Density estimates for the solutions of backward stochastic differential equations driven by Gaussian processes

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Abstract. The aim of this paper is twofold. Firstly, we derive upper and lower non-Gaussian bounds for the densities of the marginal laws of the solutions to backward stochastic differential equations (BSDEs) driven by fractional Brownian motions. Our arguments consist of utilising a relationship between fractional BSDEs and quasilinear partial differential equations of mixed type, together with the profound Nourdin-Viens formula. In the linear case, upper and lower Gaussian bounds for the densities and the tail probabilities of solutions are obtained with simple arguments by their explicit expressions in terms of the quasi-conditional expectation. Secondly, we are concerned with Gaussian estimates for the densities of a BSDE driven by a Gaussian process in the manner that the solution can be established via an auxiliary BSDE driven by a Brownian motion. Using the transfer theorem we succeed in deriving Gaussian estimates for the solutions.

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1 Introduction

The problem of density estimates for solutions of stochastic equations has been extensively studied in recent years, see e.g. the monograph [27] and references therein. Remarkably, the celebrated Bouleau-Hirsch criterion (see [27, Theorem 2.1.2]) provides a sufficient condition for a random variable possessing a density. Moreover, in [26], Nourdin and Viens derive a formula for a Malliavin differentiable random variable to admit a density with lower and upper Gaussian estimates. These results have been further extended and applied to solutions of stochastic differential equations (SDEs) and stochastic partial differential equations (SPDEs), among which let us just mention, for examples, the works by Debussche and Romito [8], Delarue, Menozzi and Nualart [10], Millet and Sanz-Solé [23], Mueller and Nualart [24], Nualart and Quer-Sardanyons [28], and the references therein.

On the other hand, in the seminal paper [29] Pardoux and Peng initiated the theory of nonlinear backward stochastic differential equations (BSDEs), which is of increasing importance in stochastic control and mathematical finance (see, e.g., [12] and most recently [33]). This class of equations is of the following form

$$y_t = \xi + \int_t^T f(s, y_s, z_s)\mathrm{d}s - \int_t^T z_s \mathrm{d}B_s, \quad t \in [0, T] \tag{1.1}$$

on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\in[0,T]})$, where $T > 0$ is arbitrarily fixed, ξ is a $\mathcal{F}_T$-measurable random variable, the generator $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a jointly measurable map, and $B = (B_t)_{t\geq 0}$ is a Brownian motion adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$ or simply $\{\mathcal{F}_t\}_{t\in[0,T]}$ is taken as the natural filtration of B. Recall that a solution to the BSDE (1.1) is a pair of predictable processes (y, z) with suitable integrability conditions such that (1.1) holds $\mathbb{P}$-a.s.. To date, there is a wealth of existence and uniqueness results under various assumptions on the generators f including, for instnace, the cases of Lipschitz or (super-)quadratic growth [11, 12, 18, 29, 30]. When dealing with applications such as the numerical approximation of the solutions, one needs to investigate the existence and regularity of densities for the marginal laws of (y, z). As far as we know, there are comparably only a few works to study this problem. The first results have been derived by Antonelli and Kohatsu-Higa [3], in which they study the existence and the estimates of the density for y_t at a fixed time $t \in [0, T]$ via the Bouleau-Hirsch criterion. Then, based on the Nourdin-Viens formula, Aboura and Bourguin [1] have proved the existence of the density for z_t under the condition that the generator f is linear with respect to its z variable, and further obtained the estimates on the densities of the laws of y_t and z_t. Recently, Mastrolia, Possamaï and Réveillac in [21] have studied the existence of densities for marginal laws of the solution (y, z) to (1.1) with a quadratic growth generator, and derived the estimates on these densities. Afterwards, Mastrolia [20] has extended the results to the case of non-Markovian BSDEs.

