On the asymptotic behavior of highly nonlinear hybrid stochastic delay differential equations[1]

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Abstract. In this paper, the existence and uniqueness, the stability analysis for the global solution of highly nonlinear stochastic differential equations with time-varying delay and Markovian switching are analyzed under a locally Lipschitz condition and a monotonicity condition. In order to overcome a difficulty stemming from the existence of the time-varying delay, one integral lemma is established. It should be mentioned that the time-varying delay is a bounded measurable function. By utilizing the integral inequality, the Lyapunov function and some stochastic analysis techniques, some sufficient conditions are proposed to guarantee the stability in both moment and almost sure senses for such equations, which can be also used to yield the almost surely asymptotic behavior. As a by-product, the exponential stability in pth($p \geq 1$)-moment and the almost sure exponential stability are analyzed. Finally, two examples are given to show the usefulness of the results obtained.

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1 Introduction

Many dynamical systems not only depend on the present state but also the past ones, which are described by differential delay equations (DDEs) [1]. Since DDEs have been used in many fields, such as the population ecology, steam or water pipes, heat exchangers, lossless transmission lines, and the mass-spring-damper model, etc, the dynamical behavior for DDEs has been widely investigated. When DDEs are subject to the environmental disturbances, it can be characterized by stochastic delay differential equations (SDDEs)(see [2]-[7] and the references therein). One of the important issues in the study of SDDEs is automatic control, with consequent emphasis being placed on the stability analysis. Many papers on the stochastic stability analysis have been published. For instance, in [8], the dynamical behavior for stochastic delay Lotka-Volterra model as a particularly important application of SDDEs was analyzed. In [9], the exponential stability analysis for linear stochastic delay differential equation has been investigated by one useful and advanced method such as the comparison principle. In [10], by establishing the LaSalle theorem, the stability analysis for SDDEs has been investigated.

Hybrid systems driven by continuous-time Markov chains have been used to describe many practical systems, in which they may experience abrupt changes in their structure and parameters, for example, electric power systems, manufacturing systems, financial systems, etc. The hybrid systems comprise two parts: one is that the state takes values continuously, and the other is that the state takes discrete values. Recently, the stability analysis for SDDEs with Markovian switching has been extensively studied. For example, in [13], the comparison principle was used to study the stability for SDDEs with Markovian switching. In [15], by using the Lyapunov functional approach, the exponential stability in pth($p \geq 1$)-moment and the almost sure exponential stability for SDDEs with Markovian switching have been investigated under one monotonicity condition, which likes (2.6) (see *Hypothesis IV*). In [18], by utilizing a linear matrix inequality approach, the delay-dependent exponential stability of stochastic systems with time-varying delays, Markovian switching and nonlinearities has been discussed. In [14], by using the Lyapunov functional approach, the delay feedback control was designed to achieve the stabilization of hybrid SDDEs. In [16, 17], in order to reduce the control cost, the feedback control based on discrete-time state observations was designed to guarantee the stabilization of SDDEs with Markovian switching.

Note that there are some results on the stability analysis of SDDEs with Markovian switching, in which the diffusion term and the drift term of the SDDEs obey the local Lipschitz condition and the linear growth condition. Usually, for many nonlinear SDDEs, these two terms often do not satisfy the linear growth condition, but satisfy the local Lipschitz condition. When the linear growth condition is replaced with the monotonicity condition, one of the most powerful technique used in the study of stability of SDDEs with Markovian switching is based on a stochastic version of the Lyapunov direct method, and there are some representative works on the stability analysis for highly nonlinear SDDEs with Markovian switching. For example, In [19], the delay-dependent stability criteria for highly nonlinear SDDEs with Markovian switching have been derived by using the Lyapunov function approach. Without the linear growth condition, the existence and uniqueness, the stability analysis and boundedness for the global solution of highly nonlinear SDDEs with Markovian switching were considered in [20, 21].

However, the obtained results in the literature are only suitable for the constant delay or the time-varying delay with its derivative value being less than one. It is well known that in most industrial process involving transportation of materials, delay variation is one among the well-known structural time variations in the process plants. Since the transportation time varies frequently according to varying flow rates, time-varying delay is an inherent characteristics of these processes, which varies around a constant value and depends on the frequency of the external excitation [23]. Thus, we will analyze the existence and uniqueness of the global solution as well as its stability properties when this restrictive condition imposed on the time-varying delay is removed, the local Lipschitz condition is satisfied for the drift term and the diffusion term, and the linear growth condition is replaced by the monotonicity condition.

In this paper, the existence and uniqueness theorem for the global solution of highly nonlinear SDDEs with Markovian switching is primarily considered under a local Lipschitz condition and a monotonicity condition. Without the derivative value of the time-varying delay being less than one, the exponential stability in pth$(p \geq 1)$-moment for such equations is discussed by using the integral inequality, and the almost sure exponential stability is analyzed by employing the nonnegative semi-martingale convergence theorem. The almost sure asymptotical stability for the global solution of highly nonlinear SDDEs with Markovian switching is also investigated by virtue of some stochastic analysis technique. Finally, two examples including one coupled systems consisting of a mass-spring-damper with the nonlinear external random forces are provided to validate the effectiveness of the theoretical findings obtained.

Notations: Throughout this paper, unless otherwise specified, we use the following notation. Let $|\cdot|$ denote the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ represent a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (*i.e.*, it is increasing and right-continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = \text{col}[B_1(t), B_2(t), \ldots, B_m(t)]$ be an m-dimensional Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. For $\tau > 0$, let $\mathcal{C}([-\tau, 0]; R^n)$ represent the family of all continuous R^n-valued functions on $[-\tau, 0]$ with norm $\|\varphi\|_{\mathcal{C}} = \sup\{|\varphi(\theta)| : -\tau \leq \theta \leq 0\}$ for any $\varphi \in \mathcal{C}([-\tau, 0]; R^n)$. $\mathcal{C}_{\mathcal{F}_t}([-\tau, 0]; R^n)$ denotes the family of all $\mathcal{F}_t$-measurable and $\mathcal{C}([-\tau, 0]; R^n)$-valued random variables $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$. $\mathcal{C}(R^n; [0, \infty))$ means the family of all nonnegative continuous functions defined on R^n. Let $\mathbb{E}\{\cdot\}$ stand for the expectation operator. For any two numbers a, b, $a \vee b$ and $a \wedge b$ denote the maximum value and the minimum value between a and b, respectively. $H(a-)$ denotes the left-hand limit of the function $H(\cdot)$ at a, *i. e.* $H(a-) = \lim_{u\to 0-} H(a + u)$. '*a.s.*' stands for 'almost surely'. $Ker(U(x)) = \{x \in R^n : U(x) = 0\}$.

2 Problem statement and preliminaries

Let $r(t)(t \geq 0)$ be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathcal{S} = \{1, 2, ..., N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\triangle) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\triangle + o(\triangle), & \text{if} \quad i \neq j, \\ 1 + \gamma_{ij}\triangle + o(\triangle), & \text{if} \quad i = j, \end{cases}$$

where $\lim_{\triangle \downarrow 0} \frac{o(\triangle)}{\triangle} = 0$. Here, $\gamma_{ij} \geq 0$ is the transition rate from i to j, if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$.

For a continuous-time Markov chain $r(t)$ with its generator Γ, it can be given as one stochastic integral with respect to a Poisson random measure

$$dr(t) = \int_R \bar{h}(r(t-), y)\nu(dt, dy), \quad t \geq 0$$

with the initial value $r(0) = i_0 \in \mathcal{S}$, where $\nu(dt, dy)$ is a Poisson random measure with intensity $dt \times m(dy)$ in which m is the Lebesgue measure on R, while the explicit definition of $\bar{h} : \ \mathcal{S} \times R \to R$ can be founded in [12].

Consider the following highly nonlinear hybrid stochastic delay differential equations:

$$dx(t) = f(t, x(t), x(t-\tau(t)), r(t))dt + g(t, x(t), x(t-\tau(t)), r(t))dB(t), \quad t \geq 0, \quad (2.1)$$

with the initial condition $\{x(\theta) : -\tau \leq \theta \leq 0\} = \varphi \in \mathcal{C}_{\mathcal{F}_0}([-\tau, 0]; R^n)$ and $r(0) = i_0 \in \mathcal{S}$, where $x(t) = \text{col}[x_1(t),\ x_2(t),\ \ldots,\ x_n(t)] \in R^n$ is the state vector. The time-varying delay $\tau(\cdot)\ : [0,\infty) \to [0, \tau]$ is a bounded measurable function. $f(\cdot,\cdot,\cdot,\cdot) :\ [0,\infty)\times R^n \times R^n \times \mathcal{S} \to R^n$ is the drift coefficient vector, and $g(\cdot,\cdot,\cdot,\cdot) :\ [0,\infty) \times R^n \times R^n \times \mathcal{S} \to R^{n\times m}$ is the diffusion coefficient matrix. In this paper, it is also assumed that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. Let $x(t, 0, \varphi, i_0)$ be the solution of Eq. (2.1). For simplicity, $x(t) = x(t, 0, \varphi, r(0))$.

