous" rating by stochastic differential equation with reflections.

4.2 Credit Derivative pricing model

Let $X(t)_{t\in[0,\infty]}$ be a rating process. Default occurs when the rating reaches 0, which is an absorbing state. Non-defaultbale bonds have rating 1, which is unreachable when starting from other rating. The "continuous" rating of a bond issuer has a rather intuitive meaning: it can be seen as an interpolation of rating provided by agencies. More precisely, one can specify the model in such a way that a given agencies rating correspond to some sub-interval $[\alpha_i, \alpha_{i+1}] \subset [0, 1]$. Rating migrations correspond to crossing one threshold $n_i(0, 1)$.

At any time t, the bond is valued as the sum of its scheduled payment, which are proportional to "defaultable discount factors" which rating $R_B(t)$. The defaultable discount factor with time to maturity x and rating R is denoted $D(t, x, R)$ and decomposed as follows:

$$D(t, x, R) = \exp(-l(t, x) - \psi(t, x, R)). \tag{4.1}$$

In other words, this quantity consists of two parts: the risk-free part $l(t, x)$ and the risky part $\psi(t, x, X(t))$. The non-default yield $y(t, x, 1) = l(t, x)/x$ follows a traditional interest rate model. The spread field $\psi(t, x, R)$ is a positive random function of two variable x and R, which is decreasing with respect of R.

4.2.1 Non-defaultbale Bond Pricing

Non-defaultable bonds have rating 1. We assume that the dynamics of default-free interest rate are given via a HJM or BGM model (see Heath-Jarrow-Morton [24]). More precisely, the default free zero-coupon bond $D(t, x, 1)$ with face value 1 and time to maturity x is given by:

$$D(t, x, 1) = e^{-l(t,x)}.$$

The function $l(t,x)$ stands for the opposite logarithm of discount factor, which is a convenient structure representation for stochastic modeling. The parameter 1 in $D(t,x,1)$ makes precise that we are dealing with non-defaultable bonds. One has

$$l(t,x) = \int_t^{t+x} f(t,s)ds = y(t,x)x,$$

where $f(t,s)$ is the forward rate at date s and $y(t,x)$ is the zero-coupon yield. We assume that the interest rate dynamics, under a measure $\mathcal{P}$, is given by:

$$dl(t,x) = \mu(t,x,l_t)dt + \sum_{j=1}^{m} \nu^j(t,x,l_t)dZ_t^j,$$

where $Z = (Z_1, \ldots, Z_m)^T$ is an m-dimensional Brownian motion. The drift μ and the volatility factor ν^j not only depend on the time and maturity, but also on the whole yield curve $l_t = l(t,\cdot)$

Remark 4.2.1. *Let $r(t) = f(t,t)$ be the short term rate. It is well known, from Arbitrage Pricing Theorem, that there exists an equivalent probability to the given P (the real probability) which is risk-neutral for non-defaultable bond. In case P is the risk neutral probability, by the celebrated Girsanov theorem, we can get*

$$\mu(t,x,l_t) = f(t,t+x) - r(t) + \frac{1}{2}\sum_{j=1}^{m}(\nu_t^j)^2$$

where $\nu_t^j = \nu^j(t,x,l_t)$.

4.2.2 Spread Field Process

Defaultable discount factors also depend on the spread field process. A bond issued by a company depends on the default-free yield curve and on its yield spread over default-free bond, which is a function of the company rating and of the recovery rate in case of default. The long-ratio between the actual market price and the "would be" default-free value, which may be different for each bond and, in particular, depends on the bond seniority, is itself modeled as a random function $\psi(t,x,R)$, which we call the spread field.

Let $D(t, x, R)$ be the price of a defaultable zero-coupon bond with rating R such that

$$D(t, x, R) = \exp(-l(t, x) - \psi(t, x, R)),$$

where the spread field ψ is defined by:

$$\psi(t, x, R) = \log\left(\frac{D(t, x, 1)}{D(t, x, R)}\right).$$

This is a random function of x and R, which for fixed (t, x), should decrease when R increases and vanish for $R = 1$.

The spread field properties allow us to write it in the form:

$$\psi(t, x, R) = \int_R^1 \varphi(t, x, u)du,$$

where φ is non-negative random field, called the derived spread field. Following the above remark, φ must satisfy $\varphi(t, 0, u) \equiv 0$, with the same comment about payment even possible discounts.

If the company defaults at time t, the value of the bond is a percentage of the default-free bond:

$$D(t, x, 0) = D(t, x, 1)e^{-\chi(t,x)}.$$

Here, the spread field value for $R = 0$ is linked to the recovery rate by the equation:

$$\psi(t, x, 0) = \int_0^1 \varphi(t, x, u)du = \chi(t, x) = \log\frac{D(t, x, 1)}{D(t, x, 0)}.$$

A formal zero recovery rate would correspond to a function ψ that is singular in $R = 0$, so that $\chi = +\infty$.

The dynamics of the derived spread field for fixed (x, u) is given by a multi-factor diffusion:

$$d\varphi(t, x, u) = \gamma(t, x, u, \varphi_t)dt + \sum_{i=1}^{n} \xi_i(t, x, u, \varphi_t)dW_t^i, \tag{4.2}$$

where $W = (W_1, \ldots, W_n)$ is an n-dimensional Brownian motion. In this formulation, the drift γ and the volatility ξ^i may depend on the whole derived spread field $\varphi_t = \varphi(t, \cdot, \cdot)$.

Further assumptions on the correlation between the different Brownian motion are made, i.e.

$$d\langle W^i, B\rangle_t = w^i dt \quad d\langle W^i, Z^j\rangle = \rho^{ij} dt \quad d\langle Z^j, B\rangle = \frac{\rho^{ij}}{w^i} dt, \ \ i \neq j.$$

Let us make the assumption of $\partial_x \gamma$ and $\partial_R \gamma$, for every i, $\partial_x \xi^i$ and $\partial_R \xi^i$ are almost surely bounded and continuous with an at most uniform linear growth with respect to φ. If we assume the same property with the initial derived spread field $\varphi_0(x,u) = \varphi(0,x,u)$, then $\partial_x \varphi$ and $\partial_R \varphi$ remain a.s. bounded and continuous for all times. More discussion on the dynamics of interest rate and derived spread field can be found in [17].

Lemma 4.2.1. *([17, Lemma2.1 p7]) For fixed T, the dynamics of the composed spread process Ψ defined by $\Psi_t = \psi(t, T-t, X(t))$ is given by the following formula, in which $x = T - t$:*

$$\begin{aligned} d\Psi_t = & \int_{X(t)}^{1} d\varphi(t,x,u)du - \varphi(t,x,X(t))dX(t) \\ & - \Big(\int_{X(t)}^{1} \partial_x \varphi(t,x,u)du\Big)dt - d\langle R, \varphi\rangle_t - \frac{1}{2}\partial_R \varphi(t,x,X(t))d\langle R\rangle_t. \end{aligned} \tag{4.3}$$

In this formula, $\langle R, \varphi_t\rangle$ stands for the bracket of $X(t)$ with the process $\varphi(t,x,u)$ for fixed (x,u), evaluated at $u = X(t)$, whereas $\langle R\rangle_t$ is the usual bracket of the process $X(t)$.

4.2.3 Rating Diffusion Process

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be a given probability space, we consider the simplest case which is one bond issuer in the market and assume the issuer of a bond suffer possible default risk. Here the default is evaluated by a "continuous" rating $X(t) \in [0,1], t > 0$ that possesses a feature of reflections between 0 and 1. We propose the rating process $X(t)$, $t \geq 0$ which determined by the following stochastic differential equation of Markovian type

$$dX(t) = \theta X(t)dt + \sigma X(t)dB(t) + d\eta(t) - \bar{\eta}(t), \tag{4.4}$$

where coefficients θ, σ, are positive constants. B is a Brownian motion.

