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Estimation of intrinsic growth factors in a class of stochastic population model

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Abstract

This paper discusses the problem of parameter estimation with nonlinear mean-reversion type stochastic differential equations (SDEs) driven by Brownian motion for population growth model. The estimator in the population model is the climate effects, population policy and environmental circumstances which affect the intrinsic rate of growth r. The consistency and asymptotic distribution of the estimator θ is studied in our general setting. In the calculation method, unlike previous study, since the nonlinear feature of the model, it is difficult to obtain an explicit formula for the estimator. To solve this, some criteria are used to derive an asymptotically consistent estimator. Furthermore Girsanov transformation is used to simplify the equations, which then gives rise to the corresponding convergence of the estimator being with respect to a family of probability measures indexed by the dispersion parameter, while in the literature the existing results have dealt with convergence with respect to a given probability measure.

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1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space endowed with a usual filtration $\{\mathcal{F}_t\}_{t\geq 0}$, i.e., $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $0 \leq s \leq t \leq 1$ and $\mathcal{F}_0$ contains all null sets of P. The stochastic

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process $X = (X_t, 0 \leq t \leq 1)$ with a given initial value $X_0 = x \in \mathbb{R}$, is determined by the following mean reversion stochastic differential equation (SDE)

$$dX_t = [r(\theta) + \alpha(X_t, t, \varepsilon)]X_t dt + \varepsilon X_t dB_t, \qquad 0 \leq t \leq 1, \tag{1.1}$$

where $\varepsilon \in (0, 1]$ is a parameter which stands for the scaling of the volatility; α is the mean correction with the function: $\mathbb{R} \times [0, 1] \times (0, 1] \to \mathbb{R}$ being twice differentiable with respect to x and differentiable with respect to t; B_t is a one dimensional $\{\mathcal{F}_t\}-$Brownian motion defined on the probability space $\{\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{0\leq t\leq 1}\}$; mean value $r(\theta)$ is a $\mathcal{C}^2$-function of parameter θ with $\inf_{\theta\in\Theta} |r'(\theta)| > 0$ for all $\theta \in \Theta = \bar{\Theta}_0$(the closure of Θ_0) with Θ_0 being an open bounded subset of $\mathbb{R}$.

In fact, Equation (1.1) should be a population model with the stochastic inference, see Mao [32]. Here, r is the intrinsic rate of growth determined the parameter θ which denotes the climate effects [6], [15], [10], population policy [4] and environmental circumstances [48] that affect the growth of the biological population. With the development of modern industry and agriculture, climate warming, environmental pollution and population policy continue to change the ecosystem, these factors have seriously affected the growth rate of the population, and even have led to extinction of many species.

In the present paper, we aim to investigate the least squares estimator for the true value of θ based on the (discrete) sampling data $(X_{t_i})_{i=1}^n$. It is important to study the factors that influence the growth rate of biological population, and it can effectively evaluate the environment. In our model, the impact of factors on growth rates is monotonically increasing or monotonically decreasing and meets the convex or concave, that is $r'(\theta) \neq 0$ and $r''(\theta) \neq 0$.

In the past decades, the theory of SDEs has played an important role in modeling uncertain and volatile systems arising in economics, finance and biology see, e.g. Basak, Ghosh and Mukherjee [3], Dogan-Ciftci and Allen [7], Kunitomo and Takahashi [19], Long [29],Takahashi [42], Takahashi and Yoshida [43], Uchida and Yoshida [44], and Yoshida [51]. Especially after the mid-1970s, the SDEstheory are widely used in the study of population ecology, population genetics, neurobiology, epidemiology, immunology, physiology and environmental pollution. So that biological mathematics has been development fast. In 1969, Levins [24] analyzed the effects of the growth of biological populations on different types of random variables. Compared with the deterministic environment, May [33] studied the stability of the population growth model under random perturbation. After that, Braumann [5] gave a detailed summary of all the previous results on population growth in a random environment. The study of the population model with random disturbance attracts the attention of many scholars, and many of the existing literatures have explored this aspect in depth and have made many remarkable achievements. However, compared with the deterministic environment, the stochastic model has a very large difference in theory and research method in the environment with stochastic perturbation.

Fundamental issues of parameter estimation theory for stochastic differential equations(SDEs) is to estimate certain parameters (i.e., deterministic quantities) appearing in the random models by certain observations (or by experimental data). There have been two main methods for the case of parameter estimation: maximum likelihood estimator(MLE) method, see for instance Andrade and Tavares [1], Kutoyants [22], Liptser and Shiryaev [27], Mishra and Prakasa Rao[34], Moummou [35],Prakasa Rao [38], Wen [47] and least square estimator (LSE) method, see Breton [23], Dorogoveev [8], and Kasonga [18]. Eventually, it turns out that both the MLE and the LSE are asymptotically equivalent and the LSE enjoys the strong consistency property under some regularity conditions. Moreover, Prakasa Rao [37] gave a study of the asymptotic distribution. Further, Shimizu and Yoshida [39] considered a multidimensional diffusion process with jumps whose jump term is driven by a compound Poisson process. Shimizu [40] considered a similar case and proposed an estimating function with more complicated situation.

The asymptotic theory of parametric estimation based on continuous time observations is well developed, references can be found in e.g., Kutoyants [21], Kutoyants [20], Uchida and Yoshida [45], Yoshida [53] and Yoshida [52].

On the other side, parametric estimation based on discrete-time observations have been studied. For instant, Sϕrensen [41] gave an excellent survey of existing estimation techniques for stationary and ergodic diffusion processes observed at discrete points in time. Long [28] gave an investigation of the parameter estimation for discretely observations on one dimensional Ornstein-Uhlenbeck (O-U) processes driven by small Lévy noise. Since the actual data is obtained in discrete time, it is more realistic and interesting to consider the parameter estimation for diffusion processes based on discrete observation, We begin with our study from this point of view.

Nowadays, the parameter estimation for mean-reversion type SDEs has received a lot of attention. In Long, Shimizu and Sun [30], the author consider the problem of parameter estimation for discretely observed stochastic processes driven by additive small Lévy noises.

$$dX_t = b(X_t, \theta)dt + \varepsilon dL_t, \quad 0 \le t \le 1.$$

In that case, the drift function $b(X_t, \theta)$ and the consistency of θ are studied. In this framework, the author established the consistency and asymptotic normality for the proposed estimators.

Recently, Li and Wu [25] give a study on drift parameter estimation for mean-reversion type stochastic differential equations with discrete observations. In that paper, we consider the model

$$dX_t = [r + \alpha(X_t, t, \varepsilon)]b(X_t, t)dt + \varepsilon\sigma(X_t, t)dB_t, \quad 0 \le t \le 1.$$

In order to study the consistency of the parameter r, we use a novel idea called Girsanov transformation to simplify drift term $\alpha(X_t, t, \varepsilon)$. Then the consistency of the estimator r and the associated asymptotic distribution are studied with high frequency and small dispersion simultaneously.

