The effect of noise intensity on parabolic equations

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Abstract

In this paper, the effect of noise intensity on parabolic equations is considered. We focus on the effect of noise on the energy solutions of stochastic parabolic equations. By utilising Ito's formula and energy estimate method, we obtain excitation indices of the solution u at time t. Furthermore, we verify the existing results in the literature by a comparably simpler method.

Keywords: Itô's formula; stochastic parabolic equation; energy estimate method.

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1 Introduction

In recent years, many authors attempt to explore the role of the noise in various dynamical equations in both analytical and numerical aspects. For example, noise can make the solution smooth [12], can prevent singularities in linear transport equations [11], can prevent collapse of Vlasov-Poisson point charges [8], and also can induce singularities (finite time blow up of solutions) [3, 4, 22]. In the present paper, we focus on the effect of noise on stochastic parabolic equations driven by space-time white noise.

The concept of "Intermittency" is the property that the solution $u(t,x)$ develops extreme oscillations at certain values of x, typically when t is going to be large. Intermittency was announced first (1949) by Batchelor and Townsend in a WHO conference in Vienna [1], and slightly later by Emmons [10] in the context of boundary layer turbulence. Meanwhile, intermittency has been observed in an enormous number of scientific disciplines. For example, intermittency is observed as "spikes" and "shocks" in neural activity and in finance, respectively. Tuckwell [27] contains a gentle introduction to SPDEs in neuroscience.

Recently, Khoshnevisan-Kim [18, 19] considered the following stochastic heat equation

$$\frac{\partial}{\partial t}u = \mathcal{L}u + \lambda\sigma(u)\xi, \tag{1.1}$$

where $t > 0$ denotes the time variable, $x \in G$ is the space variable for a nice state space G–such as $\mathbb{R}$, $\mathbb{Z}$ (a discrete set) or a finite interval like $[0,1]$–and the initial data value $u_0 : G \to \mathbb{R}$ is non random and is well behaved. The operator $\mathcal{L}$ acts on the spatial variable $x \in G$ only, and is nothing but the generator of a nice Markov process on G, and ξ denotes *space-time white noise* on $(0,\infty) \times G$. Here, $\lambda > 0$ is a constant and $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function.

Let u be a mild solution of (1.1). Define

$$\mathscr{E}_t(\lambda) := \sqrt{\mathbb{E}\left(\|u(t)\|^2_{L^2(G)}\right)}, \tag{1.2}$$

which stands for the energy of the solution at time t. In papers [18, 19, 13, 15], the authors considered the energy $\mathscr{E}_t(\lambda)$ behaves as $exp(\text{const } \cdot \lambda^q)$, for a fixed positive constant q, as $\lambda \uparrow \infty$. In order to do so, the following was introduced. Let

$$\underline{\mathbb{e}}(t) := \liminf_{\lambda\uparrow\infty} \frac{\log\log \mathscr{E}_t(\lambda)}{\log \lambda}, \quad \bar{\mathbb{e}}(t) := \limsup_{\lambda\uparrow\infty} \frac{\log\log \mathscr{E}_t(\lambda)}{\log \lambda}. \tag{1.3}$$

Clearly, $\underline{\mathbb{e}}$ and $\bar{\mathbb{e}}$ represent the lower and upper excitation indices of u at time t, respectively. In many interesting cases, $\underline{\mathbb{e}}(t)$ and $\bar{\mathbb{e}}(t)$ are equal and do not depend on the time variable $t > 0$. In such situations, we tacitly write $\mathbb{e}$ for that common value, just for simplicity.

In paper [18], Khoshnevisan-Kim showed that

(i) If G is discrete, then $\bar{\mathbb{e}}(t) \le 2$ for all $t \ge 0$. $\bar{\mathbb{e}} = 2$ if

$$l_\sigma := \inf_{z\in\mathbb{R}\setminus\{0\}} |\frac{\sigma(z)}{z}| > 0. \tag{1.4}$$

(ii) Suppose that G is connected and (1.4) holds, then $\underline{\mathbb{e}}(t) \ge 4$ for all $t \ge 0$, provided that in addition either G is non compact or G is compact, metrizable, and has more than one element.

(iii) For every $\theta \ge 4$ there are models of the triple $(G, \mathcal{L}, u_0)$ for which $\mathbb{e} = \theta$. The models is $\mathcal{L} := -(-\Delta)^{\frac{\alpha}{2}}$ (the generator of a symmetric stable Lévy process), $1 < \alpha \le 2$.

In [19], Khoshnevisan-Kim considered the following

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \dfrac{\partial^2}{\partial x^2}u(t,x) + \lambda\sigma(u(t,x))\dot{w}(t,x), & 0 < x < L,\ t > 0, \\ u(t,0) = u(t,L) = 0, & t > 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.5}$$

where $\dot{w}$ is a space-time white noise, $L > 0$ is fixed, $u_0(x) \ge 0$ is non-random bounded continuous function and $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with $\sigma(0) = 0$. Let

$$l_\sigma := \inf_{z\in\mathbb{R}\setminus\{0\}} |\frac{\sigma(z)}{z}| > 0, \quad L_\sigma := \sup_{z\in\mathbb{R}\setminus\{0\}} |\frac{\sigma(z)}{z}| > 0. \tag{1.6}$$

