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On the Difference of Coefficients of Bazilevič Functions

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Abstract
Let f be analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{S}$ be the subclass of normalized univalent functions given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ for $z \in \mathbb{D}$. We give bounds for $||a_3| - |a_2||$ for the subclass $\mathcal{B}(\alpha, i\beta)$ of generalized Bazilevič functions when $\alpha \geq 0$, and β is real.

Keywords Univalent function · Close-to-convex function · Bazilevič function · Difference of coefficients

Mathematics Subject Classification 30C45 · 30C50 · 30C55

1 Introduction

Let $\mathcal{A}$ denote the class of analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f'(0) - 1$. Then for $z \in \mathbb{D}$, $f \in \mathcal{A}$ has the following representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

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Let $\mathcal{S}$ denote the subclass of all univalent (i.e., one-to-one) functions in $\mathcal{A}$.

In 1985, de Branges [2] solved the famous Bieberbach conjecture by showing that if $f \in \mathcal{S}$, then $|a_n| \le n$ for $n \ge 2$, with equality when $f(z) = k(z) := z/(1-z)^2$, or a rotation. It was therefore natural to ask if for $f \in \mathcal{S}$, the inequality $||a_{n+1}| - |a_n|| \le 1$ is true when $n \ge 2$. This was shown not to be the case even when $n = 2$ [4], and that the following sharp bounds hold.

$$-1 \le |a_3| - |a_2| \le \frac{3}{4} + e^{-\lambda_0}(2e^{-\lambda_0} - 1) = 1.029\ldots,$$

where λ_0 is the unique value of λ in $0 < \lambda < 1$, satisfying the equation $4\lambda = e^{\lambda}$.

Hayman [6] showed that if $f \in \mathcal{S}$, then $||a_{n+1}| - |a_n|| \le C$, where C is an absolute constant. The exact value of C is unknown, best estimate to date being $C = 3.61\ldots$ [5], which because of the sharp estimate above when $n = 2$, cannot be reduced to 1.

Denote by $\mathcal{S}^*$ the subclass of $\mathcal{S}$ consisting of starlike functions, i.e. functions f which map $\mathbb{D}$ onto a set which is star-shaped with respect to the origin. Then it is well-known that a function $f \in \mathcal{S}^*$ if, and only if, for $z \in \mathbb{D}$

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0.$$

It was shown in [8], that when $f \in \mathcal{S}^*$, then $||a_{n+1}| - |a_n|| \le 1$ is true when $n \ge 2$.

Next denote by $\mathcal{K}$ the subclass of $\mathcal{S}$ consisting of functions which are close-to-convex, i.e. functions f which map $\mathbb{D}$ onto a close-to-convex domain. Then again it is well-known that a function $f \in \mathcal{K}$ if, and only if, there exists $g \in \mathcal{S}^*$ such that for $z \in \mathbb{D}$

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > 0. \tag{1.2}$$

Koepf [7] showed that if $f \in \mathcal{K}$, then $||a_{n+1}| - |a_n|| \le 1$, when $n = 2$, but establishing this inequality when $n \ge 3$ remains an open problem.

In 1955, Bazilevič [1] extended the notion of starlike and close-to-convex functions by showing that if $f \in \mathcal{A}$, and is given by (1.1), then if $\alpha > 0$ and $\beta \in \mathbb{R}$, f given by

$$f(z) = \left((\alpha + i\beta)\int_0^z g^{\alpha}(t)p(t)t^{i\beta-1}dt\right)^{1/(\alpha+i\beta)}, \tag{1.3}$$

where $g \in \mathcal{S}^*$, and $p \in \mathcal{P}$, the class of functions with positive real part in $\mathbb{D}$, then functions defined by (1.3) form a subset of $\mathcal{S}$. Such functions are known as Bazilevič functions.

We note that in the original definition of Bazilevič functions [1], Bazilevič assumed that $\alpha > 0$, however Sheil-Small [10], subsequently showed that when $\alpha = 0$, such functions also belong to $\mathcal{S}$, and satisfy

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{i\beta} = p(z), \tag{1.4}$$

where $p \in \mathcal{P}$.

