Paper:
Realdon, M. (2018). Discounting earnings with stochastic discount rates. *The European Journal of Finance*, 1-27.
http://dx.doi.org/10.1080/1351847X.2018.1548368

DISCOUNTING EARNINGS WITH STOCHASTIC DISCOUNT RATES

Dr Marco Realdon
Swansea School of Management
Swansea University, Bay Campus, Fabian Way, Swansea, UK
e-mail: marco.realdon@gmail.com; tel.: 0044/07951893423.

28/12/2014 (this version 25/10/2018)

Abstract

This paper presents new equity valuation formulae in closed form that extend the abnormal earnings growth (AEG) valuation of Ohlson and Juettner-Nauroth (2005) to the cases of time-varying or stochastic cost of capital as in Ang and Liu (2004) or **to cases of** stochastic interest rates as in Ang and Liu (2001). Interest rates are modeled by quadratic term structure models, which are not hindered by restrictions to factors correlation or by other shortcomings of affine term structure models in discounting long term earnings. This is crucial since valuation can be very sensitive to the correlation between the factors driving earnings and interest rates. Positive correlation reduces price-earnings ratios according to US data. Valuation is also sensitive to the "volatility" of abnormal earnings growth. The residual earnings risk-neutral valuation of Ang and Liu (2001) is adapted to quadratic term structure models.

Key words: abnormal earnings growth valuation, discounted dividends valuation, risk-neutral valuation, discrete time quadratic term structure models.

JEL classification: G12.

1 Introduction and literature review

Much of industry practice bases equity valuation on discounting earnings and also the accounting academic literature on valuation concentrates on discounting earnings, as for example in Ohlson and Juettner-Nauroth (2005), Ohlson (2005), Ohlson and Gao (2006). These accounting based valuations have **several** merits. **These valuations focus** on earnings, which analysts forecast. **These valuations are independent** of dividend policies despite being consistent with discounted dividends valuations**, and are also independent** of accounting policies despite being based on accounting numbers (so called "value conservation principle"). **These accounting based valuations are also applicable**

on a per-share basis **and require** minimal assumptions, **in particular they do not assume** the clean surplus relation or knowledge of equity book value, **as explained by Ohlson (2005)**. However these accounting based valuations make the restrictive assumption of using one single discount rate for all future earnings, as if the discount rate were constant over time. A notable exception is Ohlson and Gode (2004), who provide a valuation model that discounts earnings under stochastic interest rates while requiring no assumption about the dynamics of interest rates, but their model relies on a specific assumption of dividend irrelevance.

A more finance-oriented literature provides tractable equity valuations under realistic stochastic discount rates, as in Ang and Liu (2004) or in Hughes, Liu and Liu (2009), who show how valuation should reflect the term structure of expected stock returns. However these valuations discount dividends or other cash flows, not earnings, and do not share important merits of accounting based valuations. One notable exception is Ang and Liu (2001) who discount residual earnings.

Pure earnings-based valuations under stochastic discount rates seem to be missing in the literature. When discount rates change over time or are stochastic, as is the case in reality, we do not yet have a valuation with the merits of the abnormal earnings growth (AEG) valuation of Ohlson and Juettner-Nauroth (2005) and of Ohlson (2005). This paper addresses this issue by providing valuation formulae that discount a type of abnormal earnings growth under time-varying or stochastic discount rates. These formulae seem to be practical valuation tools and to provide insight.

The valuations, unlike AEG valuation, are not constrained to a single discount rate for earnings of all future maturities and can be "reverse engineered" for estimating the implied equity-risk premium even when the term structure of interest rates is not flat, as is typically the case. The valuations, unlike AEG valuation, need not capitalise infinite streams of earnings and are not constrained **by** specific assumptions about continuation value at the end of the forecast horizon. The valuations require an adjustment to the definition of abnormal earnings growth to account for changes in the discount rate over time. The valuation formulae are in closed form even under the realistic assumptions that risk premia are stochastic and that quadratic term structure models describe interest rates. Risk-neutral valuation reveals that price-earnings ratios can be very sensitive to the "volatility" of, and correlation between, the factors driving interest rates and earnings, despite the fact that equity value is a linear function of the factors driving earnings. The valuation effect of growth in AEG interacts with the valuation effect of the correlation between interest rates and earnings.

Also Ang and Liu (2001) provide equity risk-neutral valuation formulae under stochastic interest rates and show that the interest rate model can materially impact equity valuation, but they discount residual earnings and assume a discrete time "square root" affine term structure model similar to Sun (1992). **Instead** this paper focuses on discounting abnormal earnings growth **and uses** quadratic term structure models. While the "square root" affine models in Ang and Liu (2001) require severe restrictions to the **correlation** between factors driving interest rates and earnings, quadratic models require no such restriction. This seems crucial, **since the evidence in this paper shows that** US interest rates imply that price-earnings ratios can materially decrease as the correlation between earnings and interest rate factors increases. Moreover earnings and interest rates are both linked to the macroeconomic cycle and the earnings of firms such as banks are directly linked to interest rates. Assuming that earnings be independent of interest rates seems too restrictive.

Quadratic term models can not only model the said correlation, but also rule out negative interest rates, unlike affine Gaussian models. While affine Gaussian models too require no restriction to the correlation between earnings and interest rates, they predict too high chances of negative yields, which is an issue especially when discounting long term earnings and interest rates are low. For these reasons, and also because of their tractability, quadratic models seem preferable for equity valuation. **This** paper also adapts the residual earnings risk-neutral valuation of Ang and Liu (2001) to quadratic term structure models.

This paper is close in spirit to Bakshi and Chen (2005). They propose equity risk-neutral valuations in continuous time, which is not a natural setting for accounting variables, whose observations are well spaced in time. Instead the valuations of this paper are in discrete time. Bakshi and Chen assume a Vasicek interest rate term structure model and find that "modeling the discounting dynamics properly also makes a significant difference". This paper builds on quadratic term structure models, which seem preferable to Vasicek-like models. Bakshi and Chen (2005) assume that earnings and dividends are strictly proportional at all times. The valuations in this paper make no such assumption as they are independent of dividend policy, much like **the** AEG valuation **of** Ohlson and Juettner-Nauroth (2005).

An Appendix also shows that the valuation model in this paper is in closed form even when we assume the stochastic equity risk premium of Ang and Liu (2004).

2 Discounted earnings valuation under time-varying discount rates

We employ the following notation: V_t^e is the intrinsic value of equity at time t; we assume that all intrinsic values coincide with market prices; $r_{e,t}$ is the equity cost of capital during the period $[t, t+1]$; d_{t+1} are the net dividends paid during $[t, t+1]$; even if net dividends are paid during the period $[t, t+1]$, we simply assume they are paid at $t+1$; net dividends are distributions to equity holders minus new capital contributions made by equity holders; x_{t+1} denotes comprehensive net earnings produced during $[t, t+1]$. We can re-write dividend discount valuation through some algebraic manipulations as

$$
\begin{aligned}
V_0^e &= \frac{V_1^e + d_1}{1 + r_{e,0}} = \frac{x_1 + V_1^e - x_1 + d_1}{1 + r_{e,0}} = \frac{x_1 - (x_1 - d_1)}{1 + r_{e,0}} + \frac{x_2 + V_2^e - (x_2 - d_2)}{(1 + r_{e,0})(1 + r_{e,1})} \\
&= \frac{x_1}{1 + r_{e,0}} + \frac{x_2 - (x_1 - d_1) \cdot r_{e,1} + V_2^e - \sum_{i=1}^{2} (x_i - d_i)}{(1 + r_{e,0})(1 + r_{e,1})} \\
&= \frac{x_1}{1 + r_{e,0}} + \frac{x_2 - (x_1 - d_1) \cdot r_{e,1}}{(1 + r_{e,0})(1 + r_{e,1})} + \frac{x_3 - r_{e,2} \cdot \sum_{i=1}^{2} (x_i - d_i) + V_3^e - \sum_{i=1}^{3} (x_i - d_i)}{(1 + r_{e,0})(1 + r_{e,1})(1 + r_{e,2})}
\end{aligned}
$$

an so forth. Inserting the time 0 conditional expectation operator and repeating the algebraic manipulations up to the end of period T, we obtain the following proposition.

Proposition 1 *When the discount rate varies over time, the discounted dividends valuation of equity can re-written as the following discounted earnings valuation*

$$
\begin{aligned}
V_0^e &= E_0 \left[\sum_{t=1}^{T} \frac{x_t}{\prod_{i=0}^{t-1} (1 + r_{e,i})} - \sum_{t=2}^{T} \frac{r_{e,t-1} \cdot \sum_{i=1}^{t-1} (x_i - d_i)}{\prod_{i=0}^{t-1} (1 + r_{e,i})} + \frac{V_T^e - \sum_{i=1}^{T} (x_i - d_i)}{\prod_{i=0}^{T-1} (1 + r_{e,i})} \right] \quad (1) \\
&= E_0 \left[\sum_{t=1}^{\infty} \frac{x_t}{\prod_{i=0}^{t-1} (1 + r_{e,i})} - \sum_{t=2}^{\infty} \frac{r_{e,t-1} \cdot \sum_{i=1}^{t-1} (x_i - d_i)}{\prod_{i=0}^{t-1} (1 + r_{e,i})} \right].
\end{aligned}
$$

The first line of equation 1 assumes a forecast horizon of T years, while the second line assumes an infinite forecast horizon. Equation 1 makes no assumption other than discounted dividends valuation. However the formula relies on forecasting earnings rather than dividends, which is helpful since forecasting earnings seems more natural than forecasting dividends. Equation 1 is also valid on a per-share basis as it assumes no clean surplus relation, which may not hold on a per share basis as Juettner-Nauroth and Ohlson (2005) point out. By clean surplus relation we mean the accounting identity

$$b_t + x_{t+1} - d_{t+1} = b_{t+1}$$

where b_t and b_{t+1} are the book values of equity at times t and $t+1$. The clean surplus relation may not hold on a per share basis for example when the issuer buys back some of its shares and only some of the shareholders sell their shares to the issuer.

Equation 1 abides by the value conservation principle, according to which the valuation is unaffected by accounting policies that do not alter (the amount or timing of) cash flows. Discounted dividends valuation is unaffected by accounting policies and equally unaffected by the accounting must be all valuations that are equivalent to discounted dividends valuation, even though such valuations may be based on accounting data.

The term $V_T^e - \sum_{i=1}^{T} (x_i - d_i)$ in equation 1 is "continuation value" at T, i.e. the part of equity value V_0^e that is due to the firm's continuation as a going concern even after T. Thus continuation value at time T is V_T^e minus the cumulated retained earnings of periods 1 to T. No forecast beyond time T is needed when $V_T^e = \sum_{i=1}^{T}(x_i - d_i)$. **To** compute V_T^e we can employ any valuation model that is consistent with discounted dividends valuation. This freedom in computing continuation value is absent in the abnormal earnings growth (AEG) valuation of Ohlson and Juettner-Nauroth (2005) and of Ohlson (2005), which discounts infinite streams of earnings or abnormal earnings increments at the constant cost of capital r_e. Freedom in computing continuation value seems welcome because Jorgensen, Lee and Yoo (2011) found that the assumptions about earnings growth after the forecast horizon hamper the accuracy of AEG valuation.

If $V_T^e = b_T$ no forecasting beyond time T is needed. If, moreover, the clean surplus relation holds so that $b_T = \sum_{i=1}^{T}(x_i - d_i) + b_0$, it follows that $V_T^e - \sum_{i=1}^{T}(x_i - d_i) = b_0$, which means that time T continuation value equals b_0; in this case continuation value requires no forecast, because b_0 can be observed at the time of the valuation. However this result only holds on a per share basis if the clean surplus relation holds on a per share basis.

Appendix A.1 shows that the AEG valuation of Ohlson and Juettner-Nauroth (2005) and Ohlson (2005) is a special case of equation 1 when $r_{e,t} = r_e$ for all t, since

$$\begin{aligned} V_0^e &= \sum_{t=1}^{\infty} \frac{x_1 + (x_t - x_1)}{(1+r_e)^t} - \sum_{t=2}^{\infty} \frac{r_e \cdot \sum_{i=1}^{t-1}(x_i - d_i)}{(1+r_e)^t} \\ &= \frac{x_1}{r_e} + \frac{1}{r_e}\sum_{t=1}^{\infty} \frac{(x_{t+1} - x_t - r_e \cdot (x_t - d_t))}{(1+r_e)^t}. \end{aligned}$$

The first line is equation 1, while the second line is AEG valuation. AEG valuation assumes an infinite forecast horizon, in that it capitalises streams of earnings that stretch into the infinite future at a constant discount rate. **On the other hand** Hughes, Liu and Liu (2009) highlight that expected returns are time varying, stochastic, depend on leverage and are correlated with the firm's cash flows. **Therefore a constant discount rate seems too restrictive and may at times lead to gross valuation errors, as shown for example in Ang and Liu (2004).** Ang and Liu (2004) conclude that "practical valuation" should use "an analytic term structure of discount rates, with different discount rates applied to expected cash flows at different horizons".

Equation 1 can also be used for reverse engineering the equity risk premium when the default-free term structure of interest rates is not flat.

2.1 Implied cost of equity capital and implied equity risk premium

A vast literature has attempted to determine the implied cost of equity capital form observed stock prices. The cost of capital is the sum of a Government bond yield, one for each maturity, and a risk premium. Implying the cost of equity capital requires an equity valuation model that discounts earnings or cash flows over future decades, while the term structure of Government bond yields can be far from flat. Then assuming, as is often done, that the cost of equity capital implied from stock prices is the same for all future earnings seems questionable. For example Ang and Liu (2004, page 2775) conclude that "investors should be most concerned with the impact of time-varying interest rates and risk premiums for discounting cash flows. At long horizons, the time variation of risk-free interest rates or beta is more important". Therefore a first step in the direction of the results of Ang and Liu is to assume that only the equity risk premium be constant over time, rather than the entire cost of equity capital. Under this less restrictive assumption, equation 1 can be used to determine the implied equity risk premium, even while the term structure of Government bond yields is not flat. For example, let $G_{0,t}$ denote the time 0 price of a default-free Government discount bond that pays 1 at time t such that

$$G_{0,t} = \prod_{i=0}^{t-1} \frac{1}{1 + r_{g,i}}.$$

$r_{g,i}$ is the forward interest rate over the period $[i, i+1]$. Given $G_{0,t}$ for all t, we can "bootstrap" $r_{g,i}$ for $i = 0, 1, .., t-1$, and in equation 1 we can substitute

$$r_{e,i} = p + r_{g,i}$$

where p is the equity risk premium assumed constant over time. **Then, once we observe the stock price in the market, we can reverse engineer equation 1 to determine** p.

2.2 Abnormal earnings growth (AEG) when the cost of capital is not constant over time

The definition of AEG is different when the cost of equity capital is not constant over time. **When** the cost of capital is constant

$$z_{t+1} = x_{t+1} - x_t - r_e \cdot (x_t - d_t)$$

where z_{t+1} denotes abnormal earnings growth over the period $[t, t+1]$ and $r_e \cdot (x_t - d_t)$ denotes the change in "required" earnings for the same period $[t, t+1]$ to remunerate the change in equity $(x_t - d_t)$ of the previous period $[t-1, t]$. However when the cost of capital is not constant over time, the change in "required" earnings for the period $[t, t+1]$ becomes

$$r_{e,t} \cdot (x_t - d_t) + (r_{e,t} - r_{e,t-1}) \cdot \sum_{i=0}^{t-1} (x_i - d_i) \cdot 1_{t \geq 1}$$

where again $t = 0$ is the time of the valuation. $1_{t \geq 1}$ is the indicator function of the condition $t \geq 1$. The change in required earnings for the period $[0, 1]$ is $r_{e,0} \cdot (x_0 - d_0)$. The term $(r_{e,t} - r_{e,t-1}) \cdot \sum_{i=0}^{t-1} (x_i - d_i)$ is the change in required earnings that is due to the change in the cost of equity capital from $r_{e,t-1}$ to

$r_{e,t}$. **Such change in the cost of capital** is multiplied by the total change in equity over the period $[0, t-1]$ as measured by $\sum_{i=0}^{t-1} (x_i - d_i)$. Therefore when the cost of capital varies over time, AEG over the period $[t, t+1]$ can be defined as

$$z_{t+1} = x_{t+1} - x_t - \left(r_{e,t} \cdot (x_t - d_t) + (r_{e,t} - r_{e,t-1}) \cdot \sum_{i=0}^{t-1} (x_i - d_i) \cdot 1_{t \geq 1} \right). \quad (2)$$

Here AEG is defined as the difference between the actual change in earnings during $[t, t+1]$ and the change in required earnings for the same period $[t, t+1]$. We notice that $z_1 = x_1 - x_0 - r_{e,0} \cdot (x_0 - d_0)$. Then Appendix A.2 shows that equation 1 can be re-written as

$$V_0^e = E_0 \left[\sum_{t=1}^{T} \frac{v_t}{\prod_{i=0}^{t-1} (1 + r_{e,i})} \right]$$
$$v_t = v_{t-1} + z_t \text{ with } v_0 = x_0.$$

Later we refer to this formula. Appendix A.3 shows that

$$re_t = x_t - r_{e,t-1} \cdot b_{t-1} = v_t - r_{e,t-1} \cdot b_{-1} \quad (3)$$

where re_t are residual earnings produced over the period $[t-1, t]$ and b_{-1} is the book value of equity at time $t = -1$, **which is** one period before the valuation date $t = 0$. It follows that

$$re_t - re_{t-1} = z_t - (r_{e,t-1} - r_{e,t-2}) \cdot b_{-1}.$$

Therefore abnormal earnings growth z_t, defined in equation 2 as the difference between the actual change in earnings and the change in required earnings for the same period, differs from the change in residual earnings $re_t - re_{t-1}$ whenever $r_{e,t-1} \neq r_{e,t-2}$. We recall that instead, when the cost of capital is constant over time, abnormal earnings growth coincides with the change in residual earnings. **Appendix A.4 presents a firm valuation model similar to the equity valuation so far presented.**

3 Discounting earnings under risk-neutral valuation (RNV) and stochastic interest rates

So far we have relied on "classic" valuation under the real probability measure. Hereafter we reconsider the above valuation **model** under risk-neutral valuation (RNV)**. RNV** discounts expected payoffs under the risk-neutral probability measure at the default-free **short** interest rate. RNV and "classic" valuation under the real measure are equivalent. We switch to RNV to focus on the link between equity valuation and the term structure of interest rates, and also because the default-free **short** interest rate is independent of the firm's financial leverage, a welcome simplification. Instead in valuations under the real probability the discount rate is the cost of equity capital, which depends on leverage, as for example in the models of Hughes, Liu and Liu (2009) or **of** Jennergren and Skogsvik (2011) or in the **model** presented above.

