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# IRREDUCIBILITY AND ASYMPTOTICS OF STOCHASTIC BURGERS EQUATION DRIVEN BY $\alpha$ -STABLE PROCESSES

ZHAO DONG, FENG-YU WANG, AND LIHU XU

## Abstract

The irreducibility, moderate deviation principle and  $\psi$ -uniformly exponential ergodicity with  $\psi(x) := 1 + \|x\|_0$  are proved for stochastic Burgers equation driven by the  $\alpha$ -stable processes for  $\alpha \in (1, 2)$ , where the first two are new for the present model, and the last strengthens the exponential ergodicity under total variational norm derived in [21].

**Keywords:** stochastic Burgers equation;  $\alpha$ -stable noises; Irreducibility,  $\psi$ -uniformly ergodicity, moderate deviation

**Mathematics Subject Classification (2000):** 60F10, 60H15, 60J75.

## 1. INTRODUCTION

In [21], the strongly Feller property and exponential ergodicity have been proved for the stochastic Burgers equation driven by rotationally symmetric  $\alpha$ -stable processes with  $\alpha \in (1, 2)$ . In this paper, we prove a stronger  $\psi$ -uniformly exponential ergodicity, the irreducibility, and the moderate deviation principle for occupation measures. Before state our main results, we briefly recall the framework of the study and results derived in [21].

Let  $\mathbb{H}$  be the space of all square integrable functions on the torus  $\mathbb{T} = [0, 2\pi)$  with vanishing mean values. Let  $Au = -u''$  be the second order differential operator. Then  $A$  is a positive self-adjoint operator on  $\mathbb{H}$ . Let  $\lambda_{2k} := \lambda_{2k+1} := k^2$  and

$$e_{2k}(x) := \pi^{-\frac{1}{2}} \cos(kx), \quad e_{2k+1}(x) := \pi^{-\frac{1}{2}} \sin(kx).$$

It is easy to see that  $\{e_k, k \in \mathbb{N}\}$  forms an orthogonal basis of  $\mathbb{H}$  and

$$Ae_k = \lambda_k e_k, \quad k \in \mathbb{N}.$$

The norm in  $\mathbb{H}$  is denoted by  $\|\cdot\|_0$ .

For  $\gamma > 0$ , let  $\mathbb{H}^\gamma$  be the domain of the fractional operator  $A^{\frac{\gamma}{2}}$ :

$$\mathbb{H}^\gamma := A^{-\frac{\gamma}{2}}(\mathbb{H}) = \left\{ \sum_k \lambda_k^{-\frac{\gamma}{2}} a_k e_k : (a_k)_{k \in \mathbb{N}} \subset \mathbb{R}, \sum_k a_k^2 < +\infty \right\}.$$

It is a separable Hilbert space with the inner product

$$\langle u, v \rangle_\gamma := \langle A^{\frac{\gamma}{2}} u, A^{\frac{\gamma}{2}} v \rangle_0 = \sum_k \lambda_k^\gamma \langle u, e_k \rangle_0 \langle v, e_k \rangle_0.$$

For  $u \in \mathbb{H}$ , let  $\|u\|_\gamma = \sqrt{\langle u, u \rangle_\gamma}$  if  $u \in \mathbb{H}^\gamma$ , and  $\|u\|_\gamma = \infty$  otherwise. The  $C_0$ -contraction semigroup  $e^{-tA}$  generated by  $-A$  reads

$$e^{-tA}u := \sum_k e^{-t\lambda_k} \langle u, e_k \rangle_0 e_k, \quad t \geq 0.$$

Obviously,

$$(1.1) \quad \|A^\gamma e^{-tA}u\|_0 \leq \sup_{x>0} (x^\gamma e^{-x}) t^{-\gamma} \|u\|_0 = \gamma^\gamma e^{-\gamma} t^{-\gamma} \|u\|_0, \quad \gamma > 0.$$

Let  $\{W_t^k, t \geq 0\}_{k \in \mathbb{N}}$  be a sequence of independent standard one-dimensional Brownian motion on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . The cylindrical Brownian motion on  $\mathbb{H}$  is defined by

$$W_t := \sum_k W_t^k e_k.$$

For  $\alpha \in (0, 2)$ , let  $S_t$  be an independent  $\alpha/2$ -stable subordinator, i.e., an increasing one dimensional Lévy process with Laplace transform

$$\mathbb{E}e^{-\eta S_t} = e^{-t|\eta|^{\alpha/2}}, \quad \eta > 0.$$

The subordinated cylindrical Brownian motion  $\{L_t\}_{t \geq 0}$  on  $\mathbb{H}$  is defined by

$$L_t := W_{S_t}.$$

Notice that in general  $L_t$  does not belong to  $\mathbb{H}$ .

We are concerned about the following stochastic Burgers equation in the Hilbert space  $\mathbb{H}$ :

$$(1.2) \quad dX_t = [-AX_t - B(X_t)]dt + QdL_t, \quad X_0 = x \in \mathbb{H},$$

where  $B(u) := B(u, u)$  for the bilinear operator  $b$  defined by  $B(u, v) := uv'$  for  $v \in \mathbb{H}^1$  and  $u \in \mathbb{H}$ , and  $Q \in \mathcal{L}(\mathbb{H})$  is given by

$$Qu := \sum_{k=1}^{\infty} \beta_k \langle u, e_k \rangle_0 e_k, \quad u \in \mathbb{H},$$

with  $\beta = (\beta_k)_{k \in \mathbb{N}}$  such that there exist some  $\delta \in (0, 1)$  and  $\frac{3}{2} < \theta' \leq \theta < 2$  satisfying

$$(1.3) \quad \delta \lambda_k^{-\frac{\theta}{2}} \leq |\beta_k| \leq \delta^{-1} \lambda_k^{-\frac{\theta'}{2}}, \quad k \in \mathbb{N}.$$

By [25, Lemma 2.1], we have

$$(1.4) \quad \langle B(u, v), w \rangle_0 \leq C \|u\|_{\sigma_1} \|v\|_{\sigma_2+1} \|w\|_{\sigma_3}, \quad \sigma_1 + \sigma_2 + \sigma_3 > 1/2, u, w \in \mathbb{H}, v \in \mathbb{H}^1.$$

Moreover, let

$$(1.5) \quad Z_t := \int_0^t e^{-(t-s)A} Q dL_s \quad t \geq 0$$

satisfies  $Z. \in \mathcal{D}([0, \infty); \mathbb{H}^1)$  and

$$(1.6) \quad \mathbb{E} \left[ \sup_{0 \leq t \leq T} \|Z_t\|_1 \right] < \infty, \quad T > 0,$$

see e.g. [21, (4.5)]. Recall that for a topology space  $E$ ,  $\mathcal{C}([0, \infty); E)$  (resp.  $\mathcal{D}([0, \infty); E)$ ) stands for the space of the continuous (resp. right continuous with left limits) maps from

$[0, T]$  to  $E$ . The following result is due to [21, Theorem 4.2]. For a  $\sigma$ -finite measure  $\mu$  on  $E$  we denote  $\mu(f) = \int_E f d\mu$ ,  $f \in L^1(\mu)$ .

**Theorem 1.1** ([21]). *Let  $\alpha \in (1, 2)$  and the assumption (1.3) hold for some  $\delta \in (0, 1)$  and  $\frac{3}{2} < \theta' \leq \theta < 2$ .*

(1) *For any  $x \in \mathbb{H}$ , (1.2) has a unique solution  $(X_t^x)_{t \geq 0}$  starting at  $x$ , and*

$$X_t^x - Z_t \in \mathcal{C}([0, \infty), \mathbb{H}) \cap \mathcal{C}((0, \infty), \mathbb{H}^1).$$

*In particular,  $(t, x) \mapsto X_t^x$  is a Markov process on  $\mathbb{H}$ .*

(2) *The Markov semigroup  $P_t$  for  $X_t^x$  is strong Feller, and has a unique invariant probability measure  $\mu_0$  such that*

$$(1.7) \quad \sup_{|f| \leq 1} |P_t \Phi(x) - \mu_0(f)| \leq C(1 + \|x\|_0) e^{-\gamma t}, \quad t \geq 0, x \in \mathbb{H}$$

*holds for some constants  $C, \gamma > 0$ .*

In this paper, we prove the following two theorems on the irreducibility, moderate deviation principle of occupation measures for solutions to (1.2), and the  $\psi$ -uniformly exponential ergodicity for  $\psi(x) := 1 + \|x\|_0$ . The first two properties are new for the present model, and the third strengthens the exponential ergodicity (1.7) with  $|f| \leq \psi$  replacing  $|f| \leq 1$ .

**Theorem 1.2.** *In the situation of Theorem 1.1, for any  $x \in \mathbb{H}$ , the solution  $(X_t^x)_{t \geq 0}$  of (1.2) is irreducible in  $\mathbb{H}$ , i.e.*

$$\mathbb{P}(\|X_T^x - a\|_0 < \varepsilon) > 0, \quad \varepsilon > 0, T > 0, a \in \mathbb{H}.$$

To state our second result, we recall the notion of moderate deviations (MDP). Let  $\mathcal{M}_b(\mathbb{H})$  be the space of signed  $\sigma$ -additive measures of bounded variation on  $H$ , equipped with the  $\tau$ -topology  $\tau := \sigma(\mathcal{M}_b(\mathbb{H}), \mathcal{B}_b(\mathbb{H}))$  of convergence against all bounded Borel functions, which is stronger than the usual weak convergence topology  $\sigma(\mathcal{M}_b(\mathbb{H}), C_b(\mathbb{H}))$ . We denote  $\mathcal{M}_1(\mathbb{H})$  the space of probability measures on  $\mathbb{H}$ . Given a  $\psi : \mathbb{H} \rightarrow \mathbb{R}_+$ , define

$$\mathcal{B}_\psi := \mathcal{B}_\psi(\mathbb{H}, \mathbb{R}) = \{f \in \mathcal{B}(\mathbb{H}, \mathbb{R}) : |f(x)| \leq \psi(x)\}.$$

Let  $b(t) : \mathbb{R}^+ \rightarrow (0, +\infty)$  be an increasing function verifying

$$(1.8) \quad \lim_{t \rightarrow \infty} b(t) = +\infty, \quad \lim_{t \rightarrow \infty} \frac{b(t)}{\sqrt{t}} = 0,$$

and let

$$\mathfrak{M}_t := \frac{1}{b(t)\sqrt{t}} \int_0^t (\delta_{X_s} - \mu) ds.$$

To characterize *moderate deviations* of  $X_t$  from its *asymptotic limit*  $\mu$ , one estimates the long time behaviours of

$$(1.9) \quad \mathbb{P}_\mu(\mathfrak{M}_t \in A),$$

where  $A \in \tau$  is a given domain of deviation, and  $\mathbb{P}_\mu$  is the probability measure taken for the system  $X$  with initial distribution  $\mu$ . This problem refers to the central limit theorem for  $b(t) = 1$ , the large deviation principle (LDP) for  $b(t) = \sqrt{t}$ , and the moderate deviation principle (MDP) for  $b(t)$  satisfying (1.8), see [4]. We say that  $\mathbb{P}_\mu(\mathfrak{M}_t \in \cdot)$  satisfies the MDP with a rate function  $I$  on  $\mathcal{M}_1(\mathbb{H})$ , if the following three properties hold for any  $b$  satisfying (1.8):

- (a1) for any  $a \geq 0$ ,  $\{\nu \in \mathcal{M}_1(\mathbb{H}); I(\nu) \leq a\}$  is compact in  $(\mathcal{M}_1(\mathbb{H}), \tau)$ ;  
(a2) (the upper bound) for any closed set  $F$  in  $(\mathcal{M}_1(\mathbb{H}), \tau)$ ,

$$\limsup_{T \rightarrow \infty} \frac{1}{b^2(T)} \log \mathbb{P}_\mu(\mathfrak{M}_T \in F) \leq - \inf_F I;$$