For $\alpha \geq 0$ and $\beta \in \mathbb{R}$, we denote functions defined as in (1.3) and (1.4) by $\mathcal{B}(\alpha, i\beta)$, and note that the class $\mathcal{B}(\alpha, 0) \equiv \mathcal{B}(\alpha)$ has been extensively studied, and that $\mathcal{B}(0, 0) \equiv \mathcal{S}^*$ and $\mathcal{B}(1, 0) \equiv \mathcal{K}$.

Another well studied subclass of $\mathcal{B}(\alpha, i\beta)$ is the class $\mathcal{B}_1(\alpha, i\beta)$, where $\beta = 0$ and the starlike function $g(z) \equiv z$, (see e.g. [11]). This class is usually denoted by $\mathcal{B}_1(\alpha)$. Although much is known about the initial coefficients of functions in $\mathcal{B}_1(\alpha)$, there appears to be no published information concerning the difference of coefficients. We also note that $\mathcal{B}_1(1, 0)$ reduces to the class of functions in $\mathcal{R}$ such that their derivatives have positive real part for $z \in \mathbb{D}$, and that the class $\mathcal{B}_1(1, i\beta)$ has been little studied.

In this paper we present some inequalities for $||a_3| - |a_2||$ when $f \in \mathcal{B}(\alpha, i\beta)$, obtaining sharp bounds when $f \in \mathcal{B}(\alpha)$, and $f \in \mathcal{B}_1(\alpha, i\beta)$ when $\alpha \geq 0$ and $\beta \in \mathbb{R}$. We also give the sharp bounds for $||a_3| - |a_2||$, when $f \in \mathcal{B}(0, i\beta)$.

2 Preliminary Lemmas

Denote by $\mathcal{P}$, the class of analytic functions p with positive real part on $\mathbb{D}$ given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{2.1}$$

We will use the following properties for the coefficients of functions $\mathcal{P}$, given by (2.1).

Lemma 2.1 [9] *For $p \in \mathcal{P}$ and $\nu \in \mathbb{C}$,*

$$\left| p_2 - \frac{\nu}{2} p_1^2 \right| \leq 2 \max \{|\nu - 1|; 1\},$$

and

$$\left| p_2 - \frac{1}{2} p_1^2 \right| \leq 2 - \frac{1}{2} |p_1|^2.$$

Both inequalities are sharp.

Lemma 2.2 [3] *If $p \in \mathcal{P}$, then*

$$p_1 = 2\zeta_1 \tag{2.2}$$

and

$$p_2 = 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2 \tag{2.3}$$

for some $\zeta_i \in \overline{\mathbb{D}}$, $i \in \{1, 2\}$. For $\zeta_1 \in \mathbb{T}$, the boundary of $\mathbb{D}$, there is a unique function $p \in \mathcal{P}$ with p_1 as in (2.2), namely,

$$p(z) = \frac{1+\zeta_1 z}{1-\zeta_1 z} \quad (z \in \mathbb{D}).$$

For $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with p_1 and p_2 as in (2.2) and (2.3), namely,

$$p(z) = \frac{1+(\overline{\zeta}_1\zeta_2+\zeta_1)z+\zeta_2 z^2}{1+(\overline{\zeta}_1\zeta_2-\zeta_1)z-\zeta_2 z^2} \quad (z \in \mathbb{D}).$$

We will also need the following well-known result.

Lemma 2.3 [7, Lem. 3] *Let $g \in \mathcal{S}^*$ and be given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then for any $\lambda \in \mathbb{C}$,*

$$\left| b_3 - \lambda b_2^2 \right| \le \max\{1; |3-4\lambda|\}.$$

The inequality is sharp when $g(z) = k(z)$ if $|3-4\lambda| \ge 1$, and when $g(z) = (k(z^2))^{1/2}$ if $|3-4\lambda| < 1$.

3 The class $\mathcal{B}(\alpha, i\beta)$

We begin by proving the following inequalities for $f \in \mathcal{B}(\alpha, i\beta)$.

Theorem 3.1 *Let $f \in \mathcal{B}(\alpha, i\beta)$ and be given by (1.1). If $0 \le \alpha \le (\sqrt{17}-1)/2$ and $\beta \in \mathbb{R}$, then*

$$-1 \le |a_3| - |a_2| \le \frac{2+\alpha}{|2+\alpha+i\beta|}. \tag{3.1}$$

Proof Recall that $|a_2| - |a_3| \le 1$ for all $f \in \mathcal{S}$ [4, Thm. 3.11]. So, since $\mathcal{B}(\alpha, i\beta) \subset \mathcal{S}$ for all $\alpha \ge 0$ and $\beta \in \mathbb{R}$, it is sufficient to prove the upper bound in (3.1).

