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QUANTITATIVE SYMMETRY BREAKING OF GROUNDSTATES FOR A CLASS OF WEIGHTED EMDEN-FOWLER EQUATIONS

CARLO MERCURI AND EDERSON MOREIRA DOS SANTOS

ABSTRACT. We consider a class of weighted Emden-Fowler equations

$$(\mathcal{P}_\alpha) \quad \begin{cases} -\Delta u = V_\alpha(x) u^p & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

posed on the unit ball $B = B(0, 1) \subset \mathbb{R}^N$, $N \geq 1$. We prove that symmetry breaking occurs for the groundstate solutions as the parameter $\alpha \rightarrow \infty$. The above problem reads as a possibly large perturbation of the classical Hénon equation. We consider a radial function V_α having a spherical shell of zeroes at $|x| = R \in (0, 1]$. For $N \geq 3$, a quantitative condition on R for this phenomenon to occur is given by means of universal constants, such as the best constant for the subcritical Sobolev's embedding $H_0^1(B) \subset L^{p+1}(B)$. In the case $N = 2$ we highlight a similar phenomenon when $R = R(\alpha)$ is a function with a suitable decay. Moreover, combining energy estimates and Liouville type theorems we study some qualitative and quantitative properties of the groundstate solutions to (\mathcal{P}_α) as $\alpha \rightarrow \infty$.

1. INTRODUCTION AND RESULTS

For $R \in (0, 1)$ and $\alpha > 0$ we set

$$(V) \quad V(r) = \begin{cases} \left(1 - \frac{r}{R}\right)^\alpha & \text{if } 0 \leq r < R, \\ \left(1 - \frac{1-r}{1-R}\right)^\alpha & \text{if } R \leq r \leq 1. \end{cases}$$

We also set $V(|x|) = |x|^\alpha$ for $R = 0$ and $V(|x|) = (1 - |x|)^\alpha$ for $R = 1$. We will use R -subscripts and α -subscripts in those instances where the dependence on R and α plays any role. We consider the boundary value problem

$$(\mathcal{P}) \quad \begin{cases} -\Delta u = V(|x|) |u|^{p-1} u & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

on the unit ball $B = B(0, 1) \subset \mathbb{R}^N$. Throughout the paper we shall consider p superlinear and Sobolev-subcritical, namely $p \in \left(1, \frac{N+2}{N-2}\right)$ if $N \geq 3$ and $p > 1$ if $N = 1, 2$.

Note that (\mathcal{P}) interpolates two opposite situations: with $R = 1$ we obtain

$$(\mathcal{D}) \quad \begin{cases} -\Delta u = (1 - |x|)^\alpha |u|^{p-1} u & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

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whose positive solutions, in particular groundstates, are radially symmetric by the celebrated result of Gidas-Ni-Nirenberg [12], whereas with $R = 0$ we are led to the Hénon equation [14]

$$(\mathcal{H}) \quad \begin{cases} -\Delta u = |x|^\alpha |u|^{p-1}u & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

whose groundstate solutions (which we know are positive) [8, 22] are nonradial for large enough values of α . Here we highlight, quantitatively, a stability phenomenon for the symmetry breaking of groundstate solutions. In fact, we find a condition on R , by means of universal constants, for the symmetry breaking of groundstate solutions of (\mathcal{P}) to occur for any α suitably large. In the present context N and p are parameters which are fixed from the start. We recall that if u is a groundstate solution of (\mathcal{P}) , with $\alpha \geq 0$, then $|u|$ is also a groundstate solution. Then, by the strong maximum principle, $|u| > 0$ in B and hence either $u > 0$ in B or $u < 0$ in B . So, throughout this paper, we will work with positive groundstate solutions. We know from [5, Theorem 2.6] that any groundstate solution of (\mathcal{P}) is foliated Schwarz symmetric if $R \in [0, 1)$ and radially symmetric and radially decreasing if $R = 1$. Hence, we may assume that the maximum of any groundstate solution of (\mathcal{P}) is attained at a point of the form $(s_\alpha, 0, \dots, 0) \in B$ where $s_\alpha \in [0, 1)$.

Non-autonomous equations like (\mathcal{P}) naturally arise in Astrophysics when describing the stability of stellar structures. One of the most remarkable predictions in General Relativity, see e.g. [26] and [13], is the existence of black holes. These were known just as a mathematical entity since Schwarzschild's stationary and spherically symmetric singular solution to Einstein's equations, until recent years when in fact black hole like objects have been perceived [17], and remarkably, photographed [27].

In the last few decades astrophysicists and mathematicians have devoted intensive effort to detect and understand the existence of black holes in globular clusters. In 1972, Peebles [20, 21] published seminal works describing a stationary distribution of stars near a massive collapsed object, such as a black hole, located at the cluster centre.

As shown in [2–4, 15, 16], the admissibility of stationary and spherically symmetric stellar models, whose dynamics is based on coupling Vlasov and Poisson equations, is equivalent to the solvability of an Emden-Fowler type equation

$$-\Delta U(x) = f(|x|, U(x)) \quad \text{in } \mathbb{R}^3.$$

These equations are extensively considered in Astrophysics as a class of nonlinear Poisson equations describing certain self-gravitating, spherically symmetric stellar systems: the well-known Lane-Emden equation, written in PDE form as

$$-\Delta U(x) = |U(x)|^{p-2}U(x) \quad \text{in } \mathbb{R}^3,$$

describes stellar systems modelled as *polytropic* fluids, namely those where a nonlinear (polynomial) relation between the mass density, related to a certain power of U , and the pressure holds; see e.g. the classical book of the 1983 Nobel in Physics laureate S. Chandrasekhar [10]. The particular case $f(|x|, U(x)) = |x|^\alpha |U(x)|^{p-2}U(x)$, $\alpha > 0$, $p > 2$, namely

$$-\Delta U(x) = |x|^\alpha |U(x)|^{p-2}U(x) \quad \text{in } \mathbb{R}^3,$$

has been proposed in 1973 by Hénon [14] in the context of the so-called concentric shell model, as a correction/generalisation to the Lane-Emden equation, to investigate numerically the stability of spherical steady state stellar systems.

When considering (\mathcal{H}) , namely the Dirichlet problem associated with the Hénon equation, in [22] it has been shown that at least two positive solutions exist for large values of α . Namely, a radial solution, and surprisingly a least energy solution which is not radially symmetric. Moreover in [8, 9] it has been shown that the presence of $|x|^\alpha$, as α gets large, forces the least energy solutions to concentrate as far as possible from the origin (in fact near the boundary); see also [18] and references therein. In the case of (\mathcal{P}) for $R \in (0, 1]$ in line with these results, one may naturally expect that, as α gets large, concentration should occur far away from the shell of zeroes of V , $\{x : |x| = R\}$, namely either at the origin or at the boundary.