- (a3) (the lower bound) for any open set  $G$  in  $(\mathcal{M}_1(\mathbb{H}), \tau)$ ,

$$\liminf_{T \rightarrow \infty} \frac{1}{b^2(T)} \log \mathbb{P}_\mu(\mathfrak{M}_T \in G) \geq - \inf_G I.$$

**Theorem 1.3.** *In the situation of Theorem 1.1, let  $\psi(x) = 1 + \|x\|_0$ . Then the following statements hold.*

- (1) *The Markov semigroup  $P_t$  associated with (1.2) has a unique invariant measure  $\mu_0$  with  $\mu_0(\|\cdot\|_0) := \int_{\mathbb{H}} \|x\|_0 \mu_0(dx) < \infty$  and*

$$\sup_{f \in \mathcal{B}_\psi} |P_t f(x) - \mu_0(f)| \leq C e^{-\gamma t} (1 + \|x\|_0), \quad x \in \mathbb{H}, t \geq 0$$

*holds for some constants  $C, \gamma > 0$ .*

- (2) *For any initial distribution  $\nu$  with  $\mu(\|\cdot\|_0) < +\infty$  and any measurable function  $f$  with  $|f\psi^{-1}|_\infty := \sup_{\mathbb{H}} |f\psi^{-1}| < \infty$ , the limit*

$$\sigma^2(f) := \lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}^\mu \left( \int_0^t (f(X_s) - \mu(f)) ds \right)^2 \in \mathbb{R}$$

*exists. Moreover, the family  $\{\mathbb{P}_\mu(\mathfrak{M}_t \in \cdot) : t \geq 0\}$  satisfies the MDP with rate function*

$$I(\mu) := \sup \left\{ \mu(f) - \frac{1}{2} \sigma^2(f) : f \in \mathcal{B}_b(\mathbb{H}) \right\}.$$

To prove the irreducibility using a standard argument developed in [] for SDEs driven by cylindrical  $\alpha$ -stable process, we will solve a control problem for the associated deterministic system in Section 2, and establish a maximum inequality for stochastic convolution in Section 3. Unlike the cylindrical  $\alpha$ -stable process where components processes are independent, the rotationally  $\alpha$ -stable process we considered has strong correlations between any two components, which leads to essential difficulty to follow the line of []. To overcome the difficulty, we propose a new procedure including the following three steps: taking a sample path of  $\alpha/2$ -stable subordinator  $\ell$ , solving a new control problem by mollifying  $\ell$  as in [], and proving the irreducibility by showing that for the stochastic systems driven by  $W_{\ell_t}$ . With these preparations, Theorems 1.2 and 1.3 will be proved in Sections 4 and 5 respectively.

## 2. A CONTROL PROBLEM FOR THE ASSOCIATED DETERMINISTIC SYSTEM

Consider the path space of the subordinator  $S_t$ :

$$\mathcal{S} = \{\ell : [0, \infty) \rightarrow [0, \infty); \ell \text{ is strictly increasing, right continuous and has left limit}\}.$$

For any  $\ell \in \mathcal{S}$ , the set of jumps

$$\mathcal{J}(\ell) := \{t \geq 0 : \ell_{t-} \neq \ell_t\}$$

is at most countable. Let

$$\gamma_t = \inf\{s \geq 0 : \ell_s \geq t\}, \quad t \geq 0.$$

Consider the following deterministic system in  $\mathbb{H}$ :

$$(2.1) \quad dx_t^\ell + [Ax_t^\ell + B(x_t^\ell)] dt = Qdu_{\ell_t}, \quad x_0^\ell = x_0,$$

where  $u : [0, \infty) \rightarrow \mathbb{H}$  is the controller to be chosen later. Let

$$(2.2) \quad z_t^\ell = \int_0^t e^{-A(t-s)} Qdu_{\ell_s}, \quad y_t^\ell = x_t^\ell - z_t^\ell, \quad t \geq 0.$$

Then

$$(2.3) \quad \frac{dy_t^\ell}{dt} + Ay_t^\ell + B(y_t^\ell + z_t^\ell) = 0, \quad x_0^\ell = x_0.$$

Define

$$(2.4) \quad t_\varepsilon(a, T) = \sup \left\{ t < \frac{T}{2} : \|e^{-At}a - a\|_0 < \frac{\varepsilon}{2} \right\}, \quad T > 0, \varepsilon > 0, a \in \mathbb{H}.$$

It is easy to see that  $t_\varepsilon(a, T) \in (0, T/2]$ . For notational simplicity, we often write  $t_\varepsilon = t_\varepsilon(a, T)$ . The main result in this section is the following.

**Proposition 2.1.** *Let  $\ell \in \mathcal{S}$  and  $x_0 \in \mathbb{H}^1$ . For any  $\varepsilon > 0$ ,  $T > 0$  and  $a \in \mathbb{H}$ , there exist  $u \in \mathcal{C}([0, \ell_T]; \mathbb{H}^2)$  with bounded total variation and  $x^\ell \in D([0, T]; \mathbb{H}^1)$  solving (2.1) such that*

$$\|x_T^\ell - a\|_0 \leq \varepsilon, \quad T \notin \mathcal{J}(\ell).$$

Moreover,

$$\|z_t^\ell\|_2 \leq C_T(1 + \|e^{-At_\varepsilon}a\|_6^2 + \|x_{t_\varepsilon}\|_6^2), \quad 0 \leq t \leq T,$$

where  $t_\varepsilon$  is defined by (2.4) and  $x_{t_\varepsilon}$  is determined by (2.1) with  $u_{\ell_t} = 0$  for  $t \in [0, t_\varepsilon]$ .

To prove this result, we regularize  $\ell \in \mathcal{S}$  by

$$\ell_t^\delta = \frac{1}{\delta} \int_0^\delta \ell_{t+r} dr, \quad t \geq 0, \delta > 0,$$

and prove the assertion for  $\ell_t^\delta$  replacing  $\ell$ . It is clear that  $\ell_t^\delta$  is strictly increasing and continuous. Let  $\gamma_t^\delta$  be the inverse of  $\ell_t^\delta$ .

**Lemma 2.2.** *For all  $\delta > 0$ , we have*

$$\gamma_t^\delta \leq \gamma_t \leq \gamma_t^\delta + \delta, \quad \forall t \geq 0.$$

*Proof.* Denote  $t_0 = \gamma_t$  and  $t_1 = \gamma_t^\delta$ , it is easy to see  $\ell_{t_1}^\delta = t$  and  $\ell_{t_0} \geq t$ . Observe  $\ell_{t_0}^\delta = \frac{1}{\delta} \int_0^\delta \ell_{t_0+r} dr > t$  since  $\ell_{t_0+r} > t$  for  $r > 0$ . If  $t_0 < t_1$ , then  $t < \ell_{t_0}^\delta < \ell_{t_1}^\delta = t$ . Contradiction. If  $t_0 > t_1 + \delta$ , we have  $\ell_{t_1+\delta} < t$ , otherwise  $t_0 \leq t_1 + \delta$ . Consequently,  $\ell_{t_1}^\delta = \frac{1}{\delta} \int_0^\delta \ell_{t_1+r} dr < t$  since  $\ell_{t_1+r} < t$  for all  $r \in [0, \delta]$ , but  $\ell_{t_1}^\delta = t$ , contradiction. Hence,  $t_0 \in [t_1, t_1 + \delta]$ .  $\square$

**Lemma 2.3.** *For any  $T > 0, \varepsilon > 0, \delta > 0, a \in \mathbb{H}$ , let  $t_\varepsilon = t_\varepsilon(a, T)$  is defined by (2.4) and take*

$$(2.5) \quad u_t := 1_{[\ell_{t_\varepsilon}^\delta, \ell_T^\delta]}(t) Q^{-1} F(\gamma_t^\delta), \quad t \in [0, \ell_T^\delta],$$

where  $\gamma_t^\delta$  is the inverse function of  $\ell_t^\delta$  and

$$(2.6) \quad F(t) := x_t^{\ell^\delta} - x_{t_\varepsilon}^{\ell^\delta} + \int_{t_\varepsilon}^t Ax_s^{\ell^\delta} ds + \int_{t_\varepsilon}^t B(x_s^{\ell^\delta}) ds, \quad t \in [t_\varepsilon, T].$$

Then  $u \in \mathcal{C}([0, \ell_T^\delta]; \mathbb{H}^2)$  and  $F \in \mathcal{C}([t_\varepsilon, T]; \mathbb{H}^4)$  with

$$(2.7) \quad \|F(t)\|_4 \leq C_T(1 + \|e^{-At_\varepsilon} a\|_6^2 + \|x_{t_\varepsilon}^{\ell^\delta}\|_6^2) < \infty, \quad t \in [t_\varepsilon, T],$$

$$(2.8) \quad \|F(t_1) - F(t_2)\|_4 \leq C_T(1 + \|e^{-At_\varepsilon} a\|_6^2 + \|x_{t_\varepsilon}^{\ell^\delta}\|_6^2)|t_1 - t_2|, \quad t_1, t_2 \in [t_\varepsilon, T].$$

Moreover, let  $x^{\ell^\delta} \in \mathcal{C}([0, T]; \mathbb{H}^1)$  solve the system (2.1) for  $\ell^\delta$  replacing  $\ell$ . Then

$$\|x_T^{\ell^\delta} - a\|_0 < \varepsilon/2.$$

*Proof.* We first observe that  $x_t^{\ell^\delta}$  has the representation

$$(2.9) \quad x_t^{\ell^\delta} = e^{-At}x_0 + \int_0^t e^{-A(t-s)}B(x_s^{\ell^\delta})ds, \quad 0 \leq t \leq t_\varepsilon,$$

$$(2.10) \quad x_t^{\ell^\delta} = \frac{t - t_\varepsilon}{T - t_\varepsilon}e^{-At_\varepsilon}a + \frac{T - t}{T - t_\varepsilon}x_{t_\varepsilon}^{\ell^\delta}, \quad t_\varepsilon \leq t \leq T.$$

Indeed, by (2.5),  $u_t = 0$  for all  $t \in [0, \ell_{t_\varepsilon}^\delta]$ , the system (2.1) is a deterministic Burgers equation, which admits a unique solution  $x^{\ell^\delta} \in \mathcal{C}([0, t_\varepsilon]; \mathbb{H}^1)$  given by (2.9). On the other hand, for  $t \in [t_\varepsilon, T]$ , substituting  $x_t^{\ell^\delta}$  with the form (2.10) into the left hand of the system (2.1), we obtain

$$Qu_{\ell_t^\delta} = F(t), \quad t \in [t_\varepsilon, T],$$

where  $F(t)$  is defined by (2.6). Taking

$$u_t = Q^{-1}F(\gamma_t), \quad t \in [\ell_{t_\varepsilon}^\delta, \ell_T^\delta],$$

we immediately obtain that  $(x, u)$  solves the system (2.1) for  $t \in [t_\varepsilon, T]$ .