Let $f \in \mathcal{B}(\alpha, i\beta)$ be of the form (1.1). Then from (1.3) we have

$$\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta} = p(z),$$

for some $g \in \mathcal{S}^*$ and $p \in \mathcal{P}$. Writing

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad \text{and} \quad p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

and equating the coefficients, we obtain

$$a_2 = \frac{\alpha b_2 + p_1}{1+\alpha+i\beta} \tag{3.2}$$

and

$$a_3 = \frac{p_2}{2+\alpha+i\beta} - \frac{(-1+\alpha+i\beta)p_1^2}{2(1+\alpha+i\beta)^2} + \frac{\alpha(3+\alpha+i\beta)b_2p_1}{(1+\alpha+i\beta)^2(2+\alpha+i\beta)} + \frac{\alpha b_3}{2+\alpha+i\beta} + \frac{\alpha(-1+\alpha-2i\beta-i\alpha\beta+\beta^2)b_2^2}{2(2+\alpha+i\beta)(1+\alpha+i\beta)^2}. \tag{3.3}$$

Let $\mu_1 = (3+\alpha+i\beta)/(2(2+\alpha+i\beta))$, and suppose that $|a_2| \le 1/|\mu_1|$. Then by Lemmas 2.1 and 2.3 we have

$$\begin{aligned} |a_3 - \mu_1 a_2^2| &= \left| \frac{1}{2+\alpha+i\beta}\left(p_2 - \frac{1}{2}p_1^2 + \alpha\left(b_3 - \frac{1}{2}b_2^2\right)\right)\right| \\ &\le \frac{2+\alpha}{|2+\alpha+i\beta|}. \end{aligned} \tag{3.4}$$

Thus from (3.4) we obtain

$$|a_3| - |a_2| \le |a_3| - |\mu_1||a_2|^2 \le |a_3 - \mu_1 a_2^2| \le \frac{2+\alpha}{|2+\alpha+i\beta|}.$$

Now assume that $1/|\mu_1| \le |a_2| \le 2$, and let $\mu_2 = 1/(2+\alpha+i\beta)$. Then

$$a_3 - \mu_2 a_2^2 = \Psi_1 + \frac{1}{2+\alpha+i\beta}\Psi_2, \tag{3.5}$$

where

$$\Psi_1 = \frac{\alpha b_3}{2+\alpha+i\beta} - \frac{\alpha(1+i\beta)b_2^2}{2(1+\alpha+i\beta)(2+\alpha+i\beta)},$$

and

$$\Psi_2 = \frac{\alpha b_2 p_1}{(1+\alpha+i\beta)} - \frac{(\alpha+i\beta)p_1^2}{2(1+\alpha+i\beta)} + p_2.$$

Put $\mu = (1+i\beta)/(2(1+\alpha+i\beta))$. Then it is easily seen that $|3-4\mu| = |1+3\alpha+i\beta|/|1+\alpha+i\beta| \ge 1$. Thus Lemma 2.3 gives

$$|\Psi_1| \le \frac{\alpha}{|2+\alpha+i\beta|}|3-4\mu| = \frac{\alpha|1+3\alpha+i\beta|}{|2+\alpha+i\beta||1+\alpha+i\beta|}. \tag{3.6}$$

Next use (2.2) and (2.3) in Lemma 2.2 to obtain

$$\Psi_2 = \frac{2\alpha b_2 \zeta_1}{1+\alpha+i\beta} + \frac{2\zeta_1^2}{1+\alpha+i\beta} + 2\left(1-|\zeta_1|^2\right)\zeta_2,$$

where $\zeta_i \in \overline{\mathbb{D}}$ $(i = 1, 2)$. The triangle inequality and $|b_2| \leq 2$ then gives

$$|\Psi_2| \leq \psi(|\zeta_1|), \tag{3.7}$$

where

$$\psi(x) = 2 + \frac{4\alpha}{|1+\alpha+i\beta|}x + 2\left(\frac{1-|1+\alpha+i\beta|}{|1+\alpha+i\beta|}\right)x^2$$

with $x \in [0, 1]$.