They obtained the following

$$\frac{l_\sigma^2 t}{2} \le \liminf_{\lambda\to\infty} \frac{1}{\lambda^2} \log \mathscr{E}_t(\lambda), \quad \limsup_{\lambda\to\infty} \frac{1}{\lambda^4} \log \mathscr{E}_t(\lambda) \le 8L_\sigma^4 t.$$

More recently, Foondun-Joseph [13] complemented the results of [19], that is, they obtained $\mathbb{e} = 4$. It is easy to see that a mild solution u of (1.5) which is adapted to the filtration generated by the white noise and satisfies the following evolution equation

$$u(t,x) = (\mathcal{G}_D u)(t,x) + \lambda \int_0^t \int_0^L p_D(t-s,x,y)\sigma(u(s,y))w(dsdy), \tag{1.7}$$

where

$$(\mathcal{G}_D u)(t,x) := \int_0^L u_0(y) p_D(t,x,y)dy,$$

and $p_D(t,x,y)$ denotes the Dirichlet heat kernel, $D := [0,L]$. They used the estimate of kernel $p_D(t,x,y)$ and a new Gronwall's inequality to prove that $\mathbb{e} = 4$. Using similar method, Foondun-Liu-Tian [15] considered the fractional Laplacian on a bounded domain.

A natural question arises: is there any other type solution of (1.5) whose excitation indices is different from that in [19], that is to say, $\mathbb{e} \neq 4$. Let us first consider the case of SDEs. Now, given a complete probability space endowed with a filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, let us consider the following linear stochastic differential equation (SDE)

$$dX_t = \lambda X_t dB_t, \quad t > 0, \quad X_0 = x \in D.$$

For simplicity, we assume that $B(t)$ is a standard one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. It is easy to see that the unique solution of the above SDE is explicitly given by

$$X_t = xe^{-\frac{\lambda^2}{2}t} e^{\lambda B(t)}, \qquad t > 0.$$

Direct calculations then show that

$$\begin{aligned}
&\mathbb{E}[X_t] = xe^{-\frac{\lambda^2}{2}t} e^{\frac{\lambda^2}{2}t} = x; \quad \mathbb{E}[X_t^2] = x^2 e^{-\lambda^2 t} e^{2\lambda^2 t} = x^2 e^{\lambda^2 t};\\
&\mathbb{E}[X_t^p] = x^p e^{-\frac{\lambda^2 p}{2}t} e^{\frac{\lambda^2 p^2}{2}t} = x^p e^{\frac{\lambda^2 p(p-1)}{2}t}
\end{aligned}$$

for $p > 1$. This then implies that for $p > 1$

$$\lim_{\lambda\to\infty} \frac{\log\log\left(\mathbb{E}[X_t^p]\right)}{\log\lambda} = 2,$$

which yields that the excitation index of X_t is 2. This is clearly different from the results obtained in [13, 19, 28], where the authors proved the excitation indice of $u(t,x)$ of (1.5) is 4 for $x \in [\epsilon, L-\epsilon]$ (ϵ is a sufficiently small constant). Definitely, the case considered in [13, 19, 28] is stochastic partial differential equations, and the above example is stochastic ordinary differential equations. In this paper, our aim is to generalize the above case to the stochastic partial differential equations, that is to say, to find some kind of solutions of (1.5) with the associated indices being 2.

Another consideration of this paper is inspired by [13]. We consider the following stochastic parabolic equation

$$\begin{cases} du(t,x) = \Delta u(t,x)dt + \lambda u(t,x)dB_t, & x \in D,\ t > 0,\\ u|_{\partial D} = 0, & t > 0,\\ u(0,x) = u_0(x), \end{cases} \tag{1.8}$$

where $D \subset \mathbb{R}^n$ $(n \geq 1)$, B_t is a standard one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ as given above. We want to derive certain more interesting results than those obtained in [13]. It is remarked that in deriving those earlier results, the authors have applied the Itô formula to u^2 (in order to get the estimates of $\mathscr{E}_t(\lambda)$). And it is well-known that the Itô formula for $|u|^p$ does not hold with $0 < p < 1$. In order to overcome this difficulty, we first change the stochastic parabolic equations into random parabolic equations, then obtain the exact solution and further get the desired results by using the properties of Brownian motion. More precisely, we can get the index of $\left[\mathbb{E}\left(\|u(t)\|^p_{L^p(G)}\right)\right]^{1/p}$, $p > 0$, which is clearly an interesting extension.

In this paper, we will study the noise excitability of energy solution for some parabolic equations. We obtain a new result about the noise excitability, that is, $\mathbb{e} = 2$ under the same condition as in [13] when the noise is only the time perturbation (not space-time noise).

The rest of our paper is organized as follows. In Section 2, some preliminaries and main results are given. Section 3 is devoted to the proofs of the main results. In Section 4, we are concerned with a special case (1.7) and a noise excitability of a nonlocal operator.

2 Preliminaries and main results

In this section, we first recall some known results about the noise excitability, and then state our main results.