Next, since  $x_T^{\ell^\delta} = e^{-At_\varepsilon}a$  and  $\|e^{-At_\varepsilon}a - a\|_0 \leq \varepsilon/2$ , we have  $\|x_T^{\ell^\delta} - a\|_0 \leq \varepsilon/2$ . It remains to verify the claimed properties of  $u$  and  $F$ . By the regularity of Burgers equation (see the appendix below) and  $e^{-At_\varepsilon}$  respectively,  $x_{t_\varepsilon}^{\ell^\delta} \in \mathbb{H}^6$  and  $e^{-At_\varepsilon}a \in \mathbb{H}^6$ . For all  $t \in [t_\varepsilon, T]$ , we have

$$\|x_t^{\ell^\delta}\|_4 \leq \|e^{-At_\varepsilon}a\|_6 + \|x_{t_\varepsilon}^{\ell^\delta}\|_6^2,$$

$$\|B(x_t^{\ell^\delta})\|_4 \leq C\|x_t^{\ell^\delta}\|_6^2 \leq C\left(\|e^{-At_\varepsilon}a\|_6^2 + \|x_{t_\varepsilon}^{\ell^\delta}\|_6^2\right),$$

$$\|Ax_t^{\ell^\delta}\|_4 \leq C\left(\|e^{-At_\varepsilon}a\|_6 + \|x_{t_\varepsilon}^{\ell^\delta}\|_6\right) \leq C\left(1 + \|e^{-At_\varepsilon}a\|_6^2 + \|x_{t_\varepsilon}^{\ell^\delta}\|_6^2\right),$$

where the second inequality is by [25, Lemma 2.1]. Combining the above inequalities, we immediately get (2.7) and (2.8), as desired. Therefore,  $F \in \mathcal{C}([t_\varepsilon, T]; \mathbb{H}^4)$ , which, together with the assumption of  $Q$  and (2.5), yields  $u \in \mathcal{C}([0, \ell_T^\delta]; \mathbb{H}^2)$ .

Finally, it is easy to see that  $\|x_{t_\varepsilon}^{\ell^\delta}\|_6 < \infty$ . Below we present a proof for completeness. Noting that  $x_t^{\ell^\delta} \in \mathbb{H}^1$  for all  $t \in [0, t_\varepsilon]$ , letting  $t_1 = t_\varepsilon/3, t_2 = 2t_\varepsilon/3, t_3 = t_\varepsilon$  and taking

$\delta \in (0, \frac{1}{4})$ , we have

$$\begin{aligned}
 \|x_t^{\ell^\delta}\|_2 &\leq \|e^{-At}x_0\|_2 + \int_0^t \|A^{1-\delta}e^{-A(t-s)}\| \|B(x_s^{\ell^\delta})\|_{2\delta} ds \\
 (2.11) \quad &\leq Ct^{-\frac{1}{2}}\|x_0\|_1 + C \int_0^t (t-s)^{-1+\delta} \|x_s^{\ell^\delta}\|_1^2 ds \\
 &\leq C \left( t^{-\frac{1}{2}}\|x_0\|_1 + t^\delta \sup_{0 \leq s \leq t_3} \|x_s^{\ell^\delta}\|_1^2 \right), \quad t \in (0, t_3],
 \end{aligned}$$

where the last inequality is by (1.1) and (1.4). Now taking  $x_{t_1}^{\ell^\delta}$  as the initial data, we obtain

$$\begin{aligned}
 \|x_t^{\ell^\delta}\|_4 &\leq \|e^{-A(t-t_1)}x_{t_1}^{\ell^\delta}\|_4 + \int_{t_1}^t \|A^{1-\delta}e^{-A(t-t_1-s)}\| \|B(x_s^{\ell^\delta})\|_{2+2\delta} ds \\
 (2.12) \quad &\leq C(t-t_1)^{-1}\|x_{t_1}^{\ell^\delta}\|_2 + C \int_{t_1}^t (t-s)^{-1+\delta} \|x_s^{\ell^\delta}\|_2^2 ds \\
 &\leq C \left( (t-t_1)^{-1}\|x_{t_1}^{\ell^\delta}\|_2 + (t-t_1)^\delta \sup_{t_1 \leq s \leq t_3} \|x_s^{\ell^\delta}\|_2^2 \right), \quad t \in (t_1, t_3].
 \end{aligned}$$

Similarly, taking  $x_{t_2}^{\ell^\delta}$  as the initial data we get

$$(2.13) \quad \|x_t^{\ell^\delta}\|_6 \leq C \left( (t-t_2)^{-1}\|x_{t_1}^{\ell^\delta}\|_4 + (t-t_2)^\delta \sup_{t_2 \leq s \leq t_3} \|x_s^{\ell^\delta}\|_4^2 \right), \quad t \in (t_2, t_3].$$

This completes the proof.  $\square$

**Lemma 2.4.** For all  $t > 0$ , let

$$z_t^\ell = \int_0^t e^{-A(t-s)} Q du_{\ell_s}, \quad z_t^{\ell^\delta} = \int_0^t e^{-A(t-s)} Q du_{\ell_s^\delta}.$$

Then

$$(2.14) \quad \|z_t^{\ell^\delta} - z_t^\ell\|_2 \leq C_T(1 + \|e^{-At_\varepsilon}a\|_6^2 + \|x_{t_\varepsilon}\|_6^2)\delta, \quad t \in [0, T] \setminus \mathcal{J}(\ell).$$

*Proof.* By (2.5), we have  $u_t = 0$  for all  $0 \leq t \leq \ell_{t_\varepsilon}^\delta$ . Since  $\ell_t \leq \ell_t^\delta$ ,

$$(2.15) \quad z_t^\ell = z_t^{\ell^\delta} = 0, \quad t \in [0, t_\varepsilon].$$

Using integration by parts, we get

$$(2.16) \quad z_t^\ell = Qu_{\ell_t} - \int_0^t Ae^{-A(t-s)} Qu_{\ell_s} ds.$$

It is easy to see by (2.5) and (2.7) that for all  $0 \leq t \leq T$ ,

$$\|Qu_{\ell_t}\|_2 = \|F(\gamma_{\ell_t}^\delta)\|_2 \leq \sup_{0 \leq t \leq T} \|F(\gamma_{\ell_t}^\delta)\|_2 \leq C_T(1 + \|e^{-At_\varepsilon}a\|_6^2 + \|x_{t_\varepsilon}^\delta\|_6^2),$$

and that for all  $0 \leq t \leq T$  and  $0 \leq s \leq t$ ,

$$\begin{aligned}
 (2.17) \quad \|Ae^{-A(t-s)} Qu_{\ell_s}\|_2 &= \|e^{-A(t-s)} Qu_{\ell_s}\|_4 \leq \|Qu_{\ell_s}\|_4 = \|F(\gamma_{\ell_s}^\delta)\|_4 \\
 &\leq C_T(1 + \|e^{-At_\varepsilon}a\|_6^2 + \|x_{t_\varepsilon}^\delta\|_6^2).
 \end{aligned}$$



Hence,

$$\|z_t^\ell\|_2 \leq C_T(1 + \|e^{-At_\varepsilon}a\|_6^2 + \|x_{t_\varepsilon}\|_6^2), \quad 0 \leq t \leq T.$$

Similarly,

$$\|z_t^{\ell^\delta}\|_2 \leq C_T(1 + \|e^{-At_\varepsilon}a\|_6^2 + \|x_{t_\varepsilon}\|_6^2), \quad 0 \leq t \leq T.$$

Using integration by parts again, we further get

$$z_t^{\ell^\delta} - z_t^\ell = Q(u_{\ell_t^\delta} - u_{\ell_t}) - \int_0^t Ae^{-A(t-s)}Q(u_{\ell_s^\delta} - u_{\ell_s})ds$$

which, together with (2.5) and (2.8), yields

$$\begin{aligned} \|z_t^{\ell^\delta} - z_t^\ell\|_2 &\leq \|F(\gamma_{\ell_t^\delta}) - F(\gamma_{\ell_t})\|_2 + \int_0^t \|Q(u_{\ell_s^\delta} - u_{\ell_s})\|_4 ds \\ &\leq \|F(\gamma_{\ell_t^\delta}) - F(\gamma_{\ell_t})\|_2 + \int_0^t \|F(\gamma_{\ell_s^\delta}) - F(\gamma_{\ell_s})\|_4 ds \\ &\leq C_T(1 + \|e^{-At_\varepsilon}a\|_6^2 + \|x_{t_\varepsilon}\|_6^2) \left[ |\gamma_{\ell_t^\delta}^\delta - \gamma_{\ell_t}^\delta| + \int_0^t |\gamma_{\ell_s^\delta}^\delta - \gamma_{\ell_s}^\delta| ds \right] \\ &= C_T(1 + \|e^{-At_\varepsilon}a\|_6^2 + \|x_{t_\varepsilon}\|_6^2) \left[ |t - \gamma_{\ell_t}^\delta| + \int_0^t |s - \gamma_{\ell_s}^\delta| ds \right], \end{aligned}$$

where the last equality is by  $\gamma_{\ell_t^\delta}^\delta = t$  for all  $t \geq 0$ . By the definition of  $\gamma$ , if  $t \notin \mathcal{J}(\ell)$ , i.e.  $t$  is a continuous point of  $\ell$ , we have  $\gamma_{\ell_t} = t$ . Therefore, by Lemma 2.2, we have

$$|t - \gamma_{\ell_t}^\delta| \leq |t - \gamma_{\ell_t}| + |\gamma_{\ell_t}^\delta - \gamma_{\ell_t}| \leq |t - \gamma_{\ell_t}| + \delta \leq \delta, \quad t \in [0, T] \setminus \mathcal{J}(\ell).$$

Since  $\ell$  has at most countably infinite jump points, Lebesgue measure of  $\mathcal{J}(\ell)$  is zero. Thus,

$$\int_0^t |s - \gamma_{\ell_s}^\delta| ds \leq T\delta, \quad t \in [0, T]$$

and

$$\|z_t^{\ell^\delta} - z_t^\ell\|_2 \leq C_T(1 + \|e^{-At_\varepsilon}a\|_6^2 + \|x_{t_\varepsilon}\|_6^2)\delta, \quad t \in [0, T] \setminus \mathcal{J}(\ell).$$

□

We are now at the position to prove Proposition 2.1. t

*Proof of Proposition 2.1.* Let  $\delta > 0$  be small enough to be chosen. By Lemma 2.3, the equation

$$(2.18) \quad dx_t^{\ell^\delta} + [Ax_t^{\ell^\delta} + B(x_t^{\ell^\delta})] dt = Qdu_{\ell_t^\delta}, \quad x_0^{\ell^\delta} = x_0$$

is solved by  $u \in \mathcal{C}([0, \ell_T^\delta]; \mathbb{H}^2)$  and  $x^{\ell^\delta} \in \mathcal{C}([0, T]; \mathbb{H}^1)$ , which have the forms (2.9)-(2.6) and

$$\|x_T^{\ell^\delta} - a\|_0 \leq \varepsilon/2.$$

We will compare Eq. (2.18) with the following equation:

$$(2.19) \quad dx_t^\ell + [Ax_t^\ell + B(x_t^\ell)] dt = Qdu_{\ell_t}, \quad x_0 = x_0.$$

Denote  $y_t^\ell = x_t^\ell - z_t^\ell$  and  $y_t^{\ell\delta} = x_t^{\ell\delta} - z_t^{\ell\delta}$ . Then

$$\begin{aligned} \frac{dy_t^{\ell\delta}}{dt} + Ay_t^{\ell\delta} + B(x_t^{\ell\delta}) &= 0, & y_0^{\ell\delta} &= x_0, \\ \frac{dy_t^\ell}{dt} + Ay_t^\ell + B(x_t^\ell) &= 0, & y_0^\ell &= x_0. \end{aligned}$$

By (2.15), we have

$$y_t^{\ell\delta} - y_t^\ell = 0, \quad t \in [0, t_\varepsilon].$$

Write  $\Delta y_t^\ell = y_t^\ell - y_t^{\ell\delta}$ ,  $\Delta x_t^\ell = x_t^\ell - x_t^{\ell\delta}$  and  $\Delta z_t^\ell = z_t^\ell - z_t^{\ell\delta}$  for  $t \in [t_\varepsilon, T]$ . Then

$$(2.20) \quad \|\Delta y_t^\ell\|_0^2 + 2 \int_{t_\varepsilon}^t \|\Delta y_s^\ell\|_1^2 ds \leq 2 \left| \int_{t_\varepsilon}^t \langle \Delta y_s^\ell, B(x_s^{\ell\delta}) - B(x_s^\ell) \rangle_0 ds \right|.$$