Comparing to Long's case in [30], the dispersion part in our case (1.1) contains $\varepsilon X_t dB_t$ term which called bilinear noise perturbation [See [9]]. In control system, it is not purely external, but depends on system states, and more difficult to stabilize the system than Long's case. According to the existing studies in the literature, there are two major difficulties in our case. The first one is the appearance of the item $\alpha(X_t, t, \varepsilon)$ in the drift coefficient of

(1.1). In our case here, Girsanov Transformation is applied to get rid of the term $\alpha(X_t, t, \varepsilon)$, which changes the original probability measure P to a family of (equivalent) probability measures Q_ε. The other difficulty is to gain the least square estimator explicitly, due to the nonlinear feature of our case. Some criteria in statistical inference (See Chapter 5 of Vaart [46]) is used to solve it. After these works, the consistency of the least square estimators is derived under the family Q_ε to a limit which turns out to be the true value of the parameter θ. Finally, the asymptotic distribution is gained under a new equivalent probability measure.

The paper is divided into four sections. In Section 2, Girsanov Transformation will be used to simplify our system (1.1). Meanwhile, an explicit form of our estimator will be defined. Moreover, some preliminaries and auxiliary results for subsequent developments will be presented. In Section 3, the convergence of the estimators will be demonstrated with high frequency and small dispersion simultaneously. The rate of the relevant convergence and the associated asymptotic distribution are derived in Section 4. To finish, the last section draws a conclusion.

2 Preliminaries and auxiliary results

In this section, we start with an introduction of the SDEs of mean-reversion type and then we impose some assumptions on our SDE (1.1) to ensure the existence and uniqueness of the solution. Furthermore, we introduce Girsanov transformation to simplify our equation (1.1). On the other hand, in order to obtain the consistency and asymptotics of our LSE $\hat{\theta}_{n,\varepsilon}$, an explicit form of $\hat{\theta}_{n,\varepsilon}$ is defined. In addition, the following notations and preliminaries will be given, which are useful for the development of our result. Throughout the paper, we use notation "$\rightarrow_Q$" to denote "convergence in probability Q"; notation "$\rightarrow_P$" to denote "convergence in probability P" and notation "$\Rightarrow$" to denote "convergence in distribution".

We assume equation (1.1) satisfies the following condition:
(1) $\alpha(x, t, \varepsilon)x - \alpha(y, t, \varepsilon)y \leq H|x - y|$, where $H > 0$ is a constant, $x, y \in \mathbb{R}$.
The above conditions guarantee (see, e.g., [49]) the existence of a unique solution of (1.1) for a given initial data $X_0 = x \in \mathbb{R}$. The celebrated Girsanov transformation (also called the transformation of the drift) provides a very useful and efficient approach to solve (1.1). The transformation says the following. Since (1) implies that α is bounded, In order to get rid of the term $\alpha(X_t, t, \varepsilon)$, for $\varepsilon \in (0, 1]$, we can specify u such that

$$u_\varepsilon(X_t, t) = \frac{\alpha(X_t, t, \varepsilon)}{\varepsilon}. \tag{2.1}$$

Let $u : \mathbb{R} \times [0, 1] \to \mathbb{R}$ satisfy the following Novikov condition

$$E\left[\exp\left(\frac{1}{2}\int_0^t |u_\varepsilon(X_s, s)|^2 ds\right)\right] < \infty, \quad \forall t \in [0, 1]. \tag{2.2}$$

Then, we define

$$M_t^\varepsilon = \exp\left(-\int_0^t u_\varepsilon(X_s, s)dB_s - \frac{1}{2}\int_0^t u_\varepsilon^2(X_s, s)ds\right), \quad \forall t \in [0, 1] \tag{2.3}$$

by the Girsanov theorem (cf e.g. Theorem IV 4.1 of [14]), M_t^ε is an $\{\mathcal{F}_t\}_{t\in[0,1]}$martingale. For each $\varepsilon \in (0,1]$, let Q_ε be a probability measure on $\mathcal{F}_1$, satisfying

$$dQ_\varepsilon := M_1^\varepsilon dP. \tag{2.4}$$

Or equivalently, in terms of the Radon-Nikodym derivative

$$\frac{dQ_t}{dP} = \exp\left(\int_0^t u(X_s,s)dB_s - \frac{1}{2}\int_0^t u^2(X_s,s)ds\right).$$

We say Q_ε is absolutely continuous with respect to $\mathcal{F}_t$ and P. Moreover, we define

$$\hat{B}_t^\varepsilon := \int_0^t u_\varepsilon(X_s,s)ds + B_t \tag{2.5}$$

where $\hat{B}_t^\varepsilon$ is an $\{\mathcal{F}_t\}_{t\in[0,1]}$ Brownian motion with respect to Q_ε. Then we arrive at the equation (1.1) for X_t, that is

$$dX_t = r(\theta)X_t dt + \varepsilon X_t d\hat{B}_t^\varepsilon \quad \forall t \in [0,1]. \tag{2.6}$$

which, from now, we will focus on. We denote the true value of the parameter $r(\theta)$ by $r(\theta_0)$ and the least square estimator of $r(\theta)$ by $r(\hat{\theta})$. As mentioned before we focus on investigation of the least squares estimator for the true value r_0 based on the (discrete) sampling data $(X_{t_i})_{i=1}^n$ obtained by the Euler-Maruyama numerical scheme for the Cauchy problem of the equation (2.6) with initial $X_0 = x$. For simplicity, we assume that the process X_t is observed at regularly spaced time points $0 = t_0 < t_1 < ... < t_{i-1} < t_i < ... < t_n = 1$ with $\{t_i = \frac{i}{n}, i = 0,1,2,...,n\}$, where $n \in \mathbb{N}$ is arbitrarily fixed. That is,

$$X_{t_i} = x + r(\theta)\sum_{i=1}^n X_{t_{i-1}}\Delta t_i + \varepsilon\sum_{i=1}^n X_{t_{i-1}}(\hat{B}_{t_i}^\varepsilon - \hat{B}_{t_{i-1}}^\varepsilon)$$

where $\Delta t_i = t_i - t_{i-1} = \frac{1}{n}$; $\Delta\hat{B}_{t_i}^\varepsilon = \hat{B}_{t_i}^\varepsilon - \hat{B}_{t_{i-1}}^\varepsilon$ is the increment of Brownian motion. Let us start with the use of the least square method to get a consistent estimator. First of all, we consider the following contrast function

$$\rho_{n,\varepsilon}(\theta) = \sum_{i=1}^n \frac{|X_{t_i} - X_{t_{i-1}} - r(\theta)X_{t_{i-1}}\Delta t_i|^2}{\varepsilon^2 X_{t_{i-1}}^2 \Delta t_i}.$$

Then the least square estimator $\hat{\theta}_{n,\varepsilon}$ is defined as

$$\hat{\theta}_{n,\varepsilon} := arg\min \rho_{n,\varepsilon}(\theta).$$

We study the least square estimator for the true value θ_0 based on the sampling data $(X_{t_i})_{i=1}^n$ with small dispersion ε and large sample size n. The following lemma is a reformulation of Theorem 7.3 of [32] (see also [14] or [36]), which is nothing but the Burkholder-Davis-Gundy inequality in our setting

Lemma 2.1. *Let $g \in \mathcal{L}^2(\mathbb{R}_+;\mathbb{R})$. Define, for $0 \le t \le T$,*

$$x(t) = \int_0^t g(s)dB_s$$

and

$$A(t) = \int_0^t |g(s)|^2 ds.$$

Then for every $p > 0$, there exist universal positive constants c_p, C_p(depending only on p), such that

$$c_p E|A(t)|^{\frac{p}{2}} \le E\left(\sup_{0\le s\le t} |x(s)|^p\right) \le C_p E|A(t)|^{\frac{p}{2}}$$

for all $t \ge 0$. In particular, one may take $c_p = (p/2)^p, C_p = (32/p)^{p/2}$, if $0 < p < 2$; $c_p = 1, C_p = 4$, if $p = 2$; $c_p = (2p)^{-p/2}, C_p = [p^{p+1}/2(p-1)^{p-1}]^{p/2}$, if $p > 2$.