One of the main objective of the present paper concerns the problem of density estimates for the following BSDE

$$y_t = h(\eta_T) + \int_t^T f(s, \eta_s, y_s, z_s)\mathrm{d}s - \int_t^T z_s \mathrm{d}B_s^H, \quad t \in [0, T], \tag{1.2}$$

where $\eta_t = \eta_0 + \int_0^t b_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s^H$ with η_0, b_s and σ_s being respectively a constant and deterministic functions, and B^H is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$, the stochastic integral is the divergence-type integral. Precise assumptions on the (deterministic and joint measurable) generator $f : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ will be specified in later sections. Let us recall that, B^H with Hurst parameter $H \in (0, 1)$ is a center Gaussian process with covariance

$$R_H(t, s) := \mathbb{E}\left(B_t^H B_s^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right),\ t, s \in [0, T].$$

This implies that for each $p \geq 1$, there holds $\mathbb{E}(|B_t^H - B_s^H|^p) = C(p)|t - s|^{pH}$. Then B^H is $(H - \epsilon)$-order Hölder continuous for any $\epsilon > 0$ and is an H-self similar process. This, together with the fact that $B^{1/2}$ is a Brownian motion, converts fractional Brownian motion into a natural generalization of Brownian motion and leads to many applications in modelling physical phenomena and finance behaviours.

We mention that there are several papers concerning the existence and uniqueness results of solutions for (1.2). Biagini, Hu, Øksendal and Sulem in [6] first studied linear fractional BSDE with $H \in (1/2, 1)$ which are based on fractional Clark-Ocone formula and the Girsanov transformation. In the spirit of the four step scheme introduced by Ma, Protter and Yong [19] for BSDEs perturbed by a standard Brownian motion, Bender [4] constructed an explicit solutions for a kind of linear fractional BSDEs with $H \in (0, 1)$ via the solution of some PDE and fractional Itô formula. In the case of nonlinear fractional BSDEs with $H \in (1/2, 1)$, Hu and Peng [17] first proved the existence and uniqueness of the solution through the notion of quasi-conditional expectation introduced in [16]. Then, based on [17], Maticiuc and Nie in [22] made some improvements of analysis and extended to the case of fractional backward stochastic variational inequalities, and Fei, Xia and Zhang in [13] generalized the investigation to BSDEs driven by both standard and fractional Brownian motions. In a multivariate setting where each of the components is an independent fractional Brownian motion $B_i^{H_i}$ with $H_i \in (1/2, 1)$, Hu, Ocone and Song in [15] solved fractional BSDEs by using their relation to PDEs, and they further derived a comparison theorem.

In the present paper, with the help of the connection between the solution to BSDE (1.2) and the solution to its associated PDE of mixed type, we shall give some sufficient conditions to ensure the existence of densities for marginal laws of the solution (y, z) to BSDE (1.2). Moreover, we will derive non-Gaussian tail estimates of densities. To the best of our knowledge these kind of estimates for BSDE (1.2) are not available in the existing literature. When

$$f(s, \eta_s, y_s, z_s) = \alpha_s + \beta_s y_s + \gamma_t z_s,$$

i.e., BSDE (1.2) is linear, the Gaussian bounds for the densities and the tail probabilities of solutions will be derived with a direct and simpler arguments by their explicit expressions in terms of the quasi-conditional expectation.

Our paper is also dedicated to obtaining Gaussian bounds for the densities of the solution to BSDE

$$y_t = h(X_T) + \int_t^T f(s, X_s, y_s, z_s) \mathrm{d}V(t) - \int_t^T z_s \mathrm{d}^\diamond X_s, \tag{1.3}$$

where X is a centered Gaussian process with a strictly increasing continuous variance function $V(t) = \mathrm{Var} X_t$, the stochastic integral is the Wick-Itô integral defined via the S-transformation and the Wick product. When X is a Brownian motion, the above Wick-Itô integral coincides with the classical Itô integral. In [5] Bender shows the existence and uniqueness results and then obtains a strict comparison theorem for BSDE (1.3) using the transfer theorem which can transfer the concerned problems to an auxiliary BSDE driven by a Brownian motion. In [5] the author also compares this type of equations with other BSDEs driven by Gaussian non-semimartingales, especially BSDEs driven by fractional Brownian motion B^H with $H \in (1/2, 1)$. The final objective of the present paper is to deepen the investigation of BSDE (1.3). We study the Gaussian bounds for marginal laws of the solution (y, z) to BSDE (1.3) via the transfer theorem.

The rest of our paper is structured as follows. Section 2, the next section, presents some basic elements of stochastic calculus with respect to fractional Brownian motion which are needed in later sections. We investigate the non-Gaussian bounds for the densities of the nonlinear fractional BSDEs in Section 3. Section 4 is devoted to the derivation of the Gaussian bounds for the densities and the tail probabilities of linear fractional BSDEs.

