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Convergence rates of theta-method for NSDDEs under non-globally Lipschitz continuous coefficients

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This paper is concerned with strong convergence and almost sure convergence for neutral stochastic differential delay equations under non-globally Lipschitz continuous coefficients. Convergence rates of θ-EM schemes are given for these equations driven by Brownian motion and pure jumps, respectively, where the drift terms satisfy locally one-sided Lipschitz conditions, and diffusion coefficients obey locally Lipschitz conditions, and the corresponding coefficients are highly nonlinear with respect to the delay terms.

Keywords: Neutral stochastic differential delay equations; θ-EM scheme; strong convergence; almost sure convergence; highly nonlinear.

Mathematics Subject Classification: 65C30, 65L20

1. Introduction

With the development of computer technology, numerical analyses have witnessed rapid growth since most equations cannot be solved explicitly. There is an extensive literature concerned with numerical solutions for stochastic differential equations (SDEs) and stochastic differential delay equations (SDDEs). In 1955,

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Maruyama [13] put forward Euler–Maruyama (EM) scheme for SDEs. After that, there has been a strong interest in numerical methods to all kinds of differential equations. Gikhman and Skorokhod [2] showed that under global Lipschitz and linear growth condition, the EM scheme converges to the exact solution with order 1/2 for SDEs, while for additive noise case, the convergence rate is 1. Kloeden and Platen [8] also studied numerical methods under global Lipschitz conditions. However, global conditions sometimes are too strict. In order to cover a larger part of SDEs, Higham *et al.* [3] studied strong convergence of Euler-type methods for nonlinear SDEs. They gave convergence rate of EM scheme and backward Euler scheme for SDEs under local Lipschitz and one-sided Lipschitz condition. Later, Mao and Sabanis [10] showed that the EM scheme will converge to exact solutions for SDDEs under local Lipschitz condition. Higham and Kloeden [4] presented and analyzed two implicit methods for Itô SDEs with Poisson-driven jumps where coefficients satisfy local Lipschitz conditions. Bao and Yuan [1] investigated convergence rate of EM scheme for SDDEs, where the corresponding coefficients may be highly nonlinear with respect to the delay variables. There is also some other literature concerned with strong convergence of explicit and implicit Euler-type methods to SDEs or SDDEs under non-global Lipschitz conditions, see [5, 6, 11, 18] for example.

Neutral stochastic differential delay equations (NSDDEs) play an important role in stochastic analysis. As to its numerical analysis, Wu and Mao [16] examined numerical solutions of neutral stochastic functional differential equations and established the strong mean square convergence theory of EM scheme under local Lipschitz condition; Zhou [17] established a criterion on exponential stability of EM scheme and backward scheme to NSDDES; Zong and Huang [19] were concerned with pth moment and almost sure exponential stability of the exact and EM-scheme solutions of NSDDEs; Ji *et al.* [7] generalized the results of [1] to NSDDEs; Tan and Yuan [15] proposed a tamed θ-EM scheme and gave convergence rate for NSDDEs driven by Brownian motion and pure jumps under one-sided Lipschitz condition. Motivated by Bao and Yuan [1] and Zong *et al.* [18], the drift and diffusion coefficients may be highly nonlinear with respect to delay variables. Will the θ-EM scheme converge to exact solutions strongly and almost surely for NSDDEs if the drift terms satisfy locally one-sided Lipschitz condition and diffusion coefficients obey locally Lipschitz conditions, and the corresponding coefficients are highly nonlinear? In this paper, we shall give a positive answer when the corresponding coefficients are highly nonlinear with respect to the delay terms.

The rest of paper is organized as follows: in Sec. 2, strong convergence rate and almost sure convergence rate are given for NSDDEs driven by Brownian motion under nonglobally Lipschitz condition, while in Sec. 3, the Brownian motion is replaced by pure jumps, under similar conditions, the convergence rates are also provided.

2. Convergence Rates for Brownian Motion Case

2.1. *Preliminaries*

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space such that $\{\mathscr{F}_t\}_{t\geq 0}$ is right continuous and increasing while $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets. $(\mathbb{R}^n, \langle\cdot,\cdot\rangle, |\cdot|)$ is an n-dimensional Euclidean space. Denote $\mathbb{R}^{n\times d}$ by the set of all $n\times d$ matrices A with trace norm $\|A\| = \sqrt{\mathrm{trace}(A^TA)}$, where A^T is the transpose of matrix A. For given $\tau \in (0,\infty)$, define the uniform norm $\|\zeta\|_\infty := \sup_{-\tau\leq\theta\leq 0}|\zeta(\theta)|$ for $\zeta \in \mathcal{C}([-\tau,0];\mathbb{R}^n)$ which denotes all continuous functions from $[-\tau,0]$ to $\mathbb{R}^n$. $W(t)$ is a d-dimensional Brownian motion defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. In this section, we consider the following NSDDE on $\mathbb{R}^n$:

$$\mathrm{d}[X(t) - D(X(t-\tau))] = b(X(t), X(t-\tau))\mathrm{d}t + \sigma(X(t), X(t-\tau))\mathrm{d}W(t), \quad t \geq 0, \tag{2.1}$$

with initial data $X(t) = \xi(t) \in \mathcal{L}^p_{\mathscr{F}_0}([-\tau,0];\mathbb{R}^n)$ for $t \in [-\tau,0]$, that is, ξ is an $\mathscr{F}_0$-measurable $\mathcal{C}([-\tau,0];\mathbb{R}^n)$-valued random vector such that $\mathbb{E}\|\xi\|^p_\infty < \infty$ for $p \geq 2$. Here, $D : \mathbb{R}^n \to \mathbb{R}^n$, and $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ are continuous functions. In order to guarantee the existence and uniqueness of solutions to (2.1), we first introduce functions $V_i, i = 1,2,3$ such that for any $x, y \in \mathbb{R}^n$,

$$0 \leq V_i(x,y) \leq L_i(1 + |x|^{l_i} + |y|^{l_i}), \quad i = 1,2,3, \tag{2.2}$$

for some $L_i > 0, l_i \geq 1$. Furthermore, in the sequel, for any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$, we shall assume that

(A1) There exists a positive constant K_1 such that

$$\langle x - D(y) - \bar{x} + D(\bar{y}), b(x,y) - b(\bar{x},\bar{y})\rangle \leq K_1^2|x-\bar{x}|^2 + |V_1(y,\bar{y})|^2|y-\bar{y}|^2,$$

and

$$|b(x,y) - b(x,\bar{y})| \leq V_1(y,\bar{y})|y-\bar{y}|.$$

(A2) There exists a positive constant K_2 such that

$$\|\sigma(x,y) - \sigma(\bar{x},\bar{y})\| \leq K_2|x-\bar{x}| + V_2(y,\bar{y})|y-\bar{y}|.$$

(A3) $D(0) = 0$ and $|D(y) - D(\bar{y})| \leq V_3(y,\bar{y})|y-\bar{y}|$.

Remark 2.1. There are some examples such that (A1)–(A3) hold. For example, set

$$D(y) = -y^3, \quad b(x,y) = x - x^3 + y^3, \quad \sigma(x,y) = x + y^4$$

for any $x, y \in \mathbb{R}$. It is to easy to check that (A1)–(A3) hold with $K_1 = \frac{2}{3}, K_2 = 1$, $V_i(x,y) = 1 + |x|^2 + |y|^2, i = 1,3$ and $V_2(x,y) = 2(1 + |x|^3 + |y|^3)$ for arbitrary $x, y \in \mathbb{R}$.

Remark 2.2. Assumption (A1) is much weaker than the following local Lipschitz condition:

$$|b(x,y)-b(\bar{x},\bar{y})|^2 \le K_1^2|x-\bar{x}|^2 + |V_1(y,\bar{y})|^2|y-\bar{y}|^2, \tag{2.3}$$

which is studied in [7]. There are examples satisfying (A1) but not local Lipschitz condition (2.3). For example, define $D(y)=0$, $b(x,y)=x-|x|^{\frac{1}{2}}+y^3, x\ge 0$, let $\bar{x}=0,\bar{y}=0$, one can see that $b(x,y)$ does not satisfy local Lipschitz condition (2.3) as x tends to 0, however we can compute

$$\begin{aligned}
&\langle x-D(y)-\bar{x}+D(\bar{y}), b(x,y)-b(\bar{x},\bar{y})\rangle\\
&\quad = (x-\bar{x})^2-(x-\bar{x})(x^{\frac{1}{2}}-\bar{x}^{\frac{1}{2}})+(x-\bar{x})(y^3-\bar{y}^3)\\
&\quad \le (x-\bar{x})^2+(x-\bar{x})(y^3-\bar{y}^3) \le \frac{3}{2}(x-\bar{x})^2+\frac{1}{2}(y^3-\bar{y}^3)^2\\
&\quad \le \frac{3}{2}|x-\bar{x}|^2+V_1(y,\bar{y})^2|y-\bar{y}|^2,
\end{aligned}$$

where in the first inequality we use the fact that $x-\bar{x}$ and $x^{\frac{1}{2}}-\bar{x}^{\frac{1}{2}}$ have the same sign. This means that (A1) holds. Thus, our assumption covers more cases than [7].

Remark 2.3. With assumption (A3), we immediately arrive at

$$|D(y)| \le V_3(y,0)|y| \le L_3(1+|y|+|y|^{l_3+1}). \tag{2.4}$$

With assumptions (A1)–(A3), we have

$$\begin{aligned}
\langle x-D(y), b(x,y)\rangle &= \langle x-D(y)-0+D(0), b(x,y)-b(0,0)\rangle + \langle x-D(y), b(0,0)\rangle\\
&\le K_1^2|x|^2+|V_1(y,0)|^2|y|^2+\frac{1}{2}|x-D(y)|^2+\frac{1}{2}|b(0,0)|^2\\
&\le (K_1^2+1)|x|^2+|V_1(y,0)|^2|y|^2+|V_3(y,0)|^2|y|^2+\frac{1}{2}|b(0,0)|^2,
\end{aligned}$$

and

$$\begin{aligned}
\|\sigma(x,y)\|^2 &\le 2\|\sigma(x,y)-\sigma(0,0)\|^2+2\|\sigma(0,0)\|^2\\
&\le 4K_2^2|x|^2+4|V_2(y,0)|^2|y|^2+2\|\sigma(0,0)\|^2.
\end{aligned}$$

Denote $K=\max\{2(K_1^2+1), 4K_2^2, |b(0,0)|^2, 2\|\sigma(0,0)\|^2\}$, and $|V(y,0)|^2 = 2\max\{|V_1(y,0)|^2+|V_3(y,0)|^2, 2|V_2(y,0)|^2\}$, we can rewrite the above inequalities as

$$2\langle x-D(y), b(x,y)\rangle \vee \|\sigma(x,y)\|^2 \le K(1+|x|^2)+|V(y,0)|^2|y|^2. \tag{2.5}$$

Lemma 2.1. *Let (A1)–(A3) hold. Then there exists a unique global solution to (2.1), moreover, the solution has the properties that for any $p\ge 2$, $T>0$,*

$$\mathbb{E}\left(\sup_{0\le t\le T}|X(t)|^p\right) \le C, \tag{2.6}$$

where $C=C(\xi,p,T)$ is a positive constant depending on the initial data ξ and p,T.

