Paper:
Wang, F. & Wu, B. (2018). Pointwise characterizations of curvature and second fundamental form on Riemannian manifolds. *Science China Mathematics, 61*(8), 1407-1420.
http://dx.doi.org/10.1007/s11425-017-9296-8

SCIENCE CHINA
Mathematics

• ARTICLES •

https://doi.org/10.1007/s11425-017-9296-8

Pointwise characterizations of curvature and second fundamental form on Riemannian manifolds

Fengyu Wang[1,2] & Bo Wu[3,*]

[1]*Center for Applied Mathematics, Tianjin University, Tianjin* 300072*, China;*
[2]*Department of Mathematics, Swansea University, Singleton Park SA*2 8*PP, UK;*
[3]*School of Mathematical Sciences, Fudan University, Shanghai* 200433*, China*

Email: wangfy@tju.edu.cn, wubo@fudan.edu.cn

Received October 7, 2017; accepted April 3, 2018

Abstract Let M be a complete Riemannian manifold possibly with a boundary ∂M. For any C^1-vector field Z, by using gradient/functional inequalities of the (reflecting) diffusion process generated by $L := \Delta + Z$, pointwise characterizations are presented for the Bakry-Emery curvature of L and the second fundamental form of ∂M if it exists. These characterizations extend and strengthen the recent results derived by Naber for the uniform norm $\|\mathrm{Ric}_Z\|_\infty$ on manifolds without boundaries. A key point of the present study is to apply the asymptotic formulas for these two tensors found by the first author, such that the proofs are significantly simplified.

Keywords curvature, second fundamental form, diffusion process, path space

MSC(2010) 58J60, 58J58

Citation: Wang F Y, Wu B. Pointwise characterizations of curvature and second fundamental form on Riemannian manifolds. Sci China Math, 2018, 61, https://doi.org/10.1007/s11425-017-9296-8

1 Introduction

Let M be a d-dimensional complete Riemannian manifold possibly with a boundary ∂M. Let $L = \Delta + Z$ for a C^1 vector field Z. We intend to characterize the Bakry-Emery curvature $\mathrm{Ric}_Z := \mathrm{Ric} - \nabla Z$ and the second fundamental form $\mathbb{I}$ of the boundary ∂M using the (reflecting) diffusion process generated by L. When $\partial M = \emptyset$, we set $\mathbb{I} = 0$.

There are many equivalent characterizations for the (pointwise or uniform) lower bound of Ric_Z and $\mathbb{I}$ using gradient/functional inequalities of the (Neumann) semigroup generated by L, see e.g., [13] and the references within. However, the corresponding upper bound characterizations are still open. It is known that for stochastic analysis on the path space, one needs conditions on the norm of Ric_Z, see [3–6, 8, 11, 12] and the references within. Recently, Naber [7, 10] proved that the uniform bounded condition on Ric_Z for $Z = -\nabla f$ is equivalent to some gradient/functional inequalities on the path space, and thus clarified the necessity of bounded conditions used in the above mentioned references. In this study, we aim to present pointwise characterizations for the norm of Ric_Z and $\mathbb{I}$ when $\partial M \neq \emptyset$, which allow these quantities to be unbounded on the manifold.

* Corresponding author

Let $(X_t^x)_{t\geqslant 0}$ be the (reflecting if ∂M exists) diffusion process generated by $L=\Delta+Z$ on M starting at point x, and let $(U_t^x)_{t\geqslant 0}$ be the horizontal lift onto the frame bundle $O(M):=\bigcup_{x\in M}O_x(M)$, where $O_x(M)$ is the set of all orthonormal basis of the tangent space T_xM at point x. It is well known that $(X_t^x,U_t^x)_{t\geqslant 0}$ can be constructed as the unique solution to the SDEs:

$$\begin{aligned} dX_t^x &= \sqrt{2}\,U_t^x\circ dW_t+Z(X_t^x)dt+N(X_t^x)dl_t^x, \quad X_0^x=x,\\ dU_t^x &= \sqrt{2}\,H_{U_t^x}(U_t^x)\circ dW_t+H_Z(U_t^x)dt+H_N(U_t^x)dl_t^x, \quad U_0^x\in O_x(M), \end{aligned}\tag{1.1}$$

where W_t is the d-dimensional Brownian motion on a complete filtration probability space $(\Omega,\{\mathcal{F}_t\}_{t\geqslant 0},\mathrm{P})$; N is the inward unit normal vector field of ∂M; $H_\cdot: TM\to TO(M)$ is the horizontal lift, $H_u:=(H_{ue_i})_{1\leqslant i\leqslant d}$ for $u\in O(M)$ and the canonical orthonormal basis $\{e_i\}_{1\leqslant i\leqslant d}$ on $\mathbb{R}^d$. Further, l_t is an adapted increasing process which increases only when $X_t^x\in\partial M$ which is called the local time of X_t^x on ∂M. In the first part of this paper, we assume that the solution is non-explosive, so that the (Neumann) semigroup P_t generated by L is given by

$$P_tf(x)=\mathrm{E}f(X_t^x),\quad x\in M,\quad f\in\mathcal{B}_b(M),\quad t\geqslant 0.$$

For a fixed $T>0$, consider the path space $W_T(M):=C([0,T];M)$ and the class of smooth cylinder functions

$$\mathcal{F}C_T^\infty:=\{F(\gamma)=f(\gamma_{t_1},\dots,\gamma_{t_m}): m\geqslant 1,\, \gamma\in W_T(M),\, 0<t_1<t_2<\cdots<t_m\leqslant T,\, f\in C_0^\infty(M^m)\}.$$

Let

$$\mathbb{H}_T=\bigg\{h\in C([0,T];\mathbb{R}^d): h(0)=0,\ \|h\|_{\mathbb{H}_T}^2:=\int_0^T|h_s'|^2ds<\infty\bigg\}.$$

For any $F\in\mathcal{F}C_T^\infty$ with $F(\gamma)=f(\gamma(t_1),\dots,\gamma(t_m))$, the Malliavin gradient $DF(X_{[0,T]}^x)$ is an $\mathbb{H}_T$-valued random variable satisfying

$$\begin{aligned} \dot{D}_sF(X_{[0,T]}^x) &:= \frac{d}{ds}DF(X_{[0,T]}^x)\\ &= \sum_{t_i>s}(U_{t_i}^x)^{-1}\nabla_i f(X_{t_1}^x,\dots,X_{t_m}^x),\quad s\in[0,T], \end{aligned}\tag{1.2}$$

where ∇_i is the (distributional) gradient operator for the i-th component on M^m, and $P_u:\mathbb{R}^d\to\mathbb{R}^d$ is the projection along $u^{-1}N$, i.e.,

$$\langle P_ua,b\rangle:=\langle ua,N\rangle\langle ub,N\rangle,\quad a,b\in\mathbb{R}^d,\quad u\in\bigcup_{x\in\partial M}O_x(M).$$

For $K\in C(M;[0,\infty))$ and $\sigma\in C(\partial M;[0,\infty))$, we introduce the following random measure $\mu_{x,T}$ on $[0,T]$:

$$\mu_{x,T}(ds):=\mathrm{e}^{\int_0^s K(X_r^x)dr+\int_0^s\sigma(X_r^x)dl_r^x}\{K(X_s^x)ds+\sigma(X_s^x)dl_s^x\}.\tag{1.3}$$

For any $t\in[0,T]$, consider the energy form

$$\mathcal{E}_{t,T}^{K,\sigma}(F,F)=\mathrm{E}\bigg\{(1+\mu_{x,T}([t,T]))\bigg(|\dot{D}_tF(X_{[0,T]}^x)|^2+\int_t^T|\dot{D}_sF(X_{[0,T]}^x)|^2\mu_{x,T}(ds)\bigg)\bigg\}$$

for $F\in\mathcal{F}C_T^\infty$. Our main result is follows.