Let r_t **be** the continuously compounded **short** interest rate at time t for the period $[t, t+1]$, i.e. $r_t = -\ln P_{t,1}$, where 1 is the length of one time period and $P_{t,1}$ is the time t price of a default-free discount bond that matures at time $t+1$. RNV applied to discounted dividends valuation implies that

$$V_t^e = E_t^{\mathbb{Q}}\left[\exp\left(-r_t\right)\cdot\left(V_{t+1}^e + d_{t+1}\right)\right]. \tag{4}$$

$E_t^{\mathbb{Q}}[..]$ denotes time t conditional expectation under the risk-neutral measure $\mathbb{Q}$. Then we can re-write **the valuation of** Proposition 1 under RNV and continuously compounded interest rates as

$$V_0^e = E_0^{\mathbb{Q}}\left[\sum_{t=1}^{T}\frac{x_t}{e^{\sum_{i=0}^{t-1} r_i}} - \sum_{t=2}^{T}\frac{\left(e^{r_{t-1}}-1\right)\cdot\sum_{i=1}^{t-1}\left(x_i - d_i\right)}{e^{\sum_{i=0}^{t-1} r_i}} + \frac{V_T^e - \sum_{i=1}^{T}\left(x_i - d_i\right)}{e^{\sum_{i=0}^{T-1} r_i}}\right]. \tag{5}$$

Equation 5 can accommodate a non-flat **term** structure of interest rates. If the **future short** interest rate is a deterministic function of time, effectively equal to the **current** set of forward rates, RNV reduces to discounting expected payoffs using today's **term** structure of interest rates.

With a slight abuse of notation, hereafter we redefine z_{t+1}, i.e. the abnormal earnings growth produced over the period $[t, t+1]$, under RNV and continuous discounting as

$$z_{t+1} = x_{t+1} - x_t - \left(\exp(r_t) - 1\right)\cdot\left(x_t - d_t\right) - \left(\exp(r_t) - \exp(r_{t-1})\right)\cdot\sum_{i=0}^{t-1}\left(x_i - d_i\right)\cdot 1_{t\geq 1}. \tag{6}$$

Hereafter we assume that $V_T^e - \sum_{i=1}^{T}\left(x_i - d_i\right) = 0$ in equation 5**. It follows that**

$$\begin{aligned}
&V_0^e = E_0^{\mathbb{Q}}\left[\sum_{t=1}^{T}\frac{x_t}{e^{\sum_{i=0}^{t-1} r_i}} - \sum_{t=2}^{T}\frac{\left(e^{r_{t-1}}-1\right)\cdot\sum_{i=1}^{t-1}\left(x_i - d_i\right)}{e^{\sum_{i=0}^{t-1} r_i}}\right] = \sum_{t=1}^{T} V_{0,t}^v \\
&V_{0,t}^v = E_0^{\mathbb{Q}}\left[v_t \cdot e^{-\sum_{i=0}^{t-1} r_i \cdot 1_{t\geq 1}}\right] \\
&v_{t+1} = v_t + z_{t+1} \text{ with } v_0 = x_0.
\end{aligned} \tag{7}$$

We now introduce a parametric model to compute $V_{0,t}^v$ **in equation 7. As** in Jennergren and Skogsvik (2011) among others, we assume a stochastic process for abnormal earnings growth. The time of the valuation is $t = 0$. For $t \geq 0$ we assume the following Gaussian vector autoregressive process under the risk-neutral measure $\mathbb{Q}$

$$v_{t+1} = v_t + z_{t+1} \tag{8}$$

$$z_{t+1} = g\left(\mathbf{m}_z'\mathbf{h}_t\right) + (1-g)\,z_t + \sigma_z\xi_{z,t+1}, \qquad \xi_{z,t+1} \backsim \mathbb{N}(0,1) \tag{9}$$

$$\mathbf{h}_{t+1} = \mathbf{G}_h\mathbf{m}_h + \left(\mathbf{I}_n - \mathbf{G}_h\right)\mathbf{h}_t + \mathbf{\Sigma}_h\boldsymbol{\xi}_{h,t+1}, \qquad \boldsymbol{\xi}_{h,t+1} \backsim \mathbb{N}\left(\mathbf{0}_{n\times 1}, \underline{\mathbf{\Upsilon}}\right), \qquad \mathbf{\Sigma}_h = diag\left(\sigma_{h_1}, .., \sigma_{h_n}\right) \tag{10}$$

$$\underline{\mathbf{\Upsilon}} = \begin{bmatrix} 1 & .. & un \\ .. & .. & .. \\ un & .. & 1 \end{bmatrix}$$

where: g, σ_z are scalar parameters; **as z denotes (adjusted) abnormal earnings growth, σ_z is the volatility of abnormal earnings growth;** $\mathbf{m}_z$ and $\mathbf{m}_h$ are $n\times 1$ vectors of parameters; $\mathbf{\Sigma}_h$ and $\mathbf{G}_h$ are $n\times n$ matrixes of parameters; $diag\left(\sigma_{h_1}, .., \sigma_{h_n}\right)$ is the diagonal matrix with elements $\sigma_{h_1}, .., \sigma_{h_n}$ on the diagonal, **which are volatility parameters**; the modulus of all eigenvalues of $\left(\mathbf{I}_n - \mathbf{G}_h\right)$ should be less than 1 in order for the factors $\mathbf{h}_t$ to follow stationary processes; $\mathbf{h}_t = \left(h_{1,t}, .., h_{n,t}\right)'$ is a vector of n factors driving the conditional expectation of z_{t+1} and may be interpreted as (latent) factors that drive market expectations about future abnormal earnings growth and that therefore also drive stock price changes; $\mathbf{h}_t$ may also be observable macroeconomic variables, such as inflation or output gap, which may drive both the default-free term structure of interest rates as well as expectations of future abnormal earnings growth; $\xi_{z,t+1}$ is a scalar Gaussian shock under $\mathbb{Q}$ with mean 0 and variance 1, which is the meaning of the symbol $\mathbb{N}(0,1)$. $\xi_{h,t+1} = \left(\xi_{h_1,t+1}, .., \xi_{h_n,t+1}\right)'$ is an $n\times 1$ vector of Gaussian shocks under $\mathbb{Q}$ **with** mean $\mathbf{0}_{n\times 1}$ and with **covariance** matrix $\underline{\mathbf{\Upsilon}}$. **The diagonal elements of $\underline{\mathbf{\Upsilon}}$** are all equal to 1 and **the** off-diagonal elements are unspecified; "un" stands for "unspecified" and signifies that we need not specify those matrix elements. When $0 < g < 1$, z_{t+1} tends to revert to the level $\mathbf{m}_z'\mathbf{h}_t$, which is itself driven by the factors $\mathbf{h}_t$. g **determines the speed of such mean-reversion**.The factors $\mathbf{h}_t$ determine the expected long term level of abnormal earnings. The dynamics of earnings x_t implied by this parametric model are compatible with dividend policy irrelevance provided dividend distributions are zero-net-present-value transactions.

3.1 Valuation with quadratic models

Now the term structure of interest rates is stochastic and described by quadratic term structure models. Here we **first** introduce the one factor discrete time quadratic model and then extend it to multiple factors. According to the one factor quadratic **model**

$$r_t = y_t^2$$
$$y_{1,t+1} = \left(1-\Phi_y\right)y_{1,t} + \Phi_y\mu_y + \Sigma_y\xi_{y,t+1}, \qquad \xi_{y,t+1} \backsim \mathbb{N}(0,1)$$
$$P_{t,m} = E_t^{\mathbb{Q}}\left[e^{-\sum_{i=0}^{m-1} r_{t+i}}\right] = e^{A_m + B_m y_t + y_t C_m y_t}$$

where y is a scalar latent stochastic factor following an AR(1) Gaussian process under the risk-neutral measure $\mathbb{Q}$. Φ_y, μ_y, Σ_y are scalar parameters. The random shocks $\xi_{y,t+1}$ are serially independent and independent of other shocks in the model. $\mathbb{N}(0,1)$ denotes the Gaussian density with mean 0 and variance 1. $P_{t,m}$ is the time t price of a discount bond with face value 1 and maturity at **time** $t+m$. A_m, B_m, C_m are scalar functions of m, i.e. of number of time steps to the bond maturity, and satisfy Riccati difference equations presented below. The quadratic model has the merits that $r_t \geq 0$, **that it** has simple formulae for $P_{t,m}$, **it** has heteroschedastic bond yields and especially **it** requires no restriction to the correlation between the random shocks $\xi_{y,t+1}$ and the random shocks that **affect** a firm's earnings. Such correlation is crucial to valuation.

Now we turn to equity valuation when the term structure of interest rates is driven by multi-factor quadratic term structure models, which are much more realistic than **the one factor model**. **We then assume that under the risk-neutral measure** $\mathbb{Q}$

$$
\begin{aligned}
r_t &= \alpha + \boldsymbol{\beta}'\mathbf{y}_t + \mathbf{y}_t'\Psi\mathbf{y}_t && (11)\\
\mathbf{y}_{t+1} &= (\mathbf{I}_3 - \boldsymbol{\Phi}_y)\,\mathbf{y}_t + \boldsymbol{\Phi}_y\boldsymbol{\mu}_y + \boldsymbol{\Sigma}_y\boldsymbol{\xi}_{y,t+1}, \quad \boldsymbol{\xi}_{y,t+1} \backsim \mathbb{N}(\mathbf{0}_{3\times1}, \mathbf{I}_3) && (12)\\
\mathbf{y}_t &= (y_{1,t}, y_{2,t}, y_{3,t})' \\
\mathbf{x}_{t+1} &= (\mathbf{I}_N - \boldsymbol{\Phi})\,\mathbf{x}_t + \boldsymbol{\eta} + \boldsymbol{\Sigma}\boldsymbol{\xi}_{t+1}, \quad \boldsymbol{\xi}_{t+1} \backsim \mathbb{N}(\mathbf{0}_{N\times1}, \mathbf{I}_N) \\
\mathbf{x}_t &= (y_{1,t}, y_{2,t}, y_{3,t}, v_t, z_t, \mathbf{h}_t')' \\
V^v_{0,m} &= (d_m + \mathbf{D}_m'\mathbf{x}_0)\cdot e^{A_m + \mathbf{B}_m'\mathbf{y}_0 + \mathbf{y}_0'\mathbf{C}_m\mathbf{y}_0} && (13)
\end{aligned}
$$

$$
\boldsymbol{\eta} = \begin{pmatrix} \boldsymbol{\Phi}_y\boldsymbol{\mu}_y \\ 0 \\ 0 \\ \mathbf{G}_h\mathbf{m}_h \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} \mathbf{y}_0 \\ v_0 \\ z_0 \\ \mathbf{h}_0 \end{pmatrix}, \quad \boldsymbol{\xi}_{t+1} = \begin{pmatrix} \boldsymbol{\xi}_{y,t+1} \\ 0 \\ \xi_{z,t+1} \\ \boldsymbol{\xi}_{h,t+1} \end{pmatrix}, \quad \boldsymbol{\xi}_{y,t+1} = \begin{pmatrix} \xi_{1,t+1} \\ \xi_{2,t+1} \\ \xi_{3,t+1} \end{pmatrix},
$$

$$
\mathbf{D}_m = \begin{pmatrix} \mathbf{D}^y_m \\ D^v_m \\ D^z_m \\ \mathbf{D}^h_m \end{pmatrix}, \quad \boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\Phi}_y & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times n} \\ \mathbf{0}_{1\times3} & 0 & -(1-g) & -g\mathbf{m}_z' \\ \mathbf{0}_{1\times3} & 0 & g & -g\mathbf{m}_z' \\ \mathbf{0}_{n\times3} & \mathbf{0}_{n\times1} & \mathbf{0}_{n\times1} & \mathbf{G}_h \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_y & \mathbf{0}_{3\times(n+2)} \\ \mathbf{H} & \boldsymbol{\Upsilon} \end{bmatrix},
$$

$$
\mathbf{H} = E_t^{\mathbb{Q}}\left[\begin{pmatrix} 0 \\ \sigma_z\xi_{z,t+1} \\ \boldsymbol{\Sigma}_h\boldsymbol{\xi}_{h,t+1} \end{pmatrix}\cdot\boldsymbol{\xi}'_{y,t+1}\right] = \begin{bmatrix} 0 & 0 & 0 \\ \sigma_z\rho_{z,1} & \sigma_z\rho_{z,2} & \sigma_z\rho_{z,3} \\ \sigma_{h_1}\rho_{h_1,1} & \sigma_{h_1}\rho_{h_1,2} & \sigma_{h_1}\rho_{h_1,3} \\ .. & .. & .. \\ \sigma_{h_n}\rho_{h_n,1} & \sigma_{h_n}\rho_{h_n,2} & \sigma_{h_n}\rho_{h_n,3} \end{bmatrix},
$$

$$\mathbf{\Upsilon} = \left[\begin{array}{ccccc} 0 & 0 & 0 & .. & 0 \\ 0 & \sigma_z \sqrt{1 - \sum_{i=1}^{3} \rho_{z,i}^2} & un & .. & un \\ 0 & un & \sigma_{h_1} \sqrt{1 - \sum_{i=1}^{3} \rho_{h_1,i}^2} & .. & un \\ .. & .. & .. & .. & .. \\ 0 & un & un & .. & \sigma_{h_n} \sqrt{1 - \sum_{i=1}^{3} \rho_{h_n,i}^2} \end{array} \right].$$

Again "un" stands for "unspecified" and signifies that we need not specify those matrix elements, $\mathbf{I}_N$ is the $N \times N$ identity matrix, $\mathbf{\Upsilon}$ is an $(n+2) \times (n+2)$ matrix, $\mathbf{H}$ is an $(n+2) \times 3$ matrix, D_m^v and D_m^z are scalars, $\mathbf{D}_m^h$ is an $n \times 1$ vector **and**

$$\alpha = 0, \quad \boldsymbol{\beta} = \mathbf{0}_{3\times1}, \quad \mathbf{\Psi} = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right].$$

$\mathbf{x}$ collects all stochastic factors in the model and ξ collects all Gaussian random shocks in the model under the risk-neutral measure $\mathbb{Q}$. All shocks are serially and mutually independent. $\mathbf{y}$ are latent factors driving interest rates and follow **Gaussian** vector autoregressive processes with shocks $\boldsymbol{\xi}_y$. **Under the risk-neutral measure $\mathbb{Q}$ the shocks $\boldsymbol{\xi}_{t+1}$ and $\boldsymbol{\xi}_{y,t+1}$ are distributed according to Gaussian densities with zero expected values and covariance matrix equal to the identity matrix.** $\mathbf{0}_{N\times1}$ **denotes an** $N \times 1$ **vector whose elements are all equal to** 0. $\mathbf{I}_N$ **is the** $N \times N$ **identity matrix**. $\mathbf{D}_m$, $\mathbf{B}_m$, $\mathbf{C}_m$, A_m, d_m **are** functions of m that **are determined later and that drive equity value through equation** 13. Without loss in generality $\mathbf{C}_m$ is symmetric. $\mathbf{x}_t$, ξ_{t+1}, $\mathbf{D}_m$, $\mathbf{B}_m$ are $N \times 1$ vectors with $N = n + 5$; $\mathbf{y}_t$, $\boldsymbol{\beta}$, $\boldsymbol{\eta}$, $\boldsymbol{\xi}_{y,t+1}$ are 3×1 vectors; $\mathbf{\Psi}$, $\mathbf{C}_m$, $\mathbf{\Sigma}_y$, $\mathbf{\Phi}_y$ are 3×3 matrixes**;** $\mathbf{\Phi}$, $\mathbf{\Sigma}$ are $N \times N$ matrixes; r_t, A_m, α, d_m, $y_{1,t}$, $y_{2,t}$, $y_{3,t}$, v_t, z_t are scalars. $\boldsymbol{\beta}$, $\boldsymbol{\eta}$, $\mathbf{\Sigma}_y$, $\mathbf{\Phi}_y$,$\mathbf{\Phi}$, $\mathbf{\Sigma}$, $\rho_{z,1}, \rho_{z,2}, \rho_{z,3}$ and $\rho_{h_j,1}, \rho_{h_j,2}, \rho_{h_j,3}, \sigma_{h_j}$ for $j = 1, .., n$ are parameters.