In this paper, the existence-uniqueness theorem, and the asymptotic behavior for the global solution of Eq. (2.1) will be checked. In general, the following assumptions are imposed.

Hypothesis I (*Local Lipschitz condition*): For each $k = 1, 2, \ldots$, there exists a positive constant c_k such that

$$|f(t, x, y, i) - f(t, \bar{x}, \bar{y}, i)| \vee |g(t, x, y, i) - g(t, \bar{x}, \bar{y}, i)| \leq c_k(|x - \bar{x}| + |y - \bar{y}|)$$

for any $(t, i) \in [0, T] \times \mathcal{S}$ $(T > 0)$, $x, y, \bar{x}, \bar{y} \in R^n$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq k$. In addition, $f(t, 0, 0, i) = 0$ and $g(t, 0, 0, i) = 0$.

Hypothesis II (*Linear growth condition*): There is a positive constant L such that

$$|f(t, x, y, i)| \vee |g(t, x, y, i)| \leq L(1 + |x| + |y|)$$

for any $(t, x, y, i) \in [0, T] \times R^n \times R^n \times \mathcal{S}$.

Note that *Hypothesis II* is a conservative condition to check the existence of the global solution. For example, when $\mathcal{S} = \{1, 2\}$, $f(t, x, y, 1) = -0.15x - 2x^3 + 0.4y$, $f(t, x, y, 2) = -2x - 0.5xy^4 + 0.82y$, $g(t, x, y, 1) = 2x^2$, and $g(t, x, y, 2) = xy^2$, for any $t \geq 0$, *Hypothesis II* does not hold for $f(\cdot, \cdot, \cdot, \cdot)$ and $g(\cdot, \cdot, \cdot, \cdot)$. Here, we shall persist *Hypothesis I* but replace *Hypothesis II* by a more general condition to guarantee the existence of the unique global solution to Eq. (2.1). To state a general condition, we need a few notations. Let $\mathcal{C}^{1,2} \equiv \mathcal{C}^{1,2}([0, \infty) \times R^n \times \mathcal{S}; [0, \infty))$ denote the family of all continuous nonnegative functions $V(t, x, i)$ defined on $[0, \infty) \times R^n \times \mathcal{S}$, such that for each $i \in \mathcal{S}$, they are continuously once differentiable in t and twice in x. Given $V \in \mathcal{C}^{1,2}$, then we define the Itô operator $LV : [0, \infty) \times R^n \times R^n \times \mathcal{S} \longrightarrow R$ by

$$\begin{aligned}
& LV(t, x, y, i) \\
= \; & V_t(t, x, i) + V_x(t, x, i) f(t, x, y, i) + \frac{1}{2}\text{trace}[g^T(t, x, y, i) V_{xx}(t, x, i) g(t, x, y, i)] \\
& + \sum_{j=1}^{N} \gamma_{ij} V(t, x, j).
\end{aligned}$$

where

$$V_t(t, x, i) = \frac{\partial V(t, x, i)}{\partial t}, \quad V_x(t, x, i) = \left(\frac{\partial V(t, x, i)}{\partial x_1}, \frac{\partial V(t, x, i)}{\partial x_2}, \ldots, \frac{\partial V(t, x, i)}{\partial x_n} \right),$$

and

$$V_{xx}(t, x, i) = \left(\frac{\partial^2 V(t, x, i)}{\partial x_l \partial x_m} \right)_{n \times n}.$$

To obtain the main results, one more general condition is presented as follows:
Hypothesis III (*Monotonicity condition*): There exist one Lyapunov function $V \in \mathcal{C}^{1,2}$, one function $U \in \mathcal{C}(R^n; [0, \infty))$ and some positive constants c_1, c_2, λ_1 and λ_2 such that for any $x, \ y \in R^n$, $t \geq 0$, and $i \in \mathcal{S}$,

$$c_1 U(x) \leq V(t, x, i) \leq c_2 U(x), \tag{2.2}$$

and

$$LV(t, x, y, i) \leq -\lambda_1 U(x) + \lambda_2 U(y), \tag{2.3}$$

and

$$\lim_{|x| \to \infty} U(x) = \infty. \tag{2.4}$$

When $U(x) = |x|^p$, *Hypothesis III* can be written as the following form:
Hypothesis IV: There exist one Lyapunov function $V \in \mathcal{C}^{1,2}$, and some positive constants p, c_1, c_2, λ_1 and λ_2 with $\lambda_2 c_2 < \lambda_1 c_1$ such that for any $x, y \in R^n$, $t \geq 0$, and $i \in \mathcal{S}$,

$$c_1 |x|^p \leq V(t, x, i) \leq c_2 |x|^p, \tag{2.5}$$

and

$$LV(t,x,y,i) \le -\lambda_1|x|^p + \lambda_2|y|^p, \tag{2.6}$$

where $p \ge 1$.

Remark 2.1 *In [9, 11, 12, 15], Hypothesis IV has been imposed with $\tau(t) \equiv \tau$ or $\frac{d\tau(t)}{dt} \in (0,1)$. It should be mentioned that the restrictive condition that the derivative value of time-varying delay is less than one is not required in this paper. Thus, the proposed methods in [9, 11, 12, 15] can not be used here. The asymptotic behavior for high nonlinear SDDEs with Markovian switching has been considered under the general monotonicity condition [19, 20, 21, 22], but this restrictive condition is also imposed.*

Definition 2.2 *Let $x(t) : -\tau \le t < \sigma_\infty$ be a continuous $\mathcal{F}_t$-adapted R^n-valued local process, where σ_∞ is a stopping time and we set $\mathcal{F}_t = \mathcal{F}_0$ for $t \in [-\tau, 0]$. It is called a local solution of Eq. (2.1) with initial condition $\varphi \in \mathcal{C}_{\mathcal{F}_0}([-\tau,0];R^n)$. If $x_0 = \varphi = \{x(\theta) : -\tau \le \theta \le 0\}$ and for all $t \ge 0$*

$$\begin{aligned} x(t\wedge\sigma_k) =& \varphi(0) + \int_0^{t\wedge\sigma_k} f(s,x(s),x(s-\tau(s)),r(s))ds \\ &+ \int_0^{t\wedge\sigma_k} g(s,x(s),x(s-\tau(s)),r(s))dB(s) \end{aligned}$$

holds for any $k \ge 1$, where $\{\sigma_k\}_{k\ge1}$ is a nondecreasing sequence of finite stopping times such that $\sigma_k \uparrow \sigma_\infty$, *a.s.* Furthermore, if $\limsup_{k\to\infty}|x(\sigma_k)| = \infty$ is satisfied whenever $\sigma_\infty < \infty$, it is called a maximal solution and σ_∞ is called the explosion time. A maximal local solution $x(t) : -\tau \le t < \sigma_\infty$, is said to be unique if for any other maximal local solution $\hat{x}(t) : -\tau \le t < \hat{\sigma}_\infty$, we have $\sigma_\infty = \hat{\sigma}_\infty$ *a.s.*, and $x(t) = \hat{x}(t)$ for all $-\tau \le t < \sigma_\infty$ a.s.

Definition 2.3 *The global solution $x(t)$ of Eq. (2.1) is said to be exponentially stable in $pth(p \ge 1)$ moment with decay e^t of order γ, if there exists a positive constant γ such that*

$$\limsup_{t\to\infty} \frac{\log(\mathbb{E}|x(t)|^p)}{t} \le -\gamma$$

holds for any $\varphi \in \mathcal{C}_{\mathcal{F}_0}([-\tau,0];R^n)$. Furthermore, the global solution $x(t)$ of Eq. (2.1) is said to be almost surely exponentially stable with exponential decay e^t of order γ', if

$$\limsup_{t\to\infty} \frac{\log(|x(t)|)}{t} \le -\gamma', \quad a.s.$$

holds for any $\varphi \in \mathcal{C}_{\mathcal{F}_0}([-\tau,0];R^n)$.