Theorem 1.1. *Let $K\in C(M;[0,\infty))$ and $\sigma\in C(\partial M;[0,\infty))$ be such that for any $T>0, x\in M$,*

$$\mathrm{E}\mathrm{e}^{(2+\varepsilon)\int_0^T\{K(X_s^x)ds+\sigma(X_s^x)dl_s^x\}}<\infty\tag{1.4}$$

holds for some $\varepsilon>0$. For any $p,q\in[1,2]$, the following statements are equivalent to each other:

(1) *For any $x\in M$ and $y\in\partial M$,*

$$\|\mathrm{Ric}_Z\|(x):=\sup_{X\in T_xM,|X|=1}|\mathrm{Ric}(X,X)-\langle\nabla_XZ,X\rangle|(x)\leqslant K(x),$$

$$\|\mathbb{I}\|(y) := \sup_{Y \in T_y\partial M, |Y|=1} |\mathbb{I}(Y,Y)|(y) \leqslant \sigma(y).$$

(2) *For any $f \in C_0^\infty(M)$, $T > 0$, and $x \in M$,*

$$\begin{aligned}&|\nabla P_T f|^p(x) \leqslant \mathrm{E}[(1+\mu_{x,T}([0,T]))^p |\nabla f|^p(X_T^x)],\\ &\left|\nabla f(x) - \frac{1}{2}\nabla P_T f(x)\right|^q \leqslant \mathrm{E}\bigg[(1+\mu_{x,T}([0,T]))^{q-1}\bigg(\left|\nabla f(x) - \frac{1}{2}U_0^x(U_T^x)^{-1}\nabla f(X_T^x)\right|^q \\ &\qquad\qquad + \frac{\mu_{x,T}([0,T])}{2^q}|\nabla f(X_T^x)|^q\bigg)\bigg].\end{aligned}$$

(3) *For any $F \in \mathcal{F}C_T^\infty, x \in M$ and $T > 0$,*

$$|\nabla_x \mathrm{E} F(X_{[0,T]}^x)|^q \leqslant \mathrm{E}\bigg[(1+\mu_{x,T}([0,T]))^{q-1}\bigg(|\dot{D}_0 F(X_{[0,T]}^x)|^q + \int_0^T |\dot{D}_s F(X_{[0,T]}^x)|^q \mu_{x,T}(ds)\bigg)\bigg].$$

(4) *For any $t_0, t_1 \in [0,T]$ with $t_1 > t_0$, and any $x \in M$, the following log-Sobolev inequality holds*:

$$\begin{aligned}&\mathrm{E}[\mathrm{E}(F^2(X_{[0,T]}^x)\,|\,\mathcal{F}_{t_1})\log \mathrm{E}(F^2(X_{[0,T]}^x)\,|\,\mathcal{F}_{t_1})] \\ &\quad - \mathrm{E}[\mathrm{E}(F^2(X_{[0,T]}^x)\,|\,\mathcal{F}_{t_0})\log \mathrm{E}(F^2(X_{[0,T]}^x)\,|\,\mathcal{F}_{t_0})] \leqslant 4\int_{t_0}^{t_1} \mathcal{E}_{s,T}^{K,\sigma}(F,F)ds, \quad F \in \mathcal{F}C_T^\infty.\end{aligned}$$

(5) *For any $t \in [0,T]$ and $x \in M$, the following Poincaré inequality holds*:

$$\mathrm{E}[\{\mathrm{E}(F(X_{[0,T]}^x)\,|\,\mathcal{F}_t)\}^2] - \{\mathrm{E}[F(X_{[0,T]}^x)]\}^2 \leqslant 2\int_0^t \mathcal{E}_{s,T}^{K,\sigma}(F,F)ds, \quad F \in \mathcal{F}C_T^\infty.$$

Remark 1.1. (1) When $\partial M = \emptyset, Z = -\nabla f$ and K is a constant, it is proved in [10, Theorem 2.1] that $\|\mathrm{Ric}_Z\|_\infty \leqslant K$ is equivalent to each of Theorem 1.1(3)–(5) with $\sigma = 0$ and a slightly different formulation of $\mathcal{E}_{s,T}^{K,0}$. A comparison with these equivalent statements using references functions on the path space shows that the statement (2) only depends on the reference functions on M and is thus easier to verify.

(2) An important problem in geometry is to identify the Ricci curvature, for example, to characterize Einstein manifolds where Ric is a constant tensor. According to Theorem 1.1, Ric is identified by ∇Z if and only if all/some of Theorem 1.1(2)–(5) hold for $K = 0$.

We prove this result in Section 2. In Section 3, the equivalence of Theorem 1.1(1), (4) and (5) is proved without (1.4) but using the class of truncated cylindrical functions replacing $\mathcal{F}C_T^\infty$.

2 Proof of Theorem 1.1

We first introduce some known results from the monograph [13] which hold under a condition weaker than (1.4).

Let $f \in C_0^\infty(M)$ with $|\nabla f(x)| = 1$ and $\mathrm{Hess}_f(x) = 0$. According to [13, Theorem 3.2.3], if $x \in M \setminus \partial M$ then for any $p > 0$ we have

$$\begin{aligned}\mathrm{Ric}_Z(\nabla f, \nabla f)(x) &= \lim_{t\downarrow 0} \frac{P_t|\nabla f|^p(x) - |\nabla P_t f|^p(x)}{pt} \\ &= \lim_{t\downarrow 0} \frac{1}{t}\bigg(\frac{P_t f^2(x) - (P_t f)^2(x)}{2t} - |\nabla P_t f(x)|^2\bigg);\end{aligned} \tag{2.1}$$

and by [13, Theorem 3.2.3], if $x \in \partial M$ and $\nabla f \in T_x\partial M$ then

$$\begin{aligned}\mathbb{I}(\nabla f, \nabla f)(x) &= \lim_{t\downarrow 0} \frac{\sqrt{\pi}}{2p\sqrt{t}}\{P_t|\nabla f|^p(x) - |\nabla P_t f|^p(x)\} \\ &= \lim_{t\downarrow 0} \frac{3\sqrt{\pi}}{8\sqrt{t}}\bigg(\frac{P_t f^2(x) - (P_t f)^2(x)}{2t} - |\nabla P_t f|^2(x)\bigg).\end{aligned} \tag{2.2}$$

We note that in [13, (3.2.9)], $\sqrt{\pi}$ is misprinted as π.

Next, for each $u \in O(M)$ and for each $\tilde{u} \in \bigcup_{x\in\partial M} O_xM$, the matrix-valued functions $\mathrm{Ric}_Z(u)$, $\mathbb{I}(\tilde{u})$ and $P_{\tilde{u}}$ are given by

$$\begin{aligned}
&\langle \mathrm{Ric}_Z(u)a, b\rangle := \mathrm{Ric}_Z(ua, ub),\\
&\langle P_{\tilde{u}}a, b\rangle := \langle \tilde{u}a, N\rangle\langle \tilde{u}b, N\rangle,\\
&\langle \mathbb{I}(\tilde{u})a, b\rangle := \mathbb{I}(\tilde{u}a - \langle \tilde{u}a, N\rangle N, \tilde{u}b - \langle \tilde{u}b, N\rangle N), \quad a, b \in \mathbb{R}^d.
\end{aligned}$$

According to [13, Lemma 4.2.3], for any $F \in \mathcal{F}C_T^\infty$ with $F(\gamma) = f(\gamma_{t_1}, \dots, \gamma_{t_N}), f \in C_0^\infty(M)$ and $0 \leqslant t_1 < \cdots \leqslant t_N$,

$$(U_0^x)^{-1}\nabla_x \mathrm{E}[F(X_{[0,T]}^x)] = \sum_{i=1}^N \mathrm{E}[Q_{0,t_i}^x (U_{t_i}^x)^{-1}\nabla_i f(X_{t_1}^x, \dots, X_{t_N}^x)], \tag{2.3}$$

where ∇_x denotes the gradient in $x \in M$ and ∇_i is the gradient with respect to the i-th component, and for any $s \geqslant 0$, $(Q_{s,t}^x)_{t\geqslant s}$ is an adapted right-continuous process on $\mathbb{R}^d \otimes \mathbb{R}^d$ satisfies $Q_{s,t}^x P_{U_t^x} = 0$ if $X_t^x \in \partial M$ and

$$Q_{s,t}^x = \bigg(I - \int_s^t Q_{s,r}^x \{\mathrm{Ric}_Z(U_r^x)dr + \mathbb{I}(U_r^x)dl_r^x\}\bigg)(I - 1_{\{X_t^x\in\partial M\}}P_{U_t^x}). \tag{2.4}$$

The multiplicative functional $Q_{s,t}^x$ was introduced by Hsu [9] to investigate gradient estimate on P_t. For convenience, let $Q_t^x := Q_{0,t}^x$. In particular, taking $F(\gamma) = f(\gamma_t)$ in (2.3), we obtain

$$\nabla P_t f(x) = U_0^x \mathrm{E}[Q_t^x (U_t^x)^{-1}\nabla f(X_t^x)], \quad x \in M, \quad f \in C_0^\infty(M), \quad t \geqslant 0. \tag{2.5}$$

Finally, for the above $F \in \mathcal{F}C_T^\infty$, let

$$\tilde{D}_t F(X_{[0,T]}^x) = \sum_{i:t_i>t} Q_{t,t_i}^x U_{t_i}^{-1}\nabla_i f(X_{t_1}^x, \dots, X_{t_N}^x), \quad t \in [0, T]. \tag{2.6}$$

Then [13, Lemma 4.3.2] (see also [12]) implies that

$$\mathrm{E}(F(X_{[0,T]}^x) \,|\, \mathcal{F}_t) = \mathrm{E}[F(X_{[0,T]}^x)] + \sqrt{2}\int_0^t \langle \mathrm{E}(\tilde{D}_s F(X_{[0,T]}^x) \,|\, \mathcal{F}_s), dW_s\rangle, \quad t \in [0, T]. \tag{2.7}$$

Proof of Theorem 1.1. It is well known that the log-Sobolev inequality in (4) implies the Poincaré inequality in (5). We prove the theorem by verifying the following implications: (1) $\Rightarrow$ (3) for all $q \geqslant 1$; (3) $\Rightarrow$ (2) for all $p = q$; (2) for some $p \geqslant 1$ and $q \in [1, 2] \Rightarrow$ (1); (5) $\Rightarrow$ (1); and (1) $\Rightarrow$ (4).