Proof. Under assumptions (A1)–(A3) and Remark 2.3, similar to the proof of [9, Theorem 3.4], we can show that that (2.1) has a unique local solution by using truncated method. To verify that, (2.1) admits a unique global solution, it is sufficient to show (2.6). In the following, generic constants will be denoted by C, which may be different from one place to another. Applying the Itô formula and using (2.5), we have

$$\begin{aligned}
&|X(t)-D(X(t-\tau))|^p \\
&\quad \le |\xi(0)-D(\xi(-\tau))|^p \\
&\qquad + p\int_0^t |X(s)-D(X(s-\tau))|^{p-2}\langle X(s)-D(X(s-\tau)), \\
&\qquad b(X(s),X(s-\tau))\rangle \mathrm{d}s \\
&\qquad + \frac{p(p-1)}{2}\int_0^t |X(s)-D(X(s-\tau))|^{p-2}\|\sigma(X(s),X(s-\tau))\|^2\mathrm{d}s \\
&\qquad + p\int_0^t |X(s)-D(X(s-\tau))|^{p-2}\langle X(s)-D(X(s-\tau)), \\
&\qquad \sigma(X(s),X(s-\tau))\mathrm{d}W(s)\rangle, \\
&\quad \le |\xi(0)-D(\xi(-\tau))|^p + \frac{p^2K}{2}\int_0^t |X(s)-D(X(s-\tau))|^{p-2}(1+|X(s)|^2)\mathrm{d}s \\
&\qquad + \frac{p^2}{2}\int_0^t |X(s)-D(X(s-\tau))|^{p-2}|V(X(s-\tau),0)|^2|X(s-\tau)|^2\mathrm{d}s \\
&\qquad + p\int_0^t |X(s)-D(X(s-\tau))|^{p-2}\langle X(s)-D(X(s-\tau)), \\
&\qquad \sigma(X(s),X(s-\tau))\mathrm{d}W(s)\rangle \\
&\quad =: |\xi(0)-D(\xi(-\tau))|^p + I_1(t)+I_2(t)+I_3(t).
\end{aligned} \tag{2.7}$$

By the Young inequality

$$|a|^\beta|b|^{1-\beta} \le \beta|a|+(1-\beta)|b|, \quad \forall\, a,b\in\mathbb{R},\ \beta\in[0,1],$$

we see

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le u\le t}|I_1(u)|\right) \\
&\quad \le C\mathbb{E}\int_0^t |X(s)-D(X(s-\tau))|^p\mathrm{d}s + C\mathbb{E}\int_0^t (1+|X(s)|^p)\mathrm{d}s \\
&\quad \le C\mathbb{E}\int_0^t [1+|X(s)|^p+|V_3(X(s-\tau),0)|^p|X(s-\tau)|^p]\mathrm{d}s.
\end{aligned}$$

Using the Young inequality again, we obtain

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le u\le t}|I_2(u)|\right) &\le C\mathbb{E}\int_0^t |X(s)-D(X(s-\tau))|^p\mathrm{d}s\\ &\quad + C\mathbb{E}\int_0^t |V(X(s-\tau),0)|^p|X(s-\tau)|^p\mathrm{d}s.\\ &\le C\mathbb{E}\int_0^t [|X(s)|^p+|V(X(s-\tau),0)|^p|X(s-\tau)|^p]\mathrm{d}s.\end{aligned}$$

Application of the Burkholder–Davis–Gundy (BDG) inequality, the Young inequality and (2.5) yields

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le u\le t}|I_3(u)|\right)\\ &\le C\mathbb{E}\left(\int_0^t |X(s)-D(X(s-\tau))|^{2p-2}\|\sigma(X(s),X(s-\tau))\|^2\mathrm{d}s\right)^{\frac{1}{2}}\\ &\le C\mathbb{E}\left(\sup_{0\le u\le t}|X(u)-D(X(u-\tau))|^{2p-2}\int_0^t \|\sigma(X(s),X(s-\tau))\|^2\mathrm{d}s\right)^{\frac{1}{2}}\\ &= \mathbb{E}\left(\frac{1}{4}\sup_{0\le u\le t}|X(u)-D(X(u-\tau))|^p\right)^{\frac{p-1}{p}}\\ &\quad\times\left((C\int_0^t \|\sigma(X(s),X(s-\tau))\|^2\mathrm{d}s)^{\frac{p}{2}}\right)^{\frac{1}{p}}\\ &\le \frac{1}{4}\mathbb{E}\left(\sup_{0\le u\le t}|X(u)-D(X(u-\tau))|^p\right)+C\mathbb{E}\left(\int_0^t \|\sigma(X(s),X(s-\tau))\|^2\mathrm{d}s\right)^{\frac{p}{2}}\\ &\le \frac{1}{4}\mathbb{E}\left(\sup_{0\le u\le t}|X(u)-D(X(u-\tau))|^p\right)\\ &\quad + C\mathbb{E}\int_0^t [1+|X(s)|^p+|V(X(s-\tau),0)|^p|X(s-\tau)|^p]\mathrm{d}s. \qquad (2.8)\end{aligned}$$

That is, after taking sup of (2.7) and using (2.4), we obtain

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le u\le t}|X(u)-D(X(u-\tau))|^p\right) &\le C\|\xi\|_\infty^p + C|D(\xi(-\tau))|^p + C\mathbb{E}\int_0^t |X(s)|^p\mathrm{d}s\\ &\quad + C\mathbb{E}\int_0^t |V(X(s-\tau),0)|^p|X(s-\tau)|^p\mathrm{d}s\\ &\quad + C\mathbb{E}\int_0^t |V_3(X(s-\tau),0)|^p|X(s-\tau)|^p\mathrm{d}s\end{aligned}$$

$$\begin{aligned}&\le C(\|\xi\|_\infty^{(l_3+1)p}, p) + C\mathbb{E}\int_0^t |X(s)|^p \mathrm{d}s\\&\quad + C\mathbb{E}\int_0^t |V(X(s-\tau),0)|^p |X(s-\tau)|^p \mathrm{d}s\\&\quad + C\mathbb{E}\int_0^t |V_3(X(s-\tau),0)|^p |X(s-\tau)|^p \mathrm{d}s.\end{aligned}$$

By (2.2), we see that

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le u\le t}|X(u)-D(X(u-\tau))|^p\right) &\le C(\|\xi\|_\infty^{(l_3+1)p}, p) + C\mathbb{E}\int_0^t |X(s)|^p \mathrm{d}s\\&\quad + C\mathbb{E}\int_0^t |X(s-\tau)|^{(l+1)p}\mathrm{d}s, \qquad (2.9)\end{aligned}$$

where $l = l_1 \vee l_2 \vee l_3$. Then, with (2.4), we derive from (2.9) that

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le u\le t}|X(u)|^p\right)\\&\quad\le C\mathbb{E}\left(\sup_{0\le u\le t}|D(X(u-\tau))|^p\right) + C\mathbb{E}\left(\sup_{0\le u\le t}|X(u)-D(X(u-\tau))|^p\right)\\&\quad\le C(\|\xi\|_\infty^{(l_3+1)p}, p) + C\mathbb{E}\left(\sup_{-\tau\le u\le t-\tau}|X(u)|^{(l_3+1)p}\right)\\&\qquad + C\mathbb{E}\int_0^t |X(s)|^p\mathrm{d}s + C\mathbb{E}\int_0^t |X(s-\tau)|^{(l+1)p}\mathrm{d}s\\&\quad\le C(\|\xi\|_\infty^{(l+1)p}, p) + C\mathbb{E}\left(\sup_{-\tau\le u\le t-\tau}|X(u)|^{(l+1)p}\right)\\&\qquad + C\int_0^t \mathbb{E}\left(\sup_{0\le u\le s}|X(u)|^p\right)\mathrm{d}s,\end{aligned}$$

where in the last step we have used the Young inequality. The Gronwall inequality then leads to

$$\mathbb{E}\left(\sup_{0\le u\le t}|X(u)|^p\right) \le C(\|\xi\|_\infty^{(l+1)p}, p) + C\mathbb{E}\left(\sup_{0\le u\le (t-\tau)\vee 0}|X(u)|^{(l+1)p}\right). \qquad (2.10)$$

For $t\in[0,\tau]$, the above inequality implies

$$\mathbb{E}\left(\sup_{0\le t\le \tau}|X(t)|^p\right) \le C(\|\xi\|_\infty^{(l+1)p}, p).$$

For $t\in[0,2\tau]$, (2.10) gives

$$\mathbb{E}\left(\sup_{0\le t\le 2\tau}|X(t)|^p\right) \le C(\|\xi\|_\infty^{(l+1)p}, p) + C\mathbb{E}\left(\sup_{0\le t\le \tau}|X(t)|^{(l+1)p}\right) \le C(\|\xi\|_\infty^{(l+1)^2 p}, p).$$

By induction, for $k = 1, 2, \ldots, \lceil T/\tau \rceil$, we have

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le k\tau}|X(t)|^p\right) &\le C(\|\xi\|_\infty^{(l+1)p}, p) + C\mathbb{E}\left(\sup_{0\le t\le (k-1)\tau}|X(t)|^{(l+1)p}\right)\\ &\le C(\|\xi\|_\infty^{(l+1)^{k-1}p}, p) + C\mathbb{E}\left(\sup_{0\le t\le \tau}|X(t)|^{(l+1)^k p}\right) \le C.\end{aligned}$$

The desired result then follows. □

We now introduce θ-EM scheme for (2.1). Given any time $T > \tau > 0$, without loss of generality, assume that T and τ are rational numbers, and there exist two positive integers such that $\Delta = \frac{\tau}{m} = \frac{T}{M}$, where $\Delta \in (0,1)$ is the step size. For $k = -m, \ldots, 0$, set $y_{t_k} = \xi(k\Delta)$, for $k = 0, 1, \ldots, M-1$, we form

$$\begin{aligned}y_{t_{k+1}} - D(y_{t_{k+1-m}}) &= y_{t_k} - D(y_{t_{k-m}}) + \theta b(y_{t_{k+1}}, y_{t_{k+1-m}})\Delta\\ &\quad + (1-\theta)b(y_{t_k}, y_{t_{k-m}})\Delta + \sigma(y_{t_k}, y_{t_{k-m}})\Delta W_{t_k}, \qquad (2.11)\end{aligned}$$

where $t_k = k\Delta$, $\Delta W_{t_k} = W(t_{k+1}) - W(t_k)$. Here $\theta \in [0,1]$ is an additional parameter that allows us to control the implicitness of the numerical scheme. For $\theta = 0$, the θ-EM scheme reduces to the EM scheme, and for $\theta = 1$, it is exactly the backward EM scheme. For a given y_{t_k}, in order to guarantee a unique solution $y_{t_{k+1}}$ to (2.11), the step size is required to satisfy $\theta\Delta < \frac{1}{4K_1^2}$ according to a fixed point theorem (see Mao and Szpruch [12] for more information), where K_1 is defined as in assumption (A1). In order for simplicity, we introduce the corresponding split-step theta scheme to (2.1) as follows: For $k = -m, \ldots, -1$, set $z_{t_k} = y_{t_k} = \xi(k\Delta)$, and for $k = 0, \ldots, M-1$,

$$\begin{cases} y_{t_k} = D(y_{t_{k-m}}) + z_{t_k} - D(z_{t_{k-m}}) + \theta b(y_{t_k}, y_{t_{k-m}})\Delta,\\ z_{t_{k+1}} = D(z_{t_{k+1-m}}) + z_{t_k} - D(z_{t_{k-m}}) + b(y_{t_k}, y_{t_{k-m}})\Delta + \sigma(y_{t_k}, y_{t_{k-m}})\Delta W_{t_k}. \end{cases} \qquad (2.12)$$

Through computation, we can easily deduce that $y_{t_{k+1}}$ in (2.12) can be rewritten as the form of (2.11). Due to the implicitness of θ-EM scheme, we also require $\theta\Delta < \frac{1}{2K}$, where K is defined as in Remark 2.3. Thus, throughout this paper, for $\theta = 0$, we let $\Delta \in (0,1)$ and for $\theta \in (0,1]$, we set $\Delta^* \in (0, (2K \vee 4K_1^2)^{-1}\theta^{-1})$, and $0 < \Delta \le \Delta^*$.

