An unsymmetric 8-node hexahedral solid-shell element with high distortion tolerance: Linear formulations

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Abstract

A locking-free unsymmetric 8-node solid-shell element with high distortion tolerance is proposed for general shell analysis, which is equipped with translational dofs only. The prototype of this new model is a recent solid element US-ATFH8 developed by combining the unsymmetric finite element method and the analytical solutions in 3D local oblique coordinates. By introducing proper shell assumptions and assumed natural strain modifications for transverse strains, the new solid-shell element US-ATFHS8 is successfully formulated. This element is able to give highly accurate predictions for shells with different geometric features and loading conditions, and quite insensitive to mesh distortions. Especially, the excellent performance of US-ATFH8 under membrane load is well inherited, which is an outstanding advantage over other shell elements.

KEY WORDS: finite element methods; unsymmetric finite elements; solid-shell element; mesh distortions; analytical trial functions; oblique coordinates

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1. Introduction

Although the finite element method (FEM) is one significant tool and has been widely applied in various engineering and scientific fields, some key technical challenges remain outstanding. To date, many researchers are still making great efforts in developing novel high-performance models that can overcome the drawbacks existing in the traditional FEM [1]. This article is mainly about an application of the unsymmetric finite element method [2] together with the analytical trial function method [3] in shell analysis. The unsymmetric finite element method can effectively reduce the sensitivity problems to mesh distortions by successfully eliminating the Jacobian determinant after employing two different sets of interpolation for displacement fields. The analytical trial function method makes use of the solutions of governing equations of elasticity as trial functions for finite element discretization, which is similar to the Trefftz methods [4]. Because of these merits, the resulting models can achieve high accuracy and naturally avoid many locking problems caused by traditional isoparametric interpolation techniques [5].

Recently, the above strategy brought some interesting breakthroughs in low-order finite elements. It is well known that, MacNeal proved an important theorem about 30 years ago [6], which claims that any 4-node, 8-dof quadrilateral membrane element must either present trapezoidal locking results for MacNeal's thin beam problem or fail to guarantee its convergence. This theorem frustrates many scholars who devote themselves to developing low-order finite element models. In 2015, Cen et al. [7] proposed an unsymmetric 4-node, 8-dof membrane element US-ATFQ4 that can circumvent this issue. A similar scheme has been already generalized to the three-dimensional case, and an 8-node, 24-dof hexahedral element US-ATFH8 was successfully formulated [8]. Both elements are verified to be not only accurate for most standard benchmark problems, but also insensitive to severe mesh distortions.

Similar models that can only deal with isotropic cases were also proposed by Xie et al. [9]. However, it is found that, when the 3D element US-ATFH8 is used for analysis of shell structures, locking phenomena will still appear. Therefore, the purpose of this article is to eliminate this weakness and develop a high-performance solid-shell finite element model based on the original 3D element US-ATFH8.

The FEM is an effective way to simulate the complicated behaviors of shell, one kind of complex and important engineering structure forms in 3D space. Among all the shell element categories, degenerated shell elements are usually considered as the most popular models for their simplicity in kinematic and geometric description, and validity in locking elimination techniques [10, 20]. The Assumed Natural Strain (ANS) interpolation plays a significant role in the formulation of applicable degenerated shell elements. The Mixed Interpolated Tensorial Components (MITC) technique is an outstanding representative in the ANS schemes [11-19], and many MITC-based degenerated shell elements are widely applied in engineering practice. However, the general 3D constitutive laws have to be modified in degenerated shell applications, and special transition elements or extra constraints are needed when solid elements are used together with shell elements. In geometric nonlinear problems, extra difficulty is also found in the update of finite rotations [21]. Meanwhile, solid-shell model demonstrates its advantage by using translational dofs only, so that it can be directly used together with solid elements.

A prototype of the kinematic description in solid-shell models was proposed by Schoop in 1986 [36], which is also called the "double-node-model". Another early work regarding the development of solid-shell concept is Sansour's work [22] in constructing shell element without rotational dofs. General 3D constitutive laws can be applied in his models, but the dofs of the resulting models differ from those of solid elements and extra transition to solid elements is needed. General 3D constitutive laws are also considered in the work of Büchter et al. [23], in which a 7-parameter degenerated shell model is developed based upon the Enhanced Assumed Strain (EAS) method [24], and has been successfully applied in problems with hyperelasticity and large strain plasticity. Some other recent efforts in solid-shell elements include the work of Klinkel et al. [38], in which the EAS method is adopted based on Hu-Washizu variational principle, and the solid-shell model with very effective reduced integration proposed by Reese [39], etc.

Although the solid-shell model possesses some advantages over the degenerated shell model, it suffers from new locking problem, i.e., the thickness locking. The thickness locking is caused by the linear displacement interpolation in the thickness direction and the coupling between in-plane and transverse strains [21], which is critical in solid-shell formulations. This problem may be handled by adding quadratic terms in the transverse displacement field [21, 25], decoupling the in-plane and normal strains in the constitutive laws [26], or using hybrid-stress formulations [27, 28]. The presented work will demonstrate that the unsymmetric formulations and the analytical trial function method are also useful for eliminating this locking problem.

In this article, a locking-free unsymmetric solid-shell element with high distortion tolerance is proposed for general shell analysis. First, we briefly review the 8-node, 24-dof hexahedral element US-ATFH8 [8], which was constructed by combining the unsymmetric finite element method and the analytical solutions in 3D local oblique coordinate system. Then, based on the element US-ATFH8, proper shell assumptions and assumed natural strain modifications for transverse strains are introduced to formulate an unsymmetric solid-shell element US-ATFHS8, which presents locking-free results for many shell benchmark problems. The proposed element is able to provide highly accurate predictions for shells with different geometric features and loading conditions, and quite insensitive to mesh distortions. Especially, the excellent performance of US-ATFH8 under membrane forces is well inherited, which is an outstanding advantage over other existing solid-shell models.

2. Element Formulations

2.1 The unsymmetric solid element based on analytical solutions [8]

2.1.1 Three-dimensional local oblique coordinates and the corresponding analytical solutions

To construct elements that are invariant to coordinate directions, local coordinates should be used. In presented models, the oblique coordinate system proposed by Yuan et al. [29, 30] is utilized due to its simple linear relationship with the global Cartesian coordinate system. As shown in Figure 1, the 3D oblique coordinates are briefly depicted.

The relationship between the global Cartesian coordinates and natural coordinates is given by

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^{8} \bar{N}_i \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} = \begin{Bmatrix} x_0 + \bar{a}_1\xi + \bar{a}_2\eta + \bar{a}_3\zeta + \bar{a}_4\xi\eta + \bar{a}_5\eta\zeta + \bar{a}_6\xi\zeta + \bar{a}_7\xi\eta\zeta \\ y_0 + \bar{b}_1\xi + \bar{b}_2\eta + \bar{b}_3\zeta + \bar{b}_4\xi\eta + \bar{b}_5\eta\zeta + \bar{b}_6\xi\zeta + \bar{b}_7\xi\eta\zeta \\ z_0 + \bar{c}_1\xi + \bar{c}_2\eta + \bar{c}_3\zeta + \bar{c}_4\xi\eta + \bar{c}_5\eta\zeta + \bar{c}_6\xi\zeta + \bar{c}_7\xi\eta\zeta \end{Bmatrix} \tag{1}$$

in which

$$\bar{N}_i = \frac{1}{8}(1+\xi_i\xi)(1+\eta_i\eta)(1+\zeta_i\zeta), \qquad i = 1, 2, \ldots, 8 \tag{2}$$

are the shape functions of the 8-node tri-linear isoparametric element, and

$$\begin{gathered} \begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix} = \frac{1}{8}\sum_{i=1}^{8} \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}, \begin{Bmatrix} \bar{a}_1 \\ \bar{b}_1 \\ \bar{c}_1 \end{Bmatrix} = \frac{1}{8}\sum_{i=1}^{8} \xi_i \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} \\ \begin{Bmatrix} \bar{a}_2 \\ \bar{b}_2 \\ \bar{c}_2 \end{Bmatrix} = \frac{1}{8}\sum_{i=1}^{8} \eta_i \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}, \begin{Bmatrix} \bar{a}_3 \\ \bar{b}_3 \\ \bar{c}_3 \end{Bmatrix} = \frac{1}{8}\sum_{i=1}^{8} \zeta_i \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} \end{gathered} \tag{3}$$

where ξ_i, η_i, ζ_i, x_i, y_i, and z_i are the natural coordinates and global coordinates at each node, respectively.

The linear relationship between oblique coordinates and Cartesian coordinates is determined by the Jacobian matrix $\mathbf{J}_0$ at the origin of the natural coordinate system [29, 30]:

$$\begin{Bmatrix} R \\ S \\ T \end{Bmatrix} = (\mathbf{J}_0^{-1})^{\mathrm{T}} \left(\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} - \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_{\xi=\eta=\zeta=0} \right) = \frac{1}{J_0} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{Bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{Bmatrix} \tag{4}$$

$$\mathbf{J}_0 = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \\ \dfrac{\partial x}{\partial \zeta} & \dfrac{\partial y}{\partial \zeta} & \dfrac{\partial z}{\partial \zeta} \end{bmatrix}_{\xi=\eta=\zeta=0} = \begin{bmatrix} \bar{a}_1 & \bar{b}_1 & \bar{c}_1 \\ \bar{a}_2 & \bar{b}_2 & \bar{c}_2 \\ \bar{a}_3 & \bar{b}_3 & \bar{c}_3 \end{bmatrix} \tag{5}$$

$$\begin{aligned} J_0 &= |\mathbf{J}_0| = \bar{a}_1(\bar{b}_2\bar{c}_3 - \bar{b}_3\bar{c}_2) + \bar{a}_2(\bar{b}_3\bar{c}_1 - \bar{b}_1\bar{c}_3) + \bar{a}_3(\bar{b}_1\bar{c}_2 - \bar{b}_2\bar{c}_1) \\ &= \bar{a}_1 a_1 + \bar{a}_2 a_2 + \bar{a}_3 a_3 = \bar{b}_1 b_1 + \bar{b}_2 b_2 + \bar{b}_3 b_3 = \bar{c}_1 c_1 + \bar{c}_2 c_2 + \bar{c}_3 c_3 \\ a_1 &= \bar{b}_2\bar{c}_3 - \bar{b}_3\bar{c}_2, \qquad b_1 = \bar{a}_3\bar{c}_2 - \bar{a}_2\bar{c}_3, \qquad c_1 = \bar{a}_2\bar{b}_3 - \bar{a}_3\bar{b}_2 \\ a_2 &= \bar{b}_3\bar{c}_1 - \bar{b}_1\bar{c}_3, \qquad b_2 = \bar{a}_1\bar{c}_3 - \bar{a}_3\bar{c}_1, \qquad c_2 = \bar{a}_3\bar{b}_1 - \bar{a}_1\bar{b}_3 \\ a_3 &= \bar{b}_1\bar{c}_2 - \bar{b}_2\bar{c}_1, \qquad b_3 = \bar{a}_2\bar{c}_1 - \bar{a}_1\bar{c}_2, \qquad c_3 = \bar{a}_1\bar{b}_2 - \bar{a}_2\bar{b}_1 \end{aligned} \tag{6}$$

It can be found that the linear relationship between the Cartesian coordinates and the natural coordinates at the natural origin is inherited by the oblique coordinates. So, this local coordinate system has several advantages over natural coordinates: only linear transform is needed, and important element geometrical features are also easily retained.