Lemma 2.4 *([24]) For $\gamma > 0$, there exist two positive constants: λ, λ' with $\lambda' < \gamma$, and a function $y: [-\tau, \infty) \to [0, \infty)$. If the inequality*

$$y(t) \leq \begin{cases} \lambda e^{-\gamma t} + \lambda' \int_0^t e^{-\gamma(t-s)} \sup_{\theta \in [-\tau,0]} y(s+\theta) ds, & for \;\; t \geq 0, \\ \lambda e^{-\gamma t}, & for \;\; t \in [-\tau, 0], \end{cases} \tag{2.7}$$

holds, then we have $y(t) \leq \tilde{M} e^{-\mu t}$, for any $t \in [-\tau, \infty)$, where μ is a unique positive root of the algebra equation: $\frac{\lambda' e^{\mu\tau}}{\gamma - \mu} = 1$ and $\tilde{M} = \max\left\{ \frac{\lambda(\gamma-\mu)}{\lambda' e^{\mu\tau}}, \lambda \right\} > 0$.

3 Main results

Lemma 3.1 *Let $x(t)$ be a solution to Eq. (2.1) with the initial condition φ. Suppose that Hypotheses I and III hold. Assume that the inequality*

$$\lambda_2 c_2 < \lambda_1 c_1,$$

holds, then we have

$$\Delta(\varepsilon) = \int_0^\infty e^{\varepsilon t} \sup_{\theta \in [-\tau,0]} \mathbb{E} U(x(t+\theta)) dt < \infty, \tag{3.1}$$

where $\varepsilon \in (0, \varepsilon_0)$, ε_0 is a unique positive solution of the algebraic equation:

$$\frac{\lambda_2 c_2 e^{\varepsilon\tau}}{\lambda_1 c_1 - c_1 c_2 \varepsilon} = 1.$$

Proof: Define a function: $H(\varepsilon) = \frac{\lambda_2 c_2 e^{\varepsilon\tau}}{\lambda_1 c_1 - c_1 c_2 \varepsilon} - 1$. It can be proved that $H(0) < 0$, $H((\frac{\lambda_1}{c_2})-) = \infty$, and $H(\varepsilon)$ is a nondecreasing function on $(0, \frac{\lambda_1}{c_2})$. Therefore, there exists a scalar $\varepsilon_0 \in (0, \frac{\lambda_1}{c_2})$ satisfying $H(\varepsilon_0) = 0$. That is, for any $\varepsilon \in (0, \varepsilon_0)$, we have

$$\Lambda(\varepsilon) \equiv \frac{\lambda_2 c_2 e^{\varepsilon\tau}}{\lambda_1 c_1 - c_1 c_2 \varepsilon} < 1. \tag{3.2}$$

Using the Itô formula, for any $t \geq 0$, it follows

$$\begin{aligned} & e^{\frac{\lambda_1}{c_2} t} V(t, x(t), r(t)) \\ & \leq V(0, x(0), r(0)) + \int_0^t e^{\frac{\lambda_1}{c_2} s} \left[\frac{\lambda_1}{c_2} V(s, x(s), r(s)) + LV(s, x(s), x(s-\tau(s)), r(s)) \right] ds \\ & \quad + \int_0^t e^{\frac{\lambda_1}{c_2} s} V_x(s, x(s), r(s)) g(s, x(s), x(s-\tau(s)), r(s)) dB(s) \\ & \quad + \int_0^t \int_R e^{\frac{\lambda_1}{c_2} s} [V(s, x(s), i_0 + \bar{h}(r(s-), l) - V(s, x(s), r(s))] \mu(ds, dl), \end{aligned} \tag{3.3}$$

where $\mu(ds, dl) = \nu(ds, dl) - m(dl)$ is a martingale measure, which is related to the Markov chain but not the Brownian motion.

From conditions (2.2) and (2.3), we obtain

$$\frac{\lambda_1}{c_2}V(s,x(s),r(s)) + LV(s,x(s),x(s-\tau(s)),r(s)) \le \lambda_2 U(x(s-\tau(s)). \tag{3.4}$$

Substituting (3.4) into (3.3), and then taking the expectation, it yields

$$e^{\frac{\lambda_1}{c_2}t}\mathbb{E}V(t,x(t),r(t)) \le \mathbb{E}V(0,x(0),r(0)) + \lambda_2\int_0^t e^{\frac{\lambda_1}{c_2}s}\mathbb{E}U(x(s-\tau(s))ds.$$

By using condition (2.2), it concludes that for any $t \ge 0$,

$$\begin{aligned}\mathbb{E}U(x(t)) &\le \frac{\mathbb{E}V(0,x(0),r(0))}{c_1}e^{-\frac{\lambda_1}{c_2}t} + \frac{\lambda_2}{c_1}\int_0^t e^{-\frac{\lambda_1}{c_2}(t-s)}\mathbb{E}U(x(s-\tau(s))ds \\ &\le M'e^{-\frac{\lambda_1}{c_2}t} + \frac{\lambda_2}{c_1}\int_0^t e^{-\frac{\lambda_1}{c_2}(t-s)}\mathbb{E}U(x(s-\tau(s))ds,\end{aligned} \tag{3.5}$$

where $M' = \frac{\mathbb{E}V(0,x(0),r(0))}{c_1} > 0$.

For any $t \ge \tau$ and $\theta \in [-\tau, 0]$, from (3.5), we have

$$\begin{aligned}\mathbb{E}U(x(t+\theta)) &\le M'e^{-\frac{\lambda_1}{c_2}(t+\theta)} + \frac{\lambda_2}{c_1}\int_0^{t+\theta} e^{-\frac{\lambda_1}{c_2}(t+\theta-s)}\mathbb{E}U(x(s-\tau(s))ds \\ &\le M'e^{-\frac{\lambda_1}{c_2}(t+\theta)} + \frac{\lambda_2}{c_1}\int_0^{t+\theta} e^{-\frac{\lambda_1}{c_2}(t+\theta-s)}\sup_{u\in[-\tau,0]}\mathbb{E}U(x(s+u))ds.\end{aligned}$$

Multiplying by $e^{\varepsilon t}(\varepsilon \in (0,\varepsilon_0))$ on both sides of inequality above in turn, and then integrating with τ to $T(T>\tau)$, it follows

$$\begin{aligned}&\int_\tau^T e^{\varepsilon t}\mathbb{E}U(x(t+\theta))dt \\ &\le M'\int_\tau^T e^{\varepsilon t-\frac{\lambda_1}{c_2}(t+\theta)}dt + \frac{\lambda_2}{c_1}\int_\tau^T\int_0^{t+\theta} e^{\varepsilon t-\frac{\lambda_1}{c_2}(t+\theta-s)}\sup_{u\in[-\tau,0]}\mathbb{E}U(x(s+u))dsdt.\end{aligned} \tag{3.6}$$

Note that for any $\theta \in [-\tau, 0]$ and $t \ge \tau$, the formula of integration by parts implies

$$\begin{aligned}&\int_\tau^T\int_0^{t+\theta} e^{\varepsilon t-\frac{\lambda_1}{c_2}(t+\theta-s)}\sup_{u\in[-\tau,0]}\mathbb{E}U(x(s+u))dsdt \\ &\le e^{\varepsilon\tau}\int_\tau^T e^{-(\frac{\lambda_1}{c_2}-\varepsilon)(t+\theta)}\int_0^{t+\theta} e^{\frac{\lambda_1}{c_2}s}\sup_{u\in[-\tau,0]}\mathbb{E}U(x(s+u))dsdt \\ &\le \frac{e^{\frac{\lambda_1}{c_2}\tau}}{\frac{\lambda_1}{c_2}-\varepsilon}\int_0^\tau e^{\frac{\lambda_1}{c_2}s}\sup_{u\in[-\tau,0]}\mathbb{E}U(x(s+u))ds \\ &\quad + \frac{e^{\varepsilon\tau}}{\frac{\lambda_1}{c_2}-\varepsilon}\int_0^T e^{\varepsilon s}\sup_{u\in[-\tau,0]}\mathbb{E}U(x(s+u))ds.\end{aligned} \tag{3.7}$$