2.2. *Moment bounds*

Lemma 2.2. *Let (A1)–(A3) hold. Then for $\theta \in [\frac{1}{2}, 1]$, there exists a positive constant C independent of Δ such that for $p \ge 2$,*

$$\mathbb{E}\left(\sup_{0\le k\le M}|y_{t_k}|^p\right) \le C.$$

Proof. By (2.12), we see

$$\begin{aligned}
|z_{t_{k+1}} - D(z_{t_{k+1-m}})|^2 &= |z_{t_k} - D(z_{t_{k-m}})|^2 + 2\langle z_{t_k} - D(z_{t_{k-m}}), b(y_{t_k}, y_{t_{k-m}})\Delta\rangle \\
&\quad + |b(y_{t_k}, y_{t_{k-m}})|^2\Delta^2 + |\sigma(y_{t_k}, y_{t_{k-m}})\Delta W_{t_k}|^2 \\
&\quad + 2\langle z_{t_k} - D(z_{t_{k-m}}) + b(y_{t_k}, y_{t_{k-m}})\Delta, \sigma(y_{t_k}, y_{t_{k-m}})\Delta W_{t_k}\rangle \\
&= |z_{t_k} - D(z_{t_{k-m}})|^2 + 2\langle y_{t_k} - D(y_{t_{k-m}}), b(y_{t_k}, y_{t_{k-m}})\Delta\rangle \\
&\quad + (1-2\theta)|b(y_{t_k}, y_{t_{k-m}})|^2\Delta^2 + |\sigma(y_{t_k}, y_{t_{k-m}})\Delta W_{t_k}|^2 \\
&\quad + 2\langle y_{t_k} - D(y_{t_{k-m}}) + (1-\theta)b(y_{t_k}, y_{t_{k-m}})\Delta, \\
&\qquad \sigma(y_{t_k}, y_{t_{k-m}})\Delta W_{t_k}\rangle.
\end{aligned}$$

Noting that $\theta \geq \frac{1}{2}$ and substituting $b(y_{t_k}, y_{t_{k-m}})\Delta = \frac{1}{\theta}[y_{t_k} - D(y_{t_{k-m}}) - z_{t_k} + D(z_{t_{k-m}})]$ into the last term, and using (2.5) yields

$$\begin{aligned}
&|z_{t_{k+1}} - D(z_{t_{k+1-m}})|^2 \\
&\quad \leq |z_{t_k} - D(z_{t_{k-m}})|^2 + 2\Delta\langle y_{t_k} - D(y_{t_{k-m}}), b(y_{t_k}, y_{t_{k-m}})\rangle \\
&\qquad + |\sigma(y_{t_k}, y_{t_{k-m}})\Delta W_{t_k}|^2 + \frac{2}{\theta}\langle y_{t_k} - D(y_{t_{k-m}}), \sigma(y_{t_k}, y_{t_{k-m}})\Delta W_{t_k}\rangle \\
&\qquad - 2\frac{1-\theta}{\theta}\langle z_{t_k} - D(z_{t_{k-m}}), \sigma(y_{t_k}, y_{t_{k-m}})\Delta W_{t_k}\rangle \\
&\quad \leq |z_{t_k} - D(z_{t_{k-m}})|^2 + \Delta K(1 + |y_{t_k}|^2) + \Delta|V(y_{t_{k-m}}, 0)|^2|y_{t_{k-m}}|^2 \\
&\qquad + |\sigma(y_{t_k}, y_{t_{k-m}})\Delta W_{t_k}|^2 + \frac{2}{\theta}\langle y_{t_k} - D(y_{t_{k-m}}), \sigma(y_{t_k}, y_{t_{k-m}})\Delta W_{t_k}\rangle \\
&\qquad - 2\frac{1-\theta}{\theta}\langle z_{t_k} - D(z_{t_{k-m}}), \sigma(y_{t_k}, y_{t_{k-m}})\Delta W_{t_k}\rangle.
\end{aligned}$$

Summing both sides from 0 to k, we get

$$\begin{aligned}
&|z_{t_{k+1}} - D(z_{t_{k+1-m}})|^2 \\
&\quad \leq |z_{t_0} - D(z_{t_{-m}})|^2 + KT + \Delta K\sum_{i=0}^{k}|y_{t_i}|^2 + \Delta\sum_{i=0}^{k}|V(y_{t_{i-m}}, 0)|^2|y_{t_{i-m}}|^2 \\
&\qquad + \sum_{i=0}^{k}|\sigma(y_{t_i}, y_{t_{i-m}})\Delta W_{t_i}|^2 + \frac{2}{\theta}\sum_{i=0}^{k}\langle y_{t_i} - D(y_{t_{i-m}}), \sigma(y_{t_i}, y_{t_{i-m}})\Delta W_{t_i}\rangle \\
&\qquad - 2\frac{1-\theta}{\theta}\sum_{i=0}^{k}\langle z_{t_i} - D(z_{t_{i-m}}), \sigma(y_{t_i}, y_{t_{i-m}})\Delta W_{t_i}\rangle. \qquad (2.13)
\end{aligned}$$

Using the elementary inequality

$$\left|\sum_{i=1}^{n} x_i\right|^p \leq n^{p-1}\sum_{i=1}^{n}|x_i|^p, \quad p \geq 1, \qquad (2.14)$$

we then have

$$\begin{aligned}
&|z_{t_{k+1}} - D(z_{t_{k+1-m}})|^{2p} \\
&\quad \le 6^{p-1}(|z_{t_0} - D(z_{t_{-m}})|^2 + KT)^p + 6^{p-1}K^p\Delta^p\left(\sum_{i=0}^{k}|y_{t_i}|^2\right)^p \\
&\qquad + 6^{p-1}\Delta^p\left(\sum_{i=0}^{k}|V(y_{t_{i-m}},0)|^2|y_{t_{i-m}}|^2\right)^p + 6^{p-1}\left(\sum_{i=0}^{k}|\sigma(y_{t_i},y_{t_{i-m}})\Delta W_{t_i}|^2\right)^p \\
&\qquad + 6^{p-1}4^p\left|\sum_{i=0}^{k}\langle y_{t_i} - D(y_{t_{i-m}}), \sigma(y_{t_i},y_{t_{i-m}})\Delta W_{t_i}\rangle\right|^p \\
&\qquad + 6^{p-1}2^p\left|\sum_{i=0}^{k}\langle z_{t_i} - D(z_{t_{i-m}}), \sigma(y_{t_i},y_{t_{i-m}})\Delta W_{t_i}\rangle\right|^p. \qquad (2.15)
\end{aligned}$$

For $0 < j < M$, it is easy to observe that

$$\mathbb{E}\left[\sup_{0\le k\le j}\left(\sum_{i=0}^{k}|y_{t_i}|^2\right)^p\right] \le M^{p-1}\sum_{i=0}^{j}\mathbb{E}|y_{t_i}|^{2p}, \qquad (2.16)$$

and

$$\mathbb{E}\left[\sup_{0\le k\le j}\left(\sum_{i=0}^{k}|V(y_{t_{i-m}},0)|^2|y_{t_{i-m}}|^2\right)^p\right] \le M^{p-1}\sum_{i=0}^{j}\mathbb{E}(|V(y_{t_{i-m}},0)|^{2p}|y_{t_{i-m}}|^{2p}). \qquad (2.17)$$

Since we have $\mathbb{E}|\Delta W_{t_i}|^{2p} = (2p-1)!!\Delta^p$, then, by assumption (A2) and (2.5), we compute

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le k\le j}\left(\sum_{i=0}^{k}|\sigma(y_{t_i},y_{t_{i-m}})\Delta W_{t_i}|^2\right)^p\right] \\
&\quad \le M^{p-1}\mathbb{E}\left(\sum_{i=0}^{j}\|\sigma(y_{t_i},y_{t_{i-m}})\|^{2p}|\Delta W_{t_i}|^{2p}\right) \\
&\quad \le M^{p-1}\sum_{i=0}^{j}\mathbb{E}\|\sigma(y_{t_i},y_{t_{i-m}})\|^{2p}\mathbb{E}|\Delta W_{t_i}|^{2p} \\
&\quad \le M^{p-1}(2p-1)!!\Delta^p\sum_{i=0}^{j}\mathbb{E}[K(1+|y_{t_i}|^2) + |V(y_{t_{i-m}},0)|^2|y_{t_{i-m}}|^2]^p \\
&\quad \le C + C\sum_{i=0}^{j}\mathbb{E}|y_{t_i}|^{2p} + C\sum_{i=0}^{j}\mathbb{E}(|V(y_{t_{i-m}},0)|^{2p}|y_{t_{i-m}}|^{2p}). \qquad (2.18)
\end{aligned}$$

With (A2)–(A3) and (2.5), the Hölder inequality and the BDG inequality, we get

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le k\le j}\left|\sum_{i=0}^{k}\langle y_{t_i}-D(y_{t_{i-m}}),\sigma(y_{t_i},y_{t_{i-m}})\Delta W_{t_i}\rangle\right|^p\right]\\
&\le C\mathbb{E}\left(\sum_{i=0}^{j}|y_{t_i}-D(y_{t_{i-m}})|^2\|\sigma(y_{t_i},y_{t_{i-m}})\|^2\Delta\right)^{\frac{p}{2}}\\
&\le C\Delta^{\frac{p}{2}}(j+1)^{\frac{p}{2}-1}\mathbb{E}\sum_{i=0}^{j}|y_{t_i}-D(y_{t_{i-m}})|^p[K(1+|y_{t_i}|^2)\\
&\quad+|V(y_{t_{i-m}},0)|^2|y_{t_{i-m}}|^2]^{\frac{p}{2}}\\
&\le C\mathbb{E}\sum_{i=0}^{j}(|y_{t_i}|^p+|V_3(y_{t_{i-m}},0)|^p|y_{t_{i-m}}|^p)[K(1+|y_{t_i}|^2)\\
&\quad+|V(y_{t_{i-m}},0)|^2|y_{t_{i-m}}|^2]^{\frac{p}{2}}\\
&\le C+C\sum_{i=0}^{j}\mathbb{E}|y_{t_i}|^{2p}+C\sum_{i=0}^{j}\mathbb{E}(|V_3(y_{t_{i-m}},0)|^{2p}|y_{t_{i-m}}|^{2p})\\
&\quad+C\sum_{i=0}^{j}\mathbb{E}(|V(y_{t_{i-m}},0)|^{2p}|y_{t_{i-m}}|^{2p}).
\end{aligned}\tag{2.19}$$

Similarly, with (A2), the Young inequality and the BDG inequality again

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le k\le j}\left|\sum_{i=0}^{k}\langle z_{t_i}-D(z_{t_{i-m}}),\sigma(y_{t_i},y_{t_{i-m}})\Delta W_{t_i}\rangle\right|^p\right]\\
&\le C\mathbb{E}\left(\sum_{i=0}^{j}|z_{t_i}-D(z_{t_{i-m}})|^2\|\sigma(y_{t_i},y_{t_{i-m}})\|^2\Delta\right)^{\frac{p}{2}}\\
&\le C\Delta^{\frac{p}{2}}(j+1)^{\frac{p}{2}-1}\mathbb{E}\left(\sum_{i=0}^{j}|z_{t_i}-D(z_{t_{i-m}})|^p[K(1+|y_{t_i}|^2)\right.\\
&\quad\left.+|V(y_{t_{i-m}},0)|^2|y_{t_{i-m}}|^2]^{\frac{p}{2}}\right)\\
&\le C\Delta^{\frac{p}{2}}(j+1)^{\frac{p}{2}-1}\mathbb{E}\left(\sup_{0\le k\le j+1}|z_{t_k}-D(z_{t_{k-m}})|^p\sum_{i=0}^{j}[K(1+|y_{t_i}|^2)\right.\\
&\quad\left.+|V(y_{t_{i-m}},0)|^2|y_{t_{i-m}}|^2]^{\frac{p}{2}}\right)
\end{aligned}$$

$$= \mathbb{E}\left(\frac{1}{2}\sup_{0\le k\le j+1}|z_{t_k} - D(z_{t_{k-m}})|^p\right)^{\frac{p-1}{p}} \left(C\sum_{i=0}^{j}[K(1+|y_{t_i}|^2) + |V(y_{t_{i-m}},0)|^2|y_{t_{i-m}}|^2]^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

$$\le \frac{1}{2}\mathbb{E}\left(\sup_{0\le k\le j+1}|z_{t_k} - D(z_{t_{k-m}})|^p\right) + C\sum_{i=0}^{j}\mathbb{E}|y_{t_i}|^{2p} + C + C\sum_{i=0}^{j}\mathbb{E}(|V(y_{t_{i-m}},0)|^{2p}|y_{t_{i-m}}|^{2p}). \tag{2.20}$$

Applying (2.16)–(2.20) to (2.15), we get

$$\mathbb{E}\left(\sup_{0\le k\le j+1}|z_{t_k} - D(z_{t_{k-m}})|^{2p}\right)$$
$$\le C + C\sum_{i=0}^{j}\mathbb{E}|y_{t_i}|^{2p} + C\sum_{i=0}^{j}\mathbb{E}(|V(y_{t_{i-m}},0)|^{2p}|y_{t_{i-m}}|^{2p}) + C\sum_{i=0}^{j}\mathbb{E}(|V_3(y_{t_{i-m}},0)|^{2p}|y_{t_{i-m}}|^{2p})$$
$$\le C + C\sum_{i=0}^{j}\mathbb{E}|y_{t_i}|^{2p} + C\sum_{i=0}^{j}\mathbb{E}|y_{t_{i-m}}|^{2p(l+1)}. \tag{2.21}$$