We start with the outline of the proof in [13] for the Dirichlet-Cauchy problem (1.5). First, it follows from the properties of kernels $p_D(t,x,y)$ and $p(t,x,y)$, where $p(t,x,y)$ stands for the kernel of whole space, that is, $p(t,x,y) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}}$, that for fixed $\epsilon > 0$, there exists $t_0 > 0$ depending on ϵ such that for $t \le t_0$ and for $x, y \in [\epsilon, L-\epsilon]$

$$\frac{1}{2} p(t,x,y) \le p_D(t,x,y) \le p(t,x,y). \tag{2.1}$$

Then utilising Itô formula, the following can be derived

$$\mathbb{E}|u(t,x)|^2 = |(\mathcal{G}_D u)(t,x)|^2 + \lambda^2 \int_0^t \int_0^L p_D^2(t-s,x,y) \mathbb{E}|\sigma(u(s,y))|^2 dyds. \tag{2.2}$$

Using (2.1) and (2.2), one obtains that $\mathbb{e} = 4$. It is easy to see that the method used in [18, 19, 13, 15] is mainly analyzing the kernel $p(t,x,y)$. In this paper, we will use a different approach to study the "intermittency" of energy solutions.

Next, let us recall the existence of the energy solution, i.e., the results of [7, 23]. For the space-time white noise, Dalang et al. [7] established the existence of energy solution to (1.5) with $x \in [0,1]$, see the Definition 1.3 of [7] for the definition of the energy solution. When the noise is just Brownian motion, we have the following known results. Let $(H, \langle \cdot, \cdot \rangle_H)$ be a separable Hilbert space and identified with its dual space H^* by the Riesz isomorphism, and let $(V, \langle \cdot, \cdot \rangle_V)$ be a Hilbert space such that it is continuously and densely embedded into H. More precisely,

$$V \subset H \equiv H^* \subset V^*.$$

Let (W_t) be a $\mathbb{R}^n$-valued standard Wiener process on $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$. We use $[\cdot, \cdot]$ to denote the scalar product in $\mathbb{R}^d$. Pardoux [23] considered the following problem

$$du(t) + A(t)u(t)dt = G(t)u(t)dW_t \tag{2.3}$$

with initial data $u_0 \in H$, for some fixed time T, where both $A(t)$ and $G(t)$ are linear operators, satisfying

$$A(\cdot) \in L^\infty(0,T; \mathscr{L}(V, V^*)), \qquad G(\cdot) \in L^\infty(0,T; \mathscr{L}(V, H^d)),$$

and the following coercivity hypothesis: $\exists \alpha > 0$ and λ such that for any $u \in V$,

$$2\langle A(t)u, u \rangle_{V^*,V} + \lambda \|u\|_H^2 \ge \alpha \|u\|_V^2 + \|G(t)u\|_H^2.$$

Under the above assumptions and using Galerkin finite dimension approximations, Pardoux obtained the following result.

Proposition 2.1 [23, Theorem 1.3] *Equation* (2.3) *has a unique solution* u*, which satisfies*
i) $u \in L^2(\Omega; C([0,T]; H))$*;*
ii)

$$\|u(t)\|_H^2 + 2\int_0^t \langle Au, u\rangle_{V^*,V} ds = \|u_0\|_H^2 + 2\int_0^t [\langle Gu, u\rangle_H, dW_s] + \int_0^t \|Gu\|_H^2 ds, \quad a.s..$$

Later, Liu [20] (see also [21]) used the same method as in [23] and obtained the well-posedness of (2.3) under some weak assumptions on A and G. About the well-posedness of stochastic parabolic equations, also see Theorem 7.2 in [2] and Section 7.2 in [5].

For further discussions, we will assume that K is another separable Hilbert space with the inner product $(\cdot,\cdot)_K$. For the convenience of the readers we present the general case. Let $L(K,H)$ denote a space of all bounded linear operators from K to H. Let $Q \in L(K,K)$ be nonnegative self-adjoint operator. Furthermore, $L_2^0(K,H)$ denotes the space of all $\xi \in L(K,H)$ such that $\xi\sqrt{Q}$ is a Hilbert-Schmidt operator and so $tr(\xi Q \xi^*) < \infty$. The norm is given by

$$\|\xi\|_{L_2^0}^2 := \|\xi\sqrt{Q}\|_{HS}^2 = tr(\xi Q \xi^*).$$

Then ξ is called a Q-Hilbert-Schmidt operator from K to H. We note that if $Q = I$, then $L_2^0(K,H)$ implies $L^2(K,H)$.

Recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is a given complete probability space endowed with a filtration. Let $\beta_n(t)\,(n = 1, 2, \cdots)$ be a sequence of real valued one dimensional standard Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$. We considers the following series

$$\sum_{n=1}^{\infty} \beta_n(t) e_n, \qquad t \geq 0,$$

where $\{e_n\}$ $(n = 1, 2, \cdots)$ is a complete orthonormal basis in K. Usually, this series does not necessarily converge in the space K. Thus we consider a K-valued stochastic process $w(t)$ given formally by the following series:

$$w(t) := \sum_{n=1}^{\infty} \beta_n(t)\sqrt{Q} e_n, \quad t \geq 0, \; Q \in L(K,K). \tag{2.4}$$

If $Q = I$, we have to assume that there exists a Hilbert space $K_1 \supset K$ such that it converges in K_1 containing K with a Hilbert-Schmidt embedding. Moreover, Q is the covariance operator with kernel q.

Recently, Taniguchi [26] generalized the results of [23] and obtained the existence of energy solution of the following equations

$$\begin{cases} du(t) = [A(t,u(t)) + f(t,u(t))]dt + g(t,u(t))dw(t), & t > 0, \\ u(0) = u_0 \in H, \end{cases} \tag{2.5}$$

where $w(t)$ is given by (2.4). Under the condition that f and g satisfy some local Lipschitz condition, the author obtained the existence of local energy solution, see [26, Theorem 2].