Substituting (3.7) to (3.6) implies

$$
\begin{aligned}
&\int_{\tau}^{T} e^{\varepsilon t}\mathbb{E}U(x(t+\theta))dt \\
\le\ & M' e^{\frac{\lambda_1}{c_2}\tau}\int_{\tau}^{T} e^{\varepsilon t-\frac{\lambda_1}{c_2}t}dt + \frac{\lambda_2 c_2 e^{\frac{\lambda_1}{c_2}\tau}}{\lambda_1 c_1 - c_1 c_2\varepsilon}\int_{0}^{\tau} e^{\frac{\lambda_1}{c_2}s}\sup_{u\in[-\tau,0]}\mathbb{E}U(x(s+u))ds \\
&+ \frac{\lambda_2 c_2 e^{\varepsilon\tau}}{\lambda_1 c_1 - c_1 c_2\varepsilon}\int_{0}^{T} e^{\varepsilon s}\sup_{u\in[-\tau,0]}\mathbb{E}U(x(s+u))ds.
\end{aligned}
\tag{3.8}
$$

From (3.8), we have

$$
\begin{aligned}
&\int_{0}^{T} e^{\varepsilon t}\mathbb{E}U(x(t+\theta))dt \\
=\ & \int_{0}^{\tau} e^{\varepsilon t}\mathbb{E}U(x(t+\theta))dt + \int_{\tau}^{T} e^{\varepsilon t}\mathbb{E}U(x(t+\theta))dt \\
\le\ & \bar{M} + \Lambda(\varepsilon)\int_{0}^{T} e^{\varepsilon s}\sup_{u\in[-\tau,0]}\mathbb{E}U(x(s+u))ds,
\end{aligned}
\tag{3.9}
$$

where $\bar{M} = \int_0^\tau e^{\varepsilon t}\mathbb{E}U(x(t+\theta))dt + \frac{c_2 M' e^{\varepsilon\tau}}{\lambda_1 - c_2\varepsilon} + \frac{\lambda_2 c_2 e^{\frac{\lambda_1}{c_2}\tau}}{\lambda_1 c_1 - c_1 c_2\varepsilon}\int_0^\tau e^{\frac{\lambda_1}{c_2}s}\sup_{u\in[-\tau,0]}\mathbb{E}U(x(s+u))ds$.

Combing (3.2) and (3.9), it gives

$$
\int_{0}^{T} e^{\varepsilon t}\sup_{\theta\in[-\tau,0]}\mathbb{E}U(x(t+\theta))dt \le \frac{\bar{M}}{1-\Lambda(\varepsilon)} < \infty.
$$

Let $T\to\infty$, the desired result (3.1) is obtained. □

Remark 3.2 *From (3.1), it follows that*

$$
\Delta = \int_{0}^{\infty}\sup_{\theta\in[-\tau,0]}\mathbb{E}U(x(t+\theta))dt < \infty.
\tag{3.10}
$$

Theorem 3.3 *Suppose that the conditions of Lemma 3.1 hold for any initial condition $\varphi\in\mathcal{C}_{\mathcal{F}_0}([-\tau,0];R^n)$, there is a unique global solution $x(t)$ to Eq. (2.1) on $t\in[-\tau,\infty)$ with probability one.*

Proof: From *Hypothesis I*, for any initial data $\varphi\in\mathcal{C}_{\mathcal{F}_0}([-\tau,0];R^n)$, by using Theorem 7.12 (see, pp. 278 [12]), it is shown that there exist a unique maximal local strong solution $x(t)$ on $[-\tau,\sigma_e]$, where σ_e is the explosion time. To show that this solution is global, we only need to prove $\sigma_e=\infty$, *a.s.* Note that $\varphi\in\mathcal{C}_{\mathcal{F}_0}([-\tau,0];R^n)$, consequently, there must exist a positive number k_0 such that $||\varphi||_{\mathcal{C}}\le k_0$. For each integer $k>k_0$, define the stopping time

$$
\tau_k = \inf\{t\in[0,\sigma_e) : |x(t)|\ge k\},
$$

with the traditional convention $\inf \emptyset = \infty$, where $\emptyset$ denotes the empty set. Clearly, τ_k is increasing and $\tau_k \to \ \tau_\infty \le \sigma_e$ *a.s.* (as $k \to \infty$). If we can show $\tau_\infty = \infty$ *a.s.*, then $\sigma_e = \infty$ *a.s.*, which implies that $x(t)$ is actually global. This is equivalent to proving that for any $t > 0$, $\mathbb{P}(\tau_k \le t) \to 0$, as $k \to \infty$.

By using Itô formula, it yields that for any $t \ge 0$,

$$\begin{aligned}
&V(t \wedge \tau_k, x(t \wedge \tau_k), r(t \wedge \tau_k)) \\
&= V(0, x(0), r(0)) + \int_0^{t\wedge\tau_k} LV(s, x(s), x(s - \tau(s)), r(s))ds \\
&\quad + \int_0^{t\wedge\tau_k} V_x(s, x(s), r(s)) g(s, x(s), x(s - \tau(s)), r(s))dB(s) \\
&\quad + \int_0^{t\wedge\tau_k} \int_R [V(s, x(s), i_0 + \bar{h}(r(s-), l) - V(s, x(s), r(s))]\mu(ds, dl).
\end{aligned} \tag{3.11}$$

Taking the expectation on both sides of inequality (3.11), it yields from (2.3) that

$$\begin{aligned}
&\mathbb{E}V(t \wedge \tau_k, x(t \wedge \tau_k), r(t \wedge \tau_k)) \\
&\le \mathbb{E}V(0, x(0), r(0)) + \lambda_2 \int_0^{t\wedge\tau_k} \sup_{u\in[-\tau,0]} \mathbb{E}U(x(s+u))ds \\
&\le M,
\end{aligned} \tag{3.12}$$

where $M = \mathbb{E}V(0, x(0), r(0)) + \lambda_2\Delta > 0$, and Δ is given in (3.10).

Define, for each $k \ge 0$,

$$\psi(k) = \inf\{U(x) : x \in R^n, \text{ with } |x| \ge k\}$$

By condition (2.4), we note that $\lim_{k\to\infty} \psi(k) = \infty$. On the other hand, using (2.2), we have

$$\begin{aligned}
&\mathbb{E}V(t \wedge \tau_k, x(t \wedge \tau_k), r(t \wedge \tau_k)) \\
&\ge \mathbb{E}\{I_{\{\tau_k \le t\}} V(t \wedge \tau_k, x(t \wedge \tau_k), r(t \wedge \tau_k))\} \\
&\ge c_1\psi(k)\mathbb{P}\{\tau_k \le t\},
\end{aligned} \tag{3.13}$$

where $|x(\tau_k)| = k$ by the definition of stopping time τ_k.

From (3.12) and (3.13), it follows

$$c_1\psi(k)\mathbb{P}\{\tau_k \le t\} \le M, \tag{3.14}$$

For any $t \ge 0$, when $k \to \infty$, $\psi(k) \to \infty$. Then by (3.14), we can conclude that $\mathbb{P}\{\tau_\infty \le t\} = 0$. Since $t \ge 0$ is arbitrary, $\mathbb{P}\{\tau_\infty < \infty\} = 0$, which implies that $\tau_\infty = \infty$, *a.s.* That is, Eq. (2.1) almost surely admits a unique global solution $x(t)$ on $[-\tau, \infty)$. □

Theorem 3.4 *Let the conditions of Lemma 3.1 hold, for any initial data* $\varphi \in \mathcal{C}_{\mathcal{F}_0}([-\tau, 0]; R^n)$*, the global solution* $x(t)$ *of the Eq.* (2.1) *obeys*

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}U(x(t)) \le -\bar{\mu},$$

where $\bar{\mu} \in (0, \frac{\lambda_1}{c_2})$ *is a unique root of the algebra equation :* $\frac{\lambda_2 c_2 e^{\mu\tau}}{\lambda_1 c_1 - c_1 c_2 \mu} = 1$.