Since $y_{t_k} - D(y_{t_{k-m}}) = z_{t_k} - D(z_{t_{k-m}}) + \theta b(y_{t_k}, y_{t_{k-m}})\Delta$, we deduce from (A1)–(A3), (2.5) and the fact that $|x-y|^p \ge 2^{1-p}|x|^p - |y|^p$, then

$$|z_{t_k} - D(z_{t_{k-m}})|^2$$
$$= |y_{t_k} - D(y_{t_{k-m}})|^2 + \theta^2\Delta^2|b(y_{t_k}, y_{t_{k-m}})|^2 - 2\theta\Delta\langle y_{t_k} - D(y_{t_{k-m}}), b(y_{t_k}, y_{t_{k-m}})\rangle$$
$$\ge \frac{1}{2}|y_{t_k}|^2 - |V_3(y_{t_{k-m}},0)|^2|y_{t_{k-m}}|^2 - \theta\Delta[K(1+|y_{t_k}|^2) + |V(y_{t_{k-m}},0)|^2|y_{t_{k-m}}|^2]$$
$$= \left(\frac{1}{2} - \theta K\Delta\right)|y_{t_k}|^2 - [|V_3(y_{t_{k-m}},0)|^2 + \theta\Delta|V(y_{t_{k-m}},0)|^2]|y_{t_{k-m}}|^2 - \theta K\Delta, \tag{2.22}$$

this implies

$$\begin{aligned}|y_{t_k}|^2 &\leq \left(\frac{1}{2}-\theta K\Delta\right)^{-1}[|z_{t_k}-D(z_{t_{k-m}})|^2+[|V_3(y_{t_{k-m}},0)|^2\\&\quad+\theta\Delta|V(y_{t_{k-m}},0)|^2]|y_{t_{k-m}}|^2+\theta K\Delta]\\&\leq \left(\frac{1}{2}-\theta K\Delta\right)^{-1}[|z_{t_k}-D(z_{t_{k-m}})|^2+2|V(y_{t_{k-m}},0)|^2|y_{t_{k-m}}|^2+\theta K\Delta].\end{aligned}$$

By the elementary inequality (2.14) again, we derive from (2.2) and (2.21) that

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\leq k\leq j+1}|y_{t_k}|^{2p}\right)\\&\quad\leq\left(\frac{1}{2}-\theta\Delta K\right)^{-p}3^{p-1}\left[\mathbb{E}\left(\sup_{0\leq k\leq j+1}|z_{t_k}-D(z_{t_{k-m}})|^{2p}\right)\right.\\&\qquad\left.+\ 2^p\mathbb{E}\left(\sup_{0\leq k\leq j+1}|V(y_{t_{k-m}},0)|^{2p}|y_{t_{k-m}}|^{2p}\right)+(\theta\Delta K)^p\right]\\&\quad\leq C+C\sum_{i=0}^{j}\mathbb{E}|y_{t_i}|^{2p}+C\sum_{i=-m}^{j-m}\mathbb{E}|y_{t_i}|^{2p(l+1)}\\&\qquad+2^p\mathbb{E}\left(\sup_{0\leq k\leq j+1}|V(y_{t_{k-m}},0)|^{2p}|y_{t_{k-m}}|^{2p}\right)\\&\quad\leq C+C\sum_{i=0}^{j}\mathbb{E}|y_{t_i}|^{2p}+C\sum_{i=-m}^{j-m}\mathbb{E}|y_{t_i}|^{2p(l+1)}+C\mathbb{E}\left(\sup_{0\leq k\leq j+1}|y_{t_{k-m}}|^{2p(l+1)}\right)\\&\quad\leq C+C\sum_{i=0}^{j}\mathbb{E}\left(\sup_{0\leq k\leq i}|y_{t_k}|^{2p}\right)+C\mathbb{E}\left(\sup_{0\leq k\leq(j+1-m)\vee 0}|y_{t_k}|^{2p(l+1)}\right).\end{aligned}$$

In case of $j\leq m-1$, we see

$$\mathbb{E}\left(\sup_{0\leq k\leq m}|y_{t_k}|^{2p}\right)\leq C+C\sum_{i=0}^{j}\mathbb{E}\left(\sup_{0\leq k\leq i}|y_{t_k}|^{2p}\right)\leq C.$$

Further, for $j\leq 2m-1$, it follows by the Gronwall inequality that

$$\mathbb{E}\left(\sup_{0\leq k\leq 2m}|y_{t_k}|^{2p}\right)\leq C+C\mathbb{E}\left(\sup_{0\leq k\leq m}|y_{t_k}|^{2p(l+1)}\right)\leq C.$$

The desired assertion follows by the method of induction. □

Remark 2.4. By [18], for $\theta \in [0, \frac{1}{2})$, besides assumptions (A1)–(A3), if we further assume that there exists a positive constant $\bar{K}$ such that for any $x \in \mathbb{R}^n$,

$$|b(x,0)| \leq \bar{K}(1+|x|),$$

we can also show that pth moment of θ-EM scheme is bounded by a positive constant independent of Δ.

2.3. *Convergence rates*

We find it is convenient to work with a continuous form of a numerical method. Noting that the split-step θ-EM scheme (2.12) can be rewritten as

$$\begin{aligned} z_{t_{k+1}} - D(z_{t_{k+1-m}}) &= z_{t_0} - D(z_{t_{-m}}) + \sum_{i=0}^{k} b(y_{t_i}, y_{t_{i-m}})\Delta + \sum_{i=0}^{k} \sigma(y_{t_i}, y_{t_{i-m}})\Delta W_{t_i} \\ &= \xi(0) - D(\xi(-\tau)) - \theta b(\xi(0), \xi(-\tau))\Delta + \sum_{i=0}^{k} b(y_{t_i}, y_{t_{i-m}})\Delta \\ &\quad + \sum_{i=0}^{k} \sigma(y_{t_i}, y_{t_{i-m}})\Delta W_{t_i}. \end{aligned}$$

Hence, we define the corresponding continuous-time split-step θ-EM solution $Z_\Delta(t)$ as follows: For any $t \in [-\tau, 0)$, $Z_\Delta(t) = \xi(t)$, $Z_\Delta(0) = \xi(0) - \theta b(\xi(0), \xi(-\tau))\Delta$;

For any $t \in [0, T]$,

$$\mathrm{d}[Z_\Delta(t) - D(Z_\Delta(t-\tau))] = b(\bar{Y}_\Delta(t), \bar{Y}_\Delta(t-\tau))\mathrm{d}t + \sigma(\bar{Y}_\Delta(t), \bar{Y}_\Delta(t-\tau))\mathrm{d}W(t), \tag{2.23}$$

where $\bar{Y}_\Delta(t)$ is defined by

$$\bar{Y}_\Delta(t) := y_{t_k} \quad \text{for } t \in [t_k, t_{k+1}),$$

thus $\bar{Y}_\Delta(t-\tau) = y_{t_{k-m}}$. We now define the continuous θ-EM solution $Y_\Delta(t)$ as follows:

$$Y_\Delta(t) - D(Y_\Delta(t-\tau)) = Z_\Delta(t) - D(Z_\Delta(t-\tau)) + \theta b(Y_\Delta(t), Y_\Delta(t-\tau))\Delta. \tag{2.24}$$

It can be verified that $Y_\Delta(t_k) = y_{t_k}$, $k = -m, \ldots, M$. In order to obtain convergence rates, we impose another assumption as follows:

(A4) For $x, \bar{x}, y \in \mathbb{R}^n$, $|b(x,y) - b(\bar{x}, y)| \leq V_1(x, \bar{x})|x - \bar{x}|$.

Remark 2.5. From assumptions (A1) and (A4), one sees that

$$\begin{aligned} |b(x,y)| &\leq |b(x,y) - b(x,0)| + |b(x,0) - b(0,0)| + |b(0,0)| \\ &\leq V_1(x,0)|x| + V_1(y,0)|y| + |b(0,0)|, \end{aligned}$$

and further,

$$\begin{aligned}|b(x,y)-b(\bar{x},\bar{y})| &\le |b(x,y)-b(\bar{x},y)|+|b(\bar{x},y)-b(\bar{x},\bar{y})|\\ &\le V_1(x,\bar{x})|x-\bar{x}|+V_1(y,\bar{y})|y-\bar{y}|.\end{aligned}$$

Lemma 2.3. *Consider the θ-EM scheme* (2.11), *and let* (A1)–(A4) *hold. Then, for any $p\ge 2$, the continuous form of θ-EM scheme solution $Y_\Delta(t)$ satisfies that*

$$\mathbb{E}\left(\sup_{0\le t\le T}|Y_\Delta(t)|^p\right)\le C,$$

and

$$\mathbb{E}\left(\sup_{0\le t\le T}|Y_\Delta(t)-\bar{Y}_\Delta(t)|^p\right)\le C\Delta^{\frac{p}{2}},$$

where C is a constant independent of Δ.

Proof. For any $p\ge 2$, by the elementary inequality (2.14), we have

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le u\le t}|Z_\Delta(u)-D(Z_\Delta(u-\tau))|^p\right)\\ &\quad\le 3^{p-1}|Z_\Delta(0)-D(Z_\Delta(-\tau))|^p+3^{p-1}\mathbb{E}\left(\sup_{0\le u\le t}\left|\int_0^u b(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau))\mathrm{d}s\right|^p\right)\\ &\qquad+3^{p-1}\mathbb{E}\left(\sup_{0\le u\le t}\left|\int_0^u \sigma(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau))\mathrm{d}W(s)\right|^p\right).\end{aligned}$$

Using the Hölder inequality, the BDG inequality, and together with (A2)–(A4), (2.2) and Lemma 2.2 yields

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le u\le t}|Z_\Delta(u)-D(Z_\Delta(u-\tau))|^p\right)\\ &\quad\le 3^{p-1}|Z_\Delta(0)-D(Z_\Delta(-\tau))|^p+3^{p-1}t^{p-1}\mathbb{E}\int_0^t|b(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau))|^p\mathrm{d}s\\ &\qquad+C\mathbb{E}\left(\int_0^t\|\sigma(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau))\|^2\mathrm{d}s\right)^{\frac{p}{2}}\\ &\quad\le C+C\mathbb{E}\int_0^t[|V_1(\bar{Y}_\Delta(s),0)|^p|\bar{Y}_\Delta(s)|^p\\ &\qquad+|V_1(\bar{Y}_\Delta(s-\tau),0)|^p|\bar{Y}_\Delta(s-\tau)|^p+|b(0,0)|^p]\mathrm{d}s\\ &\qquad+C\mathbb{E}\int_0^t[|\bar{Y}_\Delta(s)|^p+|V(\bar{Y}_\Delta(s-\tau))|^p|\bar{Y}_\Delta(s-\tau)|^p]\mathrm{d}s\\ &\quad\le C+C\mathbb{E}\int_0^t|\bar{Y}_\Delta(s)|^p\mathrm{d}s+C\mathbb{E}\int_0^t|\bar{Y}_\Delta(s)|^{(l+1)p}\mathrm{d}s\le C.\end{aligned}\tag{2.25}$$

With the relationship (2.24), similar to (2.22), we get

$$
\begin{aligned}
|Y_\Delta(t)|^2 \le \left(\frac{1}{2} - \theta K\Delta\right)^{-1} &[|Z_\Delta(t) - D(Z_\Delta(t-\tau))|^2 \\
&+ 2|V(Y_\Delta(t-\tau),0)|^2|Y_\Delta(t-\tau)|^2 + \theta K\Delta].
\end{aligned}
$$

We then derive from (2.25) that

$$
\begin{aligned}
\mathbb{E}\left(\sup_{0\le u\le t}|Y_\Delta(u)|^p\right) &\le C + C\mathbb{E}\left(\sup_{0\le u\le t}|Z_\Delta(u) - D(Z_\Delta(u-\tau))|^p\right) \\
&\quad + C\mathbb{E}\left(\sup_{0\le u\le t}|Y_\Delta(u-\tau)|^{(l+1)p}\right) \\
&\le C + C\mathbb{E}\left(\sup_{0\le u\le (t-\tau)\vee 0}|Y_\Delta(u)|^{(l+1)p}\right).
\end{aligned}
$$

Following the process of Lemma 2.1, we can show that the pth moment of $Y_\Delta(t)$ is bounded by a positive constant C. Denote by $\bar{Z}_\Delta(t) := z_{t_k}$ for $t \in [t_k, t_{k+1})$, we see from (2.23) that

$$
\begin{aligned}
&Z_\Delta(t) - D(Z_\Delta(t-\tau)) - \bar{Z}_\Delta(t) + D(\bar{Z}_\Delta(t-\tau)) \\
&\quad = \int_{t_k}^t b(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\mathrm{d}s + \int_{t_k}^t \sigma(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\mathrm{d}W(s),
\end{aligned}
$$

Denote by $\Phi(Z_\Delta(t), \bar{Z}_\Delta(t)) := Z_\Delta(t) - D(Z_\Delta(t-\tau)) - \bar{Z}_\Delta(t) + D(\bar{Z}_\Delta(t-\tau))$, then