3.2 The stochastic discount factor (SDF) and the valuation formulae

To complete the equity valuation model and rule out arbitrage, this section formulates the SDF of the valuation model. We assume that under the real measure $\mathbb{P}$

$$\mathbf{x}_{t+1} = \left(\mathbf{I}_N - \mathbf{\Phi}^{\mathbb{P}}\right) \mathbf{x}_t + \boldsymbol{\eta}^{\mathbb{P}} + \mathbf{\Sigma} \boldsymbol{\xi}_{t+1}^{\mathbb{P}}$$

where $\boldsymbol{\eta}^{\mathbb{P}}$ is an $n \times 1$ vector and $\mathbf{\Phi}^{\mathbb{P}}$ an $n \times n$ matrix of parameters. $\boldsymbol{\xi}_{t+1}^{\mathbb{P}} \backsim \mathbb{N}\left(\mathbf{0}_{N\times1}, \mathbf{I}_N\right)$ are N independent Gaussian shocks under the real measure $\mathbb{P}$.

The time t stochastic discount factor $\mathbb{M}_t$ is such that

$$\begin{aligned}\mathbb{M}_{t+1} &= \mathbb{M}_t \cdot e^{-r_t} \cdot e^{-\frac{1}{2}\boldsymbol{\Lambda}_t'\boldsymbol{\Lambda}_t - \boldsymbol{\Lambda}_t'\boldsymbol{\xi}_{t+1}^{\mathbb{P}}} \\ \boldsymbol{\Lambda}_t &= \boldsymbol{\Lambda}_0 + \boldsymbol{\Lambda}_1 \mathbf{x}_t\end{aligned}$$

where $e^{-\frac{1}{2}\boldsymbol{\Lambda}_t'\boldsymbol{\Lambda}_t - \boldsymbol{\Lambda}_t'\boldsymbol{\xi}_{t+1}^{\mathbb{P}}}$ denotes the Radon-Nykodim derivative and $\boldsymbol{\Lambda}_t$ is an $N \times 1$ vector of the market prices of risk. $\boldsymbol{\Lambda}_0$ is an $N \times 1$ vector **of parameters** and $\boldsymbol{\Lambda}_1$ an $N \times N$ matrix of parameters. Then it can be shown that

$$\ln E_t\left[e^{-\boldsymbol{\Lambda}_t'\boldsymbol{\xi}_{t+1}^{\mathbb{P}}}\right] = \frac{1}{2}\boldsymbol{\Lambda}_t'\boldsymbol{\Lambda}_t, \quad E_t\left[\frac{\mathbb{M}_{t+1}}{\mathbb{M}_t}\right] = e^{-r_t}, \quad \boldsymbol{\xi}_{t+1} = \boldsymbol{\xi}_{t+1}^{\mathbb{P}} + \boldsymbol{\Lambda}_t.$$

$E_t[..]$ is the time t conditional expectation under the $\mathbb{P}$ measure. $\boldsymbol{\xi}_{t+1} \backsim \mathbb{N}(\mathbf{0}_{N\times 1}, \mathbf{I}_N)$ are the N independent Gaussian shocks under the $\mathbb{Q}$ measure that we assumed above. **Then it can be shown** that $\mathbf{x}_{t+1} = (\mathbf{I}_N - \boldsymbol{\Phi})\,\mathbf{x}_t + \boldsymbol{\eta} + \boldsymbol{\Sigma}\boldsymbol{\xi}_{t+1}$ under $\mathbb{Q}$, as we assumed above, and that $\boldsymbol{\eta} = \boldsymbol{\eta}^{\mathbb{P}} - \boldsymbol{\Sigma}\boldsymbol{\Lambda}_0$ and $\boldsymbol{\Phi} = \boldsymbol{\Phi}^{\mathbb{P}} + \boldsymbol{\Sigma}\boldsymbol{\Lambda}_1$. Therefore $\boldsymbol{\Lambda}_t$ entails that **x follow** Gaussian autoregressive **processes** under both the real measure $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$. Under $\mathbb{Q}$ the process $\mathbf{x}$ is stationary as long as all the eigenvalues of $(\mathbf{I}_N - \boldsymbol{\Phi})$ are smaller than 1 in absolute value, which we assume. This implies that all factors $\mathbf{x}$ are mean-reverting and stationary. **The empirical application below estimates the quadratic term structure model using US Treasury bond yields**, **with no constraint to ensure the mean reversion of y**, and finds that the unconstrained parameter estimates indicate that indeed the three latent factors $\mathbf{y}$ mean revert under the risk-neutral measure. Mean-reversion in $\mathbf{y}$ implies that the term $(\exp(r_t) - \exp(r_{t-1}))$ in equation 6 is stationary, which is consistent with the assumption of a mean-reverting autoregressive process for z as per equation 9. If $\mathbf{y}$ were mean averting, the term $(\exp(r_t) - \exp(r_{t-1}))$ in equation 6 would be non-stationary, which would be difficult to reconcile with the assumption of equation 9 for z. For simplicity we also assume that $\boldsymbol{\Phi}$ is block-diagonal, in the sense that the factors $\mathbf{y}_t$ do not affect the conditional mean of $\left(v_{t+1}, z_{t+1}, \mathbf{h}_{t+1}'\right)$ and vice versa the factors $(v_t, z_t, \mathbf{h}_t')$ do not affect the conditional mean of $\mathbf{y}_{t+1}$.

Appendix A.7 shows that, under the above assumptions, to compute $V_{0,m}^v$ in equation 13 we solve the following system of Riccati equations

$$\begin{aligned}
d_m &= d_{m-1} + \mathbf{D}_{m-1}^{y\prime}\boldsymbol{\Phi}_y\boldsymbol{\mu}_y + \mathbf{D}_{m-1}^{h\prime}\mathbf{G}_h\mathbf{m}_h + \mathbf{K}_{m-1}'\boldsymbol{\gamma}\boldsymbol{\gamma}'\left(\mathbf{B}_{m-1} + 2\mathbf{C}_{m-1}'\boldsymbol{\Phi}_y\boldsymbol{\mu}_y\right) \\
\mathbf{D}_m^{y\prime} &= \mathbf{D}_{m-1}^{y\prime}\left(\mathbf{I}_3 - \boldsymbol{\Phi}_y\right) + 2\cdot\mathbf{K}_{m-1}'\boldsymbol{\gamma}\boldsymbol{\gamma}'\mathbf{C}_{m-1}'\left(\mathbf{I}_3 - \boldsymbol{\Phi}_y\right) \\
D_m^v &= D_{m-1}^v \\
D_m^z &= \left(D_{m-1}^z + D_{m-1}^v\right)(1-g) \\
\mathbf{D}_m^{h\prime} &= \left(\mathbf{D}_{m-1}^{h\prime}\left(\mathbf{I}_n - \mathbf{G}_h\right) + \left(D_{m-1}^v + D_{m-1}^z\right)g\cdot\mathbf{m}_z'\right)
\end{aligned}$$

$$\mathbf{K}_{m-1}' = \mathbf{D}_{m-1}^{y\prime} + \left(D_{m-1}^z\sigma_z\left(\rho_{z,1}, \rho_{z,2}, \rho_{z,3}\right) + \sum_{j=1}^n D_{m-1}^{h_j}\sigma_{h_j}\left(\rho_{h_j,1}, \rho_{h_j,2}, \rho_{h_j,3}\right)\right)\boldsymbol{\Sigma}_y^{-1}$$

$$A_m = -\alpha + A_{m-1} + \mathbf{B}'_{m-1}\mathbf{\Phi}_y\boldsymbol{\mu}_y + \left(\mathbf{\Phi}_y\boldsymbol{\mu}_y\right)'\mathbf{C}_{m-1}\mathbf{\Phi}_y\boldsymbol{\mu}_y + \ln\frac{|\boldsymbol{\gamma}|}{abs\,|\mathbf{\Sigma}_y|} + \tag{14}$$
$$+\frac{1}{2}\left(\mathbf{B}'_{m-1} + 2\left(\mathbf{\Phi}_y\boldsymbol{\mu}_y\right)'\mathbf{C}_{m-1}\right)\boldsymbol{\gamma}\boldsymbol{\gamma}'\left(\mathbf{B}_{m-1} + 2\mathbf{C}'_{m-1}\mathbf{\Phi}_y\boldsymbol{\mu}_y\right)$$

$$\mathbf{B}'_m = -\boldsymbol{\beta}' + \left(\mathbf{B}'_{m-1} + 2\left(\mathbf{\Phi}_y\boldsymbol{\mu}_y\right)'\mathbf{C}_{m-1}\right)\left(\mathbf{I}_3 + 2\boldsymbol{\gamma}\boldsymbol{\gamma}'\mathbf{C}_{m-1}\right)\left(\mathbf{I}_3 - \mathbf{\Phi}_y\right) \tag{15}$$

$$\mathbf{C}_m = -\Psi + \left(\mathbf{I}_3 - \mathbf{\Phi}_y\right)'\mathbf{C}_{m-1}\left(\mathbf{I}_3 + 2\boldsymbol{\gamma}\boldsymbol{\gamma}'\mathbf{C}'_{m-1}\right)\left(\mathbf{I}_3 - \mathbf{\Phi}_y\right) \tag{16}$$

subject to the terminal conditions $d_0 = A_0 = 0$, $\mathbf{B}_0 = \mathbf{0}_{3\times1}$, $\mathbf{D}'_0 = \left(\mathbf{0}'_{3\times1}, 1, \mathbf{0}'_{n+1\times1}\right)$ and $\mathbf{C}_0 = \mathbf{0}_{3\times3}$, where $\mathbf{0}_{3\times3}$ is an 3×3 matrix of zeros. These terminal conditions imply that $V^v_{t,0} = v_t$. $D^v_m = 1$ for all m. $r_t \geq 0$ as long as $\alpha \geq \frac{1}{4}\boldsymbol{\beta}'\mathbf{\Psi}^{-1}\boldsymbol{\beta}$. Equations 14, 15 and 16 had already appeared in Realdon (2006) for pricing discount bonds according to the formula $P_{0,m} = e^{A_m + \mathbf{B}'_m\mathbf{y}_0 + \mathbf{y}'_0\mathbf{C}_m\mathbf{y}_0}$. As $\mathbf{y}_t$ is not observable, parameter identification requires that:

- $\boldsymbol{\mu}_y \geq \mathbf{0}_{3\times1}$, $\alpha \geq 0$, $\boldsymbol{\beta} = \mathbf{0}_{3\times1}$;
- Σ_y be diagonal (triangular) and $\mathbf{\Phi}_y$ be triangular (diagonal).

We can summarise the formulae in this section as follows.

Proposition 2 *Under risk-neutral valuation, continuous discounting, stochastic interest rates and assumptions 8, 9, 10, 11, 12, equation 1 for equity value becomes*

$$V^e_0 = \textstyle\sum_{m=1}^{T} V^v_{0,m}$$

where $V^v_{0,m}$ is given by 13.

Equity value depends on the correlation between factors driving earnings and factors driving interest rates, but not on the correlation between factors driving earnings, because $V^v_{0,m}$ is linear in these latter factors. Therefore the covariances $E^{\mathbb{Q}}_t\left[\xi_{h_j,t+1}\xi_{h_l,t+1}\right]$ for $j \neq l$ or the covariances $E^{\mathbb{Q}}_t\left[\xi_{h_j,t+1}\xi_{z,t+1}\right]$ are irrelevant to this valuation model. This is a convenient simplification that is not possible when the payoffs to be discounted, be they earnings or dividends or free cash flows, are specified as exponential affine functions of the factors, as in Ang and Liu (2004), D'Addona and Kind (2006) or Hughes, Liu and Liu (2009) among others.

Another simplification occurs when $\boldsymbol{\xi}_{h,t+1}$ and $\xi_{z,t+1}$ are independent of $\boldsymbol{\xi}_{y,t+1}$, in which case $\rho_{z,1} = \rho_{z,2} = \rho_{z,3} = \rho_{h_j,1} = \rho_{h_j,2} = \rho_{h_j,3} = 0$ for $j = 1,..,n$. In this case equity value no longer depends on σ_z and σ_{h_j} for $j = 1,..n$ and $\mathbf{D}^{y\prime}_m = \mathbf{K}'_{m-1} = \mathbf{0}_{1\times3}$. This means that, absent any correlation between factors driving earnings and factors driving interest rates, the valuation becomes independent of the volatility of factors driving earnings.

The term $\mathbf{K}'_{m-1}$ in the equations for d_m and $\mathbf{D}^{y\prime}_m$ entails that equity valuation depends on the correlation between the shocks to factors driving interest rates $\boldsymbol{\xi}_{y,t+1}$ and the shocks to factors driving earnings, which are $\boldsymbol{\xi}_{z,t+1}$ and $\boldsymbol{\xi}_{h,t+1}$. $\mathbf{K}'_{m-1}$ and equity valuation also depend on the "volatilities" of the factors driving interest rates ($\boldsymbol{\Sigma}_y$) and earnings (σ_z and σ_{h_j}). **Moreover** $\mathbf{D}^{y\prime}_m$ depends on $\boldsymbol{\Phi}_y$ **and** $\boldsymbol{\Phi}_y$ **determines** the unconditional variance of interest rates. **The volatilities and correlations** may increase or decrease equity value. This ambivalence is absent when the term structure model is an affine one and is due to the fact that $\mathbf{D}^{y\prime}_m\mathbf{y}_t$ may be positive or negative, because $\mathbf{y}_t$ may be positive or negative. When yields are close to zero, as **they have been** in many countries, it is particularly likely that $\mathbf{y}_t$ may switch from positive to negative and vice versa.

On the other hand when $\alpha = 0$, $\boldsymbol{\Psi} = \mathbf{0}_{3\times 3}$ and $\boldsymbol{\beta} \neq \mathbf{0}_{3\times 1}$, then $\mathbf{C}_m = \mathbf{0}_{3\times 3}$, $\mathbf{D}^{y\prime}_m = \mathbf{0}_{1\times 3}$ and the term structure model becomes an affine Gaussian model since $r_t = \boldsymbol{\beta}' \cdot \mathbf{y}_t$. **Appendix A.8 discusses this special case.** Then higher factor correlations and volatilities do not affect $\mathbf{D}^{y\prime}_m$, but reduce d_m and equity value, since $\mathbf{D}_m > \mathbf{0}_{(n+2)\times 1}$ (at least when $\mathbf{G}_h\mathbf{m}_h > \mathbf{0}_{n\times 1}$ and $g \cdot \mathbf{m}_z > \mathbf{0}_{n\times 1}$) and since $\mathbf{B}_{m-1} < \mathbf{0}_{3\times 1}$ for realistic parameters of the affine Gaussian term structure model. The intuition is that when earnings tend to rise, the affine yield curve (i.e. $-\frac{A_m + \mathbf{B}'_m \mathbf{y}_0}{m}$) tends to rise. **Therefore** it is the higher earnings that are discounted at higher interest rates, which reduces the expected present value of earnings. **This point was** already made in the residual earnings valuation of Pope and Wang (2000), among others. When $r_t = \boldsymbol{\beta}' \cdot \mathbf{y}_t$, r_t may turn negative and has no lower bound. This shortcoming is non-negligible when interest rates are close to zero and when we discount earnings expected in ten, twenty or thirty years.

3.3 The shortcomings of "square root" affine term structure models

The valuation model of Proposition 2 is close in spirit to Ang and Liu (2001), but their model discounts residual earnings and assumes a discrete time "square root" affine term structure model similar to Sun**'s** (1992). **The quadratic term structure model seems preferable to Sun's term structure model** for the following reasons. "Square root" affine models suffer from the admissibility restrictions illustrated in Dai and Singleton (2000), which restrict the correlation between factors. For example **if, other things as above, under the measure** $\mathbb{Q}$

$$
\begin{aligned}
& r_t = y_{1,t} + y_{2,t} \\
& y_{1,t+1} = y_{1,t}\left(1 - \phi_{y_1}\right) + \phi_{y_1}\mu_{y_1} + \sigma^2_{y_1}\sqrt{y_{1,t}}\xi_{1,t+1}, \qquad \xi_{1,t+1} \backsim \mathbb{N}(0,1) \\
& y_{2,t+1} = y_{2,t}\left(1 - \phi_{y_2}\right) + \phi_{y_2}\mu_{y_2} + \sigma^2_{y_2}\sqrt{y_{2,t}}\xi_{2,t+1}, \qquad \xi_{2,t+1} \backsim \mathbb{N}(0,1) \\
& z_{t+1} = g\left(\mathbf{m}'_z\mathbf{h}_t\right) + (1-g)\, z_t + \sigma_z \xi_{z,t+1}, \qquad \xi_{z,t+1} \backsim \mathbb{N}(0,1)
\end{aligned}
$$

where $\phi_{y_1}, \phi_{y_2}, \sigma^2_{y_1}, \sigma^2_{y_2}, \mu_{y_1}\mu_{y_2}$ **are scalar parameters**, then closed form valuations **are** possible only if

$$E_t^{\mathbb{Q}}\left[\xi_{1,t+1}\xi_{2,t+1}\right] = E_t^{\mathbb{Q}}\left[\xi_{1,t+1}\xi_{z,t+1}\right] = E_t^{\mathbb{Q}}\left[\xi_{2,t+1}\xi_{z,t+1}\right] = 0$$
$$E_t^{\mathbb{Q}}\left[\xi_{1,t+1}\cdot\boldsymbol{\xi}_{h,t+1}\right] = \mathbf{0}_{n\times 1}.$$

These are severe restrictions to factors correlation. Moreover, especially when interest rates are close to zero, r_t, $y_{1,t}$ and $y_{2,t}$ may turn negative, so that $\sqrt{y_{1,t}}$ and $\sqrt{y_{2,t}}$ may be complex **numbers. Instead quadratic** models rule out negative interest rates, imply heteroschedastic yields as do "square root" affine models, and need no admissibility restrictions, so that factors correlation is unrestricted as explained in Ahn, Dittmar and Gallant (2002). Therefore for equity risk-neutral valuation based on discounting earnings, discrete time quadratic models seem preferable to both discrete time affine Gaussian models and affine "square root" term structure models. This is no minor detail, as the correlation between factors driving earnings and factors driving interest rates can materially **affect** equity valuation, **as shown below**. **Moreover also the term structure model and its parameters can materially affect valuation**, as shown by Ang and Liu (2001).