Proof: From (3.5), it follows that for any $t \geq 0$,

$$\begin{aligned}\mathbb{E}U(x(t)) &\leq M' e^{-\frac{\lambda_1}{c_2}t} + \frac{\lambda_2}{c_1}\int_0^t e^{-\frac{\lambda_1}{c_2}(t-s)}\mathbb{E}U(x(s-\tau(s))ds \\ &\leq M' e^{-\frac{\lambda_1}{c_2}t} + \frac{\lambda_2}{c_1}\int_0^t e^{-\frac{\lambda_1}{c_2}(t-s)}\sup_{u\in[-\tau,0]}\mathbb{E}U(x(s+u))ds.\end{aligned}$$

Note that, for any $t \in [-\tau, 0]$, we have $\mathbb{E}U(x(t)) \leq M' e^{-\frac{\lambda_1}{c_2}t}$. Therefore, by using Lemma 2.4, we can obtain

$$\mathbb{E}U(x(t)) \leq \hat{M} e^{-\bar{\mu}t},$$

for any $t \geq -\tau$, where $\bar{\mu} \in (0, \frac{\lambda_1}{c_2})$ and $\hat{M} = \{\frac{M'(\lambda_1 c_1 - c_1 c_2 \bar{\mu})}{\lambda_2 c_2 e^{\bar{\mu}\tau}}, M'\} > 0$, which implies that $\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{E}U(x(t)) \leq -\bar{\mu}$. □

Remark 3.5 *From (3.1) in Lemma 3.1, the Chebyshev inequality, and the Fubini theorem, it follows*

$$\int_0^\infty e^{\varepsilon t}\sup_{\theta\in[-\tau,0]} U(x(t+\theta))dt < \infty, \quad a.s.,$$

for any $\varepsilon \in (0, \varepsilon_0)$, where ε_0 is given in Lemma 3.1. In addition, from (3.1) in Lemma 3.1 and the Fubini theorem, we have

$$\mathbb{E}\int_0^\infty U(x(t))dt < \infty.$$

Theorem 3.6 *Suppose that the conditions of Lemma 3.1 are satisfied. Then the global solution $x(t)$ of Eq. (2.1) with the initial condition $\varphi \in \mathcal{C}_{\mathcal{F}_0}([-\tau, 0]; R^n)$ obeys the following property*

$$\limsup_{t\to\infty} \frac{1}{t}\log U(x(t)) \leq -\varepsilon, \quad a.s.,$$

for $\varepsilon \in (0, \varepsilon_0)$, where ε_0 is given in Lemma 3.1.

Proof: For any $\varepsilon \in (0, \varepsilon_0)$, applying the Itô formula, we obtain that for any $t \geq 0$,

$$\begin{aligned}&e^{\varepsilon t}V(t, x(t), r(t)) \\ &= V(0, x(0), r(0)) + \int_0^t e^{\varepsilon s}[\varepsilon V(s, x(s), r(s)) + LV(s, x(s), x(s-\tau(s)), r(s))]ds \\ &\quad + \int_0^t e^{\varepsilon s}V_x(s, x(s), r(s))g(s, x(s), x(s-\tau(s)), r(s))dB(s) \\ &\quad + \int_0^t\int_R e^{\varepsilon s}[V(s, x(s), i_0 + \bar{h}(r(s-), l) - V(s, x(s), r(s))]\mu(ds, dl), \quad a.s.\end{aligned}$$

Then, from (2.2) and (2.3), we have

$$\begin{aligned} e^{\varepsilon t}V(t,x(t),r(t)) &\le V(0,x(0),r(0)) + \lambda_2\int_0^t e^{\varepsilon s}\sup_{u\in[-\tau,0]}U(x(s+u))ds + M(t) \\ &\le \xi_0 + A(t) + M(t), \quad a.s., \end{aligned} \tag{3.15}$$

where $\xi_0 = V(0,x(0),r(0))$ is a nonnegative bounded F_0-measurable random variable,

$$A(t) = \lambda_2\int_0^t e^{\varepsilon s}\sup_{u\in[-\tau,0]}U(x(s+u))ds, \quad a.s.,$$

and

$$\begin{aligned} M(t) = &\int_0^t e^{\varepsilon s}V_x(s,x(s),r(s))g(s,x(s),x(s-\tau(s)),r(s))dB(s) \\ &+\int_0^t\int_R e^{\varepsilon s}[V(s,x(s),i_0+\bar{h}(r(s-),l) - V(s,x(s),r(s))]\mu(ds,dl) \end{aligned}$$

is a local continuous martingale with $M(0)=0$.

Applying the nonnegative semi-martingale convergence theorem [25], it deduces from Remark 3.5 and (3.15) that

$$\limsup_{t\to\infty}[e^{\varepsilon t}V(t,x(t),r(t))] < \infty, \quad a.s.$$

Therefore, there exists a finite positive random variable ζ such that

$$e^{\varepsilon t}V(t,x(t),r(t)) \le \zeta, \qquad a.s. \tag{3.16}$$

From (2.2) and (3.16), it gives

$$U(x(t)) \le \frac{\zeta}{c_1}e^{-\varepsilon t}, \quad a.s.$$

for any $t\ge 0$ holds, which follows that $\limsup_{t\to\infty}\frac{1}{t}\log U(x(t)) \le -\varepsilon$. $a.s.$ □

Theorem 3.7 *Suppose that the conditions of Lemma 3.1 hold. Then the global solution $x(t)$ of Eq. (2.1) satisfies*

$$\lim_{t\to\infty} d(x(t),\ Ker(U)) = 0, \quad a.s.,$$

and $Ker(U)\ne\emptyset$. In particular, if U has the property that

$$U(x) = 0 \quad \text{if and only if} \ \ x = 0,$$

then the solution obeys that

$$\lim_{t\to\infty} x(t) = 0, \quad a.s.,$$

for all initial condition $\varphi\in\mathcal{C}_{\mathcal{F}_0}([-\tau,0];R^n)$. That is, the global solution of Eq. (2.1) is almost surely asymptotically stable.

Proof: The proof is very technical and follows the same steps in [16], which is split into five step.

Step 1: By using Remark 3.5 and the Chebyshev inequality, it implies

$$\int_0^\infty U(x(t))dt < \infty, \quad a.s.$$

Furthermore,

$$\liminf_{t\to\infty} U(x(t)) = 0, \quad a.s. \tag{3.17}$$

Now, we claim that

$$\lim_{t\to\infty} U(x(t)) = 0, \quad a.s. \tag{3.18}$$

If this is false, then

$$\mathbb{P}(\limsup_{t\to\infty} U(x(t)) > 0) > 0.$$

Hence, we can find a positive number ε sufficiently small, such that

$$\mathbb{P}(\Omega_1) \geq 3\varepsilon, \tag{3.19}$$

where

$$\Omega_1 = \{\limsup_{t\to\infty} U(x(t)) > 2\varepsilon\}.$$

Step 2: Let $h > ||\varphi||_{\mathcal{C}}$ be a number. Define the stopping time

$$\beta_h = \inf\{t \geq 0 : |x(t)| \geq h\}.$$

Similar to the derivation of inequality (3.12), from (2.2), it deduces

$$\begin{aligned} \mathbb{E}U(x(t\wedge\beta_h)) &\leq \frac{\mathbb{E}V(0,x(0),r(0))}{c_1} + \frac{\lambda_2}{c_1}\int_0^{t\wedge\beta_h} \sup_{u\in[-\tau,0]} \mathbb{E}U(x(s+u))ds \\ &\leq \frac{M}{c_1}, \end{aligned} \tag{3.20}$$

where $M = \mathbb{E}V(0,x(0),r(0)) + \lambda_2\Delta > 0$, where Δ is given in (3.10).

According to the definition of the function $\psi(\cdot)$, we have

$$\psi(h) = \inf\{U(x) : \ x \in R^n, \ \text{ with } |x| \geq h\} \tag{3.21}$$

From (3.20) and (3.21), it yields

$$\psi(h)\mathbb{P}(\beta_h \leq t) \leq \frac{M}{c_1}.$$

where $|x(\beta_h)| = h$ by the definition of stopping time β_h.