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{t_k\le t<t_{k+1}}|\Phi(Z_\Delta(t), \bar{Z}_\Delta(t))|^p\right) \\
&\quad \le 2^{p-1}\mathbb{E}\left(\sup_{t_k\le t<t_{k+1}}\left|\int_{t_k}^t b(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\mathrm{d}s\right|^p\right) \\
&\qquad + 2^{p-1}\mathbb{E}\left(\sup_{t_k\le t<t_{k+1}}\left|\int_{t_k}^t \sigma(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\mathrm{d}W(s)\right|^p\right).
\end{aligned}
$$

With (A2), (A4) and Lemma 2.2, the Hölder inequality, and the BDG inequality, we get

$$
\begin{aligned}
\mathbb{E}\left(\sup_{t_k\le t<t_{k+1}}|\Phi(Z_\Delta(t), \bar{Z}_\Delta(t))|^p\right) &\le 2^{p-1}\Delta^{p-1}\mathbb{E}\left[\int_{t_k}^{t_{k+1}}\left|b(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\right|^p\mathrm{d}s\right] \\
&\quad + C\mathbb{E}\left[\int_{t_k}^{t_{k+1}}\left\|\sigma(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\right\|^2\mathrm{d}s\right]^{\frac{p}{2}} \\
&\le C\Delta^p + C\Delta^{\frac{p}{2}} \le C\Delta^{\frac{p}{2}}. \qquad (2.26)
\end{aligned}
$$

On the other hand, using relation (2.11) and definitions of $\bar{Y}_\Delta(t)$ and $\bar{Z}_\Delta(t)$, we have the following relationship between $\bar{Y}_\Delta(t)$ and $\bar{Z}_\Delta(t)$,

$$\bar{Y}_\Delta(t) - D(\bar{Y}_\Delta(t-\tau)) = \bar{Z}_\Delta(t) - D(\bar{Z}_\Delta(t-\tau)) + \theta b(\bar{Y}_\Delta(t), \bar{Y}_\Delta(t-\tau))\Delta. \tag{2.27}$$

Combining (2.24) and (2.27) gives

$$\begin{aligned} Y_\Delta(t) - \bar{Y}_\Delta(t) &= D(Y_\Delta(t-\tau)) - D(\bar{Y}_\Delta(t-\tau)) + \Phi(Z_\Delta(t), \bar{Z}_\Delta(t)) \\ &\quad + \theta[b(Y_\Delta(t), Y_\Delta(t-\tau)) - b(\bar{Y}_\Delta(t), \bar{Y}_\Delta(t-\tau))]\Delta. \end{aligned}$$

Using similar skills of (2.22), we derive from (A1) and (A3)

$$\begin{aligned} &|Y_\Delta(t) - \bar{Y}_\Delta(t)|^2 \\ &\quad \le \left(\frac{1}{2} - 2\theta K_1^2 \Delta\right)^{-1} \{|\Phi(Z_\Delta(t), \bar{Z}_\Delta(t))|^2 + [|V_3(Y_\Delta(t-\tau), \bar{Y}_\Delta(t-\tau))|^2 \\ &\qquad + 2\theta\Delta|V(Y_\Delta(t-\tau), \bar{Y}_\Delta(t-\tau))|^2]|Y_\Delta(t-\tau) - \bar{Y}_\Delta(t-\tau)|^2 + 2\theta\Delta K_1^2\}. \end{aligned}$$

Obviously, due to (2.26),

$$\begin{aligned} &\mathbb{E}\left(\sup_{0\le u\le t} |Y_\Delta(u) - \bar{Y}_\Delta(u)|^p\right) \\ &\quad \le C\mathbb{E}\left(\sup_{0\le u\le t} |\Phi(Z_\Delta(u), \bar{Z}(u))|^p\right) + C\mathbb{E}\left(\sup_{0\le u\le t} |Y_\Delta(u-\tau) - \bar{Y}_\Delta(u-\tau)|^p\right) \\ &\quad \le C\Delta^{\frac{p}{2}} + C\mathbb{E}\left(\sup_{0\le u\le (t-\tau)\vee 0} |Y_\Delta(u) - \bar{Y}_\Delta(u)|^p\right). \end{aligned}$$

The desired result follows by repeating the techniques of Lemma 2.1. □

Theorem 2.4. *Let assumptions (A1)–(A4) hold and $\theta \in \left[\frac{1}{2}, 1\right]$. Then for $p \ge 2$, we have*

$$\mathbb{E}\left(\sup_{0\le t\le T} |Y_\Delta(t) - X(t)|^p\right) \le C\Delta^{\frac{p}{2}},$$

that is, the convergence rate of θ-EM is $\frac{1}{2}$.

Proof. Denote by $e(t) := Z_\Delta(t) - D(Z_\Delta(t-\tau)) - X(t) + D(X(t-\tau))$, then

$$\begin{aligned} e(t) = e(0) &+ \int_0^t [b(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) - b(X(s), X(s-\tau))]\mathrm{d}s \\ &+ \int_0^t [\sigma(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) - \sigma(X(s), X(s-\tau))]\mathrm{d}W(s), \end{aligned}$$

where $e(0) = -\theta b(\xi(0), \xi(-\tau))\Delta$. Application of the Itô formula yields

$$\begin{aligned}|e(t)|^p \le |e(0)|^p &+ p\int_0^t |e(s)|^{p-2}\langle e(s), b(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) - b(X(s), X(s-\tau))\rangle \mathrm{d}s\\ &+ \frac{1}{2}p(p-1)\int_0^t |e(s)|^{p-2}\|\sigma(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) - \sigma(X(s), X(s-\tau))\|^2 \mathrm{d}s\\ &+ p\int_0^t |e(s)|^{p-2}\langle e(s), \sigma(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) - \sigma(X(s), X(s-\tau))\mathrm{d}W(s)\rangle.\end{aligned}$$

Rewriting $|e(t)|^p$ as

$$\begin{aligned}|e(t)|^p \ \le\ & |e(0)|^p + p\int_0^t |e(s)|^{p-2}\langle e(s), b(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) - b(Y_\Delta(s), \bar{Y}_\Delta(s-\tau))\rangle \mathrm{d}s\\ &+ p\int_0^t |e(s)|^{p-2}\langle e(s), b(Y_\Delta(s), \bar{Y}_\Delta(s-\tau)) - b(Y_\Delta(s), Y_\Delta(s-\tau))\rangle \mathrm{d}s\\ &+ p\int_0^t |e(s)|^{p-2}\langle e(s), b(Y_\Delta(s), Y_\Delta(s-\tau)) - b(X(s), X(s-\tau))\rangle \mathrm{d}s\\ &+ \frac{3}{2}p(p-1)\int_0^t |e(s)|^{p-2}\|\sigma(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) - \sigma(Y_\Delta(s), \bar{Y}_\Delta(s-\tau))\|^2 \mathrm{d}s\\ &+ \frac{3}{2}p(p-1)\int_0^t |e(s)|^{p-2}\|\sigma(Y_\Delta(s), \bar{Y}_\Delta(s-\tau)) - \sigma(Y_\Delta(s), Y_\Delta(s-\tau))\|^2 \mathrm{d}s\\ &+ \frac{3}{2}p(p-1)\int_0^t |e(s)|^{p-2}\|\sigma(Y_\Delta(s), Y_\Delta(s-\tau)) - \sigma(X(s), X(s-\tau))\|^2 \mathrm{d}s\\ &+ p\int_0^t |e(s)|^{p-2}\langle e(s), \sigma(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) - \sigma(X(s), X(s-\tau))\mathrm{d}W(s)\rangle\\ =:\ & |e(0)|^p + H_1(t) + H_2(t) + H_3(t) + H_4(t) + H_5(t) + H_6(t) + H_7(t).\end{aligned}$$

By (A3), (A4), (2.24), Lemma 2.3, the Young inequality and the Hölder inequality

$$\mathbb{E}|XY| \le (\mathbb{E}|X|^p)^{\frac{1}{p}}(\mathbb{E}|Y|^q)^{\frac{1}{q}}, \quad p, q > 1, \quad 1/p + 1/q = 1,$$

we have

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le u\le t}|H_1(u)|\right)\\ &\quad\le C\mathbb{E}\int_0^t |e(s)|^p \mathrm{d}s + C\mathbb{E}\int_0^t |b(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) - b(Y_\Delta(s), \bar{Y}_\Delta(s-\tau))|^p \mathrm{d}s\\ &\quad\le C\mathbb{E}\int_0^t [|Y_\Delta(s) - X(s)|^p + |V_3(Y_\Delta(s-\tau), X(s-\tau))|^p\\ &\qquad\times |Y_\Delta(s-\tau) - X(s-\tau)|^p + \theta^p\Delta^p|b(Y_\Delta(s), Y_\Delta(s-\tau))|^p]\mathrm{d}s\\ &\qquad + C\mathbb{E}\int_0^t |V_1(\bar{Y}_\Delta(s), Y_\Delta(s))|^p|\bar{Y}_\Delta(s) - Y_\Delta(s)|^p \mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\leq C\mathbb{E}\int_0^t |Y_\Delta(s)-X(s)|^p\mathrm{d}s \\
&\quad + C\int_0^t [\mathbb{E}|V_3(Y_\Delta(s-\tau),X(s-\tau))|^{2p}]^{\frac{1}{2}}[\mathbb{E}|Y_\Delta(s-\tau)-X(s-\tau)|^{2p}]^{\frac{1}{2}}\mathrm{d}s \\
&\quad + C\Delta^p\mathbb{E}\int_0^t [V_1(Y_\Delta(s),0)|Y_\Delta(s)| + V_1(Y_\Delta(s-\tau),0)|Y_\Delta(s-\tau)| \\
&\quad + |b(0,0)|]^p\mathrm{d}s \\
&\quad + C\int_0^t [\mathbb{E}|V_1(\bar{Y}_\Delta(s),Y_\Delta(s))|^{2p}]^{\frac{1}{2}}[\mathbb{E}|\bar{Y}_\Delta(s)-Y_\Delta(s)|^{2p}]^{\frac{1}{2}}\mathrm{d}s \\
&\leq C\int_0^t \mathbb{E}\left(\sup_{0\leq u\leq s}|Y_\Delta(u)-X(u)|^p\right)\mathrm{d}s + C\Delta^p + C\Delta^{\frac{p}{2}}.
\end{aligned}
$$

By (A1), Lemma 2.3, the Young inequality and the Hölder inequality,

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq u\leq t}|H_2(u)|\right) \\
&\quad\leq C\mathbb{E}\int_0^t |e(s)|^p\mathrm{d}s + C\mathbb{E}\int_0^t |b(Y_\Delta(s),\bar{Y}_\Delta(s-\tau)) - b(Y_\Delta(s),Y_\Delta(s-\tau))|^p\mathrm{d}s \\
&\quad\leq C\mathbb{E}\int_0^t |e(s)|^p\mathrm{d}s \\
&\qquad + C\mathbb{E}\int_0^t |V(\bar{Y}_\Delta(s-\tau),Y_\Delta(s-\tau))|^p|\bar{Y}_\Delta(s-\tau)-Y_\Delta(s-\tau)|^p\mathrm{d}s \\
&\quad\leq C\int_0^t \mathbb{E}\left(\sup_{0\leq u\leq s}|Y_\Delta(u)-X(u)|^p\right)\mathrm{d}s + C\Delta^p + C\Delta^{\frac{p}{2}}.
\end{aligned}
$$

Due to (A1) and (A2), Lemma 2.3 and the Young inequality,

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq u\leq t}|H_3(u)+H_6(u)|\right) \\
&\quad\leq C\mathbb{E}\int_0^t |e(s)|^{p-2}|Y_\Delta(s)-X(s)|^2\mathrm{d}s \\
&\qquad + C\mathbb{E}\int_0^t |e(s)|^{p-2}|V_1(Y_\Delta(s-\tau),X(s-\tau))|^2|Y_\Delta(s-\tau)-X(s-\tau)|^2\mathrm{d}s \\
&\qquad + C\mathbb{E}\int_0^t |e(s)|^{p-2}|V_2(Y_\Delta(s-\tau),X(s-\tau))|^2|Y_\Delta(s-\tau)-X(s-\tau)|^2\mathrm{d}s \\
&\qquad + C\mathbb{E}\int_0^t |e(s)|^{p-2}|\theta b(Y_\Delta(s),Y_\Delta(s-\tau))\Delta||b(Y_\Delta(s),Y_\Delta(s-\tau)) \\
&\qquad - b(X(s),X(s-\tau))|\mathrm{d}s
\end{aligned}
$$

$$\begin{aligned}&\le C\mathbb{E}\int_0^t |Y_\Delta(s)-X(s)|^p \mathrm{d}s + C\Delta^p\\ &\le C\int_0^t \mathbb{E}\left(\sup_{0\le u\le s}|Y_\Delta(u)-X(u)|^p\right)\mathrm{d}s + C\Delta^p.\end{aligned}$$