Notation. $B_\rho = B(0, \rho) \subset \mathbb{R}^N$ is the ball centred at the origin with radius $\rho > 0$. We set $B = B_1$, whose boundary has $(N - 1)$ -dimensional Lebesgue measure given in terms of the classical Gamma-function by

$$|\partial B| = |\mathbb{S}^{N-1}| = \frac{2\pi^{N/2}}{\Gamma(N/2)}.$$

Solutions to (\mathcal{P}) are critical points of

$$(I) \quad I(u) = \frac{1}{2} \int_B |Du|^2 dx - \frac{1}{p+1} \int_B V(|x|)|u|^{p+1} dx, \quad u \in H_0^1(B),$$

and its groundstates can be found by suitable rescaling of optimisers to the best constant

$$(S_\alpha) \quad S_\alpha = \inf_{0 \neq u \in H_0^1(B)} \frac{\int_B |Du|^2 dx}{\left(\int_B V(|x|)|u|^{p+1} dx \right)^{\frac{2}{p+1}}},$$

and its least energy radial solutions correspond to rescaling of optimisers to the constant

$$(S_{\alpha, \text{rad}}) \quad S_{\alpha, \text{rad}} = \inf_{0 \neq u \in H_{0, \text{rad}}^1(B)} \frac{\int_B |Du|^2 dx}{\left(\int_B V(|x|)|u|^{p+1} dx \right)^{\frac{2}{p+1}}},$$

which are achieved by standard direct minimisation in $H_0^1(B)$ and, respectively in $H_{0, \text{rad}}^1(B)$. In particular, we consider a positive optimiser $u_{\alpha, \text{rad}}^*$ to $S_{\alpha, \text{rad}}$ such that

$$\int_B V(|x|)|u_{\alpha, \text{rad}}^*|^{p+1} dx = 1.$$

Then by the Palais symmetric criticality principle

$$(1.1) \quad -\Delta u_{\alpha, \text{rad}}^* = S_{\alpha, \text{rad}} V(|x|) (u_{\alpha, \text{rad}}^*)^p$$

holds weakly in $H_0^1(B)$. Since the rescaled function $u_{\alpha, \text{rad}} = (S_{\alpha, \text{rad}})^{\frac{1}{p-1}} u_{\alpha, \text{rad}}^*$ solves

$$(1.2) \quad -\Delta u_{\alpha, \text{rad}} = V(|x|) (u_{\alpha, \text{rad}})^p$$

weakly in $H_0^1(B)$, it is easy to see that the relation between the energy level

$$(C_{\alpha, \text{rad}}) \quad C_{\alpha, \text{rad}} = I_\alpha(u_{\alpha, \text{rad}})$$

and $S_{\alpha, \text{rad}}$ is

$$(1.3) \quad S_{\alpha, \text{rad}} = \left(\frac{2(p+1)}{p-1} \right)^{\frac{p-1}{p+1}} (C_{\alpha, \text{rad}})^{\frac{p-1}{p+1}}.$$

We will extensively refer to this relation, as well as to such a positive radial solution $u_{\alpha, \text{rad}}$ to (\mathcal{P}) . The same scaling argument using a positive optimiser u_{α}^* to S_{α} allows one to define $u_{\alpha} = S_{\alpha}^{\frac{1}{p-1}} u_{\alpha}^*$. We refer to such u_{α} as a *groundstate solution* to (\mathcal{P}) . It is well-known that, among all the nontrivial critical points of I_{α} , u_{α} has least energy, and that $C_{\alpha} = I_{\alpha}(u_{\alpha})$ is in particular a mountain pass level for I_{α} . Finally we set

$$(S) \quad S = \inf_{0 \neq u \in H_0^1(B)} \frac{\int_B |Du|^2 dx}{\left(\int_B |u|^{p+1} dx \right)^{\frac{2}{p+1}}},$$

i.e. the best constant for the subcritical Sobolev's embedding.

Asymptotic notation. Throughout the paper we use the following asymptotic notation for $\alpha \rightarrow +\infty$. For real valued functions $f(\alpha), g(\alpha) \geq 0$ defined for α large we write:

- $f(\alpha) = o(g(\alpha))$ as $\alpha \rightarrow +\infty$ if $g(\alpha) \neq 0$ for α large and $\lim_{\alpha \rightarrow +\infty} \frac{f(\alpha)}{g(\alpha)} = 0$;
- $f(\alpha) = O(g(\alpha))$ as $\alpha \rightarrow +\infty$ if there exists $C > 0$ such that $f(\alpha) \leq Cg(\alpha)$ for α large;
- $f(\alpha) \asymp g(\alpha)$ if it holds that $f(\alpha) = O(g(\alpha))$ and $g(\alpha) = O(f(\alpha))$;
- $f(\alpha) \sim g(\alpha)$ if $g(\alpha) \neq 0$ and $\lim_{\alpha \rightarrow +\infty} \frac{f(\alpha)}{g(\alpha)} = 1$.

Main results. Using the convergence result in Proposition 3.1 below and the symmetry breaking result for groundstate solutions for the Hénon equation [22], we obtain the following preliminary observation.

Proposition 1.1 (An embryonic symmetry breaking result). *Let $N \geq 1$ and $\alpha^* > 0$ be such that no groundstate of the Hénon equation (\mathcal{H}) is radially symmetric for all $\alpha > \alpha^*$. Then, given any $\alpha > \alpha^*$, there exists $R_0 = R_0(\alpha) \leq 1$ such that no groundstate of (\mathcal{P}) is radially symmetric if $0 \leq R < R_0$.*

One of our main results is a quantitative estimate for a radius R_0 which is sufficient for symmetry breaking to occur for *all* the groundstate solutions relative to $R < R_0$. The price we pay for such a quantitative bound on R is a *non-uniform* symmetry breaking result with respect to α . More precisely, we prove the following result.

Theorem 1.2 (Quantitative symmetry breaking). *Let $N \geq 3$, $1 < p < \frac{N+2}{N-2}$ and set*

$$(1.4) \quad K(N, p) = (N-2)^{-\frac{p+1}{2}} |\mathbb{S}^{N-1}|^{\frac{1-p}{2}} \Gamma \left(N - \frac{(p+1)(N-2)}{2} \right).$$

For any $R \in [0, 1)$ with

$$(1.5) \quad R^{\frac{2N}{p+1} - (N-2)} \min \left\{ e^{\frac{4}{R(p+1)}}, e^{\frac{4}{(1-R)(p+1)}} \right\} < K(N, p)^{-\frac{2}{p+1}} S^{-1},$$

there exists $\alpha(R)$ such that

$$S_{\alpha} < S_{\alpha, \text{rad}}$$

and groundstates of (\mathcal{P}) are nonradial for all $\alpha > \alpha(R)$.

Remark 1.3. Since $\min \left\{ e^{\frac{4}{R(p+1)}}, e^{\frac{4}{(1-R)(p+1)}} \right\} \leq e^{\frac{8}{p+1}}$ for $R \in (0, 1)$, (1.5) yields a radius

$$R_0 := \left[e^{-\frac{8}{p+1}} K(N, p)^{-\frac{2}{p+1}} S^{-1} \right]^{\frac{p+1}{2N-(p+1)(N-2)}}$$

such that for any $R < R_0$, a value $\alpha(R)$ exists such that groundstates of (\mathcal{P}) are nonradial for all $\alpha > \alpha(R)$.

Remark 1.4. We stress that a symmetry breaking result, uniform with respect to $R \in [0, 1)$, based on the growth of S_α and $S_{\alpha,rad}$, cannot be obtained. Indeed, this follows from the convergence (3.4), combined with

$$S_\alpha = S_{\alpha,rad} \asymp \alpha^{\frac{2N}{p+1}-(N-2)} \quad \text{as } \alpha \rightarrow \infty, \quad \text{for } R = 1,$$

which is a consequence of (2.1) and (5.4), and

$$S_\alpha \asymp \alpha^{\frac{2N}{p+1}-(N-2)} \quad \text{and} \quad S_{\alpha,rad} \asymp \alpha^{1+\frac{2}{p+1}} \quad \text{as } \alpha \rightarrow \infty \quad \text{for } R = 0,$$

which have been proved in [22, eq. (5)] and [9, Proposition 3.1].