3.4 Special case under constant interest rates

To gain insight into **the model of Proposition 2**, we now focus on the special case where r_t is constant over time and equal to r. Then we define

$$\mathfrak{r} = \exp(r) - 1$$
$$\underline{\mathbf{x}}_t = (v_t, z_t, \mathbf{h}_t')'.$$

Under these assumptions, Appendix A.9 shows that V_t^e in Proposition 2 becomes

$$V_t^e = \mathfrak{d} + \mathfrak{D}'\underline{\mathbf{x}}_t = E_t^{\mathbb{Q}}\left[\frac{v_{t+1} + V_{t+1}^e}{1+\mathfrak{r}}\right] = \frac{E_t^{\mathbb{Q}}\left[v_{t+1} + \mathfrak{d} + \mathfrak{D}'\underline{\mathbf{x}}_{t+1}\right]}{1+\mathfrak{r}}$$
$$\mathfrak{D} = \left(\mathfrak{D}^v, \mathfrak{D}^z, \mathfrak{D}^{h\prime}\right)'$$
$$\mathfrak{d} = \frac{\mathfrak{D}^{h\prime}\mathbf{G}_h\mathbf{m}_h}{\mathfrak{r}}$$
$$\mathfrak{D}^v = \frac{1}{\mathfrak{r}}$$
$$\mathfrak{D}^z = \frac{1-g}{\mathfrak{r}+g}\left(1+\mathfrak{D}^v\right)$$
$$\mathfrak{D}^{h\prime} = \left(1+\mathfrak{D}^v+\mathfrak{D}^z\right)\cdot g\cdot\mathbf{m}_z'\left(\mathfrak{r}\mathbf{I}_{n\times n}+\mathbf{G}_h\right)^{-1}$$

where $\mathfrak{d}$, $\mathfrak{D}^v$ and $\mathfrak{D}^z$ are scalar constants and $\mathfrak{D}^h$ is a $n\times 1$ vector of constants. If $g = 0$ then $z_{t+1} = z_t + \sigma_z\xi_{z,t+1}$ so that there is no expected change in abnormal earnings growth and equity value becomes $V_0^e = \frac{v_0}{\mathfrak{r}} + \frac{1}{\mathfrak{r}}\frac{\mathfrak{r}+1}{\mathfrak{r}}z_0$. In this case equity value is independent of $\mathbf{h}_t$. Similarly, if $\mathbf{m}_z = \mathbf{0}_{n\times 1}$ then $z_{t+1} = (1-g)\, z_t + \sigma_z \xi_{z,t+1}$ and $V_0^e = \frac{v_0}{\mathfrak{r}} + \frac{1-g}{\mathfrak{r}+g}\frac{\mathfrak{r}+1}{\mathfrak{r}} z_0$, so that equity value is again independent of $\mathbf{h}_t$ and $-g$ can be interpreted as the expected growth rate in abnormal earnings growth. $\mathbf{m}_z'\mathbf{h}_t$ is the central tendency of z_{t+1} when $g > 0$. Fluctuations of $\mathbf{h}_t$ over time can drive price changes even as accounting information, summarised by v_t, does not change. When $\mathbf{m}_h = \mathbf{0}_{n\times 1}$, abnormal earnings growth in the long term tends to zero, which may often be a reasonable assumption, and $\mathfrak{d} = 0$.

Changes in $\mathbf{h}_t$ and z_t over time can explain changes in price-earnings ratios over time. **Note that, as** $v_0 = x_0$, the value-to-earnings ratio is

$$\frac{V_0^e}{v_0} = \frac{1}{\mathfrak{r}}\left(\frac{\mathfrak{D}^{h\prime}\mathbf{G}_h\mathbf{m}_h + \frac{\mathfrak{r}+1}{\mathfrak{r}+g}\Big((1-g)\, z_0 + (\mathfrak{r}+1)\, g\cdot \mathbf{m}_z' \left(\mathfrak{r}\mathbf{I}_{n\times n} + \mathbf{G}_h\right)^{-1}\mathbf{h}_0\Big)}{v_0} + 1\right). \tag{17}$$

According to this formula the ratio $\frac{V_0^e}{v_0}$ is driven by the ratios $\frac{z_0}{v_0}$ and $\frac{\mathbf{h}_0}{v_0}$, i.e. by the ratio between current abnormal earnings growth and current earnings and by the ratio between the drivers $\mathbf{h}_0$ of expected future abnormal earnings growth and current earnings.

3.5 Residual earnings valuation with quadratic term structure models

The main focus of this paper is on AEG valuation, but to complete the analysis this section also provides formulae for residual earnings valuation under quadratic term structure models. **These formulae are later applied and their predictions are compared with those of the valuation of Proposition 2**. re_{t+1} **denotes** the residual earnings produced over the period $[t, t+1]$, which we defined above as $re_{t+1} = x_{t+1} - r_{e,t}\cdot b_t$, where b_t denotes the book value of equity at time t. Then under risk-neutral valuation and continuous discounting we can re-define residual earnings as $re_{t+1} = x_{t+1} - (\exp(r_t) - 1)\cdot b_t$. According to residual earnings risk-neutral valuation under continuous discounting and stochastic interest rates

$$V_0^e = b_0 + \textstyle\sum_{t=1}^{\infty} E_0^{\mathbb{Q}}\left[re_t \cdot e^{-\sum_{i=0}^{t-1} r_i}\right].$$

Then we assume the following parametric process for residual earnings under the risk-neutral measure

$$re_{t+1} = g_{re}\left(\mathbf{m}_{re}'\mathbf{h}_t\right) + (1-g_{re})\, re_t + \sigma_{re}\xi_{re,t+1}$$

where $g_{re}, \sigma_{re}, \rho_{re,1}, \rho_{re,2}, \rho_{re,3}$ are scalar constants, $\mathbf{m}_{re}$ is an $n\times 1$ vector of parameters and $E_t^{\mathbb{Q}}\left[\xi_{re,t+1}\xi_{1,t+1}\right] = \rho_{re,1}$, $E_t^{\mathbb{Q}}\left[\xi_{re,t+1}\xi_{2,t+1}\right] = \rho_{re,2}$, $E_t^{\mathbb{Q}}\left[\xi_{re,t+1}\xi_{3,t+1}\right] = \rho_{re,3}$. Moreover under the additional assumptions of equations 10 and 11 residual earnings valuation becomes

$$V_0^e = b_0 + \textstyle\sum_{m=1}^{\infty} E_0^{\mathbb{Q}}\left[re_m \cdot e^{-\sum_{i=0}^{m-1} r_i}\right] = b_0 + \sum_{m=1}^{\infty} V_{0,m}^{re} \tag{18}$$

where $V_{0,m}^{re}$ is the same as $V_{0,m}^{v}$ in equation 13 if only we make the following substitutions: g_{re} **replaces** g, $\mathbf{m}'_{re}$ **replaces** $\mathbf{m}'_{z}$, σ_{re} **replaces** σ_{z}, $\rho_{re,1}$ **replaces** $\rho_{z,1}$, $\rho_{re,2}$ **replaces** $\rho_{z,2}$, $\rho_{re,3}$ **replaces** $\rho_{z,3}$, $\mathbf{D}'_0 = \left(\mathbf{0}'_{3\times1}, 1, \mathbf{0}'_{(n+1)\times1}\right)$ **replaces** $\mathbf{D}'_0 = \left(\mathbf{0}'_{4\times1}, 1, \mathbf{0}'_{n\times1}\right)$. **This** last substitution means that $V_{t,0}^{re} = re_t$ replaces the terminal condition $V_{t,0}^{v} = v_t$. The advantages of quadratic term structure models are retained also when discounting residual earnings.

3.6 Comparison with Bakshi and Chen (2005) and with Ang and Liu (2004), the normality assumption and the time step

The spirit of the equity risk-neutral valuation of **Proposition 2 is also similar to** Bakshi and Chen (2005). Inspired by the literature on derivatives valuation, their paper presents a risk-neutral valuation model for equities that avoids the problem of estimating the cost of equity capital and that discounts expected earnings under the risk-neutral probability measure. However their model is in continuous time, which is not a natural setting for earnings and other accounting variables. **The** above discrete time setting seems more suitable for accounting based valuations.

Bakshi and Chen discount expected earnings under the assumption that dividends equal a fixed fraction of earnings. However, when this assumption does not hold, the analysis of Ohlson and Juettner-Nauroth (2005) and Ohlson (2005) implies that simply discounting expected earnings may lead to incorrect valuations, because discounting earnings is generally not equivalent to discounting dividends. Instead discounting earnings and abnormal earnings growth is equivalent to discounting dividends. **Propositions 1 and 2 build** on this insight.

Bakshi and Chen assume that the earnings growth rate follows a stochastic mean reverting process and find that "modeling earnings growth dynamics properly is the most crucial" for equity valuation. The model presented above captures this insight through the auto-regressive mean-reverting processes of z and of the $\mathbf{h}$ factors.

Bakshi and Chen assume a Vasicek model for the term structure of interest rates. Various studies report that quadratic term structure models such as that proposed above seem to describe the dynamics of the term structure better than affine Vasicek-like Gaussian models. Moreover quadratic models encompass affine Gaussian models.

Bakshi and Chen calibrate the parameters of the risk-neutral factor processes in their valuation model by minimising the sum of squared differences between observed price-earnings (PE) ratios and model predicted PE ratios. The valuation model of Proposition 2 can be calibrated in a similar way, **when the h factors are absent or when they are present and are observable**.

Proposition 2 uses risk-neutral valuation. However Appendix A.10 shows that the equity valuation of Proposition 2 is in closed form even

under the real probability measure when the equity risk premium is stochastic as in Ang and Liu (2004).

The valuation formula of Proposition 2 relies on normally distributed random shocks. Alternatively we may approximate the empirical distribution of the random shocks using a mixture of Gaussian densities and we would still have tractable valuation formulae.

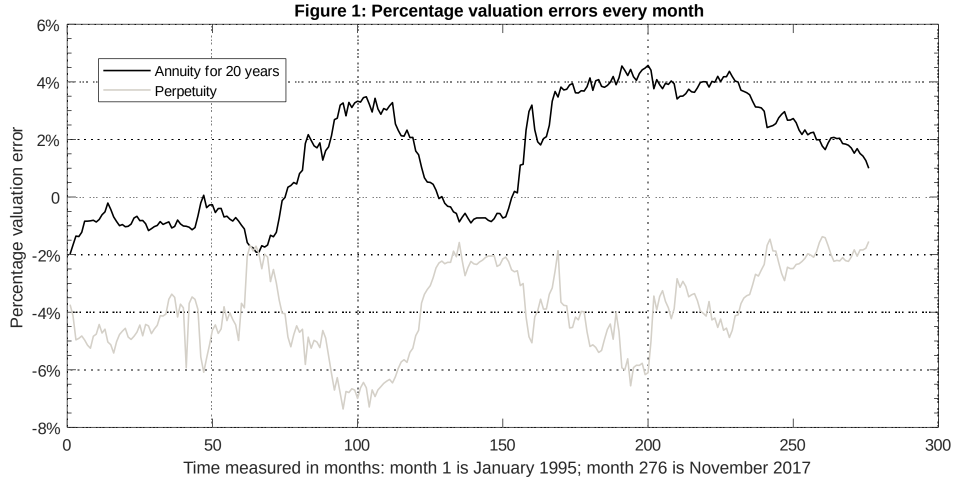
The valuation in Proposition 2 assumes that the short interest rate only changes at the end of the accounting period, e.g. every quarter. The accounting period is also the time step of all stochastic processes in the model. This assumption simplifies the illustration of the model. In some practical applications we may have daily interest rate observations, while the h factors may drive daily price changes. Then we can assume daily time steps for the short rate and for the h factors, while the accounting periods can be quarterly. The coexistence of different time steps, daily and quarterly, only requires minor changes to the above valuation formulae.

4 Empirical application

This section uses US Treasury bond yield curves during the 1995 to 2017 period to show the valuation errors due to using one single discount rate instead of the whole yield curve. This section also shows how the correlation between earnings and interest rate factors drives the PE ratio predicted by the valuation model of Proposition 2.

We use a panel of US Treasury continuously compounded discount bond yields for maturities from one year to twenty years observed on the last trading day of each month included in the period from 3/1/1995 to 29/11/2017. Therefore we use yields for twenty different maturities for every month in the sample. The yields are computed from discount factors for each maturity and date provided by Thompson-Reuters Eikon.

Figure 1: Percentage valuation errors every month



The black line in Figure 1 shows the percentage valuation error for each month τ of the 276 months in the sample, which is computed as $\left(Q^{20}_{avg,\tau} - Q^{20}_{\tau}\right)/Q^{20}_{\tau}$. $Q^{20}_{\tau} = \sum_{\mathfrak{m}=1}^{20} \exp\left(-\mathfrak{m} \cdot i_{\tau,\mathfrak{m}}\right)$ is the present value of an annuity that pays one dollar each year end for twenty years, discounted using the continuously compounded Treasury bond yields $i_{\tau,\mathfrak{m}}$ in month τ for each maturity $\mathfrak{m}$. $Q^{20}_{avg,\tau} = \sum_{\mathfrak{m}=1}^{20} \exp\left(-\mathfrak{m} \cdot i_{\tau,avg}\right)$ is an approximation to Q^{20}_{τ} where every annuity payment is discounted using the average yield $i_{\tau,avg} = \frac{1}{20}\sum_{\mathfrak{m}=1}^{20} i_{\tau,\mathfrak{m}}$. The valuation errors, often exceeding 3%, are due to using the single discount rate $i_{\tau,avg}$ for all maturities. The grey line in Figure 1 is the percentage valuation error for each month computed as $\left(Q^{p}_{20,\tau} - Q^{p}_{\tau}\right)/Q^{p}_{\tau}$. $Q^{p}_{20,\tau} = 1/\left(\exp\left(i_{\tau,20}\right) - 1\right)$ is the present value of a perpetuity that pays one dollar each year end**, and each payment** is discounted using the twenty year continuously compounded yield $i_{\tau,20}$ observed in month τ. $Q^{p}_{\tau} = Q^{20}_{\tau} + \exp\left(-20 \cdot i_{\tau,20}\right)/\left(\exp\left(i_{\tau,20}\right) - 1\right)$ is the present value of the same perpetuity where the present value of the payments of the first twenty years is again Q^{20}_{τ} and the payments thereafter are discounted at the constant rate $i_{\tau,20}$. The valuation error, often more than 3% in absolute value, is now due to discounting the payments in the first twenty years at the single rate $i_{\tau,20}$. Figure 1 highlights the valuation errors due to using one single discount rate for earnings at different maturities as per "classic" AEG valuation, as opposed to valuation as per Proposition 1. However it is Proposition 2 that shows the valuation effects of factors volatilities and correlation. Therefore we **now** turn to valuation under stochastic interest rates as per Proposition 2.

First we follow the quasi-maximum-likelihood approach of Realdon (2017) in estimating **the** three factor quadratic term structure model using the sample of US Treasury yields. This approach provides the latent factors $\mathbf{y}_\tau$ for every month τ, which we need for valuation. We assume that the one year, ten year and twenty year yields are observed without error. Table 1 provides summary statistics of the yields in the sample. The quadratic model assumes $r_t = y^2_{3,t}$, while $y^2_{1,t}$ and $y^2_{2,t}$ drive the central tendency of $y^2_{3,t}$. The model has monthly time steps and its estimated parameters under the risk-neutral measure $\mathbb{Q}$ are

$$\mathbf{\Phi}_y = \begin{pmatrix} 0.279 & 0 & 0 \\ 0 & 2.720 & 0 \\ -0.198 & -0.198 & 0.198 \end{pmatrix}, \quad \boldsymbol{\mu}_y = (0, 0, 0.58)', \quad \mathbf{\Sigma}_y = \begin{pmatrix} 0.135 & 0 & 0 \\ 3.3 \cdot 10^{-6} & 0.033 & 0 \\ -0.04078 & -0.0167 & 0.06 \end{pmatrix}.$$

Then we assume for simplicity that $\mathbf{m}_z$ is a vector of zeros, so that $\mathbf{m}_z' \mathbf{h}_t = 0$ for all t. Therefore the valuation is independent of the factors $\mathbf{h}$ and the valuation model of Proposition 2 simplifies **to**

$$\begin{aligned}
& z_{t+1} = (1-g)\, z_t + \sigma_z \xi_{z,t+1} \\
& d_m = d_{m-1} + \mathbf{D}_{m-1}^{y\prime} \mathbf{\Phi}_y \boldsymbol{\mu}_y + \mathbf{K}_{m-1}' \boldsymbol{\gamma}\boldsymbol{\gamma}' \left(\mathbf{B}_{m-1} + 2\mathbf{C}_{m-1}' \mathbf{\Phi}_y \boldsymbol{\mu}_y \right) \\
& \mathbf{D}_m^{y\prime} = \mathbf{D}_{m-1}^{y\prime} \left(\mathbf{I}_3 - \mathbf{\Phi}_y \right) + 2 \cdot \mathbf{K}_{m-1}' \boldsymbol{\gamma}\boldsymbol{\gamma}' \mathbf{C}_{m-1}' \left(\mathbf{I}_3 - \mathbf{\Phi}_y \right) \\
& D_m^v = 1 \\
& D_m^z = \left(D_{m-1}^z + 1 \right) (1-g) \\
& \mathbf{K}_{m-1}' = \mathbf{D}_{m-1}^{y\prime} + D_{m-1}^z \sigma_z \left(\rho_{z,1}, \rho_{z,2}, \rho_{z,3} \right) \mathbf{\Sigma}_y^{-1}
\end{aligned}$$

while the model-implied PE ratio is

$$\frac{V_0^e}{v_0} = \sum\nolimits_{m=1}^{T} \left(\frac{d_m + \mathbf{D}_m^{y\prime} \mathbf{y}_0 + D_m^z z_0}{v_0} + 1 \right) e^{A_m + \mathbf{B}_m' \mathbf{y}_0 + \mathbf{y}_0' \mathbf{C}_m \mathbf{y}_0}. \tag{19}$$

We recall that $z_0 = x_0 - x_{-1} - (\exp(r_{-1}) - 1)(x_{-1} - d_{-1})$ and $v_0 = x_0$. For simplicity and with little loss in accuracy, we assume monthly time steps in the valuation model of Proposition 2, even through earnings in the US market are reported quarterly, because we have monthly yield curve observations. Therefore x_0 are monthly earnings, z_0 is monthly abnormal earnings growth and r_t is the one month interest rate at time t. The valuation horizon is very long since T is 12.000 months (1.000 years times 12 months) so that continuation value is of negligible importance. Such long streams of earnings are meant to approximate the infinite streams of earning of "classic" AEG valuation. When T is halved to 6.000 months (i.e. 500 years) the valuation results are virtually the same. When $r_t = r$ for all t the discount rate is constant over time and, since $\mathbf{m}_z$ is a vector of zeros, equation 17 reduces to

$$\frac{V_0^e}{v_0} = \frac{1}{\mathfrak{r}} + \frac{1-g}{\mathfrak{r}+g} \left(1 + \frac{1}{\mathfrak{r}} \right) \frac{z_0}{v_0} \tag{20}$$

with $\mathfrak{r} = \exp(r) - 1$. We consider a base case with $v_0 = 1/12$, $z_0 = 0$, $\rho_{z,1} = \rho_{z,2} = \rho_{z,3} = 0$, which effectively assumes a perpetual stream of expected risk-neutral earnings of 1 dollar every year, more precisely 1/12 dollars every month, that are independent of interest rates. In the base case the valuation is independent of the parameters g and σ_z.