Noting that

$$\begin{aligned} B(x_s^\ell) - B(x_s^{\ell\delta}) &= B(x_s^\ell, \Delta x_s^\ell) + B(\Delta x_s^\ell, x_s^{\ell\delta}) \\ &= B(\Delta x_s^\ell) + B(\Delta x_s^\ell, x_s^{\ell\delta}) + B(x_s^{\ell\delta}, \Delta x_s^\ell) \\ &= B(\Delta y_s^\ell) + B(\Delta z_s^\ell) + B(\Delta y_s^\ell, \Delta z_s^\ell) + B(\Delta z_s^\ell, \Delta y_s^\ell) + B(\Delta x_s^\ell, x_s^{\ell\delta}) + B(x_s^{\ell\delta}, \Delta x_s^\ell), \end{aligned}$$

and that  $\langle x, B(x, x) \rangle_0 = 0$  for  $x \in \mathbb{H}^1$ , we obtain

$$\begin{aligned} |\langle \Delta y_s^\ell, B(x_s^\ell) - B(x_s^{\ell\delta}) \rangle_0| &\leq \|\Delta y_s^\ell\|_0 \left[ \|B(\Delta z_s^\ell)\|_0 + \|B(\Delta y_s^\ell, \Delta z_s^\ell)\|_0 + \|B(\Delta z_s^\ell, \Delta y_s^\ell)\|_0 \right. \\ &\quad \left. + \|B(\Delta x_s^\ell, x_s^{\ell\delta})\|_0 + \|B(x_s^{\ell\delta}, \Delta x_s^\ell)\|_0 \right]. \end{aligned}$$

Combining this with (1.4) and the inequality  $2ab \leq a^2 + b^2$  for  $a \geq 0$  and  $b \geq 0$ , we arrive at

$$\begin{aligned} |\langle \Delta y_s^\ell, B(x_s^\ell) - B(x_s^{\ell\delta}) \rangle_0| &\leq C \|\Delta y_s^\ell\|_0 \left[ \|\Delta z_s^\ell\|_1^2 + \|\Delta y_s^\ell\|_1 \|\Delta z_s^\ell\|_1 + \|\Delta x_s^\ell\|_1 \|x_s^{\ell\delta}\|_1 \right] \\ &\leq C \|\Delta y_s^\ell\|_0 \left[ \|\Delta z_s^\ell\|_1^2 + \|\Delta y_s^\ell\|_1 \|\Delta z_s^\ell\|_1 + \|\Delta y_s^\ell\|_1 \|x_s^{\ell\delta}\|_1 + \|\Delta z_s^\ell\|_1 \|x_s^{\ell\delta}\|_1 \right] \\ &\leq \|\Delta y_s^\ell\|_1^2 + C \|\Delta y_s^\ell\|_0^2 \left( \|\Delta z_s^\ell\|_1^2 + \|x_s^{\ell\delta}\|_1^2 \right) + C \|\Delta z_s^\ell\|_1^2. \end{aligned}$$

This, together with (2.20) and (2.14), implies

$$\begin{aligned} \|\Delta y_t^\ell\|_0^2 &\leq C \int_{t_\varepsilon}^t \|\Delta y_s^\ell\|_0^2 \left( \|\Delta z_s^\ell\|_1^2 + \|x_s^{\ell\delta}\|_1^2 \right) ds + C \int_{t_\varepsilon}^t \|\Delta z_s^\ell\|_1^2 ds \\ &\leq C \int_{t_\varepsilon}^t \|\Delta y_s^\ell\|_0^2 \left( \|\Delta z_s^\ell\|_1^2 + \|x_s^{\ell\delta}\|_1^2 \right) ds + C_T (1 + \|e^{-At_\varepsilon} a\|_6^4 + \|x_{t_\varepsilon}\|_6^4) \delta^2, \quad t \in [t_\varepsilon, T]. \end{aligned}$$

By Gronwall's inequality, we obtain

$$\|\Delta y_T^\ell\|_0^2 \leq C_T \exp \left[ C \int_{t_\varepsilon}^T \left( \|\Delta z_s^\ell\|_1^2 + \|x_s^{\ell\delta}\|_1^2 \right) ds \right] (1 + \|e^{-At_\varepsilon} a\|_6^2 + \|x_{t_\varepsilon}\|_6^2) \delta^2.$$

On the other hand, (2.10) implies

$$\|x_t^{\ell\delta}\|_1 \leq \|e^{-At_\varepsilon} a\|_1 + \|x_{t_\varepsilon}^{\ell\delta}\|_1 \leq C \left( \|e^{-At_\varepsilon} a\|_6 + \|x_{t_\varepsilon}^{\ell\delta}\|_6 \right), \quad t \in [t_\varepsilon, T],$$

which, together with (2.14), leads to

$$\int_{t_\varepsilon}^T \left( \|\Delta z_s^\ell\|_1^2 + \|x_s^{\ell\delta}\|_1^2 \right) ds \leq C_T (1 + \|e^{-At_\varepsilon} a\|_6^4 + \|x_{t_\varepsilon}^{\ell\delta}\|_6^4)$$

Hence,

$$\|\Delta y_T^\ell\|_0^2 \leq C_T \exp \left[ C_T (1 + \|e^{-At_\varepsilon} a\|_6^4 + \|x_{t_\varepsilon}^{\ell\delta}\|_6^4) \right] (1 + \|e^{-At_\varepsilon} a\|_6^4 + \|x_{t_\varepsilon}^{\ell\delta}\|_6^4) \delta^2.$$

Combining this with (2.14), as long as  $\delta > 0$  is chosen to be sufficiently small we obtain

$$\|\Delta x_T^\ell\|_0^2 \leq 2\|\Delta y_T^\ell\|_0^2 + 2\|\Delta z_T^\ell\|_0^2 \leq \frac{\varepsilon^2}{4}, \quad T \notin \mathcal{J}(\ell).$$

Therefore, it follows from Lemma 2.3 that

$$\|x_T^\ell - a\|_0 \leq \|\Delta x_T^\ell\|_0 + \|x_T^{\ell\delta} - a\|_0 \leq \varepsilon, \quad T \in \mathcal{J}(\ell).$$

The proof is then complete.  $\square$

### 3. ESTIMATE OF CONVOLUTIONS

For  $\ell \in \mathcal{S}$ ,  $T > 0$  and  $u \in \mathcal{C}([0, \ell_T])$ , let  $z_t^\ell$  be given in (2.2), and define

$$(3.1) \quad Z_t^\ell := \int_0^t e^{-(t-s)A} Q dW_{\lambda_s} \quad t \geq 0.$$

**Lemma 3.1.** *For any  $T > 0$ ,  $\gamma \in [1, \theta' - \frac{1}{2})$  and  $p \geq 1$ , there exists a constant  $C > 0$  such that*

$$(3.2) \quad \mathbb{E} \left[ \sup_{0 \leq t \leq T} \|Z_t^\ell\|_\gamma^p \right] \leq C \ell_T^{p/2}, \quad \ell \in \mathcal{S}.$$

*Proof.* Using integration by parts, we have

$$Z_t^\ell = \int_0^t e^{-A(t-s)} Q dW_{\lambda_s} = QW_{\ell_t} + \int_0^t A e^{-A(t-s)} QW_{\lambda_s} ds.$$

By (1.3) and the martingale inequality, we obtain

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq T} \|QW_{\ell_t}\|_\gamma^p &\leq \mathbb{E} \sup_{0 \leq t \leq \ell_T} \|QW_t\|_\gamma^p \\ &\leq C_{\gamma, \theta'} \mathbb{E} \sup_{0 \leq t \leq \ell_T} \|W_t\|_{\gamma - \theta'}^p \\ &\leq C_{\gamma, \theta', p} \mathbb{E} \|W_{\ell_T}\|_{\gamma - \theta'}^p \leq C_{\gamma, \theta', p} \ell_T^{p/2}. \end{aligned}$$

For  $\gamma' \in (\gamma, \theta' - \frac{1}{2})$ , (2.1) implies

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t Ae^{-A(t-s)} QW_{\ell_s} ds \right\|_{\gamma}^p &\leq \mathbb{E} \sup_{0 \leq t \leq T} \left( \int_0^t \|Ae^{-A(t-s)} QW_{\ell_s}\|_{\gamma} ds \right)^p \\ &= \mathbb{E} \sup_{0 \leq t \leq T} \left( \int_0^t \|A^{1+\gamma-\gamma'} e^{-A(t-s)} QA^{\gamma'-\gamma} W_{\ell_s}\|_{\gamma} ds \right)^p \\ &\leq C_{\gamma, \gamma'} \mathbb{E} \sup_{0 \leq t \leq T} \left( \int_0^t (t-s)^{-1-\gamma+\gamma'} \|QA^{\gamma'-\gamma} W_{\ell_s}\|_{\gamma} ds \right)^p \\ &\leq C_{\gamma, \gamma', \theta'} \mathbb{E} \sup_{0 \leq t \leq T} \left( \int_0^t (t-s)^{-1-\gamma+\gamma'} \|W_{\ell_s}\|_{\gamma'-\theta'} ds \right)^p. \end{aligned}$$

Since

$$\begin{aligned} \int_0^t (t-s)^{-1-\gamma+\gamma'} \|W_{\ell_s}\|_{\gamma'-\theta'} ds &\leq \sup_{0 \leq t \leq T} \|W_{\ell_s}\|_{\gamma'-\theta'} \int_0^t (t-s)^{-1-\gamma+\gamma'} ds \\ &\leq C_{\gamma, \gamma', T} \sup_{0 \leq t \leq T} \|W_{\ell_s}\|_{\gamma'-\theta'}, \end{aligned}$$

by the same argument as the above we get

$$\mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t Ae^{-A(t-s)} QW_{\ell_s} ds \right\|_{\gamma}^p \leq C_{\gamma, \gamma', \theta', p, T} \ell_T^{p/2}.$$

Collecting the above inequalities, we obtain the desired estimate.  $\square$

**Lemma 3.2.** For any  $\ell \in \mathcal{S}$ ,  $T > 0$  and  $\varepsilon > 0$ ,

$$\mathbb{P} \left( \sup_{0 \leq t \leq T} \|Z_t^{\ell} - z_t^{\ell}\|_1 \leq \varepsilon \right) > 0.$$

*Proof.* For any  $N \in \mathbb{N}$ , let  $\mathcal{H}_N = \text{span}\{e_i : i \leq N\}$  and let  $\mathcal{H}^N$  be its orthogonal complementary. Let  $\Pi_N : \mathbb{H} \rightarrow \mathcal{H}_N$  and  $\Pi^N : \mathbb{H} \rightarrow \mathcal{H}^N$  to be the corresponding orthogonal projections. We have

$$\begin{aligned} &\mathbb{P} \left( \sup_{0 \leq t \leq T} \|Z_t^{\ell} - z_t^{\ell}\|_1 \leq \varepsilon \right) \\ &\geq \mathbb{P} \left( \sup_{0 \leq t \leq T} \|\Pi_N(Z_t^{\ell} - z_t^{\ell})\|_1 \leq \frac{\varepsilon}{2}, \sup_{0 \leq t \leq T} \|\Pi^N(Z_t^{\ell} - z_t^{\ell})\|_1 \leq \frac{\varepsilon}{2} \right) \\ &= \mathbb{P} \left( \sup_{0 \leq t \leq T} \|\Pi_N(Z_t^{\ell} - z_t^{\ell})\|_1 \leq \frac{\varepsilon}{2} \right) \mathbb{P} \left( \sup_{0 \leq t \leq T} \|\Pi^N(Z_t^{\ell} - z_t^{\ell})\|_1 \leq \frac{\varepsilon}{2} \right), \end{aligned}$$

where the last inequality follows from the independence of  $\Pi_N Z_t^{\ell}$  and  $\Pi^N Z_t^{\ell}$ . Below, we estimate these two probabilities respectively.