In the same way as the estimation of $H_1(t)$ and $H_2(t)$, we get

$$\mathbb{E}\left(\sup_{0\le u\le t}|H_4(u)|\right) \le C\int_0^t \mathbb{E}\left(\sup_{0\le u\le s}|Y_\Delta(u)-X(u)|^p\right)\mathrm{d}s + C\Delta^p + C\Delta^{\frac{p}{2}},$$

and

$$\mathbb{E}\left(\sup_{0\le u\le t}|H_5(u)|\right) \le C\int_0^t \mathbb{E}\left(\sup_{0\le u\le s}|Y_\Delta(u)-X(u)|^p\right)\mathrm{d}s + C\Delta^p + C\Delta^{\frac{p}{2}}.$$

Furthermore, by (A3), Lemma 2.3, the BDG inequality and the Young inequality, similar to the process of (2.8), we compute

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le u\le t}|H_7(u)|\right)\\ &\quad\le C\mathbb{E}\left(\int_0^t |e(s)|^{2p-2}\|\sigma(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau))-\sigma(X(s),X(s-\tau))\|^2\mathrm{d}s\right)^{\frac{1}{2}}\\ &\quad\le \mathbb{E}\left(\frac{1}{4}\sup_{0\le u\le t}|e(u)|^p\right)^{\frac{p-1}{p}}\left(\left(C\int_0^t \|\sigma(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau))\right.\right.\\ &\quad\quad\left.\left.-\ \sigma(X(s),X(s-\tau))\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}\\ &\quad\le \frac{1}{4}\mathbb{E}\left(\sup_{0\le u\le t}|e(u)|^p\right) + C\int_0^t \mathbb{E}\left(\sup_{0\le u\le s}|Y_\Delta(u)-X(u)|^p\right)\mathrm{d}s\\ &\quad\quad + C\Delta^{\frac{p}{2}} + C\Delta^p.\end{aligned}$$

Consequently, by sorting $H_1(t)$–$H_7(t)$ together, we arrive at

$$\mathbb{E}\left(\sup_{0\le u\le t}|e(u)|^p\right) \le C\int_0^t \mathbb{E}\left(\sup_{0\le u\le s}|Y_\Delta(u)-X(u)|^p\right)\mathrm{d}s + C\Delta^{\frac{p}{2}}.$$

By the definition of $e(t)$, we derive from (A3) that

$$\begin{aligned}|Y_\Delta(t)-X(t)|^p &\le 3^{p-1}|e(t)|^p + 3^{p-1}|\theta b(Y_\Delta(t),Y_\Delta(t-\tau))\Delta|^p\\ &\quad + 3^{p-1}|D(Y_\Delta(t-\tau))-D(X(t-\tau))|^p\\ &\le 3^{p-1}|e(t)|^p + 3^{p-1}\theta^p\Delta^p|b(Y_\Delta(t),Y_\Delta(t-\tau))|^p\\ &\quad + 3^{p-1}|V_3(Y_\Delta(t-\tau),X(t-\tau))|^p|Y_\Delta(t-\tau)-X(t-\tau)|^p.\end{aligned}$$

Taking (A1) and Lemma 2.3 into consideration,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le u\le t}|Y_\Delta(u)-X(u)|^p\right)\\
&\quad\le C\mathbb{E}\left(\sup_{0\le u\le t}|e(u)|^p\right)+C\Delta^p+C\mathbb{E}\left(\sup_{0\le u\le t}|Y_\Delta(u-\tau)-X(u-\tau)|^p\right)\\
&\quad\le C\int_0^t\mathbb{E}\left(\sup_{0\le u\le s}|Y_\Delta(u)-X(u)|^p\right)\mathrm{d}s+C\Delta^{\frac{p}{2}}\\
&\qquad+C\Delta^p+C\mathbb{E}\left(\sup_{0\le u\le (t-\tau)\vee 0}|Y_\Delta(u)-X(u)|^p\right).
\end{aligned}$$

The Gronwall inequality yields

$$\mathbb{E}\left(\sup_{0\le u\le t}|Y_\Delta(u)-X(u)|^p\right)\le C\Delta^{\frac{p}{2}}+C\mathbb{E}\left(\sup_{0\le u\le (t-\tau)\vee 0}|Y_\Delta(u)-X(u)|^p\right).$$

Again, the desired result is followed by an induction. □

With strong convergence rate given in Theorem 2.4, we can easily show the following result on almost sure convergence.

Theorem 2.5. *Let* (A1)–(A4) *hold and* $\theta\in\left[\frac{1}{2},1\right]$, *then the continuous form of* θ*-EM scheme* (2.11) *converges to the exact solution of* (2.1) *almost surely with order* $\alpha<\frac{1}{2}$, *i.e. there exists a finite random variable* ζ_α *such that*

$$\sup_{0\le t\le T}|Y_\Delta(t)-X(t)|\le\zeta_\alpha\Delta^\alpha$$

for $\alpha\in(0,\frac{1}{2})$.

Proof. Define a sequence $\Delta_k, k=1,2,\ldots$ such that $\Delta_1>\Delta_2>\cdots$ and assume $\sum_k\Delta_k^{(\frac{1}{2}-\alpha)p}<\infty$. By the Chebyshev inequality and Theorem 2.4, for sufficiently large p and $\alpha<\frac{1}{2}$

$$\begin{aligned}
\sum_k\mathbb{P}\left(\sup_{0\le t\le T}|Y_{\Delta_k}(t)-X(t)|>\Delta_k^\alpha\right)&\le\sum_k\mathbb{E}\left(\sup_{0\le t\le T}|Y_{\Delta_k}(t)-X(t)|^p\right)\Delta_k^{-\alpha p}\\
&\le C\sum_k\Delta_k^{\left(\frac{1}{2}-\alpha\right)p}<\infty.
\end{aligned}$$

The Borel–Cantelli lemma implies that there exists a finite random variable ζ_α such that

$$\sup_{0\le t\le T}|Y_{\Delta_k}(t)-X(t)|\le\zeta_\alpha\Delta_k^\alpha.$$

□

3. Convergence Rates for Pure Jumps Case

In this section, we further impose some notation. Let $N(\cdot,\cdot)$ be a Poisson random measure with characteristic measure λ on a measurable subset U of $[0,\infty)$ with $\lambda(U) < \infty$, and $\tilde{N}(\mathrm{d}u,\mathrm{d}t) = N(\mathrm{d}u,\mathrm{d}t) - \lambda(\mathrm{d}u)\mathrm{d}t$ is a compensated martingale process. We consider the following NSDDE with jumps on $\mathbb{R}^n$:

$$\begin{aligned}\mathrm{d}[X(t) - D(X(t-\tau))] &= b(X(t), X(t-\tau))\mathrm{d}t \\ &\quad + \int_U h(X(t), X(t-\tau), u)\tilde{N}(\mathrm{d}u,\mathrm{d}t), \quad t \geq 0, \end{aligned} \tag{3.1}$$

with initial data $X(\theta) = \xi(\theta) \in \mathcal{L}^p_{\mathscr{F}_0}([-\tau,0];\mathbb{R}^n)$ for $\theta \in [-\tau,0]$, i.e. ξ is an $\mathscr{F}_0$-measurable $\mathcal{D}([-\tau,0];\mathbb{R}^n)$-valued random variable such that $\mathbb{E}\|\xi\|^p_\infty < \infty$ for $p \geq 2$, where $\mathcal{D}([-\tau,0];\mathbb{R}^n)$ denotes the space of all cádlág paths $\zeta : [-\tau,0] \to \mathbb{R}^n$ with uniform norm $\|\zeta\|_\infty := \sup_{-\tau\leq\theta\leq 0}|\zeta(\theta)|$. Here, $D : \mathbb{R}^n \to \mathbb{R}^n$, and $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \times \mathbb{R}^n \times U \to \mathbb{R}^n$ are measurable functions. We further assume that b is a continuous function and $\int_U |u|^p\lambda(\mathrm{d}u) < \infty$ for $p \geq 2$. Similar to Brownian motion case, for $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$, we shall assume that:

(A5) There exist positive constants $\bar{K}_2$ and $r \geq 1$ such that

$$\begin{aligned}&|h(x,y,u) - h(\bar{x},\bar{y},u)| \\ &\quad \leq [\bar{K}_2|x-\bar{x}| + V_2(y,\bar{y})|y-\bar{y}|]|u|^r, \quad \text{and} \quad |h(0,0,u)| \leq |u|^r.\end{aligned}$$

Remark 3.1. With assumption (A5), we have

$$|h(x,y,u)| \leq |h(x,y,u) - h(0,0,u)| + |h(0,0,u)| \leq [1 + \bar{K}_2|x| + V_2(y,0)|y|]|u|^r.$$

Lemma 3.1. *Let (A1), (A3) and (A5) hold. Then there exists a unique global solution to* (3.1), *moreover, the solution has the properties that for any $p \geq 2$, $T > 0$,*

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|X(t)|^p\right) \leq C, \tag{3.2}$$

where $C = C(\xi,p,T)$ is a positive constant depending on the initial data ξ and p, T.

Proof. We omit the proof here since it is similar to that of Lemma 2.1. □

We now introduce the θ-EM scheme for (3.1). Given any time $T > \tau > 0$, assume that T and τ are rational numbers, and there exists two positive integers such that $\Delta = \frac{\tau}{m} = \frac{T}{M}$, where $\Delta \in (0,1)$ is the step size. For $k = -m,\ldots,0$, set $y_{t_k} = \xi(k\Delta)$; For $k = 0,1,\ldots,M-1$, we form

$$\begin{aligned}y_{t_{k+1}} - D(y_{t_{k+1-m}}) &= y_{t_k} - D(y_{t_{k-m}}) + \theta b(y_{t_{k+1}}, y_{t_{k+1-m}})\Delta + (1-\theta)b(y_{t_k}, y_{t_{k-m}})\Delta \\ &\quad + \int_U h(y_{t_k}, y_{t_{k-m}}, u)\Delta\tilde{N}_k(\mathrm{d}u),\end{aligned} \tag{3.3}$$

where $t_k = k\Delta$, and $\Delta\tilde{N}_k(\mathrm{d}u) = \tilde{N}(t_{k+1}, \mathrm{d}u) - \tilde{N}(t_k, \mathrm{d}u)$. Here $\theta \in [0, 1]$ is an additional parameter that allows us to control the implicitness of the numerical scheme. For $\theta = 0$, the θ-EM scheme reduces to the EM scheme, and for $\theta = 1$, it is exactly the backward EM scheme. Here, we always assume $\theta \geq 1/2$. The corresponding split-step θ-EM scheme to (3.1) is defined as follows: For $k = -m, \ldots, -1$, set $z_{t_k} = y_{t_k} = \xi(k\Delta)$; For $k = 0, 1, \ldots, M-1$,

$$\begin{cases} y_{t_k} = D(y_{t_{k-m}}) + z_{t_k} - D(z_{t_{k-m}}) + \theta b(y_{t_k}, y_{t_{k-m}})\Delta, \\ z_{t_{k+1}} = D(z_{t_{k+1-m}}) + z_{t_k} - D(z_{t_{k-m}}) + b(y_{t_k}, y_{t_{k-m}})\Delta \\ \qquad\qquad + \displaystyle\int_U h(y_{t_k}, y_{t_{k-m}}, u)\Delta\tilde{N}_k(\mathrm{d}u). \end{cases} \tag{3.4}$$

It is easy to see $y_{t_{k+1}}$ in (3.4) can be rewritten as the form of (3.3). Due to the implicitness of θ-EM scheme, we require $0 < \Delta \leq \Delta^*$, where $\Delta^* \in (0, (2K \vee 4K_1^2)^{-1}\theta^{-1})$, K_1 and K are defined as in (A1) and Remark 2.3 with $\sigma \equiv \mathbf{0}$, respectively.

3.1. *Moment bounds*

Firstly, we introduce an important lemma coming from [14].

Lemma 3.2. *Let $\phi : \mathbb{R}_+ \times U \to \mathbb{R}^n$ be progressively measurable and assume that the right side is finite. Then, there exists a positive constant C such that*

$$\mathbb{E}\left(\sup_{0\leq s\leq t}\left|\int_0^s\int_U \phi(r-, u)\tilde{N}(du, dr)\right|^p\right) \leq C\mathbb{E}\int_0^t\int_U |\phi(s, u)|^p\lambda(du)ds,$$

for $p \geq 2$.