Let D be the domain of $X(t)$ which is $[0,1]$ and ∂D be the boundary of D. $(X(t), \eta(t), \bar{\eta}(t))$ is the solution of (4.4). $\eta(t)$ and $\bar{\eta}(t)$ are increasing processes which increase only when $X(t)$

is on the boundary ∂D. $d\eta(t)$ and $d\bar{\eta}(t)$ act when $X(t) \in \partial D$ and cause reflections. $d\eta(t) > 0$ if and only if $X(t) = 0$, $d\bar{\eta}(t) > 0$ if and only if $X(t) = 1$.

In what follows, we shall discuss the rating processes with two reflecting barriers in more detail. Suppose we are given $\sigma(x) = (\sigma^{i,k}(x)) : R^d \to R^d \times R^r$, and $\theta(x) = (\theta^i(x)) : R^d \to R^d$. Consider the following stochastic differential equation

$$dX^i(t) = \sum_{k=1}^{r} \sigma^{i,k}(X(t))dB^k(t) + \theta^i(X(t))dt, \qquad i = 1, \dots, d.$$

Here we consider the case of a domain with boundary, a diffusion is described by a second order differential operator. For simplicity we only consider diffusion processes on the upper half space R^d_+, $d \geq 2$.

So let D be a convex domain in R^d and $\bar{D}$ its closure. For instance, $D = R^d_+ = \{x = (x^1, x^2, \dots, x^d); x^d \geq 0\}$, $\partial D = \{x \in D; x^d = 0\}$ be the boundary of D and $\dot{D} = \{x \in D; x^d > 0\}$ be the interior of D. Suppose we are given a second order differential operator on D acting on $C^2_k(D)$, where $C^2_k(D)$ is the set of all twice continuously differentiable functions with compact support.

$$Af(x) = \frac{1}{2}\sum_{i,j=1}^{d} a^{i,j}(x)\frac{\partial^2 f}{\partial x^i \partial x^j}(x) + \sum_{i=1}^{d} b^i(x)\frac{\partial f}{\partial x^i}(x),$$

where $a^{ij}(x)$ and $b^i(x)$ are bounded continuous functions on D and $(a^{ij}(x))$ is symmetric and non-negative definite. Assume that a boundary operator of the Wentzell type is also given,it has been introduced by Feller and Wentzell in the context of diffusion processes, see [60], i.e., a mapping from $C^2_k(D)$ to the space of continuous functions on ∂D given as follows:

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{d-1} \alpha^{i,j}(x)\frac{\partial^2 f}{\partial x^i \partial x^j}(x) + \sum_{i=1}^{d-1} \beta^i(x)\frac{\partial f}{\partial x^i}(x) + \delta(x)\frac{\partial f}{\partial x^d}(x) - \rho(x)Af(x),$$

where $x \in \partial D$, and $\alpha^{ij}(x)$, $\beta^i(x)$, $\rho(x)$ are bounded continuous functions on ∂D and such that $(\alpha^{i,j}(x))$ is symmetric and non-negative definite, $\delta(x) \geq 0$ and $\rho(x) \geq 0$. A diffusion process is generated by the pair of operator (A, L), or simply (A, L)-diffusion process.

Next we will formulate a stochastic differential equation which describes an (A, L)-diffusion process. For this, we choose $\sigma(x) = (\sigma^{i,k}(x)) : D \to R^d \otimes R^r$ and $\tau(x) = (\tau^{i,l}(x)) :$

$\partial D \to R^{d-1} \otimes R^s$ which are continuous and

$$a^{i,j}(x) = \sum_{k=1}^{r} \sigma^{i,k}(x)\sigma^{j,k}(x), \qquad i,j = 1,2,\ldots,d,$$

and

$$\alpha^{i,j}(x) = \sum_{l=1}^{s} \tau^{i,k}(x)\tau^{j,k}(x), \qquad i,j = 1,2,\ldots,d-1.$$

Now, let us consider stochastic differential equation with reflection:

$$\begin{aligned} dX^i(t) &= \sum_{k=1}^{r} \sigma^{i,k}(X(t))\chi_{\dot{D}} dB(t)^k + \theta^i(X(t))\chi_{\dot{D}} dt \\ &+ \sum_{l=1}^{s} \tau_1^{i,l}(X(t))\chi_{\partial D} dM_1^l(t) + \beta_1^i(X(t))\chi_{\partial D} d\eta(t) \\ &- \sum_{l=1}^{s} \tau_2^{i,l}(X(t))\chi_{\partial D} dM_2^l(t) + \beta_2^i(X(t))\chi_{\partial D} d\bar{\eta}(t). \end{aligned} \tag{4.5}$$

An intuitive meaning of the equation is as follows. Here $d\eta(t)$, $d\bar{\eta}(t)$, are increasing processes which increase only when $X(t)$ is on the boundary ∂D and is called the local time of $X(t)$ on ∂D. $d\eta(t)$, and $d\bar{\eta}(t)$, act only when $X(t) \in \partial D$ and causes the reflection at ∂D. $\{B^k(t), M^l(t)$ is a orthogonal system of martingales such that $d\langle B^k\rangle(t) = dt$, $k = 1,2,\ldots,r$ and $d\langle M^l\rangle(t) = d\eta(t)$, $d\langle \bar{M}^l\rangle(t) = d\bar{\eta}(t)$, $l = 1,2,\ldots,d$. $d\langle B^k, M^l\rangle_t = 0$. i.e., B is an r-dimensional Brownian motion in the ordinary time, and M is an s-dimensional Brownian motion if the time is measured by the local time.

We assume that if the stochastic differential equation (4.5) satisfies the following conditions: σ and b are linear growth and Lipschitz continuous on D, i.e., there exists a constant $K > 0$ such that

$$|b(x) - b(y)| \le K|x-y|, \|\sigma(x) - \sigma(y)\| \le K|x-y|, x, y \in R^d.$$

$$\|b(x) + \sigma(x)\| \le K(1 + |x|).$$

Then the uniqueness and existence of solutions hold for equation (4.5). A proof of this result can be found in Hiroshi Tanaka [55] (cf. Section 4 p174).

Now consider a special case of (4.5), let $d = 1$, we only consider 1-dimensional condition because the rating $X(t) \in [0,1]$. Then the rating process $X(t)$, $t \ge 0$ is modeled as a reflected diffusion process (4.4) with $D = [0,1]$ and $\partial D = \{0,1\}$.

Our main result is as follows:

Theorem 4.2.1. *Let P be a risk-neutral probability for non-defaultable discount factor $D(t,x,1)$. Assume that $\varphi(t,x,u)$, almost surely and that the process $\tilde{\theta}$ defined by:*

$$\varphi_t\tilde{\theta}X(t) = \Gamma_t - \partial_x\psi_t - \sum_{i=1}^{n}\sigma X(t)\xi_t^i w^i - \frac{1}{2}\varphi_t'\sigma^2 X(t)^2 - \varphi_t\frac{d\eta(t)}{dt} + \varphi_t\frac{d\bar{\eta}(t)}{dt} + \sum_{i=1}^{n}\sum_{i=1}^{m}\nu_t^j\Xi_t^i\rho^{i,j} - \sum_{i=1}^{m}\nu_i^j\varphi_t\sigma X(t)\frac{\rho^{i,j}}{w^i}$$

is such that:

$$\forall T>0, \qquad E_P[\exp(\frac{1}{2}\int_0^T(\theta-\tilde{\theta})^2 dt)] < +\infty. \tag{4.6}$$

Then the probability $\tilde{P}$ defined by

$$\frac{d\tilde{P}}{dP} = \exp\left(\int_0^T(\theta-\tilde{\theta})dB(t) - \frac{1}{2}\int_0^T(\theta-\tilde{\theta})^2 dt\right).$$

is a risk-neutral probability for defaultbale zero-coupon bonds.