For simplicity, we will write F and f for $F(X_{[0,T]}^x)$ and $f(X_{t_1}^x, \dots, X_{t_N}^x)$, respectively.

(a) (1) $\Rightarrow$ (3) for all $q \geqslant 1$. From (1.2), (2.3) and (2.4) we have

$$\begin{aligned}
U_0^{-1}\nabla_x \mathrm{E}[F] &= \mathrm{E}\bigg[\sum_{i=1}^N Q_{t_i}^x (U_{t_i}^x)^{-1}\nabla_i f\bigg]\\
&= \mathrm{E}\bigg[\sum_{i=1}^N \bigg(I - \int_0^{t_i} Q_s^x \mathrm{Ric}_Z(U_s)ds - \int_0^{t_i} Q_s^x \mathbb{I}_{U_s^x} dl_s^x\bigg)(U_{t_i}^x)^{-1}\nabla_i f\bigg]\\
&= \mathrm{E}\bigg[\sum_{i=1}^N (U_{t_i}^x)^{-1}\nabla_i f - \sum_{i=1}^N \bigg(\int_0^{t_i} Q_s^x \mathrm{Ric}_Z(U_s^x)ds + \int_0^{t_i} Q_s^x \mathbb{I}_{U_s^x} dl_s^x\bigg)(U_{t_i}^x)^{-1}\nabla_i f\bigg]\\
&= \mathrm{E}\bigg[\dot{D}_0 F - \int_0^T \{Q_s^x \mathrm{Ric}_Z(U_s^x)\dot{D}_s F\}ds - \int_0^T \{Q_s^x \mathbb{I}(U_s^x)\dot{D}_s F\}dl_s^x\bigg].
\end{aligned}$$

By [13, Theorem 3.2.1], we have

$$\|Q_s^x\| \leqslant \exp\bigg[\int_0^s K(X_r)dr + \int_0^s \sigma(X_r)dl_r^x\bigg]. \tag{2.8}$$

Combining these with (1), (1.3), and using Hölder's inequality twice, we obtain

$$\begin{aligned}|\nabla_x \mathrm{E}[F]|^q &\leqslant \left\{\mathrm{E}|\dot{D}_0 F| + \mathrm{E}\int_0^T |\dot{D}_s F|\mu_{x,T}(ds)\right\}^q\\ &\leqslant \mathrm{E}\left\{|\dot{D}_0 F| + \int_0^T |\dot{D}_s F|\mu_{x,T}(ds)\right\}^q\\ &\leqslant \mathrm{E}\left\{\left(|\dot{D}_0 F|^q + \frac{(\int_0^T |\dot{D}_s F(X^x_{[0,T]})|\mu_{x,T}(ds))^q}{\{\mu_{x,T}([0,T])\}^{q-1}}\right)(1+\mu_{x,T}([0,T]))^{q-1}\right\}\\ &\leqslant \mathrm{E}\left\{\left(|\dot{D}_0 F|^q + \int_0^T |\dot{D}_s F(X^x_{[0,T]})|^q\mu_{x,T}(ds)\right)(1+\mu_{x,T}([0,T]))^{q-1}\right\}.\end{aligned}$$

Thus, the inequality in (3) holds.

(b) (3) $\Rightarrow$ (2) for all $p = q$. Take $F(\gamma) = f(\gamma_T)$. Then $\mathrm{E}F(X^x_{[0,T]}) = P_T f(x)$ and by (1.2), $|\dot{D}_s F| \leqslant |\nabla f(X_T)|$ for $s \in [0,T]$. So, the first inequality in (2) with $p = q$ follows from (3) immediately. Similarly, by taking $F(\gamma) = f(\gamma_0) - \frac{1}{2}f(\gamma_T)$, we have

$$\mathrm{E}F = f(x) - \frac{1}{2}P_T f(x)$$

and

$$\begin{aligned}|\dot{D}_0 F| &= \left|\nabla f(x) - \frac{1}{2}U^x_0 (U^x_T)^{-1}\nabla f(X^x_T)\right|,\\ |\dot{D}_s F| &\leqslant \frac{1}{2}|\nabla f(X^x_T)|, \quad s \in (0,T].\end{aligned}$$

Then the second inequality in (2) is implied by (3).

(c) (2) for some $p \geqslant 1$ and $q \in [1,2] \Rightarrow$ (1). Let $x \in M \setminus \partial M$. There exists $r > 0$ such that

$$B(x,r) := \{y \in M : \rho(x,y) \leqslant r\} \subset M \setminus \partial M,$$

where ρ is the Riemannian distance. Let $\tau_r = \inf\{t \geqslant 0 : \rho(x, X^x_t) \geqslant r\}$. By [13, Lemma 3.1.1] (see also [1, Lemma 2.3]), there exists a constant $c > 0$ such that

$$\mathrm{P}(\tau_r \leqslant T) \leqslant \mathrm{e}^{-c/T}, \quad T \in (0,1]. \tag{2.9}$$

Then $\mathrm{P}(l^x_T > 0) \leqslant \mathrm{e}^{-c/T}$ so that for each $n \geqslant 1$,

$$\lim_{T\to 0} T^{-n} l^x_T = 0, \quad \text{P-a.s.} \tag{2.10}$$

Combining this with (1.3) we obtain

$$\lim_{T\to 0} \frac{\mu_{x,T}([0,T])}{T} = K(x). \tag{2.11}$$

Therefore, by the dominated convergence theorem attributed to (1.4), the first inequality in (2) and (2.1) yield

$$\begin{aligned}-\mathrm{Ric}_Z(\nabla f, \nabla f)(x) &= \lim_{T\to 0} \frac{|\nabla P_T f|^p(x) - P_T|\nabla f|^p(x)}{pT}\\ &\leqslant \lim_{T\to 0} \frac{\mathrm{E}\{[(1+\mu_{x,T}([0,T]))^p - 1]|\nabla f|^p(X^x_T)\}}{pT} = K(x),\end{aligned} \tag{2.12}$$

where $f \in C^\infty_0(M)$ with $\mathrm{Hess}_f(x) = 0$ and $|\nabla f(x)| = 1$. This implies $\mathrm{Ric}_Z(X,X) \geqslant -K(x)$ for any $X \in T_x M$ with $|X| = 1$.

Next, we prove that the second inequality in (2) implies $\mathrm{Ric}_Z \leqslant K$. By Hölder's inequality, the second inequality in (2) for some $q \in [1,2]$ implies the same inequality for $q = 2$:

$$\begin{aligned}&\left|\nabla f(x) - \frac{1}{2}\nabla P_T f(x)\right|^2\\&\quad\leqslant \mathrm{E}\left[(1+\mu_{x,T}([0,T]))\left(\left|\nabla f(x) - \frac{1}{2}U_0^x(U_T^x)^{-1}\nabla f(X_T^x)\right|^2 + \frac{\mu_{x,T}([0,T])}{4}|\nabla f(X_T^x)|^2\right)\right].\end{aligned}$$

Then

$$\begin{aligned}\frac{|\nabla P_T f(x)|^2 - P_T|\nabla f(x)|^2}{4T} \leqslant \frac{1}{T}\mathrm{E}\bigg\{&\langle \nabla f(x), \nabla P_T f(x) - \mathrm{E}[U_0^x(U_T^x)^{-1}\nabla f(X_T^x)]\rangle\\&+\mu_{x,T}([0,T])\left|\nabla f(x) - \frac{1}{2}U_0^x(U_T^x)^{-1}\nabla f(X_T^x)\right|^2\\&+\frac{(1+\mu_{x,T}([0,T]))\mu_{x,T}([0,T])}{4}|\nabla f(X_T^x)|^2\bigg\}.\end{aligned}\tag{2.13}$$