Figure 2 shows how yield curve changes in the sample period drive changes in the model predicted PE ratio, even as fundamentals, namely $v_0 = 1/12$ and $z_0 = 0$, do not change from month to month. Figure 2 shows six time series of PE ratios computed as per equation 19 each month from February 1995 to November 2017. In all months all six valuations assume the same stream of expected risk-neutral earnings. The thick black line is the base case, which assumes independence between earnings and interest rate factors. Every month the predicted PE ratio changes only because of changes in **bond** yields. The other cases are like the base case except for: $\sigma_z = 0.01, \rho_{z,1} = 0.01$ in case 2; $\sigma_z = 0.02, \rho_{z,1} = 0.01$ in case 3; $\sigma_z = 0.01, \rho_{z,1} = 0.01, \rho_{z,2} = 0.01$ in case 4; $\sigma_z = 0.01, \rho_{z,1} = 0.01, \rho_{z,2} = 0.01, \rho_{z,3} = 0.01$ in case 5; the PE ratio in the constant discount rate case is computed as in 20 with $r = i_{\tau,20}/12$ every month τ in the sample. The difference in PE between the constant discount rate case and the base case is significant. Moreover a constant discount rate is inconsistent with the data, as model implied PE's are not constant over time, even as the stream of expected risk-neutral earnings is the same all months. The upward trend in PE ratios is mainly due to the lower interest rates of the more recent past. The dominant effect of a rise in $\rho_{z,1}, \rho_{z,2}, \rho_{z,3}$ in our sample is to decrease d_m, while the change in $\mathbf{D}_m^{y\prime}\mathbf{y}_0$ is of lesser importance. This is why, irrespective of $\mathbf{y}_0$, the PE ratio decreases every month in the sample as $\rho_{z,1}, \rho_{z,2}, \rho_{z,3}$ increase. Cases 2 and 3 show that a rise in AEG volatility σ_z decreases the PE ratio when correlations are positive, and it can be shown that it increases the PE ratio when correlations are negative. This is due to the term $\sigma_z \left(\rho_{z,1}, \rho_{z,2}, \rho_{z,3}\right) \mathbf{\Sigma}_y^{-1}$ in the formula for $\mathbf{K}'_{m-1}$. σ_z multiplies the the correlation parameters and its valuation effect depends on the sign of such parameters. The PE ratio is very sensitive to correlations and to AEG volatility (i.e. σ_z) and this is due to the discounting of long term earnings.

Figure 3 shows the PB ratio predicted by residual earnings **risk-neutral valuation equation 18 for** essentially the same cases as Figure 2. **Figure 3 assumes** $b_0 = 100$, $g_{re} = 0$ **and** $re_0 = 2$. In the base case $\sigma_{re} = \rho_{re,1} = \rho_{re,2} = \rho_{re,3} = 0$. The other cases are like the base case, except for $\sigma_{re} = 0.01, \rho_{re,1} = 0.01$ in case 2, $\sigma_{re} = 0.02, \rho_{re,1} = 0.01$ in case 3, $\sigma_{re} = 0.01, \rho_{re,1} = 0.01, \rho_{re,2} = 0.01$ in case 4, $\sigma_{re} = 0.01, \rho_{re,1} = 0.01, \rho_{re,2} = 0.01, \rho_{re,3} = 0.01$ in case 5. The PB ratio in the constant discount rate case is computed assuming $r = i_{\tau,20}/12$ every month τ in the sample. Figure 3 is effectively a re-scaling of Figure 2 and confirms the intuition in Figure 2.

Figure 4 shows the yearly yields predicted by the quadratic model for maturities up to one thousand years in every December in the sample, i.e. from December 1995 to December 2016. Figure 4 shows that during these more than twenty years yields for maturities longer than two hundred years tend to converge to levels around 3%. In this sense the estimated quadratic model seems "well-behaved" in discounting perpetual streams of earnings.

Overall PE ratios tend to drop when the correlation between AEG (i.e. z_t) and US term structure factors increases. PE ratios are very sensitive to such correlation and also to the conditional variance of AEG. **Such conditional variance** amplifies the valuation impact of the correlations. Other comparative statics can show that also the quadratic model parameters $\mathbf{\Phi}_y, \mu_3, \mathbf{\Sigma}_y$ affect the predicted PE ratio by altering the interest rates used to discount earnings.

Table 1: Descriptive statistics of monthly continuously compounded yields implied by US Treasury bonds (1995-2017)

Yield maturity	1 year	2 years	3 years	4 years	5 years	6 years	7 years	8 years	9 years	10 years
Average	2.64%	2.86%	3.09%	3.30%	3.50%	3.68%	3.84%	3.97%	4.08%	4.18%
Std deviation	6.26%	12.02%	17.93%	23.87%	29.84%	35.81%	41.79%	47.77%	53.76%	59.74%
Max	7.20%	7.59%	7.72%	7.74%	7.72%	7.71%	7.72%	7.74%	7.76%	7.79%
Min	0.10%	0.19%	0.31%	0.45%	0.63%	0.82%	1.02%	1.22%	1.41%	1.50%
Yield maturity	11 years	12 years	13 years	14 years	15 years	16 years	17 years	18 years	19 years	20 years
Average	4.29%	4.39%	4.49%	4.58%	4.65%	4.71%	4.75%	4.78%	4.80%	4.81%
Std deviation	65.73%	71.72%	77.71%	83.70%	89.70%	95.69%	101.69%	107.68%	113.68%	119.67%
Max	7.83%	7.88%	7.93%	7.98%	8.02%	8.03%	8.04%	8.02%	8.00%	7.97%
Min	1.54%	1.58%	1.61%	1.64%	1.68%	1.72%	1.76%	1.81%	1.87%	1.92%

Figure 5 plots $\frac{V_0^e}{v_0}$ as per equation 20**, which refers to the constant discount rate case.** The thick black line is a base case, which assumes $\frac{z_0}{v_0} = 0.01$ **and** $\mathfrak{r} = 0.03$**. The thick black line** shows that PE decreases with g, since higher g entails that AEG (i.e. z_0) reverts from 0.01 to zero more quickly. The bright line assumes $\frac{z_0}{v_0} = -0.01$ **and** $\mathfrak{r} = 0.03$ and shows that PE increases with g, since higher g entails that AEG reverts from -0.01 to zero more quickly. The thin black line assumes $\frac{z_0}{v_0} = 0.01$ **and** $\mathfrak{r} = 0.04$ and shows that PE decreases with $\mathfrak{r}$ **as** earnings are discounted at a higher rate. The dashed line assumes $z_0 = 0$ and $\mathfrak{r} = 0.03$. Equation 20 implies that we have to impose $\mathfrak{r} > -g$ to prevent negative equity prices and that $\lim_{g \to (-\mathfrak{r})^+} \left(\frac{1}{\mathfrak{r}} + \frac{1-g}{\mathfrak{r}+g} \left(1 + \frac{1}{\mathfrak{r}}\right) \frac{z_0}{v_0} \right) \to \pm\infty$; in this right limit the PE ratio explodes to $+\infty$ if $\frac{z_0}{v_0} > 0$ and to $-\infty$ if $\frac{z_0}{v_0} < 0$.

Figure 4: yield curves for maturities from 1 month to 12000 months (i.e. 1000 years times 12 months) predicted by the estimated quadratic term structure model every december from 1995 to 2016

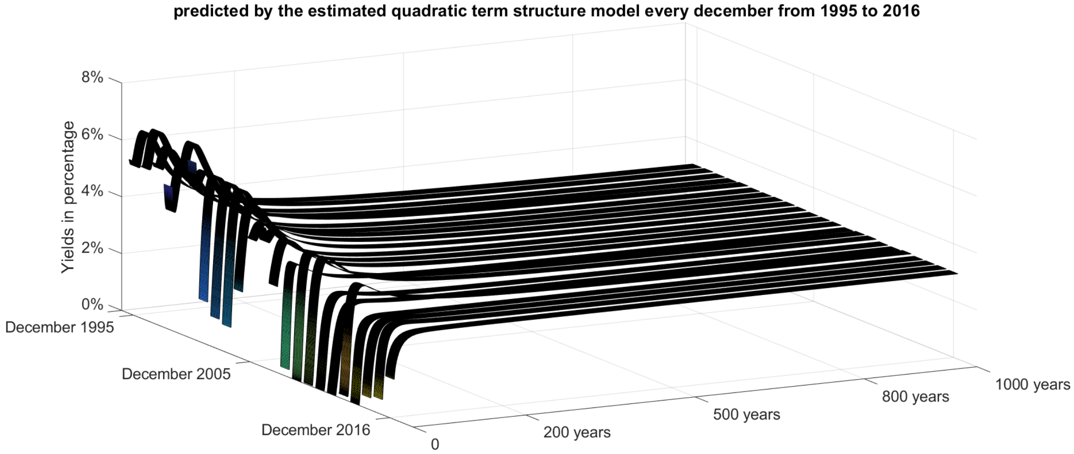
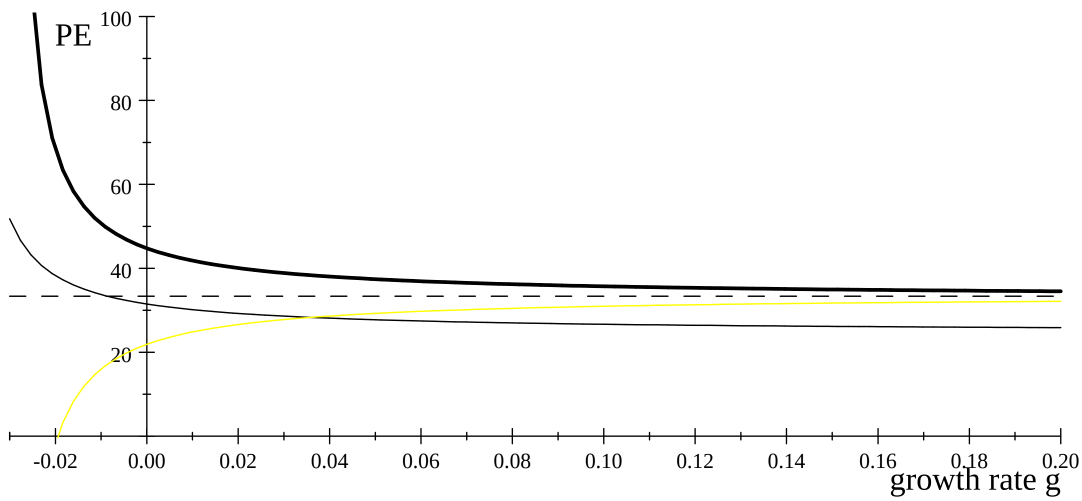


Figure 5: PE ratio and the growth rate g



This Figure shows the model predicted PE ratio under constant discount rate $\mathfrak{r}$ as a function of the growth rate g.

Similar comparative statics with respect to g and z_0 are possible for equation 19 under stochastic interest rates and confirm that large positive or negative values **of** g can cause the valuation to "explode" to plus or minus infinity. These comparative statics concern the effect of g on PE when $z_0 \neq 0$. However equation 19 **implies** that g affects PE even when AEG is zero, i.e. even when $z_0 = 0$. The reason is that g still affects D_m^z and $\mathbf{K}'_{m-1}$ through the term $D_{m-1}^z \sigma_z \left(\rho_{z,1}, \rho_{z,2}, \rho_{z,3}\right) \mathbf{\Sigma}_y^{-1}$, and therefore g **still affects** d_m and $\mathbf{D}_m^{y\prime}$. When $z_0 = 0$ **and** g rises and the correlation parameters $\left(\rho_{z,1}, \rho_{z,2}, \rho_{z,3}\right)$ are positive (negative), **then** PE rises (decreases). These effects are amplified by the volatility of AEG σ_z. The intuition is that, after **future random shocks will have driven** z away from its mean, as g rises, z **will revert** to its mean more quickly and the unconditional variance of z **therefore** decreases, which dampens the effect on PE of positive (negative) correlation between interest rate factors and earnings. **We recall that such positive (negative) correlation reduces (increase) PE, as seen above, and it is the dampening of such reduction** (increase) in PE that explains the rise (decrease) in PE as g rises. For example in February 1995 in case 5 of Figure 2 the model predicts PE of 7.26. Case 5 assumes $z_0 = 0$ and $g = 0$, but when, other things as in case 5, $g = 0.01$ then PE is 13.24, when $g = 0.02$ then PE is 13.92, when $g = 0.03$ then PE is 14.17. Similarly in November 2017 in case 5 of Figure 2 the model predicts PE of 17.5, but when, other things as in case 5, $g = 0.01$ then PE is 31.7, when

$g = 0.02$ then PE is 33.13, when $g = 0.03$ then PE is 33.65. The valuation effect of growth in AEG interacts with the valuation effect of the correlation between interest **rate factors** and earnings.

5 Conclusion

This paper has presented earnings based equity valuations in closed form that extend the popular AEG valuation to cases of time-varying or stochastic discount rates. These valuations do no rely on specific assumptions about continuation value and can be used to reverse engineer the equity risk premium when the term structure of interest rates is not flat.

Some of the valuation formulae use discrete time quadratic Gaussian term structure models, which overcome the limitations of "square root" affine term structure models and can rule out negative yields. Quadratic models require no restriction to the correlation between stochastic factors, which seems crucial, as valuation can be very sensitive to the correlation between factors driving interest rates and factors driving earnings, as well as to the "volatility" of these factors. **Also** the residual earnings valuation of Ang and Liu (2001) has been adapted to quadratic term structure models. **The** valuation formulae seem capable of explaining complex links between stock value and interest rates, while providing consistent valuations of stocks and bonds.

A Appendix

A.1 Proof that AEG valuation is a special case of equation 1

We notice that

$$\begin{aligned}
V_0^e &= \sum_{t=1}^{\infty} \frac{x_1 + (x_t - x_1)}{(1+r_e)^t} - \sum_{t=2}^{\infty} \frac{r_e \cdot \sum_{i=1}^{t-1} (x_i - d_i)}{(1+r_e)^t} \\
&= \sum_{t=1}^{\infty} \frac{x_1}{(1+r_e)^t} + \sum_{t=2}^{\infty} \frac{x_t - x_1}{(1+r_e)^t} - \sum_{t=2}^{\infty} \frac{r_e \cdot \sum_{i=1}^{t-1} (x_i - d_i)}{(1+r_e)^t} \\
&= \sum_{t=1}^{\infty} \frac{x_1}{(1+r_e)^t} + \sum_{t=2}^{\infty} \frac{\sum_{i=1}^{t-1} (x_{i+1} - x_i)}{(1+r_e)^t} - \sum_{t=2}^{\infty} \frac{r_e \cdot \sum_{i=1}^{t-1} (x_i - d_i)}{(1+r_e)^t} \\
&= \frac{x_1}{r_e} + \sum_{t=2}^{\infty} \frac{\sum_{i=1}^{t-1} (x_{i+1} - x_i - r_e \cdot (x_i - d_i))}{(1+r_e)^t} \\
&= \frac{x_1}{r_e} + \frac{1}{r_e} \sum_{t=1}^{\infty} \frac{(x_{t+1} - x_t - r_e \cdot (x_t - d_t))}{(1+r_e)^t}.
\end{aligned}$$

The first line is equation 1, while the last line is AEG valuation.