Proof. It has been proved in Lemma 4.2.1, the dynamics of the composed spread process $\Psi_t = \psi(t,T-t,X(t))$, in which $x = T-t$. From (4.3), together with (4.4) (4.2), we have

$$\begin{aligned} d\Psi_t &= \int_{X(t)}^1 d\varphi(t,x,u)du - \varphi(t,x,X(t))dX(t) \\ &- (\int_{X(t)}^1 \partial_x\varphi(t,x,u)du) - d\langle R,\varphi\rangle_t - \frac{1}{2}\partial_R\varphi(t,x,X(t))d\langle R\rangle_t \\ &= \int_{X(t)}^1\Big(r(t,x,u,\varphi_t)dt + \sum_{i=1}^{n}\xi^i(t,x,u,\varphi_t)dW_t^i\Big)du \\ &- \varphi(t,x,X(t))\Big(\theta X(t)dt + \sigma X(t)dB(t) + \eta(t) - \bar{\eta}(t)\Big) \\ &- \Big(\int_{X(t)}^1 \partial_x\varphi(t,x,u)du\Big)dt - \sum_{i=1}^{n}\sigma X(t)dB(t)\xi^i(t,x,u,\varphi_t)dW_t^i \\ &- \frac{1}{2}\partial_R\varphi(t,x,X(t))\sigma^2 X(t)^2 dt. \end{aligned} \tag{4.7}$$

Let us now denote:

$$\begin{aligned}
&\nu_t^j = \nu^j(t, x, l_t) \\
&\Gamma_t = \Gamma(t, x, X(t)) = \int_{X(t)}^{1} \gamma(t, x, u, \varphi_t) \\
&\Xi_t^i = \Xi^i(t, x, X(t)) = \int_{X(t)}^{1} \xi^i(t, x, u, \varphi_t) du \\
&\xi_t^i = \xi^i(t, x, X(t), \varphi_t) \qquad \varphi_t = \varphi(t, x, X(t)) \\
&\varphi_t' = \partial_R \varphi(t, x, X(t)) \qquad \partial_x \psi_t = \int_{X(t)}^{1} \partial_x \varphi(t, x, u) du,
\end{aligned}$$

then (4.7) is equivalent to

$$\begin{aligned}
&\Gamma_t dt + \sum_{i=1}^{n} \Xi_t^i dW_t^i - \varphi_t \Big(\theta X(t) dt + \sigma X(t) dB(t) + d\eta(t) - d\bar{\eta}(t) \Big) \\
&- \partial_x \psi_t dt - \sum_{i=1}^{n} \sigma X(t) \xi_t^i w^i dt - \frac{1}{2} \varphi_t' \sigma^2 X(t)^2 dt \\
&= \Big(\Gamma_t - \varphi_t \theta_t X(t) - \partial_x \psi_t - \sum_{i=1}^{n} \sigma X(t) \xi_{i,t} w^i - \frac{1}{2} \varphi_t' \sigma^2 X(t)^2 \Big) dt \\
&- \varphi_t d\eta(t) + \varphi_t d\bar{\eta}(t) + \sum_{i=1}^{n} \Xi_t^i dW_t^i - \varphi_t \sigma X(t) dB(t).
\end{aligned}$$

Using the Itô formula to (4.1), for fixed T, the defaultable discount factor dynamics is given by

$$
\begin{aligned}
\frac{dD_t}{D_t} &= -dl_t - d\Psi_t + \frac{1}{2}d\langle l\rangle_t + \frac{1}{2}d\langle\Psi\rangle_t + \frac{1}{2}d\langle l,\Psi\rangle_t \\
&= -dl_t + \frac{1}{2}d\langle l\rangle_t \\
&- \Big[\Big(\Gamma_t - \varphi_t\theta_t X(t) - \partial_x\psi_t - \sum_{i=1}^{n}\sigma X(t)\xi_t^i w^i - \frac{1}{2}\varphi_t'\sigma^2 X(t)^2\Big)dt - \varphi_t d\eta(t) + \varphi_t) d\bar{\eta}(t) \\
&+ \sum_{i=1}^{n}\Xi_t^i dW_t^i - \varphi_t\sigma X(t)dB(t)\Big] \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{m}\nu_{j,t}\Xi_{i,t}\rho_{i,j}dt - \sum_{j=1}^{m}\nu_t^j\varphi_t X(t)\frac{\rho^{i,j}}{w^i}dt \\
&= -dl_t + \frac{1}{2}d\langle l\rangle_t \\
&+ \Big[-\Gamma_t + \varphi_t\theta_t X(t) + \partial_x\psi_t + \sum_{i=1}^{n}\sigma X(t)\xi_t^i w^i + \frac{1}{2}\varphi_t'\sigma^2 X(t)^2 + \varphi_t\frac{d\eta(t)}{dt} \\
&- \varphi_t\frac{d\bar{\eta}(t)}{dt} + \sum_{i=1}^{n}\sum_{j=1}^{m}\nu_t^j\Xi_t^i\rho_{i,j} - \sum_{j=1}^{m}\nu_t^j\varphi_t X(t)\frac{\rho^{i,j}}{w^i}\Big]dt \\
&+ \text{martingale}.
\end{aligned}
$$

Let us recall that, for P to be a risk-neutral probability for non-defaultable bonds, one must have, for fixed T :

$$-dl_t + \frac{1}{2}d\langle l\rangle_t = X(t) + \text{martinglae}$$

Equating the drift of $\frac{dD_t}{D_t}$ to $X(t)$, we get the "risk-neutral drift" θ of the rating process and the proof is completed. □

The Girsanov theorem shows that, under condition (4.6), $\tilde{P}$ is probability measure equivalent to P and that defaultable zero-coupon bonds are martingales with respect to $\tilde{P}$. We should point out here that the price we obtain is not necessarily "arbitrage price" but the risk premium. We summarise our conclusion as the following:

Theorem 4.2.2. *In the defaultable context, with φ being given and positive a.s., there exists an equivalent probability to the real world probability which is risk neutral for the defaultable bonds.*

Chapter 5

A sufficiency theorem for the path-independent property

5.1 Introduction

The aim of this paper is to derive a link of (Markovian type) semi-linear stochastic differential equations (SDEs) in infinite dimensions to nonlinear partial differential equations (PDEs) of Burgers-KPZ type which gives a characterization of the path-independence property of the density process of Girsanov transformation for the infinite-dimensional SDEs. The above link for finite-dimensional SDEs was considered in [63, 56] where the simple case of one-dimensional SDEs was discussed in [63] in which a (generalized) Burgers equation has been derived from SDEs on R. In [56], a complete link of finite-dimensional SDEs on R^d as well as on connected complete differential manifolds to Burgers-KPZ equations has been established.

The motivation comes from the mathematical study of economics and finance in conjunction with optimization problems. In recent years, due to the necessity of stochastic volatility as the measurement of uncertainty in modeling of financial markets, stochastic differential equations have received huge attention from both theoretical and practical aspects cf. e.g. [29, 39, 44]. The primary point here is to model the price dynamics or the wealth growth by utilising SDEs, after having established a so-called real world probability space (e.g., the seminal paper [4] by Black and Scholes). To an equilibrium financial market, there must exist a so-called risk neutral probability measure which is absolutely continuous with the given real world probability measure and it is pivotal to determine the path-independence property for the associated density process defined by the Radon-Nikodym derivative [25, 26]. It is often encountered in the economical and financial market models that one should consider agents in large scale that there are (at least) countably many stocks are treated together so that their pricing dynamics form an infinite-dimensional SDEs. From the view point of variational calculus, optimization problems – either in the pattern of maximizing the utility functions (and/or profits) or in the formulation of minimizing the cost functions (and/or risk factors) – are in fact linked with the path-independent property of the pricing trajectories, cf. e.g., [20, 67]. Hence, characterizing the relevant path-independence of the SDEs in terms of (non-linear) PDEs would be interesting and useful.