3 Consistency of the least square estimator

This section is devoted to prove the consistency of the least squares estimator $\hat{\theta}_{n,\varepsilon}$. The following theorem is our first main result of this section.

Theorem 3.1. *Let $n \to \infty$ and $\varepsilon \to 0$ with $\varepsilon n^{\frac{1}{2}} \to 0$, we have $\hat{\theta}_{n,\varepsilon} \to_{Q_\varepsilon} \theta_0$.*

Before we give a proof, we derive an explicit decomposition for $\hat{\theta}_{n,\varepsilon}$. Since minimizing $\rho_{n,\varepsilon}(\theta)$ is equivalent to minimizing

$$\phi_{n,\varepsilon}(\theta) := \varepsilon^2(\rho_{n,\varepsilon}(\theta) - \rho_{n,\varepsilon}(\theta_0)). \tag{3.1}$$

We have

$$\begin{aligned}
\phi_{n,\varepsilon}(\theta) &= \varepsilon^2(\rho_{n,\varepsilon}(\theta) - \rho_{n,\varepsilon}(\theta_0)) \\
&= \sum_{i=1}^n \frac{|X_{t_i} - X_{t_{i-1}} - X_{t_{i-1}} r(\theta)\Delta t_{i-1}|^2 - |X_{t_i} - X_{t_{i-1}} - X_{t_{i-1}} r(\theta_0)\Delta t_{i-1}|^2}{X_{t_{i-1}}^2 \Delta t_i} \\
&= \sum_{i=1}^n \frac{|X_{t_i} - X_{t_{i-1}}|^2 - 2X_{t_{i-1}} r(\theta)\Delta t_{i-1}(X_{t_i} - X_{t_{i-1}}) + |X_{t_{i-1}} r(\theta)\Delta t_{i-1}|^2}{X_{t_{i-1}}^2 \Delta t_i} \\
&\quad - \sum_{i=1}^n \frac{|X_{t_i} - X_{t_{i-1}}|^2 - 2X_{t_{i-1}} r(\theta_0)\Delta t_{i-1}(X_{t_i} - X_{t_{i-1}}) + |X_{t_{i-1}} r(\theta_0)\Delta t_{i-1}|^2}{X_{t_{i-1}}^2 \Delta t_i} \\
&= \sum_{i=1}^n \frac{2(X_{t_i} - X_{t_{i-1}})X_{t_{i-1}}\Delta t_{i-1}(r(\theta_0) - r(\theta)) + X_{t_{i-1}}^2 \Delta t_{i-1}^2(|r(\theta)|^2 - |r(\theta_0)|^2)}{X_{t_{i-1}}^2 \Delta t_i} \\
&= 2(r(\theta_0) - r(\theta)) \sum_{i=1}^n \frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}} + (r^2(\theta) - r^2(\theta_0)).
\end{aligned} \tag{3.2}$$

In what follows, we first let

$$\Phi_{n,\varepsilon}(\theta) = \sum_{i=1}^{n} \frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}} \tag{3.3}$$

which is the part of above decomposition. Since

$$X_{t_i} - X_{t_{i-1}} = r(\theta_0) \int_{t_{i-1}}^{t_i} X_s ds + \varepsilon \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s^{\varepsilon}. \tag{3.4}$$

We have

$$\begin{aligned}
\Phi_{n,\varepsilon}(\theta) &= \sum_{i=1}^{n} \frac{r(\theta_0) \int_{t_{i-1}}^{t_i} X_s ds + \varepsilon \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s^{\varepsilon}}{X_{t_{i-1}}} \\
&= \sum_{i=1}^{n} \frac{r(\theta_0) \int_{t_{i-1}}^{t_i} (X_s - X_{t_{i-1}}) ds + r(\theta_0) \int_{t_{i-1}}^{t_i} X_{t_{i-1}} ds + \varepsilon \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s^{\varepsilon}}{X_{t_{i-1}}} \\
&= r(\theta_0) + \sum_{i=1}^{n} \frac{r(\theta_0) \int_{t_{i-1}}^{t_i} (X_s - X_{t_{i-1}}) ds}{X_{t_{i-1}}} + \sum_{i=1}^{n} \frac{\varepsilon \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s^{\varepsilon}}{X_{t_{i-1}}} \\
&:= r(\theta_0) + \Phi_1(n,\varepsilon) + \Phi_2(n,\varepsilon).
\end{aligned}$$

To prove Theorem 3.1, the following Proposition is needed. We shall study the asymptotic behavior of $\Phi_{n,\varepsilon}(\theta)$.

Proposition 3.1. *Let $n \to \infty$ and $\varepsilon \to 0$ with $\varepsilon n^{\frac{1}{2}} \to 0$, we have $\Phi_{n,\varepsilon}(\theta) \to_{Q_\varepsilon} r(\theta_0)$.*

Proof. This result follows from the following lemmas.

Lemma 3.1. *We have $\Phi_1(n,\varepsilon) \to_{Q_\varepsilon} 0$ as $n \to \infty$ and $\varepsilon \to 0$.*

Proof. From (3.4), we get

$$\begin{aligned}
|X_t - X_{t_{i-1}}| &\leq \int_{t_{i-1}}^{t} |r(\theta_0)||X_s| ds + |\varepsilon \int_{t_{i-1}}^{t} X_s d\hat{B}_s^{\varepsilon}| \\
&\leq |r(\theta_0)| \int_{t_{i-1}}^{t} |X_s - X_{t_{i-1}}| + |X_{t_{i-1}}| ds + |\varepsilon \int_{t_{i-1}}^{t} X_s d\hat{B}_s^{\varepsilon}|.
\end{aligned}$$

By Gronwall's Inequality

$$|X_t - X_{t_{i-1}}| \leq e^{|r(\theta_0)|(t-t_{i-1})} \Big(n^{-1} |r(\theta_0)||X_{t_{i-1}}| + \varepsilon \sup_{t_{i-1} \leq t \leq t_i} |\int_{t_{i-1}}^{t} X_s d\hat{B}_s^{\varepsilon}| \Big).$$