Let $t \to \infty$ and then choose h sufficiently large, we have

$$\mathbb{P}(\beta_h < \infty) \leq \varepsilon,$$

which implies

$$\mathbb{P}(\Omega_2) \geq 1 - \varepsilon, \tag{3.22}$$

where

$$\Omega_2 = \{|x(t)| < h, \text{ for all } \ 0 \leq t < \infty\}.$$

Then, it follows from (3.19) and (3.22) that

$$\mathbb{P}(\Omega_1 \cap \Omega_2) \geq 2\varepsilon. \tag{3.23}$$

Step 3: Define a sequence of stopping times:

$$\begin{aligned} \alpha_1 &= \inf\{t \geq 0 : U(x(t)) \geq 2\varepsilon\}, \\ \alpha_{2i} &= \inf\{t \geq \alpha_{2i-1} : U(x(t)) \leq \varepsilon\}, \quad i = 1, 2, \dots, \\ \alpha_{2i+1} &= \inf\{t \geq \alpha_{2i} : U(x(t)) \geq 2\varepsilon\}, \quad i = 1, 2, \dots. \end{aligned}$$

It is observed that from (3.17) and the definition of Ω_1 and Ω_2, we have $\alpha_{2i} < \infty$ when $\alpha_{2i-1} < \infty$. Moreover,

$$\beta_h(\omega) = \infty \ \text{ and } \ \alpha_{2i}(\omega) < \infty, \text{ for all } \ i \geq 1, \ \text{ whenever } \ \omega \in \Omega_1 \cap \Omega_2. \tag{3.24}$$

From Remark 3.5, we obtain

$$\begin{aligned} \infty > \mathbb{E}\int_0^\infty U(x(t))dt &\geq \sum_{i=1}^{\infty} \mathbb{E}\Big\{ I_{\{\alpha_{2i}<\infty,\, \beta_h=\infty\}} \int_{\alpha_{2i-1}}^{\alpha_{2i}} U(x(t))dt \Big\} \\ &\geq \varepsilon \sum_{i=1}^{\infty} \mathbb{E}\{I_{\{\alpha_{2i}<\infty, \beta_h=\infty\}}[\alpha_{2i} - \alpha_{2i-1}]\}. \end{aligned} \tag{3.25}$$

From *Hypothesis I*, we have

$$|f(t, x, y, i)|^2 \vee |g(t, x, y, i)|^2 \leq C_h \quad \forall\, t \geq 0,$$

for any $|x| \vee |y| \leq h$, where C_h is positive constant.

By the Hölder inequality and the Doob's martingale inequality, it is derived that for any $T > 0$,

$$
\begin{aligned}
&\mathbb{E}\{I_{\{\beta_h\wedge\alpha_{2i-1}<\infty\}}\sup_{0\le t\le T}|x(\beta_h\wedge(\alpha_{2i-1}+t))-x(\beta_h\wedge\alpha_{2i-1})|^2\}\\
&\le 2\mathbb{E}\Big\{I_{\{\beta_h\wedge\alpha_{2i-1}<\infty\}}\sup_{0\le t\le T}|\int_{\beta_h\wedge\alpha_{2i-1}}^{\beta_h\wedge(\alpha_{2i-1}+t)} f(s,x(s),x(s-\tau(s)),r(s))ds|^2\Big\}\\
&\quad + 2\mathbb{E}\Big\{I_{\{\beta_h\wedge\alpha_{2i-1}<\infty\}}\sup_{0\le t\le T}|\int_{\beta_h\wedge\alpha_{2i-1}}^{\beta_h\wedge(\alpha_{2i-1}+t)} g(s,x(s),x(s-\tau(s)),r(s))dB(s)|^2\Big\}\\
&\le 2T\mathbb{E}\Big\{I_{\{\beta_h\wedge\alpha_{2i-1}<\infty\}}\int_{\beta_h\wedge\alpha_{2i-1}}^{\beta_h\wedge(\alpha_{2i-1}+T)} |f(s,x(s),x(s-\tau(s)),r(s))|^2ds\Big\}\\
&\quad + 8\mathbb{E}\Big\{I_{\{\beta_h\wedge\alpha_{2i-1}<\infty\}}\int_{\beta_h\wedge\alpha_{2i-1}}^{\beta_h\wedge(\alpha_{2i-1}+T)} |g(s,x(s),x(s-\tau(s)),r(s))|^2ds\Big\}\\
&\le 2C_hT(T+4).
\end{aligned}
\tag{3.26}
$$

Since $U(x)$ is continuous in R^n, it must be uniformly continuous. That is, when $|x|\vee|y|\le h$, we can therefore choose $\delta=\delta(\varepsilon)$ satisfying

$$
|U(x)-U(y)|<\varepsilon, \quad \text{whenever } |x-y|<\delta,\ |x|\vee|y|\le h. \tag{3.27}
$$

Choose T sufficiently small such that

$$
\frac{2C_hT(T+4)}{\delta^2}<\varepsilon, \tag{3.28}
$$

From (3.26) and (3.28), it follows that

$$
\begin{aligned}
&\mathbb{P}(\{\beta_h\wedge\alpha_{2i-1}<\infty\}\cap\{\sup_{0\le t\le T}|x(\beta_h\wedge(\alpha_{2i-1}+t))-x(\beta_h\wedge\alpha_{2i-1})|\ge\delta\})\\
&\le\frac{2C_hT(T+4)}{\delta^2}<\varepsilon.
\end{aligned}
$$

Accordingly, we have

$$
\begin{aligned}
&\mathbb{P}(\{\alpha_{2i-1}<\infty,\beta_h=\infty\}\cap\{\sup_{0\le t\le T}|x(\alpha_{2i-1}+t)-x(\alpha_{2i-1})|\ge\delta\})\\
&=\mathbb{P}(\{\beta_h\wedge\alpha_{2i-1}<\infty,\beta_h=\infty\}\cap\{\sup_{0\le t\le T}|x(\beta_h\wedge(\alpha_{2i-1}+t))-x(\beta_h\wedge\alpha_{2i-1})|\ge\delta\})\\
&\le\mathbb{P}(\{\beta_h\wedge\alpha_{2i-1}<\infty\}\cap\{\sup_{0\le t\le T}|x(\beta_h\wedge(\alpha_{2i-1}+t))-x(\beta_h\wedge\alpha_{2i-1})|\ge\delta\})\\
&<\varepsilon.
\end{aligned}
$$

Using (3.23) and (3.24), we have

$$
\begin{aligned}
&\mathbb{P}(\{\alpha_{2i-1}<\infty,\beta_h=\infty\}\cap\{\sup_{0\le t\le T}|x(\alpha_{2i-1}+t)-x(\alpha_{2i-1})|<\delta\})\\
&=\mathbb{P}(\{\alpha_{2i-1}<\infty,\beta_h=\infty\})\\
&\quad-\mathbb{P}(\{\alpha_{2i-1}<\infty,\beta_h=\infty\}\cap\{\sup_{0\le t\le T}|x(\alpha_{2i-1}+t)-x(\alpha_{2i-1})|\ge\delta\})\\
&>2\varepsilon-\varepsilon=\varepsilon.
\end{aligned}
$$

From (3.27), it yields

$$\begin{aligned}
&\mathbb{P}(\{\alpha_{2i-1} < \infty, \beta_h = \infty\} \cap \{\sup_{0 \le t \le T} |U(x(\alpha_{2i-1} + t)) - U(y(\alpha_{2i-1}))| < \varepsilon\} \\
&\ge \mathbb{P}(\{\alpha_{2i-1} < \infty, \beta_h = \infty\} \cap \{\sup_{0 \le t \le T} |x(\alpha_{2i-1} + t) - x(\alpha_{2i-1})| < \delta\}) \\
&> \varepsilon.
\end{aligned} \tag{3.29}$$

Set

$$\hat{\Omega}_i = \{\sup_{0 \le t \le T} |U(x(\alpha_{2i-1} + t)) - U(y(\alpha_{2i-1}))| < \varepsilon\},$$

and note that

$$\alpha_{2i}(\omega) - \alpha_{2i-1}(\omega) \ge T, \quad \text{if } \omega \in \{\alpha_{2i-1} < \infty, \beta_h = \infty\} \cap \hat{\Omega}_i.$$

Using (3.25) and (3.29), we have

$$\begin{aligned}
\infty &\ge \varepsilon \sum_{i=1}^{\infty} \mathbb{E}\{I_{\{\alpha_{2i} < \infty, \beta_h = \infty\}}[\alpha_{2i} - \alpha_{2i-1}]\} \\
&\ge \varepsilon \sum_{i=1}^{\infty} \mathbb{E}\{I_{\{\alpha_{2i} < \infty, \beta_h = \infty\} \cap \hat{\Omega}_i}[\alpha_{2i} - \alpha_{2i-1}]\} \\
&\ge \varepsilon T \sum_{i=1}^{\infty} \mathbb{P}(\{\alpha_{2i} < \infty, \beta_h = \infty\} \cap \hat{\Omega}_i) \\
&> \varepsilon T \sum_{i=1}^{\infty} \varepsilon = \infty,
\end{aligned}$$

which is a contradiction. Hence, (3.18) holds.

Step 4: Now, it is necessary to show that $Ker(U) \neq \emptyset$. From (3.18), it is seen that there exists an $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that

$$\lim_{t \to \infty} U(x(t)) = 0 \quad \text{and} \quad \sup_{0 \le t < \infty} |x(t)| < \infty, \quad \text{for any } \omega \in \Omega_0. \tag{3.30}$$

Choose any $\omega \in \Omega_0$, then $\{x(t)\}_{t \ge 0}$ is bounded in R^n. Then, there must be an increasing sequence $\{t_k\}_{k \ge 1}$ such that $t_k \to \infty$ and $\{x(t_k)\}_{k \ge 1}$ converges to some $\bar{x} \in R^n$. Thus,

$$U(\bar{x}) = \lim_{k \to \infty} U(x(t_k)) = 0,$$

which implies that $\bar{x} \in Ker(U)$. That is, $Ker(U) \neq \emptyset$.