Going a step further, it is well known that a fairly rich class of the large scale systems is modeled by infinite-dimensional Markovian type semi-linear SDEs and the associated scaling limits of such systems are determined by KZP type nonlinear PDEs, cf. e.g. [34, 52, 62]. Thus, it is very natural to reveal an intrinsic link between the infinite-dimensional SDEs and nonlinear Burgers-KPZ type PDEs. In fact, our main result obtained in this paper does provide a direct link between infinite-dimensional stochastic equations and parabolic nonlinear PDEs in a persuasive manner, which shows that certain intrinsic properties of the (infinite) stochastic dynamical systems are indeed characterized by Burgers-KPZ type equations. This indicates in certain sense that the Burgers-KPZ type equations is ubiquitous for infinite systems of stochastically dynamical motions. Actually, this point inspired our investigation of the present work.

In this chapter, we will consider SDEs on a separable Hilbert space. To our aim, we notice that the methods employed in [63] and in [56] are the Itô formula and Girsanov transformation. However, it is not straightforward to have Itô formula in infinite-dimensional so we have to use the finite-dimensional approximation approach here. We will derive a complete link of infinite-dimensional semi-linear SDEs to Burgers-KPZ nonlinear PDEs infinite dimensions. Extensions to more general infinite-dimensional spaces like Banach spaces, multi-Hilbertian spaces as well as locally convex topological vector spaces are interesting and will be considered in the forthcoming works.

The rest of the chapter is organized as follows. In the next section, we first give a brief account of Girsanov transformation of SDEs on (infinite-dimensional) a separable Hilbert space H. Then we prove our main result on the characterization of path-independence of the Girsanov density of the SDEs. The final section is devoted ti a consideration of parabolic stochastic partial differential equations as an example where we demonstrate application of our main result of Section 5.3.

5.2 The sufficiency theorem for the path-independent property

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a given filtered probability space satisfying the usual conditions that $(\Omega, \mathcal{F}, P)$ is a complete probability space and for each $t \geq 0, \mathcal{F}_t$ contains all P-null sets of $\mathcal{F}$ and $\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t$. We use E to denote the expectation with respect to P.

Given a real separable Hilbert space $(H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H)$. Let $\{W_t\}_{t\geq 0}$ be a cylindrical Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with the following expression

$$W_t := W_t(\omega) := \sum_{i=1}^{\infty} \beta_i(t, \omega) e_i, \quad \omega \in \Omega, \ t \in [0, \infty)$$

where $\{\beta_i(t, \omega)\}_{i\geq 1}$ is a family of independent one-dimensional Brownian motions and $\{e_i\}_{i\geq 1}$ is a complete orthonormal basis for H which is fixed throughout the paper. We have

$$E(\langle W_t, x \rangle_H \langle W_s, y \rangle_H) = (t \wedge s) \langle x, y \rangle_H, \quad t, s \in [0, \infty), \quad x, y \in H.$$

Notice that the covariance operator of our cylindrical Brownian motion is just the identity operator I on H.

Let $L(H)$ be the collection of all bounded linear operators $L : H \to H$ equipped with the usual operator norm

$$\|L\| := \sup_{\|x\|=1} \|Lx\|_H.$$

Clearly, $(L(H), \|\cdot\|)$ is a Banach space.

Furthermore, we use $L_{HS}(H)$ for the family of all Hilbert-Schmidt operators $L : H \to H$ endowed with the norm

$$\|L\|_{HS} := (\sum_{i=1}^{\infty} \|Le_i\|_H^2)^{\frac{1}{2}},$$

then $(L_{HS}(H), \|\cdot\|_{HS})$ is a Hilbert space.

Before proceeding further, let us introduce the notion of Fréchet differentiation for infinite-dimensional spaces which is crucial in our paper. We state it in a little general form. Given two Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, we let $L(\mathbb{X}, \mathbb{Y})$ denote the totality of all bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$. $L(\mathbb{X}, \mathbb{Y})$ is a Banch space endowed with the usual operator norm. A function $f : \mathbb{X} \to \mathbb{Y}$ is called *Fréchet differentiable* at $x \in \mathbb{X}$, if there exists a bounded linear operator $A_x : \mathbb{X} \to \mathbb{Y}$ such that

$$\lim_{\|h\|_{\mathbb{X}} \to 0} \frac{\|f(x+h) - f(x) - A_x h\|_{\mathbb{Y}}}{\|h\|_{\mathbb{X}}} = 0.$$

If the limit exists, we write $\nabla f(x) := A_x$ and call it the *Fréchet derivative* of f at x. A function $f : \mathbb{X} \to \mathbb{Y}$ that Fréchet differentiable for any point $x \in \mathbb{X}$ is said to be C^1 if the function

$$\nabla f : x \in \mathbb{X} \mapsto Df(x) \in L(\mathbb{X}, \mathbb{Y})$$

is continuous. Furthermore, $f : \mathbb{X} \to \mathbb{Y}$ is called a C^2 function if $\nabla f : \mathbb{X} \to L(\mathbb{X}, \mathbb{Y})$ is a C^1 function. Moreover, we let $\mathrm{Dom}(\nabla)$ denote the totality of all Fréchet differentiable functions $f : \mathbb{X} \to \mathbb{Y}$.

We would like to follow [58] to introduce the stochastic equation we are concerned. Let $(A, \mathcal{D}(A))$ be a linear, unbounded, negative definite, self-adjoint operator on H generating a contraction C_0-semigroup $\{e^{tA}\}_{t\geq 0}$. Let $L_A(H)$ be the totality of all densely defined closed linear operators $L : H \to H$ with domain $Dom(L) \subset H$ such that for every $t > 0, e^{tA}L$ extends to a unique Hilbert-Schmidt operator from H to H, while we use the same notation for the extension so $e^{tA}L \in L_{HS}(H)$. Namely,

$$L_A(H) := \{L : H \to H \mid e^{tA}L \in L_{HS}(H), \ \forall t > 0\}.$$

We endow $L_A(H)$ with the σ-algebra induced by the family

$$\{L \to < e^{tA}Lx, y >_H \mid t > 0, x, y \in H\}$$

from $\mathcal{B}(R)$ so that $L_A(H)$ is a measurable space.

We are concerned with the following initial value problem for a semi-linear stochastic differential equation on H

$$\begin{cases} dX_t = \{AX_t + b(t, X_t)\}dt + \sigma(t, X_t)dW_t, & t > 0 \\ X_0 = x \in H, \end{cases} \tag{5.1}$$

where $b : [0, \infty) \times H \to H$ and $\sigma : [0, \infty) \times H \to L_A(H)$ are measurable mappings. In this paper, we require the two coefficients fulfill further that $b : [0, \infty) \times H \to H$ and $(t, x) \in [0, \infty) \times H \mapsto e^{tA}\sigma(t, x) \in L_{HS}(H)$ are C^1 with respect to the first variable and C^2 with respect to the second variable respectively. Here we would like to point out that one should interpret $([0, \infty), |\cdot|)$ and $(R, |\cdot|)$ as Banach spaces and the differentiation with respect to $t \in [0, \infty)$ or for R-valued functions on any Banach space follows from above description. Throughout the paper we shall assume the following two conditions:

(H1) Assume that $-A$ has discrete spectrum with eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \ldots$$

counting multiplicities such that

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j} < \infty.$$

We let $\{e_j\}_{j \in N}$ be the corresponding eigen-basis of $-A$ throughout the paper.