Combining this with (2.1) and (2.11), we arrive at

$$\begin{aligned}&-\frac{1}{2}\mathrm{Ric}_Z(\nabla f, \nabla f)(x)\\&\quad\leqslant \frac{1}{2}K(x)|\nabla f(x)|^2 + \limsup_{T\to 0}\frac{1}{T}\mathrm{E}\langle \nabla f(x), \nabla P_T f(x) - \mathrm{E}[U_0^x(U_T^x)^{-1}\nabla f(X_T^x)]\rangle.\end{aligned}$$

Since by (2.5), (2.4) and (2.10) we have

$$\begin{aligned}&\langle \nabla f(x), \nabla P_T f(x) - \mathrm{E}[U_0^x(U_T^x)^{-1}\nabla f(X_T^x)]\rangle\\&\quad= -\int_0^T \langle \nabla f(x), U_0^x \mathrm{Ric}_Z(U_r^x)(U_T^x)^{-1}\nabla f(X_T^x)\rangle dr\\&\quad= -T\mathrm{Ric}_Z(\nabla f, \nabla f)(x) + \mathrm{o}(T)\end{aligned}$$

for small $T > 0$, this implies $\mathrm{Ric}_Z(\nabla f, \nabla f)(x) \leqslant K(x)$.

On the other hand, to prove the desired bound on $\|\mathbb{I}\|$, we let $x \in \partial M$, $f \in C_0^\infty(M)$ with $\langle \nabla f, N\rangle(x) = 0$, $|\nabla f(x)| = 1$ and $\mathrm{Hess}_f(x) = 0$. By [13, Lemma 3.1.2],

$$\mathrm{E}e^{\lambda l^x_{T\wedge\tau_1}} < \infty, \quad \mathrm{E}l^x_{T\wedge\tau_1} = \frac{2\sqrt{T}}{\sqrt{\pi}} + \mathrm{O}(T^{3/2})$$

for all $\lambda > 0$ and small $T > 0$. Combining this with (1.3), (1.4) and (2.9), we obtain

$$\lim_{T\to 0}\frac{\mathrm{E}\mu_{x,T}([0,T])}{\sqrt{T}} = \frac{2\sigma(x)}{\sqrt{\pi}}, \quad \lim_{T\to 0}\frac{[\mathrm{E}\mu_{x,T}([0,T])]^2}{\sqrt{T}} = 0.\tag{2.14}$$

Then repeating the above argument with (2.2) replacing (2.1), we prove

$$|\mathbb{I}(\nabla f, \nabla f)(x)| \leqslant \sigma(x).$$

Indeed, by (2.2) and (2.14), instead of (2.12) we have

$$-\mathbb{I}(\nabla f, \nabla f)(x) \leqslant \frac{\sqrt{\pi}}{2}\lim_{T\to\infty}\frac{|\nabla P_T f|^p(x) - P_T|\nabla f|^p(x)}{p\sqrt{T}} = \sigma(x),$$

while multiplying (2.13) by $\sqrt{T}$ and letting $T \to \infty$ leads to

$$-\frac{1}{\sqrt{\pi}}\mathbb{I}(\nabla f, \nabla f)(x) \leqslant \frac{\sigma(x)}{\sqrt{\pi}} - \frac{2}{\sqrt{\pi}}\mathbb{I}(\nabla f, \nabla f)(x).$$

(d) (5) $\Rightarrow$ (1). Let $F(\gamma) = f(\gamma_T)$. Then (5) implies

$$P_T f^2(x) - (P_T f(x))^2 \leqslant 2\int_0^T \mathrm{E}[(1+\mu_{x,T}([s,T]))^2|\nabla f(X_T^x)|^2]ds. \tag{2.15}$$

For f in (2.1), by combining this with (2.1) and (2.11), we obtain

$$\begin{aligned}\mathrm{Ric}_Z(\nabla f,\nabla f)(x) &= \lim_{T\to 0}\frac{1}{T}\bigg(\frac{P_T f^2(x)-(P_T f)^2(x)}{2T} - |\nabla P_T f|^2\bigg)\\ &\leqslant \lim_{T\to 0}\frac{1}{T}\bigg\{\frac{1}{T}\int_0^T \{\mathrm{E}[(1+\mu([s,T]))^2|\nabla f(X_T^x)|^2] - |\nabla P_T f(x)|^2\}ds\bigg\}\\ &= \lim_{T\to 0}\frac{1}{T}\bigg\{P_T|\nabla f|^2(x) - |\nabla P_T f|^2(x) + \frac{2|\nabla f|^2(x)}{T}\int_0^T (T-s)K(x)ds\bigg\}\\ &= 2\mathrm{Ric}_Z(\nabla f,\nabla f)(x) + K(x)|\nabla f|^2(x).\end{aligned}$$

This implies

$$\mathrm{Ric}_Z(\nabla f,\nabla f)(x) \geqslant -K(x)|\nabla f(x)|^2.$$

Next, for f in (2.2), by combining (2.15) with (2.2) and (2.14), we obtain

$$\begin{aligned}\mathbb{I}(\nabla f,\nabla f)(x) &= \lim_{T\to 0}\frac{3\sqrt{\pi}}{8\sqrt{T}}\bigg(\frac{P_T f^2(x)-(P_T f)^2(x)}{2T} - |\nabla P_T f(x)|^2\bigg)\\ &\leqslant \lim_{T\to 0}\frac{3\sqrt{\pi}}{8\sqrt{T}}\bigg\{\frac{1}{T}\int_0^T \{\mathrm{E}[(1+\mu([s,T]))^2|\nabla f(X_T^x)|^2] - |\nabla P_T f(x)|^2\}ds\bigg\}\\ &= \lim_{T\to 0}\frac{3\sqrt{\pi}}{8\sqrt{T}}\bigg\{P_T|\nabla f|^2(x) - |\nabla P_T f|^2(x) + \frac{2|\nabla f(x)|^2}{T}\int_0^T \frac{2\sigma(x)(\sqrt{T}-\sqrt{s})}{\sqrt{\pi}}ds + \mathrm{o}(\sqrt{T})\bigg\}\\ &= \frac{3}{2}\mathbb{I}(\nabla f,\nabla f)(x) + \frac{1}{2}\sigma(x).\end{aligned}$$

Hence, $\mathbb{I}(\nabla f,\nabla f)(x) \geqslant -\sigma(x)|\nabla f(x)|^2$.

On the other hand, to prove the upper bound estimates, we take $F(\gamma) = f(\gamma_\varepsilon) - \frac{1}{2}f(\gamma_T)$ for $\varepsilon\in(0,T)$. By (1.2),

$$|\dot{D}_t F| = \bigg|\nabla f(X_\varepsilon) - \frac{1}{2}U_\varepsilon^x(U_T^x)^{-1}\nabla f(X_T^x)\bigg|1_{[0,\varepsilon)}(t) + \frac{1}{2}|\nabla f(X_T^x)|1_{[\varepsilon,T]}(t).$$

Then (5) implies

$$\begin{aligned}I_\varepsilon &:= \mathrm{E}\bigg[f(X_\varepsilon^x) - \frac{1}{2}\mathrm{E}(f(X_T^x)\,|\,\mathcal{F}_\varepsilon)\bigg]^2 - \bigg(P_\varepsilon f(x) - \frac{1}{2}P_T f(x)\bigg)^2\\ &\leqslant 2\varepsilon\mathrm{E}\bigg\{(1+\mu_{x,T}([0,T]))\bigg|\nabla f(X_\varepsilon^x) - \frac{1}{2}U_\varepsilon^x(U_T^x)^{-1}\nabla f(X_T^x)\bigg|^2\\ &\quad + \frac{\mu_{x,T}([0,T])|\nabla f(X_T^x)|^2}{4}\bigg\} + c\varepsilon^2\\ &=: J_\varepsilon, \quad \varepsilon\in(0,T)\end{aligned} \tag{2.16}$$

for some constant $c>0$. Obviously,

$$\lim_{\varepsilon\to 0}\frac{J_\varepsilon}{\varepsilon} = \mathrm{E}\bigg\{(1+\mu_{x,T}([0,T]))\bigg(\bigg|\nabla f(x) - \frac{1}{2}U_0^x(U_T^x)^{-1}\nabla f(X_T^x)\bigg|^2 + \frac{\mu_{x,T}([0,T])}{4}|\nabla f|^2(X_T^x)\bigg)\bigg\}. \tag{2.17}$$