Let $x_0 = \alpha/(|1+\alpha+i\beta| - 1)$, so that $x_0 \in [0, 1]$, and ψ has a unique critical point at $x = x_0$. Since ψ has a negative leading coefficient, it follows from (3.7) that for all $x \in [0, 1]$,

$$|\Psi_2| \leq \psi(x_0) = 2 + \frac{2\alpha^2}{|1+\alpha+i\beta|(|1+\alpha+i\beta|-1)} \quad (x \in [0, 1]). \tag{3.8}$$

Therefore from (3.5), (3.6) and (3.10) we obtain

$$\begin{aligned}|a_3 - \mu_2 a_2^2| &\leq \frac{1}{|2+\alpha+i\beta|}\left(2 + \frac{\alpha|1+3\alpha+i\beta|}{|1+\alpha+i\beta|} + \frac{2\alpha^2}{|1+\alpha+i\beta|(|1+\alpha+i\beta|-1)}\right)\\ &=: \Psi(\alpha, \beta).\end{aligned}$$

Next write $y := |a_2|$, and assume that $y \in [1/|\mu_1|, \tilde{x}]$, where

$$\tilde{x} = \frac{2\alpha+2}{|1+\alpha+i\beta|}, \tag{3.9}$$

so that

$$|a_3| - |a_2| \leq |a_3 - \mu_2 a_2^2| + |\mu_2||a_2|^2 - |a_2| \leq \Psi(\alpha, \beta) + \varphi(y), \tag{3.10}$$

where φ is defined by

$$\varphi(y) = \frac{1}{|2+\alpha+i\beta|}y^2 - y \quad (y \in [1/|\mu_1|, \tilde{x}]).$$

Since φ is convex on $[1/|\mu_1|, \tilde{x}]$,

$$\varphi(y) \leq \max\{\varphi(1/|\mu_1|); \varphi(\tilde{x})\} \tag{3.11}$$

for all $y \in [1/|\mu_1|, \tilde{x}]$.

Thus in order to establish the upper bound in (3.1), we use (3.10) and (3.11), and need to show that

$$\Psi(\alpha, \beta) + \varphi\left(\frac{1}{|\mu_1|}\right) \leq \frac{2+\alpha}{|2+\alpha+i\beta|} \tag{3.12}$$

and

$$\Psi(\alpha,\beta)+\varphi(\tilde{x})\leq\frac{2+\alpha}{|2+\alpha+i\beta|}. \tag{3.13}$$

We first obtain (3.12).
Since

$$\frac{4}{|3+\alpha+i\beta|}-2<0 \quad\text{and}\quad \frac{|2+\alpha+i\beta|}{|3+\alpha+i\beta|}\geq\frac{2+\alpha}{3+\alpha},$$

(3.12) holds provided

$$\begin{aligned}A_1 &:= \frac{\alpha|1+3\alpha+i\beta|}{|1+\alpha+i\beta|}+\frac{2\alpha^2}{|1+\alpha+i\beta|(|1+\alpha+i\beta|-1)}\\&\quad+\frac{4(2+\alpha)|2+\alpha+i\beta|}{(3+\alpha)|3+\alpha+i\beta|}-\alpha\\&\leq\frac{2(2+\alpha)|2+\alpha+i\beta|}{3+\alpha}=:A_2.\end{aligned}$$

Clearly $A_1\leq A_2$ is true when $\alpha=0$. For $\alpha>0$, using the inequalities

$$\frac{|1+3\alpha+i\beta|}{|1+\alpha+i\beta|}\leq\frac{1+3\alpha}{1+\alpha},\quad \frac{1}{|1+\alpha+i\beta|}\leq\frac{1}{1+\alpha}$$

and

$$\frac{1}{|1+\alpha+i\beta|-1}\leq\frac{1}{\alpha},$$

it follows that

$$\frac{1}{2}(A_1-A_2)\leq|2+\alpha+i\beta|\left(\frac{\alpha}{|2+\alpha+i\beta|}+\frac{2(2+\alpha)}{(3+\alpha)|3+\alpha+i\beta|}-\frac{2+\alpha}{3+\alpha}\right). \tag{3.14}$$

We next note that the following is valid provided $\alpha\in[0,(\sqrt{17}-1)/2]$.

$$\frac{\alpha}{|2+\alpha+i\beta|}+\frac{2(2+\alpha)}{(3+\alpha)|3+\alpha+i\beta|}\leq\frac{\alpha}{2+\alpha}+\frac{2(2+\alpha)}{(3+\alpha)^2}\leq\frac{2+\alpha}{3+\alpha}. \tag{3.15}$$

Thus from (3.15) and (3.14), $A_1\leq A_2$ and (3.12) is established, providing $\alpha\in[0,(\sqrt{17}-1)/2]$.