(H2) There exist a constant $\epsilon \in (0, 1)$ and an increasing function $L : [0, \infty) \to (0, \infty)$ such that

$$\sup_{t \in [0,T]} \left\{ \|b(t, 0)\|_H^2 + \int_0^t \|e^{(t-s)A}\sigma(s, 0)\|_{HS}^2 s^{-\epsilon} ds \right\} < \infty, \quad \forall T > 0$$

and

$$\|b(t,x)-b(t,y)\|_H + \|e^{tA}\left(\sigma(t,x)-\sigma(t,y)\right)\|_{HS} \le L(t)\|x-y\|_H, \quad \forall t\ge 0, \quad \forall x,y\in H.$$

Remark 5.2.1. *Under the assumption (H1), it clear that the space $L_A(H)$ allows to have invertible operators from H to H, such as the identity operator.*

It is well known by [12, 6] and most recently [58] that (H1) and (H2) imply the existence and uniqueness of the mild solution to (1.1), that is,for any $x \in H$ there exists a unique H-valued $\{\mathcal{F}_t\}_{t\ge 0}$-adapted continuous process $X_t, t \ge 0$, such that $\mathbb{P}$-a.s.

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A} b(s,X_s)ds + \int_0^t e^{(t-s)A}\sigma(s,X_s)dW_s\,, \quad t\ge 0. \tag{5.2}$$

Moreover, we have

$$E\left(\sup_{t\in[0,T]} \|X_t\|_H^2\right) < \infty, \quad \forall T>0.$$

For our purpose, we need a finite dimensional approximation to (5.1) so that we can link the characterization theorem for finite-dimensional SDEs obtained in [63, 56] to the present infinite-dimensional problem (5.1). To be more precise, we want to set a Galerkin approximation to (5.1), which is classical and efficient to get existence an uniqueness results for infinite-dimensional equations (see, e.g., Chapter 6 of [6]). So let us follow [6] to set up the Galerkin approximation for (5.1). We notice that our assumption (H1) indicates that the operator A satisfies the coercivity condition and the monotonicity condition in [6] (see page 178 there). For simplicity, we assume that $\sigma : [0,\infty)\times H \to L_A(H)$ is diagonal with respect to the eigen-base $\{e_i\}_{i\ge 1}$.

For any $n \ge 1$, let $\pi_n : H \to H_n := \text{span}\{e_1,\cdots,e_n\}$ be the (orthogonal) projection operator, that is

$$\pi_n x := \sum_{i=1}^n \langle x, e_i\rangle_H e_i, \quad x\in H.$$

We note that the projection operator π_n commutes with the semigroup $e^{tA}, t\ge 0$. Furthermore, we let $A_n := A\mid_{H_n}, b_n := \pi_n b$ and $\sigma_n := \pi_n\sigma$. We consider the following stochastic differential equation in H_n

$$\begin{cases} dX_t^n = \{A_n X_t^n + b_n(t, X_t^n)\}dt + \sigma_n(t, X_t^n)dW_t, \\ X^n(0) = \pi_n x. \end{cases} \tag{5.3}$$

As illustrated in [58], the assumption (H2) implies that the coefficients b_n and σ_n fulfill the usuall growth and Lipschitz conditions so that there exists a unique strong solution $X_t^n \in H_n, t \in [0, \infty)$ to (5.3). Furthermore, by Theorem 3.1.2 of [58], one has

$$\lim_{n\to\infty} E\|X_t^n - X_t\|_H^2 = 0, \quad t \geq 0. \tag{5.4}$$

Before we present our result, let us recall the Girsanov theorem in infinite-dimensions (see 10.2.1 page 290 in [12]). Notice that the covariance operator of our cylindrical Brownian motion $\{W_t\}_{t\geq 0}$ is the identity operator I on $(H, \|\cdot\|_H)$. One can then determine the infinite dimensional Brownian motion on Itô's universal Wiener space with the reproducing kernel space H, cf. e.g. [22].

Next, assume that $\gamma : [0, \infty) \times H \to H$ is measurable such that for every $T > 0$ (note here T could take to be ∞ as well)

$$E\left(\exp\left[\frac{1}{2}\int_0^T \|\gamma(s, X_s)\|_H^2 ds\right]\right) < \infty, \tag{5.5}$$

which is known as the Novikov condition. Then the process

$$\tilde{W}_t := W_t - \int_0^t \gamma(s, X_s)ds, \quad t \in [0, T]$$

is a cylindrical Brownian motion (i..e, having the identity operator I on H as its covariance operator) with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$ on the probability space $(\Omega, \mathcal{F}, \tilde{P}_T)$, where $\tilde{P}_T$ is defined via the Radon-Nikodym derivative

$$\frac{d\tilde{P}_T}{dP}(\omega) := \exp\left(\int_0^T \langle\gamma(s, X_s(\omega)), dW_s(\omega)\rangle_H - \frac{1}{2}\int_0^T \|\gamma(s, X_s(\omega))\|_H^2 ds\right).$$

We refer the reader, e.g., to Proposition 10.17 of [12] (see page 295 there) for an alternative sufficient condition instead of of (5.5). The relation between W_t and $\tilde{W}_t$ in the stochastic differentiation form is

$$d\tilde{W}_t = dW_t - \gamma(t, X_t)dt$$

from which, in terms of the new cylindrical Brownian motion $\tilde{W}_t$, the SDE in (5.1) reads

$$dX_t = \{AX_t + b(t, X_t) + \sigma(t, X_t)\gamma(t, X_t)\}dt + \sigma(t, X_t)d\tilde{W}_t, \quad t \in (0, T].$$

Furthermore, if $\sigma(t, x)$ is invertible for each $(t, x) \in [0, \infty) \times H$, we can specify

$$\gamma(t, x) := -\sigma^{-1}(t, x)b(t, x), \quad (t, x) \in [0, \infty) \times H.$$

Thus, if the coefficients b and σ in our equation (5.1) fulfill the following condition

$$E\left(\exp\left[\frac{1}{2}\int_0^T \|\sigma^{-1}(s, X_s)b(s, X_s)\|_H^2 ds\right]\right) < \infty, \quad \forall T > 0$$

or equivalently,

$$E\left(\exp\left[-\int_0^T \langle\sigma^{-1}(s, X_s)b(s, X_s), dW_s\rangle_H - \frac{1}{2}\int_0^T \|\sigma^{-1}(s, X_s)b(s, X_s)\|_H^2 ds\right]\right) = 1$$

for $T > 0$, then our SDE in (5.1) becomes simply

$$dX_t = AX_t dt + \sigma(t, X_t)d\tilde{W}_t, \quad t \in (0, T].$$

From now on, we assume further the following condition throughout the rest of the paper:

(H3) The operator $\sigma(t, x)$ is invertible for each $(t, x) \in [0, \infty) \times H$ and the two coefficients b, σ in Equation (1.1) fulfill

$$E\left(\exp\left\{\frac{1}{2}\int_0^T \|\sigma^{-1}(t, X_t)b(t, X_t)\|_H^2 dt\right\}\right) < \infty, \quad \forall T > 0.$$

To summarize the above discussion, we conclude that under (H1), (H2) and (H3), the *Girsanov density*

$$\begin{aligned} \frac{d\tilde{P}_t}{dP}(\omega) \;:=\; & \exp\Big\{-\int_0^t \langle\sigma^{-1}(s, X_s(\omega))b(s, X_s(\omega)), dW_s(\omega)\rangle_H \\ & -\frac{1}{2}\int_0^t \|\sigma^{-1}(s, X_s(\omega))b(s, X_s(\omega))\|_H^2 ds\Big\}, \quad t \geq 0 \end{aligned} \tag{5.6}$$

is a well-defined process for the SDE in (5.1).