It yields

$$\sup_{t_{i-1} \leq t \leq t_i} |X_t - X_{t_{i-1}}| \leq e^{n^{-1}|r(\theta_0)|} \Big(n^{-1} |r(\theta_0)||X_{t_{i-1}}| + \varepsilon \sup_{t_{i-1} \leq t \leq t_i} |\int_{t_{i-1}}^{t} X_s d\hat{B}_s^{\varepsilon}| \Big). \tag{3.5}$$

On the other side, from $\Phi_1(n,\varepsilon)$, it is seen that

$$\begin{aligned}|\Phi_1(n,\varepsilon)| &\leq |r(\theta_0)|\sum_{i=1}^{n}\frac{\int_{t_{i-1}}^{t_i}|X_s-X_{t_{i-1}}|ds}{|X_{t_{i-1}}|}\\ &\leq |r(\theta_0)|\sum_{i=1}^{n}\frac{n^{-1}\sup\limits_{t_{i-1}\leq t\leq t_i}|X_t-X_{t_{i-1}}|}{|X_{t_{i-1}}|}.\end{aligned}$$

From (3.5), note that

$$\begin{aligned}|\Phi_1(n,\varepsilon)| &\leq |r(\theta_0)|\sum_{i=1}^{n}\frac{n^{-1}e^{\frac{|r(\theta_0)|}{n}}\left(n^{-1}|r(\theta_0)||X_{t_{i-1}}|+\varepsilon\sup\limits_{t_{i-1}\leq t\leq t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s^{\varepsilon}|\right)}{|X_{t_{i-1}}|}\\ &=\frac{|r(\theta_0)|^2 e^{\frac{|r(\theta_0)|}{n}}}{n}+\sum_{i=1}^{n}\frac{|r(\theta_0)|n^{-1}e^{\frac{|r(\theta_0)|}{n}}\varepsilon\sup\limits_{t_{i-1}\leq t\leq t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s^{\varepsilon}|}{|X_{t_{i-1}}|}.\end{aligned}$$

We set
$\Phi_1^{(1)}(n,\varepsilon):=\frac{|r(\theta_0)|^2 e^{\frac{|r(\theta_0)|}{n}}}{n}$;
$\Phi_1^{(2)}(n,\varepsilon):=\sum_{i=1}^{n}\frac{|r(\theta_0)|n^{-1}e^{\frac{|r(\theta_0)|}{n}}\varepsilon\sup\limits_{t_{i-1}\leq t\leq t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s^{\varepsilon}|}{|X_{t_{i-1}}|}$.
It is clear that $\Phi_1^{(1)}(n,\varepsilon)\to 0$, as $n\to\infty$. Then we consider $\Phi_1^{(2)}(n,\varepsilon)$, by Holder's Inequality, Markov Inequality and Lemma 2.1, for $\delta>0$, we have

$$\begin{aligned}Q_\varepsilon(|\Phi_1^{(2)}(n,\varepsilon)|>\delta) &\leq \frac{E_{Q_\varepsilon}|\Phi_1^{(2)}(n,\varepsilon)|}{\delta}\\ &\leq \frac{1}{\delta}E_{Q_\varepsilon}\sum_{i=1}^{n}\frac{|r(\theta_0)|n^{-1}e^{\frac{|r(\theta_0)|}{n}}\varepsilon\sup\limits_{t_{i-1}\leq t\leq t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s^{\varepsilon}|}{|X_{t_{i-1}}|}\\ &=\frac{1}{\delta}\sum_{i=1}^{n}E_{Q_\varepsilon}X_{t_{i-1}}^{-1}\left(|r(\theta_0)|n^{-1}e^{\frac{|r(\theta_0)|}{n}}\varepsilon\sup_{t_{i-1}\leq t\leq t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s^{\varepsilon}|\right)\\ &\leq \frac{1}{\delta}\sum_{i=1}^{n}(E_{Q_\varepsilon}X_{t_{i-1}}^{-2})^{\frac{1}{2}}\left[E_{Q_\varepsilon}\left(|r(\theta_0)|n^{-1}e^{\frac{|r(\theta_0)|}{n}}\varepsilon\sup_{t_{i-1}\leq t\leq t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s^{\varepsilon}|\right)^2\right]^{\frac{1}{2}}\\ &=\frac{1}{\delta}\sum_{i=1}^{n}(E_{Q_\varepsilon}X_{t_{i-1}}^{-2})^{\frac{1}{2}}\left[|r(\theta_0)|^2 n^{-2}e^{\frac{|2r(\theta_0)|}{n}}\varepsilon^2 E_{Q_\varepsilon}\left(\sup_{t_{i-1}\leq t\leq t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s^{\varepsilon}|\right)^2\right]^{\frac{1}{2}}\\ &=\frac{1}{\delta}\sum_{i=1}^{n}X_0^{-1}e^{\frac{3\varepsilon^2}{2}t_{i-1}-rt_{i-1}}\left[|r(\theta_0)|^2 n^{-2}e^{\frac{|2r(\theta_0)|}{n}}\varepsilon^2 E_{Q_\varepsilon}\left(\sup_{t_{i-1}\leq t\leq t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s^{\varepsilon}|\right)^2\right]^{\frac{1}{2}}\\ &:=A.\end{aligned}$$

We set $E_{Q_\varepsilon}(\sup\limits_{t_{i-1}\leq t\leq t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s^{\varepsilon}|^2)=\vartheta$.

It yields

$$
\begin{aligned}
\vartheta &\le 4E_{Q_\varepsilon}\Big(\int_{t_{i-1}}^{t_i}|X_s|^2ds\Big)\\
&\le 4\int_{t_{i-1}}^{t_i}E_{Q_\varepsilon}|X_s|^2ds \qquad (3.6)\\
&= 4n^{-1}X_0^2exp(\varepsilon^2 s+2r(\theta)s).
\end{aligned}
$$

So that, we have

$$
\begin{aligned}
A &\le \frac{1}{\delta}\sum_{i=1}^{n}X_0^{-1}\exp\Big(\frac{3\varepsilon^2}{2}t_{i-1}-r(\theta)t_{i-1}\Big)|r_0|n^{-1}e^{\frac{|r_0|}{n}}\varepsilon 2n^{-\frac{1}{2}}X_0\exp\Big(\frac{1}{2}\varepsilon^2 s+r(\theta)s\Big)\\
&= |r(\theta_0)|\frac{2}{\delta}\exp\Big(\frac{3\varepsilon^2}{2}t_{i-1}-r(\theta)t_{i-1}+\frac{|r(\theta_0)|}{n}+\frac{1}{2}\varepsilon^2 s+r(\theta)s\Big)\varepsilon n^{-\frac{1}{2}}
\end{aligned}
$$

which imply $A\to_{Q_\varepsilon}0$ as $n\to\infty$ and $\varepsilon\to 0$. Then we have $\phi_1^{(2)}(n,\varepsilon)\to_{Q_\varepsilon}0$ as $n\to\infty$ and $\varepsilon\to 0$. Finally, we get $\Phi_1(n,\varepsilon)\to_{Q_\varepsilon}0$ as $n\to\infty$ and $\varepsilon\to 0$. □

Lemma 3.2. *Let $n\to\infty$ and $\varepsilon\to 0$ with $\varepsilon n^{\frac{1}{2}}\to 0$, we have $\Phi_2(n,\varepsilon)\to_{Q_\varepsilon}0$.*

Proof. Since $\Phi_2(n,\varepsilon)=\sum\limits_{i=1}^{n}\frac{\varepsilon\int_{t_{i-1}}^{t_i}X_s d\hat{B}_s^\varepsilon}{X_{t_{i-1}}}$.