The three coordinate axes and corresponding base vectors are also introduced in Equation (4): the base vector of R-coordinate points from the natural origin to the middle point of the 2-3-7-6 surface; the base vector of S-coordinate points from the natural origin to the middle point of the 3-4-8-7 surface; and the base vector of T-coordinate points from the natural origin to the middle point of the 5-6-7-8 surface. These base vectors are denoted by $\mathbf{g}^R$, $\mathbf{g}^S$, and $\mathbf{g}^T$, respectively, which means that these vectors are considered as contravariant base vectors in following formulae, while the consideration of covariant vectors yields the same formulation.

The equilibrium equations of the covariant stress components are

$$\nabla\cdot\boldsymbol{\sigma}+\mathbf{f}=\mathbf{0}\Leftrightarrow\begin{cases}\dfrac{\partial\sigma_{RR}}{\partial R}+\dfrac{\partial\sigma_{SR}}{\partial S}+\dfrac{\partial\sigma_{TR}}{\partial T}+f_R=0\\ \dfrac{\partial\sigma_{SS}}{\partial S}+\dfrac{\partial\sigma_{TS}}{\partial T}+\dfrac{\partial\sigma_{RS}}{\partial R}+f_S=0\\ \dfrac{\partial\sigma_{TT}}{\partial T}+\dfrac{\partial\sigma_{RT}}{\partial R}+\dfrac{\partial\sigma_{ST}}{\partial S}+f_T=0\end{cases} \tag{7}$$

in which $\mathbf{f}=f_R\mathbf{g}^R+f_S\mathbf{g}^S+f_T\mathbf{g}^T$ is the body force (Einstein summation convention is not used unless it is explicitly stated). The compatibility equations expressed by stress components under constant body load are the second order partial differential equations (B-M equations) as follows:

$$\begin{cases}(1+\mu)\nabla^2\sigma_{xx}+\dfrac{\partial^2\Theta}{\partial x^2}=0\\ (1+\mu)\nabla^2\sigma_{yy}+\dfrac{\partial^2\Theta}{\partial y^2}=0\\ (1+\mu)\nabla^2\sigma_{zz}+\dfrac{\partial^2\Theta}{\partial z^2}=0\\ (1+\mu)\nabla^2\sigma_{yz}+\dfrac{\partial^2\Theta}{\partial y\partial z}=0\\ (1+\mu)\nabla^2\sigma_{zx}+\dfrac{\partial^2\Theta}{\partial z\partial x}=0\\ (1+\mu)\nabla^2\sigma_{xy}+\dfrac{\partial^2\Theta}{\partial x\partial y}=0\end{cases},\quad \Theta=\sigma_{xx}+\sigma_{yy}+\sigma_{zz} \tag{8}$$

Thus, linear stress fields will automatically satisfy the B-M equations, and the linear analytical solutions of stress which satisfy all governing equations may be obtained if the constraints in Equation (7) are imposed.

The solutions of Equation (7) may be divided into the general solution part and the particular solution part, and one set of the particular solutions for constant body force can be written as

$$\boldsymbol{\sigma}^{*} = \begin{Bmatrix} \sigma_{RR} \\ \sigma_{SS} \\ \sigma_{TT} \\ \sigma_{RS} \\ \sigma_{ST} \\ \sigma_{RT} \end{Bmatrix}^{*} = \begin{bmatrix} -f_R R \\ -f_S S \\ -f_T T \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

Constant general solutions for stress components are trivial and will not be detailed. Fifteen linear general solutions are listed together with constant general solutions in Table 1, in which the 13th~21st solutions describe the conditions of pure bending and twisting.

Then the solutions for stress components in Cartesian coordinates can be obtained with the following transform:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \bar{a}_1 & \bar{b}_1 & \bar{c}_1 \\ \bar{a}_2 & \bar{b}_2 & \bar{c}_2 \\ \bar{a}_3 & \bar{b}_3 & \bar{c}_3 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \sigma_{RR} & \sigma_{RS} & \sigma_{RT} \\ \sigma_{SR} & \sigma_{SS} & \sigma_{ST} \\ \sigma_{TR} & \sigma_{TS} & \sigma_{TT} \end{bmatrix} \begin{bmatrix} \bar{a}_1 & \bar{b}_1 & \bar{c}_1 \\ \bar{a}_2 & \bar{b}_2 & \bar{c}_2 \\ \bar{a}_3 & \bar{b}_3 & \bar{c}_3 \end{bmatrix} \tag{10}$$

For linear-elastic material, corresponding strain solutions and displacement solutions can be easily derived. In shell analysis, body forces are usually replaced by surface tractions, so the particular solution in Equation (9) is not considered in the following discussion. Detailed strain and displacement solutions for isotropic linear-elastic material are given in Appendix.

2.1.2 Formulation of the unsymmetric 8-node, 24-dof hexahedral solid element

The unsymmetric finite element method belongs to the Petrov-Galerkin finite element method family [31], and its concept originates from the virtual work principle for a single element:

$$\int_{\Omega^e} \delta\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}\hat{\boldsymbol{\sigma}}\mathrm{d}V - \int_{\Omega^e} \delta\bar{\mathbf{u}}^{\mathrm{T}}\mathbf{b}\mathrm{d}V - \int_{S^e} \delta\bar{\mathbf{u}}^{\mathrm{T}}\mathbf{T}\mathrm{d}S - \delta\bar{\mathbf{u}}_{\mathrm{c}}^{\mathrm{T}}\mathbf{f}_{\mathrm{c}} = 0 \tag{11}$$

in which $\delta\bar{\boldsymbol{\varepsilon}}$ and $\delta\bar{\mathbf{u}}$ are the virtual strain and virtual displacement fields, respectively, and they are the test functions. The trial function part, $\hat{\boldsymbol{\sigma}}$, is interpolated by the analytical solutions introduced in the last section. **b**, **T**, and $\mathbf{f}_{\mathrm{c}}$ are the body force, boundary distributed force, and concentrated force, respectively.

In the unsymmetric finite element method, the element stiffness matrix is not symmetric because two different interpolation schemes are employed for test functions and trial functions, respectively. The test functions must meet the inter-element compatibility requirement [2], so that the traditional isoparametric interpolation is adopted:

$$\delta\bar{\mathbf{u}} = \bar{\mathbf{N}}\delta\mathbf{q}^e \tag{12}$$

$$\bar{\mathbf{N}} = \begin{bmatrix} \bar{N}_1 & 0 & 0 & \cdots & \bar{N}_8 & 0 & 0 \\ 0 & \bar{N}_1 & 0 & \cdots & 0 & \bar{N}_8 & 0 \\ 0 & 0 & \bar{N}_1 & \cdots & 0 & 0 & \bar{N}_8 \end{bmatrix} \tag{13}$$

$$\delta\bar{\boldsymbol{\varepsilon}} = \bar{\mathbf{B}}\delta\mathbf{q}^e \tag{14}$$

where the shape functions are the same as given in Equation (2), $\bar{\mathbf{B}}$ is the corresponding strain matrix, and $\mathbf{q}^e$ is the nodal displacement vector with the numbering depicted in Figure 1:

$$\begin{gathered} \bar{\mathbf{B}} = \mathbf{L}\bar{\mathbf{N}} = \frac{1}{|\mathbf{J}|}\bar{\mathbf{B}}^* \\ \mathbf{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}^{\mathrm{T}} \end{gathered} \tag{15}$$

$$\mathbf{q}^e = [u_1 \quad v_1 \quad w_1 \quad \cdots \quad u_8 \quad v_8 \quad w_8]^{\mathrm{T}} \tag{16}$$

in which $\mathbf{J}$ is the Jacobian matrix, and the Jacobian determinant $|\mathbf{J}|$ will be cancelled out by its counterpart in the volume element.

The trial function part (stress) in Equation (11) is usually required to satisfy the equilibrium conditions, with the force boundary conditions and body forces neglected. In practical implementation, trial functions are interpolated by the analytical solutions given in Table 1, but as non-analytical high order terms are inevitable in presented scheme as can be seen in Equation (17) (the *RST* terms do not satisfy the equilibrium equations), only quasi-equilibrium is achieved.

The corresponding trial functions for displacement fields are

$$\hat{\mathbf{u}} = \begin{Bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{Bmatrix} = \left[\begin{array}{cccccccccccc:ccc:ccc} 1 & 0 & 0 & x & 0 & 0 & y & 0 & 0 & z & 0 & 0 & U_{13} & \cdots & U_{21} & RST & 0 & 0 \\ 0 & 1 & 0 & 0 & x & 0 & 0 & y & 0 & 0 & z & 0 & V_{13} & \cdots & V_{21} & 0 & RST & 0 \\ 0 & 0 & 1 & 0 & 0 & x & 0 & 0 & y & 0 & 0 & z & W_{13} & \cdots & W_{21} & 0 & 0 & RST \end{array}\right] \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_{24} \end{Bmatrix} = \mathbf{P}\boldsymbol{\alpha} \tag{17}$$

in which the first twelve terms in this interpolation are related to 3 translational motions and 9 linear displacement fields, and the 13th~21st terms (see Appendix) are displacement solutions related to the 13th~21st stress solutions listed in Table 1. The last three *RST*-terms are non-analytical high-order terms which are employed to ensure the feasibility of the presented model, so that the unknown parameters α_i can be fully determined by 24 nodal dofs. As all interpolation terms are complete or expressed by local coordinates, this model is able to present unique solution independent of the global coordinate directions.

Substitution of nodal coordinates into Equation (17) yields

$$\hat{\mathbf{d}}\,\boldsymbol{\alpha} = \mathbf{q}^e \tag{18}$$

where

$$\hat{\mathbf{d}} = \begin{bmatrix} \mathbf{P}(x_1, y_1, z_1) \\ \mathbf{P}(x_2, y_2, z_2) \\ \vdots \\ \mathbf{P}(x_8, y_8, z_8) \end{bmatrix} \tag{19}$$

The invertibility of matrix $\hat{\mathbf{d}}$ is guaranteed by the linear independence between the trial functions, then α_i can be solved and the trial displacement fields are completely determined:

$$\hat{\mathbf{u}} = \begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{Bmatrix} = \mathbf{P}\boldsymbol{\alpha} = \mathbf{P}\hat{\mathbf{d}}^{-1}\mathbf{q}^e = \hat{\mathbf{N}}\mathbf{q}^e \tag{20}$$

And the trial strain fields are given by

$$\hat{\boldsymbol{\varepsilon}} = \begin{Bmatrix} \hat{\varepsilon}_x \\ \hat{\varepsilon}_y \\ \hat{\varepsilon}_z \\ \hat{\gamma}_{xy} \\ \hat{\gamma}_{yz} \\ \hat{\gamma}_{zx} \end{Bmatrix} = (\mathbf{LP})\hat{\mathbf{d}}^{-1}\mathbf{q}^e = \tilde{\mathbf{P}}\hat{\mathbf{d}}^{-1}\mathbf{q}^e = \hat{\mathbf{B}}\mathbf{q}^e \tag{21}$$

These related expressions can be detailed in Appendix.