Theorem 1.2 is achieved by the method which we may outline as follows. We first deduce an asymptotic upper bound on S_α in Section 2. Then we obtain asymptotic lower bound estimates on $S_{\alpha,rad}$ by means of intermediate inequalities developed in the main body of the paper, Section 4. These are obtained by Nehari's and Pohozaev's identities, expressing integrals of radial functions in terms of Euler's Beta functions (involving α) after using Ni's inequality [19]; see also [23]. In performing this we pay attention to the contribution of the boundary to integrals of the form $\int_{B \setminus B_R} \dots dx$, which are the technical obstruction for symmetry breaking to occur for all $R < 1$. This is expected, as we have recalled that symmetry holds when $R = 1$. The aforementioned intermediate estimates take into account possibly different asymptotic contributions, namely $A_\alpha(R)$ and B_α (independent on R) in Lemma 4.4, which may be regarded as a weighted influence produced by both the (\mathcal{D}) and the (\mathcal{H}) limiting PDEs; see also Remark 6.1. In fact the proof of Theorem 1.2 in Section 5 consists in analysing the (least favourable) case where the lower bound for $S_{\alpha,rad}$ is asymptotically equivalent to the upper bound for S_α . This corresponds to $A_\alpha(R)$ being asymptotically strong: a phenomenon which does not occur when dealing with the Hénon equation [22]. In this case the condition $R < R_0$ is found comparing the constants so that $S_{\alpha,rad} > S_\alpha$ holds asymptotically and strictly. We believe that this analysis relates in a nontrivial way to that for the classical Hénon equation as given in [22]. As a byproduct of this method we obtain a lower bound on the subcritical best constant S ; see also Remark 6.1.

Remark 1.5 (A lower bound for S). Let $N \geq 3$, $S = S(N, p)$ be the best constant defined in (\mathcal{S}) and $K(N, p)$ as in (1.4). Then by [12] no symmetry breaking occurs for $R = 1$ and so

$$(1.6) \quad (N-2) |\mathbb{S}^{N-1}|^{\frac{p-1}{p+1}} \left(\Gamma \left(N - \frac{(p+1)(N-2)}{2} \right) \right)^{-\frac{2}{p+1}} e^{-\frac{4}{p+1}} \leq S(N, p).$$

In a recent paper [1] it has been proved that the function

$$p \mapsto S(N, p) |\mathbb{S}^{N-1}|^{2/(p+1)}$$

is decreasing on $(0, 2^* - 1)$, for $N \geq 3$ and on $(0, \infty)$ for $N = 1, 2$. Using this result one has a lower bound by means of the explicitly known value of the classical Sobolev constant

for $p = 2^* - 1$ [24]. We wonder if our method would yield an improved lower bound with a possibly different choice of V .

The case $N = 2$ is more tricky. In this case we use an *a priori* logarithmic radial estimate from [7, Lemma 2.5] which is the low dimensional analog of Ni's classical estimate in [19]. Following the scheme of the proof of Theorem 1.2 for $N = 2$ one finds that, as a consequence of the 'log' factor, the asymptotic bounds for S_α and $S_{\alpha, \text{rad}}$ do not match, see Remark 5.2. Nevertheless, allowing the radius R to be a suitable function of α yields the following.

Theorem 1.6 ('Moving shell' symmetry breaking). *Let $N = 2$ and $p > 1$. Let $1 > R(\cdot) \geq 0$ be a function such that for some $\delta > 0$*

$$(1.7) \quad \limsup_{\alpha \rightarrow \infty} \alpha^\delta R(\alpha) < \infty.$$

Then, there exists $\alpha_ > 0$ such that*

$$S_\alpha < S_{\alpha, \text{rad}}$$

and groundstates of (\mathcal{P}) are nonradial for all $\alpha > \alpha_$.*

The bound provided by Lemma 2.1 below is also used to achieve the following qualitative result, which essentially states that under a certain growth assumption on the picks of the groundstates, as α gets large, maximum points are allowed to accumulate only at the boundary or at the origin, as heuristically pointed out earlier in this introduction. The proof also combines in an essential way a blow-up argument with Liouville theorems, such as Gidas-Spruck's [11].

Theorem 1.7 (Boundary or origin concentration). *Let $N \geq 3$ and $R \in (0, 1)$ be fixed. Given $\alpha > 0$, let u_α be a groundstate solution of (\mathcal{P}) . Set $\beta_\alpha := \max_{x \in \bar{B}} u_\alpha = u_\alpha(x_\alpha)$, where $x_\alpha = (s_\alpha, 0, \dots, 0)$ with $s_\alpha \in [0, 1)$. Assume that*

$$(1.8) \quad \liminf_{\alpha \rightarrow \infty} \alpha^{\frac{2}{p-1}} \beta_\alpha^{-1} < \infty.$$

Then (x_α) has at most two accumulation points, namely either the origin or $(1, 0, \dots, 0)$.

Since for the Hénon equation, namely for $R = 0$, it holds that $\beta_\alpha \asymp \alpha^{2/(p-1)}$ (see e.g. [8] and [9]) and therefore

$$\int_{\mathbb{R}^N} |Dv_\alpha|^2 dx = O(1),$$

with $v_\alpha(x) := \frac{1}{\beta_\alpha} u_\alpha(x_\alpha + \beta_\alpha^{\frac{1-p}{2}} x)$, it is reasonable to believe that (1.8) holds for groundstates when $R \in (0, 1)$. In fact, as we show in the following proposition, (1.8) holds for groundstates also in the case of the endpoint $R = 1$. The proof is based on the estimates developed in the main body of the paper, Section 4, together with the symmetry result of Gidas-Ni-Nirenberg [12]; see also Remark 6.1.

Proposition 1.8. *Let $N \geq 3$. Given $\alpha > 0$, let u_α be a groundstate solution of (\mathcal{D}) . Set $\beta_\alpha := \max_{x \in \bar{B}} u_\alpha$. Then, as $\alpha \rightarrow \infty$, it holds that $\beta_\alpha \asymp \alpha^{2/(p-1)}$.*

Remark 1.9. Based on the proof of Lemma 2.1 it is natural to expect that when $R < 1/2$ concentration holds at the boundary, whereas when $R > 1/2$ concentration may occur at the origin. In the case $R = 1/2$ a groundstate may find equally convenient to concentrate around either the origin, the boundary, or both: in principle one may have different maximum points $(x_\alpha), (x'_\alpha)$ which converge respectively to the origin and to $(1, 0, \dots, 0)$.

2. ESTIMATE FOR S_α

Here we write S_α as $S_{\alpha,R}$ to emphasize its dependence on R .

Lemma 2.1. *Let $N \geq 1$. Given any $R \in [0, 1]$,*

$$(2.1) \quad \limsup_{\alpha \rightarrow \infty} S_{\alpha,R} \alpha^{N-2-\frac{2N}{p+1}} \leq \min \left\{ e^{\frac{4}{R(p+1)}}, e^{\frac{4}{(1-R)(p+1)}} \right\} S \leq e^{\frac{8}{p+1}} S.$$

Proof. Let $R \in [0, 1]$. Setting $x_\alpha = (1 - \frac{1}{\alpha}, 0, \dots, 0)$, for any $\omega \in \mathcal{D}(B) \setminus \{0\}$ the function $w_\alpha = \omega(\alpha(\cdot - x_\alpha))$ is supported in $B \setminus B_R$ for all $\alpha > \frac{2}{1-R}$. Moreover since the radial profile $V_R(r)$ is increasing for $r > R$ we obtain for $\alpha > \frac{2}{1-R}$

$$S_{\alpha,R} \leq \frac{\int_B |Dw_\alpha|^2 dx}{\left(\int_B V_R(x) |w_\alpha|^{p+1} dx \right)^{\frac{2}{p+1}}} \leq \alpha^{2-N+\frac{2N}{p+1}} \frac{\int_B |D\omega|^2 dx}{\left(\int_B \left(1 - \frac{2}{(1-R)\alpha} \right)^\alpha |\omega|^{p+1} dx \right)^{\frac{2}{p+1}}}.$$