A.2 Re-writing equation 1 in terms of AEG under time-varying discount rate

Since from the definition of AEG with time-varying cost of capital we know that $x_{t+1} = x_t + z_{t+1} + r_{e,t}(x_t - d_t) + (r_{e,t} - r_{e,t-1})\sum_{i=0}^{t-1}(x_i - d_i) \cdot 1_{t\geq 1}$, it follows that

$$\begin{aligned} x_1 &= x_0 + z_1 + r_{e,0}(x_0 - d_0) \\ x_2 &= x_1 + z_2 + r_{e,1}(x_1 - d_1) + (r_{e,1} - r_{e,0})(x_0 - d_0) \end{aligned}$$

and so on. Then substituting into equation 1, we obtain

$$\begin{aligned} V_0^e &= E_0\left[\begin{array}{c} \frac{r_{e,0}(x_0-d_0)+x_0+z_1-r_{e,0}(x_0-d_0)}{1+r_{e,0}} + \\ + \frac{r_{e,1}(x_1-d_1)+(r_{e,1}-r_{e,0})(x_0-d_0)+x_0+z_1+r_{e,0}(x_0-d_0)+z_2-r_{e,1}\sum_{i=0}^{1}(x_i-d_i)}{(1+r_{e,0})(1+r_{e,1})} + .. \end{array}\right] \\ &= E_0\left[\frac{x_0+z_1}{1+r_{e,0}} + \frac{x_0+z_1+z_2}{(1+r_{e,0})(1+r_{e,1})} + ..\right]. \end{aligned}$$

This gives the result in the text.

A.3 Proof of the link between residual earnings and abnormal earnings growth under time-varying discount rate

This section proves that $re_t = v_t - r_{e,t-1} \cdot b_{-1}$ where re_t are residual earnings produced over the period $[t-1, t]$ and b_{-1} is the book value of equity at time $t = -1$. Given that $z_{t+1} = x_{t+1} - x_t - \left(r_{e,t} \cdot (x_t - d_t) + (r_{e,t} - r_{e,t-1}) \cdot \sum_{i=0}^{t-1}(x_i - d_i) \cdot 1_{t\geq 1}\right)$, given the definition of residual earnings $re_t = x_{t+1} - r_{e,t} \cdot b_t$ and given that $v_{t+1} = v_t + z_{t+1}$ with $v_0 = x_0$, it follows that

$$\begin{aligned} &z_1 = x_1 - x_0 - r_{e,0} \cdot (x_0 - d_0) \\ &z_2 = x_2 - x_1 - r_{e,1} \cdot (x_1 - d_1) - (r_{e,1} - r_{e,0}) \cdot (x_0 - d_0) \\ &z_3 = x_3 - x_2 - (r_{e,2} \cdot (x_2 - d_2) + (r_{e,2} - r_{e,1}) \cdot (x_0 + x_1 - d_0 - d_1)) \\ &v_1 = x_0 + x_1 - x_0 - r_{e,0} \cdot (x_0 - d_0) \\ &v_1 - r_{e,0} \cdot b_{-1} = x_0 + x_1 - x_0 - r_{e,0} \cdot (x_0 - d_0) - r_{e,0} \cdot b_{-1} = x_1 - r_{e,0} \cdot b_0 = re_1 \\ &v_2 = x_1 - r_{e,0} \cdot (x_0 - d_0) + x_2 - x_1 - r_{e,1} \cdot (x_1 - d_1) - (r_{e,1} - r_{e,0}) \cdot (x_0 - d_0) = \\ &= x_2 - r_{e,1} \cdot (x_0 + x_1 - d_0 - d_1) \\ &v_2 - r_{e,1} \cdot b_{-1} = x_2 - r_{e,1} \cdot (x_0 + x_1 - d_0 - d_1) - r_{e,1} \cdot b_{-1} = x_2 - r_{e,1} \cdot b_1 = re_2 \\ &v_3 = x_2 - r_{e,1} \cdot (x_0 + x_1 - d_0 - d_1) + x_3 - x_2 - r_{e,2} \cdot (x_2 - d_2) + \\ &- (r_{e,2} - r_{e,1}) \cdot (x_0 + x_1 - d_0 - d_1) = x_3 - r_{e,2} \cdot (x_0 + x_1 + x_2 - d_0 - d_1 - d_2) \\ &v_3 - r_{e,2} \cdot b_{-1} = x_3 - r_{e,2} \cdot (x_0 + x_1 + x_2 - d_0 - d_1 - d_2) - r_{e,2} \cdot b_{-1} = x_3 - r_{e,2} \cdot b_2 = re_3 \end{aligned}$$

and so on for later time periods. This proves that $re_t = v_t - r_{e,t-1} \cdot b_{-1}$.

A.4 Discounting operating income under time-varying discount rates

This **Appendix** presents a firm valuation model that parallels the equity valuation above. We employ the following notation: V_t^{noa} is the intrinsic value of net operating assets, assumed to coincide with the fair value of the firm; $r_{f,t}$ is the cost of capital of the firm during $[t, t+1]$; c_{t+1} and i_{t+1} are respectively net cash flow from operations and net cash investment in fixed operating assets during the period $[t, t+1]$, so that $c_{t+1} - i_{t+1}$ summarises free cash flows; oi_{t+1} is operating income produced during the period $[t, t+1]$. Absent arbitrage, according to discounted free cash flow valuation the value of firm's net operating assets can be written as

$$V_t^{noa} = E_t\left[\frac{V_{t+1}^{noa} + c_{t+1} - i_{t+1}}{1 + r_{f,t}}\right] = E_t\left[\frac{oi_{t+1} + V_{t+1}^{noa} - oi_{t+1} + c_{t+1} - i_{t+1}}{1 + r_{f,t}}\right]$$

for all t. Then manipulations of this formula similar to those that led to Proposition 1 above give in the following corollary.

Corollary 3 *When the discount rate varies over time, discounted free cash flow valuation of the firm can re-written as the following discounted operating income valuation*

$$V_0^{noa} = E_0\left[\sum_{t=1}^{T} \frac{oi_t}{\prod_{i=0}^{t-1}(1 + r_{f,i})} - \sum_{t=2}^{T} \frac{r_{f,t-1} \cdot \sum_{i=1}^{t-1}(oi_i - (c_i - i_i))}{\prod_{i=0}^{t-1}(1 + r_{f,i})} + \frac{V_T^{noa} - \sum_{i=1}^{T}(oi_i - (c_i - i_i))}{\prod_{i=0}^{T-1}(1 + r_{f,i})}\right] \tag{21}$$

$$= E_0\left[\sum_{t=1}^{\infty} \frac{oi_t}{\prod_{i=0}^{t-1}(1 + r_{f,i})} - \sum_{t=2}^{\infty} \frac{r_{f,t-1} \cdot \sum_{i=1}^{t-1}(oi_i - (c_i - i_i))}{\prod_{i=0}^{t-1}(1 + r_{f,i})}\right].$$

As before $E_0[..]$ is the time 0 conditional expectation operator. The intuition of this formula is that the operating income oi_t to be produced by the firm adds value to the firm only after it has rewarded, according to the cost of capital $r_{f,t-1}$, the investment in net operating assets measured by $\sum_{i=1}^{t-1}(oi_i - (c_i - i_i))$ to be made by re-investing operating income. $oi_i - (c_i - i_i)$ is the part of operating income that is re-invested in the firm's operations, while $(c_i - i_i)$ is "free" cash flow to be paid out to the firm's financing activities. Proposition 1 does not rely on any accounting identity such as the clean surplus relation. Similarly Corollary 3 does not rely on any accounting identity. Proposition 1 is applicable on a per share basis. Also Corollary 3 is applicable on a per-share basis to determine firm value per share. No forecast beyond time T is needed when $V_T^{noa} = \sum_{i=1}^{T}(oi_i - (c_i - i_i))$. Again T is the end of the forecast horizon.

The abnormal operating income growth (AOIG) valuation of Ohlson and Gao (2006) is a special case of formula 21 when $r_{f,t} = r_f$ for all t, since

$$\begin{aligned} V_0^{noa} &= \sum_{t=1}^{\infty} \frac{oi_1 + (oi_t - oi_1)}{(1 + r_f)^t} - \sum_{t=2}^{\infty} \frac{r_f \cdot \sum_{i=1}^{t-1}(oi_i - (c_i - i_i))}{(1 + r_f)^t} \\ &= \frac{oi_1}{r_f} + \frac{1}{r_f}\sum_{t=1}^{\infty} \frac{oi_{t+1} - oi_t - r_f \cdot (oi_t - (c_t - i_t))}{(1 + r_f)^t}. \end{aligned}$$

The first line of this equation is equation 21, while the second line is AOIG valuation. AOIG valuation discounts, at the constant cost of capital r_f, streams of operating income and operating income increments that stretch into the infinite future. Instead Corollary 3 only requires a finite forecast horizon, provides full freedom in computing continuation value at time T and accommodates discount rates that change over time. To compute V_T^{noa} we can employ any valuation model that is consistent with discounted free cash flow valuation.

Formula 21 only depends on operating income and free cash flow. In particular this formula does not even depend on the accounting identity

$$c_{t+1} - i_{t+1} = oi_{t+1} - (noa_{t+1} - noa_t), \tag{22}$$

which is the clean surplus relation for a firm with no financial assets and no financial liabilities. noa_t denotes the book value of net operating assets at time t, with $noa_t = oa_t - ol_t$ where oa_t and ol_t are respectively the time t book values of operating assets and operating liabilities. Identity 22 may not always hold, for example when the accounting standards envisage that fixed assets may be revalued without affecting the income statement, but only the balance sheet. When identity 22 holds for all t, so that $noa_T = \sum_{i=1}^{T} (oi_i - (c_i - i_i)) + noa_0$, and when $V_T^{noa} = noa_T$, then

$$V_T^{noa} - \sum_{i=1}^{T} (oi_i - (c_i - i_i)) = noa_0.$$

In this case continuation value equals noa_0 and requires no forecast, because noa_0 can be observed at the time of the valuation. For this result to hold on a per share basis, identity 22 needs to hold on a per share basis.

The advantages of valuation as per Corollary 3 over AOIG valuation mirror those of valuation as per Proposition 1 over AEG valuation. Appendix A.5 discusses the definition of abnormal operating income growth (AOIG) when the cost of capital is not constant over time. Appendix A.6 discusses how Proposition 1 and Corollary 3 are **linked.**

A.5 The definition of abnormal operating income growth (AOIG) when the cost of capital is not constant over time

When the cost of capital for the firm is constant over time

$$z_{o,t+1} = oi_{t+1} - oi_t - r_f \cdot (oi_t - (c_t - i_t))$$

where $z_{o,t+1}$ denotes abnormal operating income growth over the period $[t, t+1]$ and where $r_f \cdot (oi_t - (c_t - i_t))$ denotes the change in "required" operating income for the same period $[t, t+1]$ to remunerate the change of investment in operations $(oi_t - (c_t - i_t))$ of the previous period $[t-1, t]$. However when the cost of capital is not constant over time, the change in "required" operating income over the period $[t, t+1]$ becomes

$$r_{f,t} \cdot (oi_t - (c_t - i_t)) + (r_{f,t} - r_{f,t-1}) \cdot \sum_{i=0}^{t-1} (oi_i - (c_i - i_i)) \cdot 1_{t \geq 1}$$

where again $t = 0$ is the time of the valuation. $1_{t \geq 1}$ is the indicator function of the condition $t \geq 1$. The term $(r_{f,t} - r_{f,t-1}) \cdot \sum_{i=0}^{t-1} (oi_i - (c_i - i_i)) \cdot 1_{t \geq 1}$ is the change in required operating income due to the change in the cost of capital from $r_{f,t-1}$ to $r_{f,t}$ multiplied by the total increase in capital invested in the operations over the period $[0, t-1]$ as measured by $\sum_{i=0}^{t-1} (oi_i - (c_i - i_i)) \cdot 1_{t \geq 1}$. Then when the cost of capital varies over time, AOIG over the period $[t, t+1]$ can be defined as

$$z_{o,t+1} = oi_{t+1} - oi_t - r_{f,t} \cdot (oi_t - (c_t - i_t)) + (r_{f,t} - r_{f,t-1}) \cdot \sum_{i=0}^{t-1} (oi_i - (c_i - i_i)) \cdot 1_{t \geq 1}. \tag{23}$$

This equation defines AOIG as the difference between the actual change in operating income and the change in required operating income during $[t, t+1]$. Note that $z_{o,1} = oi_1 - oi_0 - r_{f,0} \cdot (oi_0 - (c_0 - i_0))$. Then equation 21 can be re-written as

$$V_0^{noa} = E_0 \left[\sum_{t=1}^{\infty} \frac{oi_0 + \sum_{i=1}^{t} z_{o,i}}{\prod_{i=0}^{t-1} (1 + r_{f,i})} \right] = E_0 \left[\sum_{t=1}^{\infty} \frac{v_{o,t}}{\prod_{i=0}^{t-1} (1 + r_{f,i})} \right]$$

with $v_{o,t} = v_{o,t-1} + z_{o,t}$ and $v_{o,0} = oi_0$. **It** can be shown that

$$ro_t = oi_t - r_{f,t-1} \cdot noa_{t-1} = v_{o,t} - r_{f,t-1} \cdot noa_{-1} \tag{24}$$

where ro_t is residual operating income produced over the period $[t-1, t]$ and noa_{-1} is the book value of equity at time $t = -1$. Equation 24 implies that

$$ro_t - ro_{t-1} = z_{o,t} - (r_{f,t-1} - r_{f,t-2}) \cdot noa_{-1}.$$

Therefore abnormal operating income growth $z_{o,t}$, defined in equation 23 as the difference between the actual change in operating income and the change in required operating income for the same period, differs from the change in residual operating income $ro_t - ro_{t-1}$ whenever $r_{f,t-1} \neq r_{f,t-2}$. Instead when the cost of capital is constant over time abnormal operating income growth coincides with the change in residual operating income.

A.6 Articulating Proposition 1 and Corollary 3

We have derived Proposition 1 as a re-writing of dividend discount valuation, with no additional assumptions. We have also derived Corollary 3 as a re-writing of discounted free cash flow valuation, with no additional assumptions. Therefore, in order for both Proposition 1 and Corollary 3 to be true at the same time, we only need to impose the necessary assumptions for dividend discount valuation and discounted free cash flow valuation to be true at the same time in such a way that

$$V_t^e = V_t^{noa} - V_t^{nfo}$$

where V_t^{nfo} is the fair value value of net financial obligations. These necessary assumptions are the cash flow statement identity

$$c_{t+1} - i_{t+1} = F_{t+1} + d_{t+1}$$

and a "time-dependent" version of Modigliani and Miller's Proposition II, i.e.

$$r_{e,t} = r_{f,t} + \frac{V_t^{nfo}}{V_t^{noa} - V_t^{nfo}} \left(r_{f,t} - r_{d,t}\right). \tag{25}$$

F_{t+1} is the net cash flow to and from borrowers and lenders during $[t, t+1]$ and $r_{d,t}$ is the cost of capital of net financial obligations during $[t, t+1]$. $F_{t+1} = -\left(nfo_{t+1} - nfo_t\right) + nfe_{t+1}$ if additionally we assume that the reformulated financial statements articulate for all t so that

$$\begin{aligned}
&b_t = noa_t - nfo_t \\
&noa_t = oa_t - ol_t, \;\; nfo_t = fo_t - fa_t \\
&oi_{t+1} - nfe_{t+1} = x_{t+1}
\end{aligned}$$

where: oa_t and ol_t are the time t book values of operating assets and operating liabilities, fa_t and fo_t are the time t book values of financial assets and financial obligations, noa_t and nfo_t are net operating assets and net financial obligations at time t on the balance sheet, b_t is the time t book value of common shareholders' equity, nfe_{t+1} is net financial expense incurred during $[t, t+1]$. In passing we notice that these assumptions also imply that

$$\begin{aligned}
V_t^{nfo} &= E_t\left[\frac{nfe_{t+1} - nfo_{t+1} + nfo_t + V_{t+1}^{nfo}}{1 + r_{d,t}}\right] \\
&= E_0\left[\sum_{t=1}^{\infty} \frac{nfe_t}{\prod_{i=0}^{t-1}\left(1 + r_{d,i}\right)} - \sum_{t=2}^{\infty} \frac{r_{d,t-1} \cdot \sum_{i=1}^{t-1}\left(nfe_i - F_i\right)}{\prod_{i=0}^{t-1}\left(1 + r_{d,i}\right)}\right].
\end{aligned}$$

The first line is a way to write the discounted cash flows valuation of net financial obligations, while second line is just a re-writing of the first line, which is similar in spirit to Proposition 1, but is applied to the valuation of net financial obligations. However in this paper we assume that $V_t^{nfo} = nfo_t$ for all t, which is also a common assumption in practice.