Considering the arbitrariness of the virtual displacement, substitution of test functions and trial functions into the virtual work principle Equation (11) yields the expressions of element stiffness matrix and element equivalent nodal load:

$$\begin{aligned} \mathbf{K}^e &= \int_{\Omega^e} \bar{\mathbf{B}}^{\mathrm{T}} \mathbf{D} \hat{\mathbf{B}} \mathrm{d}V = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \frac{\bar{\mathbf{B}}^{*\mathrm{T}}}{|\mathbf{J}|} \mathbf{D} \hat{\mathbf{B}} |\mathbf{J}| \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}\zeta \\ &= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \bar{\mathbf{B}}^{*\mathrm{T}} \mathbf{D} \hat{\mathbf{B}} \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}\zeta \end{aligned} \tag{22}$$

$$\mathbf{F}^e = \int_{\Omega^e} \bar{\mathbf{N}}^{\mathrm{T}} \mathbf{b} \mathrm{d}V + \int_{S^e} \bar{\mathbf{N}}^{\mathrm{T}} \mathbf{T} \mathrm{d}S + \bar{\mathbf{N}}_{\mathrm{c}}^{\mathrm{T}} \mathbf{f}_{\mathrm{c}} \tag{23}$$

where $\mathbf{D}$ is the elasticity matrix. It can be found from Equation (22) that the inversion of Jacobian determinant, which is the reason leading to many numerical problems when the mesh is distorted, is avoided. The element stiffness matrix is evaluated with 2×2×2 Gauss quadrature rule, and resulting solid element is denoted by US-ATFH8.

The results of general 3D elastic problems show that element US-ATFH8 presents highly accurate solutions, and is insensitive to severe mesh distortion, especially for low-order problems such as beam loaded by pure bending, in which exact solutions are reached regardless of the mesh conditions [8]. However, inevitable locking problems are observed when this element is applied for thin shell structure analysis, so the main work in this article is to eliminate this weakness of US-ATFH8 and formulate a locking-free solid-shell element which can also retain the mesh distortion tolerance of the unsymmetric finite element models.

2.2 The unsymmetric solid-shell element based on analytical solutions and ANS interpolations

In order to avoid the locking phenomena for thin shell problems, extra modifications are needed for the transverse strains of the solid element. Although locking-free solutions of standard 3D problems are obtained by US-ATFH8 under different loading and mesh conditions, the coupling between membrane stress and bending stress and the complexity of geometry in thin shell structures lead to severe shear, trapezoidal and thickness locking problems.

A systematic study shows that the source of locking problems in US-ATFH8 comes from the test function part (isoparametric interpolation). Therefore, the assumed natural strain interpolations are employed in this section to eliminate the existing locking problems, and an effective solid-shell element US-ATFHS8 is formulated.

The natural coordinates and element nodal numbering are depicted in Figure 2, in which ζ denotes the thickness direction. The 1-2-3-4 and 5-6-7-8 surfaces are the bottom surface and top surface of the shell, respectively. For description convenience, the isoparametric interpolation is rewritten as follows:

$$\mathbf{u} = \mathbf{N}\mathbf{q}_0 + \zeta\mathbf{N}\mathbf{q}_n \tag{24}$$

where

$$\mathbf{N} = \begin{bmatrix} N_1\mathbf{I}_3 & N_2\mathbf{I}_3 & N_3\mathbf{I}_3 & N_4\mathbf{I}_3 \end{bmatrix} \tag{25}$$

$$\mathbf{I}_3 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad \begin{cases} N_1 = \dfrac{1}{4}(1-\xi)(1-\eta) \\ N_2 = \dfrac{1}{4}(1+\xi)(1-\eta) \\ N_3 = \dfrac{1}{4}(1+\xi)(1+\eta) \\ N_4 = \dfrac{1}{4}(1-\xi)(1+\eta) \end{cases} \tag{26}$$

$$\mathbf{q}_0=\begin{Bmatrix}\frac{1}{2}\left(\mathbf{u}_1+\mathbf{u}_5\right)\\\frac{1}{2}\left(\mathbf{u}_2+\mathbf{u}_6\right)\\\frac{1}{2}\left(\mathbf{u}_3+\mathbf{u}_7\right)\\\frac{1}{2}\left(\mathbf{u}_4+\mathbf{u}_8\right)\end{Bmatrix},\ \mathbf{q}_n=\begin{Bmatrix}\frac{1}{2}\left(-\mathbf{u}_1+\mathbf{u}_5\right)\\\frac{1}{2}\left(-\mathbf{u}_2+\mathbf{u}_6\right)\\\frac{1}{2}\left(-\mathbf{u}_3+\mathbf{u}_7\right)\\\frac{1}{2}\left(-\mathbf{u}_4+\mathbf{u}_8\right)\end{Bmatrix} \tag{27}$$

The geometric description is similar:

$$\mathbf{X}=\mathbf{N}\mathbf{X}_0+\zeta\mathbf{N}\mathbf{X}_n,\ \mathbf{X}_0=\begin{Bmatrix}\frac{1}{2}\left(\mathbf{X}_1+\mathbf{X}_5\right)\\\frac{1}{2}\left(\mathbf{X}_2+\mathbf{X}_6\right)\\\frac{1}{2}\left(\mathbf{X}_3+\mathbf{X}_7\right)\\\frac{1}{2}\left(\mathbf{X}_4+\mathbf{X}_8\right)\end{Bmatrix},\ \mathbf{X}_n=\begin{Bmatrix}\frac{1}{2}\left(-\mathbf{X}_1+\mathbf{X}_5\right)\\\frac{1}{2}\left(-\mathbf{X}_2+\mathbf{X}_6\right)\\\frac{1}{2}\left(-\mathbf{X}_3+\mathbf{X}_7\right)\\\frac{1}{2}\left(-\mathbf{X}_4+\mathbf{X}_8\right)\end{Bmatrix} \tag{28}$$

in which $\mathbf{X}$ is the coordinate vector, and $\mathbf{X}_i$ is the coordinate vector of the i-th node.

The strain tensor is given as follows:

$$\boldsymbol{\varepsilon}=\frac{1}{2}\left(\nabla\mathbf{u}+\mathbf{u}\nabla\right)\Rightarrow\varepsilon_{ij}\mathbf{g}^i\mathbf{g}^j=\frac{1}{2}\left[\mathbf{g}^i\frac{\partial\mathbf{u}}{\partial\xi^i}+\frac{\partial\mathbf{u}}{\partial\xi^i}\mathbf{g}^i\right]=\frac{1}{2}\left[\mathbf{g}^i\mathbf{g}^j\left(\mathbf{g}_j\cdot\frac{\partial\mathbf{u}}{\partial\xi^i}\right)+\left(\frac{\partial\mathbf{u}}{\partial\xi^i}\cdot\mathbf{g}_j\right)\mathbf{g}^j\mathbf{g}^i\right] \tag{29}$$

in which Einstein summation convention is employed, and ξ^1, ξ^2, and ξ^3 denote ξ, η, and ζ, respectively.

So the covariant strain components are

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial\mathbf{X}}{\partial\xi^i}\cdot\frac{\partial\mathbf{u}}{\partial\xi^j}+\frac{\partial\mathbf{X}}{\partial\xi^j}\cdot\frac{\partial\mathbf{u}}{\partial\xi^i}\right)=\frac{1}{2}\left[\left(\frac{\partial\mathbf{X}}{\partial\xi^i}\right)^{\mathrm{T}}\frac{\partial\mathbf{u}}{\partial\xi^j}+\left(\frac{\partial\mathbf{X}}{\partial\xi^j}\right)^{\mathrm{T}}\frac{\partial\mathbf{u}}{\partial\xi^i}\right] \tag{30}$$

Substitution of Equation (24) into Equation (30) yields:

$$
\left\{
\begin{aligned}
\varepsilon_{\xi\xi} &= \mathbf{X}_{,\xi}^{\mathrm{T}}\mathbf{u}_{,\xi} = \left(\mathbf{X}_0^{\mathrm{T}}\mathbf{N}_{,\xi}^{\mathrm{T}} + \zeta\mathbf{X}_n^{\mathrm{T}}\mathbf{N}_{,\xi}^{\mathrm{T}}\right)\left(\mathbf{N}_{,\xi}\mathbf{q}_0 + \zeta\mathbf{N}_{,\xi}\mathbf{q}_n\right)\\
\varepsilon_{\eta\eta} &= \mathbf{X}_{,\eta}^{\mathrm{T}}\mathbf{u}_{,\eta} = \left(\mathbf{X}_0^{\mathrm{T}}\mathbf{N}_{,\eta}^{\mathrm{T}} + \zeta\mathbf{X}_n^{\mathrm{T}}\mathbf{N}_{,\eta}^{\mathrm{T}}\right)\left(\mathbf{N}_{,\eta}\mathbf{q}_0 + \zeta\mathbf{N}_{,\eta}\mathbf{q}_n\right)\\
\varepsilon_{\zeta\zeta} &= \mathbf{X}_{,\zeta}^{\mathrm{T}}\mathbf{u}_{,\zeta} = \mathbf{X}_n^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}\mathbf{N}\mathbf{q}_n\\
\varepsilon_{\xi\eta} &= \frac{1}{2}\left(\mathbf{X}_{,\xi}^{\mathrm{T}}\mathbf{u}_{,\eta} + \mathbf{X}_{,\eta}^{\mathrm{T}}\mathbf{u}_{,\xi}\right) = \frac{1}{2}\left(\mathbf{X}_0^{\mathrm{T}}\mathbf{N}_{,\xi}^{\mathrm{T}} + \zeta\mathbf{X}_n^{\mathrm{T}}\mathbf{N}_{,\xi}^{\mathrm{T}}\right)\left(\mathbf{N}_{,\eta}\mathbf{q}_0 + \zeta\mathbf{N}_{,\eta}\mathbf{q}_n\right)\\
&\qquad + \frac{1}{2}\left(\mathbf{X}_0^{\mathrm{T}}\mathbf{N}_{,\eta}^{\mathrm{T}} + \zeta\mathbf{X}_n^{\mathrm{T}}\mathbf{N}_{,\eta}^{\mathrm{T}}\right)\left(\mathbf{N}_{,\xi}\mathbf{q}_0 + \zeta\mathbf{N}_{,\xi}\mathbf{q}_n\right)\\
\varepsilon_{\eta\zeta} &= \frac{1}{2}\left(\mathbf{X}_{,\eta}^{\mathrm{T}}\mathbf{u}_{,\zeta} + \mathbf{X}_{,\zeta}^{\mathrm{T}}\mathbf{u}_{,\eta}\right) = \frac{1}{2}\left[\left(\mathbf{X}_0^{\mathrm{T}}\mathbf{N}_{,\eta}^{\mathrm{T}} + \zeta\mathbf{X}_n^{\mathrm{T}}\mathbf{N}_{,\eta}^{\mathrm{T}}\right)\mathbf{N}\mathbf{q}_n + \mathbf{X}_n^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}\left(\mathbf{N}_{,\eta}\mathbf{q}_0 + \zeta\mathbf{N}_{,\eta}\mathbf{q}_n\right)\right]\\
\varepsilon_{\xi\zeta} &= \frac{1}{2}\left(\mathbf{X}_{,\xi}^{\mathrm{T}}\mathbf{u}_{,\zeta} + \mathbf{X}_{,\zeta}^{\mathrm{T}}\mathbf{u}_{,\xi}\right) = \frac{1}{2}\left[\left(\mathbf{X}_0^{\mathrm{T}}\mathbf{N}_{,\xi}^{\mathrm{T}} + \zeta\mathbf{X}_n^{\mathrm{T}}\mathbf{N}_{,\xi}^{\mathrm{T}}\right)\mathbf{N}\mathbf{q}_n + \mathbf{X}_n^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}\left(\mathbf{N}_{,\xi}\mathbf{q}_0 + \zeta\mathbf{N}_{,\xi}\mathbf{q}_n\right)\right]
\end{aligned}
\right. \tag{31}
$$

Following the fiber assumption in practical shell element formulation, the transverse shear strains ($\varepsilon_{\xi\zeta}$ and $\varepsilon_{\eta\zeta}$) should be constant and the membrane strains ($\varepsilon_{\xi\xi}$, $\varepsilon_{\eta\eta}$, and $\varepsilon_{\xi\eta}$) should be linear with respect to the thickness coordinate ζ. Thus, the higher order terms are removed.