Together with Holder's Inequality, Markov Inequality and Lemma 2.3.1, we obtain, for δ

$$
\begin{aligned}
Q_\varepsilon(|\Phi_2(n,\varepsilon)|>\delta) &\le \frac{E_{Q_\varepsilon}|(\Phi_2(n,\varepsilon))|}{\delta}\\
&= \frac{1}{\delta}\sum_{i=1}^{n}E_{Q_\varepsilon}X_{t_{i-1}}^{-1}\varepsilon\Big|\int_{t_{i-1}}^{t_i}X_s d\hat{B}_s^\varepsilon\Big|\\
&\le \frac{1}{\delta}\sum_{i=1}^{n}\Big(E_{Q_\varepsilon}X_{t_{i-1}}^{-2}\Big)^{\frac{1}{2}}\Big[E_{Q_\varepsilon}(\varepsilon\sup_{t_{i-1}\le t\le t_i}|\int_{t_{i-1}}^{t_i}X_s d\hat{B}_s^\varepsilon)|^2\Big]^{\frac{1}{2}}\\
&\le \frac{1}{\delta}\sum_{i=1}^{n}X_0^{-1}exp\Big(\frac{3\varepsilon^2}{2}t_{i-1}-r(\theta)t_{i-1}\Big)2\varepsilon n^{-\frac{1}{2}}X_0 exp\Big(\frac{1}{2}\varepsilon^2 s+r(\theta)s\Big)\\
&= \frac{2}{\delta}\exp\Big(\frac{3\varepsilon^2}{2}t_{i-1}-r(\theta)t_{i-1}+\frac{1}{2}\varepsilon^2 s+r(\theta)s\Big)\varepsilon n^{\frac{1}{2}}
\end{aligned}
$$

which implies that $\Phi_2(n,\varepsilon)\to_{Q_\varepsilon}0$ as $n\to\infty$, $\varepsilon\to 0$ and $\varepsilon n^{\frac{1}{2}}\to 0$. □

Then by using Lemma 3.1 and 3.2, we have

$$
\Phi(n,\varepsilon):=r(\theta_0)+\phi_1(n,\varepsilon)+\phi_2(n,\varepsilon)\to_{Q_\varepsilon}r(\theta_0)
$$

as $n\to\infty$ and $\varepsilon\to 0$ with $\varepsilon n^{\frac{1}{2}}\to 0$. □

Proof. Proof of Theorem 3.1.

We recall that $\phi_{n,\varepsilon}(\theta) = 2(r(\theta_0) - r(\theta)) \sum_{i=1}^{n} \frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}} + (r^2(\theta) - r^2(\theta_0))$, by Proposition 3.1, as $n \to \infty$, $\varepsilon \to 0$ and $\varepsilon n^{\frac{1}{2}} \to 0$, we have

$$\phi_{n,\varepsilon}(\theta) \to_{Q_\varepsilon} (r(\theta) - r(\theta_0))^2. \tag{3.7}$$

Recall that our contrast function is

$$\rho_{n,\varepsilon}(\theta) = \sum_{i=1}^{n} \frac{|X_{t_i} - X_{t_{i-1}} - r(\theta) X_{t_{i-1}} \Delta t_i|^2}{\varepsilon^2 X_{t_{i-1}}^2 \Delta t_i}.$$

In order to obtain the least square estimator $r(\hat{\theta}_{n,\varepsilon})$, we let

$$\frac{\partial \rho_{n,\varepsilon}(r(\theta))}{\partial \theta} = r'(\theta).$$

Since $r'(\theta) \neq 0$, we get

$$\frac{\partial \rho_{n,\varepsilon}(r(\hat{\theta}))}{\partial r(\hat{\theta})} = 0$$

So that

$$r(\hat{\theta}_{n,\varepsilon}) = \sum_{i=1}^{n} \frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}}. \tag{3.8}$$

From Proposition 3.1, we know $\Phi_{n,\varepsilon}(\theta) \to_{Q_\varepsilon} r(\theta_0)$, as $n \to \infty$, $\varepsilon \to 0$ and $\varepsilon n^{\frac{1}{2}} \to 0$. Together with (3.3) and (3.8) we get $r(\hat{\theta}_{n,\varepsilon}) \to_{Q_\varepsilon} r(\theta_0)$ as $n \to \infty$, $\varepsilon \to 0$ and $\varepsilon n^{\frac{1}{2}} \to 0$. Since $r(\theta)$ is a $\mathcal{C}^2$-function of θ with $r'(\theta) \neq 0$, we have

$$\{|\hat{\theta}_{n,\varepsilon} - \theta_0| > \eta\} \subset \left\{ |r(\hat{\theta}_{n,\varepsilon})| > \frac{\eta}{\inf_{\theta \in \Theta} |r'(\theta)|} \right\}$$

for $\eta > 0$. This implies $\hat{\theta}_{n,\varepsilon} \to_{Q_\varepsilon} \theta_0$ as $n \to \infty$, $\varepsilon \to 0$ and $\varepsilon n^{\frac{1}{2}} \to 0$. This completes the proof. $\square$

4 Asymptotic of The Least Square Estimator

We aim to study the asymptotic behavior of the least square estimator in this section. For the sake of simplicity, we assume that $\alpha(x,t,\varepsilon)=\varepsilon\alpha(x,t)$ such that $Q=Q_\varepsilon$ is independent of ε. Then we formulate our main result of this section as following theorem.

Theorem 4.1. *Let $n\to\infty$ and $\varepsilon\to 0$ with $\varepsilon n^{\frac{1}{2}}\to 0$, we have*

$$\varepsilon^{-1}(\hat{\theta}_{n,\varepsilon}-\theta_0)\to_Q \Big(r'(\theta_0)\Big)^{-1}U$$

where U is a Q-random variable with standard normal distribution $N(0,1)$.