Next we prove (3.13), which is satisfied if $B_1\leq B_2$, where

$$B_1:=\alpha(|1+3\alpha+i\beta|-|1+\alpha+i\beta|)+\frac{2\alpha^2}{|1+\alpha+i\beta|-1}+\frac{(2\alpha+2)^2}{|1+\alpha+i\beta|}$$

and

$$B_2 := 2(1+\alpha)|2+\alpha+i\beta|.$$

A similar process to the above gives

$$B_1 \le 2\alpha^2 + 2\alpha + \frac{(2\alpha+2)^2}{1+\alpha} = 2(1+a)(2+a) \le B_2,$$

which proves inequality (3.13), and so the proof of Theorem 3.1 is complete. □

When $\beta = 0$, we deduce the following, noting that when $\alpha = 1$, we obtain the inequality $||a_3| - |a_2|| \le 1$ for $f \in \mathcal{K}$ obtained in [7].

Corollary 3.1 *Let $f \in \mathcal{B}(\alpha)$. Then $||a_3|-|a_2|| \le 1$ provided $0 \le \alpha \le (\sqrt{17}-1)/2] = 1.56\ldots$.*
The inequality is sharp when both the functions f and g are the Koebe function.

We end this section by noting from the definition, since $\mathcal{B}_1(0, i\beta) \equiv \mathcal{B}(0, i\beta)$, the following is an immediate consequence of Theorem 4.1 below.

Theorem 3.2 *Let $f \in \mathcal{B}(0, i\beta)$, and be given by (1.1) with $\beta \in \mathbb{R}$. Then*

$$-\frac{2}{\sqrt{|1+i\beta|^2 + |3+i\beta|}} \le |a_3| - |a_2| \le \frac{2}{|2+i\beta|}. \tag{3.16}$$

Both inequalities are sharp.

4 The class $\mathcal{B}_1(\alpha, i\beta)$,

We next consider the class $\mathcal{B}_1(\alpha, i\beta)$, recalling that $f \in \mathcal{B}_1(\alpha, i\beta)$ if, and only if, for $\alpha \ge 0$ and $\beta \in \mathbb{R}$,

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha+i\beta}\right\} > 0 \quad (z \in \mathbb{D}).$$

We find the sharp upper and lower bounds of $|a_3| - |a_2|$ over the class $\mathcal{B}_1(\alpha, i\beta)$.

Theorem 4.1 *Let $f \in \mathcal{B}_1(\alpha, i\beta)$ for $\alpha \ge 0$ and $\beta \in \mathbb{R}$, and be given by (1.1). Then*

$$-\frac{2}{\sqrt{|1+\alpha+i\beta|^2 + |3+\alpha+i\beta|}} \le |a_3| - |a_2| \le \frac{2}{|2+\alpha+i\beta|}. \tag{4.1}$$

Both inequalities are sharp.

Proof From (3.2), (3.3) (with $b_2 = b_3 = 0$), and Lemma 2.2, we obtain

$$a_2 = \frac{2\zeta_1}{1 + \alpha + i\beta}$$

and

$$a_3 = \left(\frac{2}{2 + \alpha + i\beta} - \frac{2(-1 + \alpha + i\beta)}{(1 + \alpha + i\beta)^2}\right)\zeta_1^2 + \frac{2}{2 + \alpha + i\beta}\left(1 - |\zeta_1|^2\right)\zeta_2$$

for some $\zeta_i \in \overline{\mathbb{D}}$ $(i = 1, 2)$. The triangle inequality gives

$$|a_3| - |a_2| \le \psi(|\zeta_1|), \tag{4.2}$$

where

$$\psi(x) = \kappa_2 x^2 + \kappa_1 x + \kappa_0 \quad (x \in [0, 1])$$

with

$$\kappa_2 = \left|\frac{2}{2 + \alpha + i\beta} - \frac{2(-1 + \alpha + i\beta)}{(1 + \alpha + i\beta)^2}\right| - \frac{2}{|2 + \alpha + i\beta|},$$
$$\kappa_1 = -\frac{2}{|1 + \alpha + i\beta|}, \quad \text{and} \quad \kappa_0 = \frac{2}{|2 + \alpha + i\beta|}.$$

We first prove the upper bound in (4.1).