Definition 2.1 *An* $\mathcal{F}_t$*-adapted stochastic process* $u(t)$ *is called the energy solution to (2.5) if*

$$u \in L^2(\Omega \times [0,T]; V) \cap L^2(\Omega; C([0,T]; H))$$

and the following are satisfied

(1) the following holds in V^* almost surely, $t \in [0,T]$,

$$u(t) = u_0 + \int_0^t [A(s,u(s)) + f(s,u(s))]ds + \int_0^t g(s,u(s))dw(s);$$

(2) the following energy equality holds

$$\begin{aligned} \|u(t)\|_H^2 &= \|u_0\|_H^2 + 2\int_0^t \langle A(s,u(s)),u(s)\rangle_{V^*,V} ds + 2\int_0^t \langle f(s,u(s)),u(s)\rangle_H ds \\ &\quad +2\langle \int_0^t g(s,u(s)),u(s)dw(s)\rangle_H + \int_0^t \|g(s,u(s))\|_{L_2^0}^2 ds, \quad t\in[0,T]. \end{aligned}$$

Inspired by [18, 19, 13, 23, 26], in this paper, we consider the following problem

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \Delta u(t,x) + \lambda\sigma(u(t,x))\dot{w}(t), & x\in D,\, t>0, \\ u|_{\partial D} = 0, & t>0, \\ u|_{t=0} = u_0(x), & x\in D, \end{cases} \tag{2.6}$$

where $D \subset \mathbb{R}^n$ $(n \geq 1)$, and $\dot{w}$ denotes the space-time white noise or cylindrical Brownian motion. Here we take $H = L^2(D)$, $V = H_0^1(D)$.

Our main results are formulated in the following two theorems.

Theorem 2.1 *Assume that (1.6) holds and let w be the one-dimensional Brownian motion. The noise excitation index of the energy solution to (2.6) with initial data $u_0(x) \geq 0, \not\equiv 0$ is also 2.*

In the earlier results, many authors considered (2.6) by using the properties of heat kernel, but the method will be not suitable to the following problem

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = a(t)\Delta u(t,x) + \lambda\sigma(u(t,x))\dot{w}(t,x), & x\in D,\, t>0, \\ u|_{\partial D} = 0, & t>0, \\ u|_{t=0} = u_0(x), & x\in D, \end{cases} \tag{2.7}$$

where $a(t)$ is a stochastic process satisfying $0 < a_0 \leq a(t) \leq a_1$ (a_0 and a_1 are two positive constants), $D \subset \mathbb{R}^n$ $(n \geq 1)$ and $\dot{w}$ denotes the space-time white noise or cylindrical Brownian motion. Since the stochastic process $a(t)$ has uniformly bounded, the Galerkin method used in [23, 26] is also suitable for the existence of the energy solution. We have our further main result that

Theorem 2.2 *Assume that (1.6) holds. Let $w(t,x)$ be a Q-Wiener process, such that*

$$\mathbb{E}w(t,x)w(s,y) = (t\wedge s)q(x,y), \quad t,s>0,\, x,y\in D.$$

Assume that $0 < \sup_{x\in D} q(x,x) \leq q_1 < \infty$, then, the upper excitation index of the solution to (2.7) with initial data $u_0(x) \geq 0$ is 2. Furthermore, if $\sigma \geq 0$ (or ≤ 0) and there is a positive real number $q_0 > 0$ such that $q_0 < \inf_{x,y\in D} q(x,y)$, then the excitation index of the solution to (2.7) with initial data $u_0(x) \geq 0, \not\equiv 0$ is 2.

Remark 2.1 *1. We give the reason why we can not consider the case that $g(u)$ satisfies* **local Lipschitz** *condition. More precisely, consider the following general case*

$$\begin{cases} du(t) = [Au(t) + f(u(t))]dt + \lambda\sigma(u(t))dw(t), & t>0, \\ u(0) = u_0, \end{cases} \tag{2.8}$$

where A is a divergence operator, f and σ satisfy the local Lipschitz condition. For example, let $f(u) \geq au^{1+\alpha}$ and $\sigma(u) = u^m$. Then the solutions of (2.8) *will blow up in finite time (see [4, 22]). Moreover, the largest existence time $T \to 0$ as $\lambda \to \infty$. So we cannot consider problem* (2.8).

2. We give the relationship between mild solution, weak solution and energy solution. In order to find the relationship, we first recall the definition of weak solution (coincide with Definition 2.1): $u(t) \in C(0,T;H)$ is called a weak solution if the following equality holds almost surely

$$(u,\phi)_H + \int_0^t (u, -\Delta\phi)_H ds = (u_0,\phi)_H + \lambda(\int_0^t \sigma(u)dW_s, \phi)_H$$

for all $t \in [0,T]$ and all $\phi \in C_c^\infty(D)$.

It follows from the definitions of weak and mild solution that one can prove the two definitions are equivalent, see [29, Proposition 3.5]. For energy solution, firstly it is a special weak solution. On the other hand, following the definition 2.1, we find the assumptions about the energy solution is stronger than those of weak solution. And thus we can not say a weak solution is some kind of energy solution.