Finally in Section 5, we provide the Gaussian bounds for the densities of BSDEs driven by Gaussian processes.

2 Preliminaries

In this section, we shall give some basic elements of stochastic calculus with respect to fractional Brownian motion. For a deeper and detailed discussion, we refer the reader to [2, 7, 9] and [27].

Let Ω be the canonical probability space $C_0([0,T],\mathbb{R})$, i.e., the Banach space of continuous functions on $[0,T]$ vanishing at time 0, equipped with the supremum norm, and $\mathcal{F}$ is taken to be the Borel σ-algebra. Let $\mathbb{P}$ be the unique probability measure on Ω such that the canonical process $B^H=(B_t^H)_{t\in[0,T]}$ is a fractional Brownian motion with Hurst parameter $H\in(0,1)$.

Let $\mathscr{E}$ be the space of step functions on $[0,T]$, and $\mathcal{H}$ the closure of $\mathscr{E}$ with respect to the following scalar product determined by the covariance R_H of B^H

$$\langle I_{[0,t]}, I_{[0,s]}\rangle_{\mathcal{H}} = R_H(t,s).$$

By the bounded linear transformation theorem, the mapping $I_{[0,t]}\mapsto B_t^H$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $\mathcal{H}_1$ associated with B^H. Denote this isometry by $\phi\mapsto B^H(\phi)$. When $H\in(1/2,1)$ it can be shown that $L^{1/H}([0,T])\subset\mathcal{H}$, and when $H\in(0,1/2)$ there holds $\mathcal{H}\subset L^2([0,T])$. For $H\in(1/2,1)$, we shall use the following representation of the inner product in $\mathcal{H}$:

$$\langle \phi,\psi\rangle_{\mathcal{H}} = C_H\int_0^T\int_0^T \phi_u\psi_v|u-v|^{2H-2}\mathrm{d}u\mathrm{d}v, \tag{2.1}$$

where $C_H:=H(2H-1)$.

Let $\mathcal{S}$ denote the totality of smooth and cylindrical random variables of the form

$$F=f(B^H(\phi_1),\cdots,B^H(\phi_n)),$$

where $n\geq 1, f\in C_b^\infty(\mathbb{R}^n)$, the set of f and all its partial derivatives are bounded, $\phi_i\in\mathcal{H}, 1\leq i\leq n$. The Malliavin derivative of F, denoted by D^HF, is defined as the $\mathcal{H}$-valued random variable

$$D^HF=\sum_{i=1}^n\frac{\partial f}{\partial x_i}(B^H(\phi_1),\cdots,B^H(\phi_n))\phi_i.$$

For any $p\geq 1$, we define the Sobolev space $\mathbb{D}_H^{1,p}$ as the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p}^p=\mathbb{E}|F|^p+\mathbb{E}\|D^HF\|_{\mathcal{H}}^p.$$

Next, let F be in $\mathbb{D}_H^{1,2}$ and write $D^HF:=\Phi_F(B^H)$, where $\Phi_F:\mathbb{R}^{\mathcal{H}}\to\mathcal{H}$ is a measurable mapping. Set

$$g_F(x)=\int_0^\infty \mathrm{e}^{-\theta}\mathbb{E}\left[\mathbb{E}'\left[\langle\Phi_F(B^H),\widetilde{\Phi_F^\theta}(B^H)\rangle_{\mathcal{H}}\right]|F-\mathbb{E}F=x\right]\mathrm{d}\theta,\ x\in\mathbb{R} \tag{2.2}$$

where $\widetilde{\Phi_F^\theta}(B^H) := \Phi_F(\mathrm{e}^{-\theta}B^H + \sqrt{1-\mathrm{e}^{-2\theta}}B'^H)$ with B'^H an independent copy of B^H such that B^H and B'^H are defined on the product probability space $(\Omega\times\Omega', \mathcal{F}\times\mathcal{F}', \mathbb{P}\times\mathbb{P}')$. We recall the following result, cf. [26, Theorem 3.1 and Proposition 3.7], which presents a criterion for a Malliavin differentiable random variable to have a density with Gaussian bounds based on the above function g.