Proof. Before we give a proof, we introduce

$$I(\theta)=\Big(r'(\theta)\Big)^2$$

and

$$D(\theta)=-\Big(r'(\theta)\Big)^2. \tag{4.1}$$

Since

$$\phi_{n,\varepsilon}(\theta)=2(r(\theta_0)-r(\theta))\sum_{i=1}^{n}\frac{X_{t_i}-X_{t_{i-1}}}{X_{t_{i-1}}}+(r^2(\theta)-r^2(\theta_0)).$$

We have

$$\phi'_{n,\varepsilon}(\theta)=-2\sum_{i=1}^{n}\frac{X_{t_i}-X_{t_{i-1}}}{X_{t_{i-1}}}r'(\theta)+2r(\theta)r'(\theta).$$

Set

$$f_{n,\varepsilon}(\theta)=\sum_{i=1}^{n}\frac{X_{t_i}-X_{t_{i-1}}}{X_{t_{i-1}}}r'(\theta)-r(\theta)r'(\theta)$$

and

$$\begin{aligned}D_{n,\varepsilon}(\theta)&=f'_{n,\varepsilon}(\theta)\\&=\sum_{i=1}^{n}\frac{X_{t_i}-X_{t_{i-1}}}{X_{t_{i-1}}}r''(\theta)-\Big(r'(\theta)\Big)^2-r(\theta)r''(\theta).\end{aligned}$$

Let $B(\theta_0;\rho)=\{\theta:|\theta-\theta_0|\le\rho\}$ for $\rho>0$. Then, by the consistency of $\hat{\theta}_{n,\varepsilon}$, there exists a sequence $\eta_{n,\varepsilon}\to 0$ as $n\to\infty$ and $\varepsilon\to 0$ such that $B(\theta_0;\eta_{n,\varepsilon})\subset\Theta_0$, and $P_{\theta_0}[\hat{\theta}_{n,\varepsilon}\in B(\theta_0;\eta_{n,\varepsilon})]\to 1$. When $\hat{\theta}_{n,\varepsilon}\in B(\theta_0;\eta_{n,\varepsilon})$, we have

$$\begin{aligned}&\varepsilon^{-1}\{f_{n,\varepsilon}(\hat{\theta}_{n,\varepsilon})-f_{n,\varepsilon}(\theta_0)\}\\=&\varepsilon^{-1}(\hat{\theta}_{n,\varepsilon}-\theta_0)\int_0^1 D_{n,\varepsilon}(\theta_0+u(\hat{\theta}_{n,\varepsilon}-\theta_0))du.\end{aligned}$$

Now, we will consider $\varepsilon^{-1}f_{n,\varepsilon}(\hat{\theta}_{n,\varepsilon})$ and $\varepsilon^{-1}f_{n,\varepsilon}(\theta_0)$ respectively. For $\varepsilon^{-1}f_{n,\varepsilon}(\hat{\theta}_{n,\varepsilon})$, we have

$$\begin{aligned}\varepsilon^{-1}f_{n,\varepsilon}(\hat{\theta}_{n,\varepsilon}) &= \varepsilon^{-1}\sum_{k=1}^{n}\frac{X_{t_i}-X_{t_{i-1}}}{X_{t_{i-1}}}r'(\hat{\theta}_{n,\varepsilon}) - \varepsilon^{-1}r(\hat{\theta}_{n,\varepsilon})r'(\hat{\theta}_{n,\varepsilon})\\ &= \varepsilon^{-1}r(\hat{\theta}_{n,\varepsilon}))r'(\hat{\theta}_{n,\varepsilon}) - \varepsilon^{-1}r(\hat{\theta}_{n,\varepsilon})r'(\hat{\theta}_{n,\varepsilon})\\ &= 0\end{aligned}$$

since $r(\hat{\theta}_{n,\varepsilon}) = \sum\limits_{i=1}^{n}\frac{X_{t_i}-X_{t_{i-1}}}{X_{t_{i-1}}}$. by (3.8). For $\varepsilon^{-1}f_{n,\varepsilon}(\theta_0)$, we present following Proposition.

Proposition 4.1. *We have*

$$\varepsilon^{-1}f_{n,\varepsilon}(\theta_0) \to_Q r'(\theta_0)U$$

as $n \to \infty$ and $\varepsilon \to 0$.

Proof. Note that from (3.4)

$$\begin{aligned}\varepsilon^{-1}f_{n,\varepsilon}(\theta_0) &= \varepsilon^{-1}\Big(\sum_{i=1}^{n}\frac{X_{t_i}-X_{t_{i-1}}}{X_{t_{i-1}}}r'(\theta_0) - r(\theta_0)r'(\theta_0)\Big)\\ &= \varepsilon^{-1}\Big(\sum_{i=1}^{n}\frac{r(\theta_0)\int_{t_{i-1}}^{t_i}(X_s - X_{t_{i-1}})ds}{X_{t_{i-1}}} + \sum_{i=1}^{n}\frac{\varepsilon\int_{t_{i-1}}^{t_i}X_s d\hat{B}_s}{X_{t_{i-1}}}\Big)r'(\theta_0).\end{aligned} \tag{4.2}$$

We set

$$\begin{aligned}C(n,\varepsilon) &= \varepsilon^{-1}\Big(\sum_{i=1}^{n}\frac{r(\theta_0)\int_{t_{i-1}}^{t_i}(X_s - X_{t_{i-1}})ds}{X_{t_{i-1}}} + \sum_{i=1}^{n}\frac{\varepsilon\int_{t_{i-1}}^{t_i}X_s d\hat{B}_s}{X_{t_{i-1}}}\Big)\\ &= \varepsilon^{-1}\sum_{i=1}^{n}\frac{r(\theta_0)\int_{t_{i-1}}^{t_i}(X_s - X_{t_{i-1}})ds}{X_{t_{i-1}}} + \sum_{i=1}^{n}\frac{\int_{t_{i-1}}^{t_i}X_s d\hat{B}_s}{X_{t_{i-1}}}\\ &:= C_1(n,\varepsilon) + C_2(n,\varepsilon).\end{aligned}$$

To prove Proposition 4.1, we give the following two lemmas.

Lemma 4.1. *We have $C_1(n,\varepsilon) \to_Q 0$ as $n \to \infty$ and $\varepsilon \to 0$.*

Proof. From $C_1(n,\varepsilon)$, we have

$$\begin{aligned}|C_1(n,\varepsilon)| &\le |\varepsilon^{-1}||r(\theta_0)|\sum_{i=1}^{n}\frac{\int_{t_{i-1}}^{t_i}|X_s - X_{t_{i-1}}|ds}{|X_{t_{i-1}}|}\\ &\le |\varepsilon^{-1}||r(\theta_0)|\sum_{i=1}^{n}\frac{n^{-1}\sup\limits_{t_{i-1}\le t\le t_i}|X_t - X_{ti-1}|}{|X_{t_{i-1}}|}.\end{aligned}$$

From (3.5), we have

$$
\begin{aligned}
|C_1(n,\varepsilon)| &\le |\varepsilon^{-1}||r(\theta_0)|\sum_{i=1}^{n}\frac{n^{-1}e^{\frac{|r(\theta_0)|}{n}}(n^{-1}|r(\theta_0)||X_{t_{i-1}}|+\varepsilon\sup\limits_{t_{i-1}\le t\le t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s|)}{|X_{t_{i-1}}|}\\
&= \frac{|\varepsilon^{-1}||r(\theta_0)|^2 e^{\frac{|r(\theta_0)|}{n}}}{n}+|r(\theta_0)|\sum_{i=1}^{n}\frac{n^{-1}e^{\frac{|r(\theta_0)|}{n}}\sup\limits_{t_{i-1}\le t\le t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s|}{|X_{t_{i-1}}|}\\
&:= C_1^1(n,\varepsilon)+C_1^2(n,\varepsilon).
\end{aligned}
$$