Since $\lim_{\alpha \rightarrow \infty} \left(1 - \frac{2}{(1-R)\alpha} \right)^\alpha = e^{-\frac{2}{1-R}}$, we infer that

$$(2.2) \quad \limsup_{\alpha \rightarrow \infty} S_{\alpha,R} \alpha^{N-2-\frac{2N}{p+1}} \leq e^{\frac{4}{(1-R)(p+1)}} S \quad \text{for all } R \in [0, 1].$$

On the other hand, let $R \in (0, 1]$. In this case we have V_R is decreasing for $r < R$. Setting $x_\alpha = (\frac{1}{\alpha}, 0, \dots, 0)$, for any $\omega \in \mathcal{D}(B) \setminus \{0\}$ the function $w_\alpha = \omega(\alpha(\cdot - x_\alpha))$ is supported in B_R for all $\alpha > \frac{2}{R}$. Hence, for $\alpha > \frac{2}{R}$, we obtain

$$S_{\alpha,R} \leq \frac{\int_B |Dw_\alpha|^2 dx}{\left(\int_B V_R(|x|) |w_\alpha|^{p+1} dx \right)^{\frac{2}{p+1}}} \leq \alpha^{2-N+\frac{2N}{p+1}} \frac{\int_B |D\omega|^2 dx}{\left(\int_B \left(1 - \frac{2}{\alpha R} \right)^\alpha |\omega|^{p+1} dx \right)^{\frac{2}{p+1}}}.$$

Since $\lim_{\alpha \rightarrow \infty} \left(1 - \frac{2}{\alpha R} \right)^\alpha = e^{-\frac{2}{R}}$, we infer that

$$(2.3) \quad \limsup_{\alpha \rightarrow \infty} S_{\alpha,R} \alpha^{N-2-\frac{2N}{p+1}} \leq e^{\frac{4}{R(p+1)}} S \quad \text{for all } R \in (0, 1].$$

Then the conclusion follows from (2.2) and (2.3). \square

Remark 2.2. Note that for $R = 1/2$ the same estimate can be proved using a convex combination $\omega_\alpha = t\omega(\alpha(\cdot - x_\alpha)) + (1-t)\omega(\alpha(\cdot - x'_\alpha))$ with any $t \in [0, 1]$ where $x_\alpha = (\frac{1}{\alpha}, 0, \dots, 0)$, and $x'_\alpha = (1 - \frac{1}{\alpha}, 0, \dots, 0)$.

3. RELATION WITH (\mathcal{D}) AND (\mathcal{H}) EQUATIONS AND PROOF OF PROPOSITION 1.1

A first consequence of the bound (2.1) on the best constant S_α is the following proposition. Here we write S_α as $S_{\alpha,R}$ and V as V_R to emphasize their dependence on R .

Proposition 3.1 (Relation with the Hénon equation). *Let $N \geq 1$. If $\alpha > 0$ is fixed, then*

$$(3.1) \quad S_{\alpha,0} = \lim_{R \rightarrow 0^+} S_{\alpha,R}.$$

Moreover, given any positive sequence $(R_n)_{n \in \mathbb{N}}$ converging to zero, if u_{α, R_n}^* is an optimiser to S_{α, R_n} with $\int_B V_{R_n}(|x|)|u_{\alpha, R_n}^*|^{p+1} dx = 1$, then $u_{\alpha, R_n}^* \rightarrow U_0$ in $H_0^1(B)$ and in $C^{2, \delta}(\bar{B})$, up to a subsequence, where U_0 is an optimiser to the Hénon quotient $S_{\alpha, 0}$ with $\int_B |x|^\alpha |U_0|^{p+1} dx = 1$.

Proof. Pick a sequence $(R_n)_{n \in \mathbb{N}}$ converging to zero and consider optimisers u_{α, R_n}^* to S_{α, R_n} with

$$\int_B V_{R_n}(|x|)|u_{\alpha, n}^*|^{p+1} dx = 1.$$

The bound (2.1) on S_{α, R_n} implies that

$$\|u_{\alpha, R_n}^*\|_{H_0^1(B)} < C(\alpha, p, N).$$

Passing if necessary to a subsequence we have that

$$\begin{aligned} u_{\alpha, R_n}^* &\rightharpoonup U_0 && \text{in } H_0^1(B), \\ u_{\alpha, R_n}^* &\rightarrow U_0 && \text{in } L^{p+1}(B), \\ u_{\alpha, R_n}^* &\rightarrow U_0 && \text{almost everywhere on } B. \end{aligned}$$

Since $V_{R_n} \rightarrow |x|^\alpha$, by weakly lower semicontinuity we have that

$$(3.2) \quad S_{\alpha, 0} \leq \frac{\int_B |DU_0|^2 dx}{\left(\int_B |x|^\alpha |U_0|^{p+1} dx\right)^{\frac{2}{p+1}}} \leq \liminf_{n \rightarrow \infty} \frac{\int_B |Du_{\alpha, R_n}^*|^2 dx}{\left(\int_B V_{R_n}(|x|)|u_{\alpha, R_n}^*|^{p+1} dx\right)^{\frac{2}{p+1}}} = \liminf_{n \rightarrow \infty} S_{\alpha, R_n}.$$

On the other hand, pick $U \in H_0^1(B)$ which minimises the Hénon quotient, namely

$$(S_{\alpha, 0}) \quad \frac{\int_B |DU|^2 dx}{\left(\int_B |x|^\alpha |U|^{p+1} dx\right)^{\frac{2}{p+1}}} = S_{\alpha, 0} := \inf_{0 \neq u \in H_0^1(B)} \frac{\int_B |Du|^2 dx}{\left(\int_B |x|^\alpha |u|^{p+1} dx\right)^{\frac{2}{p+1}}}.$$

Since

$$\int_B |x|^\alpha |U|^{p+1} dx = \lim_{n \rightarrow \infty} \int_B V_{R_n}(|x|)|U|^{p+1} dx,$$

and using (3.2), we have

$$(3.3) \quad \limsup_{n \rightarrow \infty} S_{\alpha, R_n} \leq \limsup_{n \rightarrow \infty} \frac{\int_B |DU|^2 dx}{\left(\int_B V_{R_n}(|x|)|U|^{p+1} dx\right)^{\frac{2}{p+1}}} = S_{\alpha, 0}.$$

Therefore, from (3.2) and (3.3), $S_{\alpha, 0} = \lim_{n \rightarrow \infty} S_{\alpha, R_n}$, $u_{\alpha, R_n}^* \rightarrow U_0$ in $H_0^1(B)$ and since the sequence (R_n) converging to zero was arbitrary we infer that $S_{\alpha, 0} = \lim_{R \rightarrow 0^+} S_{\alpha, R}$. The sub-criticality of p allows to perform a classical bootstrap argument to show that the convergence is in $C^{2, \delta}(\bar{B})$, and this concludes the proof. \square

Proof of Proposition 1.1. It follows directly from Proposition 3.1 combined with the symmetry breaking result in [8, 22]. \square

We end this section with a proposition which highlights the connection with the limiting problem (D).

Proposition 3.2 (Relation with the equation (D)). *Let $N \geq 1$. For any $\alpha > 0$, it holds that*

$$(3.4) \quad S_{\alpha,1} = \inf_{0 \neq u \in H_0^1(B)} \frac{\int_B |Du|^2 dx}{\left(\int_B (1-|x|)^\alpha |u|^{p+1} dx \right)^{\frac{2}{p+1}}} = \lim_{R \rightarrow 1^-} S_{\alpha,R}.$$

Moreover, given any sequence $(R_n)_{n \in \mathbb{N}}$ converging to 1^- , if u_{α,R_n}^* is an optimiser to S_{α,R_n} with $\int_B V_{R_n}(|x|) |u_{\alpha,R_n}^*|^{p+1} dx = 1$, then $u_{\alpha,R_n}^* \rightarrow U_1$ in $H_0^1(B)$ and in $C^{2,\delta}(\overline{B})$, up to a subsequence, where U_1 is an optimiser to the quotient $S_{\alpha,1}$ with $\int_B (1-|x|)^\alpha |U_1|^{p+1} dx = 1$.