On the other hand, we have

$$\begin{aligned}\frac{I_\varepsilon}{\varepsilon} &= \frac{P_\varepsilon f^2 - (P_\varepsilon f)^2}{\varepsilon} + \frac{1}{4\varepsilon}\mathrm{E}[\{\mathrm{E}(f(X_T^x)\,|\,\mathcal{F}_\varepsilon)\}^2 - (P_T f)^2(x)]\\ &\quad + \frac{\mathrm{E}[f(X_T^x)\{P_\varepsilon f(x) - f(X_\varepsilon^x)\}]}{\varepsilon}.\end{aligned} \tag{2.18}$$

Let $f \in C_0^\infty(M)$ satisfy the Neumann boundary condition. Then we have

$$\lim_{\varepsilon\to 0} \frac{P_\varepsilon f^2 - (P_\varepsilon f)^2}{\varepsilon} = 2|\nabla f|^2(x). \tag{2.19}$$

Next, (2.6) and (2.7) yield

$$\mathrm{E}(f(X_T^x) \,|\, \mathcal{F}_\varepsilon) = P_T f(x) + \sqrt{2}\int_0^\varepsilon \langle \mathrm{E}(Q_{s,T}^x (U_T^x)^{-1}\nabla f(X_T^x) \,|\, \mathcal{F}_s), dW_s\rangle. \tag{2.20}$$

Then

$$\mathrm{E}[\mathrm{E}(f(X_T^x) \,|\, \mathcal{F}_\varepsilon)]^2 = (P_T f)^2 + 2\int_0^\varepsilon \mathrm{E}|Q_{0,T}^x (U_T^x)^{-1}\nabla f(X_T^x)|^2 ds.$$

Along with (2.5), this leads to

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{4\varepsilon}\mathrm{E}[\{\mathrm{E}(f(X_T^x) \,|\, \mathcal{F}_\varepsilon)\}^2 - (P_T f)^2(x)] \\
&\quad = \frac{1}{2}|\mathrm{E}[Q_{0,T}^x (U_T^x)^{-1}\nabla f(X_T^x)]|^2 = \frac{1}{2}|\nabla P_T f(x)|^2.
\end{aligned} \tag{2.21}$$

Finally, using Itô's formula we have

$$\begin{aligned}
P_\varepsilon f(x) - f(X_\varepsilon^x) &= P_\varepsilon f(x) - f(x) - \int_0^\varepsilon L f(X_s^x) ds - \sqrt{2}\int_0^\varepsilon \langle \nabla f(X_s^x), U_s^x dW_s\rangle \\
&= \mathrm{o}(\varepsilon) - \sqrt{2}\int_0^\varepsilon \langle \nabla f(X_s^x), U_s^x dW_s\rangle.
\end{aligned}$$

Combining this with (2.20) and (2.5), we arrive at

$$\lim_{\varepsilon\to 0}\frac{\mathrm{E}[f(X_T^x)\{P_\varepsilon f(x) - f(X_\varepsilon^x)\}]}{\varepsilon} = -2\langle \nabla f(x), \nabla P_t f(x)\rangle.$$

Substituting this and (2.19)–(2.21) into (2.18), we obtain

$$\lim_{\varepsilon\to 0}\frac{I_\varepsilon}{\varepsilon} = 2\left|\nabla f(x) - \frac{1}{2}\nabla P_T f(x)\right|^2.$$

Combining this with (2.16) and (2.17), we prove the second inequality in (2) for $q = 2$, which implies $\mathrm{Ric}_Z \leqslant K$ and $\mathbb{I} \leqslant \sigma$ as shown in Step (c).

(e) (1) $\Rightarrow$ (4). According to (2.7),

$$G_t := \mathrm{E}(F^2 \,|\, \mathcal{F}_t) = \mathrm{E}(F^2) + \sqrt{2}\int_0^t \langle \mathrm{E}(\tilde{D}_s F^2 \,|\, \mathcal{F}_s), dW_s\rangle, \quad t \in [0, T]. \tag{2.22}$$

By Itô's formula,

$$\begin{aligned}
d(G_t \log G_t) &= (1 + \log G_t) dG_t + \frac{|\mathrm{E}(\tilde{D}_s F^2 \,|\, \mathcal{F}_s)|^2}{G_t} dt \\
&\leqslant (1 + \log G_t) dG_t + 4\mathrm{E}(|\tilde{D}_s F|^2 \,|\, \mathcal{F}_s) dt.
\end{aligned} \tag{2.23}$$

Then

$$\mathrm{E}[G_{t_1}\log G_{t_1}] - \mathrm{E}[G_{t_0}\log G_{t_0}] \leqslant 4\int_{t_0}^{t_1} \mathrm{E}|\tilde{D}_s F|^2 ds. \tag{2.24}$$

By (2.6) we have

$$\tilde{D}_s F = \sum_{i=1}^N 1_{\{s<t_i\}} Q_{s,t_i}^x (U_{t_i}^x)^{-1}\nabla_i f$$

$$= \sum_{i=1}^{N} 1_{\{s<t_i\}} \bigg(I - \int_s^{t_i} Q_{s,t}^x \{\mathrm{Ric}_V(U_t^x)dt + \mathbb{I}_{U_t^x} dl_t^x\} \bigg) (I - 1_{\{X_{t_i}^x \in \partial M\}} P_{U_{t_i}^x}) (U_{t_i}^x)^{-1} \nabla_i f$$
$$= \dot{D}_0 F - \int_s^T Q_{s,t}^x \{\mathrm{Ric}_Z(U_t^x)dt + \mathbb{I}(U_t^x) dl_t^x\}.$$

Combining this with (1), (2.8) and (2.11), and using the Schwarz inequality, we prove

$$|\tilde{D}_s F|^2 \leqslant (1 + \mu_{x,T}([s,T])) \bigg(|\dot{D}_0 F|^2 + \int_s^T |\dot{D}_s F|^2 \mu_{x,T}(ds) \bigg). \tag{2.25}$$

This together with (2.24) implies the log-Sobolev inequality in (4). □

3 Extension of Theorem 1.1

In this section, we aim to remove the condition (1.4) in Theorem 1.1 and allow the (reflecting) diffusion process generated by L to be explosive. The idea is to make a conformal change of metric such that the condition (1.4) holds on the new Riemannian manifold. Since both Ric_Z and $\mathbb{I}$ are local quantities, they do not change at x if the new metric coincides with the original one around point x.

Let (M, g) be a Riemannian manifold with boundary, and let N be the inward pointing unit normal vector field of ∂M. Let $\phi \in C_0^\infty(M)$ be non-negative with non-empty $M_\phi := \{\phi > 0\}$. Then, M_ϕ is a complete Riemannian manifold under the metric $g_\phi := \phi^{-2} g$. Let $\nabla^\phi, \Delta^\phi, \mathrm{Ric}^\phi$ and $\mathbb{I}^\phi$ be the associated Laplacian, gradient, Ricci curvature and the second fundamental form of ∂M_ϕ respectively. By e.g., [2, Theorem 1.159 d)],

$$\nabla_X^\phi Y = \nabla_X Y - \langle X, \nabla \log \phi \rangle Y - \langle Y, \nabla \log \phi \rangle X + \langle X, Y \rangle \nabla \log \phi.$$

Moreover, according to [13, Theorem 1.2.4] and the proof of [13, Theorem 1.2.5], we have

$$\mathrm{Ric}_\phi = \mathrm{Ric} + (d-2)\phi^{-1} \mathrm{Hess}_\phi + (\phi^{-1} \Delta \phi - (d-3) |\nabla \log \phi|) g,$$
$$\mathbb{I}^\phi = \phi^{-1} \mathbb{I} + (N \log \phi) g.$$

Note that $|X| = 1$ if and only if $g_\phi(\phi X, \phi X) = 1$, we obtain

$$\|\mathbb{I}_g\|_\infty = \sup_{X \in T\partial M_\phi, |X|=1} |\mathbb{I}_\phi(\phi X, \phi X)| < \infty,$$

and for $\mathrm{Ric}_{\phi Z}^\phi$ the curvature of $L^\phi := \Delta^\phi + \phi Z$,

$$\|\mathrm{Ric}_{\phi Z}^\phi\|_\infty = \sup_{X \in TM_\phi, |X|=1} |\mathrm{Ric}^\phi(\phi X, \phi X) - g_\phi(\nabla_{\phi X}(\phi Z), \phi X)| < \infty.$$

Therefore, Theorem 1.1 applies to L^ϕ on the manifold M_ϕ. In particular, by taking ϕ such that $\phi = 1$ around a point x, we have $\mathrm{Ric}_Z = \mathrm{Ric}^\phi$ and $\mathbb{I} = \mathbb{I}^\phi$ at point x. Thus we characterize these two quantities at x. To this end, we will take $\phi = \ell(\rho_x)$, where ρ_x is the Riemannian distance to x and $\ell \in C_0^\infty(\mathbb{R})$ is such that $0 \leqslant \ell \leqslant 1$, $\ell(s) = 1$ for $s \leqslant r$ and $\ell(s) = 0$ for $s \geqslant 2r$ for some constant $r > 0$ with compact $B_{2r}(x) := \{\rho_x \leqslant 2r\}$.