For the first one, using integration by parts, we get

$$Z_t^{\ell} - z_t^{\ell} = Q(W_{\ell_t} - u_{\ell_t}) + \int_0^t Ae^{-A(t-s)} Q(W_{\ell_s} - u_{\ell_s}) ds.$$

Obviously, there exist a constant  $C_N > 0$  such that

$$\|\Pi_N [Q(W_{\ell_t} - u_{\ell_t})]\|_1 \leq C_N \|\Pi_N [W_{\ell_t} - u_{\ell_t}]\|_0,$$

and

$$\begin{aligned} \left\| \Pi_N \int_0^t A e^{-A(t-s)} Q(W_{\ell_s} - u_{\ell_s}) ds \right\|_1 &\leq \int_0^t \left\| \Pi_N \int_0^s A e^{-A(s-r)} Q(W_{\ell_r} - u_{\ell_r}) \right\|_1 ds \\ &\leq C_N \int_0^t \|\Pi_N [W_{\ell_s} - u_{\ell_s}]\|_0 ds \\ &\leq TC_N \sup_{0 \leq t \leq \ell_T} \|\Pi_N [W_t - u_t]\|_0. \end{aligned}$$

Hence,

$$\begin{aligned} \sup_{0 \leq t \leq T} \|\Pi^N(Z_t^\ell - z_t^\ell)\|_1 &\leq TC_N \sup_{0 \leq t \leq T} \|\Pi_N [W_{\ell_t} - u_{\ell_t}]\|_0 \\ &\leq TC_N \sup_{0 \leq t \leq \ell_T} \|\Pi_N [W_t - u_t]\|_0. \end{aligned}$$

It is clear  $(\Pi_N W_t)_{t \geq 0}$  and  $(\Pi_N u_t)_{t \geq 0}$  can be identified with an  $N$  dimensional standard Wiener process and a continuous function in  $\mathcal{C}([0, \infty); \mathbb{R}^N)$ . Since the support of a Brownian motion is the whole continuous function space, we have

$$\mathbb{P}\left(\sup_{0 \leq t \leq \ell_T} \|\Pi_N (W_t - u_t)\|_0 \leq \delta\right) > 0, \quad \delta > 0.$$

Therefore,

$$(3.3) \quad \mathbb{P}\left(\sup_{0 \leq t \leq T} \|\Pi_N(Z_t^\ell - z_t^\ell)\|_1 \leq \frac{\varepsilon}{2}\right) > 0.$$

On the other hand, by (3.2) with  $\gamma \in (1, \theta' - \frac{1}{2})$ , Chebyshev's inequality and the spectral inequality  $\|\Pi^N x\|_1 \leq \lambda_N^{\gamma-1} \|x\|_\gamma$  for  $x \in \mathbb{H}^\gamma$ , we have

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq t \leq T} \|\Pi^N(Z_t^\ell - z_t^\ell)\|_1 \geq \frac{\varepsilon}{2}\right) &\leq \mathbb{P}\left(\sup_{0 \leq t \leq T} \|(Z_t^\ell - z_t^\ell)\|_\gamma \geq \frac{\varepsilon}{2} \lambda_N^{\gamma-1}\right) \\ &\leq \frac{2\mathbb{E}\left[\sup_{0 \leq t \leq T} \|Z_t^\ell\|_\gamma\right] + 2 \sup_{0 \leq t \leq T} \|z_t^\ell\|_\gamma}{\varepsilon \lambda_N^{\gamma-1}}. \end{aligned}$$

From the previous inequality and (3.2), choose a sufficiently large  $N$ , we get

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} \|\Pi^N(Z_t^\ell - z_t^\ell)\|_1 \geq \frac{\varepsilon}{2}\right) < 1,$$

equivalently,

$$(3.4) \quad \mathbb{P}\left(\sup_{0 \leq t \leq T} \|\Pi^N(Z_t^\ell - z_t^\ell)\|_1 < \frac{\varepsilon}{2}\right) > 0.$$

Combining (3.3), (3.3) and (3.4), we finish the proof.  $\square$

## 4. PROOF OF THEOREM 1.2

For  $\ell \in \mathcal{S}$ , let  $Z_t^\ell$  be in (3.1), and let  $X_t^\ell$  solve

$$(4.1) \quad dX_t^\ell = [-AX_t^\ell - B(X_t^\ell)]dt + QdW_{\ell t}, \quad X_0^\ell = x_0 \in \mathbb{H}.$$

Then  $Y_t^\ell := X_t^\ell - Z_t^\ell$  satisfies

$$(4.2) \quad \frac{dY_t^\ell}{dt} + AY_t^\ell + B(Y_t^\ell + Z_t^\ell) = 0, \quad Y_0^\ell = x_0.$$

**Proof of Theorem 1.2.** Since  $S \in \mathcal{S}$  a.s., it suffices to show that for each  $\ell \in \mathcal{S}$ ,

$$(4.3) \quad \mathbb{P}(\|X_T^\ell - a\|_0 \leq \varepsilon) > 0.$$

Since  $X_t^\ell \in \mathbb{H}^1$  for  $t > 0$ , by the Markov property, we may and do assume that  $x_0 \in \mathbb{H}^1$ . Below, we prove (4.3) for  $x_0 \in \mathbb{H}^1$ .

By Proposition 2.1, there exist  $u \in \mathcal{C}([0, T]; \mathbb{H}^4)$  with bounded total variation and  $x^\ell \in \mathcal{D}([0, T]; \mathbb{H}^1)$  solving

$$dx_t^\ell + [Ax_t^\ell + B(x_t^\ell)] dt = Qdu_{\ell t}, \quad x_0^\ell = x_0,$$

such that

$$\|x_T^\ell - a\|_0 \leq \varepsilon/2, \quad T \notin \mathcal{J}(\ell).$$

So, when  $T \notin \mathcal{J}(\ell)$  we have

$$(4.4) \quad \begin{aligned} \mathbb{P}(\|X_T^\ell - a\|_0 \leq \varepsilon) &\geq \mathbb{P}\left(\|X_T^\ell - x_T^\ell\|_0 \leq \frac{\varepsilon}{2}, \|X_T^\ell - a\|_0 \leq \frac{\varepsilon}{2}\right) \\ &= \mathbb{P}\left(\|X_T^\ell - x_T^\ell\|_0 \leq \frac{\varepsilon}{2}\right) \geq \mathbb{P}\left(\|Y_T^\ell - y_T^\ell\|_0 \leq \frac{\varepsilon}{4}, \|Z_T^\ell - z_T^\ell\|_0 \leq \frac{\varepsilon}{4}\right) \\ &\geq \mathbb{P}\left(\|Y_T^\ell - y_T^\ell\|_0 \leq \frac{\varepsilon}{4}, \sup_{0 \leq t \leq T} \|Z_t^\ell - z_t^\ell\|_0 \leq \varepsilon'\right), \quad \varepsilon' \in (0, \varepsilon/4), \end{aligned}$$

where  $z_t^\ell = \int_0^t e^{-A(t-s)} Qdu_{\ell s}$  and  $y_t^\ell$  are in (2.2).

Write  $\Delta Y_t^\ell = Y_t^\ell - y_t^\ell$ ,  $\Delta X_t^\ell = X_t^\ell - x_t^\ell$  and  $\Delta Z_t^\ell = Z_t^\ell - z_t^\ell$ . Then (2.3) and (4.2) yield

$$\frac{d\Delta Y_t^\ell}{dt} + A\Delta Y_t^\ell + B(X_t^\ell) - B(x_t^\ell) = 0, \quad \Delta Y_0^\ell = 0,$$

which clearly implies

$$\|\Delta Y_t^\ell\|_0^2 + 2 \int_0^t \|\Delta Y_s^\ell\|_1^2 ds \leq 2 \int_0^t |\langle \Delta Y_s^\ell, B(X_s^\ell) - B(x_s^\ell) \rangle_0| ds.$$

Since  $\langle x, B(x, x) \rangle_0 = 0$  for  $x \in \mathbb{H}^1$ , we have

$$\begin{aligned} &|\langle \Delta Y_s^\ell, B(X_s^\ell) - B(x_s^\ell) \rangle_0| \\ &= \langle \Delta Y_s^\ell, B(\Delta X_s^\ell) \rangle_0 + \langle \Delta Y_s^\ell, B(\Delta X_s^\ell, x_s^\ell) \rangle_0 + \langle \Delta Y_s^\ell, B(x_s^\ell, \Delta X_s^\ell) \rangle_0 \\ &= \langle \Delta Y_s^\ell, B(\Delta Y_s^\ell, \Delta Z_s^\ell) \rangle_0 + \langle \Delta Y_s^\ell, B(\Delta Z_s^\ell, \Delta Y_s^\ell) \rangle_0 + \langle \Delta Y_s^\ell, B(\Delta Z_s^\ell, \Delta Z_s^\ell) \rangle_0 \\ &\quad + \langle \Delta Y_s^\ell, B(\Delta X_s^\ell, x_s^\ell) \rangle_0 + \langle \Delta Y_s^\ell, B(x_s^\ell, \Delta X_s^\ell) \rangle_0, \end{aligned}$$

which, together with (1.4) and the inequality  $2ab \leq a^2 + b^2$  for  $a, b \geq 0$ , implies

$$\begin{aligned} & |\langle Y_s^\ell, B(X_s^\ell) - B(x_s^\ell) \rangle_0| \\ & \leq C(\|\Delta Y_s^\ell\|_0 \|\Delta Y_s^\ell\|_1 \|\Delta Z_s^\ell\|_1 + \|\Delta Y_s^\ell\|_0 \|\Delta Z_s^\ell\|_1^2 + \|x_s^\ell\|_1 \|\Delta Y_s^\ell\|_0 \|\Delta X_s^\ell\|_1) \\ & \leq C(\|\Delta Z_s^\ell\|_1^2 + \|x_s^\ell\|_1^2) \|\Delta Y_s^\ell\|_0^2 + C\|\Delta Z_s^\ell\|_1^2 + \left(\frac{1}{2}\|\Delta Y_s^\ell\|_1^2 + \frac{1}{4}\|\Delta X_s^\ell\|_1^2\right) \\ & \leq C(\|\Delta Z_s^\ell\|_1^2 + \|x_s^\ell\|_1^2) \|\Delta Y_s^\ell\|_0^2 + \|\Delta Y_s^\ell\|_1^2 + C\|\Delta Z_s^\ell\|_1^2 \end{aligned}$$

for some constant  $C > 0$ . Hence,

$$\begin{aligned} \|\Delta Y_t^\ell\|^2 & \leq C \int_0^t (\|\Delta Z_s^\ell\|_1^2 + \|x_s^\ell\|_1^2) \|\Delta Y_s^\ell\|_0^2 ds + C \int_0^t \|\Delta Z_s^\ell\|_1^2 ds \\ & \leq C(\sup_{0 \leq t \leq T} \|\Delta Z_t^\ell\|_1^2 + \sup_{0 \leq t \leq T} \|x_t^\ell\|_1^2) \int_0^t \|\Delta Y_s^\ell\|_0^2 ds + CT \sup_{0 \leq t \leq T} \|\Delta Z_t^\ell\|_1^2, \quad 0 \leq t \leq T. \end{aligned}$$

When  $\sup_{0 \leq t \leq T} \|\Delta Z_t^\ell\|_0 \leq \varepsilon'$ , we have

$$\|\Delta Y_t^\ell\|^2 \leq C((\varepsilon')^2 + \sup_{0 \leq t \leq T} \|x_t^\ell\|_1^2) \int_0^t \|\Delta Y_s^\ell\|_0^2 ds + CT(\varepsilon')^2.$$

By Gronwall's inequality,

$$\|\Delta Y_T^\ell\|^2 \leq CT \exp \left[ C(\varepsilon' + \sup_{0 \leq t \leq T} \|x_t^\ell\|_1) T \right] (\varepsilon')^2, \quad \text{if } \sup_{0 \leq t \leq T} \|\Delta Z_t^\ell\|_0 \leq \varepsilon'.$$