Proposition 2.1 *The law of F has a density ρ_F with respect to the Lebesgue measure if and only if $g_F(F-\mathbb{E}F) > 0$ a.s. In this case, $\mathrm{Supp}(\rho_F)$ is a closed interval of $\mathbb{R}$ and for all $z \in \mathrm{Supp}(\rho_F)$, there holds*

$$\rho_F(z) = \frac{\mathbb{E}|F-\mathbb{E}F|}{2g_F(z-\mathbb{E}F)}\exp\left(-\int_0^{z-\mathbb{E}F}\frac{u\mathrm{d}u}{g_F(u)}\right).$$

Furthermore, if there exist constants $c_1, c_2 > 0$ such that

$$c_1 \le g_F(x) \le c_2, \quad \mathbb{P}-\text{a.s.},$$

then the law of F has a density ρ satisfying, for almost all $x\in\mathbb{R}$,

$$\frac{\mathbb{E}|F-\mathbb{E}F|}{2c_2}\exp\left(-\frac{(x-\mathbb{E}F)^2}{2c_1}\right) \le \rho(x) \le \frac{\mathbb{E}|F-\mathbb{E}F|}{2c_1}\exp\left(-\frac{(x-\mathbb{E}F)^2}{2c_2}\right).$$

Besides, by [26, Theorem 4.1] (see also [25, Proposition 2.2]) we then have the following tail estimates.

Proposition 2.2 *Let $F\in\mathbb{D}_H^{1,2}$ with $\mathbb{E}F = 0$. If $0 < g_F(x) \le a_1x + a_2$, a.s. for some $a_1 \ge 0$ and $a_2 > 0$, then*

$$\mathbb{P}(F\ge x) \le \exp\left(-\frac{x^2}{2a_1x+2a_2}\right) \quad\text{and}\quad \mathbb{P}(F\le -x)\le \exp\left(-\frac{x^2}{2a_2}\right), \quad x>0.$$

3 BSDEs driven by fractional Brownian motions

The objective of this section is to study the non-Gaussian densities estimates for the solution of the following BSDE driven by fractional Brownian motion

$$\begin{cases} \mathrm{d}y_t = f(t,\eta_t,y_t,z_t)\mathrm{d}t - z_t\mathrm{d}B_t^H, \\ y_T = h(\eta_T), \end{cases} \tag{3.1}$$

with

$$\eta_t := \eta_0 + \int_0^t b_s\mathrm{d}s + \int_0^t \sigma_s\mathrm{d}B_s^H.$$

Here η_0 is a given constant, b and σ are bounded deterministic functions such that $\sigma_t \ne 0$ for all $t\in[0,T], H\in(\frac{1}{2},1)$. A pair of $\mathcal{F}_t$-adapted stochastic processes (y,z) is called a solution to the equation (3.1) if

$$y_t = h(\eta_T) + \int_t^T f(s,\eta_s,y_s,z_s)\mathrm{d}s - \int_t^T z_s\mathrm{d}B_s^H, \quad t\in[0,T].$$

We begin with the assumption (H1) below

(i) $f:[0,T]\times\mathbb{R}^3\to\mathbb{R}$ is differentiable with respect to the third component and there exists a nonnegative constant K such that, for all $t\in[0,T], x, y_1, y_2, z_1, z_2\in\mathbb{R}$,

$$|f(t,x,y_1,z_1)-f(t,x,y_2,z_2)|+|f_y(t,x,y_1,z_1)-f_y(t,x,y_2,z_2)|\le K(|y_1-y_2|+|z_1-z_2|).$$

(ii) $h:\mathbb{R}\to\mathbb{R}$ is continuously differentiable and of polynomial growth.