$$
\left\{
\begin{aligned}
\varepsilon_{\xi\xi} &= \mathbf{X}_{,\xi}^{\mathrm{T}}\mathbf{u}_{,\xi} = \mathbf{X}_0^{\mathrm{T}}\mathbf{N}_{,\xi}^{\mathrm{T}}\mathbf{N}_{,\xi}\mathbf{q}_0 + \zeta\left(\mathbf{X}_0^{\mathrm{T}}\mathbf{N}_{,\xi}^{\mathrm{T}}\mathbf{N}_{,\xi}\mathbf{q}_n + \mathbf{X}_n^{\mathrm{T}}\mathbf{N}_{,\xi}^{\mathrm{T}}\mathbf{N}_{,\xi}\mathbf{q}_0\right)\\
\varepsilon_{\eta\eta} &= \mathbf{X}_{,\eta}^{\mathrm{T}}\mathbf{u}_{,\eta} = \mathbf{X}_0^{\mathrm{T}}\mathbf{N}_{,\eta}^{\mathrm{T}}\mathbf{N}_{,\eta}\mathbf{q}_0 + \zeta\left(\mathbf{X}_0^{\mathrm{T}}\mathbf{N}_{,\eta}^{\mathrm{T}}\mathbf{N}_{,\eta}\mathbf{q}_n + \mathbf{X}_n^{\mathrm{T}}\mathbf{N}_{,\eta}^{\mathrm{T}}\mathbf{N}_{,\eta}\mathbf{q}_0\right)\\
\varepsilon_{\zeta\zeta} &= \mathbf{X}_{,\zeta}^{\mathrm{T}}\mathbf{u}_{,\zeta} = \mathbf{X}_n^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}\mathbf{N}\mathbf{q}_n\\
\varepsilon_{\xi\eta} &= \frac{1}{2}\left(\mathbf{X}_{,\xi}^{\mathrm{T}}\mathbf{u}_{,\eta} + \mathbf{X}_{,\eta}^{\mathrm{T}}\mathbf{u}_{,\xi}\right) = \frac{1}{2}\left[\mathbf{X}_0^{\mathrm{T}}\mathbf{N}_{,\xi}^{\mathrm{T}}\mathbf{N}_{,\eta}\mathbf{q}_0 + \zeta\left(\mathbf{X}_n^{\mathrm{T}}\mathbf{N}_{,\xi}^{\mathrm{T}}\mathbf{N}_{,\eta}\mathbf{q}_0 + \mathbf{X}_0^{\mathrm{T}}\mathbf{N}_{,\xi}^{\mathrm{T}}\mathbf{N}_{,\eta}\mathbf{q}_n\right)\right]\\
&\qquad + \frac{1}{2}\left[\mathbf{X}_0^{\mathrm{T}}\mathbf{N}_{,\eta}^{\mathrm{T}}\mathbf{N}_{,\xi}\mathbf{q}_0 + \zeta\left(\mathbf{X}_n^{\mathrm{T}}\mathbf{N}_{,\eta}^{\mathrm{T}}\mathbf{N}_{,\xi}\mathbf{q}_0 + \mathbf{X}_0^{\mathrm{T}}\mathbf{N}_{,\eta}^{\mathrm{T}}\mathbf{N}_{,\xi}\mathbf{q}_n\right)\right]\\
\varepsilon_{\eta\zeta} &= \frac{1}{2}\left(\mathbf{X}_{,\eta}^{\mathrm{T}}\mathbf{u}_{,\zeta} + \mathbf{X}_{,\zeta}^{\mathrm{T}}\mathbf{u}_{,\eta}\right) = \frac{1}{2}\left[\mathbf{X}_0^{\mathrm{T}}\mathbf{N}_{,\eta}^{\mathrm{T}}\mathbf{N}\mathbf{q}_n + \mathbf{X}_n^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}\mathbf{N}_{,\eta}\mathbf{q}_0\right]\\
\varepsilon_{\xi\zeta} &= \frac{1}{2}\left(\mathbf{X}_{,\xi}^{\mathrm{T}}\mathbf{u}_{,\zeta} + \mathbf{X}_{,\zeta}^{\mathrm{T}}\mathbf{u}_{,\xi}\right) = \frac{1}{2}\left[\mathbf{X}_0^{\mathrm{T}}\mathbf{N}_{,\xi}^{\mathrm{T}}\mathbf{N}\mathbf{q}_n + \mathbf{X}_n^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}\mathbf{N}_{,\xi}\mathbf{q}_0\right]
\end{aligned}
\right. \tag{32}
$$

According to the MITC interpolation technique [11-19], the modification of transverse shear strains is given as follows:

$$
\begin{aligned}
\tilde{\varepsilon}_{\xi\zeta} &= \frac{1-\eta}{2}\varepsilon_{\xi\zeta}\Big|_{\xi=0,\,\eta=-1} + \frac{1+\eta}{2}\varepsilon_{\xi\zeta}\Big|_{\xi=0,\,\eta=+1}\\
\tilde{\varepsilon}_{\eta\zeta} &= \frac{1-\xi}{2}\varepsilon_{\eta\zeta}\Big|_{\xi=-1,\,\eta=0} + \frac{1+\xi}{2}\varepsilon_{\eta\zeta}\Big|_{\xi=+1,\,\eta=0}
\end{aligned} \tag{33}
$$

Due to the geometric complexity of shell structures, “trapezoidal” elements may not be avoided, e.g., in cylindrical and spherical shell cases. By analyzing a beam problem meshed with trapezoidal elements, a substituted strain interpolation scheme was suggested by Sze et al. [28]. The transverse normal strain is given by:

$$\tilde{\varepsilon}_{\zeta\zeta} = N_1\, \varepsilon_{\zeta\zeta}\Big|_{\xi=-1,\,\eta=-1} + N_2\, \varepsilon_{\zeta\zeta}\Big|_{\xi=+1,\,\eta=-1} + N_3\, \varepsilon_{\zeta\zeta}\Big|_{\xi=+1,\,\eta=+1} + N_4\, \varepsilon_{\zeta\zeta}\Big|_{\xi=-1,\,\eta=+1} \tag{34}$$

where the shape function N_i (i=1,…,4) are the isoparametric interpolation functions given in Equation (26). Although trapezoidal locking can be overcome by US-ATFH8 in simple benchmark problems, such as MacNeal’s thin beam under pure bending and linear bending load, the modification given in Equation (34) is of great importance for constructing locking-free solid-shell element for general shell structures.

The strains in global Cartesian coordinates can be obtained by tensor transformation:

$$\begin{bmatrix} \tilde{\varepsilon}_{xx} & \tilde{\varepsilon}_{xy} & \tilde{\varepsilon}_{xz} \\ \tilde{\varepsilon}_{xy} & \tilde{\varepsilon}_{yy} & \tilde{\varepsilon}_{yz} \\ \tilde{\varepsilon}_{xz} & \tilde{\varepsilon}_{yz} & \tilde{\varepsilon}_{zz} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \varepsilon_{\xi\xi} & \varepsilon_{\xi\eta} & \tilde{\varepsilon}_{\xi\zeta} \\ \varepsilon_{\xi\eta} & \varepsilon_{\eta\eta} & \tilde{\varepsilon}_{\eta\zeta} \\ \tilde{\varepsilon}_{\xi\zeta} & \tilde{\varepsilon}_{\eta\zeta} & \tilde{\varepsilon}_{\zeta\zeta} \end{bmatrix} \mathbf{J}^{-\mathrm{T}} \tag{35}$$

in which $\mathbf{J}$ is the Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \\ \dfrac{\partial x}{\partial \zeta} & \dfrac{\partial y}{\partial \zeta} & \dfrac{\partial z}{\partial \zeta} \end{bmatrix} \tag{36}$$

Thus, the virtual strain interpolation for unsymmetric solid-shell element can be written in the following matrix form:

$$\delta\tilde{\boldsymbol{\varepsilon}} = \tilde{\mathbf{B}}\delta\mathbf{q}^{e} \tag{37}$$

The test function part in Equation (22) is replaced by the modified formulae, and the trial function part (stress interpolation) remains the same. Hence, the element stiffness matrix of solid-shell element (denoted by US-ATFHS8) is obtained:

$$\mathbf{K}^e = \int_{\Omega^e} \tilde{\mathbf{B}}^{\mathrm{T}} \mathbf{D} \hat{\mathbf{B}} \mathrm{d}V = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \tilde{\mathbf{B}}^{\mathrm{T}} \mathbf{D} \hat{\mathbf{B}} |\mathbf{J}| \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}\zeta \tag{38}$$

It should be noted that a $1/|\mathbf{J}|^2$ term is contained in the $\tilde{\mathbf{B}}$ matrix, so that inversion of Jacobian determinant is needed in computing the element stiffness matrix, which will weaken the mesh distortion tolerance of the unsymmetric finite elements. Such imperfection cannot be overcome unless the modification is conducted without the help of natural coordinates. Despite of this weakness, remarkable mesh distortion tolerance is still presented by this locking-free solid-shell model, and this will be shown in the next section.

2×2×2 Gauss quadrature rule for Equation (38) is still proved to be sufficient in computer coding. Although remarkable mesh distortion tolerance is inherited from US-ATFH8, US-ATFHS8 is not recommended for the elements of which the top and bottom surfaces are degenerated triangles or concave quadrilaterals to prevent the Jacobian determinant from vanishing.

3. Numerical Examples

Nine numerical tests are presented in this section to demonstrate the performance of the proposed element US-ATFHS8. And results obtained by some other elements are also given for comparison. These elements referred to are as follows:

ANSγε: A displacement-based solid-shell model using assumed natural strain techniques introduced by Sze et al. [28].

ANSγε-HS: A hybrid-stress solid-shell model based on the ANSγε model proposed by Sze et al. [28].

SC8R: A (reduced integration) solid-shell element assembled in commercial software Abaqus

[32].

It should be noted that the displacement solutions solved by ANSγε and ANSγε-HS are calculated by the authors using the formulation in reference [28]. Due to some distinctions in modeling and programming, for example, the number of Gaussian points employed in one element (2×2 points for presented results), the presented data are not completely identical to those given by Sze et al. [28], but these differences are negligible and reasonable (and these two elements are also verified by patch tests).

3.1 Patch tests

As shown in Figure 3, a square shell is meshed into five elements. Proper constraints are imposed on the left-side boundary to eliminate rigid body motions. Membrane tests, bending/twisting tests, and transverse shear test are performed for both thin and moderately thick shells. The geometry and material parameters can be found in Figure 3.

Membrane tests: Boundary tractions corresponding to constant membrane stress conditions are imposed on the boundaries. For the fact that US-ATFHS8 is formulated by modifying transverse strain components of a solid element US-ATFH8, and the membrane strains are not changed, such tests are passed without any problem.

Bending/twisting tests: As used for Mindlin-Reissner plate elements, pure bending/twisting boundary stresses are imposed and the numerical solutions are compared with corresponding analytical solutions. The proposed element fulfills the requirement as well.

Transverse shear test: Such test aims to reveal that the present element can exactly reproduce constant transverse shear deformations. In this test, all x, y-dofs are constrained, and a constant shear stress state is imposed on the right side of the patch. Linear z-displacement distribution can be obtained, so this test is also satisfied.

All above tests demonstrate that the validity of the coding, i.e., the present element converges to right answers.

3.2 Mesh distortion sensitivity study for in-plane bending

Membrane-dominated problems are of great importance in engineering practice. As an example, shear walls are applied in many civil structures, which are mainly loaded by membrane forces. So it is a critical indicator to evaluate shell element models under membrane loading conditions.