Since  $\sup_{0 \leq t \leq T} \|x_t^\ell\|_1 < \infty$ , when  $\varepsilon'$  is sufficiently this implies

$$\|\Delta Y_T^\ell\|_0 \leq \frac{\varepsilon}{4}, \quad \text{if } \sup_{0 \leq t \leq T} \|\Delta Z_t^\ell\|_0 \leq \varepsilon'.$$

Hence, for small enough  $\varepsilon' > 0$ ,

$$\mathbb{P} \left( \left\| Y_T^\ell - y_T^\ell \right\|_0 \leq \frac{\varepsilon}{4}, \sup_{0 \leq t \leq T} \left\| Z_T^\ell - z_T^\ell \right\|_0 \leq \varepsilon' \right) = \mathbb{P} \left( \left\| Z_T^\ell - z_T^\ell \right\|_0 \leq \varepsilon' \right) > 0.$$

This and (4.4) yield that (4.3) holds for  $T \notin \mathcal{J}(\ell)$ . Since  $X_t$  is right continuous and the set  $[0, \infty) \setminus \mathcal{J}(\ell)$  is dense, (4.3) holds for all  $T > 0$ . Then the proof is finished.  $\square$

## 5. $\psi$ -UNIFORMLY EXPONENTIAL ERGODICITY AND MODERATE DEVIATION

**5.1. Galerkin approximation.** Recall that  $\{e_k\}_{k \in \mathbb{N}}$  is an orthonormal basis of  $\mathbb{H}$ . For any  $m \in \mathbb{N}$ , let  $\mathcal{H}_m := \text{span}\{e_k : k \leq m\}$  with orthogonal projection  $\Pi_m : \mathbb{H} \rightarrow \mathcal{H}_m$ . Then the Galerkin approximation of (1.2) reads

$$(5.1) \quad d\tilde{X}_t^m + [A\tilde{X}_t^m + B^m(\tilde{X}_t^m)]dt = QdL_t^m, \quad \tilde{X}_0^m = x^m,$$

where  $x^m = \Pi_m x$ ,  $B^m(x) = \Pi_m[B(x)]$  for  $x \in \mathbb{H}$ , and  $L_t^m = \Pi_m L_t = W_{S_t}^m$  with  $W_t^m$  being an  $m$ -dimensional standard Brownian motion.

Since the Lévy measure of  $W_{S_t}$  can not be approximated by those of  $W_{S_t}^m$ , the approximation procedure in [] does not apply. Alternatively, we show that  $\Delta X_t^m = \tilde{X}_t^m - X_t^m$  converges to zero. The advantage of this new procedure is that the approximation of  $W_{S_t}$  is avoided.

**Theorem 5.1.** *For all  $t > 0$ ,  $\mathbb{P}$ -a.s.*

$$(5.2) \quad \lim_{m \rightarrow \infty} \|\tilde{X}_t^m - X_t\|_1 = 0.$$

*Proof.* Let  $X_t$  solve (1.2) with  $X_0 = x$ , and denote  $X_t^m = \Pi_m X_t$ . Then

$$(5.3) \quad dX_t^m + [AX_t^m + B^m(X_t)]dt = QdL_t^m, \quad X_0^m = x^m.$$

By (1.6) and Theorem 1.1,

$$\lim_{m \rightarrow \infty} \|X_t^m - X_t\|_1 = 0, \quad t > 0.$$

Combining this with Lemma 5.2 below, we finish the proof. □

**Lemma 5.2.** *Let  $\Delta X_t^m = \tilde{X}_t^m - X_t^m$ . Then  $\mathbb{P}$ -a.s.*

$$\lim_{m \rightarrow \infty} \|\Delta X_t^m\|_1 = 0, \quad t \geq 0.$$

*Proof.* (1) We first prove that for some constant  $C > 0$ ,

$$(5.4) \quad \sup_{0 \leq t \leq T, m \in \mathbb{N}} \|\tilde{X}_t^m\|_0^2 \leq A_T, \quad T > 0, m \in \mathbb{N},$$

holds for

$$A_T := 2 \exp \left( C \int_0^T (1 + \|Z_s\|_1^2) ds \right) \left[ \|x\|_0^2 + T \sup_{0 \leq t \leq T} \|Z_t\|_1^4 \right] + 2 \sup_{0 \leq t \leq T} \|Z_t\|_1^2.$$

For  $\ell \in \mathcal{S}$ , let

$$Z_t^{m,\ell} = \int_0^t e^{-A(t-s)} Q dW_{\ell_s}^m.$$

Then

$$\|Z_t^{m,\ell}\|_\gamma \leq \|Z_t^\ell\|_\gamma, \quad \gamma \in \mathbb{R}.$$

By (3.2) with  $\gamma = 1$ , we have  $\mathbb{P}$ -a.s.

$$(5.5) \quad \sup_{0 \leq t \leq T, m \in \mathbb{N}} \|Z_t^{m,\ell}\|_0 \leq \sup_{0 \leq t \leq T, m \in \mathbb{N}} \|Z_t^{m,\ell}\|_1 \leq \sup_{0 \leq t \leq T} \|Z_t^\ell\|_1 < \infty.$$

It is easy to see that  $\tilde{Y}_t^{m,\ell} := \tilde{X}_t^{m,\ell} - Z_t^{m,\ell}$  solves the equation

$$(5.6) \quad \partial_t \tilde{Y}_t^{m,\ell} + A \tilde{Y}_t^{m,\ell} + B^m(\tilde{Y}_t^{m,\ell} + Z_t^{m,\ell}) = 0, \quad \tilde{X}_0^{m,\ell} = x^m.$$

Applying the chain rule to  $\|\tilde{Y}_t^{m,\ell}\|_0^2$  gives

$$(5.7) \quad \|\tilde{Y}_t^{m,\ell}\|_0^2 + 2 \int_0^t \|\tilde{Y}_s^{m,\ell}\|_1^2 ds = \|x^m\|_0^2 + 2 \int_0^t \langle \tilde{Y}_s^{m,\ell}, B^m(\tilde{Y}_s^{m,\ell} + Z_s^{m,\ell}) \rangle ds.$$

Letting  $\tilde{B}^m(x, y) = B^m(x, y) + B^m(y, x)$ , the relation  $\langle \tilde{Y}_s^{m,\ell}, B^m(\tilde{Y}_s^{m,\ell}) \rangle = 0$  implies

$$\begin{aligned} & |\langle \tilde{Y}_s^{m,\ell}, B^m(\tilde{Y}_s^{m,\ell} + Z_s^{m,\ell}) \rangle| \\ &= |\langle \tilde{Y}_s^{m,\ell}, \tilde{B}^m(\tilde{Y}_s^{m,\ell}, Z_s^{m,\ell}) + B^m(Z_s^{m,\ell}) \rangle| \\ &\leq C \|\tilde{Y}_s^{m,\ell}\|_0 \|\tilde{Y}_s^{m,\ell}\|_1 \|Z_s^{m,\ell}\|_1 + C \|\tilde{Y}_s^{m,\ell}\|_0 \|Z_s^{m,\ell}\|_1^2 \\ &\leq C(1 + \|Z_s^{m,\ell}\|_1^2) \|\tilde{Y}_s^{m,\ell}\|_0^2 + \|\tilde{Y}_s^{m,\ell}\|_1^2 + \|Z_s^{m,\ell}\|_1^4 \\ &\leq C(1 + \|Z_s^\ell\|_1^2) \|\tilde{Y}_s^{m,\ell}\|_0^2 + \|\tilde{Y}_s^{m,\ell}\|_1^2 + \|Z_s^\ell\|_1^4, \end{aligned}$$



for some constant  $C > 0$  independent of  $m$  and  $T$ . Combining this with (5.7) and  $\|x^m\|_0 \leq \|x\|_0$ , we arrive at

$$\|\tilde{Y}_t^{m,\ell}\|_0^2 \leq \|x\|_0^2 + C \int_0^t (1 + \|Z_s^\ell\|_1^2) \|\tilde{Y}_s^{m,\ell}\|_0^2 ds + \int_0^t \|Z_s^\ell\|_1^4 ds.$$

By Gronwall's lemma this implies

$$\|\tilde{Y}_t^{m,\ell}\|_0^2 \leq \exp\left(C \int_0^t (1 + \|Z_s^\ell\|_1^2) ds\right) \|x\|_0^2 + \int_0^t \exp\left[C \int_s^t (1 + \|Z_r^\ell\|_1^2) dr\right] \|Z_s^\ell\|_1^4 ds,$$

so that (5.4) holds.

(2) By the equations (5.6) and (5.3), we have

$$\partial_t \Delta X_t^m + A X_t^m + B^m(\tilde{X}_t^m) - B^m(X_t) = 0, \quad \Delta X_0^m = 0.$$

Then there exists a constant  $C > 0$  such that

$$\begin{aligned} \|\Delta X_t^m\|_0 &\leq \int_0^t \|e^{-(t-s)} [B_m(\tilde{X}_s^m) - B_m(X_s)]\|_0 ds \\ (5.8) \quad &= \int_0^t \|e^{-(t-s)} [B(\tilde{X}_s^m) - B(X_s)]\|_0 ds \\ &\leq C \int_0^t (t-s)^{-\frac{5}{6}} \|B(\tilde{X}_s^m) - B(X_s)\|_{-\frac{5}{3}} ds \end{aligned}$$

Since  $B(x) = B(x^m + (x - x^m))$  for  $x \in \mathbb{H}^1$ , it follows that

$$B(\tilde{X}_s^m) - B(X_s) = B(\tilde{X}_s^m) - B(X_s^m) - \tilde{B}(X_s^m, X_s - X_s^m) - B(X_s - X_s^m),$$

where  $\tilde{B}(x, y) = B(x, y) + B(y, x)$  for  $x, y \in \mathbb{H}^1$ . Applying Eq. (1.4) with  $\sigma_1 = \frac{5}{3}$ ,  $\sigma_2 = -1$ ,  $\sigma_3 = 0$ , we obtain

$$\begin{aligned} \|B(\tilde{X}_s^m) - B(X_s^m)\|_{-\frac{5}{3}} &\leq \|B(\Delta X_s^m, \tilde{X}_s^m)\|_{-\frac{5}{3}} + \|B(X_s^m, \Delta X_s^m)\|_{-\frac{5}{3}} \\ &\leq \|\Delta X_s^m\|_0 \|\tilde{X}_s^m\|_0 + \|\Delta X_s^m\|_0 \|X_s^m\|_0 \\ &\leq \left(\sqrt{A_T} + \sup_{0 \leq t \leq T} \|X_t\|_0\right) \|\Delta X_s^m\|_0. \end{aligned}$$

Combining this with (5.8) gives

$$\begin{aligned} \|\Delta X_t^m\|_0^2 &\leq C \int_0^t (t-s)^{-\frac{5}{6}} \left(\sqrt{A_T} + \sup_{0 \leq t \leq T} \|X_t\|_0\right) \|\Delta X_s^m\|_0 ds \\ &\quad + C \int_0^t (t-s)^{-\frac{5}{6}} (\|X_s\|_0 \|X_s - X_s^m\|_0 + \|X_s - X_s^m\|_0^2) ds. \end{aligned}$$

Noting that

$$\|\Delta X_t^m\|_0 \leq \|X_t^m\|_0 + \|\tilde{X}_t^m\|_0 \leq \sup_{0 \leq t \leq T} \|X_t\|_0 + \sqrt{A_T} < \infty, \quad t \in [0, T],$$

by Fatou's lemma we get

$$\limsup_{m \rightarrow \infty} \|\Delta X_t^m\|_0^2 \leq C \int_0^t (t-s)^{-\frac{5}{6}} \left(\sqrt{A_T} + \sup_{0 \leq t \leq T} \|X_t\|_0\right) \limsup_{m \rightarrow \infty} \|\Delta X_s^m\|_0 ds, \quad 0 \leq t \leq T,$$

so that by Gronwall's inequality,

$$\limsup_{m \rightarrow \infty} \|\Delta X_t^m\|_0 = 0, \quad t \in [0, T].$$

□

**5.2.  $\psi$ -uniformly exponential ergodicity and moderate deviation.** We will use the following exponential ergodicity result in [9].