The mild solution (see (1.7)) is some kind of weak solution. Actually, we only need to take the test function as $\int_D p(t,x-y)\phi(x)dx$ but may be not energy solution because the test function may not stay in desired space. For example, in Definition 2.1, we need the test function belongs to $L^2(\Omega \times [0,T];V) \cap L^2(\Omega; C([0,T];H))$, but we can not conclude that $(p * u)(t,x,\omega)$ stay in the space. We remark that we consider problem (2.6) on a bounded domain and if we consider problem (2.6) in the whole space, then it is easy to prove that $(p * u)(t,x,\omega)$ belongs to $L^2(\Omega \times [0,T];V) \cap L^2(\Omega; C([0,T];H))$ because the properties of heat kernel on a bounded domain are different from that in the whole space.

Therefore, both mild solution and energy solution are weak solutions but they are two different types of the weak solution.

3. For problem (2.7), we did not use the properties of heat kernel to get the index because the heat kernel is a stochastic process and the noise term $\int_0^t \int_D p(t-s,x-y)\sigma(u)dydW_s$ is no longer a martingale, see [17] for the similar reason.

3 Proof of our main results

In this section, we will prove Theorem 2.1 and Theorem 2.2 by using energy method. Let us first prove Theorem 2.1.

Proof of Theorem 2.1. By using the idea of [23, 26], one can prove that there exists a unique energy solution. It follows from the results of [22] that the energy solution will keep positive if the initial data $u_0 \geq 0$ almost surely. We divide the proof into two steps.

Step 1: $\mathbb{E}(t) = 2$.

By Itô formula, we have

$$\begin{aligned} \|u(t)\|_{L^2}^2 &= \|u_0\|_{L^2}^2 + 2\int_0^t \langle \Delta u(s,x), u(s,x)\rangle ds + 2\lambda \int_0^t \int_D u(s,x)\sigma(u(s,x)dxdw_s \\ &\quad + \lambda^2 \int_0^t \int_D \sigma^2(u(s,x))dxds. \end{aligned} \tag{3.1}$$

Integrating by parts shows that

$$\begin{aligned}
\|u(t)\|_{L^2}^2 &= \|u_0\|_{L^2}^2 - 2\int_0^t \|\nabla u(s)\|_{L^2}^2 ds + 2\lambda \int_0^t \int_D u(s,x)\sigma(u(s,x))dxdw_s \\
&\quad +\lambda^2 \int_0^t \int_D \sigma^2(u(s,x))dxds \\
&\leq \|u_0\|_{L^2}^2 + 2\lambda \int_0^t \int_D u(s,x)\sigma(u(s,x))dxdw_s + \lambda^2 \int_0^t \int_D \sigma^2(u(s,x))dxds,
\end{aligned}$$

which implies

$$\begin{aligned}
\mathbb{E}\|u(t)\|_{L^2}^2 &\leq \mathbb{E}\|u_0\|_{L^2}^2 + \lambda^2 \mathbb{E}\int_0^t \int_D \sigma^2(u(s,x))dxds \\
&\leq \mathbb{E}\|u_0\|_{L^2}^2 + L_\sigma \lambda^2 \int_0^t \mathbb{E}\|u(s)\|_{L^2}^2 dxds.
\end{aligned}$$

It follows from Gronwall's inequality that

$$\mathbb{E}\|u(t)\|_{L^2}^2 \leq \mathbb{E}\|u_0\|_{L^2}^2 e^{L_\sigma \lambda^2 t}, \tag{3.2}$$

which implies that $\bar{\mathbb{e}}(t) \leq 2$.

Step 2: $\underline{\mathbb{e}}(t) = 2$.

In order to get the lower bounded, we will consider the eigenvalue problem for the elliptic equation

$$\begin{cases} -\Delta\phi = \lambda\phi, & \text{in } D, \\ \phi = 0, & \text{on } \partial D. \end{cases} \tag{3.3}$$

Then, since all the eigenvalues are strictly positive, increasing and the eigenfunction ϕ corresponding to the smallest eigenvalue λ_1 does not change sign in domain D, as shown in [16]. Therefore, we normalize it in such a way that

$$\phi(x) > 0 \text{ in } D, \quad \int_D \phi(x)dx = 1.$$

Noting that under the assumptions of Theorem 2.1, the solutions of (1.5) will keep positive, thus we can consider (u,ϕ) due to $(u,\phi) > 0$. Denote $\hat{u}(t) := (u,\phi)$. By applying Itô's formula to $\hat{u}^2(t)$ and making use of (3.3), we get

$$\begin{aligned}
\hat{u}^2(t) &= (u_0,\phi)^2 - 2\lambda_1 \int_0^t \hat{u}^2(s)ds + 2\lambda \int_0^t \int_D \hat{u}(s)\sigma(u_s(x))\phi(x)dxdw_s \\
&\quad +\lambda^2 \int_0^t \int_D \sigma^2(u_s(x))\phi^2(x)dxds \\
&\geq (u_0,\phi)^2 - 2\lambda_1 \int_0^t \hat{u}^2(s)ds + 2\lambda \int_0^t \int_D \hat{u}(s)\sigma(u(s,x))\phi(x)dxdw_s \\
&\quad +\lambda^2 l_\sigma^2 \int_0^t \int_D u^2(s,x)\phi^2(x)dxds \\
&\geq (u_0,\phi)^2 - 2\lambda_1 \int_0^t \hat{u}^2(s)ds + 2\lambda \int_0^t \int_D \hat{u}(s)\sigma(u(s,x))\phi(x)dxdw_s \\
&\quad +\lambda^2 l_\sigma^2 \int_0^t \hat{u}^2(s)ds.
\end{aligned} \tag{3.4}$$