Due to [15, Theorem 3.4], the condition (H1) ensures that there exists a unique solution (y,z) to BSDE (3.1), which is given by $y_t=u(t,\eta_t)$ and $z_t=\sigma_t u_x(t,\eta_t)$ via some deterministic function $u:[0,T]\times\mathbb{R}\to\mathbb{R}$, where $u_x(t,x):=\frac{\partial}{\partial x}u(t,x)$. The argument in [15] is based on a connection between this equation and a quasilinear PDE of mixed type. In the remaining part of this section, we assume (H1) holds and moreover the unique solution is of the above form. We aim to show the non-Gaussian densities estimates for the marginal laws of (y,z) at a fixed time $t\in(0,T)$. To this end, we let

$$\varrho_t=C_H\int_0^t\int_0^t\sigma_u\sigma_v|u-v|^{2H-2}\mathrm{d}u\mathrm{d}v,\quad p_{\varrho_t}(x)=\frac{1}{\sqrt{2\pi\varrho_t}}\mathrm{e}^{-\frac{x^2}{2\varrho_t}}$$

and for each $h\in L^0(\mathbb{R})$, define

$$\overline{h}:=\inf\{\gamma>0:\limsup_{|x|\to+\infty}\frac{|h(x)|}{|x|^\gamma}<\infty\},\quad \underline{h}:=\inf\{\gamma>0:\liminf_{|x|\to+\infty}\frac{|h(x)|}{|x|^\gamma}<\infty\}.$$

Clearly, the above $\overline{h}$ and $\underline{h}$ can describe the asymptotic growth of h in the neighborhood of $+\infty$ and $-\infty$.

Our main result of this section reads as follows

Theorem 3.1 *Let $t\in(0,T]$. Suppose that $0<\underline{u(t,\cdot)}<+\infty,\overline{u_x(t,\cdot)}<+\infty$ and there exist positive constants L,λ satisfying $u_x(t,\cdot)\ge\frac{1}{L(1+|\cdot|^\lambda)}$. Then, the law of $y_t=u(t,\eta_t)$ has a density ρ_{y_t}, and for any $\epsilon,\delta>0$ there exist positive constants $C_{\epsilon,t}$ and $C_{\delta,t}$ depending on ϵ,t and δ,t, respectively, such that*

$$\begin{aligned}&\frac{\mathbb{E}|y_t-\mathbb{E}y_t|}{2C(\epsilon,\delta)\varrho_t\left(1+|z|^{\bar\epsilon\bar\delta}\right)\left(1+|z|^{\bar\epsilon\bar\delta}+\varrho_t^{\frac{\bar\epsilon}{2}}\right)}\exp\left(-\frac{1}{\tilde C(\delta)\varrho_t}\int_0^{z-\mathbb{E}y_t}u\left(1+|u+\mathbb{E}y_t|^{2\lambda\bar\delta}\right)\mathrm{d}u\right)\le\rho_{y_t}(z)\\ \le\;&\frac{\mathbb{E}|y_t-\mathbb{E}y_t|}{2\tilde C(\delta)\varrho_t}\left(1+|z|^{2\lambda\bar\delta}\right)\exp\left(-\frac{1}{C(\epsilon,\delta)\varrho_t}\int_0^{z-\mathbb{E}y_t}\frac{u\mathrm{d}u}{\left(1+|u+\mathbb{E}y_t|^{\bar\epsilon\bar\delta}\right)\left(1+|u+\mathbb{E}y_t|^{\bar\epsilon\bar\delta}+\varrho_t^{\frac{\bar\epsilon}{2}}\right)}\right),\end{aligned}\tag{3.2}$$

where

$$\begin{aligned}C(\epsilon,\delta)\;:=\;&\sup_{t\in[0,T]}\Bigg[C_{\epsilon,t}^2\left(1+C_{\delta,t}^{\bar\epsilon}2^{(\bar\epsilon-1)_+}\right)\\&\cdot\left(\left(1+3^{(\bar\epsilon-1)_+}\left|\eta_0+\int_0^t b_s\mathrm{d}s\right|^{\bar\epsilon}+\frac{C_{\delta,t}^{\bar\epsilon}6^{(\bar\epsilon-1)_+}}{1+\bar\epsilon}\right)\vee\left(\frac{\Gamma\left(\frac{1+\bar\epsilon}{2}\right)2^{\frac{\bar\epsilon}{2}}}{\sqrt{\pi}}\right)\right)\Bigg]\end{aligned}$$

and

$$\tilde C(\delta):=\frac{1}{3^{(\lambda-1)_+}2L^2(1+2^{(2\lambda-1)_+}C_{\delta,t}^{2\lambda})}\cdot\sup_{t\in[0,T]}\int_{\mathbb{R}}\frac{p_{\varrho_t}(z)}{1+\left|\eta_0+\int_0^t b_s\mathrm{d}s\right|^\lambda+|z|^\lambda}\mathrm{d}z$$

with $\bar\epsilon:=\overline{u_x(t,\cdot)}+\epsilon$ and $\bar\delta:=\overline{u^{-1}(t,\cdot)}+\delta$.