As shown in Figure 4, a beam is loading by a moment M=40. This numerical test is suggested by Pian et al. to study the mesh distortion sensitivity of finite elements [40]. In this problem, material parameters are:

$$E=1500,\ \mu=0$$

The exact tip deflection is 1, and the numerical solutions obtained with different elements are listed in Table 2. In this particular problem, we also compare the results with the HSEE element, which is an EAS solid-shell model proposed by Klinkel et al. [38]. A direct comparison can also be seen in Figure 5.

It can be found that the proposed model gives exact results even when the mesh is severely distorted, which is much superior over other models.

3.3 MacNeal's thin beam test

A thin beam problem is presented here with the geometric parameters, mesh descriptions, loading conditions, and material parameters plotted in Figure 6. To evaluate the membrane features, the thickness direction is set to be parallel to the z-axis. Under pure bending load M and linearly bending load P, the deflection results of point A at the free end are listed in Table 3, with the reference solution proposed by MacNeal et al. [34].

The numerical results demonstrate that the US-ATFHS8 element is able to present highly accurate solutions using regular, parallelogram, and trapezoidal meshes. Such feature is inherited from the 3D element US-ATFH8, which presents excellent membrane solutions for many problems [8]. As corresponding analytical solutions are included in the trial functions, exact solution can be achieved for pure bending case using US-ATFHS8 and US-ATFH8 elements. Whereas other solid-shell models give poor results for these problems, especially when the meshes are distorted. The reduced integration element SC8R suffers from serious "hourglass" problem, which causes the membrane stiffness much too underestimated.

It should also be noted that the solid-shell element US-ATFHS8 almost presents the same solutions as the 3D element US-ATFH8, which is due to the fact that the solid-shell model is successfully formulated by modifying the transverse strain components, while the membrane part remains untouched.

We here only give two membrane examples because US-ATFHS8 and US-ATFH8 present almost the same solutions for in-plane loading problems, and the excellent membrane behavior of US-ATFH8 has been fully studied in our previous paper [8].

3.4 Curved beam

Necessary parameters and the problem description can be seen in Figure 7, the deflection of the free end along the loading direction is evaluated. A reference solution is derived using the Euler-Bernoulli beam theory for this bending-dominated problem:

$$\Delta = 3\pi \frac{FR^3}{Ebh^3} \tag{39}$$

Numerical solutions with 10 elements are listed in Table 4. As Sze et al. [28] claimed, thickness locking is observed when the given solution is about $(1-\mu^2)$ times the exact solution. So it can be found in this problem that US-ATFH8 and ANSγε suffer from the thickness locking, and other models, including the new element US-ATFHS8, can give locking-free solutions.

3.5 Twisted beam

A 90° pre-twisted thin beam with a fixed end is plotted in Figure 8. Two loading cases, in-plane loading P_1 and out-of-plane loading P_2, are considered. The deflections of the free end along the loading direction are evaluated and compared to the reference solutions suggested by Belytschko et al. [33]. Most solid elements will suffer from severe locking due to the geometric complexity and inevitable distortions. Normalized solutions are given in Table 5, where it can be seen that the 3D element US-ATFH8 presents negative numbers due to locking problems, while the new solid-shell element US-ATFHS8 gives relatively accurate results.

3.6 Hemispherical shell with an 18° circular cut

A hemispherical shell with an 18° circular cut under concentrated loads is described in Figure 9. Only 1/8 of a spherical shell is modeled due to symmetry. For the deflection at the loading points, MacNeal et al. suggested a reference solution 0.094 [34], but another reference solution 0.093 proposed by Simo et al. [35] gives better prediction as compared with several finite element models, so the latter is chosen as the reference solution here. Normalized results calculated by different elements are listed in Table 6, and it can be seen that the shell element SC8R converges slowly and the 3D element US-ATFH8 gives poor results with coarse meshes.

The strain energy results are also listed in Table 7 and plotted in Figure 10. It can be seen that the presented model also gives good energy predictions.

3.7 Scordelis-Lo roof

A roof under self-weight is described in Figure 11. On the supported boundaries, x, y-dofs are constrained, and the vertical deflection of the mid-point of a free edge is evaluated with the reference solution 0.3024 [34]. Numerical results obtained with regular meshes are listed in Table 8.

Distorted mesh cases are also considered for this problem, and two distortion modes are employed. In these distorted meshes, z-coordinates of a pair of top and bottom nodes are kept the same. The first kind of distortion is generated by randomly changing the z-coordinates of top (bottom) nodes:

$$\tilde{z} = z + 0.5\alpha\Delta z \tag{40}$$

where z is the z-coordinate in the initial regular mesh, α is a random number between -1 and 1, and Δz is the regular element size.

The second kind of distortion changes the random number α to be an assigned value for each top (bottom) node. For different nodes with the same x and y coordinates (they are in a row), the values of α are also same. For different nodes in two adjacent rows, the values of α are taken to be 1 and -1, respectively. Regular mesh and two kinds of distorted meshes discretized with 4×4 elements are also plotted in Figure 12, and the numerical results using distorted meshes are given in Table 9. To better understand the convergence, the corresponding strain energy results are presented in Table 10 and visualized in Figure 13.

It is clear that the proposed solid-shell element US-ATFHS8 gives remarkable (local and global) results under both regular and distorted meshes, and these solutions are not sensitive to mesh distortions. Rapid energy convergence is always achieved by the proposed model under various mesh conditions.

3.8 Partly clamped hyperbolic paraboloid shell

The geometric and material parameters of a partly clamped hyperbolic paraboloid shell loaded by its self-weight are plotted in Figure 14. The reference solution for this problem is given by Bathe et al. using high-order MITC model [15]. The vertical deflection at point O is evaluated and the numerical results are given in Table 11. It can be seen that SC8R model gives much larger solutions in coarse meshes, which is due to the reduced integration.

3.9 A beam with L-shaped cross-section

As depicted in Figure 15, an L-shaped section beam is loaded by two bending moments at its free end. To illustrate the mesh and thickness directions of shell elements, this beam is divided into three parts (two plates and one ridge). The material parameters are as follows:

$$E=10^7,\ \mu=0.3$$

The geometrical data are all given in the figure, and the assigned moment is implemented by using concentrated forces. The x-, y-deflections at the outer angular point, where the largest displacement should be observed, are evaluated in this test.

Four meshes are considered, of which three are coarse meshes discretized with 30 elements, and one is a fine mesh for convergence analysis. Both parallelogram mesh and trapezoidal mesh are used to check the mesh distortion sensitivity.

The reference solution is obtained by using over 120 thousand SC8R elements (4×36×400 elements for part II and part III, and 4×4×400 elements for part I). It is observed in Table 12 that although so many elements are used, the x-, y-deflections at the sample point are clearly different, while they are analytically identical due to the symmetry.

Known by other numerical tests, the bending property of SC8R is pretty reasonable, while its membrane part gives poor results, so the reference solution is suggested to be 0.69×10^{-4}, and it can also be found that the numerical solutions obtained with other elements converge to this reference solution.

The normalized numerical results given by different models are listed in Table 13, and similar conclusions can be drawn as in MacNeal's beam test. US-ATFHS8 and US-ATFH8 present highly accurate solutions that are insensitive to mesh distortions. Other finite element models show non-negligible sensitivity to, especially, the trapezoidal mesh distortion. The reduced integration element SC8R suffers from "hourglass" problem in its membrane part, especially when coarse meshes are used.

4. Conclusion

An unsymmetric 8-node hexahedral solid-shell element US-ATFHS8 with high distortion tolerance is formulated by consolidating the unsymmetric finite element method, the analytical trial function method, and the assumed natural strain modifications. The cause of the locking phenomena in unsymmetric elements, in which analytical solutions are employed as trial functions, is the isoparametric interpolation in the test function part.

The numerical examples demonstrate that the proposed element US-ATFHS8 presents locking-free and highly accurate numerical solutions for shell problems of different geometric features, mesh conditions, and loading conditions. Some classical elements give remarkable numerical solutions for bending-dominated problems, while they lose reliability for the trapezoidal locking and hourglass modes under in-plane loading cases. Thus, the proposed element is more reliable in engineering practice.

Although part of the distortion tolerance is inherited from US-ATFH8, inversion of Jacobian determinant is inevitable in the new solid-shell formulation, which causes numerical dangers and reduces the accuracy when the top and bottom surfaces of solid-shell element are distorted into degenerated triangles or concave quadrilaterals. This weakness originates from the modification of strain components in natural coordinates, and finding a better scheme is still an interesting topic.

Mesh distortion tolerance is significant in nonlinear analysis. Recently, excellent geometric nonlinear model for 2D problems has been formulated based on the unsymmetric finite element method and the analytical trial function method [37]. It is clearly evident that the proposed formulations can be extended into nonlinear problems, and the resulting models will be both accurate under even coarse meshes and insensitive to mesh distortions. Some progress for 3D solids and shells will be reported in the near future.

Appendix: Analytical general solutions for linear stresses, strains, and quadratic displacements in terms of oblique coordinates

Reference [8] has provided related solutions for both isotropic and anisotropic materials. Here, only the isotropic solutions are listed.

Denote

$$\begin{array}{lll} h_1 = \bar{b}_2\bar{c}_3 + \bar{b}_3\bar{c}_2, & h_2 = \bar{a}_2\bar{c}_3 + \bar{a}_3\bar{c}_2, & h_3 = \bar{a}_2\bar{b}_3 + \bar{a}_3\bar{b}_2 \\ h_4 = \bar{b}_1\bar{c}_3 + \bar{b}_3\bar{c}_1, & h_5 = \bar{a}_1\bar{c}_3 + \bar{a}_3\bar{c}_1, & h_6 = \bar{a}_1\bar{b}_3 + \bar{a}_3\bar{b}_1 \\ h_7 = \bar{b}_1\bar{c}_2 + \bar{b}_2\bar{c}_1, & h_8 = \bar{a}_1\bar{c}_2 + \bar{a}_2\bar{c}_1, & h_9 = \bar{a}_1\bar{b}_2 + \bar{a}_2\bar{b}_1 \end{array} \tag{A.1}$$

A1. Analytical general solutions for global linear stresses and strains in terms of R, S and T

(1) The 13th set of solutions for global stresses and strains

Stresses:

$$\sigma_{x13} = \bar{a}_2^2 R, \quad \sigma_{y13} = \bar{b}_2^2 R, \quad \sigma_{z13} = \bar{c}_2^2 R, \quad \tau_{xy13} = \bar{a}_2\bar{b}_2 R, \quad \tau_{yz13} = \bar{b}_2\bar{c}_2 R, \quad \tau_{zx13} = \bar{a}_2\bar{c}_2 R \tag{A.2}$$

Strains:

$$\begin{cases} \varepsilon_{x13} = \dfrac{1}{E}(\bar{a}_2^2 - \mu\bar{b}_2^2 - \mu\bar{c}_2^2)R = A_{x13}R, & \varepsilon_{y13} = \dfrac{1}{E}(\bar{b}_2^2 - \mu\bar{a}_2^2 - \mu\bar{c}_2^2)R = A_{y13}R \\ \varepsilon_{z13} = \dfrac{1}{E}(\bar{c}_2^2 - \mu\bar{a}_2^2 - \mu\bar{b}_2^2)R = A_{z13}R, & \gamma_{xy13} = \dfrac{2(1+\mu)}{E}\bar{a}_2\bar{b}_2R = A_{xy13}R \\ \gamma_{yz13} = \dfrac{2(1+\mu)}{E}\bar{b}_2\bar{c}_2R = A_{yz13}R, & \gamma_{zx13} = \dfrac{2(1+\mu)}{E}\bar{a}_2\bar{c}_2R = A_{zx13}R \end{cases} \tag{A.3}$$