If $\kappa_2 \le 0$, then since $\kappa_1 < 0$, we have $\psi'(x) = 2\kappa_2 x + \kappa_1 < 0$ for all $x \in [0, 1]$. Thus

$$\psi(x) \le \psi(0) = \kappa_0 \quad (x \in [0, 1]). \tag{4.3}$$

Suppose next that $\kappa_2 > 0$. We now note that $\kappa_2 + \kappa_1 \le 0$, since

$$\begin{aligned}\frac{1}{2}(\kappa_2 + \kappa_1) &\le \frac{|-1 + \alpha + i\beta|}{|1 + \alpha + i\beta|^2} - \frac{1}{|1 + \alpha + i\beta|}\\ &= \frac{1}{|1 + \alpha + i\beta|}\left(\frac{|-1 + \alpha + i\beta|}{|1 + \alpha + i\beta|} - 1\right)\end{aligned}$$

and $|1 + \alpha + i\beta| \ge |-1 + \alpha + i\beta|$.

Since $\kappa_2 > 0$, ψ is a quadratic function with positive leading coefficient, and $\psi(1) = \kappa_2 + \kappa_1 + \kappa_0 \le \kappa_0 = \psi(0)$, it follows that

$$\psi(x) \le \max\{\psi(0); \psi(1)\} = \psi(0) = \kappa_0 \quad (x \in [0, 1]). \tag{4.4}$$

Thus from (4.2), (4.3) and (4.5) we obtain

$$|a_3| - |a_2| \leq \kappa_0 = \frac{2}{|2+\alpha+i\beta|}.$$

We next prove the lower bound in (4.1).
Write

$$|a_3| - |a_2| = \frac{2}{|2+\alpha+i\beta|}\Psi, \tag{4.5}$$

where

$$\Psi = \left| R_1 e^{i\theta}\zeta_1^2 + (1-\zeta_1^2)\zeta_2 \right| - R_2\zeta_1$$

with

$$R_1 = \left| \frac{3+\alpha+i\beta}{(1+\alpha+i\beta)^2} \right|, \quad \theta = \arg\left(\frac{3+\alpha+i\beta}{(1+\alpha+i\beta)^2} \right)$$

and

$$R_2 = \left| \frac{2+\alpha+i\beta}{1+\alpha+i\beta} \right|,$$

so that we need to show that

$$\Psi \geq \frac{-R_2}{\sqrt{R_1+1}}.$$

Since both $\mathcal{B}_1(\alpha, i\beta)$ and $\mathcal{P}$ are rotationally invariant, we may assume that $\zeta_1 \in [0, 1]$.

Now write $\zeta_2 = se^{i\varphi}$ with $s \in [0, 1]$ and $\varphi \in \mathbb{R}$, so that

$$\Psi = \left| R_1 e^{i(\theta-\varphi)}\zeta_1^2 + (1-\zeta_1^2)s \right| - R_2\zeta_1.$$

Then

$$\begin{aligned} \Psi &= \sqrt{R_1^2\zeta_1^4 + 2R_1\zeta_1^2(1-\zeta_1^2)s\cos(\theta-\varphi) + (1-\zeta_1^2)^2 s^2} - R_2\zeta_1 \\ &\geq \left| R_1\zeta_1^2 - (1-\zeta_1^2)s \right| - R_2\zeta_1, \end{aligned} \tag{4.6}$$

with equality when $\cos(\theta - \varphi) = -1$.

Suppose that $R_1\zeta_1^2-(1-\zeta_1^2)s\leq 0$, then $\zeta_1\leq\sqrt{s/(R_1+s)}=:\eta_1$, and so by (4.6) it follows that

$$\begin{aligned}\Psi&\geq-(R_1+s)\zeta_1^2-R_2\zeta_1+s\\&\geq-(R_1+s)\eta_1^2-R_2\eta_1+s\\&=-R_2\sqrt{\frac{s}{R_1+s}}\\&\geq\frac{-R_2}{\sqrt{R_1+1}},\end{aligned}$$

since $s\leq 1$.