Lemma 3.3. *Let (A1), (A3) and (A5) hold. Then, there exists a positive constant C independent of Δ such that*

$$\mathbb{E}\left(\sup_{0\leq k\leq M}|y_{t_k}|^p\right) \leq C,$$

for $p \geq 2$.

Proof. It is easy to see from (3.4)

$$\begin{aligned} &|z_{t_{k+1}} - D(z_{t_{k+1-m}})|^2 \\ &\quad = |z_{t_k} - D(z_{t_{k-m}})|^2 + 2\langle z_{t_k} - D(z_{t_{k-m}}), b(y_{t_k}, y_{t_{k-m}})\Delta\rangle \\ &\qquad + |b(y_{t_k}, y_{t_{k-m}})|^2\Delta^2 + \left|\int_U h(y_{t_k}, y_{t_{k-m}}, u)\Delta\tilde{N}_k(\mathrm{d}u)\right|^2 \\ &\qquad + 2\left\langle z_{t_k} - D(z_{t_{k-m}}) + b(y_{t_k}, y_{t_{k-m}})\Delta, \int_U h(y_{t_k}, y_{t_{k-m}}, u)\Delta\tilde{N}_k(\mathrm{d}u)\right\rangle \\ &\quad = |z_{t_k} - D(z_{t_{k-m}})|^2 + 2\langle y_{t_k} - D(y_{t_{k-m}}), b(y_{t_k}, y_{t_{k-m}})\Delta\rangle \end{aligned}$$

$$+(1-2\theta)|b(y_{t_k},y_{t_{k-m}})|^2\Delta^2+\left|\int_U h(y_{t_k},y_{t_{k-m}},u)\Delta\tilde{N}_k(\mathrm{d}u)\right|^2$$
$$+2\left\langle y_{t_k}-D(y_{t_{k-m}})+(1-\theta)b(y_{t_k},y_{t_{k-m}})\Delta,\int_U h(y_{t_k},y_{t_{k-m}},u)\Delta\tilde{N}_k(\mathrm{d}u)\right\rangle.$$

Substituting $b(y_{t_k},y_{t_{k-m}})$ with y_{t_k},z_{t_k} into the last term, and using assumption (A1) leads to

$$\begin{aligned}
&|z_{t_{k+1}}-D(z_{t_{k+1-m}})|^2\\
&\quad\le |z_{t_k}-D(z_{t_{k-m}})|^2+2\Delta\langle y_{t_k}-D(y_{t_{k-m}}),b(y_{t_k},y_{t_{k-m}})\rangle\\
&\qquad+\left|\int_U h(y_{t_k},y_{t_{k-m}},u)\Delta\tilde{N}_k(\mathrm{d}u)\right|^2\\
&\qquad+\frac{2}{\theta}\left\langle y_{t_k}-D(y_{t_{k-m}}),\int_U h(y_{t_k},y_{t_{k-m}},u)\Delta\tilde{N}_k(\mathrm{d}u)\right\rangle\\
&\qquad-2\frac{1-\theta}{\theta}\left\langle z_{t_k}-D(z_{t_{k-m}}),\int_U h(y_{t_k},y_{t_{k-m}},u)\Delta\tilde{N}_k(\mathrm{d}u)\right\rangle\\
&\quad\le |z_{t_k}-D(z_{t_{k-m}})|^2+\Delta K(1+|y_{t_k}|^2)+\Delta|V(y_{t_{k-m}},0)|^2|y_{t_{k-m}}|^2\\
&\qquad+\left|\int_U h(y_{t_k},y_{t_{k-m}},u)\Delta\tilde{N}_k(\mathrm{d}u)\right|^2\\
&\qquad+\frac{2}{\theta}\left\langle y_{t_k}-D(y_{t_{k-m}}),\int_U h(y_{t_k},y_{t_{k-m}},u)\Delta\tilde{N}_k(\mathrm{d}u)\right\rangle\\
&\qquad-2\frac{1-\theta}{\theta}\left\langle z_{t_k}-D(z_{t_{k-m}}),\int_U h(y_{t_k},y_{t_{k-m}},u)\Delta\tilde{N}_k(\mathrm{d}u)\right\rangle.
\end{aligned}$$

Summing both sides from 0 to k, we deduce that

$$\begin{aligned}
&|z_{t_{k+1}}-D(z_{t_{k+1-m}})|^2\\
&\quad\le |z_{t_0}-D(z_{t_{-m}})|^2+KT+\Delta K\sum_{i=0}^k|y_{t_i}|^2+\Delta\sum_{i=0}^k|V(y_{t_{i-m}},0)|^2|y_{t_{i-m}}|^2\\
&\qquad+\sum_{i=0}^k\left|\int_U h(y_{t_i},y_{t_{i-m}},u)\Delta\tilde{N}_i(\mathrm{d}u)\right|^2\\
&\qquad+\frac{2}{\theta}\sum_{i=0}^k\left\langle y_{t_i}-D(y_{t_{i-m}}),\int_U h(y_{t_i},y_{t_{i-m}},u)\Delta\tilde{N}_i(\mathrm{d}u)\right\rangle\\
&\qquad-2\frac{1-\theta}{\theta}\sum_{i=0}^k\left\langle z_{t_i}-D(z_{t_{i-m}}),\int_U h(y_{t_i},y_{t_{i-m}},u)\Delta\tilde{N}_i(\mathrm{d}u)\right\rangle.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&|z_{t_{k+1}} - D(z_{t_{k+1-m}})|^{2p} \\
&\quad \le 6^{p-1}(|z_{t_0} - D(z_{t_{-m}})|^2 + KT)^p + 6^{p-1}K^p\Delta^p\left(\sum_{i=0}^{k}|y_{t_i}|^2\right)^p \\
&\qquad + 6^{p-1}\Delta^p\left(\sum_{i=0}^{k}|V(y_{t_{i-m}},0)|^2|y_{t_{i-m}}|^2\right)^p \\
&\qquad + 6^{p-1}\left(\sum_{i=0}^{k}\left|\int_U h(y_{t_i},y_{t_{i-m}},u)\Delta\tilde{N}_i(\mathrm{d}u)\right|^2\right)^p \\
&\qquad + 6^{p-1}4^p\left|\sum_{i=0}^{k}\left\langle y_{t_i} - D(y_{t_{i-m}}), \int_U h(y_{t_i},y_{t_{i-m}},u)\Delta\tilde{N}_i(\mathrm{d}u)\right\rangle\right|^p \\
&\qquad + 6^{p-1}2^p\left|\sum_{i=0}^{k}\left\langle z_{t_i} - D(z_{t_{i-m}}), \int_U h(y_{t_i},y_{t_{i-m}},u)\Delta\tilde{N}_i(\mathrm{d}u)\right\rangle\right|^p.
\end{aligned}$$

Here we only have to estimate the last three terms according to Lemma 2.2. With assumption (A5), we find that for $0 < j < M$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le k\le j}\left(\sum_{i=0}^{k}\left|\int_U h(y_{t_i},y_{t_{i-m}},u)\Delta\tilde{N}_i(\mathrm{d}u)\right|^2\right)^p\right] \\
&\quad \le M^{p-1}C\mathbb{E}\left(\sum_{i=0}^{j}\int_U |h(y_{t_i},y_{t_{i-m}},u)|^{2p}\lambda(\mathrm{d}u)\right) \\
&\quad \le C\sum_{i=0}^{j}\mathbb{E}\int_U([1+\bar{K}_2|y_{t_i}| + V_2(y_{t_{i-m}},0)|y_{t_{i-m}}|]^{2p}|u|^{2pr})\lambda(\mathrm{d}u) \\
&\quad \le C + C\sum_{i=0}^{j}\mathbb{E}|y_{t_i}|^{2p} + C\sum_{i=0}^{j}\mathbb{E}(|V_2(y_{t_{i-m}},0)|^{2p}|y_{t_{i-m}}|^{2p}).
\end{aligned}$$

Applying (A3), (A5), Lemma 3.2 and the Hölder inequality and noticing that $\int_U |u|^p\lambda(\mathrm{d}u) < \infty$ for $p \ge 2$, we compute

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le k\le j}\left|\sum_{i=0}^{k}\left\langle y_{t_i} - D(y_{t_{i-m}}), \int_U h(y_{t_i},y_{t_{i-m}},u)\Delta\tilde{N}_i(\mathrm{d}u)\right\rangle\right|^p\right] \\
&\quad \le C\mathbb{E}\left(\sum_{i=0}^{j}|y_{t_i} - D(y_{t_{i-m}})|^2\int_U |h(y_{t_i},y_{t_{i-m}},u)|^2\lambda(\mathrm{d}u)\right)^{\frac{p}{2}} \\
&\quad \le C\mathbb{E}\sum_{i=0}^{j}|y_{t_i} - D(y_{t_{i-m}})|^p\int_U[1+\bar{K}_2|y_{t_i}| + V_2(y_{t_{i-m}},0)|y_{t_{i-m}}|]^p|u|^{pr}\lambda(\mathrm{d}u)
\end{aligned}$$

$$
\begin{aligned}
&\leq C\mathbb{E}\sum_{i=0}^{j}(|y_{t_i}|^p + |V_3(y_{t_{i-m}},0)|^p|y_{t_{i-m}}|^p)\int_U [1+\bar{K}_2|y_{t_i}| \\
&\quad + V_2(y_{t_{i-m}},0)|y_{t_{i-m}}|]^p|u|^{pr}\lambda(\mathrm{d}u) \\
&\leq C + C\sum_{i=0}^{j}\mathbb{E}|y_{t_i}|^{2p} + C\sum_{i=0}^{j}\mathbb{E}(|V_3(y_{t_{i-m}},0)|^{2p}|y_{t_{i-m}}|^{2p}) \\
&\quad + C\sum_{i=0}^{j}\mathbb{E}(|V_2(y_{t_{i-m}},0)|^{2p}|y_{t_{i-m}}|^{2p}).
\end{aligned}
$$

Similarly, by (A5) and Lemma 3.2 again

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq k\leq j}\left|\sum_{i=0}^{k}\left\langle z_{t_i} - D(z_{t_{i-m}}), \int_U h(y_{t_i}, y_{t_{i-m}}, u)\Delta\tilde{N}_i(\mathrm{d}u)\right\rangle\right|^p\right] \\
&\quad \leq C\mathbb{E}\left(\sum_{i=0}^{j}|z_{t_i} - D(z_{t_{i-m}})|^2\int_U |h(y_{t_i}, y_{t_{i-m}}, u)|^2\lambda(\mathrm{d}u)\right)^{\frac{p}{2}} \\
&\quad \leq C\mathbb{E}\left(\sum_{i=0}^{j}|z_{t_i} - D(z_{t_{i-m}})|^p\int_U [\bar{K}_2(1+|y_{t_i}|^2)\right. \\
&\qquad \left. + |V_2(y_{t_{i-m}},0)|^2|y_{t_{i-m}}|^2]^{\frac{p}{2}}|u|^{pr}\lambda(\mathrm{d}u)\right) \\
&\quad \leq C\mathbb{E}\left(\sup_{0\leq k\leq j+1}|z_{t_k} - D(z_{t_{k-m}})|^p\sum_{i=0}^{j}[\bar{K}_2(1+|y_{t_i}|^2)\right. \\
&\qquad \left. + |V_2(y_{t_{i-m}},0)|^2|y_{t_{i-m}}|^2]^{\frac{p}{2}}\right) \\
&\quad \leq \frac{1}{2}\mathbb{E}\left[\sup_{0\leq k\leq j+1}|z_{t_k} - D(z_{t_{k-m}})|^{2p}\right] \\
&\qquad + C + C\sum_{i=0}^{j}\mathbb{E}|y_{t_i}|^{2p} + C\sum_{i=0}^{j}\mathbb{E}(|V_2(y_{t_{i-m}},0)|^{2p}|y_{t_{i-m}}|^{2p}).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq k\leq j+1}|z_{t_k} - D(z_{t_{k-m}})|^{2p}\right] \\
&\quad \leq C + C\sum_{i=0}^{j}\mathbb{E}|y_{t_i}|^{2p} + C\sum_{i=0}^{j}\mathbb{E}(|V_2(y_{t_{i-m}},0)|^{2p}|y_{t_{i-m}}|^{2p})
\end{aligned}
$$

$$
\begin{aligned}
&+ C\sum_{i=0}^{j}\mathbb{E}(|V_3(y_{t_{i-m}},0)|^{2p}|y_{t_{i-m}}|^{2p})\\
&\le C + C\sum_{i=0}^{j}\mathbb{E}|y_{t_i}|^{2p} + C\sum_{i=0}^{j}\mathbb{E}|y_{t_{i-m}}|^{2p(l+1)}.
\end{aligned}
$$

Following the steps of (2.21), the desired assertion can be derived by similar skills. □

3.2. *Convergence rates*

Firstly, we define the corresponding continuous-time split-step θ-EM solution $Z_\Delta(t)$ as follows: For any $t \in [-\tau, 0)$, $Z_\Delta(t) = \xi(t)$, $Z_\Delta(0) = \xi(0) - \theta b(\xi(0), \xi(-\tau))\Delta$; For any $t \in [0, T]$,

$$
\begin{aligned}
&\mathrm{d}[Z_\Delta(t) - D(Z_\Delta(t-\tau))]\\
&\quad = b(\bar{Y}_\Delta(t), \bar{Y}_\Delta(t-\tau))\mathrm{d}t + \int_U h(\bar{Y}_\Delta(t), \bar{Y}_\Delta(t-\tau), u)\tilde{N}(\mathrm{d}u, \mathrm{d}t), \quad (3.5)
\end{aligned}
$$

where $\bar{Y}_\Delta(t)$ is defined by

$$\bar{Y}_\Delta(t) := y_{t_k} \quad \text{for } t \in [t_k, t_{k+1}),$$

thus $\bar{Y}_\Delta(t-\tau) = y_{t_{k-m}}$. Further, the continuous form of θ-EM solution $Y_\Delta(t)$ is defined by (2.24).