A.7 Valuation under multi-factor quadratic term structure models

$V_{0,m}^{v}$ is given by equations 7 and 13 and therefore it can be shown to satisfy

$$\left(d_m + \mathbf{D}'_m \mathbf{x}_0\right) e^{A_m + \mathbf{B}'_m \mathbf{y}_0 + \mathbf{y}'_0 \mathbf{C}_m \mathbf{y}_0} = E_0^{\mathbb{Q}} \left[e^{-\alpha - \boldsymbol{\beta}' \mathbf{y}_0 - \mathbf{y}'_0 \Psi \mathbf{y}_0} \left(d_{m-1} + \mathbf{D}'_{m-1} \mathbf{x}_1\right) e^{A_{m-1} + \mathbf{B}'_{m-1} \mathbf{y}_1 + \mathbf{y}'_1 \mathbf{C}_{m-1} \mathbf{y}_1} \right]. \tag{26}$$

Then we can rewrite equation 26 as

$$\ln\left(d_m + \mathbf{D}'_m \mathbf{x}_0\right) + A_m + \mathbf{B}'_m \mathbf{y}_0 + \mathbf{y}'_0 \mathbf{C}_m \mathbf{y}_0 = -\alpha - \boldsymbol{\beta}' \mathbf{y}_0 - \mathbf{y}'_0 \Psi \mathbf{y}_0 + A_{m-1} + \tag{27}$$

$$+ \mathbf{B}'_{m-1}\left(\left(\mathbf{I}_3 - \boldsymbol{\Phi}_y\right)\mathbf{y}_0 + \boldsymbol{\Phi}_y \boldsymbol{\mu}_y\right) + \left(\left(\mathbf{I}_3 - \boldsymbol{\Phi}_y\right)\mathbf{y}_0 + \boldsymbol{\Phi}_y \boldsymbol{\mu}_y\right)' \mathbf{C}_{m-1}\left(\left(\mathbf{I}_3 - \boldsymbol{\Phi}_y\right)\mathbf{y}_0 + \boldsymbol{\Phi}_y \boldsymbol{\mu}_y\right)$$
$$+ \ln E_0^{\mathbb{Q}}\left[\left(d_{m-1} + \mathbf{D}'_{m-1}\mathbf{x}_1\right) \cdot e^{(\mathbf{B}_{m-1} + \mathbf{U})' \boldsymbol{\Sigma}_y \boldsymbol{\xi}_{y,1} + \boldsymbol{\xi}'_{y,1} \boldsymbol{\Sigma}'_y \mathbf{C}_{m-1} \boldsymbol{\Sigma}_y \boldsymbol{\xi}_{y,1}}\right]$$

where sub-Appendix A.7.1 shows that

$$\ln E_0^{\mathbb{Q}}\left[\left(d_{m-1} + \mathbf{D}'_{m-1}\left(\left(\mathbf{I}_N - \boldsymbol{\Phi}\right)\mathbf{x}_0 + \boldsymbol{\eta} + \boldsymbol{\Sigma}\boldsymbol{\xi}_1\right)\right) e^{(\mathbf{B}_{m-1} + \mathbf{U})' \boldsymbol{\Sigma}_y \boldsymbol{\xi}_{y,1} + \boldsymbol{\xi}'_{y,1} \boldsymbol{\Sigma}'_y \mathbf{C}_{m-1} \boldsymbol{\Sigma}_y \boldsymbol{\xi}_{y,1}}\right] = \tag{28}$$

$$= \ln\left(\left(d_{m-1} + \mathbf{D}'_{m-1}\left(\left(\mathbf{I}_N - \boldsymbol{\Phi}\right)\mathbf{x}_0 + \boldsymbol{\eta}\right)\right) + \Sigma_{i=1}^{3} \mathbf{K}'_{m-1} \boldsymbol{\gamma}_i \left(\mathbf{B}_{m-1} + \mathbf{U}\right)' \boldsymbol{\gamma}_i\right) +$$
$$+ \ln \frac{|\boldsymbol{\gamma}|}{abs\left|\boldsymbol{\Sigma}_y\right|} + \frac{1}{2} \sum_{i=1}^{3} \left(\left(\mathbf{B}_{m-1} + \mathbf{U}\right)' \boldsymbol{\gamma}_i\right)^2$$

with γ_i being the i-th column of the $3{\times}3$ matrix $\gamma = \left(\left(\boldsymbol{\Sigma}_y \boldsymbol{\Sigma}'_y\right)^{-1} - 2\mathbf{C}_{m-1}\right)^{-1/2}$ and with

$$\mathbf{K}'_{m-1} = \mathbf{D}^{y\prime}_{m-1} + \left(D^{z}_{m-1}\sigma_z\left(\rho_{z,1}, \rho_{z,2}, \rho_{z,3}\right) + \sum_{j=1}^{n} D^{h_j}_{m-1}\sigma_{h_j}\left(\rho_{h_j,1}, \rho_{h_j,2}, \rho_{h_j,3}\right)\right) \boldsymbol{\Sigma}_y^{-1}$$
$$\mathbf{U}' = 2\mathbf{y}'_0\left(\mathbf{I}_3 - \boldsymbol{\Phi}_y\right)' \mathbf{C}_{m-1} + 2\left(\boldsymbol{\Phi}_y \boldsymbol{\mu}_y\right)' \mathbf{C}_{m-1}.$$

A.7.1 Intermediate result

Equation 28 is derived as follows. We define $w = \Sigma_y \xi_{y,t+1}$ and notice that $w \sim N\left(\mathbf{0}_{3\times1}, \boldsymbol{\Sigma}_y \boldsymbol{\Sigma}'_y\right)$, where w is a 3×1 vector. Then we set $a = \mathbf{B}_{m-1} + \mathbf{U}$, $k_{m-1} = d_{m-1} + \mathbf{D}'_{m-1}\left(\left(\mathbf{I}_N - \phi\right)\mathbf{x}_0 + \boldsymbol{\eta}\right)$ and notice that

$$\begin{aligned}
\mathbf{D}'_{m-1}\boldsymbol{\Sigma}\boldsymbol{\xi}_{t+1} &= \mathbf{D}^{y\prime}_{m-1} w + \left(D^{z}_{m-1}\sigma_z\left(\rho_{z,1}, \rho_{z,2}, \rho_{z,3}\right) + \sum_{j=1}^{n} D^{h_j}_{m-1}\sigma_{h_j}\left(\rho_{h_j,1}, \rho_{h_j,2}, \rho_{h_j,3}\right)\right) \boldsymbol{\xi}_{y,t+1} + .. \\
&= \mathbf{K}'_{m-1} w + ..
\end{aligned}$$

where the dots .. denote elements we need not consider to our purposes and

$\mathbf{K}'_{m-1} = \mathbf{D}^{y\prime}_{m-1} + \Big(D^z_{m-1}\sigma_z\left(\rho_{z,1},\rho_{z,2},\rho_{z,3}\right) + \sum_{j=1}^{n} D^{h_j}_{m-1}\sigma_{h_j}\left(\rho_{h_j,1},\rho_{h_j,2},\rho_{h_j,3}\right)\Big)\,\boldsymbol{\Sigma}_y^{-1}.$

Then

$$
\begin{aligned}
& E_t^{\mathbb{Q}}\left[\left(k_{m-1} + \mathbf{D}'_{m-1}\boldsymbol{\Sigma}\boldsymbol{\xi}_{t+1}\right)\cdot e^{(\mathbf{B}_{m-1}+\mathbf{U})'\boldsymbol{\Sigma}_y\boldsymbol{\xi}_{y,t+1}+\boldsymbol{\xi}'_{y,t+1}\boldsymbol{\Sigma}'_y\mathbf{C}_{m-1}\boldsymbol{\Sigma}_y\boldsymbol{\xi}_{y,t+1}}\right] \\
&= \frac{1}{\sqrt{(2\pi)^3}}\int\left(k_{m-1} + \mathbf{K}'_{m-1}w\right)e^{-\frac{1}{2}\boldsymbol{\xi}'_{y,t+1}\boldsymbol{\xi}_{y,t+1}+a'w+w'\mathbf{C}_{m-1}w}\,d\boldsymbol{\xi}_{y,t+1} \\
&= \frac{1}{\sqrt{(2\pi)^3}\,abs\left|\boldsymbol{\Sigma}_y\right|}\int\left(k_{m-1} + \mathbf{K}'_{m-1}w\right)e^{-\frac{1}{2}w'\left(\boldsymbol{\Sigma}_y\boldsymbol{\Sigma}'_y\right)^{-1}w+a'w+w'\mathbf{C}_{m-1}w}\,dw \\
&= E_t^{\mathbb{Q}}\left[\left(k_{m-1} + \mathbf{K}'_{m-1}w\right)e^{a'w+w'\mathbf{C}_{m-1}w}\right]
\end{aligned}
$$

where we have made the substitutions $\boldsymbol{\xi}_{y,t+1} = \Sigma_y^{-1}w$ and $d\boldsymbol{\xi}_{y,t+1} = abs\left|\boldsymbol{\Sigma}_y^{-1}\right|dw$ and where $abs\left|\boldsymbol{\Sigma}_y^{-1}\right|$ denotes the absolute value of the determinant of Σ_y^{-1}. Then $\left(\left(\boldsymbol{\Sigma}_y\boldsymbol{\Sigma}'_y\right)^{-1} - 2\mathbf{C}_{m-1}\right)$ is positive semi-definite and symmetric. This is the case since $\Sigma_y\Sigma'_y$ is symmetric and positive semi-definite and so is $\left(\boldsymbol{\Sigma}_y\boldsymbol{\Sigma}'_y\right)^{-1}$. Then $\mathbf{C}_{m-1}$ is assumed symmetric and negative definite for our purposes without loss in generality**;** then $\gamma = \left(\left(\boldsymbol{\Sigma}_y\boldsymbol{\Sigma}'_y\right)^{-1} - 2\mathbf{C}_{m-1}\right)^{-1/2}$ exists and is symmetric. Then we can write the following

$$
\begin{aligned}
& -\frac{1}{2}w'\left(\boldsymbol{\Sigma}_y\boldsymbol{\Sigma}'_y\right)^{-1}w + a'w + w'\mathbf{C}_{m-1}w \\
&= -\frac{1}{2}w'\left(\left(\boldsymbol{\Sigma}_y\boldsymbol{\Sigma}'_y\right)^{-1} - 2\mathbf{C}_{m-1}\right)w + a'w = -\frac{1}{2}w'\boldsymbol{\gamma}^{-2}w + a'w \\
&= -\frac{1}{2}\left(\boldsymbol{\gamma}^{-1}w\right)'\boldsymbol{\gamma}^{-1}w + a'w = -\frac{1}{2}v'v + a'\boldsymbol{\gamma}v
\end{aligned}
$$

where $v = \gamma^{-1}w$. Hence, if γ is of full rank, it follows that the differential dw is such that

$$dw = abs\left|\boldsymbol{\gamma}\right|dv = \left|\boldsymbol{\gamma}\right|dv$$

where $abs\,|\boldsymbol{\gamma}|$ is the absolute value of $|\boldsymbol{\gamma}|$ and $abs\,|\boldsymbol{\gamma}| = |\boldsymbol{\gamma}|$ since γ is non-negative definite. At this point we can write

$$\frac{1}{abs\,|\boldsymbol{\Sigma}_y|\sqrt{(2\pi)^3}}\int\left(k_{m-1}+\mathbf{K}'_{m-1}w\right)e^{-\frac{1}{2}w'\left(\boldsymbol{\Sigma}_y\boldsymbol{\Sigma}'_y\right)^{-1}w+a'w+w'\mathbf{C}_{m-1}w}dw$$
$$= \frac{1}{abs\,|\boldsymbol{\Sigma}_y|\sqrt{(2\pi)^3}}\int\left(k_{m-1}+\mathbf{K}'_{m-1}\boldsymbol{\gamma}v\right)e^{-\frac{1}{2}v'v+a'\boldsymbol{\gamma}v}\,|\boldsymbol{\gamma}|\,dv$$
$$= \frac{|\boldsymbol{\gamma}|}{abs\,|\boldsymbol{\Sigma}_y|\sqrt{(2\pi)^3}}\int\left(k_{m-1}+\mathbf{K}'_{m-1}\boldsymbol{\gamma}v\right)\cdot\prod_{i=1}^3 e^{-\frac{v_i^2}{2}+a'\boldsymbol{\gamma}_iv_i}dv_i$$
$$= \frac{|\boldsymbol{\gamma}|}{abs\,|\boldsymbol{\Sigma}_y|}\left(k_{m-1}+\sum_{i=1}^3\mathbf{K}'_{m-1}\boldsymbol{\gamma}_ia'\boldsymbol{\gamma}_i\right)\cdot\prod_{i=1}^3\cdot e^{\frac{(a'\boldsymbol{\gamma}_i)^2}{2}}$$

where γ_i denotes the i-th column of γ, and substituting for $a = \mathbf{B}_{m-1}+\mathbf{U}$ into the last line we get equation 28. We notice that the last line makes use of the fact that

$$\frac{1}{\sqrt{2\pi}}\int e^{\left(-\frac{u^2}{2}+au\right)}du = \frac{1}{\sqrt{2\pi}}e^{\frac{a^2}{2}}\int e^{-\frac{(u-a)^2}{2}}du = e^{\frac{a^2}{2}}$$

and

$$\frac{1}{\sqrt{(2\pi)^3}}\int\mathbf{K}'_{m-1}\left(\boldsymbol{\gamma}_1v_1+\boldsymbol{\gamma}_2v_2+\boldsymbol{\gamma}_3v_3\right)\cdot e^{-\frac{v_1^2}{2}+a'\boldsymbol{\gamma}_1v_1-\frac{v_2^2}{2}+a'\boldsymbol{\gamma}_2v_2-\frac{v_3^2}{2}+a'\boldsymbol{\gamma}_3v_3}dv_1dv_2dv_3 =$$
$$= \mathbf{K}'_{m-1}\left(\boldsymbol{\gamma}_1a'\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2a'\boldsymbol{\gamma}_2+\boldsymbol{\gamma}_3a'\boldsymbol{\gamma}_3\right)\cdot e^{\frac{(a'\boldsymbol{\gamma}_1)^2+(a'\boldsymbol{\gamma}_2)^2+(a'\boldsymbol{\gamma}_3)^2}{2}}.$$

A.7.2 Final result

Equations 27 and 28 imply the following equation

$$\begin{aligned} d_m+\mathbf{D}'_m\mathbf{x}_0 &= \left(d_{m-1}+\mathbf{D}'_{m-1}\left(\left(\mathbf{I}_N-\phi\right)\mathbf{x}_0+\boldsymbol{\eta}\right)\right)+ \\ &\quad +\boldsymbol{\Sigma}_{i=1}^N\mathbf{K}'_{m-1}\boldsymbol{\gamma}_i\left(\mathbf{B}'_{m-1}+2\mathbf{y}'_0\left(\mathbf{I}_3-\phi_y\right)'\mathbf{C}_{m-1}+2\left(\phi_y\boldsymbol{\mu}_y\right)'\mathbf{C}_{m-1}\right)\boldsymbol{\gamma}_i\end{aligned}$$

which can be re-written as

$$\begin{aligned} & d_m+\left(\mathbf{D}_m^{y\prime},D_m^v,D_m^z,\mathbf{D}_m^{h\prime}\right)\left(\mathbf{y}'_0,v_0,z_0,\mathbf{h}'_0\right)' = \\ = \quad & d_{m-1}+\begin{pmatrix}\mathbf{D}_{m-1}^{y\prime}\\ D_{m-1}^v\\ D_{m-1}^z\\ \mathbf{D}_{m-1}^{h\prime}\end{pmatrix}'\left(\begin{bmatrix}\mathbf{I}_3-\phi_y & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times n}\\ \mathbf{0}_{1\times3} & 1 & (1-g) & g\mathbf{m}'_z\\ \mathbf{0}_{1\times3} & 0 & 1-g & g\mathbf{m}'_z\\ \mathbf{0}_{n\times3} & \mathbf{0}_{n\times1} & \mathbf{0}_{n\times1} & \mathbf{I}_n-\mathbf{G}_h\end{bmatrix}\begin{pmatrix}\mathbf{y}_0\\ v_0\\ z_0\\ \mathbf{h}_0\end{pmatrix}+\begin{pmatrix}\phi_y\boldsymbol{\mu}_y\\ 0\\ 0\\ \mathbf{G}_h\mathbf{m}_h\end{pmatrix}\right)+ \\ & +\boldsymbol{\Sigma}_{i=1}^3\mathbf{K}'_{m-1}\boldsymbol{\gamma}_i\left(\mathbf{B}'_{m-1}+2\mathbf{y}'_0\left(\mathbf{I}_3-\phi_y\right)'\mathbf{C}_{m-1}+2\left(\phi_y\boldsymbol{\mu}_y\right)'\mathbf{C}_{m-1}\right)\boldsymbol{\gamma}_i\end{aligned}$$

implying that

$$
\begin{aligned}
&d_m = d_{m-1} + \mathbf{D}^{y\prime}_{m-1}\phi_y\boldsymbol{\mu}_y + \mathbf{D}^{h\prime}_{m-1}\mathbf{G}_h\mathbf{m}_h + \textstyle\sum_{i=1}^{3}\mathbf{K}'_{m-1}\gamma_i\left(\mathbf{B}'_{m-1} + 2\left(\phi_y\boldsymbol{\mu}_y\right)'\mathbf{C}_{m-1}\right)\gamma_i \\
&\mathbf{D}^{y\prime}_m\mathbf{y}_0 = \mathbf{D}^{y\prime}_{m-1}\left(\mathbf{I}_3 - \phi_y\right)\mathbf{y}_0 + 2\cdot\boldsymbol{\Sigma}_{i=1}^{3}\mathbf{K}'_{m-1}\gamma_i\mathbf{y}'_0\left(\mathbf{I}_3 - \phi_y\right)'\mathbf{C}_{m-1}\gamma_i \\
&D^v_m v_0 = D^v_{m-1}v_0 \\
&D^z_m z_0 = \left(D^z_{m-1} + D^v_{m-1}\right)(1-g)\,z_0 \\
&\mathbf{D}^{h\prime}_m h_0 = \left(\mathbf{D}^{h\prime}_{m-1}\left(\mathbf{I}_n - \mathbf{G}_h\right) + \left(D^v_{m-1} + D^z_{m-1}\right)g\cdot\mathbf{m}'_z\right)h_0.
\end{aligned}
$$

Recalling the result $\boldsymbol{\gamma\gamma}' = \sum_{i=1}^{3}\gamma_i\gamma'_i$ gives the **Riccati** equations in the text.