Step 5: It is necessary to show that for any $\omega \in \Omega_0$,

$$\lim_{t \to \infty} d(x(t), Ker(U)) = 0. \tag{3.31}$$

If this is false, then there exists some $\bar{\omega} \in \Omega_0$ such that

$$\limsup_{t\to\infty} d(x(t,\bar{\omega}), Ker(U)) > 0.$$

Thus, there exists a subsequence $\{x(t_k,\bar{\omega})\}_{k\geq 0}$ of $\{x(t,\bar{\omega})\}_{t\geq 0}$ satisfying

$$\limsup_{k\to\infty} d(x(t_k,\bar{\omega}), Ker(U)) > \bar{\varepsilon},$$

for some $\bar{\varepsilon} > 0$. Since $\{x(t_k,\bar{\omega})\}_{k\geq 0}$ is bounded, we can find a subsequence converging to some $\tilde{x} \in R^n$. Clearly, $\tilde{x} \notin Ker(U)$ and $U(\tilde{x}) > 0$. However, from (3.30),

$$U(\tilde{x}) = \lim_{k\to\infty} U(x(t_k,\bar{\omega})) = 0.$$

Thus, there is a contradiction. Consequently, (3.31) holds. In addition, if $U(x) = 0 \Leftrightarrow x = 0$, then $Ker(U) = 0$. Consequently, from (3.31), it is deduced that

$$\lim_{t\to\infty} x(t) = 0, \quad a.s.$$

The proof is therefore complete. □

Corollary 3.8 *Suppose that Hypotheses I and IV are satisfied, then the existence and uniqueness for the global solution of Eq. (2.1) can be guaranteed. Furthermore, we have the following two results:*
i) for any initial data $\varphi \in \mathcal{C}_{\mathcal{F}_0}([-\tau,0];R^n)$, the $pth(p \geq 1)$-moment Lyapunov exponent of the solution of the Eq. (2.1) *obeys*

$$\limsup_{t\to\infty} \frac{1}{t}\log(\mathbb{E}|x(t)|^p) \leq -\bar{\mu},$$

where $\bar{\mu} \in (0, \frac{\lambda_1}{c_2})$ is a root of the algebra equation : $\frac{\lambda_2 c_2 e^{\mu\tau}}{\lambda_1 c_1 - c_1 c_2 \mu} = 1$. That is, the solution of the Eq. (2.1) *is exponentially stable in $pth(p \geq 1)$ mean;*
ii) for any initial data $\varphi \in \mathcal{C}_{\mathcal{F}_0}([-\tau,0];R^n)$, the sample Lyapunov exponent of the solution of the Eq. (2.1) *obeys*

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) \leq -\frac{\varepsilon}{p}, \quad a.s.$$

where $p \geq 1$, and $\varepsilon \in (0,\varepsilon_0)$, where ε_0 is given in Lemma 3.1. That is, the solution of the Eq. (2.1) *is almost surely exponentially stable.*

4 Two Examples

In order to illustrate the advantages of the main results, two examples are provided.

Example 4.1: Let $B(t)$ be a scalar Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Consider one dimensional stochastic differential equations with time-varying delay and Markovian switching:

$$dx(t) = f(t,x(t),x(t-\tau(t)),r(t))dt + g(t,x(t),x(t-\tau(t)),r(t))dB(t), \quad t \geq 0, \tag{4.1}$$

with the initial value $\{x(\theta) : -\tau \leq \theta \leq 0\} = \varphi \in \mathcal{C}_{\mathcal{F}_0}([-\tau, 0]; R)$ and $r(0) = i_0 \in \mathcal{S}$ and $r(0) = 1 \in \mathcal{S} = \{1, 2\}$, where $x(t)$ and $x(t-\tau(t))$ are the state scalar and the delayed state scalar, respectively. $\tau(t)$ is a bounded measurable function with $0 \leq \tau(t) \leq \tau$ $(t \geq 0,\ \tau > 0)$, and $r(t)$ is a right-continuous Markov chain taking values in $\mathcal{S}$ with the generator $\Gamma = (\gamma_{ij})_{2\times 2} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$.

In (4.1), we assume that $f,\ g :\ [0, \infty) \times R \times R \times \mathcal{S} \to R$ with

$$f(t, x, y, i) = \begin{cases} -0.15x - 2x^3 + 0.4y, & \text{if } \ i = 1, \\ -2x - 0.5xy^4 + 0.82y, & \text{if } \ i = 2, \end{cases}$$

and

$$g(t, x, y, i) = \begin{cases} 2x^2, & \text{if } \ i = 1, \\ xy^2, & \text{if } \ i = 2. \end{cases}$$

Define a Lyapunov function

$$V(t, x, i) = \begin{cases} x^2, & \text{if } \ i = 1, \\ 0.5x^2, & \text{if } \ i = 2, \end{cases}$$

then, it is computed for the Itô operator to Eq. (4.1) that

$$\begin{aligned} LV(t, x, y, 1) = & \, 2x[-0.15x - 2x^3 + 0.4y] + 4x^4 + \sum_{j=1}^{2} \gamma_{1j} V(t, x, j) \\ = & -1.3x^2 + 0.8xy \\ \leq & -0.9x^2 + 0.4y^2, \end{aligned}$$

and

$$\begin{aligned} LV(t, x, y, 2) = & \, x[-2x - 0.5xy^4 + 0.82y] + 0.5x^2y^4 + \sum_{j=1}^{2} \gamma_{2j} V(t, x, j) \\ = & -1.5x^2 + 0.82xy \\ \leq & -1.09x^2 + 0.41y^2. \end{aligned}$$

Hence, we have

$$LV(t, x, y, i) \leq -0.9x^2 + 0.41y^2,$$

with $\lambda_1 = 0.9$, $\lambda_2 = 0.41$, $c_1 = 0.5$ and $c_2 = 1$. Then, $\lambda_2 c_2 < \lambda_1 c_1$ holds, which implies that the conditions of Corollary 3.8 hold. Thus, the existence and uniqueness, the exponential stability in mean square, the almost sure exponential stability and the almost sure asymptotical stability of the global solution for Eq. (4.1) are guaranteed. When the initial condition $x(t) = -1$ $(t \in [-2.3, 0])$, $r(0) = 1$, and $\tau(t) = 1.1|\sin(t)| + 1.2$ are fixed, Fig. 1 and Fig. 2 illustrate the asymptotic behavior in mean square and in almost sure sense of the global solution for Eq. (1), respectively.

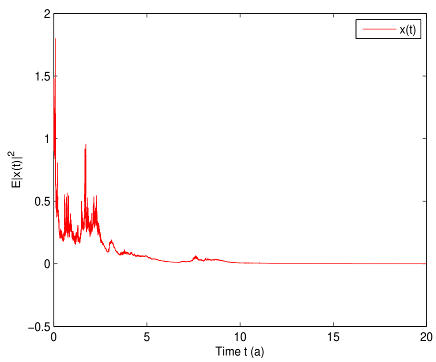

Figure 1: Asymptotic behavior in mean square of the global solution for Eq. (4.1)

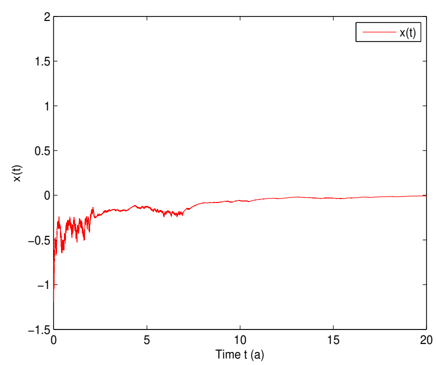

Figure 2: Asymptotic behavior in almost sure sense of the global solution for Eq. (4.1)