We are now in the position to state our main result. It gives sufficient conditions of the path-independence of the Girsanov density process for (infinite-dimensional) SDEs on separable Hilbert spaces. To illustrate our main result in its simplest manner, let us assume that the operator $\sigma(t,x)$ is diagonal for each $(t,x) \in [0,\infty) \times H$ with respect to the orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$, i.e.,

$$\sigma(t,x) = diag(\sigma_i(t,x))_{i\in\mathbb{N}}$$

with $(\sigma_i(t,x))_{i\in\mathbb{N}}$ being, for each $(t,x) \in [0,\infty) \times H$, an (infinite dimensional) $\mathbb{R}^\infty$-vector with respect to the orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$.

Theorem 5.2.1. *Assume (H1), (H2), (H3) and let* $v : [0,\infty)\times H \to \mathbb{R}$ *be in* $C_b^{1,2}([0,\infty)\times H)$ *such that* $[\nabla v(t,\cdot)]x : H \to H \in Dom(A)$ *for any* $(t,x) \in [0,\infty) \times H$ *and* $\|A\nabla v(t,\cdot)\|_H$ *is bounded locally and uniformly in* $t \in [0,\infty)$. *If* v *satisfies*

$$\frac{\partial}{\partial t}v(t,x) = -\frac{1}{2}\{Tr[(\sigma\sigma^*)\nabla^2 v](t,x) + \|\sigma^*\nabla v\|_H^2(t,x)\} - \langle x, A\nabla v(t,x)\rangle_H \qquad (5.7)$$

and

$$b(t,x) = [(\sigma\sigma^*)\nabla v](t,x), \quad \forall (t,x) \in [0,\infty) \times H, \qquad (5.8)$$

then the Girsanov density (5.6) *for* (1.1) *satisfies the following path-indendenpent property*

$$\frac{d\tilde{P}_t}{dP} = \exp\{v(0,X_0) - v(t,X_t)\}, \quad t \geq 0. \qquad (5.9)$$

Proof. We note that showing (5.9) is equivalent to verifying the following

$$\begin{aligned} v(t,X_t) &= v(0,X_0) + \frac{1}{2}\int_0^t \left\|\sigma^{-1}(s,X_s)b(s,X_s)\right\|_H^2 ds \\ &\quad + \int_0^t \langle(\sigma^{-1}(s,X_s)b(s,X_s), dW_s\rangle_H . \end{aligned} \qquad (5.10)$$

However, unlike the procedure carried out in [56], we are not able to apply Itô formula [1] directly to the real-valued function $v(t,X_t)$ of the infinite dimensional process $\{X_t, t \geq 0\}$, due to the fact that our σ in (5.1) is not Hilbert-Schmidt. Here we will use the Galerkin

[1]We refer the reader to, e.g., in [12, page 105, Theorem 4.17 of Chapter 4] or [6, page 153, Theorem 4.1 of Chapter 6] for the (infinite dimensional) Itô formula.

approximation (5.3) associated with our initial value problem (1.1) where we have derived an approximation sequence $\{X_t^n, t \geq 0\}_{n\in\mathbb{N}}$ for the solution $\{X_t, t \geq 0\}$ of (5.1), that is, $\{X_t^n, t \geq 0\}$, is indeed n-dimensional (semimartingale) process (i.e., the process $X_t^n, t \geq 0$, lives on the finite dimensional space H_n for each n, respectively) and the sequence $\{X_t^n, t \geq 0\}_{n\in\mathbb{N}}$ converges to $\{X_t, t \geq 0\}$ in $\|\cdot\|_H^2$. Furthermore, it clear that for each $t \geq 0, \|\cdot\|_H - \lim_{n\to\infty} v(t, \pi_n x) = v(t, x)$ so

$$\lim_{n\to\infty} v(t, X_t^n) = v(t, X_t).$$

Hence, we turn to the expression $v(t, X_t^n)$, which, for each fixed $n \in N$, is a real-valued function of the finite dimensional process $X_t^n, t \geq 0$ and we can apply Itô formula to $v(t, X_t^n)$. To be more precise, viewing the expression $v(t, X_t^n)$ as the composition of the deterministic $C^{1,2}$-function $v : [0, \infty) \times H_n \to R$ with the finite dimensional, continuous semi-martingale X_t^n with expression (i.e., from our previous (5.3))

$$dX_t^n = [A_n X_t^n + b_n(t, X_t^n)]dt + \sigma_n(t, X_t^n)dW_t, \quad t \geq 0\,,$$

we can apply the Itô formula (e.g., [12, page 105, Theorem 4.17 of Chapter 4] or [6, page 153, Theorem 4.1 of Chapter 6]) to $v(t, X_t^n)$ with notice that here our W_t is (standard) cylindrical Brownian motion (with mean zero and covariance given by identity, which yields the following derivation

$$\begin{aligned}
v(t, X_t^n) &= v(0, \pi_n X_0) + \int_0^t \langle(\nabla_n v(s, X_s^n), \sigma_n(s, X_s^n))dW_s\rangle_H \\
&\quad + \int_0^t [\frac{\partial}{\partial s} v(s, X_s^n) + \langle(\nabla_n v(s, X_s^n), A_n X_s^n + b_n(s, X_s^n))\rangle_H] ds \\
&\quad + \frac{1}{2}\int_0^t Tr[\nabla_n^2 v(s, X_s^n)(\sigma_n(s, X_s^n)(Id)^{\frac{1}{2}})(\sigma_n(s, X_s^n)(Id)^{\frac{1}{2}})^* ds \\
&= v(0, \pi_n X_0) + \int_0^t \langle \sigma_n^*(s, X_s^n)\nabla_n v(s, X_s^n), dW_s\rangle_H \\
&\quad + \int_0^t [\frac{\partial}{\partial s} v(s, X_s^n) + \langle(\nabla_n v(s, X_s^n), A_n X_s^n + b_n(s, X_s^n))\rangle_H] ds \\
&\quad + \frac{1}{2}\int_0^t Tr[(\sigma_n \sigma_n^*)(s, X_s^n)\nabla_n^2 v(s, X_s^n)] ds \qquad (5.11)
\end{aligned}$$

where $\nabla_n := \sum_{j=1}^n \nabla_{e_j} e_j$, $\nabla_{e_j} := \langle \nabla, e_j \rangle_H, 1 \leq j \leq n$, and we have used the following identity in the above derivation

$$\langle \nabla_n v(s, X_s^n), \sigma_n(s, X_s^n) dW_s \rangle_H = \langle \sigma_n^*(s, X_s^n) \nabla_n v(s, X_s^n), dW_t \rangle_H .$$

By our assumptions on v and that the operator A is self-adjoint, we have

$$\lim_{n\to\infty} \int_0^t \langle [\sigma_n^* \nabla_n v](s, X_s^n), dW_s \rangle_H = \int_0^t \langle [\sigma^* \nabla v](s, X_s), dW_s \rangle_H ,$$

$$\lim_{n\to\infty} \int_0^t \langle \nabla_n v(s, X_s^n), A_n X_s^n \rangle_H ds = \int_0^t \langle A \nabla v(s, X_s), X_s \rangle_H ds ,$$

$$\lim_{n\to\infty} \int_0^t \langle \nabla_n v(s, X_s^n), b_n(s, X_s^n) \rangle_H ds = \int_0^t \langle \nabla v(s, X_s), b(s, X_s) \rangle_H ds ,$$

$$\lim_{n\to\infty} \int_0^t Tr[(\sigma_n \sigma_n^*)(s, X_s^n) \nabla_n^2 v(s, X_s^n)] ds = \int_0^t Tr[(\sigma \sigma^*)(s, X_s) \nabla^2 v(s, X_s)] ds$$

and

$$\lim_{n\to\infty} \frac{\partial}{\partial s} v(s, X_s^n) = \frac{\partial}{\partial s} v(s, X_s) .$$