If $R_1\zeta_1^2-(1-\zeta_1^2)s\geq 0$, then $\zeta_1\geq\eta_1$, and define ϕ by

$$\phi(x)=(R_1+s)x^2-R_2x-s,$$

and let

$$\eta_2=\frac{R_2}{2(R_1+s)}$$

be the unique critical point of ϕ. Then by (4.6) we have

$$\Psi\geq\phi(\zeta_1). \tag{4.7}$$

The condition $\eta_2\geq\eta_1$ is equivalent to the inequality $4s^2+4R_1s-R_2^2\geq 0$, which holds for $0\leq s\leq\lambda$, where

$$\lambda=\lambda_{\alpha,\beta}:=\frac{1}{2}\left(-R_1+\sqrt{R_1^2+R_2^2}\right).$$

It is easily seen that $\lambda<1$ since

$$R_2^2=\frac{(2+\alpha)^2+\beta^2}{(1+\alpha)^2+\beta^2}\leq\left(\frac{2+\alpha}{1+\alpha}\right)^2\leq 4<4+R_1,$$

for $\alpha\geq 0$, and $\beta\in\mathbb{R}$.

We also note that $R_2-2R_1<2$, since

$$R_2-2R_1<R_2\leq\frac{2+\alpha}{1+\alpha}\leq 2.$$

We consider next the case $R_2\leq 2R_1$, where $\eta_1\leq 1$ for all $s\in[0,1]$, and distinguish two sub-cases, $\eta_2\leq\eta_1$, and $\eta_2\geq\eta_1$.

When $s\in[\lambda,1]$, we have $\eta_2\leq\eta_1$, and so from (4.7) we obtain

$$\Psi\geq\phi(\eta_1)=-R_2\sqrt{\frac{s}{R_1+s}}\geq\frac{-R_2}{\sqrt{R_1+1}} \tag{4.8}$$

since $s \in [0, 1]$. When $s \in [0, \lambda]$, we have $\eta_2 \geq \eta_1$. This, and (4.7), implies that

$$\Psi \geq \phi(\eta_2) = -\left(s + \frac{R_2^2}{4(R_1 + s)}\right) = -\frac{1}{4}h(s), \tag{4.9}$$

where h is defined by

$$h(x) = 4x + \frac{R_2^2}{R_1 + x}. \tag{4.10}$$

Differentiating h gives

$$(R_1 + x)^2 h'(x) = 4x^2 + 8R_1 x + 4R_1^2 - R_2^2.$$

Since $4R_1^2 - R_2^2 = (2R_1 + R_2)(2R_1 - R_2) \geq 0$, h is increasing on the interval $[0, \lambda]$, and so from (4.9) we have

$$\Psi \geq -\frac{1}{4}h(\lambda) = -\left(\lambda + \frac{R_2^2}{4(R_1 + \lambda)}\right). \tag{4.11}$$

Next note that

$$\frac{R_2}{\sqrt{R_1 + 1}} \geq \lambda + \frac{R_2^2}{4(R_1 + \lambda)}, \tag{4.12}$$

since

$$\lambda + \frac{R_2^2}{4(R_1 + \lambda)} \leq \frac{R_2\sqrt{\lambda}}{\sqrt{R_1 + \lambda}},$$

provided $\sqrt{\lambda(R_1 + 1)} \leq \sqrt{R_1 + \lambda}$ which is valid for all $\alpha \geq 0$ and $\beta \in \mathbb{R}$ since $\lambda < 1$.

Thus it follows from (4.8), (4.11) and (4.12) that

$$\Psi \geq \frac{-R_2}{\sqrt{R_1 + 1}}$$

is true provided $R_2 \leq 2R_1$.

Next assume that $R_2 \geq 2R_1$. In this case there exists $s \in [0, 1]$, such that $\eta_2 \geq 1$. Setting $\hat{\lambda} = (R_2 - 2R_1)/2$ it follows that $0 < \hat{\lambda} < \lambda < 1$.