Proof. By the fact that $u(t,\cdot)$ is continuous and increasing, we easily verify that y_t has a density ρ_{y_t}. In order to prove (3.2), we rely heavily on Proposition 2.1.

Notice first that for $0 < u \le t \le T$, we have $D_u^H y_t = u_x(t,\eta_t)\sigma_u$. Then, it follows that

$$\Phi_{y_t}(B^H) = u_x(t,\eta_t)\sigma.$$

and

$$\begin{aligned}\widetilde{\Phi^\theta_{y_t}}(B^H) &= \Phi_{y_t}\left(e^{-\theta}B^H + \sqrt{1-\mathrm{e}^{-2\theta}}B'^H\right)\\ &= u_x\left(t, \left(1-e^{-\theta}\right)\left(\eta_0 + \int_0^t b_s \mathrm{d}s\right) + e^{-\theta}\eta_t + \sqrt{1-\mathrm{e}^{-2\theta}}\int_0^t \sigma_s \mathrm{d}B'^H_s\right)\sigma..\end{aligned}$$

Hence, using (2.2) and (2.1), we deduce that for $y \in \mathbb{R}$,

$$\begin{aligned}& g_{y_t}(y)\\ =\ & \int_0^\infty \mathrm{e}^{-\theta}\mathbb{E}\left[\mathbb{E}'\left[\langle \Phi_{y_t}(B^H), \widetilde{\Phi^\theta_{y_t}}(B^H)\rangle_{\mathcal{H}}\right]\Big| y_t - \mathbb{E}y_t = y\right]\mathrm{d}\theta\\ =\ & \varrho_t \int_0^\infty \mathrm{e}^{-\theta}\mathbb{E}\Big[u_x(t,\eta_t)\\ & \mathbb{E}'u_x\left(t, \left(1-e^{-\theta}\right)\left(\eta_0 + \int_0^t b_s \mathrm{d}s\right) + e^{-\theta}\eta_t + \sqrt{1-\mathrm{e}^{-2\theta}}\int_0^t \sigma_s \mathrm{d}B'^H_s\right)\Big|\eta_t = u^{-1}(t, y+\mathbb{E}y_t)\Big]\mathrm{d}\theta\\ =\ & \varrho_t u_x(t, u^{-1}(t, y+\mathbb{E}y_t))\\ & \cdot \int_0^\infty \mathrm{e}^{-\theta}\int_{\mathbb{R}} u_x\left(t, \left(1-e^{-\theta}\right)\left(\eta_0 + \int_0^t b_s \mathrm{d}s\right) + e^{-\theta}u^{-1}(t, y+\mathbb{E}y_t) + \sqrt{1-\mathrm{e}^{-2\theta}}z\right)p_{\varrho_t}(z)\mathrm{d}z\mathrm{d}\theta.\end{aligned} \tag{3.3}$$

With the relation (3.3) in hand, we shall follow the strategy designed in [21] to obtain upper and lower bounds for g_{y_t}.

Upper bound. Since $\underline{u(t,\cdot)} \in (0,\infty)$, we know from [21, Lemma 5.2] that $\overline{u^{-1}(t,\cdot)} \in (0,\infty)$. Taking into account the definitions of $\overline{u_x(t,\cdot)}$ and $\overline{u^{-1}(t,\cdot)}$, we have for each $\epsilon > 0$

$$(0<)u_x(t,z) \le C_{\epsilon,t}(1+|z|^{\overline{u_x(t,\cdot)}+\epsilon}), \quad \forall z \in \mathbb{R}, \tag{3.4}$$

and for each $\delta > 0$

$$|u^{-1}(t,z)| \le C_{\delta,t}(1+|z|^{\overline{u^{-1}(t,\cdot)}+\delta}), \quad \forall z \in \mathbb{R}, \tag{3.5}$$