It is easy to see that $C_1^1(n,\varepsilon)\to_Q 0$, as $n\to\infty$.
Then we consider $C_1^2(n,\varepsilon)$, by Holder's Inequality, Markov Inequality, Lemma 2.3.1, we have

$$
\begin{aligned}
Q(|C_1^2(n,\varepsilon)|>\delta) &\le \frac{E_Q|C_1^2(n,\varepsilon)|}{\delta}\\
&\le \frac{1}{\delta}|r(\theta_0)|E_Q\sum_{i=1}^{n}\frac{n^{-1}e^{\frac{|r(\theta_0)|}{n}}\sup\limits_{t_{i-1}\le t\le t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s|}{|X_{t_{i-1}}|}\\
&= \frac{1}{\delta}|r(\theta_0)|\sum_{i=1}^{n}E_Q X_{t_{i-1}}^{-1}(n^{-1}e^{\frac{|r(\theta_0)|}{n}}\sup_{t_{i-1}\le t\le t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s|)\\
&\le \frac{1}{\delta}|r(\theta_0)|\sum_{i=1}^{n}(E_Q X_{t_{i-1}}^{-2})^{\frac{1}{2}}[E_Q(n^{-1}e^{\frac{|r(\theta_0)|}{n}}\sup_{t_{i-1}\le t\le t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s|)^2]^{\frac{1}{2}}\\
&= \frac{1}{\delta}|r(\theta_0)|\sum_{i=1}^{n}(E_Q X_{t_{i-1}}^{-2})^{\frac{1}{2}}[n^{-2}e^{\frac{|2r(\theta_0)|}{n}}E_Q(\sup_{t_{i-1}\le t\le t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s|)^2]^{\frac{1}{2}}\\
&= \frac{1}{\delta}|r(\theta_0)|\sum_{i=1}^{n}X_0^{-1}e^{\frac{3\varepsilon^2}{2}t_{i-1}-r(\theta_0)t_{i-1}}[n^{-2}e^{\frac{|2r(\theta_0)|}{n}}E_Q(\sup_{t_{i-1}\le t\le t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s|)^2]^{\frac{1}{2}}\\
&= \iota.
\end{aligned}
$$

We recall that $E_Q(\sup\limits_{t_{i-1}\le t\le t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s|^2)=\vartheta$.

By equation (3.6), we have

$$
\begin{aligned}
\iota &\le \frac{1}{\delta}|r(\theta_0)|\sum_{i=1}^{n}X_0^{-1}\exp(\frac{3\varepsilon^2}{2}t_{i-1}-r(\theta)t_{i-1})n^{-1}e^{\frac{|r(\theta_0)|}{n}}2n^{-\frac{1}{2}}X_0\exp(\frac{1}{2}\varepsilon^2 s+r(\theta_0)s)\\
&= \frac{2}{\delta}|r(\theta_0)|\exp(\frac{3\varepsilon^2}{2}t_{i-1}-r(\theta)t_{i-1}+\frac{|r(\theta_0)|}{n}+\frac{1}{2}\varepsilon^2 s+r(\theta)s)n^{-\frac{1}{2}}
\end{aligned}
$$

which implies $\iota\to_Q 0$ as $n\to\infty$. Then we have $C_1^2(n,\varepsilon)\to_Q 0$ as $n\to\infty$ and $\varepsilon\to 0$. Finally, we get $C_1(n,\varepsilon)\to_Q 0$ as $n\to\infty$ and $\varepsilon\to 0$. □

Let X_t^0 be the solution of the underlying ordinary differential equation under the true value of the drift parameter:

$$dX_t^0=r(\theta_0)X_t^0dt,\qquad X_0^0=x_0. \tag{4.3}$$

Lemma 4.2. *We have $C_2(n,\varepsilon) \to_Q U$ as $n \to \infty$ and $\varepsilon \to 0$, where U is a Q-random variable with standard normal distribution $N(0,1)$.*

Proof. Since

$$\begin{aligned}
C_2(n,\varepsilon) &= \sum_{i=1}^{n} \frac{\int_{t_{i-1}}^{t_i} X_s d\hat{B}_s}{X_{t_{i-1}}}. \\
&= \sum_{i=1}^{n} X_{t_{i-1}}^{-1} \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s \\
&= \sum_{i=1}^{n} (X_{t_{i-1}}^0)^{-1} \int_{t_{i-1}}^{t_i} X_s^0 d\hat{B}_s + \sum_{i=1}^{n} (X_{t_{i-1}}^0)^{-1} \int_{t_{i-1}}^{t_i} (X_s - X_s^0) d\hat{B}_s \\
&+ \sum_{i=1}^{n} (X_{t_{i-1}}^{-1} - (X_{t_{i-1}}^0)^{-1}) \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s \\
&:= C_2^1(n,\varepsilon) + C_2^2(n,\varepsilon) + C_2^3(n,\varepsilon).
\end{aligned}$$

Define a deterministic process $V(s)$ by $V(s) = \sum_{i=1}^{n} (X_{t_{i-1}}^0)^{-1} X_s^0 1_{(t_{i-1},t_i]}(s)$. Let $V_+(s)$ and $V_-(s)$ denote the positive and negative part of $V(s)$. By Theorem 4.1 of Kallenberg [17], there exist two independent Q-Brownian motions $\hat{B}'$, $\hat{B}''$, which have the same distribution of $\hat{B}$, such that

$$C_2^1(n,\varepsilon) = \int_0^1 V(s) d\hat{B}_s = \hat{B}' \circ \int_0^1 V_+^2(s) ds - \hat{B}'' \circ \int_0^1 V_-^2(s) ds.$$

Note that

$$V_+^2 = \sum_{i=1}^{n} (X_{t_{i-1}}^0)^{-2} (X_s^0)_+^2 1_{(t_{i-1},t_i]}(s)$$

and

$$V_-^2 = \sum_{i=1}^{n} (X_{t_{i-1}}^0)^{-2} (X_s^0)_-^2 1_{(t_{i-1},t_i]}(s).$$

Then we have

$$\int_0^1 V_+^2(s) ds \to \int_0^1 (X_s^0)^{-2} (X_s^0)_+^2 ds$$

and

$$\int_0^1 V_-^2(s) ds \to \int_0^1 (X_s^0)^{-2} (X_s^0)_-^2 ds$$

as $n \to \infty$. Then,

$$\hat{B}' \circ \int_0^1 V_+^2(s) ds \to \hat{B}' \circ \int_0^1 (X_s^0)^{-2} (X_s^0)_+^2 ds$$

and

$$\hat{B}'' \circ \int_0^1 V_-^2(s) ds \to \hat{B}'' \circ \int_0^1 (X_s^0)^{-2} (X_s^0)_-^2 ds.$$

We get

$$C_2^1(n,\varepsilon) \to_Q U_1\Big(\int_0^1 (X_s^0)^{(-2)}(X_s^0)_+^2 ds\Big)^{\frac{1}{2}} - U_2\Big(\int_0^1 (X_s^0)^{-2}(X_s^0)_-^2 ds\Big)^{\frac{1}{2}}$$

where U_1 and U_2 are two random variables with standard normal distribution $N(0,1)$ as $n \to \infty$. Since

$$X_{s+}^0 = \max(X_s^0, 0)$$

and

$$X_{s-}^0 = \max(-X_s^0, 0).$$

$$C_2^1(n,\varepsilon) \to_Q \{^{U_1, \quad X_s^0 \geq 0}_{U_2, \quad X_s^0 \leq 0}$$

as $n \to \infty$.
So it can be summarized by

$$C_2^1(n,\varepsilon) \to_Q U$$

as $n \to \infty$.