Obviously, before exiting the ball $B_r(x)$, the diffusion process generated by L coincides with that generated by L^ϕ. Therefore, to use the original diffusion process in place of the new one, we will take references functions which vanishes as soon as the diffusion exits this ball. To this end, we will truncate the cylindrical functions in terms of the uniform distance

$$\tilde{\rho_x}(\gamma) := \sup_{t \in [0,1]} \rho(\gamma(t), x).$$

To make the manifold M_ϕ complete, let $\delta : M \to (0,\infty)$ be a smooth function such that $B_R(x)$ is compact for any $R \leqslant \delta_x$. Consider the class of truncated cylindrical functions

$$\mathcal{F}C^\infty_{T,\mathrm{loc}} := \{F\ell(\tilde{\rho_x}) : F \in \mathcal{F}C^\infty_T,\ x \in M,\ \ell \in C_0^\infty(\mathbb{R}),\ \mathrm{supp}\ell \subset [0,\delta_x)\}. \tag{3.1}$$

To define $\mathcal{E}^{K,\sigma}_{t,T}(\tilde{F},\tilde{F})$ for $\tilde{F} = F\ell(\tilde{\rho_x}) \in \mathcal{F}C^\infty_{T,\mathrm{loc}}$, we take $\phi \in C_0^\infty(M)$ such that $0 \leqslant \phi \leqslant 1$, $\phi = 1$ for $\ell(\rho_x) > 0$, and $\phi = 0$ for $\rho_x \geqslant \delta_x$. Then M_ϕ is complete with bounded $\mathrm{Ric}^\phi_{\phi Z}$ and $\mathbb{I}^\phi$. Let $X^{x,\phi}_{[0,T]}$ be the (reflecting) diffusion process generated by L^ϕ. Similar to the proof of [4, Lemma 2.1] for the case without a boundary, we see that $|\dot{D}_s\tilde{F}(X^{x,\phi}_{[0,T]})|$ is well-defined and bounded for $s \in [0,T]$. Note that $\tilde{F}$ is supported on $\{\ell(\tilde{\rho}_x) > 0\} \subset W_T(M^\phi)$ and $X^{x,\phi}_{[0,T]} = X^x_{[0,T]}$ if $\ell(\tilde{\rho}_x(X^{x,\phi}_{[0,T]})) > 0$ (see (3.4)). Therefore, we conclude that $|\dot{D}_s\tilde{F}(X^x_{[0,T]})| = |\dot{D}_s\tilde{F}(X^{x,\phi}_{[0,T]})|$ is well-defined and bounded in $s \in [0,T]$ as well, which does not depend on the choice of ϕ. Again since $\tilde{F}$ is supported on $\{\ell(\tilde{\rho}_x) > 0\} \subset W_T(M^\phi)$ and M^ϕ is relatively compact in M, we have

$$\mathcal{E}^{K,\sigma}_{t,T}(\tilde{F},\tilde{F}) := \mathrm{E}\bigg\{(1+\mu_{x,T}([t,T]))\bigg(|\dot{D}_t\tilde{F}(X^x_{[0,T]})|^2 + \int_t^T |\dot{D}_s\tilde{F}(X^x_{[0,T]})|^2\mu_{x,T}(ds)\bigg)\bigg\} < \infty.$$

Theorem 3.1. *Let $K \in C(M;[0,\infty))$ and $\sigma \in C(\partial M;[0,\infty))$. The following statements are equivalent to each other:*

(1) *For any $x \in M$ and $y \in \partial M$,*

$$\begin{aligned}\|\mathrm{Ric}_Z\|(x) &:= \sup_{X\in T_xM,|X|=1} |\mathrm{Ric}(X,X) - \langle\nabla_X Z, X\rangle|(x) \leqslant K(x),\\ \|\mathbb{I}\|(y) &:= \sup_{Y\in T_y\partial M,|Y|=1} |\mathbb{I}(Y,Y)|(y) \leqslant \sigma(y).\end{aligned}$$

(2) *For any $t_0, t_1 \in [0,T]$ with $t_1 > t_0$, and any $x \in M$, the following log-Sobolev inequality holds:*

$$\begin{aligned}&\mathrm{E}[\mathrm{E}(F^2(X^x_{[0,T]})\,|\,\mathcal{F}_{t_1})\log\mathrm{E}(F^2(X^x_{[0,T]})\,|\,\mathcal{F}_{t_1})]\\ &\quad - \mathrm{E}[\mathrm{E}(F^2(X^x_{[0,T]})\,|\,\mathcal{F}_{t_0})\log\mathrm{E}(F^2(X^x_{[0,T]})\,|\,\mathcal{F}_{t_0})] \leqslant 4\int_{t_0}^{t_1}\mathcal{E}^{K,\sigma}_{s,T}(F,F)ds, \quad F \in \mathcal{F}C^\infty_{T,\mathrm{loc}}.\end{aligned}$$

(3) *For any $t \in [0,T]$ and $x \in M$, the following Poincaré inequality holds:*

$$\mathrm{E}[\{\mathrm{E}(F(X^x_{[0,T]})\,|\,\mathcal{F}_t)\}^2] - \{\mathrm{E}[F(X_{[0,T]})]\}^2 \leqslant 2\int_0^t \mathcal{E}^{K,\sigma}_{s,T}(F,F)ds, \quad F \in \mathcal{F}C^\infty_{T,\mathrm{loc}}.$$

Proof. Since (2) $\Rightarrow$ (3) is well-known, we only prove (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1).

(a) (1) $\Rightarrow$ (2). Fix $x \in M$. For any $\tilde{F} := F\ell(\tilde{\rho}_x) \in \mathcal{F}C^\infty_{T,\mathrm{loc}}$, there exists $R \in (0,\delta_x)$ such that

$$\mathrm{supp}(\ell(\tilde{\rho}_x)) \subset B_R(x) := \{y \in M : \rho(x,y) \leqslant R\}.$$

Let $\phi_R \in C_0^\infty(M)$ such that $\phi_R|_{B_R(x)} = 1$ and $0 \leqslant \phi_R \leqslant 1$. We consider the following Riemannian metric on the manifold $M_R := \{y \in M : \phi_R(y) > 0\}$:

$$g_R := \phi_R^{-2}g.$$

As explained above that (M_R, g_R) is a complete Riemannian manifold with

$$K_R := \sup_{M_R}\|\mathrm{Ric}^R_Z\|_\infty < \infty, \quad \sigma_R := \sup_{M_R}\|\mathbb{I}^R\|_\infty < \infty. \tag{3.2}$$

We consider the SDE (1.1) on M,

$$\begin{cases} dU^x_t = \sqrt{2}\,H_{U^x_t}(U^x_t)\circ dW_t + H_Z(U^x_t)dt + H_N(U^x_t)dl^x_t,\\ U_0 = u_0.\end{cases} \tag{3.3}$$

Then $X_t := \pi(U_t)$ is the (reflecting if ∂M exists) diffusion process on M generated by $L = \Delta + Z$.

Similarly, let $\{H_{i,R}\}_{i=1}^n$ and $H_{\phi_R Z,R}$ be the orthonormal basis of horizontal vector fields and horizontal lift of $\phi_R Z$ under the metric g_R. Since $g_R = g$ and $\phi_R = 1$ on $B_R(x)$, for $u \in O(M_R)$ with $\pi u \in B_R(x)$ we have $H_{i,R}(u) = H_i(u)$ and $H_{\phi Z,R}(u) = H_Z(u)$. For W_t and u_0 in (3.3), we consider the following SDE on the manifold M_R:

$$\begin{cases} dU_{t,R} = \displaystyle\sum_{i=1}^n H_{i,R}(U_{t,R}) \circ dW_t^i + H_{\phi_R Z,R}(U_t^x)dt + H_N(U_t^x)dl_{R,t}^x, \\ U_{0,R} = u_0. \end{cases}$$

Then $X_\cdot^{x,R} := \pi(U_{\cdot,R})$ is the (reflecting if ∂M_R exists) diffusion process on M_R generated by $L_R := \Delta_R + \phi_R Z$, where Δ_R is the Laplacian on M_R. Obviously,

$$U_{t,R} = U_t, \quad l_{R,t}^x = l_t^x \quad \text{for} \quad t \leqslant \tau_R := \inf\{t \geqslant 0 : X_t \notin B_R(x)\}. \tag{3.4}$$