Lemma 3.4. *Consider the θ-EM scheme* (3.3), *and let* (*A*1), (*A*3)–(*A*5) *hold. Then, for any $p \ge 2$, the continuous form $Y_\Delta(t)$ of θ-EM scheme has the following properties*:

$$\mathbb{E}\left(\sup_{0\le t\le T}|Y_\Delta(t)|^p\right) \le C,$$

and

$$\mathbb{E}\left(\sup_{0\le t\le T}|Y_\Delta(t) - \bar{Y}_\Delta(t)|^p\right) \le C\Delta,$$

where C is a constant independent of Δ.

Proof. The proof is similar to that of Lemma 2.3. Here, we only give the most critical part to show the differences to the Brownian motion case. For $t \in [t_k, t_{k+1})$, (3.5) gives

$$
\begin{aligned}
&Z_\Delta(t) - D(Z_\Delta(t-\tau)) - Z_\Delta(t_k) + D(Z_\Delta(t_{k-m}))\\
&\quad = \int_{t_k}^{t} b(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\mathrm{d}s + \int_{t_k}^{t}\int_U h(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau), u)\tilde{N}(\mathrm{d}u, \mathrm{d}s).
\end{aligned}
$$

Denote by $\Phi(Z_\Delta(t), Z_\Delta(t_k)) = Z_\Delta(t) - D(Z_\Delta(t-\tau)) - Z_\Delta(t_k) + D(Z_\Delta(t_{k-m}))$, then

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_k \le t < t_{k+1}} |\Phi(Z_\Delta(t), Z_\Delta(t_k))|^p\right) \\
&\quad \le 2^{p-1}\mathbb{E}\left(\sup_{t_k \le t < t_{k+1}} \left|\int_{t_k}^t b(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\mathrm{d}s\right|^p\right) \\
&\qquad + 2^{p-1}\mathbb{E}\left(\sup_{t_k \le t < t_{k+1}} \left|\int_{t_k}^t \int_U h(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau), u)\tilde{N}(\mathrm{d}u, \mathrm{d}s)\right|^p\right).
\end{aligned}$$

Application of (A4), Lemmas 3.2, 3.3 and the Hölder inequality gives that

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_k \le t < t_{k+1}} |\Phi(Z_\Delta(t), Z_\Delta(t_k))|^p\right) \\
&\quad \le 2^{p-1}\Delta^{p-1}\mathbb{E}\int_{t_k}^{t_{k+1}} \left|b(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\right|^p \mathrm{d}s \\
&\qquad + C\mathbb{E}\int_{t_k}^{t_{k+1}} \int_U |h(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau), u)|^p \lambda(\mathrm{d}u)\mathrm{d}s \\
&\quad \le C\Delta^p + C\Delta \le C\Delta.
\end{aligned}$$

Following the proof of Lemma 2.3, we will get the desired result. □

Theorem 3.5. *Let assumptions* (A1), (A3)–(A5) *hold, then the θ-EM solution $Y_\Delta(t)$ and the exact solution $X(t)$ has the following relationship*:

$$\mathbb{E}\left(\sup_{0 \le t \le T} |Y_\Delta(t) - X(t)|^p\right) \le C\Delta^{\frac{1}{2}},$$

for $p \ge 2$.

Proof. Let $e(t) = Z_\Delta(t) - D(Z_\Delta(t-\tau)) - X(t) + D(X(t-\tau))$, it is obvious that

$$\begin{aligned}
e(t) = e(0) &+ \int_0^t [b(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) - b(X(s), X(s-\tau))]\mathrm{d}s \\
&+ \int_0^t \int_U [h(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau), u) - h(X(s), X(s-\tau), u)]\tilde{N}(\mathrm{d}u, \mathrm{d}s),
\end{aligned}$$

where $e(0) = -\theta b(\xi(0), \xi(-\tau))\Delta$. Define

$$\mu(t) = b(\bar{Y}_\Delta(t), \bar{Y}_\Delta(t-\tau)) - b(X(t), X(t-\tau)),$$

and

$$\upsilon(t) = h(\bar{Y}_\Delta(t), \bar{Y}_\Delta(t-\tau), u) - h(X(t), X(t-\tau), u).$$

Application of the Itô formula yields

$$\begin{aligned}
|e(t)|^p \leq\ & |e(0)|^p + p\int_0^t |e(s)|^{p-2}\langle e(s),\mu(s)\rangle \mathrm{d}s \\
& + \int_0^t\int_U [|e(s)+\upsilon(s)|^p - |e(s)|^p - p|e(s)|^{p-2}\langle e(s),\upsilon(s)\rangle]\lambda(\mathrm{d}u)\mathrm{d}s \\
& + \int_0^t\int_U [|e(s)+\upsilon(s)|^p - |e(s)|^p]\tilde{N}(\mathrm{d}u,\mathrm{d}s) \\
\leq\ & |e(0)|^p + p\int_0^t |e(s)|^{p-2}\langle e(s),\mu(s)\rangle \mathrm{d}s + C\int_0^t\int_U |e(s)|^{p-2}|\upsilon(s)|^2\lambda(\mathrm{d}u)\mathrm{d}s \\
& + C\int_0^t\int_U |\upsilon(s)|^p\lambda(\mathrm{d}u)\mathrm{d}s + \int_0^t\int_U [|e(s)+\upsilon(s)|^p - |e(s)|^p]\tilde{N}(\mathrm{d}u,\mathrm{d}s) \\
=:\ & |e(0)|^p + \bar{H}_1(t) + \bar{H}_2(t) + \bar{H}_3(t) + \bar{H}_4(t).
\end{aligned}$$

Similar to the derivation of Theorem 2.4, with (A5) and the results of Lemmas 3.3 and 3.4, we calculate

$$\begin{aligned}
& \mathbb{E}\left(\sup_{0\leq u\leq t}|\bar{H}_1(u)|\right) \\
& \leq C\mathbb{E}\int_0^t |e(s)|^p \mathrm{d}s + C\mathbb{E}\int_0^t |b(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau)) - b(Y_\Delta(s),\bar{Y}_\Delta(s-\tau))|^p \mathrm{d}s \\
& \quad + C\mathbb{E}\int_0^t |b(Y_\Delta(s),\bar{Y}_\Delta(s-\tau)) - b(Y_\Delta(s),Y_\Delta(s-\tau))|^p \mathrm{d}s \\
& \quad + C\mathbb{E}\int_0^t |b(Y_\Delta(s),Y_\Delta(s-\tau)) - b(X(s),X(s-\tau))|^p \mathrm{d}s \\
& \leq C\int_0^t \mathbb{E}\left(\sup_{0\leq u\leq s}|Y_\Delta(u)-X(u)|^p\right)\mathrm{d}s + C\Delta^p \\
& \quad + C\int_0^t [\mathbb{E}(1+|\bar{Y}_\Delta(s)|^{l_1} + |Y_\Delta(s)|^{l_1})^{2p}]^{\frac{1}{2}}[\mathbb{E}|\bar{Y}_\Delta(s)-Y_\Delta(s)|^{2p}]^{\frac{1}{2}}\mathrm{d}s \\
& \quad + C\int_0^t [\mathbb{E}(1+|\bar{Y}_\Delta(s-\tau)|^{l_1} \\
& \quad + |Y_\Delta(s-\tau)|^{l_1})^{2p}]^{\frac{1}{2}}[\mathbb{E}|\bar{Y}_\Delta(s-\tau)-Y_\Delta(s-\tau)|^{2p}]^{\frac{1}{2}}\mathrm{d}s \\
& \quad + C\int_0^t [\mathbb{E}(1+|Y_\Delta(s)|^{l_1} + |X(s)|^{l_1})^{2p}]^{\frac{1}{2}}[\mathbb{E}|Y_\Delta(s)-X(s)|^{2p}]^{\frac{1}{2}}\mathrm{d}s \\
& \quad + C\int_0^t [\mathbb{E}(1+|Y_\Delta(s-\tau)|^{l_1}
\end{aligned}$$

$$
\begin{aligned}
&+ |X(s-\tau)|^{l_1})^{2p}]^{\frac{1}{2}}[\mathbb{E}|Y_\Delta(s-\tau)-X(s-\tau)|^{2p}]^{\frac{1}{2}}\mathrm{d}s \\
&\le C\int_0^t \mathbb{E}\left(\sup_{0\le u\le s}|Y_\Delta(u)-X(u)|^p\right)\mathrm{d}s + C\Delta^p + C\Delta^{\frac{1}{2}}.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le u\le t}|\bar{H}_2(u)|\right) + \mathbb{E}\left(\sup_{0\le u\le t}|\bar{H}_3(u)|\right) \\
&\quad \le C\int_0^t \mathbb{E}\left(\sup_{0\le u\le s}|Y_\Delta(u)-X(u)|^p\right)\mathrm{d}s + C\Delta^p + C\Delta^{\frac{1}{2}}.
\end{aligned}
$$

Furthermore, by Lemmas 3.2–3.4 and the Hölder inequality, we compute

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le u\le t}|\bar{H}_4(u)|\right) \\
&\quad \le \frac{1}{4}\mathbb{E}\left(\sup_{0\le u\le t}|e(u)|^p\right) + C\mathbb{E}\left(\int_0^t\int_U |\upsilon(s)|^p\lambda(\mathrm{d}u)\mathrm{d}s\right) \\
&\quad \le \frac{1}{4}\mathbb{E}\left(\sup_{0\le u\le t}|e(u)|^p\right) + C\int_0^t \mathbb{E}\left(\sup_{0\le u\le s}|Y_\Delta(u)-X(u)|^p\right)\mathrm{d}s + C\Delta^p + C\Delta^{\frac{1}{2}}.
\end{aligned}
$$

Putting $\bar{H}_1(t)$–$\bar{H}_4(t)$ together, we arrive at

$$
\mathbb{E}\left(\sup_{0\le u\le t}|e(u)|^p\right) \le C\int_0^t \mathbb{E}\left(\sup_{0\le u\le s}|Y_\Delta(u)-X(u)|^p\right)\mathrm{d}s + C\Delta^{\frac{1}{2}}.
$$

Consequently, following the process of Theorem 2.4, the desired result will be obtained. □

Remark 3.2. We see from Theorems 2.4 and 3.5 that the strong convergence rate of θ-EM scheme for NSDDEs is $\frac{1}{2}$ for the Brownian motion case, while for the pure jumps case, the order is $\frac{1}{2p}$, that is to say, lower moment has a better convergence rate for NSDDEs with jumps, whence it is better to use the mean-square convergence for jump case.

Theorem 3.6. *Let* (A1), (A3)–(A5) *hold, then the continuous form of θ-EM scheme* (3.3) *converges to the exact solution of* (3.1) *almost surely with order $\alpha < \frac{1}{2p}$, i.e. there exists a finite random variable ζ_α such that*

$$
\sup_{0\le t\le T}|Y_\Delta(t)-X(t)| \le \zeta_\alpha\Delta^\alpha,
$$

for $\alpha\in(0,\frac{1}{2p})$.

Proof. The desired result can be obtained with Theorem 3.5, and the procedure is similar to that of Theorem 2.5. □

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