Proof. The proof is identical to that of Proposition 3.1. We leave out the details. \square

4. GROWTH ESTIMATES FOR $C_{\alpha,\text{RAD}}$

The estimates which follow are essentially based on implementing Ni's inequality into a suitable rewriting of Nehari's and Pohozaev's identities associated with $u_{\alpha,\text{rad}}$.

Estimate for $\int_{\partial B} |Du_{\alpha,\text{rad}}|^2$ from above.

Lemma 4.1. *Let $N \geq 2$, $R \in [0, 1]$ and $K^*(N, p) = |\mathbb{S}^{N-1}|^{\frac{1-p}{p+1}} \left(\frac{2(p+1)}{p-1} \right)^{\frac{2p}{p+1}}$. Then*

$$(4.1) \quad \int_{\partial B} |Du_{\alpha,\text{rad}}|^2 d\sigma \leq K^*(N, p) \frac{(C_{\alpha,\text{rad}})^{2p/(p+1)}}{(\alpha+1)^{2/(p+1)}}.$$

Proof. Integrating (1.2) over B , by radial symmetry and the divergence theorem we have:

$$|\mathbb{S}^{N-1}| \int_{\partial B} |Du_{\alpha,\text{rad}}|^2 d\sigma = \left(\int_B V(|x|) (u_{\alpha,\text{rad}})^p dx \right)^2.$$

Then, by Hölder's inequality we obtain:

$$(4.2) \quad \int_{\partial B} |Du_{\alpha,\text{rad}}|^2 d\sigma \leq \frac{1}{|\mathbb{S}^{N-1}|} \left(\int_B V(|x|) (u_{\alpha,\text{rad}})^{p+1} dx \right)^{2p/(p+1)} \left(\int_B V(|x|) dx \right)^{2/(p+1)}.$$

By the definition $(C_{\alpha,\text{rad}})$ and testing (1.2) with $u_{\alpha,\text{rad}}$ we infer that

$$(4.3) \quad \int_B V(|x|) (u_{\alpha,\text{rad}})^{p+1} dx = \frac{2(p+1)}{p-1} C_{\alpha,\text{rad}}.$$

To estimate $\int_B V(|x|) dx$, write

$$\begin{aligned} \int_B V(|x|) dx &= \int_{B_R} V(|x|) dx + \int_{B \setminus B_R} V(|x|) dx \\ &= |\mathbb{S}^{N-1}| \left[\int_0^R \left(1 - \frac{r}{R}\right)^\alpha r^{N-1} dr + \int_R^1 \left(1 - \frac{1-r}{1-R}\right)^\alpha r^{N-1} dr \right] \\ &= |\mathbb{S}^{N-1}| [I + II]. \end{aligned}$$

Setting $s = 1 - r/R$ we have

$$I = \int_0^1 s^\alpha (R(1-s))^{N-1} R ds \leq R \int_0^1 s^\alpha ds = \frac{R}{\alpha+1}.$$

A similar estimate using the change of variable $s = \left(1 - \frac{1-r}{1-R}\right)$, yields

$$II \leq \int_0^1 s^\alpha (1-R) ds = \frac{1-R}{\alpha+1}.$$

It follows that

$$(4.4) \quad \int_B V(|x|) dx \leq \frac{|\mathbb{S}^{N-1}|}{\alpha+1}.$$

Finally, inserting (4.3) and (4.4) into (4.2), we obtain (4.1). This concludes the proof. \square

Estimate for $\int_{\partial B} |Du_{\alpha, \text{rad}}|^2$ from below.

Lemma 4.2. *Let $N \geq 3$ and $R \in [0, 1]$. Then*

$$(4.5) \quad \int_{\partial B} |Du_{\alpha, \text{rad}}|^2 d\sigma \geq \left(\frac{2(N+\alpha)}{p+1} - (N-2) \right) \frac{2(p+1)}{p-1} C_{\alpha, \text{rad}} \\ - K_*(N, p) R^{\beta+1} \frac{\alpha \Gamma(\alpha)}{\Gamma(\alpha + \beta + 1)} (C_{\alpha, \text{rad}})^{\frac{p+1}{2}}$$

where

$$(4.6) \quad K_*(N, p) = 2^{\frac{p+3}{2}} (p+1)^{\frac{p-1}{2}} [(p-1)(N-2)]^{-\frac{p+1}{2}} |\mathbb{S}^{N-1}|^{\frac{1-p}{2}} \Gamma(\beta+1)$$

and

$$(4.7) \quad \beta = N - 1 - (p+1) \frac{N-2}{2}.$$

Proof. From the Pohozaev identity

$$(4.8) \quad \int_{\partial B} |Du_{\alpha, \text{rad}}|^2 d\sigma = \left(\frac{2N}{p+1} - (N-2) \right) \int_B V(|x|) |u_{\alpha, \text{rad}}|^{p+1} dx \\ + \frac{2}{p+1} \int_B V'(|x|) |x| |u_{\alpha, \text{rad}}|^{p+1} dx.$$

Let $R \in [0, 1]$. By using the definition of V it is convenient to write

$$\int_B V'(|x|) |x| |u_{\alpha, \text{rad}}|^{p+1} dx = \alpha \int_B V(|x|) |u_{\alpha, \text{rad}}|^{p+1} dx - \alpha \int_{B_R} \left(1 - \frac{|x|}{R}\right)^{\alpha-1} |u_{\alpha, \text{rad}}|^{p+1} dx \\ + \frac{\alpha R}{1-R} \int_{B \setminus B_R} \left(1 - \frac{1-|x|}{1-R}\right)^{\alpha-1} |u_{\alpha, \text{rad}}|^{p+1} dx.$$

Inserting in (4.8) and taking into account the positivity of the last integral, we obtain

$$(4.9) \quad \int_{\partial B} |Du_{\alpha, \text{rad}}|^2 d\sigma \geq \left(\frac{2(N+\alpha)}{p+1} - (N-2) \right) \int_B V(|x|) |u_{\alpha, \text{rad}}|^{p+1} dx \\ - \frac{2\alpha}{p+1} \int_{B_R} \left(1 - \frac{|x|}{R}\right)^{\alpha-1} |u_{\alpha, \text{rad}}|^{p+1} dx.$$

By Ni's inequality [19, eq. (4)]

$$(4.10) \quad |u_{\alpha, \text{rad}}(x)| \leq \frac{1}{(|\mathbb{S}^{N-1}|(N-2))^{1/2}} \frac{\|Du_{\alpha, \text{rad}}\|_{L^2(B)}}{|x|^{(N-2)/2}},$$

we estimate

$$\begin{aligned} \int_{B_R} \left(1 - \frac{|x|}{R}\right)^{\alpha-1} |u_{\alpha,\text{rad}}|^{p+1} dx &\leq |\mathbb{S}^{N-1}| \left(\frac{1}{|\mathbb{S}^{N-1}|(N-2)}\right)^{\frac{p+1}{2}} \left[C_{\alpha,\text{rad}} \frac{2(p+1)}{p-1}\right]^{\frac{p+1}{2}} \\ &\quad \times \int_0^R \left(1 - \frac{r}{R}\right)^{\alpha-1} r^{N-1-(p+1)\frac{N-2}{2}} dr. \end{aligned}$$

Set

$$I_R = \int_0^R \left(1 - \frac{r}{R}\right)^{\alpha-1} r^{N-1-(p+1)\frac{N-2}{2}} dr.$$