Denote by $\mathrm{P}_{R,x}^T$ the distribution of the process $X_{[0,T]}^{x,R}$. From [13] and (2.24), the damped logarithmic Sobolev inequality holds

$$\mathrm{E}[G_{t_1} \log G_{t_1}] - \mathrm{E}[G_{t_0} \log G_{t_0}] \leqslant 4\tilde{\mathcal{E}}_R^{t_1,t_0}(G,G), \quad G \in \mathcal{F}C_T^\infty, \tag{3.5}$$

where

$$G_t := \mathrm{E}(G^2(X_{[0,T]}^{x,R}) \,|\, \mathcal{F}_t)$$

and

$$\tilde{\mathcal{E}}_R^{t_1,t_0}(H,G) = \int_{W_x^T(M_R)} \int_{t_0}^{t_1} \langle \tilde{D}_s^R F, \tilde{D}_s^R G \rangle ds d\mathbb{P}_{R,x}^T.$$

According to [13], the form $(\tilde{\mathcal{E}}_R^{t_1,t_0}, \mathcal{F}C_T^\infty)$ is closable in $L^2(\mathbb{P}_{R,x}^T)$. Let $(\tilde{\mathcal{E}}_R^{t_1,t_0}, \mathcal{D}(\tilde{\mathcal{E}}_R^{t_1,t_0}))$ be its closure. Let ρ^R be the Riemannian distance on M_R and

$$\tilde{\rho_x}^R(\gamma) := \sup_{t \in [0,1]} \rho^R(\gamma(t), x), \quad \gamma \in W_x^T(M_R).$$

We have $\tilde{\rho_x}^R(\gamma) = \tilde{\rho_x}(\gamma)$ for each $\gamma \in W_x^T(M_R) \subseteq W_x^T(M)$ satisfying $\rho_x^R(\gamma) \leqslant R$. Then [4, Lemma 2.1] implies that $\ell(\tilde{\rho_x})$ is in $\mathcal{D}(\tilde{\mathcal{E}}_{\mathbb{P}_{R,x}^T})$, and so is $\tilde{F} := F\ell(\tilde{\rho_x})$. Combining this with (3.4) and (3.5), we get

$$\begin{aligned}
&\mathrm{E}[\mathrm{E}(\tilde{F}^2(X_{[0,T]}^x) \,|\, \mathcal{F}_{t_1}) \log \mathrm{E}(\tilde{F}^2(X_{[0,T]}^x) \,|\, \mathcal{F}_{t_1})] \\
&\quad - \mathrm{E}[\mathrm{E}(\tilde{F}^2(X_{[0,T]}^x)|\mathcal{F}_{t_0}) \log \mathrm{E}(\tilde{F}^2(X_{[0,T]}^x) \,|\, \mathcal{F}_{t_0})] \\
&\quad\quad = \mathrm{E}[\mathrm{E}(\tilde{F}^2(X_{[0,T]}^{x,R}) \,|\, \mathcal{F}_{t_1}) \log \mathrm{E}(\tilde{F}^2(X_{[0,T]}^{x,R}) \,|\, \mathcal{F}_{t_1})] \\
&\quad\quad\quad - \mathrm{E}[\mathrm{E}(\tilde{F}^2(X_{[0,T]}^{x,R}) \,|\, \mathcal{F}_{t_0}) \log \mathrm{E}(\tilde{F}^2(X_{[0,T]}^{x,R}) \,|\, \mathcal{F}_{t_0})] \\
&\quad\quad \leqslant 4 \int_{W_x^T(M_R)} \int_{t_0}^{t_1} \langle \tilde{D}_s^R \tilde{F}, \tilde{D}_s^R \tilde{F} \rangle ds d\mathbb{P}_{R,x}^T \\
&\quad\quad = 4 \int_{W_x^T(M)} \int_{t_0}^{t_1} \langle \tilde{D}_s \tilde{F}, \tilde{D}_s \tilde{F} \rangle ds d\mathbb{P}_x^T.
\end{aligned} \tag{3.6}$$

By combining this with (2.25), we prove (2).

(a) (3) $\Rightarrow$ (1). We first prove the lower bound estimates. When $x \in M \setminus \partial M$, there exists $r \in (0, \frac{1}{2}\delta_x)$ such that $B_{2r}(x) \subset M \setminus \partial M$. Let $\Phi = \ell(\tilde{\rho_x})$, where $\ell \in C_0^\infty(\mathbb{R})$ such that $0 \leqslant \ell \leqslant 1$, $\ell(s) = 1$ for $s \leqslant r$ and $\ell(s) = 0$ for $s \geqslant 2r$. Let

$$\tau_s = \inf\{t \geqslant 0 : \rho(x, X_t^x) \geqslant s\} \quad \text{for} \quad s > 0.$$

Consider $\tilde{F}(\gamma) = (\Phi F)(\gamma) = \Phi(\gamma) f(\gamma_T)$ for f in (2.1). Then (3) and (2.9) imply

$$\mathrm{E}[(F\Phi)^2(X_{[0,T]}^x)] - \{\mathrm{E}[(F\Phi)(X_{[0,T]})]\}^2$$

$$
\begin{aligned}
&\leqslant 2\int_0^T \mathcal{E}_{t,T}^{K,\sigma}(\tilde F,\tilde F)dt\\
&=2\int_0^T \mathrm{E}\bigg\{(1+\mu_{x,T}([t,T]))\bigg(|\dot D_t\tilde F(X_{[0,T]}^x)|^2+\int_t^T|\dot D_s\tilde F(X_{[0,T]}^x)|^2\mu_{x,T}(ds)\bigg)\bigg\}dt\\
&\leqslant 2\int_0^T \mathrm{E}[1_{\{\tau_{2r}>T\}}(1+\mu_{x,T}([t,T]))^2|\nabla f(X_T^x)|^2]dt+C\mathrm{P}(\tau_r\leqslant T)\\
&=2\int_0^T \mathrm{E}[1_{\{\tau_{2r}>T\}}(1+\mu_{x,T}([t,T]))^2|\nabla f(X_T^x)|^2]dt+\mathrm{o}(T^3),
\end{aligned}\tag{3.7}
$$

where $C>0$ is a constant depending on f and Φ. On the other hand, by (2.1) and (2.9), we have

$$
\begin{aligned}
&\lim_{T\to 0}\frac{1}{T}\bigg(\frac{\mathrm{E}[F^2\Phi^2(X_{[0,T]}^x)]-\{\mathrm{E}[F\Phi(X_{[0,T]})]\}^2}{2T}-|\nabla P_Tf|^2\bigg)\\
&\quad=\lim_{T\to 0}\frac{1}{T}\bigg(\frac{P_Tf^2(x)-(P_Tf)^2(x)}{2T}-|\nabla P_Tf|^2\bigg)\\
&\quad=\mathrm{Ric}_Z(\nabla f,\nabla f)(x).
\end{aligned}
$$

Since $l_s^x=0$ for $s\leqslant\tau_{2r}$, these two estimates together with (2.9) and (1.3) lead to

$$
\begin{aligned}
&\mathrm{Ric}_Z(\nabla f,\nabla f)(x)\\
&\quad=\lim_{T\to 0}\frac{1}{T}\bigg(\frac{\mathrm{E}[(F\Phi)^2(X_{[0,T]}^x)]-\{\mathrm{E}[(F\Phi)(X_{[0,T]})]\}^2}{2T}-|\nabla P_Tf|^2\bigg)\\
&\quad\leqslant\lim_{T\to 0}\frac{1}{T}\bigg\{\frac{1}{T}\int_0^T\{\mathrm{E}[1_{\{\tau_{2r}>T\}}(1+\mu([s,T]))^2|\nabla f(X_T^x)|^2]-|\nabla P_Tf(x)|^2\}ds\bigg\}\\
&\quad\leqslant\lim_{T\to 0}\bigg(\frac{P_T|\nabla f|^2(x)-|\nabla P_Tf|^2(x)}{T}+\frac{\int_0^T\mathrm{E}\{1_{\{\tau_{2r}>T\}}[(1+\mu([s,T]))^2-1]|\nabla f(X_T^x)|^2\}ds}{T^2}\bigg)\\
&\quad=2\mathrm{Ric}_Z(\nabla f,\nabla f)(x)+K(x)|\nabla f|^2(x).
\end{aligned}
$$