Now, let us consider $C_2^2(n,\varepsilon)$. By using Holder's Inequality, Markov Inequality, Lemma 2.1, we get

$$\begin{aligned} Q(|C_2^2(n,\varepsilon)| > \delta) &\leq \delta^{-1} E_Q[\sum_{i=1}^n (X_{t_{i-1}}^0)^{-1} \sup_{t_{i-1}\leq t\leq t_i} |\int_{t_{i-1}}^{t_i} (X_s - X_s^0) d\hat{B}_s|] \\ &\leq \frac{1}{\delta}\sum_{i=1}^n E_Q[(X_{t_{i-1}}^0)^{-2}]^{\frac{1}{2}} [E_Q(\sup_{t_{i-1}\leq t\leq t_i} |\int_{t_{i-1}}^{t_i} (X_s - X_s^0) d\hat{B}_s)|^2]^{\frac{1}{2}} \\ &\leq \frac{1}{\delta}\sum_{i=1}^n E_Q[(X_{t_{i-1}}^0)^{-2}]^{\frac{1}{2}} [E_Q \int_{t_{i-1}}^{t_i} |X_s - X_s^0|^2 ds]^{\frac{1}{2}} \\ &\leq \frac{1}{\delta}\sum_{i=1}^n E_Q[(X_{t_{i-1}}^0)^{-2}]^{\frac{1}{2}} [2t^{\frac{1}{2}} E_Q \sup_{t_{i-1}\leq t\leq t_i} |X_s - X_s^0| ds] \end{aligned}$$

which tends to zero as $n \to \infty$ and $\varepsilon \to 0$. For $C_2^3(n,\varepsilon)$, we have

$$\begin{aligned} C_2^3(n,\varepsilon) &= \sum_{i=1}^n (X_{t_{i-1}}^{-1} - (X_{t_{i-1}}^0)^{-1}) \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s \\ &= \sum_{i=1}^n (-\frac{X_{t_{i-1}} - X_{t_{i-1}}^0}{X_{t_{i-1}}^0 X_{t_{i-1}}}) \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s \\ &\leq \sum_{i=1}^n (-\frac{\sup_{t_{i-1}\leq t\leq t_i} |X_{t_{i-1}} - X_{t_{i-1}}^0|}{X_{t_{i-1}}^0 X_{t_{i-1}}}) \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s. \end{aligned}$$

$\sup_{0\leq t\leq 1} |X_t - X_t^0| \to_{Q_\varepsilon} 0$ as $\varepsilon \to 0$.[See Lemma2.3 of [25]] we have $C_2^3(n,\varepsilon) \to_Q 0$ as $n \to \infty$.
□

Proof of Proposition 4.1, by combining Lemma 4.1 and Lemma 4.2, we have

$$\begin{aligned} C(n,\varepsilon) &= C_1(n,\varepsilon) + C_2(n,\varepsilon) \\ &\to_Q U \end{aligned}$$

as $n \to \infty$ and $\varepsilon \to 0$.
By (4.2),

$$\varepsilon^{-1} f_{n,\varepsilon}(\theta_0) \to_Q r'(\theta_0) U$$

as $n \to \infty$ and $\varepsilon \to 0$. $\square$

Now we construct

$$\begin{aligned} & \Big| \int_0^1 D_{n,\varepsilon}(\theta_0 + u(\hat{\theta}_{n,\varepsilon} - \theta_0)) du - D_{n,\varepsilon}(\theta_0) \Big| 1_{\{\theta_{n,\varepsilon} \in B(\hat{\theta}_0; \eta_{n,\varepsilon})\}} \\ \leq & \sup_{\theta \in B(\theta_0; \eta_{n,\varepsilon})} |D_{n,\varepsilon}(\theta) - D_{n,\varepsilon}(\theta_0)| \\ \leq & \sup_{\theta \in B(\theta_0; \eta_{n,\varepsilon})} |D_{n,\varepsilon}(\theta) - D(\theta)| + \sup_{\theta \in B(\theta_0; \eta_{n,\varepsilon})} |D(\theta) - D(\theta_0)| + \sup_{\theta \in B(\theta_0; \eta_{n,\varepsilon})} |D_{n,\varepsilon}(\theta_0) - D(\theta_0)| \\ := & A_1 + A_2 + A_3. \end{aligned}$$

Since

$$D_{n,\varepsilon}(\theta) = \sum_{i=1}^{n} \frac{X_{t_k} - X_{t_{k-1}}}{X_{t_{k-1}}} r''(\theta) - \Big(r'(\theta) \Big)^2 - r(\theta) r''(\theta).$$

According to Proposition 3.1, (3.3) and (4.1), let $n \to \infty$ and $\varepsilon \to 0$ with $\varepsilon n^{\frac{1}{2}} \to 0$, then, we obtain

$$D_{n,\varepsilon}(\theta) \to_Q D(\theta).$$

Consequently, we have $A_1 \to 0$, $A_2 \to 0$ and $A_3 \to 0$ as $n \to \infty$ and $\varepsilon \to 0$ with $\varepsilon n^{\frac{1}{2}} \to 0$. So that, we get

$$\int_0^1 D_{n,\varepsilon}(\theta_0 + u(\hat{\theta}_{n,\varepsilon} - \theta_0)) du \to_Q D(\theta_0)$$

Finally, we are ready to prove Theorem 4.1. Proof of Theorem 4.1. With previous proof, we have

$$\begin{aligned} \varepsilon^{-1}(\hat{\theta}_{n,\varepsilon} - \theta_0) &= -\Big(\int_0^1 D_{n,\varepsilon}(\theta_0 + u(\hat{\theta}_{n,\varepsilon} - \theta_0)) d\theta \Big)^{-1} \varepsilon^{-1} f_{n,\varepsilon}(\theta_0) \\ &\to_Q \Big(r'(\theta_0) \Big)^{-1} U \end{aligned}$$

as $n \to \infty$ and $\varepsilon \to 0$ with $\varepsilon n^{\frac{1}{2}} \to 0$. $\square$

5 Conclusion

In this paper, we give a study on the problem of parameter estimation in a class of stochastic population model. In population system, the parameter θ represent the climate effects, population policy and environmental circumstances including temperature, humidity, air quality, toxins and so on. These factors affect the rate of growth r directly. We consider the consistency and asymptotic distribution of the parameter θ in (1.1). In the calculation, some novel ideas are used. For instant, we utilise the celebrated Girsanov Transformation to simplify the drift coefficient, which then changes the originally given probability measure P to a family of equivalent probability measures $\{Q_\varepsilon\}_{\varepsilon>0}$. The significance of the parameter estimation in the model is to conduct an environmental assessment of the located environment.

6 Competing interests

The authors declare that they have no competing interests.

7 Authors' contributions

All authors have made equal contributions. All authors have read and approved the final manuscript.

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