(2) The 14th set of solutions for global stresses and strains

Stresses:

$$\sigma_{x14} = \bar{a}_3^2R, \quad \sigma_{y14} = \bar{b}_3^2R, \quad \sigma_{z14} = \bar{c}_3^2R, \quad \tau_{xy14} = \bar{a}_3\bar{b}_3R, \quad \tau_{yz14} = \bar{b}_3\bar{c}_3R, \quad \tau_{zx14} = \bar{a}_3\bar{c}_3R \tag{A.4}$$

Strains:

$$\begin{cases} \varepsilon_{x14} = \dfrac{1}{E}(\bar{a}_3^2 - \mu\bar{b}_3^2 - \mu\bar{c}_3^2)R = A_{x14}R, & \varepsilon_{y14} = \dfrac{1}{E}(\bar{b}_3^2 - \mu\bar{a}_3^2 - \mu\bar{c}_3^2)R = A_{y14}R \\ \varepsilon_{z14} = \dfrac{1}{E}(\bar{c}_3^2 - \mu\bar{a}_3^2 - \mu\bar{b}_3^2)R = A_{z14}R, & \gamma_{xy14} = \dfrac{2(1+\mu)}{E}\bar{a}_3\bar{b}_3R = A_{xy14}R \\ \gamma_{yz14} = \dfrac{2(1+\mu)}{E}\bar{b}_3\bar{c}_3R = A_{yz14}R, & \gamma_{zx14} = \dfrac{2(1+\mu)}{E}\bar{a}_3\bar{c}_3R = A_{zx14}R \end{cases} \tag{A.5}$$

(3) The 15th set of solutions for global stresses and strains

Stresses:

$$\sigma_{x15} = 2\bar{a}_2\bar{a}_3R, \quad \sigma_{y15} = 2\bar{b}_2\bar{b}_3R, \quad \sigma_{z15} = 2\bar{c}_2\bar{c}_3R, \quad \tau_{xy15} = h_3R, \quad \tau_{yz15} = h_1R, \quad \tau_{zx15} = h_2R \tag{A.6}$$

Strains:

$$\begin{cases} \varepsilon_{x15} = \dfrac{2}{E}(\bar{a}_2\bar{a}_3 - \mu\bar{b}_2\bar{b}_3 - \mu\bar{c}_2\bar{c}_3)R = A_{x15}R, & \varepsilon_{y15} = \dfrac{2}{E}(\bar{b}_2\bar{b}_3 - \mu\bar{a}_2\bar{a}_3 - \mu\bar{c}_2\bar{c}_3)R = A_{y15}R \\ \varepsilon_{z15} = \dfrac{2}{E}(\bar{c}_2\bar{c}_3 - \mu\bar{a}_2\bar{a}_3 - \mu\bar{b}_2\bar{b}_3)R = A_{z15}R, & \gamma_{xy15} = \dfrac{2(1+\mu)}{E}(\bar{a}_2\bar{b}_3 + \bar{a}_3\bar{b}_2)R = A_{xy15}R \\ \gamma_{yz15} = \dfrac{2(1+\mu)}{E}(\bar{b}_2\bar{c}_3 + \bar{b}_3\bar{c}_2)R = A_{yz15}R, & \gamma_{zx15} = \dfrac{2(1+\mu)}{E}(\bar{a}_2\bar{c}_3 + \bar{a}_3\bar{c}_2)R = A_{zx15}R \end{cases} \tag{A.7}$$

(4) The 16th set of solutions for global stresses and strains

Stresses:

$$\sigma_{x16} = \bar{a}_1^2S, \quad \sigma_{y16} = \bar{b}_1^2S, \quad \sigma_{z16} = \bar{c}_1^2S, \quad \tau_{xy16} = \bar{a}_1\bar{b}_1S, \quad \tau_{yz16} = \bar{b}_1\bar{c}_1S, \quad \tau_{zx16} = \bar{a}_1\bar{c}_1S \tag{A.8}$$

Strains:

$$\begin{cases} \varepsilon_{x16} = \dfrac{1}{E}(\bar{a}_1^2 - \mu\bar{b}_1^2 - \mu\bar{c}_1^2)S = A_{x16}S, & \varepsilon_{y16} = \dfrac{1}{E}(\bar{b}_1^2 - \mu\bar{a}_1^2 - \mu\bar{c}_1^2)S = A_{y16}S \\ \varepsilon_{z16} = \dfrac{1}{E}(\bar{c}_1^2 - \mu\bar{a}_1^2 - \mu\bar{b}_1^2)S = A_{z16}S, & \gamma_{xy16} = \dfrac{2(1+\mu)}{E}\bar{a}_1\bar{b}_1S = A_{xy16}S \\ \gamma_{yz16} = \dfrac{2(1+\mu)}{E}\bar{b}_1\bar{c}_1S = A_{yz16}S, & \gamma_{zx16} = \dfrac{2(1+\mu)}{E}\bar{a}_1\bar{c}_1S = A_{zx16}S \end{cases} \tag{A.9}$$

(5) The 17th set of solutions for global stresses and strains

Stresses:

$$\sigma_{x17} = \bar{a}_3^2S, \quad \sigma_{y17} = \bar{b}_3^2S, \quad \sigma_{z17} = \bar{c}_3^2S, \quad \tau_{xy17} = \bar{a}_3\bar{b}_3S, \quad \tau_{yz17} = \bar{b}_3\bar{c}_3S, \quad \tau_{zx17} = \bar{a}_3\bar{c}_3S \tag{A.10}$$

Strains:

$$\begin{cases} \varepsilon_{x17} = \dfrac{1}{E}(\bar{a}_3^2 - \mu\bar{b}_3^2 - \mu\bar{c}_3^2)S = A_{x17}S, & \varepsilon_{y17} = \dfrac{1}{E}(\bar{b}_3^2 - \mu\bar{a}_3^2 - \mu\bar{c}_3^2)S = A_{y17}S \\ \varepsilon_{z17} = \dfrac{1}{E}(\bar{c}_3^2 - \mu\bar{a}_3^2 - \mu\bar{b}_3^2)S = A_{z17}S, & \gamma_{xy17} = \dfrac{2(1+\mu)}{E}\bar{a}_3\bar{b}_3S = A_{xy17}S \\ \gamma_{yz17} = \dfrac{2(1+\mu)}{E}\bar{b}_3\bar{c}_3S = A_{yz17}S, & \gamma_{zx17} = \dfrac{2(1+\mu)}{E}\bar{a}_3\bar{c}_3S = A_{zx17}S \end{cases} \tag{A.11}$$

(6) The 18th set of solutions for global stresses and strains

Stresses:

$$\sigma_{x18} = 2\bar{a}_1\bar{a}_3S, \quad \sigma_{y18} = 2\bar{b}_1\bar{b}_3S, \quad \sigma_{z18} = 2\bar{c}_1\bar{c}_3S, \quad \tau_{xy18} = h_6S, \quad \tau_{yz18} = h_4S, \quad \tau_{zx18} = h_5S \tag{A.12}$$

Strains:

$$\begin{cases} \varepsilon_{x18} = \dfrac{2}{E}(\bar{a}_1\bar{a}_3 - \mu\bar{b}_1\bar{b}_3 - \mu\bar{c}_1\bar{c}_3)S = A_{x18}S, & \varepsilon_{y18} = \dfrac{2}{E}(\bar{b}_1\bar{b}_3 - \mu\bar{a}_1\bar{a}_3 - \mu\bar{c}_1\bar{c}_3)S = A_{y18}S \\ \varepsilon_{z18} = \dfrac{2}{E}(\bar{c}_1\bar{c}_3 - \mu\bar{a}_1\bar{a}_3 - \mu\bar{b}_1\bar{b}_3)S = A_{z18}S, & \gamma_{xy18} = \dfrac{2(1+\mu)}{E}(\bar{a}_1\bar{b}_3 + \bar{a}_3\bar{b}_1)S = A_{xy18}S \\ \gamma_{yz18} = \dfrac{2(1+\mu)}{E}(\bar{b}_1\bar{c}_3 + \bar{b}_3\bar{c}_1)S = A_{yz18}S, & \gamma_{zx18} = \dfrac{2(1+\mu)}{E}(\bar{a}_1\bar{c}_3 + \bar{a}_3\bar{c}_1)S = A_{zx18}S \end{cases} \tag{A.13}$$

(7) The 19th set of solutions for global stresses and strains

Stresses:

$$\sigma_{x19} = \bar{a}_1^2T, \quad \sigma_{y19} = \bar{b}_1^2T, \quad \sigma_{z19} = \bar{c}_1^2T, \quad \tau_{xy19} = \bar{a}_1\bar{b}_1T, \quad \tau_{yz19} = \bar{b}_1\bar{c}_1T, \quad \tau_{zx19} = \bar{a}_1\bar{c}_1T \tag{A.14}$$

Strains:

$$\begin{cases} \varepsilon_{x19} = \dfrac{1}{E}(\bar{a}_1^2 - \mu\bar{b}_1^2 - \mu\bar{c}_1^2)T = A_{x19}T, & \varepsilon_{y19} = \dfrac{1}{E}(\bar{b}_1^2 - \mu\bar{a}_1^2 - \mu\bar{c}_1^2)T = A_{y19}T \\ \varepsilon_{z19} = \dfrac{1}{E}(\bar{c}_1^2 - \mu\bar{a}_1^2 - \mu\bar{b}_1^2)T = A_{z19}T, & \gamma_{xy19} = \dfrac{2(1+\mu)}{E}\bar{a}_1\bar{b}_1T = A_{xy19}T \\ \gamma_{yz19} = \dfrac{2(1+\mu)}{E}\bar{b}_1\bar{c}_1T = A_{yz19}T, & \gamma_{zx19} = \dfrac{2(1+\mu)}{E}\bar{a}_1\bar{c}_1T = A_{zx19}T \end{cases} \tag{A.15}$$

(8) The 20th set of solutions for global stresses and strains

Stresses:

$$\sigma_{x20} = \bar{a}_2^2T, \quad \sigma_{y20} = \bar{b}_2^2T, \quad \sigma_{z20} = \bar{c}_2^2T, \quad \tau_{xy20} = \bar{a}_2\bar{b}_2T, \quad \tau_{yz20} = \bar{b}_2\bar{c}_2T, \quad \tau_{zx20} = \bar{a}_2\bar{c}_2T \tag{A.16}$$

Strains:

$$\begin{cases} \varepsilon_{x20} = \dfrac{1}{E}(\bar{a}_2^2 - \mu\bar{b}_2^2 - \mu\bar{c}_2^2)T = A_{x20}T, & \varepsilon_{y20} = \dfrac{1}{E}(\bar{b}_2^2 - \mu\bar{a}_2^2 - \mu\bar{c}_2^2)T = A_{y20}T \\ \varepsilon_{z20} = \dfrac{1}{E}(\bar{c}_2^2 - \mu\bar{a}_2^2 - \mu\bar{b}_2^2)T = A_{z20}T, & \gamma_{xy20} = \dfrac{2(1+\mu)}{E}\bar{a}_2\bar{b}_2T = A_{xy20}T \\ \gamma_{yz20} = \dfrac{2(1+\mu)}{E}\bar{b}_2\bar{c}_2T = A_{yz20}T, & \gamma_{zx20} = \dfrac{2(1+\mu)}{E}\bar{a}_2\bar{c}_2T = A_{zx20}T \end{cases} \tag{A.17}$$