Therefore,

$$
\mathrm{Ric}_Z(\nabla f,\nabla f)(x)\geqslant -K(x)|\nabla f(x)|^2.
$$

Next, let $x\in\partial M$. For f in (2.2), by (2.9) we have

$$
\begin{aligned}
&\lim_{T\to 0}\frac{3\sqrt{\pi}}{8\sqrt{T}}\bigg(\frac{\mathrm{E}[(F\Phi)^2(X_{[0,T]}^x)]-\{\mathrm{E}[(F\Phi)(X_{[0,T]})]\}^2}{2T}-|\nabla P_Tf|^2\bigg)\\
&\quad=\lim_{T\to 0}\frac{3\sqrt{\pi}}{8\sqrt{T}}\bigg(\frac{P_Tf^2(x)-(P_Tf)^2(x)}{2T}-|\nabla P_Tf|^2\bigg)\\
&\quad=\mathbb{I}(\nabla f,\nabla f)(x).
\end{aligned}\tag{3.8}
$$

Combining this with (3.7) and (2.14), we obtain

$$
\begin{aligned}
\mathbb{I}(\nabla f,\nabla f)(x)&=\lim_{T\to 0}\frac{3\sqrt{\pi}}{8\sqrt{T}}\bigg(\frac{\mathrm{E}[(F\Phi)^2(X_{[0,T]}^x)]-\big\{\mathrm{E}[(F\Phi)(X_{[0,T]})]\big\}^2}{2T}-|\nabla P_Tf(x)|^2\bigg)\\
&\leqslant\lim_{T\to 0}\frac{3\sqrt{\pi}}{8\sqrt{T}}\bigg(\int_0^T\frac{\mathrm{E}\{1_{\{\tau_{2r}>T\}}(1+\mu_{x,T}([t,T]))^2|\nabla F(X_T^x)|^2\}}{T}dt-|\nabla P_Tf(x)|^2\bigg)\\
&=\lim_{T\to 0}\frac{3\sqrt{\pi}}{8\sqrt{T}}\bigg\{P_T|\nabla f|^2(x)-|\nabla P_Tf|^2(x)+\frac{2|\nabla f(x)|^2}{T}\int_0^T\frac{2\sigma(x)(\sqrt{T}-\sqrt{s})}{\sqrt{\pi}}ds\bigg\}\\
&=\frac{3}{2}\mathbb{I}(\nabla f,\nabla f)(x)+\frac{1}{2}\sigma(x).
\end{aligned}
$$

Therefore,

$$
\mathbb{I}(\nabla f,\nabla f)(x)\geqslant -\sigma(x)|\nabla f(x)|^2.
$$

To prove the upper bound estimates, we take $F(\gamma) = f(\gamma_\varepsilon) - \frac{1}{2} f(\gamma_T)$ for $\varepsilon \in (0, T)$. From (1.2),

$$|\dot{D}_t F| = \left|\nabla f(X_\varepsilon) - \frac{1}{2} U_\varepsilon^x (U_T^x)^{-1} \nabla f(X_T^x)\right| 1_{[0,\varepsilon)}(t) + \frac{1}{2} |\nabla f(X_T^x)| 1_{[\varepsilon, T]}(t).$$

Moreover, from (3) and (2.9), we may find a constant $C > 0$ depending on f and Φ such that for any $\varepsilon, T \in (0, 1)$,

$$\begin{aligned} I_\varepsilon &:= \mathrm{E}\bigg[\mathrm{E}\bigg(\Phi(X_{[0,T]}^x) f(X_\varepsilon^x) - \frac{1}{2} \Phi(X_{[0,T]}^x) f(X_T^x) \,|\, \mathcal{F}_\varepsilon\bigg)\bigg]^2 \\ &\quad - \bigg[\mathrm{E}\bigg(\Phi(X_{[0,T]}^x) f(X_\varepsilon^x) - \frac{1}{2} \Phi(X_{[0,T]}^x) f(X_T^x)\bigg)\bigg]^2 \\ &\leqslant 2 \int_0^\varepsilon \mathrm{E}\bigg\{(1 + \mu_{x,T}([t, T])) |\Phi(X_{[0,T]}^x) \dot{D}_t F|^2 \\ &\quad + \int_t^T |\Phi(X_{[0,T]}^x) \dot{D}_s F|^2 \mu_{x,T}(ds)\bigg\} dt + C\varepsilon T^4. \end{aligned} \tag{3.9}$$

Then

$$\begin{aligned} \limsup_{\varepsilon \to 0} \frac{I_\varepsilon}{\varepsilon} &\leqslant \mathrm{E}\bigg\{\Phi(X_{[0,T]}^x)(1 + \mu_{x,T}([0, T])) \bigg(\bigg|\nabla f(x) - \frac{1}{2} U_0^x (U_T^x)^{-1} \nabla f(X_T^x)\bigg|^2 \\ &\quad + \frac{\Phi(X_{[0,T]}^x) \mu_{x,T}([0, T])}{4} |\nabla f|^2 (X_T^x)\bigg)\bigg\} + \mathrm{o}(T^3) \end{aligned} \tag{3.10}$$

for small $T > 0$. On the other hand, according to (d) of proof in Theorem 1.1, we have

$$\begin{aligned} \frac{I_\varepsilon}{\varepsilon} &= \frac{P_\varepsilon f^2 - (P_\varepsilon f)^2}{\varepsilon} + \frac{1}{4\varepsilon} \mathrm{E}[\{\mathrm{E}(f(X_T^x) \,|\, \mathcal{F}_\varepsilon)\}^2 - (P_T f)^2(x)] \\ &\quad + \frac{\mathrm{E}[f(X_T^x)\{P_\varepsilon f(x) - f(X_\varepsilon^x)\}]}{\varepsilon} + o(T^3) \\ &= 2\bigg|\nabla f(x) - \frac{1}{2} \nabla P_T f(x)\bigg|^2 + o(T^3). \end{aligned} \tag{3.11}$$

Combining this with (3.10), we arrive at the following:

$$\begin{aligned} &2\bigg|\nabla f(x) - \frac{1}{2} \nabla P_T f(x)\bigg|^2 \\ &\quad \leqslant \mathrm{E}\bigg\{\Phi(X_{[0,T]}^x)(1 + \mu_{x,T}([0, T])) \bigg(\bigg|\nabla f(x) - \frac{1}{2} U_0^x (U_T^x)^{-1} \nabla f(X_T^x)\bigg|^2 \\ &\qquad + \frac{\Phi(X_{[0,T]}^x) \mu_{x,T}([0, T])}{4} |\nabla f|^2 (X_T^x)\bigg)\bigg\} + o(T^3). \end{aligned} \tag{3.12}$$

With this estimate, we may repeat the last part in the proof of (2) $\Rightarrow$ (1) of Theorem 1.1 to derive the desired upper bound estimates on Ric_Z and $\mathbb{I}$ at point x. $\square$

Acknowledgements This work was supported the by National Natural Science Foundation of China (Grant Nos. 11771326 and 11431014). The authors thank the referees for the helpful comments

References

1 Arnaudon M, Thalmaier A, Wang F Y. Gradient estimates and Harnack inequalities on non-compact Riemannian manifolds. Stochastic Process Appl, 2009, 119: 3653–3670

2 Besse A L. Einstein Manifolds. Berlin: Springer, 1987

3 Capitaine B, Hsu E P, Ledoux M. Martingale representation and a simple proof of logarithmic Sobolev inequalities on path spaces. Electron Comm Probab, 1997, 2: 71–81

4 Chen X, Wu B. Functional inequality on path space over a non-compact Riemannian manifold. J Funct Anal, 2014, 266: 6753–6779

5 Driver B. A Cameron-Martin type quasi-invariant theorem for Brownian motion on a compact Riemannian manifold. J Funct Anal, 1992, 110: 272–376

6 Fang S Z. Inégalité du type de Poincaré sur l'espace des chemins riemanniens. C R Math Acad Sci Paris, 1994, 318: 257–260

7 Haslhofer R, Naber A. Ricci curvature and Bochner formulas for martingales. ArXiv:1608.04371, 2016

8 Hsu E P. Logarithmic Sobolev inequalities on path spaces over Riemannian manifolds. Comm Math Phys, 1997, 189: 9–16

9 Hsu E P. Multiplicative functional for the heat equation on manifolds with boundary. Michigan Math J, 2002, 50: 351–367

10 Naber A. Characterizations of bounded Ricci curvature on smooth and nonsmooth spaces. ArXiv:1306.6512v4, 2013

11 Wang F Y. Weak Poincaré inequalities on path spaces. Int Math Res Not IMRN, 2004, 2004: 90–108

12 Wang F Y. Analysis on path spaces over Riemannian manifolds with boundary. Commun Math Sci, 2011, 9: 1203–1212

13 Wang F Y. Analysis for Diffusion Processes on Riemannian Manifolds. Singapore: World Scientific, 2014