Taking mean norm then yields that

$$\mathbb{E}\hat{u}^2(t) \quad \geq \quad \mathbb{E}(u_0,\phi)^2 - 2\lambda_1 \int_0^t \mathbb{E}\hat{u}^2(s)ds + \lambda^2 l_\sigma^2 \int_0^t \mathbb{E}\hat{u}^2(s)ds. \tag{3.5}$$

By the comparison principle, we know that

$$\mathbb{E}\hat{u}^2(t) \geq \mathbb{E}(u_0,\phi)^2 e^{(\lambda^2 l_\sigma^2 - 2\lambda_1)t}.$$

Due to

$$\hat{u}^2(t) = (u,\phi)^2 \leq \|\phi\|_{L^\infty}\|u\|_{L^2}^2,$$

we have $\underline{\mathbb{e}}(t) \geq 2$. So we have $\mathbb{e} = 2$. $\square$

Outline of the proof of Theorem 2.2. Similar to the proof of Theorem 2.1, equation (2.6) has a unique positive energy solution.

From (3.1), we have

$$\begin{aligned}
\|u(t)\|_{L^2}^2 &= \|u_0\|_{L^2}^2 + 2\int_0^t a(s)\langle \Delta u(s,x), u(s,x)\rangle ds + 2\lambda \int_0^t \int_D u(s,x)\sigma(u(s,x))w(dxds) \\
&\quad + \lambda^2 \int_0^t \int_D q(x,x)\sigma^2(u(s,x))dxds \\
&\leq \|u_0\|_{L^2}^2 + 2\lambda \int_0^t \int_D u(s,x)\sigma(u(s,x))w(dxds) + q_1\lambda^2 L_\sigma \int_0^t \int_D u^2(s,x)dxds.
\end{aligned}$$

Then taking expectation on both sides and using Grönwall's inequality, we have $\bar{\mathbb{e}}(t) \leq 2$.

Similar to the proof of Theorem 2.1, we have further

$$\begin{aligned}
\hat{u}^2(t) &= (u_0,\phi)^2 - 2\lambda_1 \int_0^t a(s)\hat{u}^2(s)ds + 2\lambda \int_0^t \int_D \hat{u}(s)\sigma(u(s,x))\phi(x)w(dx,ds) \\
&\quad + \lambda^2 \int_0^t \int_D \int_D \sigma(u(s,x))\phi(x)q(x,y)\sigma(u(s,y))\phi(y)dxdyds \\
&\geq (u_0,\phi)^2 - 2\lambda_1 a_1 \int_0^t \hat{u}^2(s)ds + 2\lambda \int_0^t \int_D \hat{u}(s)\sigma(u(s,x))\phi(x)w(dx,ds) \\
&\quad + \lambda^2 q_0 \int_0^t \int_D \int_D \sigma(u(s,x))\phi(x)\sigma(u(s,y))\phi(y)dxdyds \\
&\geq (u_0,\phi)^2 - 2\lambda_1 a_1 \int_0^t \hat{u}^2(s)ds + 2\lambda \int_0^t \int_D \hat{u}(s)\sigma(u(s,x))\phi(x)w(dx,ds) \\
&\quad + \lambda^2 q_0 l_\sigma^2 \int_0^t \hat{u}^2(s)ds,
\end{aligned} \tag{3.6}$$

which implies that $\underline{\mathbb{e}}(t) \geq 2$. So we have $\mathbb{e} = 2$. $\square$

Remark 3.1 *1. In the above proof we note that D can be a bounded domain in $\mathbb{R}^n$, $n \geq 1$. But in papers [18, 19, 13, 15], the authors only considered one dimension.*

2. The operator Δ can be replaced by a divergent operator A.

4 A special case and the noise excitability for nonlocal equations

In this section, we consider the following special case

$$\begin{cases}
du(t,x) = \Delta u(t,x)dt + \lambda u(t,x)dB_t, & x \in D,\ t > 0, \\
u|_{\partial D} = 0, & t > 0, \\
u(0,x) = u_0(x), & x \in D,
\end{cases} \tag{4.1}$$

where $D \subset \mathbb{R}^n$ $(n \geq 1)$, B_t is a standard one-dimensional Brownian motion on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

We first give an equivalent equation to (4.1).

Lemma 4.1 *Let u be a weak solution of (4.1). Then the function v defined by*

$$v(t,x) = e^{-\lambda B_t} u(t,x), \quad t > 0, \ x \in D$$

solves

$$\begin{cases} \frac{\partial}{\partial t} v(t,x) = \Delta v(t,x) - \frac{\lambda^2}{2} v(t,x), & x \in D, \, t > 0, \\ v|_{\partial D} = 0, & t > 0, \\ v(0,x) = u_0(x), & x \in D. \end{cases} \tag{4.2}$$

The proof of this lemma is standard, see e.g. the proof of Proposition 1.1 of [9]. We therefore omit it here.

Theorem 4.1 *Let u be a weak solution of (4.1) with non-random initial data u_0 satisfying*

$$c_1 \leq u_0(x) \leq c_2, \qquad \forall x \in D, \tag{4.3}$$

where c_i, $i = 1, 2$, are positive constants. Then we have, for $p > 0$,

$$2 \leq \liminf_{\lambda \to \infty} \frac{\log \log \mathcal{E}_t(\lambda)}{\log \lambda} \leq \limsup_{\lambda \to \infty} \frac{\log \log \mathcal{E}_t(\lambda)}{\log \lambda} \leq 2, \tag{4.4}$$

where $\mathcal{E}_t(\lambda) = \left[\mathbb{E} \left(\|u_t\|^p_{L^p(D)} \right) \right]^{1/p}$.