Therefore, letting $n \to \infty$, we get from (5.11) for any fixed $t > 0$

$$\begin{aligned} v(t, X_t) &= v(0, X_0) + \int_0^t \langle \sigma^*(s, X_s) \nabla v(s, X_s), dW_s \rangle_H \\ &\quad + \int_0^t \Big[\frac{\partial}{\partial s} v(s, X_s) + \langle (\nabla v(s, X_s), b(s, X_s)) \rangle_H + \langle A \nabla v(s, X_s), X_s \rangle_H \Big] ds \\ &\quad + \frac{1}{2} \int_0^t Tr[(\sigma \sigma^*)(s, X_s) \nabla^2 v(s, X_s)] ds . \end{aligned} \tag{5.12}$$

Now from our assumption (5.8), we get

$$\begin{aligned} \|\sigma^* \nabla v\|_H^2(t, x) &= \langle [\sigma^* \nabla v](t, x), [\sigma^* \nabla v](t, x) \rangle_H \\ &= \langle [(\sigma \sigma^*) \nabla v](t, x), \nabla v(t, x) \rangle_H \\ &= \langle b(t, x), \nabla v(t, x) \rangle_H, \quad (t, x) \in [0, \infty) \times H \end{aligned} \tag{5.13}$$

and

$$\|\sigma^* \nabla v\|_H^2(t, x) = \|\sigma^{-1} b\|_H^2(t, x) . \tag{5.14}$$

Putting the identity (5.13) to (5.7) yields

$$\frac{\partial}{\partial t} v(t, x) = -\frac{1}{2} Tr[(\sigma \sigma^*) \nabla^2 v](t, x) - \frac{1}{2} \langle b(t, x), \nabla v(t, x) \rangle_H - \langle x, A \nabla v(t, x) \rangle_H$$

and further along the path $X_s, s \geq 0$

$$\frac{\partial}{\partial s}v(s,X_s) = -\frac{1}{2}Tr[(\sigma\sigma^*)\nabla^2 v](s,X_s) - \frac{1}{2}\langle b(s,X_s), \nabla v(s,X_s)\rangle_H - \langle X_s, A\nabla v(s,X_s)\rangle_H . \tag{5.15}$$

and by (5.14)

$$\|\sigma^*\nabla v\|_H^2(t,X_t) = \|\sigma^{-1}b\|_H^2(t,X_t) . \tag{5.16}$$

Putting (5.15) and (5.16) into (5.12), we obtain

$$\begin{aligned} v(t,X_t) &= v(0,X_0) + \int_0^t \langle \sigma^*(s,X_s)\nabla v(s,X_s), dW_s\rangle_H + \frac{1}{2}\int_0^t \langle(\nabla v(s,X_s), b(s,X_s))\rangle_H] ds \\ &= v(0,X_0) + \frac{1}{2}\int_0^t \|\sigma^{-1}(s,X_s)b(s,X_s)\|_H^2 ds + \int_0^t \langle(\sigma^{-1}(s,X_s)b(s,X_s), dW_s\rangle_H \end{aligned}$$

which is the exact (5.10) we wanted. This completes the proof.

□

We end up this section with two remarks on a link from finite-dimensional SDEs to infinite-dimensional SDEs.

Remark 1 Let $n \in \mathbb{N}$ be fixed. For Equation (5.7), we let $v(t,x)$ depend on the first n components of $x = (x_1, x_2, ..., x_n, ...) \in H$, that is

$$v(t,x) := v(t, x_1, x_2, \ldots, x_n).$$

Clearly, this is the similar to the case of finite-dimensions situation considered in [30]. In fact, for $x \in H_n$ recall that $Ae_i = -\lambda_i e_i$ (see our assumption (H1)), so we have

$$-\langle Ax, \nabla v(t,x)\rangle_H = \sum_{i=1}^n \lambda_i \frac{\partial}{\partial x_i} v(t,x), \quad \forall (t,x) \in [0,\infty) \times H_n. \tag{5.17}$$

Furthermore, since for $i > n$

$$\frac{\partial}{\partial x_i} v(t,x) = \frac{\partial^2}{\partial x_i^2} v(t,x) = 0,$$

we have for $b = (b_n^1, b_n^2, ..., b_n^n, b_n^{n+1}, ...)$

$$b^i(t,x) = \sigma_i(t,x)^2 \frac{\partial}{\partial x_i} v(t,x) \tag{5.18}$$

and

$$\mathrm{Tr}[(\sigma\sigma^*)\nabla^2 v](t,x) = \sum_{i=1}^{n} \sigma_i(t,x)^2 \frac{\partial^2}{\partial x_i^2} v(t,x), \qquad x \in H_n. \tag{5.19}$$

Similarly, we set $\sigma_n(t,x) = \mathrm{diag}((\sigma_n)_i(t,x))$,

$$\|\sigma_n^* \nabla v\|_H^2(t,x) = \sum_{i=1}^{n} \sigma_i^2(t,x)(\frac{\partial}{\partial x_i} v(t,x))^2 \quad x \in H_n. \tag{5.20}$$

Combining (5.17), (5.18), (5.19) and (5.20), the equation (5.7) for such special $v : [t,x) \times H^n \to R$ then becomes

$$\begin{aligned}\frac{\partial}{\partial t} v(t,x) = & -\frac{1}{2}\Big\{\sum_{i=1}^{n} \sigma_i(t,x)^2 \frac{\partial^2}{\partial x_i^2} v(t,x) + \sum_{i=1}^{n} \sigma_i^2(t,x)(\frac{\partial}{\partial x_i} v(t,x))^2\Big\} \\ & + \sum_{i=1}^{n} \lambda_i \frac{\partial}{\partial x_i} v(t,x).\end{aligned} \tag{5.21}$$

Moreover, letting $n \to \infty$, we arrive the straightforward infinite dimensional analogy of the Burgers-KPZ equation

$$\begin{aligned}\frac{\partial}{\partial t} v(t,x) = & -\frac{1}{2}\Big\{\sum_{i=1}^{\infty} \sigma_i(t,x)^2 \frac{\partial^2}{\partial x_i^2} v(t,x) + \sum_{i=1}^{\infty} \sigma_i^2(t,x)(\frac{\partial}{\partial x_i} v(t,x))^2\Big\} \\ & + \sum_{i=1}^{\infty} \lambda_i \frac{\partial}{\partial x_i} v(t,x), \quad (t,x) \in [0,\infty) \times H.\end{aligned} \tag{5.22}$$

The link to the Burgers-KPZ equation obtained in [63] (as well as from the one-dimensional equation derived in [56]) is that at there $W_t, t \geq 0$, is the standard Brownian motion with mean zero and covariance being the identity matrix, while as here our $W_t, t \geq 0$, is the (standard) cylindrical Brownian motion whose finite dimensional projects are just the standard Brownian motion with mean zero and identity matrix covariance. It would be of interest to study infinite dimensional SDEs driven by cylindrical Wiener processes with more general covariance operators Q in the framework of abstract Wiener spaces (cf., e.g., [12, 6, 46, 58]). We will consider this problem in our forthcoming work.