**Theorem 5.3** (Theorem 5.2 (b), [9]). *Let  $(X_t)_{t \geq 0}$  be an irreducible and aperiodic Markov process on a Polish space  $E$  with Markov semigroup  $P_t$ , and let  $\psi \geq 1$  be a measurable function on  $E$ . If*

$$P_t \psi(x) \leq \lambda(t) \psi(x) + b 1_{\mathcal{K}}(x), \quad t \in (0, T], x \in E$$

*holds for some constants  $T, b > 0$ , a measurable petite set  $\mathcal{K}$  on  $E$ , and a bounded function  $\lambda$  on  $[0, T]$  with  $\lambda(T) < 1$ , then  $X_t$  is  $\psi$ -uniformly ergodic, i.e., there exist constants  $C, \gamma > 0$  such that*

$$(5.9) \quad \sup_{|f| \leq \psi} |P_t f(x) - \mu_0(f)| \leq C e^{-\gamma t} \psi(x), \quad t > 0.$$

*Proof of Theorem 1.3(1).* Since  $1 + \|\cdot\|_0$  is comparable with  $\sqrt{M + \|\cdot\|_0^2}$  for any  $M \geq 1$ , we will take  $\psi(x) = \sqrt{M + \|x\|_0^2}$  instead of  $1 + \|x\|_0$  for  $M > 1$  large enough to be determined.

(1) We first observe that it suffices to find out a constant  $C > 0$  such that

$$(5.10) \quad \left| \int_{\mathcal{H}^m} (\psi(x^m + Qy) - \psi(x^m) - \langle Qy, \nabla \psi(x^m) \rangle_0) 1_{\|y\|_0 \leq 1} \nu_m(dy) \right| \leq C \left( 1 + \frac{1}{\sqrt{M}} \right), \quad x^m \in \mathcal{H}^m, \quad x^m \in \mathcal{H}_m := \text{span}\{e_i : i \leq m\}.$$

Let  $\mathcal{L}^m$  be the generator of  $\tilde{X}_t^m$  given by (5.6). Since  $\langle x^m, B_m(x^m) \rangle = 0$ , it is easy to see that

$$\begin{aligned} \mathcal{L}^m \psi(x^m) &= -\langle Ax^m + B_m(x^m), \nabla \psi(x^m) \rangle_0 \\ &\quad + \int_{\mathcal{H}^m} (\psi(x^m + Qy) - \psi(x^m) - \langle Qy, \nabla \psi(x^m) \rangle_0) 1_{\|y\|_0 \leq 1} \nu_m(dy) \\ &= -\frac{\|x^m\|_1^2}{\psi(x^m)} + \int_{\mathcal{H}^m} (\psi(x^m + Qy) - \psi(x^m) - \langle Qy, \nabla \psi(x^m) \rangle_0) 1_{\|y\|_0 \leq 1} \nu_m(dy). \end{aligned}$$

where the last equality is by  $\langle x^m, B_m(x^m) \rangle = 0$ . Let  $\mathcal{K}_m = \{x^m \in \mathcal{H}^m : \|x^m\|_1 \leq M\}$ . By (5.10) and (5.2), we have

$$\begin{aligned} \mathcal{L}^m \psi(x^m) &\leq -\frac{\|x^m\|_1^2}{\psi(x^m)} + C \left( 1 + \frac{1}{\sqrt{M}} \right) \\ &\leq -\frac{\|x^m\|_1^2 + M}{\psi(x^m)} + \frac{M}{\psi(x^m)} + C \left( 1 + \frac{1}{\sqrt{M}} \right) \\ &\leq -\psi(x^m) + \sqrt{M} + C \left( 1 + \frac{1}{\sqrt{M}} \right), \quad x^m \in \mathcal{K}_m. \end{aligned}$$

On the other hand, if  $x^m \notin \mathcal{K}_m$ , then  $e \|x^m\|_1 \geq M$  and thus,

$$\begin{aligned}
\mathcal{L}^m \psi(x^m) &\leq -\frac{\|x^m\|_1^2}{\psi(x^m)} + C_{\alpha, Q} \left(1 + \frac{1}{\sqrt{M}}\right) \\
(5.11) \quad &\leq -\frac{\frac{1}{2}(M + \|x^m\|_1^2)}{\psi(x^m)} + C_{\alpha, Q} \left(1 + \frac{1}{\sqrt{M}}\right) \\
&\leq -\frac{1}{2} \psi(x^m) + C_{\alpha, Q} \left(1 + \frac{1}{\sqrt{M}}\right) \\
&\leq -\frac{1}{4} \psi(x^m),
\end{aligned}$$

as long as we choose  $M > 1$  sufficiently large. In conclusion, when  $M > 1$  is large enough, there exists a constant  $b > 0$  such that

$$\mathcal{L}^m \psi(x^m) \leq -\frac{1}{4} \psi(x^m) + b 1_{\mathcal{K}_m}(x^m), \quad m \geq 1.$$

By [9, Theorem 5.1 (d)], this implies

$$\mathbb{E}[\psi(\tilde{X}_t^m)] \leq e^{-t/4} \psi(x^m) + b 1_{\mathcal{K}_m}(x^m), \quad t \geq 0.$$

. Since  $\lim_{m \rightarrow \infty} \|x^m - x\|_0 = 0$  and  $\lim_{m \rightarrow \infty} \|\tilde{X}_t^m - X_t\|_1 = 0$  a.s. for  $t > 0$ , by letting  $m \rightarrow \infty$  we obtain

$$\mathbb{E}[\psi(X_t)] \leq e^{-t/4} \psi(x) + b 1_{\mathcal{K}}(x), \quad t \geq 0,$$

where  $\mathcal{K} := \{x \in \mathbb{H} : \|x\|_1 \leq M\}$  is a compact (hence petite) set in  $\mathbb{H}$ . By Theorem (5.3), we prove the  $\psi$ -uniformly exponential ergodicity of  $X_t$ .

(2) It remains to prove (5.10). Obviously,

$$\begin{aligned}
(5.12) \quad &\left| \int_{\mathcal{H}^m} (\psi(x^m + Qy) - \psi(x^m) - \langle Qy, \nabla \psi(x^m) \rangle_0) 1_{\|y\|_0 \leq 1} \nu_m(dy) \right| \\
&\leq \left| \int_{\|y\|_0 \leq 1} (\psi(x^m + Qy) - \psi(x^m) - \langle Qy, \nabla \psi(x^m) \rangle_0) \nu_m(dy) \right| \\
&\quad + \left| \int_{\|y\|_0 > 1} (\psi(x^m + Qy) - \psi(x^m)) \nu_m(dy) \right|
\end{aligned}$$

By Taylor's expansion,

$$\begin{aligned}
&|\psi(x^m + Qy) - \psi(x^m) - \langle Qy, \nabla \psi(x^m) \rangle_0| \\
&\leq \sup_{\theta \in [0, 1]} \left| \frac{\|y\|_0^2}{\psi(x^m + \theta Qy)} - \frac{|\langle y, x^m + \theta Qy \rangle_0|^2}{\psi^3(x^m + \theta Qy)} \right| \leq \frac{2}{\sqrt{M}} \|y\|_0^2.
\end{aligned}$$

Since  $\nu_m$  has a density  $\frac{C_m}{\|y\|_0^{m+\alpha}}$  for  $y \in \mathcal{H}_m$  with  $C_m = \frac{\alpha 2^\alpha \Gamma(\frac{m}{2} + \frac{\alpha}{2})}{\Gamma(\frac{m}{2}) \Gamma(\frac{2-\alpha}{2})}$ , we have

$$\begin{aligned}
&\left| \int_{\|y\|_0 \leq 1} (\psi(x^m + Qy) - \psi(x^m) - \langle Qy, \nabla \psi(x^m) \rangle_0) \nu_m(dy) \right| \\
&\leq \frac{2}{\sqrt{M}} \int_{\|y\|_0 \leq 1} \|y\|_0^2 \frac{C_m}{\|y\|_0^{m+\alpha}} dy = \frac{2C_m}{\sqrt{M}} \int_0^1 \int_{\mathbb{S}_{m-1}} r^{1-\alpha} dr d\sigma_{m-1} = \frac{2C_m |\mathbb{S}_{m-1}|}{(2-\alpha)\sqrt{M}},
\end{aligned}$$

where  $|\mathbb{S}_{m-1}| = \frac{2(\pi)^{m/2}}{\Gamma(m/2)}$  is the volume of  $\mathbb{S}_{m-1}$ . Moreover,

$$\begin{aligned} C_m |\mathbb{S}_{m-1}| &= \frac{\alpha 2^\alpha \Gamma\left(\frac{m}{2} + \frac{\alpha}{2}\right) 2\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{2-\alpha}{2}\right) \Gamma(m/2)} \leq \frac{\alpha 2^\alpha \Gamma\left(\frac{m}{2} + 1\right) 2\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{2-\alpha}{2}\right) \Gamma(m/2)} \\ &= \frac{\alpha 2^\alpha \frac{m}{2} \Gamma\left(\frac{m}{2}\right) 2\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{2-\alpha}{2}\right) \Gamma(m/2)} \leq \sup_{m \geq 1} \frac{\alpha 2^\alpha m \pi^{m/2}}{\Gamma\left(\frac{2-\alpha}{2}\right) \Gamma\left(\frac{m}{2}\right)} =: C' < \infty. \end{aligned}$$

Hence,

$$\left| \int_{\|y\|_0 \leq 1} (\psi(x^m + Qy) - \psi(x^m) - \langle Qy, \nabla \psi(x^m) \rangle_0) \nu_m(dy) \right| \leq \frac{C'}{\sqrt{M}}.$$

Similarly, there exist constants  $C_Q > 0$  such that

$$\begin{aligned} &\left| \int_{\|y\|_0 > 1} (\psi(x^m + Qy) - \psi(x^m)) \nu_m(dy) \right| \\ &\leq \left| \int_{\|y\|_0 > 1} \frac{|\langle x^m + \theta Qy, Qy \rangle_0|}{\psi(x^m + \theta Qy)} \nu_m(dy) \right| \leq \left| \int_{\|y\|_0 > 1} \|Qy\|_0 \nu_m(dy) \right| \\ &\leq C_Q \left| \int_{\|y\|_0 > 1} \|y\|_0 \nu_m(dy) \right| \leq \sup_{m \geq 1} C_Q \int_1^\infty \int_{\mathbb{S}_{m-1}} \frac{C_m}{r^\alpha} dr d\sigma_{m-1} < \infty. \end{aligned}$$

Therefore, (5.10) holds for some constant  $C > 0$ . □

*Proof of Theorem 1.3(2).* We follow the argument in [18, p. 429-431]. Given  $f \in \mathcal{B}_b(\mathbb{H})$ , consider the following Feynman-Kac formula

$$P_t^{\lambda f} g(x) = \mathbb{E} \left[ \exp \left( \lambda \int_0^t f(X_s^x) ds \right) g(X_t^x) \right], \quad g \in \mathcal{B}_\psi.$$

For any  $\delta > 0$  and  $|\lambda| \leq \delta$ , we have

$$\|P_t^{\lambda f} g\|_\psi \leq e^{\delta \|f\| t} \|g\|_\psi.$$

So,  $\lambda \rightarrow P_1^{\lambda f} g \in \mathcal{B}_\psi$  is holomorphic for all  $|\lambda| < \delta$ .