Example 4.2: One coupled system consists of a mass-spring-damper (MSD) model [26]. An actuator is taken to a transfer system. The mathematical expression of the system is DDEs, which are written as

$$M\ddot{y}(t) + C\dot{y}(t) + Ky(t) = 0 \tag{4.2}$$

on $t \geq 0$, where M, C, K are the mass, stiffness and damping of a mass-spring-damper model, and $y(t)$, $\dot{y}(t)$, $\ddot{y}(t)$ denote the position, velocity and acceleration of MSD at time t. If this physical model is affected by the external force, then Eq. (4.2) is further described as

$$M\ddot{y}(t) + C\dot{y}(t) + Ky(t) + F(t) = 0 \tag{4.3}$$

on $t \geq 0$, where $F(t)$ denotes the external force, $M = 10$, $C = 25$, and $K = 15$. Assume that this external force is subject to the environmental noise and abrupt changes in the parameters, which is characterized by

$$F(t) = F_1(\dot{y}(t), \dot{y}(t-\tau(t)), r(t)) + F_2(\dot{y}(t), y(t-\tau(t)), \dot{y}(t-\tau(t)), r(t))\dot{B}(t)$$

where $\dot{B}(t)$ is a scalar white noise (i.e. $\dot{B}(t)$ is a scalar Brownian motion), $\tau(t)$ is the time-varying delay, $r(t)$ is a Markovian switching taking values in $\mathcal{S} = \{1, 2\}$ with its generator $\Gamma = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}$,

$$F_1(\dot{y}(t), \dot{y}(t-\tau(t)), r(t)) = \begin{cases} 5.4\dot{y}(t)\dot{y}^2(t-\tau(t)), & \text{if } \; i = 1, \\ 15\dot{y}^3(t)\dot{y}^2(t-\tau(t)), & \text{if } \; i = 2, \end{cases}$$

and

$$\begin{aligned} & F_2(\dot{y}(t), y(t-\tau(t)), \dot{y}(t-\tau(t)), r(t)) \\ = & \begin{cases} 6\dot{y}(t)\dot{y}(t-\tau(t)) + 3y(t-\tau(t)) + 3\dot{y}(t-\tau(t)), & \text{if } \; i = 1, \\ 10\dot{y}^2(t)\dot{y}(t-\tau(t)) + 2y(t-\tau(t)) + 2\dot{y}(t-\tau(t)), & \text{if } \; i = 2. \end{cases} \end{aligned}$$

Let $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$, Eq. (4.3) can be written as highly nonlinear SDDEs with Markovian switching:

$$dx(t) = f(t, x(t), x(t-\tau(t)), r(t))dt + g(t, x(t), x(t-\tau(t)), r(t))dB(t) \tag{4.4}$$

where $x(t) = \text{col}[x_1(t), x_2(t)]$,

$$f(t, x(t), x(t-\tau(t)), 1) = \begin{bmatrix} x_2(t) \\ -1.5x_1(t) - 2.5x_2(t) - 0.54x_2(t)x_2^2(t-\tau(t)) \end{bmatrix},$$

$$f(t, x(t), x(t-\tau(t)), 2) = \begin{bmatrix} x_2(t) \\ -1.5x_1(t) - 2.5x_2(t) - 1.5x_2^3(t)x_2^2(t-\tau(t)) \end{bmatrix},$$

$$g(t, x(t), x(t-\tau(t)), 1) = \begin{bmatrix} 0 \\ 0.6x_1(t-\tau(t)) + 0.3x_2(t-\tau(t)) + 0.3x_2(t)x_2(t-\tau(t)) \end{bmatrix},$$

and

$$g(t, x(t), x(t-\tau(t)), 2) = \begin{bmatrix} 0 \\ x_1(t-\tau(t)) + 0.2x_2(t-\tau(t)) + 0.2x_2(t)x_2(t-\tau(t)) \end{bmatrix}.$$

For Eq. (4.4), consider a Lyapunov function

$$V(t, x, i) = \begin{cases} |x|^2, & \text{if } \; i = 1, \\ 0.8|x|^2, & \text{if } \; i = 2, \end{cases}$$

with $|x|^2 = x_1^2 + x_2^2$.

Then, for Eq. (4.4), the Itô operator is computed as

$$\begin{aligned} & LV(t, x(t), x(t-\tau(t)), i) \\ = & 2q_i x^T(t) f(t, x(t), x(t-\tau(t)), r(t)) + q_i \text{trace}[g^T(t, x(t), x(t-\tau(t)), i) \\ & \times g(t, x(t), x(t-\tau(t)), i)] + \sum_{j=1}^{2} \gamma_{ij} V(t, x(t), j), \end{aligned}$$

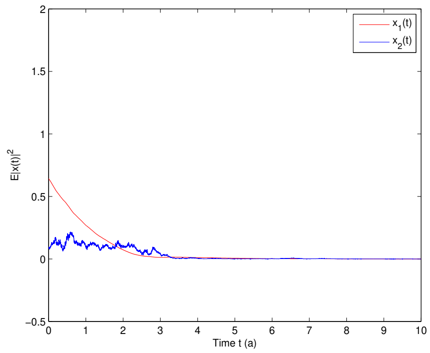

Figure 3: Asymptotic behavior in mean square of the global solution for Eq. (4.4)

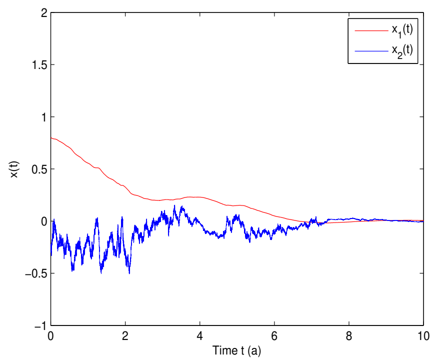

Figure 4: Asymptotic behavior in almost sure sense of the global solution for Eq. (4.4)

where $q_1 = 1$, $q_2 = 0.8$.

Consequently, when $i = 1$, we have

$$\begin{aligned} LV(t, x(t), x(t-\tau(t), 1) \leq & -2[x_1^2(t) + x_2^2(t)] - 1.08x_2^2(t)x_2^2(t-\tau(t)) \\ & + [0.6x_2(t)x_2(t-\tau(t)) + 0.3x_1(t-\tau(t)) + 0.3x_2(t-\tau(t))]^2 \\ & - 0.4[x_1^2(t) + x_2^2(t)] \\ \leq & -2.4|x(t)|^2 + 0.27|x(t-\tau(t))|^2, \end{aligned}$$

and when $i = 2$,

$$\begin{aligned} LV(t, x(t), x(t-\tau(t), 2) \leq & -1.6[x_1^2(t) + x_2^2(t)] - 2.4x_2^4(t)x_2^2(t-\tau(t)) \\ & + 0.8[x_2^2(t)x_2(t-\tau(t)) + 0.2x_1(t-\tau(t)) + 0.2x_2(t-\tau(t))]^2 \\ & + 0.6[x_1^2(t) + x_2^2(t)] \\ \leq & -|x(t)|^2 + 0.096|x(t-\tau(t))|^2. \end{aligned}$$

Thus, for any $i \in \mathcal{S}$.

$$LV(t, x(t), x(t-\tau(t)), i) \leq -|x(t)|^2 + 0.27|x(t-\tau(t))|^2.$$

with $\lambda_1 = 1$, $\lambda_2 = 0.27$, $c_1 = 0.8$ and $c_2 = 1$. Thus, $\lambda_2 c_2 < \lambda_1 c_1$ is satisfied. Consequently, by Corollary 3.8, the existence and uniqueness, the exponential stability in mean square, the almost sure exponential stability and the almost sure asymptotical stability of the global solution for Eq. (4.4) are guaranteed. When the initial condition $x(t) = \text{col}[-\sin(t),\ 0.5\cos(t)]$ $(t \in [-2.3, 0])$, $r(0) = 1$, and $\tau(t) = 1.1|\cos(t)| + 1.2$ are given, Fig. 3 and Fig. 4 show the asymptotic behavior in mean square and in almost sure sense of the global solution for Eq. (4.4), respectively.

5 Conclusion

The method of Lyapunov function has been widely used in the study for the stability of highly nonlinear stochastic differential delay equations with Markovian switching. However, so far, most of the existing results in this area usually require that the delay is a constant or the time-varying delay with its derivative value being less than one, which limits their applications to some extent. When the involved delay is time-varying with it being a bounded measurable function, one integral lemma has first been given. Then, under a locally Lipschitz condition and a monotonicity condition, the existence and uniqueness for the global solution of stochastic differential delay equations with Markovian switching has been proved; by using the integral inequality, some stochastic analysis technique and the nonnegative semi-martingale convergence theorem, the stability analysis for the global solution of highly nonlinear stochastic differential delay equations with Markovian switching have been discussed. Finally, two examples have been provided to illustrate the effectiveness of the theoretical results obtained.

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