A.8 Valuation under affine Gaussian term structure models

Other things as above, when $\alpha = 0, \boldsymbol{\Psi} = \mathbf{0}_{3\times 3}$ in equation 11, then

$$r_t = \boldsymbol{\beta}'\cdot\mathbf{y}_t \tag{29}$$

and $\mathbf{C}_m = \mathbf{0}_{3\times 3}$ for all m, so that the price of a discount bond reduces to

$$
\begin{aligned}
&P_{0,m} = e^{A_m + \mathbf{B}'_m\mathbf{y}_0} && (30)\\
&A_m = A_{m-1} + \mathbf{B}'_{m-1}\boldsymbol{\Phi}_y\boldsymbol{\mu}_y + \frac{1}{2}\cdot\mathbf{B}_{m-1}\boldsymbol{\Sigma}_y\boldsymbol{\Sigma}'_{\mathbf{y}}\mathbf{B}'_{m-1} && (31)\\
&\mathbf{B}_m = -\boldsymbol{\beta} + \left(\mathbf{I}_3 - \boldsymbol{\Phi}_y\right)'\mathbf{B}_{m-1} && (32)\\
&A_0 = 0, \quad \mathbf{B}_0 = \mathbf{0}_{3\times 1}.
\end{aligned}
$$

Then under assumption 29 equity value in Proposition 2 reduced to

$$V^e_0 = \textstyle\sum_{m=1}^{\infty}\left(d_m + \underline{\mathbf{D}}'_m\underline{\mathbf{x}}_0\right)\cdot e^{A_m + \mathbf{B}'_m\mathbf{y}_0} \tag{33}$$

where A_m and B'_m satisfy equations 31 and 32, $\mathbf{D}^y_m = \mathbf{0}_{3\times 1}$ for all m, and

$$
\begin{aligned}
&\underline{\mathbf{x}}_0 = \left(v_0, z_0, \mathbf{h}'_0\right)' \\
&d_m = d_{m-1} + \underline{\mathbf{D}}'_{m-1}\mathbf{H}\boldsymbol{\Sigma}'_y\mathbf{B}_{m-1} + \mathbf{D}^{h\prime}_{m-1}\mathbf{G}_h\mathbf{m}_h \\
&\underline{\mathbf{D}}'_m = \left(D^v_m, D^z_m, \mathbf{D}^{h\prime}_m\right) \\
&D^v_m = D^v_{m-1} \\
&D^z_m = \left(D^z_{m-1} + D^v_{m-1}\right)(1-g) \\
&\mathbf{D}^{h\prime}_m = \mathbf{D}^{h\prime}_{m-1}\left(\mathbf{I}_n - \mathbf{G}_h\right) + \left(D^v_{m-1} + D^z_{m-1}\right)g\cdot\mathbf{m}'_z \\
&d_0 = 0,\ D^v_0 = 1,\ D^z_0 = 0,\ \mathbf{D}^h_0 = \mathbf{0}_{n\times 1}.
\end{aligned}
$$

The terms $\left(d_m + \underline{\mathbf{D}}'_m\underline{\mathbf{x}}_0\right)e^{A_m + \mathbf{B}'_m\mathbf{y}_0}$ in equation 33 are special cases of $V^v_{0,m}$ in equation 13. Again $D^v_m = 1$ for all m**.** The term $\underline{\mathbf{D}}'_{m-1}\mathbf{H}\boldsymbol{\Sigma}'_y\mathbf{B}_{m-1}$ in the equation for d_m highlights that equity valuation depends on the correlation between the shocks to factors driving interest rates $\boldsymbol{\xi}_{y,t+1}$ and the shocks to factors driving abnormal earnings growth $\xi_{z,t+1}$ and $\boldsymbol{\xi}_{h,t+1}$. d_m also depends on the "volatility" of factors driving interest rates, as determined by the matrix $\boldsymbol{\Sigma}_y$, and the "volatility" of factors driving abnormal earnings growth, namely $\sigma_z, \sigma_{h_1}, .., \sigma_{h_n}$ that appear in matrix $\mathbf{H}$. **Since factors are latent**, we can impose the following restrictions to identify the parameters for the affine Gaussian term structure model of equations 30, 31, 32

$$\boldsymbol{\beta} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{\mu}_y = \begin{pmatrix} \mu_{y_1} \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{\Phi}_y = \begin{pmatrix} \phi_{y_1} & 0 & 0 \\ 0 & \phi_{y_2} & 0 \\ 0 & 0 & \phi_{y_3} \end{pmatrix}$$

All elements in these matrixes are scalar constants.

A.9 Parametric version of equation 1 with risk-neutral valuation and constant discount rate

No-arbitrage implies that

$$\begin{aligned} V_t^e &= \mathfrak{d} + \mathfrak{D}'\underline{\mathbf{x}}_t = E_t^{\mathbb{Q}}\left[\frac{v_{t+1} + V_{t+1}^e}{1+\mathfrak{r}}\right] = \frac{E_t^{\mathbb{Q}}\left[v_{t+1} + \mathfrak{d} + \mathfrak{D}'\underline{\mathbf{x}}_{t+1}\right]}{1+\mathfrak{r}} \\ &= \frac{\left(v_t + g\left(\mathbf{m}_z'\mathbf{h}_t\right) + (1-g)\, z_t + \mathfrak{d} + \mathfrak{D}' E_0^{\mathbb{Q}}\left[\underline{\mathbf{x}}_{t+1}\right]\right)}{1+\mathfrak{r}} \end{aligned}$$

giving

$$\mathfrak{d} + \mathfrak{D}' \begin{pmatrix} v_t \\ z_t \\ \mathbf{h}_t \end{pmatrix} = \frac{1}{1+\mathfrak{r}} \left(v_t + g\left(\mathbf{m}_z'\mathbf{h}_t\right) + (1-g)\, z_t + \mathfrak{d} + \mathfrak{D}' \begin{pmatrix} v_t + g\left(\mathbf{m}_z'\mathbf{h}_t\right) + (1-g)\, z_t \\ g\left(\mathbf{m}_z'\mathbf{h}_t\right) + (1-g)\, z_t \\ \mathbf{G}_h\mathbf{m}_h + \left(\mathbf{I}_n - \mathbf{G}_h\right)\mathbf{h}_t \end{pmatrix} \right).$$

Separating the variables in this equation we obtain

$$\begin{aligned} &\mathfrak{d} = \frac{1}{1+\mathfrak{r}}\left(\mathfrak{d} + \mathfrak{D}^{h\prime}\mathbf{G}_h\mathbf{m}_h\right) \\ &\mathfrak{D}^v \cdot v_t = \frac{1+\mathfrak{D}^v}{1+\mathfrak{r}} \cdot v_t \\ &\mathfrak{D}^z z_t = \frac{1-g}{1+\mathfrak{r}}\left(1 + \mathfrak{D}^v + \mathfrak{D}^z\right) z_t \\ &\mathfrak{D}^{h\prime} \cdot \mathbf{h}_t = \frac{\mathfrak{D}^{h\prime}\left(\mathbf{I}_n - \mathbf{G}_h\right) + \left(1 + \mathfrak{D}^v + \mathfrak{D}^z\right) g \cdot \mathbf{m}_z'}{1+\mathfrak{r}} \cdot \mathbf{h}_t. \end{aligned}$$

The solution to this system is

$$\begin{aligned}
\mathfrak{d} &= \frac{\mathfrak{D}^{h\prime}\mathbf{G}_h\mathbf{m}_h}{\mathfrak{r}} \\
\mathfrak{D}^v &= \frac{1}{\mathfrak{r}} \\
\mathfrak{D}^z &= \frac{1-g}{\mathfrak{r}+g}\left(1+\mathfrak{D}^v\right) \\
\mathfrak{D}^{h\prime} &= \left(1+\mathfrak{D}^v+\mathfrak{D}^z\right) g \cdot \mathbf{m}_z' \left(\mathfrak{r}\mathbf{I}_{n\times n}+\mathbf{G}_h\right)^{-1}
\end{aligned}$$

so that

$$V_t^e = \frac{1}{\mathfrak{r}}\left(\mathfrak{D}^{h\prime}\mathbf{G}_h\mathbf{m}_h + v_t + \frac{(1-g)(\mathfrak{r}+1)}{\mathfrak{r}+g}z_t + \frac{(\mathfrak{r}+1)^2}{\mathfrak{r}+g}g\cdot\mathbf{m}_z'\left(\mathfrak{r}\mathbf{I}_{n\times n}+\mathbf{G}_h\right)^{-1}\mathbf{h}_t\right).$$

A.10 Valuation under the Ang and Liu (2004) assumptions for the cost of equity capital

Proposition 2 gives an equity risk-neutral valuation formula, while most of the literature and practice rely on equity valuation under the real probability measure. The formula of Proposition 2 can be adapted to valuation under the real probability measure by dropping the assumption about the risk premia $\mathbf{\Lambda}_t = \mathbf{\Lambda}_0 + \mathbf{\Lambda}_1\mathbf{x}_t$ and by assuming instead the equity risk premium of Ang and Liu (2004). Then let $r_{e_c,t}$ denote the continuously compounded cost of equity capital, b_t^{capm} the CAPM beta and p_t^{market} the CAPM market risk premium over the period $[t, t+1]$. For simplicity of exposition we now assume

$$r_t = y_{1,t}, \qquad y_{2,t} = b_t^{capm}, \qquad y_{3,t} = p_t^{market}.$$

Therefore now the equity risk-premium is determined by CAPM, while the stock beta and the market risk premium both follow AR(1) processes. Then, other things equal, under these assumptions and under the real measure $\mathbb{P}$ the equity valuation formula of Proposition 2 becomes

$$\begin{aligned}
V_{0,m}^v &= E_0\left[v_t \cdot e^{-\sum_{i=0}^{t-1} r_{e_c,i}\cdot \mathbf{1}_{t\geq 1}}\right] \\
r_{e_c,t} &= y_{1,t} + y_{2,t}\cdot y_{3,t} = \alpha + \boldsymbol{\beta}'\mathbf{y}_t + \mathbf{y}_t'\mathbf{\Psi}\mathbf{y}_t \\
\mathbf{x}_{t+1} &= \left(\mathbf{I}_N - \mathbf{\Phi}^{\mathbb{P}}\right)\mathbf{x}_t + \boldsymbol{\eta}^{\mathbb{P}} + \mathbf{\Sigma}\boldsymbol{\xi}_{t+1}^{\mathbb{P}}, \quad \boldsymbol{\xi}_{t+1}^{\mathbb{P}} \backsim N\left(\mathbf{0}_{N\times 1}, \mathbf{I}_N\right) \\
\mathbf{x}_t &= \left(y_{1,t}, y_{2,t}, y_{3,t}, v_t, z_t, \mathbf{h}_t'\right)' \\
\mathbf{y}_{t+1} &= \left(\mathbf{I}_3 - \mathbf{\Phi}_y^{\mathbb{P}}\right)\mathbf{y}_t + \mathbf{\Phi}_y^{\mathbb{P}}\boldsymbol{\mu}_y^{\mathbb{P}} + \mathbf{\Sigma}_y\boldsymbol{\xi}_{y,t+1}^{\mathbb{P}}, \quad \boldsymbol{\xi}_{y,t+1}^{\mathbb{P}} \backsim N\left(\mathbf{0}_{3\times 1}, \mathbf{I}_3\right) \\
\mathbf{y}_t &= \left(y_{1,t}, y_{2,t}, y_{3,t}\right)' \\
V_{0,m}^v &= \left(d_m^{\mathbb{P}} + \left(\mathbf{D}_m^{\mathbb{P}}\right)'\mathbf{x}_0\right)\cdot e^{A_m^{\mathbb{P}} + \left(\mathbf{B}_m^{\mathbb{P}}\right)'\mathbf{y}_0 + \mathbf{y}_0'\mathbf{C}_m^{\mathbb{P}}\mathbf{y}_0}
\end{aligned}$$

with $\alpha = 0, \boldsymbol{\beta}' = (1,0,0), \mathbf{\Psi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$. Now $r_{e_c,t}$ effectively replaces r_t in the valuation of Proposition 2. $E_0[..]$ denotes time 0 conditional expectation under the real probability measure. The superscript $\mathbb{P}$ denotes parameters and variables under the real probability measure. $d_m^{\mathbb{P}}, \mathbf{D}_m^{\mathbb{P}}, A_m^{\mathbb{P}}, \mathbf{B}_m^{\mathbb{P}}, \mathbf{C}_m^{\mathbb{P}}$ can be determined in the same way as $d_m, \mathbf{D}_m, A_m, \mathbf{B}_m, \mathbf{C}_m$ once we replace $\mathbf{\Phi}, \boldsymbol{\eta}, \mathbf{\Phi}_y, \boldsymbol{\mu}_y$ with $\mathbf{\Phi}^{\mathbb{P}}, \boldsymbol{\eta}^{\mathbb{P}}, \mathbf{\Phi}_y^{\mathbb{P}}, \boldsymbol{\mu}_y^{\mathbb{P}}$. **The valuation formula of this Appendix** can also be extended to cases where the equity risk premium is determined by multiple systematic risk-factors in the spirit of the Arbitrage Pricing Theory, with factor loadings and factor risk premia following AR(1) processes.

Ang and Liu (2004) showed that valuations should account for stochastic interest rates and **stochastic** equity risk premia, but they discounted cash flows, not earnings. Instead this **Appendix** uses their stochastic interest rate and **stochastic** equity risk premia in valuations that discount earnings and abnormal earnings **growth**.

References

[1] Ahn D., Dittmar R. and Gallant R., 2002, "Quadratic term structure models: theory and evidence", The Review of financial studies 15, n.1, 243-288.

[2] Ang A. and Liu J., 2001, "A General Affine Earnings Valuation Model, Review of Accounting Studies, 6, 397–425.

[3] And A. and Liu J., 2004, "How to discount cash flows with time-varying expected returns", Journal of Finance LIX, n. 6, 2745-2783.

[4] Bakshi G. and Chen Z., 2005, "Stock valuation in dynamic economies", Journal of Financial Markets 8, 111–151.

[5] Brief R. P., 2007, "Accounting Valuation Models: A Short Primer", ABACUS, vol. 43, No. 4, 2007.

[6] D'Addona S. and Kind A., 2006, "International Stock-Bond Correlation in a Simple Affine Asset Pricing Model", Journal of Banking and Finance 30, n.10, 2747-2765.

[7] Dai Q. and Singleton K.J., 2000, "Specification analysis of affine term structure models", Journal of Finance 55, n. 5, 1943-1978.

[8] Easton, P. D., 'PE Ratios, PEG Ratios, and Estimating the Implied Expected Rate of Return on Equity Capital', The Accounting Review, Vol. 79, No. 1, 2004.

[9] Feltham G. and Ohlson J., 1995, "Valuation and clean surplus accounting for operating and financial activities", Contemporary Accounting research 11, Spring, 689-732.

[10] Feltham G. and Ohlson J., 1999, "Residual earnings valuation with risk and stochastic interest rates", The Accounting Review 74, n.2, 165-183.

[11] Gode D. and Ohlson J., 2004, "Accounting-based valuation with changing interest rates", Review of Accounting Studies 9, n. 4, 419 - 441.

[12] Hughes J, Liu Jing and Liu Jun, 2009, "On the relation between expected returns and implied cost of capital", Review of Accouting Studies 14, n.2, 246-259.

[13] Jennergren L.P and Skogsvik K., 2011, "The Abnormal Earnings Growth Model, Two Exogenous Discount Rates, and Taxes", Journal of Business Finance & Accounting, volume 38, issue 5-6, pages 505–535, June/July 2011.

[14] Jorgensen B.N., Lee Y.G. and Yoo Y.K., 2011, "The Valuation Accuracy of Equity Value Estimates Inferred from Conventional Empirical Implementations of the Abnormal Earnings Growth Model: US Evidence",Journal of Business Finance & Accounting, Volume 38, Issue 3-4, pages 446–471, April/May 2011.

[15] Joslin S., Singleton K.J. and Zhu H., 2011, "A New Perspective on Gaussian Dynamic Term Structure Models", Review of Financial Studies 24, 926-970.

[16] Ohlson, J. A., and Z. Gao, 'Earnings, Earnings Growth and Value', Foundations and Trends in Accounting, Vol. 1, No. 1, 2006.

[17] Ohlson, J. A., and D. Gode, 2004, "Accounting-Based Valuation with Changing Interest Rates", Review of Accounting Studies 9, 419-441.

[18] Ohlson, J. A., and B. E. Juettner-Nauroth, 2005. 'Expected EPS and EPS Growth as Determinants of Value', Review of Accounting Studies 10, Nos 2/3, 2005.

[19] Ohlson J., 2005, "On Accounting-Based Valuation Formulae", Review of Accounting Studies 10, 323–347.

[20] Penman, S. H., Discussion of 'On Accounting-Based Valuation Formulae' and 'Expected EPS and EPS Growth as Determinants of Value', Review of Accounting Studies, Vol. 10, Nos 2/3, 2005.

[21] Pope P.F. Pope and Wang P., 2000, "Risk-adjusted equity valution and accounting betas", Working paper. The department of accounting and finance, Lancaster University.

[22] Realdon M., 2006, "Quadratic term structure models in discrete time", Finance Research Letters 3, n.4, 277-289.

[23] Realdon M., 2017, "Gaussian Models for Euro High Grade Government Yields", European Journal of Finance, vol. 23, n.15, 1468-1511.

[24] Sun T.S., 1992, "Real and nominal interest rates: a discrete-time model and its continuous time limit", Review of Financial Studies 5, n.4, 581-611.