By the change of variable $s = 1 - r/R$ and by the definition of β we can express

$$I_R = R^{\beta+1} \int_0^1 s^{\alpha-1} (1-s)^\beta ds = R^{\beta+1} \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)},$$

where we have used the well-known expression of Euler's integral of the first kind in terms of Beta function $B(\alpha, \beta+1)$. Coming back to (4.9) this immediately yields the desired estimate for $R \in [0, 1)$. Finally, observing that all the calculations after (4.9) also holds in case $R = 1$, we conclude the proof. \square

Lemma 4.3. *Let $N = 2$ and $R \in [0, 1]$. Then*

$$(4.11) \quad \int_{\partial B} |Du_{\alpha,\text{rad}}|^2 d\sigma \geq \frac{4(2+\alpha)}{p-1} C_{\alpha,\text{rad}} - K_*(2, p) R^{\beta+1} \frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha+\beta+1)} (C_{\alpha,\text{rad}})^{\frac{p+1}{2}}$$

where

$$(4.12) \quad K_*(2, p) = 4\pi^{\frac{1-p}{2}} (p+1)^{\frac{p-1}{2}} (p-1)^{-\frac{p+1}{2}} \Gamma(\beta+1) c_\varepsilon^{p+1},$$

$$(4.13) \quad \beta = 1 - (p+1)\varepsilon,$$

and $c_\varepsilon = \sup_{r \in (0,1)} \left(r^\varepsilon |\ln r|^{1/2}\right)$, with $0 < \varepsilon < \frac{2}{p+1}$.

Proof. The only change with respect to the proof of Lemma 4.2 is in the use of Ni's inequality, which we replace by an estimate valid for $N = 2$, namely

$$|u_{\alpha,\text{rad}}(x)| \leq \left(\frac{|\ln|x||}{|\mathbb{S}^1|}\right)^{1/2} \|Du_{\alpha,\text{rad}}\|_{L^2(B)},$$

see [7, Lemma 2.5]. In view of the definition of c_ε we write the above inequality as

$$(4.14) \quad |u_{\alpha,\text{rad}}(x)| \leq \frac{c_\varepsilon}{(2\pi)^{1/2}} \frac{\|Du_{\alpha,\text{rad}}\|_{L^2(\mathbb{R}^2)}}{|x|^\varepsilon}.$$

Hence

$$\int_{B_R} \left(1 - \frac{|x|}{R}\right)^{\alpha-1} |u_{\alpha,\text{rad}}|^{p+1} dx \leq c_\varepsilon^{p+1} (2\pi)^{\frac{1-p}{2}} \left[C_{\alpha,\text{rad}} \frac{2(p+1)}{p-1}\right]^{\frac{p+1}{2}} \int_0^R \left(1 - \frac{r}{R}\right)^{\alpha-1} r^{1-(p+1)\varepsilon} dr.$$

Set

$$I_R = \int_0^R \left(1 - \frac{r}{R}\right)^{\alpha-1} r^{1-(p+1)\varepsilon} dr.$$

By the change of variable $s = 1 - r/R$ and setting $\beta = 1 - (p+1)\varepsilon$ we can express

$$I_R = R^{\beta+1} \int_0^1 s^{\alpha-1} (1-s)^\beta ds = R^{\beta+1} \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}.$$

Using this estimate in (4.9) gives immediately the statement for $N = 2$. And this concludes the proof. \square

Estimate for $C_{\alpha, \text{rad}}$.

Lemma 4.4. *Let $N \geq 2$ and $R \in [0, 1]$. Then*

$$(4.15) \quad \left(\frac{2(N + \alpha)}{p + 1} - (N - 2) \right) \frac{2(p + 1)}{p - 1} C_{\alpha, \text{rad}} \leq A_{\alpha}(R) + B_{\alpha}$$

where

$$(4.16) \quad A_{\alpha}(R) = K_{*}(N, p) R^{\beta+1} \frac{\alpha \Gamma(\alpha)}{\Gamma(\alpha + \beta + 1)} (C_{\alpha, \text{rad}})^{\frac{p+1}{2}}$$

and

$$(4.17) \quad B_{\alpha} = K^{*}(N, p) \frac{(C_{\alpha, \text{rad}})^{2p/(p+1)}}{(\alpha + 1)^{2/(p+1)}}.$$

Proof. Combining Lemma 4.1 with Lemma 4.2 (for $N \geq 3$) and Lemma 4.3 (for $N = 2$), we immediately obtain the estimate. \square

Estimate for $S_{\alpha, \text{rad}}$ when $R = 1$ and $p = 1$. This estimate will be useful in the proof of Proposition 1.8. In the case $p = 1$ our PDE becomes an eigenvalue problem, and since the scaling $u_{\alpha, \text{rad}} = (S_{\alpha, \text{rad}})^{\frac{1}{p-1}} u_{\alpha, \text{rad}}^{*}$ is not defined, we provide directly an estimate for $S_{\alpha, \text{rad}}$.

Lemma 4.5. *Let $N \geq 3$, $R = 1$ and $p = 1$. There exists $C > 0$ such that*

$$(4.18) \quad \liminf_{\alpha \rightarrow \infty} \alpha^{-2} S_{\alpha, \text{rad}} \geq C.$$

Proof. Since $R = 1$, by [12], $S_{\alpha, \text{rad}} = S_{\alpha}$. Pick a radial optimiser $u_{\alpha} \in H_0^1(B)$ for $S_{\alpha, \text{rad}}$. By using Ni's inequality (4.10) we have

$$\int_0^1 (1-r)^{\alpha} |u_{\alpha}(r)|^2 r^{N-1} dr \leq C' \|Du_{\alpha}\|_{L^2(B)}^2 \int_0^1 (1-r)^{\alpha} r dr = C' \frac{\|Du_{\alpha}\|_{L^2(B)}^2}{\alpha^2 + 3\alpha + 2}.$$

This immediately implies that there exists $C > 0$ such that

$$\liminf_{\alpha \rightarrow \infty} \alpha^{-2} \frac{\int_B |Du_{\alpha}|^2 dx}{\int_B (1-|x|)^{\alpha} |u_{\alpha}|^2 dx} \geq C,$$

and this concludes the proof. \square

5. QUANTITATIVE SYMMETRY BREAKING: PROOFS OF THEOREMS 1.2 AND 1.6

Proof of Theorem 1.2. Let $A_{\alpha}(R)$ and B_{α} as in Lemma 4.4 and set

$$L(R) := \limsup_{\alpha \rightarrow \infty} \frac{A_{\alpha}(R)}{B_{\alpha}},$$

to emphasize its dependence on R . We distinguish the cases $L(R)$ finite and $L(R) = +\infty$.

Case 1: $L(R) \in [0, +\infty)$. From Lemma 4.4 and using the relation

$$(5.1) \quad S_{\alpha, \text{rad}} = \left(\frac{2(p+1)}{p-1} \right)^{\frac{p-1}{p+1}} (C_{\alpha, \text{rad}})^{\frac{p-1}{p+1}}$$

we obtain a lower bound for $S_{\alpha, \text{rad}}$ of the form

$$S_{\alpha, \text{rad}} \geq C(N, p, R) \alpha^{1 + \frac{2}{p+1}} + o(1), \quad \alpha \rightarrow +\infty.$$

Since $1 < p$, by this estimate and Lemma 2.1 it follows that there exists $\alpha = \alpha(R)$ such that

$$S_{\alpha} < S_{\alpha, \text{rad}}, \quad \text{for all } \alpha > \alpha(R).$$

Case 2: $L(R) = +\infty$. Here we assume $N \geq 3$ and write the estimate (4.15) in the form

$$(5.2) \quad \left(\frac{2(N+\alpha)}{p+1} - (N-2) \right) \frac{2(p+1)}{p-1} C_{\alpha, \text{rad}} \leq A_{\alpha} \left[1 + \frac{B_{\alpha}}{A_{\alpha}} \right].$$