When $s \in [\lambda, 1]$, we have $\eta_2 \leq \eta_1$, and a similar method to that used in the case $R_2 \leq 2R_1$ gives

$$\Psi \geq \frac{-R_2}{\sqrt{R_1 + 1}}.$$

When $s \in [\hat{\lambda}, \lambda]$, we have $\eta_2 \geq \eta_1$, and so the function h, defined by (4.10), is increasing on $[\hat{\lambda}, \lambda]$ since

$$\begin{aligned}(R_1 + x)^2 h'(x) &= 4x^2 + 8R_1 x + 4R_1^2 - R_2^2 \\ &\geq 4\hat{\lambda}^2 + 8R_1\hat{\lambda} + 4R_1^2 - R_2^2 = 0 \quad (x \in [\hat{\lambda}, \lambda]).\end{aligned}$$

Thus from (4.11) and (4.12), we have

$$\Psi \geq -\frac{1}{4}h(\lambda) \geq \frac{-R_2}{\sqrt{R_1 + 1}}.$$

When $s \in [0, \hat{\lambda}]$, we have $\eta_2 \geq 1$, which implies

$$\Psi \geq \phi(1) = R_1 - R_2. \tag{4.13}$$

Finally from (4.13), in order to establish the left hand inequality in (4.1), it is enough to show that

$$\frac{R_2}{\sqrt{R_1 + 1}} \geq R_2 - R_1. \tag{4.14}$$

Since

$$R_1 - R_2 + \frac{R_2}{\sqrt{R_1 + 1}} = R_1 R_2 \left(\frac{1}{R_2} - \frac{1}{R_1 + 1 + \sqrt{R_1 + 1}} \right),$$

and since $R_1 > 0$ and $R_2 > 0$, (4.14) is satisfied, if for $\alpha \geq 0$ and $\beta \in \mathbb{R}$

$$\sqrt{R_1 + 1} > R_2 - R_1 - 1. \tag{4.15}$$

Since

$$R_2 - R_1 - 1 = \frac{1}{|1 + \alpha + i\beta|} \left(|2 + \alpha + i\beta| - |1 + \alpha + i\beta| - \frac{|3 + \alpha + i\beta|}{|1 + \alpha + i\beta|} \right)$$

and

$$|2 + \alpha + i\beta| \leq |1 + \alpha + i\beta| + 1 < |1 + \alpha + i\beta| + \frac{|3 + \alpha + i\beta|}{|1 + \alpha + i\beta|},$$

it follows that $R_2 - R_1 - 1 < 0 < \sqrt{R_1 + 1}$, which establishes (4.15), and hence (4.14).

Thus the proof of the inequalities for $|a_3| - |a_2|$ is complete.

In order to show that the inequalities are sharp, first let the function f_1 be defined by (1.3) with $g(z) = z$ and $p(z) = (1 + z^2)/(1 - z^2)$. Then $f_1 \in \mathcal{B}_1(\alpha, i\beta)$ with

$$f_1(z) = z + \frac{2}{2+\alpha+i\beta} z^3 + \cdots .$$

Thus the upper bound in (4.1) is sharp.

Next put $\zeta_1 = 1/\sqrt{R_1+1}$, and $\zeta_2 = se^{i\varphi}$ with $s = 1$ and $\varphi = \theta - \pi$. Then

$$\Psi = \left| R_1 e^{i(\theta-\varphi)} \zeta_1^2 + (1-\zeta_1^2)s \right| - R_2\zeta_1 = -\frac{R_2}{\sqrt{R_1+1}}. \tag{4.16}$$

Since $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, it follows from Lemma 2.2 that the function $\hat{p}$ defined by

$$\begin{aligned} \hat{p}(z) &= \frac{1+(\zeta_1\zeta_2+\zeta_1)z+\zeta_2 z^2}{1+(\zeta_1\zeta_2-\zeta_1)z-\zeta_2 z^2} \\ &= \frac{\sqrt{R_1+1}+(e^{i\varphi}+1)z+\sqrt{R_1+1}e^{i\varphi}z^2}{\sqrt{R_1+1}+(e^{i\varphi}-1)z-\sqrt{R_1+1}e^{i\varphi}z^2} \end{aligned}$$

belongs to $\mathcal{P}$. Now let the function f_2 be defined by (1.3) with $g(z) = z$ and $p = \hat{p}$. Then $f_2 \in \mathcal{B}_1(\alpha, i\beta)$. From (4.5) and (4.16), we obtain

$$|a_3| - |a_2| = \frac{2}{|2+\alpha+i\beta|}\Psi = -\frac{2}{\sqrt{|1+\alpha+i\beta|^2 + |3+\alpha+i\beta|}},$$

which shows that the left hand equality in (4.1) is sharp.

This completes the proof of Theorem 4.1. □

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