Proof. It follows from Lemma 4.1 that the solutions of (4.1) can be interpreted as

$$u(t,x) = e^{\lambda B_t} v(t,x).$$

It follows from the classical parabolic theory that the solutions v of (4.2) can be written as

$$\begin{aligned} v(t,x) &= e^{\frac{\lambda^2}{2} t} (e^{t\Delta} u_0)(x) \\ &= e^{\frac{\lambda^2}{2} t} \int_D p_D(t, x-y) u_0(y) dy, \quad a.s., \end{aligned}$$

where $p_D(t,x)$ is the kernel function of Δ on D. By using (4.3), we have

$$\hat{c}_1 e^{\frac{\lambda^2}{2} t} \leq v(t,x) \leq \hat{c}_2 e^{\frac{\lambda^2}{2} t}, \quad a.s.,$$

which implies that

$$\begin{aligned} \mathbb{E} \left[\|u(t)\|^p_{L^p(D)} \right] &= \mathbb{E} \left[\|v(t) e^{\lambda B_t}\|^p_{L^p(D)} \right] \\ &\geq \tilde{c}_1 e^{\frac{\lambda^2}{2} pt} \mathbb{E} \left[e^{\lambda p B_t} \right] \\ &= \tilde{c}_1 e^{\frac{\lambda^2}{2} pt} e^{\lambda^2 p^2 t} \end{aligned}$$

and

$$\begin{aligned} \mathbb{E} \left[\|u(t)\|^p_{L^p(D)} \right] &= \mathbb{E} \left[\|v(t) e^{\lambda B_t}\|^p_{L^p(D)} \right] \\ &\leq \tilde{c}_2 e^{\frac{\lambda^2}{2} pt} \mathbb{E} \left[e^{\lambda p B_t} \right] \\ &= \tilde{c}_2 e^{\frac{\lambda^2}{2} pt} e^{\lambda^2 p^2 t}. \end{aligned}$$

Combining the above two inequalities, we get the desired result. The proof is thus complete. □

Next, we will consider the following initial value problem

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = -(-\Delta)^{\frac{\alpha}{2}}u(t,x) + \lambda\sigma(u(t,x))\dot{w}(t,x), & x \in \mathbb{R},\ t > 0 \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{4.5}$$

where $\alpha \in (1,2]$, $(-\Delta)^{\frac{\alpha}{2}}$ is the L^2-generator of a symmetric stable process X_t of order α so that $\mathbb{E}\exp(i\xi \cdot X_t) = \exp(-t|\xi|^\alpha)$, $\{\dot{w}(x,t)\}_{t\geq 0, x\in\mathbb{R}}$ denotes the space-time white noise. In [15], the authors considered the equation (4.5) on a bounded domain. In this section, we generalize the result to the whole spatial space.

When σ satisfies global Lipschitz continuous condition, it is routine to show that (4.5) has a unique global mild solution, see the books [2, 24, 21], Dalang [6] and Foondun-Khoshnevisan [14]. It is easy to see that the mild solution of (4.5) can be represented by

$$u(t,x) = \int_{\mathbb{R}} p(t,x-y)u_0(y)dy + \lambda\int_0^t\int_{\mathbb{R}} p(t-s,x-y)\sigma(u(s,y))w(dsdy), \tag{4.6}$$

where $p(t,x)$ is the transition density function of a symmetric stable process of order α.

Before we state our main results, we recall some properties of kernel function (transition density function) $p(t,x)$.

Proposition 4.1 *([25]) The transition density $p(t,\cdot)$ of a strictly α-stable process satisfies*
(i) $p(st,x) = t^{-1/\alpha}p\left(s, t^{-1/\alpha}x\right)$;
(ii) For t large enough such that $p(t,0) \leq 1$ and $a > 2$, we have

$$p(t,(x-y)/a) \geq p(t,x)p(t,y), \quad \text{for all } x \in \mathbb{R};$$

(iii) $p(t,x) \asymp t^{-1/\alpha} \wedge \frac{t}{|x|^{1+\alpha}}$.

By using Proposition 4.1, it is easy to verify that

$$\int_{\mathbb{R}} p(t,x)p(s,x)dx = p(t+s,0). \tag{4.7}$$

In particular, $\|p(t,\cdot)\|^2_{L^2(\mathbb{R})} = p(2t,0)$.

Let

$$\mathscr{E}_t(\lambda) = \mathbb{E}\left[|u_t|^2\right].$$

We remark that the mild solution satisfying $\sup_{x\in\mathbb{R}}\mathscr{E}_t(\lambda) < \infty$ exists for the problem (4.5), see [14].

Theorem 4.2 *Assume that* (1.6) *holds and the initial data $u_0 \geq c_0 > 0$ is a bounded non-random function, where c_0 is a positive constant. Then the noise excitation index of solution to* (4.5) *with initial data $u_0(x)$ is $2\alpha/(\alpha-1)$.*

Remark 4.1 *Comparing with the results of [14], we take a different form $\mathscr{E}_t(\lambda)$. In [15], Foondun-Tian-Liu took*

$$\mathscr{E}_t(\lambda) = \sqrt{\mathbb{E}\left(\|u(t)\|^2_{L^2([0,1])}\right)}.$$

The reason why we can not take similar form to [15] is that we can not get the low bounded of this term $\int_{\mathbb{R}}\left(\int_{\mathbb{R}} p(t,x-y)u_0(y)dy\right)^2 dx$.