Using again (5.1), we infer that

$$(5.3) \quad 1 + \frac{1}{\alpha} \left[\frac{2N}{p+1} - (N-2) \right] \frac{p+1}{2} \leq K(N, p) R^{\beta+1} \frac{\Gamma(\alpha)}{\Gamma(\alpha + \beta + 1)} (S_{\alpha, \text{rad}})^{\frac{p+1}{2}} \left[1 + \frac{B_{\alpha}}{A_{\alpha}} \right]$$

where

$$K(N, p) = \frac{p+1}{2} \left(\frac{p-1}{2(p+1)} \right)^{\frac{p+1}{2}} K_*(N, p),$$

namely the constant given in (1.4). Classical asymptotic estimates on the Gamma function, see e.g. [25], yield

$$\frac{\Gamma(\alpha)}{\Gamma(\alpha + \beta + 1)} \sim \alpha^{-\beta-1} \quad \text{as } \alpha \rightarrow \infty,$$

that is

$$\frac{\Gamma(\alpha)}{\Gamma(\alpha + \beta + 1)} \sim \alpha^{\frac{p+1}{2}(N-2)-N} \quad \text{as } \alpha \rightarrow \infty.$$

From this and (5.3) we finally obtain

$$(5.4) \quad \liminf_{\alpha \rightarrow \infty} S_{\alpha, \text{rad}} \alpha^{N-2-\frac{2N}{p+1}} \geq K(N, p)^{-\frac{2}{p+1}} R^{N-2-\frac{2N}{p+1}}.$$

Since by Lemma 2.1 we have

$$\limsup_{\alpha \rightarrow \infty} S_{\alpha} \alpha^{N-2-\frac{2N}{p+1}} \leq \min \left\{ e^{\frac{4}{R(p+1)}}, e^{\frac{4}{(1-R)(p+1)}} \right\} S,$$

and since the hypotheses on R and the sub-criticality of p imply the strict inequality

$$\min \left\{ e^{\frac{4}{R(p+1)}}, e^{\frac{4}{(1-R)(p+1)}} \right\} S - K(N, p)^{-\frac{2}{p+1}} R^{N-2-\frac{2N}{p+1}} < 0,$$

it follows that

$$\limsup_{\alpha \rightarrow \infty} (S_{\alpha} - S_{\alpha, \text{rad}}) < 0$$

and this concludes the proof. \square

Remark 5.1. If $R = 0$, then $A_{\alpha} = 0$. So we are in *Case 1* above, whose arguments works for $N \geq 2$. Then we recover, with a slightly different method, the symmetry breaking result for the Hénon equation [22, Theorems 3.1 and 4.2], with a unified proof that works for $N \geq 2$. The idea of using the Pohozaev identity to obtain energy bounds in the radial framework has been inspired by [6].

Remark 5.2. We observe that the above argument for $N = 2$ in *Case 2* does not imply $S_{\alpha, \text{rad}} > S_\alpha$ for large values of α . In fact carrying out the proof in *Case 2*, using the expression of $\beta = 1 - (p+1)\varepsilon$ provided by Lemma 4.3, one finds

$$\liminf_{\alpha \rightarrow \infty} S_{\alpha, \text{rad}} \alpha^{2\varepsilon - \frac{4}{p+1}} \geq C(N, p, R, \varepsilon),$$

which does not match with the asymptotic estimate provided by Lemma 2.1

$$\limsup_{\alpha \rightarrow \infty} S_\alpha \alpha^{-\frac{4}{p+1}} \leq \min \left\{ e^{\frac{4}{R(p+1)}}, e^{\frac{4}{(1-R)(p+1)}} \right\} S.$$

Proof of Theorem 1.6. Let A_α and B_α as in Lemma 4.4 having replaced R with $R(\alpha)$ and set

$$L := \limsup_{\alpha \rightarrow \infty} \frac{A_\alpha}{B_\alpha}.$$

As in the proof of Theorem 1.2, symmetry breaking occurs if L is finite. So let us consider the case $L = +\infty$. In this case by using Lemma 4.3 we have for all $\varepsilon \in (0, \frac{2}{p+1})$

$$(5.5) \quad 1 + \frac{2}{\alpha} \leq C(p, \varepsilon) R(\alpha)^{2-\varepsilon(p+1)} \frac{\Gamma(\alpha)}{\Gamma(\alpha + 2 - \varepsilon(p+1))} (S_{\alpha, \text{rad}})^{\frac{p+1}{2}} \left[1 + \frac{B_\alpha}{A_\alpha} \right].$$

Using the asymptotic estimates on the Gamma function [25], we obtain that for all $\varepsilon \in (0, \frac{2}{p+1})$ it holds that

$$(5.6) \quad \liminf_{\alpha \rightarrow \infty} S_{\alpha, \text{rad}} \left(\frac{R(\alpha)}{\alpha} \right)^{\frac{4}{p+1} - 2\varepsilon} \geq C(p, \varepsilon) > 0$$

for some constant $C(p, \varepsilon)$. With the particular choice

$$\varepsilon = \frac{4\delta}{(p+1)(2\delta+3)} \in (0, \frac{2}{p+1}),$$

since

$$\left(\frac{R(\alpha)}{\alpha} \right)^{\frac{4}{p+1} - 2\varepsilon} = \alpha^{-\frac{4}{p+1} - \varepsilon} \left(\alpha^\delta R(\alpha) \right)^{\frac{4}{p+1} - 2\varepsilon}$$

we obtain from (5.6) and by the hypothesis on $R(\cdot)$ that

$$\liminf_{\alpha \rightarrow \infty} S_{\alpha, \text{rad}} \alpha^{-\frac{4}{p+1} - \varepsilon} \geq C'(p, \varepsilon) > 0.$$

This means that the growth of $S_{\alpha, \text{rad}}$ is faster than that of S_α provided by Lemma 2.1 and this concludes the proof. \square

6. NECESSARY CONDITIONS FOR CONCENTRATION OF GROUNDSTATES AND PROOFS OF THEOREM 1.7 AND OF PROPOSITION 1.8

Throughout this section we consider $N \geq 3$. Before proving Proposition 1.8 it is worth making the following remark, which is of independent interest.

Remark 6.1. From the proof of Theorem 1.2 it is clear that when $R = 1$ it holds necessarily that $L = +\infty$, as $L \in [0, +\infty)$ would imply symmetry breaking, a contradiction by the symmetry result of Gidas-Ni-Nirenberg [12]. Therefore from (5.4) we obtain

$$(6.1) \quad \liminf_{\alpha \rightarrow \infty} S_{\alpha, \text{rad}} \alpha^{N-2-\frac{2N}{p+1}} \geq K(N, p)^{-\frac{2}{p+1}}.$$

Since $S_\alpha = S_{\alpha, \text{rad}}$ in this case, the above estimate combined with Lemma 2.1 yields the lower bound on S provided in the Remark 1.5.