where $C_{\epsilon,t}$ and $C_{\delta,t}$ are both constants depending on ϵ, t and δ, t respectively.
For the convenience of the notation, we let $\bar{\epsilon} = \overline{u_x(t,\cdot)} + \epsilon$ and $\bar{\delta} = \overline{u^{-1}(t,\cdot)} + \delta$. Then

plugging the inequalities (3.4) and (3.5) and resorting to the C_r-inequality, we have

$$\begin{aligned}
& g_{y_t}(y) \\
\leq\ & \varrho_t C_{\epsilon,t}^2\left(1+|u^{-1}(t,y+\mathbb{E}y_t)|^{\bar{\epsilon}}\right) \\
& \cdot\int_0^\infty \mathrm{e}^{-\theta}\int_{\mathbb{R}}\left(1+\left|\left(1-e^{-\theta}\right)\left(\eta_0+\int_0^t b_s\mathrm{d}s\right)+e^{-\theta}u^{-1}(t,y+\mathbb{E}y_t)+\sqrt{1-\mathrm{e}^{-2\theta}}z\right|^{\bar{\epsilon}}\right)p_{\varrho_t}(z)\mathrm{d}z\mathrm{d}\theta \\
\leq\ & \varrho_t C_{\epsilon,t}^2\left(1+|u^{-1}(t,y+\mathbb{E}y_t)|^{\bar{\epsilon}}\right) \\
& \cdot\int_0^\infty \mathrm{e}^{-\theta}\int_{\mathbb{R}}\left(1+3^{(\bar{\epsilon}-1)_+}\left[\left|\eta_0+\int_0^t b_s\mathrm{d}s\right|^{\bar{\epsilon}}+e^{-\bar{\epsilon}\theta}|u^{-1}(t,y+\mathbb{E}y_t)|^{\bar{\epsilon}}+|z|^{\bar{\epsilon}}\right]\right)p_{\varrho_t}(z)\mathrm{d}z\mathrm{d}\theta \\
=\ & \varrho_t C_{\epsilon,t}^2\left(1+|u^{-1}(t,y+\mathbb{E}y_t)|^{\bar{\epsilon}}\right) \\
& \cdot\left(1+3^{(\bar{\epsilon}-1)_+}\left|\eta_0+\int_0^t b_s\mathrm{d}s\right|^{\bar{\epsilon}}+\frac{3^{(\bar{\epsilon}-1)_+}}{1+\bar{\epsilon}}|u^{-1}(t,y+\mathbb{E}y_t)|^{\bar{\epsilon}}+\frac{\Gamma\left(\frac{1+\bar{\epsilon}}{2}\right)2^{\frac{\bar{\epsilon}}{2}}}{\sqrt{\pi}}\varrho_t^{\frac{\bar{\epsilon}}{2}}\right) \\
\leq\ & \varrho_t C_{\epsilon,t}^2\left(1+C_{\delta,t}^{\bar{\epsilon}}2^{(\bar{\epsilon}-1)_+}\left(1+|y+\mathbb{E}y_t|^{\bar{\epsilon}\bar{\delta}}\right)\right) \\
& \cdot\left(1+3^{(\bar{\epsilon}-1)_+}\left|\eta_0+\int_0^t b_s\mathrm{d}s\right|^{\bar{\epsilon}}+\frac{C_{\delta,t}^{\bar{\epsilon}}6^{(\bar{\epsilon}-1)_+}}{1+\bar{\epsilon}}\left(1+|y+\mathbb{E}y_t|^{\bar{\epsilon}\bar{\delta}}\right)+\frac{\Gamma\left(\frac{1+\bar{\epsilon}}{2}\right)2^{\frac{\bar{\epsilon}}{2}}}{\sqrt{\pi}}\varrho_t^{\frac{\bar{\epsilon}}{2}}\right) \\
\leq\ & C(\epsilon,\delta)\varrho_t\left(1+|y+\mathbb{E}y_t|^{\bar{\epsilon}\bar{\delta}}\right)\left(1+|y+\mathbb{E}y_t|^{\bar{\epsilon}\bar{\delta}}+\varrho_t^{\frac{\bar{\epsilon}}{2}}\right)
\end{aligned}\tag{3.6}$$