The assumption $u_0 \geq c_0 > 0$ is necessary because we used the fact $\inf_{x\in\mathbb{R}} u_0(x) > 0$.

Lemma 4.2 ([15, Proposition 2.6]) *Let $T \le \infty$ and $\beta > 0$. Suppose that $f(t)$ is a nonnegative locally integrable function satisfying*

$$f(t) \ge c_1 + k \int_0^t (t-s)^{\beta-1} f(s) ds, \qquad \text{for all } 0 \le t \le T, \tag{4.8}$$

where c_1 is some positive constant. Then for any $t \in (0, T]$, we have the following

$$\liminf_{k\to\infty} \frac{\log\log f(t)}{\log k} \ge \frac{1}{\beta}.$$

When the inequality (4.8) is reversed with the second inequality in (4.3), we have

$$\limsup_{k\to\infty} \frac{\log\log f(t)}{\log k} \le \frac{1}{\beta}.$$

Proof of Theorem 4.2. By using the mild formulation and Itô isometry, we have the following

$$\mathbb{E}|u(t,x)|^2 = \Big| \int_{\mathbb{R}} p(t, x-y) u_0(y) dy \Big|^2 + \lambda^2 \int_0^t \int_{\mathbb{R}} p^2(t-s, x-y) \mathbb{E}|\sigma(u(s,y))|^2 dy ds, \tag{4.9}$$

which implies that

$$\begin{aligned} \sup_{x\in\mathbb{R}} \mathbb{E}|u(t)|^2 \quad &\le \quad \sup_{x\in\mathbb{R}} \Big| \int_{\mathbb{R}} p(t, x-y) u_0(y) dy \Big|^2 \\ &\quad + \lambda^2 \int_0^t \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}|\sigma(u(s, x-y))|^2 p^2(t-s, y) dy ds. \end{aligned} \tag{4.10}$$

By using the boundedness of u_0, (1.6) and (4.7), we have

$$\begin{aligned} \sup_{x\in\mathbb{R}} \mathbb{E}|u(t)|^2 \quad &\le \quad C + \lambda^2 \int_0^t p(2(t-s), 0) \sup_{x\in\mathbb{R}} \mathbb{E}|\sigma(u(s,x))|^2 ds \\ &\le \quad C + \lambda^2 L_\sigma^2 \int_0^t \frac{C}{(t-s)^{1/\alpha}} \sup_{x\in\mathbb{R}} \mathbb{E}|u(s)|^2 ds. \end{aligned} \tag{4.11}$$

By Lemma 4.2, (4.11) then implies that

$$\limsup_{\lambda\to\infty} \frac{\log\log \sup_{x\in\mathbb{R}} \mathbb{E}|u(t)|^2}{\log \lambda} \le \frac{2\alpha}{\alpha - 1}. \tag{4.12}$$

Next, we prove $\underline{\mathfrak{e}}(t) \ge 2\alpha/(\alpha-1)$. First, it follows from (4.9) that

$$\inf_{x\in\mathbb{R}} \mathbb{E}|u(t,x)|^2 \ge c + \lambda^2 \int_0^t \inf_{x\in\mathbb{R}} \int_{\mathbb{R}} p^2(t-s, y) \mathbb{E}|\sigma(u(s, x-y))|^2 dy ds,$$

where c is a positive constant depending on c_0 and the heat kernel p. Taking infimum over $\mathbb{R}$, we have

$$\begin{aligned} \inf_{x\in\mathbb{R}} \mathbb{E}|u(t)|^2 \quad &\ge \quad c + \lambda^2 \int_0^t \inf_{x\in\mathbb{R}} \mathbb{E}|\sigma(u(s,x))|^2 \left(\int_{\mathbb{R}} p^2(t-s, y) dx \right) dy ds \\ &\ge \quad c + \lambda^2 l_\sigma^2 \int_0^t \frac{C}{(t-s)^{1/\alpha}} \inf_{x\in\mathbb{R}} \mathbb{E}|u(s)|^2 ds. \end{aligned} \tag{4.13}$$

Again, by Lemma 4.2, (4.13) then implies that

$$\liminf_{\lambda\to\infty} \frac{\log\log \inf_{x\in\mathbb{R}} \mathbb{E}|u(t)|^2}{\log \lambda} \ge \frac{2\alpha}{\alpha - 1}. \tag{4.14}$$

Combining (4.12) and (4.14), we get

$$\inf_{x\in\mathbb{R}} \mathbb{E}|u(t)|^2 \geq e^{\lambda^{\frac{2\alpha}{\alpha-1}}} \quad \text{for} \quad \lambda \gg 1;$$

$$\sup_{x\in\mathbb{R}} \mathbb{E}|u(t)|^2 \leq e^{\lambda^{\frac{2\alpha}{\alpha-1}}} \quad \text{for} \quad \lambda \gg 1.$$

Thus by using

$$\inf_{x\in\mathbb{R}} \mathbb{E}|u(t)|^2 \leq \mathbb{E}|u(t)|^2 \leq \sup_{x\in\mathbb{R}} \mathbb{E}|u(t)|^2,$$

we get the desired result. The proof of Theorem 4.2 is complete. □

Remark 4.2 *When $\alpha = 2$, (4.14) was obtained in Khoshnevisan-Kim* [18].

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