Proof of Proposition 1.8. Following the Remark 6.1, the Gidas-Ni-Nirenberg result [12] implies that $S_\alpha = S_{\alpha, \text{rad}}$ and the double sided growth estimates for S_α holds, namely (6.1) in addition to Lemma 2.1. Picking a groundstate u_α , by the Nehari identity it follows

$$\int_B |Du_\alpha|^2 dx = (S_\alpha)^{\frac{p+1}{p-1}}.$$

Hence, setting $v_\alpha(x) := \alpha^{\frac{2}{1-p}} u_\alpha\left(\frac{x}{\alpha}\right)$, we obtain

$$(6.2) \quad C_1 \leq \int_{B_\alpha} |Dv_\alpha|^2 dx \leq C_2$$

for some constant $C_1, C_2 > 0$. Using (4.18) from Lemma 4.5 we have, as in [9] p. 475,

$$\int_B (1 - |x|)^\alpha |u_\alpha|^2 dx \leq C\alpha^{-2} \int_B |Du_\alpha|^2 dx.$$

This implies for some $C > 0$ that

$$\int_{B_\alpha} \left(1 - \left|\frac{x}{\alpha}\right|\right)^\alpha |v_\alpha|^2 dx \leq C \int_{B_\alpha} |Dv_\alpha|^2 dx \leq C'_2,$$

and finally, by Nehari's identity and (6.2) we obtain

$$(6.3) \quad C_1 \leq \max_{\overline{B_\alpha}} |v_\alpha|^{p-1} \int_{B_\alpha} \left(1 - \left|\frac{x}{\alpha}\right|\right)^\alpha |v_\alpha|^2 dx \leq C'_2 \max_{\overline{B_\alpha}} |v_\alpha|^{p-1}.$$

On the other hand, since $v_\alpha \in H_0^1(B_\alpha)$ satisfies $-\Delta v_\alpha \leq v_\alpha^p$, by classical Moser's iteration we can show that the uniform bound

$$(6.4) \quad \|v_\alpha\|_\infty \leq C$$

holds. The conclusion follows now immediately by (6.3) and (6.4). \square

For all $R \in [0, 1]$ the norm of groundstate solutions, $\|u_\alpha\|_\infty$, blows up as $\alpha \rightarrow \infty$. However, when $R \neq 0, 1$, we are able to show only that this occurs at a growth rate which is slower than that required by Theorem 1.7.

Proposition 6.2 ($\|u_\alpha\|_\infty$ blows up). *Let $N \geq 1$, $R \in [0, 1]$ and for any $\alpha > 0$, denote by u_α a positive groundstate solution of (\mathcal{P}) . Then $\|u_\alpha\|_\infty \rightarrow +\infty$ as $\alpha \rightarrow +\infty$.*

Proof. Set $\beta_\alpha = \|u_\alpha\|_\infty$ and recall that by (4.4), for $N \geq 2$ we have

$$\int_B V(|x|) dx \leq \frac{|\mathbb{S}^{N-1}|}{\alpha + 1}.$$

For $N = 1$ note that

$$\int_{-1}^1 V(|x|) dx = \frac{2}{\alpha + 1} = \frac{|\mathbb{S}^0|}{\alpha + 1}.$$

Combining these and Nehari's identity we infer that

$$S^{\frac{p+1}{p-1}} \leq (S_\alpha)^{\frac{p+1}{p-1}} = \int_B |\nabla u_\alpha|^2 dx = \int_B V(|x|) |u_\alpha|^{p+1} dx \leq \frac{C_N}{\alpha + 1} \beta_\alpha^{p+1},$$

which concludes the proof. \square

Proof of Theorem 1.7. Let (α_m) be a sequence with $\alpha_m \rightarrow +\infty$ and, for each m , let u_m be a positive groundstate solution of (\mathcal{P}) . Set $\beta_m := \max_{x \in \bar{B}} u_m = u_m(x_m)$ and

$$v_m(x) := \frac{1}{\beta_m} u_m(x_m + \beta_m^{\frac{1-p}{2}} x).$$

Then $v_m \in H_0^1(\Omega_m)$, where $\Omega_m := \beta_m^{\frac{p-1}{2}} (B(0,1) - x_m)$, and

$$(6.5) \quad -\Delta v_m = V_{\alpha_m} \left(|x_m + \beta_m^{\frac{1-p}{2}} x| \right) v_m^p, \quad v_m > 0, \quad |v_m| \leq 1.$$

Set $d := \liminf_{m \rightarrow +\infty} \beta_m^{\frac{p-1}{2}} \text{dist}(x_m, \partial B)$. If $d = 0$, necessarily concentration occurs at the boundary. So below we analyse the case $d > 0$.

Set

$$\gamma(x) := \liminf_{m \rightarrow +\infty} V_{\alpha_m}(|x_m + \beta_m^{\frac{1-p}{2}} x|).$$

Then γ is defined on a halfspace H or on \mathbb{R}^N according to whether $d \in (0, \infty)$ or $d = +\infty$.

If γ is not constant, then $(x_m)_{m \in \mathbb{N}}$ accumulates at a point on $\partial B \cup \{0\}$, as otherwise $\gamma = 0$.

Therefore, we consider the case $\gamma(x)$ is a constant and show that in fact this case does not occur by showing that we get a contradiction. We have sub-cases according to the different values of d . If $d = +\infty$ we see that as m gets large enough, for every fixed $\rho > 0$ any ball B_ρ is properly contained in Ω_m . Classical elliptic estimates and bootstrap show that v_m is uniformly bounded in $C_{\text{loc}}^{2,\delta}(\mathbb{R}^N)$ and, as a consequence, a suitable subsequence converges in $C_{\text{loc}}^{2,\delta'}(\mathbb{R}^N)$ to some v satisfying

$$(6.6) \quad -\Delta v = \gamma v^p \text{ on } \mathbb{R}^N, \quad v > 0 \text{ on } \mathbb{R}^N, \quad v(0) = 1.$$

Similarly, if $d \in (0, \infty)$ we find that some nonnegative $v \in C_{\text{loc}}^{2,\delta}(\bar{H})$ exists, defined on a halfspace H and satisfying

$$(6.7) \quad -\Delta v = \gamma v^p \text{ on } H, \quad v > 0 \text{ on } H, \quad v|_{\partial H} = 0, \quad v(0) = 1.$$

In the case $\gamma > 0$ we get a contradiction, since the classical result of Gidas-Spruck [11] says that both equations (6.6) and (6.7) have no solution.

If $\gamma = 0$ and $d \in (0, \infty)$ again we have contradiction, at this time using the average properties of harmonic functions.

The only possibility left is that $d = +\infty$ and $\gamma = 0$. In this case by classical classification results on positive harmonic functions we conclude that $v \equiv \text{constant}$, and therefore $v \equiv 1$ necessarily. To see that this is a contradiction note that as in (1.3) and by Nehari's identity we have

$$\int_B |Du_m|^2 dx = (S_{\alpha_m})^{\frac{p+1}{p-1}}.$$

Hence using Lemma 2.1 to estimate S_{α_m} we have

$$\begin{aligned} \int_{\mathbb{R}^N} |Dv_m|^2 dx &= \beta_m^{\frac{p(N-2)-N-2}{2}} \int_B |Du_m|^2 dx \\ &\leq C(N, p, R) \beta_m^{\frac{p(N-2)-N-2}{2}} \alpha_m^{\frac{2N+(2-N)(p+1)}{p-1}} = C(N, p, R) \left(\alpha_m^{\frac{2}{p-1}} \beta_m^{-1} \right)^{\frac{N+2-p(N-2)}{2}}. \end{aligned}$$

Then, from (1.8), it follows that up to subsequences, $(v_m)_{m \in \mathbb{N}}$ is bounded in $D^{1,2}(\mathbb{R}^N)$ and $v_m \rightharpoonup v \in D^{1,2}(\mathbb{R}^N)$. As v is harmonic, by Sobolev's inequality it follows that $v \equiv 0$, a contradiction. This concludes the proof. \square

7. FINAL REMARKS AND RELATED PROBLEMS

There are a number of questions related to our work which we believe it is worth studying in future projects. Here below we list some of them.

- How close to ∂B the zeros of V , i.e. $|x| = R$, are allowed to be in order for symmetry breaking to occur?
- To prove (or disprove) that (1.8) holds for every $R \in (0, 1)$.
- To study the limiting profile of groundstates when $\alpha \rightarrow \infty$.
- Let $R \in (0, 1)$. May groundstate solutions concentrate simultaneously at the origin and at $(1, 0, \dots, 0)$?
- Is it possible to obtain similar results replacing V with an oscillating function, e.g. $|\cos(k\pi|x|)|^\alpha$?

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