(9) The 21st set of solutions for global stresses and strains

Stresses:

$$\sigma_{x21} = 2\bar{a}_1\bar{a}_2T, \quad \sigma_{y21} = 2\bar{b}_1\bar{b}_2T, \quad \sigma_{z21} = 2\bar{c}_1\bar{c}_2T, \quad \tau_{xy21} = h_9T, \quad \tau_{yz21} = h_7T, \quad \tau_{zx21} = h_8T \tag{A.18}$$

Strains:

$$\begin{cases} \varepsilon_{x21} = \dfrac{2}{E}(\bar{a}_1\bar{a}_2 - \mu\bar{b}_1\bar{b}_2 - \mu\bar{c}_1\bar{c}_2)T = A_{x21}T, & \varepsilon_{y21} = \dfrac{2}{E}(\bar{b}_1\bar{b}_2 - \mu\bar{a}_1\bar{a}_2 - \mu\bar{c}_1\bar{c}_2)T = A_{y21}T \\ \varepsilon_{z21} = \dfrac{2}{E}(\bar{c}_1\bar{c}_2 - \mu\bar{a}_1\bar{a}_2 - \mu\bar{b}_1\bar{b}_2)T = A_{z21}T, & \gamma_{xy21} = \dfrac{2(1+\mu)}{E}(\bar{a}_1\bar{b}_2 + \bar{a}_2\bar{b}_1)T = A_{xy21}T \\ \gamma_{yz21} = \dfrac{2(1+\mu)}{E}(\bar{b}_1\bar{c}_2 + \bar{b}_2\bar{c}_1)T = A_{yz21}T, & \gamma_{zx21} = \dfrac{2(1+\mu)}{E}(\bar{a}_1\bar{c}_2 + \bar{a}_2\bar{c}_1)T = A_{zx21}T \end{cases} \tag{A.19}$$

A2. Analytical general solutions for quadratic displacements in terms of R, S and T

(1) The 13th~15th sets of solutions for displacements (i=13~15)

$$U_i=\frac{1}{2J_0}\{[\bar{a}_1J_0A_{xi}+(J_0-\bar{a}_1a_1)(\bar{a}_1A_{xi}+\bar{b}_1A_{xyi}+\bar{c}_1A_{zxi})-a_1(\bar{b}_1^2A_{yi}+\bar{c}_1^2A_{zi}+\bar{b}_1\bar{c}_1A_{yzi})]R^2-a_1(\bar{a}_2^2A_{xi}$$
$$+\bar{b}_2^2A_{yi}+\bar{c}_2^2A_{zi}+\bar{a}_2\bar{b}_2A_{xyi}+\bar{b}_2\bar{c}_2A_{yzi}+\bar{a}_2\bar{c}_2A_{zxi})S^2-a_1(\bar{a}_3^2A_{xi}+\bar{b}_3^2A_{yi}+\bar{c}_3^2A_{zi}+\bar{a}_3\bar{b}_3A_{xyi}+\bar{b}_3\bar{c}_3A_{yzi}$$
$$+\bar{a}_3\bar{c}_3A_{zxi})T^2+[J_0(2\bar{a}_2A_{xi}+\bar{b}_2A_{xyi}+\bar{c}_2A_{zxi})-2a_1(\bar{a}_1\bar{a}_2A_{xi}+\bar{b}_1\bar{b}_2A_{yi}+\bar{c}_1\bar{c}_2A_{zi})-a_1(h_9A_{xyi}+h_7A_{yzi}$$
$$+h_8A_{zxi})]RS+[J_0(2\bar{a}_3A_{xi}+\bar{b}_3A_{xyi}+\bar{c}_3A_{zxi})-2a_1(\bar{a}_1\bar{a}_3A_{xi}+\bar{b}_1\bar{b}_3A_{yi}+\bar{c}_1\bar{c}_3A_{zi})-a_1(h_6A_{xyi}+h_4A_{yzi}$$
$$+h_5A_{zxi})]RT-a_1(2\bar{a}_2\bar{a}_3A_{xi}+2\bar{b}_2\bar{b}_3A_{yi}+2\bar{c}_2\bar{c}_3A_{zi}+h_3A_{xyi}+h_1A_{yzi}+h_2A_{zxi})ST\}$$

(A.20a)

$$V_i=\frac{1}{2J_0}\{[\bar{b}_1J_0A_{yi}+(J_0-\bar{b}_1b_1)(\bar{a}_1A_{xyi}+\bar{b}_1A_{yi}+\bar{c}_1A_{yzi})-b_1(\bar{a}_1^2A_{xi}+\bar{c}_1^2A_{zi}+\bar{a}_1\bar{c}_1A_{zxi})]R^2-b_1(\bar{a}_2^2A_{xi}$$
$$+\bar{b}_2^2A_{yi}+\bar{c}_2^2A_{zi}+\bar{a}_2\bar{b}_2A_{xyi}+\bar{b}_2\bar{c}_2A_{yzi}+\bar{a}_2\bar{c}_2A_{zxi})S^2-b_1(\bar{a}_3^2A_{xi}+\bar{b}_3^2A_{yi}+\bar{c}_3^2A_{zi}+\bar{a}_3\bar{b}_3A_{xyi}+\bar{b}_3\bar{c}_3A_{yzi}$$
$$+\bar{a}_3\bar{c}_3A_{zxi})T^2+[J_0(\bar{a}_2A_{xyi}+2\bar{b}_2A_{yi}+\bar{c}_2A_{yzi})-2b_1(\bar{a}_1\bar{a}_2A_{xi}+\bar{b}_1\bar{b}_2A_{yi}+\bar{c}_1\bar{c}_2A_{zi})-b_1(h_9A_{xyi}+h_7A_{yzi}$$
$$+h_8A_{zxi})]RS+[J_0(\bar{a}_3A_{xyi}+2\bar{b}_3A_{yi}+\bar{c}_3A_{yzi})-2b_1(\bar{a}_1\bar{a}_3A_{xi}+\bar{b}_1\bar{b}_3A_{yi}+\bar{c}_1\bar{c}_3A_{zi})-b_1(h_6A_{xyi}+h_4A_{yzi}$$
$$+h_5A_{zxi})]RT-b_1(2\bar{a}_2\bar{a}_3A_{xi}+2\bar{b}_2\bar{b}_3A_{yi}+2\bar{c}_2\bar{c}_3A_{zi}+h_3A_{xyi}+h_1A_{yzi}+h_2A_{zxi})ST\}$$

(A.20b)

$$W_i=\frac{1}{2J_0}\{[\bar{c}_1J_0A_{zi}+(J_0-\bar{c}_1c_1)(\bar{a}_1A_{zxi}+\bar{b}_1A_{yzi}+\bar{c}_1A_{zi})-\bar{c}_1(\bar{a}_1^2A_{xi}+\bar{b}_1^2A_{yi}+\bar{a}_1\bar{b}_1A_{xyi})]R^2-c_1(\bar{a}_2^2A_{xi}$$
$$+\bar{b}_2^2A_{yi}+\bar{c}_2^2A_{zi}+\bar{a}_2\bar{b}_2A_{xyi}+\bar{b}_2\bar{c}_2A_{yzi}+\bar{a}_2\bar{c}_2A_{zxi})S^2-c_1(\bar{a}_3^2A_{xi}+\bar{b}_3^2A_{yi}+\bar{c}_3^2A_{zi}+\bar{a}_3\bar{b}_3A_{xyi}+\bar{b}_3\bar{c}_3A_{yzi}$$
$$+\bar{a}_3\bar{c}_3A_{zxi})T^2+[J_0(\bar{a}_2A_{zxi}+\bar{b}_2A_{yzi}+2\bar{c}_2A_{zi})-2c_1(\bar{a}_1\bar{a}_2A_{xi}+\bar{b}_1\bar{b}_2A_{yi}+\bar{c}_1\bar{c}_2A_{zi})-c_1(h_9A_{xyi}+h_7A_{yzi}$$
$$+h_8A_{zxi})]RS+[J_0(\bar{a}_3A_{zxi}+\bar{b}_3A_{yzi}+2\bar{c}_3A_{zi})-2c_1(\bar{a}_1\bar{a}_3A_{xi}+\bar{b}_1\bar{b}_3A_{yi}+\bar{c}_1\bar{c}_3A_{zi})-c_1(h_6A_{xyi}+h_4A_{yzi}$$
$$+h_5A_{zxi})]RT-c_1(2\bar{a}_2\bar{a}_3A_{xi}+2\bar{b}_2\bar{b}_3A_{yi}+2\bar{c}_2\bar{c}_3A_{zi}+h_3A_{xyi}+h_1A_{yzi}+h_2A_{zxi})ST\}$$

(A.20c)

(2) The 16th~18th sets of solutions for displacements (i=16~18)

$$U_i=\frac{1}{2J_0}\{-a_2(\bar{a}_1^2A_{xi}+\bar{b}_1^2A_{yi}+\bar{c}_1^2A_{zi}+\bar{a}_1\bar{b}_1A_{xyi}+\bar{b}_1\bar{c}_1A_{yzi}+\bar{a}_1\bar{c}_1A_{zxi})R^2+[\bar{a}_2J_0A_{xi}+(J_0-\bar{a}_2a_2)(\bar{a}_2A_{xi}$$
$$+\bar{b}_2A_{xyi}+\bar{c}_2A_{zxi})-a_2(\bar{b}_2^2A_{yi}+\bar{c}_2^2A_{zi}+\bar{b}_2\bar{c}_2A_{yzi})]S^2-a_2(\bar{a}_3^2A_{xi}+\bar{b}_3^2A_{yi}+\bar{c}_3^2A_{zi}+\bar{a}_3\bar{b}_3A_{xyi}+\bar{b}_3\bar{c}_3A_{yzi}$$
$$+\bar{a}_3\bar{c}_3A_{zxi})T^2+[J_0(2\bar{a}_1A_{xi}+\bar{b}_1A_{xyi}+\bar{c}_1A_{zxi})-2a_2(\bar{a}_1\bar{a}_2A_{xi}+\bar{b}_1\bar{b}_2A_{yi}+\bar{c}_1\bar{c}_2A_{zi})-a_2(h_9A_{xyi}+h_7A_{yzi}$$
$$+h_8A_{zxi})]RS-a_2(2\bar{a}_1\bar{a}_3A_{xi}+2\bar{b}_1\bar{b}_3A_{yi}+2\bar{c}_1\bar{c}_3A_{zi}+h_6A_{xyi}+h_4A_{yzi}+h_5A_{zxi})RT+[J_0(2\bar{a}_3A_{xi}+\bar{b}_3A_{xyi}$$
$$+\bar{c}_3A_{zxi})-2a_2(\bar{a}_2\bar{a}_3A_{xi}+\bar{b}_2\bar{b}_3A_{yi}+\bar{c}_2\bar{c}_3A_{zi})-a_2(h_3A_{xyi}+h_1A_{yzi}+h_2A_{zxi})]ST\}$$