Remark 2 Let $R : H \to L_{HS}(H)$ be a fixed operator. For $m \in N$, let $R_m : [0,\infty) \times H \mapsto R_m(t,x) \in L_{HS}(H)$ be bounded, i.e.,

$$\sup_{(t,x)\in[0,\infty)\times H} \|R_m(t,x)\|_{HS} < \infty.$$

We set for the $\sigma(t,x) \in L_{HS}(H), (t,x) \in [0,\infty) \times H$, in our Theorem 2.1 as the following perturbation

$$\sigma^m(t,x) := R + 2^{-m} R_m(t,x), \quad (t,x) \in [0,\infty) \times H.$$

That is, under the given orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$, the dependence of $\sigma^m(t,x)$ on the m-th coordinate $x_m = \langle x, e_m \rangle$ becomes weaker and weaker as m goes to sufficiently large and $\lim_{m\to\infty} \|\sigma^m(t,x) - R\|_{HS} = 0$. Next, we denote

$$(\sigma_i^m(t,x))_{i\in\mathbb{N}} := \text{diag}\left((\sigma^m(t,x))_{\mathbb{N}\times\mathbb{N}}\right) = \text{diag}\left((R + 2^{-m} R_m(t,x))_{\mathbb{N}\times\mathbb{N}}\right)$$

i.e., the real-valued coordinate

$$\sigma_i^m(t,x) := (R + 2^{-m} R_m(t,x))_{ii}$$

with $\lim_{m\to\infty} \sigma_i^m(t,x) = \langle Re_i, Re_i \rangle =: r_i \in \mathbb{R}$. Then Equation (5.7) in Theorem 5.2.1 for the $v^m(t,x)$ reads

$$\begin{aligned}\frac{\partial}{\partial t} v^m(t,x) = & -\frac{1}{2}\Big\{ \sum_{i=1}^{\infty} ((R + 2^{-m} R_m(t,x))_{ii})^2 \frac{\partial^2}{\partial x_i^2} v^m(t,x) \\ & + \sum_{i=1}^{\infty} ((R + 2^{-i} R_i(t,x))_{ii})^2 (t,x) (\frac{\partial}{\partial x_i} v^m(t,x))^2 \Big\} \\ & + \sum_{i=1}^{\infty} \lambda_i \frac{\partial}{\partial x_i} v^m(t,x).\end{aligned}$$

As $m \to \infty$, we have the real-valued (point wise) limit $v(t,x) := \lim_{m\to\infty} v^m(t,x)$ which satisfies the following infinite-dimensional Burger-KPZ equation (with constant coefficients)

$$\frac{\partial}{\partial t} v(t,x) = -\frac{1}{2}\Big\{ \sum_{i=1}^{\infty} r_i^2 \frac{\partial^2}{\partial x_i^2} v(t,x) + \sum_{i=1}^{\infty} (r_i^2 \frac{\partial}{\partial x_i} v(t,x))^2 \Big\} + \sum_{i=1}^{\infty} \lambda_i \frac{\partial}{\partial x_i} v(t,x).$$

5.3 Application to parabolic SPDEs

In this final section, we will consider an example of space time inhomogeneous parabolic SPDEs. Here, we take for granted the familiarity with the introductory account on SPDEs presented e.g. in [57, 6] or [46]. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be the given probability set-up as in Section 5.2. We consider the following problem for a parabolic SPDE on the bounded space domain $[0,1] \subset R$

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + \phi(t,x,u(t,x)) + \psi(t,x,u(t,x))\dfrac{\partial^2 B}{\partial t \partial x}(t,x), & t > 0, x \in (0,1), \\ u(t,0) = u(t,1) = 0, & t \geq 0, \\ u(0,x) = u_0(x), & x \in [0,1], \end{cases} \tag{5.23}$$

where $\phi, \psi : [0,\infty) \times [0,1] \times \mathbb{R} \to \mathbb{R}$ are space time inhomogeneous coefficients, and $\{B(t,x)\}_{(t,x)\in[0,\infty)\times[0,1]}$, is a Brownian sheet on $[0,\infty) \times [0,1]$. The heuristic derivative $\frac{\partial B}{\partial t \partial x}$ is interpreted as the space time white noise, which can be made rigorously, e.g., by utilizing generalized functions ([57]).

It is sometimes also convenient, cf. e.g., [6, 12], to link the space time white noise to an $L^2([0,1])$-valued cylindrical Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. Let us elucidate this point a bit here. First, let $B(ds, dz)$ be such that

$$B(t,x) = \int_0^t \int_0^x B(ds, dz), \quad \forall (t,x) \in [0,\infty) \times [0,1].$$

Next, it is clearly that the Hilbert space $H := L^2([0,1])$ is separable. Let $A := \frac{\partial^2}{\partial x^2}$ be the one-dimensional Laplace operator on $[0,1]$ with Dirichlet boundary condition so its domain is $D(A) = H^2([0,1]) \bigcap H_0^1([0,1])$, where $H^k([0,1])$ stands for the L^2-Sobolev space of order k and $H_0^k[0,1]()$ is the closure of $C_0^\infty([0,1])$ in H^k, for $k = 1, 2$. We denote by $\{\theta_n\}_{n\in\mathbb{N}}$ the complete orthonormal system in H consisting of the eigenfunctions of A, which is given by

$$\theta_n(x) := \sqrt{2}\sin(n\pi x), \quad n \in \mathbb{N}$$

so that $A\theta_n(x) = -n^2\pi^2\theta_n(x)$. Then

$$W_t := \sum_{n=1}^{\infty} \left(\int_0^t \int_0^1 e_n(x) B(ds, dx) \right) \theta_n,$$

defines A-cylindrical Brownian motion on H (i.e., with covariance $Q = A$).

The problem (5.23) is sovlable with a unique strong solution under the following assumption on the coefficients (cf. e.g., Chapter 6 of [6] or Chapter 7 of [12])

I) The coefficients ϕ, ψ are Lipschitz continuous with linear growth in the sense that there exists $C > 0$ such that

$$|\phi(t,x,z)|^2 + |\psi(t,x,z)|^2 \leq C(1+|z|^2),$$

and

$$|\phi(t,x,z_1) - \phi(t,x,z_2)|^2 + |\psi(t,x,z) - \psi(t,x,z_2)|^2 \leq C|z_1 - z_2|^2$$

hold for all $(t,x) \in [0,\infty) \times [0,1]$ and for abitrarily given $z, z_1, z_2 \in \mathbb{R}$;

II) The diffusion coefficient ψ is uniformly bounded from below and above, i.e., there exist positive constants C_1 and C_2 such that for all $z \in \mathbb{R}$

$$C_1 \leq |\psi(t,x,z)| \leq C_2$$

holds for all $(t,x,z) \in [0,\infty) \times [0,1] \times \mathbb{R}$.

If I) is fulfilled, one can show that (5.23) has a unique (global) mild solution $u(t,x), t \geq 0, x \in [0,1]$, i.e., u satisfies the following mild equation

$$\begin{aligned} u(t,x) = &\int_0^1 p(t,x,y)u_0(y)dy + \int_0^t \int_0^1 p(t-s,x,y)\phi(s,y,u(s,y))dsdy \\ &+ \int_0^t \int_0^1 p(t-s,x,y)\psi(s,y,u(s,y))B(ds,dy), \end{aligned}$$

with the property that $u(t) := u(t,\cdot) : [0,1] \to \mathbb{R} \in L^2([0,1]) + H$

$$E\Big[\sup_{t\in[0,\infty} \|u(t)\|_H^2\Big] < \infty.$$

where $p(t,x,y)$ stands for the fundamental solution of $\frac{\partial}{\partial t} - A$.

Now we want to reformulate the equation (5.23) in its abstract form. To this end, we set

$$X_t := u(t,\cdot), \quad b(t,X_t) := \phi(t,\cdot,u(t,\cdot)), \quad \sigma(t,X_t)(v) := \psi(t,\cdot,u(t,\cdot))v(t,\cdot) \tag{5.24}$$

for $u(t,\cdot), v(t,\cdot) \in H$ for any $t \geq 0$. Then, Equation (5.23) becomes

$$\begin{cases} dX_t = \{AX_t + b(t,X_t)\}dt + \sigma(t,X_t)dW_t, \quad t \geq 0 \\ X_0 = u_0 \in H, \end{cases} \tag{5.25}$$

which is exactly in the form of (5.1). Therefore, our Theorem 2.1 goes to verbatim for characterizing the path-independent property of the Girsanov density process for (5.25), which can be further transferred to (5.23) via the links (5.24) in the straightforward manner.

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