When  $\lambda = 0$ ,  $P_1 g = \mathbb{E}[g(X_1^x)]$  with  $g \in \mathcal{B}_\psi$ . By the exponential ergodicity result (5.9), we get that 1 is an isolated simple spectrum of  $P_1$  and the constant function is the corresponding eigenfunction. Denote  $\mathcal{P}_0$  be the projection with respect to the eigenvalue 1, which is defined by

$$\mathcal{P}_0 g = \mu(g), \quad g \in \mathcal{B}_\psi.$$

The spectrum of the  $P_1(I - \mathcal{P}_0)$  has a spectrum radius less than  $\rho$  from (5.9).

By Kato's holomorphic perturbation theorem, for any  $r \in (\rho, \frac{1+\rho}{2})$ , there exist some  $\tilde{\delta} \in (0, \delta)$  such that for all  $D_{\tilde{\delta}} = \{\lambda \in \mathbb{C} : |\lambda| \leq \tilde{\delta}\}$  the operator  $P_1^{\lambda f}$  acting on  $\mathcal{B}_\psi$  has the following properties: (1)  $P_1^{\lambda f}$  has a single simple eigenvalue  $\sigma(\lambda)$  with the largest modulus of the spectrum, moreover, there exists some number  $c \in (\frac{1}{2}, 1)$  such that  $|\sigma(\lambda)| \geq c$ ; (2)  $\mathcal{P}_\lambda$  is the projection of  $P_1^{\lambda f}$  corresponding to  $\sigma(\lambda)$ ,  $\lambda \in D_{\tilde{\delta}} \rightarrow \mathcal{P}_\lambda \in \mathcal{L}(\mathcal{B}_\psi)$  is holomorphic and  $\|\mathcal{P}_\lambda 1 - \mathcal{P}_0 1\|_\psi \leq e$  with some sufficiently small  $e \in (0, 1)$ ; (3) the spectral radius of  $P_1^{\lambda f}(I - \mathcal{P}_\lambda)$  is strictly less than  $r$ .

By (3), the following relation holds

$$N := \sup_{z \in S(\frac{1}{r}), \lambda \in D_{\tilde{\delta}}} \|(I - zP_1^{\lambda f}(I - \mathcal{P}_\lambda))^{-1}\|_{\mathcal{B}_\psi \rightarrow \mathcal{B}_\psi} < \infty,$$

where  $S(1/r) = \{z \in \mathbb{C} : |z| = \frac{1}{r}\}$ .

By Cauchy integral we have

$$\begin{aligned} (P_1^{\lambda f}(I - \mathcal{P}_\lambda))^n &= \frac{1}{n!} \frac{\partial^n}{\partial z^n} (I - zP_1^{\lambda f}(I - \mathcal{P}_\lambda))^{-1} \Big|_{z=0} \\ &= \frac{1}{2\pi i} \int_{S(\frac{1}{r})} \frac{(I - zP_1^{\lambda f}(I - \mathcal{P}_\lambda))^{-1}}{z^{n+1}} dz, \end{aligned}$$

from which we get

$$\|P_n^{\lambda f} - \sigma(\lambda)^n \mathcal{P}_\lambda\|_{\mathcal{B}_\psi \rightarrow \mathcal{B}_\psi} = \|(P_1^{\lambda f}(I - \mathcal{P}_\lambda))^n\|_{\mathcal{B}_\psi \rightarrow \mathcal{B}_\psi} \leq Nr^n.$$

Since  $\|P_t^{\lambda f}\|_{\mathcal{B}_\psi \rightarrow \mathcal{B}_\psi} \leq e^{\lambda \|f\|}$  for  $0 \leq t \leq 1$ , by a standard argument and the semigroup property of  $P_t^{\lambda f}$ , we have

$$(5.13) \quad \|P_t^{\lambda f} - \exp(t \log \sigma(\lambda)) \mathcal{P}_\lambda\|_{\mathcal{B}_\psi \rightarrow \mathcal{B}_\psi} \leq Cr^t.$$

For any probability measure  $\nu$  with  $\nu(\psi) < \infty$ , by (5.13), for all large  $t$  so that  $Cr^t < 1$ ,  $\log \int_{\mathbb{H}} P_t^{\lambda f} 1 d\nu$  are holomorphic on  $D_{\tilde{\delta}}$ . Moreover, by the inequality in (2),

$$\limsup_{t \rightarrow \infty} \sup_{|\lambda| < \tilde{\delta}} \left| \frac{1}{t} \log \int_{\mathbb{H}} P_t^{\lambda f} 1 d\nu - \log \sigma(\lambda) \right| = 0.$$

By Cauchy's theorem for holomorphic function, for any  $\epsilon \in (0, \tilde{\delta})$  we have

$$\limsup_{t \rightarrow \infty} \sup_{|\lambda| < \epsilon} \left| \frac{d^k}{d\lambda^k} \frac{1}{t} \log \int_{\mathbb{H}} P_t^{\lambda f} 1 d\nu - \frac{d^k}{d\lambda^k} \log \sigma(\lambda) \right| = 0, \quad k \in \mathbb{N}.$$

By the  $C^2$ -regularity criterion in [, Theorem 1.2], we have

$$\lim_{t \rightarrow \infty} \sup_{\nu: \nu(\psi) < \infty} \left| \frac{1}{b^2(t)} \log \mathbb{E}^\nu \exp(b^2(t) \mathfrak{M}_t(f)) - \frac{1}{2} \sigma^2(f) \right| = 0,$$

where  $\mathfrak{M}_t(f) := \frac{1}{b(t)\sqrt{t}} \left( \int_0^t f(X_s) ds - \mu(f) \right)$  with  $b(t) \rightarrow \infty$  and  $\frac{b(t)}{\sqrt{t}} \rightarrow 0$  as  $t \rightarrow \infty$ , and

$$\sigma^2(f) = \lim_{t \rightarrow \infty} \left( \frac{d^2}{d\lambda^2} \frac{1}{t} \log \int_{\mathbb{H}} P_t^{\lambda f} 1 d\mu \right) \Big|_{\lambda=0} = \lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}^\mu \left( \int_0^t (f(X_s) - \mu(f)) ds \right)^2.$$

By [4, Chapter 6], we immediately obtain the MDP result in the theorem.  $\square$

## REFERENCES

- [1] Applebaum D. (2009) *Lévy processes and stochastic calculus*. Second edition. Cambridge Studies in Advanced Mathematics, **116**, Cambridge University Press. MR 2512800
- [2] Da Prato G. (2004) *Kolmogorov equations for stochastic PDEs*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel. MR 1417491
- [3] Da Prato G. and Zabczyk J. (1996) *Ergodicity for infinite-dimensional systems*. London Mathematical Society Lecture Note Series, **229**, Cambridge University Press, Cambridge. MR 2111320
- [4] Dembo A. and Zeitouni O. (1998) *Large deviations techniques and applications*. Second edition, Applications of Mathematics, **38**, Springer-Verlag. MR 2571413

- [5] Dong Z. and Xie Y. (2011) Ergodicity of stochastic 2D Navier-Stokes equations with Lévy noise. *J. Differential Equations*, **251**, 196-222. MR 2793269
- [6] Dong Z., Xu L. and Zhang X. (2011) Invariance measures of stochastic 2D Navier-Stokes equations driven by  $\alpha$ -stable processes. *Electron. Commun. Probab.*, **16**, 678-688. MR 2853105
- [7] Dong Z., Xu T. and Zhang T. (2009) Invariant measures for stochastic evolution equations of pure jump type. *Stochastic Process. Appl.*, **119**, 410-427. MR 2493997
- [8] Doob J. L. (1948) Asymptotic properties of Markov transition probability. *Trans. Am. Math. Soc.*, **64**, 393-421. MR 0025097
- [9] Down D., Meyn S. P. and Tweedie R. L. (1995) Exponential and uniform ergodicity of Markov processes. *Ann. Probab.*, **23**, 1671-1691. MR 1379163
- [10] Eckmann J.-P. and Hairer M. (2001) Uniqueness of the invariant measure for a stochastic PDE driven by degenerate noise. *Comm. Math. Phys.*, **219**, 523-565. MR 1838749
- [11] Funaki T. and Xie B. (2009) A stochastic heat equation with the distributions of Lévy processes as its invariant measures. *Stochastic Process. Appl.*, **119**, 307-326. MR 2493992
- [12] Masuda H. (2007) Ergodicity and exponential  $\beta$ -mixing bounds for multidimensional diffusions with jumps. *Stochastic Process. Appl.*, **117**, 35-56. MR 2287102
- [13] Peszat S. and Zabczyk J. (2007) *Stochastic partial differential equations with Lévy noise. An evolution equation approach*. Encyclopedia of Mathematics and its Applications, **113**, Cambridge University Press, Cambridge. MR 2356959
- [14] Priola E., Shirikyan A., Xu L. and Zabczyk J. (2012) Exponential ergodicity and regularity for equations with Lévy noise. *Stochastic Process. Appl.*, **122**, 106-133. MR 2773026
- [15] Priola E., Xu L. and Zabczyk J. (2011) Exponential mixing for some SPDEs with Lévy noise. *Stoch. Dyn.*, **11**, 521-534. MR 2836539
- [16] Priola E. and Zabczyk J. (2011) Structural properties of semilinear SPDEs driven by cylindrical stable processes. *Probab. Theory Related Fields*, **149**, 97-137. MR 2773026
- [17] Sato K. (1999), *Lévy processes and infinite divisible distributions*. Cambridge University Press, Cambridge.
- [18] Wu L. (1995) Moderate deviations of dependent random variables related to CLT. *The Annals of Probability* 23 , no. 1, 420-445.
- [19] Wu L. (1995) Moderate deviations of dependent random variables related to CLT. *The Annals of Probability* 23 , no. 1, 420-445.
- [20] Dong Z., Xu L. and Zhang X.: Invariance measures of stochastic 2D Navier-Stokes equations driven by  $\alpha$ -stable processes. *Electronic Communications in Probability*. Vol.16 (2011), 678-688.
- [21] Dong Z., Xu L. and Zhang X.: Exponential ergodicity of stochastic Burgers equations driven by  $\alpha$ -stable processes. *Journal of Statistical Physics* 154 (2014), no. 4, 929-949.
- [22] Dong Z. and Xu T.G.: One-dimensional stochastic Burgers equation driven by Lévy processes. *J. Func. Anal.* Vol. 243 (2007), no. 2, 631-678.
- [23] Dong Z. and Xie Y.: Ergodicity of stochastic 2D Navier-Stokes equations with Lévy noise. *Journal of Differential Equations*, Vol. 251 (2011), no. 1, 196-222.
- [24] Priola E., Shirikyan A., Xu L. and Zabczyk J.: Exponential ergodicity and regularity for equations with Lévy noise, *Stoch. Proc. Appl.*, Vol. 122, 1 (2012), 106-133.
- [25] Temam R.: *Navier-Stokes equations and nonlinear functional analysis*, second ed., CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 66, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1995.
- [26] Wang F.Y.: Gradient estimate for Ornstein-Uhlenbeck jump processes. *Stoch. Proc. Appl.*, Vol.121, 3 (2011), 466-478.
- [27] Wang F.Y., Xu L. and Zhang X.: Gradient estimates for SDEs driven by multiplicative Lévy noise. arXiv:1301.4528.
- [28] Wang F.Y. and Wang J.: Coupling and strong Feller for jump processes on Banach spaces. arXiv:1111.3795v1.
- [29] Xu L.: Ergodicity of stochastic real Ginzburg-Landau equation driven by  $\alpha$ -stable noises. *Stochastic Process. Appl.* 123 (2013), no. 10, 3710-3736.

- [30] Xu L. and Zegarliński B.: Existence and exponential mixing of infinite white  $\alpha$ -stable systems with unbounded interactions. *Electron. J. Probab.* Vol. 15 (2010), 1994-2018.
- [31] Zhang X.: Derivative formula and gradient estimate for SDEs driven by  $\alpha$ -stable processes. Vol. 123, Issue 4, pp.1213-1228 (2013).

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