(A.21a)

$$
\begin{aligned}
V_i = \frac{1}{2J_0}\{ & -b_2(\bar{a}_1^2 A_{xi} + \bar{b}_1^2 A_{yi} + \bar{c}_1^2 A_{zi} + \bar{a}_1\bar{b}_1 A_{xyi} + \bar{b}_1\bar{c}_1 A_{yzi} + \bar{a}_1\bar{c}_1 A_{zxi})R^2 + [\bar{b}_2 J_0 A_{yi} + (J_0 - \bar{b}_2 b_2)(\bar{a}_2 A_{xyi} \\
& + \bar{b}_2 A_{yi} + \bar{c}_2 A_{yzi}) - b_2(\bar{a}_2^2 A_{xi} + \bar{c}_2^2 A_{zi} + \bar{a}_2\bar{c}_2 A_{zxi})]S^2 - b_2(\bar{a}_3^2 A_{xi} + \bar{b}_3^2 A_{yi} + \bar{c}_3^2 A_{zi} + \bar{a}_3\bar{b}_3 A_{xyi} + \bar{b}_3\bar{c}_3 A_{yzi} \\
& + \bar{a}_3\bar{c}_3 A_{zxi})T^2 + [J_0(\bar{a}_1 A_{xyi} + 2\bar{b}_1 A_{yi} + \bar{c}_1 A_{yzi}) - 2b_2(\bar{a}_1\bar{a}_2 A_{xi} + \bar{b}_1\bar{b}_2 A_{yi} + \bar{c}_1\bar{c}_2 A_{zi}) - b_2(h_9 A_{xyi} + h_7 A_{yzi} \\
& + h_8 A_{zxi})]RS - b_2(2\bar{a}_1\bar{a}_3 A_{xi} + 2\bar{b}_1\bar{b}_3 A_{yi} + 2\bar{c}_1\bar{c}_3 A_{zi} + h_6 A_{xyi} + h_4 A_{yzi} + h_5 A_{zxi})RT + [J_0(\bar{a}_3 A_{xyi} + 2\bar{b}_3 A_{yi} \\
& + \bar{c}_3 A_{yzi}) - 2b_2(\bar{a}_2\bar{a}_3 A_{xi} + \bar{b}_2\bar{b}_3 A_{yi} + \bar{c}_2\bar{c}_3 A_{zi}) - b_2(h_3 A_{xyi} + h_1 A_{yzi} + h_2 A_{zxi})]ST\}
\end{aligned}
\tag{A.21b}
$$

$$
\begin{aligned}
W_i = \frac{1}{2J_0}\{ & -c_2(\bar{a}_1^2 A_{xi} + \bar{b}_1^2 A_{yi} + \bar{c}_1^2 A_{zi} + \bar{a}_1\bar{b}_1 A_{xyi} + \bar{b}_1\bar{c}_1 A_{yzi} + \bar{a}_1\bar{c}_1 A_{zxi})R^2 + [\bar{c}_2 J_0 A_{zi} + (J_0 - \bar{c}_2 c_2)(\bar{a}_2 A_{zxi} \\
& + \bar{b}_2 A_{yzi} + \bar{c}_2 A_{zi}) - c_2(\bar{a}_2^2 A_{xi} + \bar{b}_2^2 A_{yi} + \bar{a}_2\bar{b}_2 A_{xyi})]S^2 - c_2(\bar{a}_3^2 A_{xi} + \bar{b}_3^2 A_{yi} + \bar{c}_3^2 A_{zi} + \bar{a}_3\bar{b}_3 A_{xyi} + \bar{b}_3\bar{c}_3 A_{yzi} \\
& + \bar{a}_3\bar{c}_3 A_{zxi})T^2 + [J_0(\bar{a}_1 A_{zxi} + \bar{b}_1 A_{yzi} + 2\bar{c}_1 A_{zi}) - 2c_2(\bar{a}_1\bar{a}_2 A_{xi} + \bar{b}_1\bar{b}_2 A_{yi} + \bar{c}_1\bar{c}_2 A_{zi}) - c_2(h_9 A_{xyi} + h_7 A_{yzi} \\
& + h_8 A_{zxi})]RS - c_2(2\bar{a}_1\bar{a}_3 A_{xi} + 2\bar{b}_1\bar{b}_3 A_{yi} + 2\bar{c}_1\bar{c}_3 A_{zi} + h_6 A_{xyi} + h_4 A_{yzi} + h_5 A_{zxi})RT + [J_0(\bar{a}_3 A_{zxi} + \bar{b}_3 A_{yzi} \\
& + 2\bar{c}_3 A_{zi}) - 2c_2(\bar{a}_2\bar{a}_3 A_{xi} + \bar{b}_2\bar{b}_3 A_{yi} + \bar{c}_2\bar{c}_3 A_{zi}) - c_2(h_3 A_{xyi} + h_1 A_{yzi} + h_2 A_{zxi})]ST\}
\end{aligned}
\tag{A.21c}
$$

(3) The 19th~21st sets of solutions for displacements (*i*=19~21)

$$
\begin{aligned}
U_i = \frac{1}{2J_0}\{ & -a_3(\bar{a}_1^2 A_{xi} + \bar{b}_1^2 A_{yi} + \bar{c}_1^2 A_{zi} + \bar{a}_1\bar{b}_1 A_{xyi} + \bar{b}_1\bar{c}_1 A_{yzi} + \bar{a}_1\bar{c}_1 A_{zxi})R^2 - a_3(\bar{a}_2^2 A_{xi} + \bar{b}_2^2 A_{yi} + \bar{c}_2^2 A_{zi} \\
& + \bar{a}_2\bar{b}_2 A_{xyi} + \bar{b}_2\bar{c}_2 A_{yzi} + \bar{a}_2\bar{c}_2 A_{zxi})S^2 + [\bar{a}_3 J_0 A_{xi} + (J_0 - \bar{a}_3 a_3)(\bar{a}_3 A_{xi} + \bar{b}_3 A_{xyi} + \bar{c}_3 A_{zxi}) - a_3(\bar{b}_3^2 A_{yi} \\
& + \bar{c}_3^2 A_{zi} + \bar{b}_3\bar{c}_3 A_{yzi})]T^2 - a_3(2\bar{a}_1\bar{a}_2 A_{xi} + 2\bar{b}_1\bar{b}_2 A_{yi} + 2\bar{c}_1\bar{c}_2 A_{zi} + h_9 A_{xyi} + h_7 A_{yzi} + h_8 A_{zxi})RS \\
& + [J_0(2\bar{a}_1 A_{xi} + \bar{b}_1 A_{xyi} + \bar{c}_1 A_{zxi}) - 2a_3(\bar{a}_1\bar{a}_3 A_{xi} + \bar{b}_1\bar{b}_3 A_{yi} + \bar{c}_1\bar{c}_3 A_{zi}) - a_3(h_6 A_{xyi} + h_4 A_{yzi} + h_5 A_{zxi})]RT \\
& + [J_0(2\bar{a}_2 A_{xi} + \bar{b}_2 A_{xyi} + \bar{c}_2 A_{zxi}) - 2a_3(\bar{a}_2\bar{a}_3 A_{xi} + \bar{b}_2\bar{b}_3 A_{yi} + \bar{c}_2\bar{c}_3 A_{zi}) - a_3(h_3 A_{xyi} + h_1 A_{yzi} + h_2 A_{zxi})]ST\}
\end{aligned}
\tag{A.22a}
$$

$$
\begin{aligned}
V_i = \frac{1}{2J_0}\{ & -b_3(\bar{a}_1^2 A_{xi} + \bar{b}_1^2 A_{yi} + \bar{c}_1^2 A_{zi} + \bar{a}_1\bar{b}_1 A_{xyi} + \bar{b}_1\bar{c}_1 A_{yzi} + \bar{a}_1\bar{c}_1 A_{zxi})R^2 - b_3(\bar{a}_2^2 A_{xi} + \bar{b}_2^2 A_{yi} + \bar{c}_2^2 A_{zi} \\
& + \bar{a}_2\bar{b}_2 A_{xyi} + \bar{b}_2\bar{c}_2 A_{yzi} + \bar{a}_2\bar{c}_2 A_{zxi})S^2 + [\bar{b}_3 J_0 A_{yi} + (J_0 - \bar{b}_3 b_3)(\bar{a}_3 A_{xyi} + \bar{b}_3 A_{yi} + \bar{c}_3 A_{yzi}) - b_3(\bar{a}_3^2 A_{xi} \\
& + \bar{c}_3^2 A_{zi} + \bar{a}_3\bar{c}_3 A_{zxi})]T^2 - b_3(2\bar{a}_1\bar{a}_2 A_{xi} + 2\bar{b}_1\bar{b}_2 A_{yi} + 2\bar{c}_1\bar{c}_2 A_{zi} + h_9 A_{xyi} + h_7 A_{yzi} + h_8 A_{zxi})RS \\
& + [J_0(\bar{a}_1 A_{xyi} + 2\bar{b}_1 A_{yi} + \bar{c}_1 A_{yzi}) - 2b_3(\bar{a}_1\bar{a}_3 A_{xi} + \bar{b}_1\bar{b}_3 A_{yi} + \bar{c}_1\bar{c}_3 A_{zi}) - b_3(h_6 A_{xyi} + h_4 A_{yzi} + h_5 A_{zxi})]RT \\
& + [J_0(\bar{a}_2 A_{xyi} + 2\bar{b}_2 A_{yi} + \bar{c}_2 A_{yzi}) - 2b_3(\bar{a}_2\bar{a}_3 A_{xi} + \bar{b}_2\bar{b}_3 A_{yi} + \bar{c}_2\bar{c}_3 A_{zi}) - b_3(h_3 A_{xyi} + h_1 A_{yzi} + h_2 A_{zxi})]ST\}
\end{aligned}
\tag{A.22b}
$$

$$
\begin{aligned}
W_i = \frac{1}{2J_0}\{ & -c_3(\bar{a}_1^2 A_{xi} + \bar{b}_1^2 A_{yi} + \bar{c}_1^2 A_{zi} + \bar{a}_1\bar{b}_1 A_{xyi} + \bar{b}_1\bar{c}_1 A_{yzi} + \bar{a}_1\bar{c}_1 A_{zxi})R^2 - c_3(\bar{a}_2^2 A_{xi} + \bar{b}_2^2 A_{yi} + \bar{c}_2^2 A_{zi} \\
& + \bar{a}_2\bar{b}_2 A_{xyi} + \bar{b}_2\bar{c}_2 A_{yzi} + \bar{a}_2\bar{c}_2 A_{zxi})S^2 + [\bar{c}_3 J_0 A_{zi} + (J_0 - \bar{c}_3 c_3)(\bar{a}_3 A_{zxi} + \bar{b}_3 A_{yzi} + \bar{c}_3 A_{zi}) - c_3(\bar{a}_3^2 A_{xi} \\
& + \bar{b}_3^2 A_{yi} + \bar{a}_3\bar{b}_3 A_{xyi})]T^2 - c_3(2\bar{a}_1\bar{a}_2 A_{xi} + 2\bar{b}_1\bar{b}_2 A_{yi} + 2\bar{c}_1\bar{c}_2 A_{zi} + h_9 A_{xyi} + h_7 A_{yzi} + h_8 A_{zxi})RS \\
& + [J_0(\bar{a}_1 A_{zxi} + \bar{b}_1 A_{yzi} + 2\bar{c}_1 A_{zi}) - 2c_3(\bar{a}_1\bar{a}_3 A_{xi} + \bar{b}_1\bar{b}_3 A_{yi} + \bar{c}_1\bar{c}_3 A_{zi}) - c_3(h_6 A_{xyi} + h_4 A_{yzi} + h_5 A_{zxi})]RT \\
& + [J_0(\bar{a}_2 A_{zxi} + \bar{b}_2 A_{yzi} + 2\bar{c}_2 A_{zi}) - 2c_3(\bar{a}_2\bar{a}_3 A_{xi} + \bar{b}_2\bar{b}_3 A_{yi} + \bar{c}_2\bar{c}_3 A_{zi}) - c_3(h_3 A_{xyi} + h_1 A_{yzi} + h_2 A_{zxi})]ST\}
\end{aligned}
\tag{A.22c}
$$

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