where

$$\begin{aligned}
C(\epsilon,\delta) =\ & \sup_{t\in[0,T]}\Bigg[C_{\epsilon,t}^2\left(1+C_{\delta,t}^{\bar{\epsilon}}2^{(\bar{\epsilon}-1)_+}\right) \\
& \cdot\left(\left(1+3^{(\bar{\epsilon}-1)_+}\left|\eta_0+\int_0^t b_s\mathrm{d}s\right|^{\bar{\epsilon}}+\frac{C_{\delta,t}^{\bar{\epsilon}}6^{(\bar{\epsilon}-1)_+}}{1+\bar{\epsilon}}\right)\vee\left(\frac{\Gamma\left(\frac{1+\bar{\epsilon}}{2}\right)2^{\frac{\bar{\epsilon}}{2}}}{\sqrt{\pi}}\right)\right)\Bigg].
\end{aligned}$$

Lower bound. Due to the condition on $u_x(t,\cdot)$, the C_r-inequality and (3.5), we obtain

$$\begin{aligned}
& g_{y_t}(y) \\
\geq\ & \frac{\varrho_t}{L^2(1+|u^{-1}(t,y+\mathbb{E}y_t)|^{\lambda})} \\
& \cdot\int_0^\infty \mathrm{e}^{-\theta}\int_{\mathbb{R}}\frac{p_{\varrho_t}(z)}{1+\left|(1-e^{-\theta})\left(\eta_0+\int_0^t b_s\mathrm{d}s\right)+e^{-\theta}u^{-1}(t,y+\mathbb{E}y_t)+\sqrt{1-\mathrm{e}^{-2\theta}}z\right|^{\lambda}}\mathrm{d}z\mathrm{d}\theta \\
\geq\ & \frac{\varrho_t}{L^2(1+|u^{-1}(t,y+\mathbb{E}y_t)|^{\lambda})}\cdot\int_{\mathbb{R}}\frac{p_{\varrho_t}(z)}{1+3^{(\lambda-1)_+}\left(\left|\eta_0+\int_0^t b_s\mathrm{d}s\right|^{\lambda}+|u^{-1}(t,y+\mathbb{E}y_t)|^{\lambda}+|z|^{\lambda}\right)}\mathrm{d}z \\
\geq\ & \frac{1}{3^{(\lambda-1)_+}L^2}\cdot\int_{\mathbb{R}}\frac{p_{\varrho_t}(z)}{1+\left|\eta_0+\int_0^t b_s\mathrm{d}s\right|^{\lambda}+|z|^{\lambda}}\mathrm{d}z\cdot\frac{\varrho_t}{(1+|u^{-1}(t,y+\mathbb{E}y_t)|^{\lambda})^2} \\
\geq\ & \frac{1}{3^{(\lambda-1)_+}2L^2}\cdot\int_{\mathbb{R}}\frac{p_{\varrho_t}(z)}{1+\left|\eta_0+\int_0^t b_s\mathrm{d}s\right|^{\lambda}+|z|^{\lambda}}\mathrm{d}z\cdot\frac{\varrho_t}{1+C_{\delta,t}^{2\lambda}\left(1+|y+\mathbb{E}y_t|^{\bar{\delta}}\right)^{2\lambda}} \\
\geq\ & \tilde{C}(\delta)\frac{\varrho_t}{1+|y+\mathbb{E}y_t|^{2\lambda\bar{\delta}}},
\end{aligned}\tag{3.7}$$

where $\tilde{C}(\delta) = \frac{1}{3^{(\lambda-1)}+2L^2(1+2^{(2\lambda-1)}+C_{\delta,t}^{2\lambda})} \cdot \sup_{t\in[0,T]} \int_{\mathbb{R}} \frac{p_{\varrho_t}(z)}{1+\left|\eta_0+\int_0^t b_s \mathrm{d}s\right|^\lambda+|z|^\lambda}\mathrm{d}z$.

Therefore, applying Proposition 2.1, together with (3.6) and (3.7), we complete the proof. □

Remark 3.2 *Let us have a close look at the lower bound of ρ_{y_t}. In fact, a direct computation shows that*

$$\begin{aligned}\int_0^{z-\mathbb{E}y_t} u\left(1+|u+\mathbb{E}y_t|^{2\lambda\bar{\delta}}\right)\mathrm{d}u &= \frac{1}{2}|z-\mathbb{E}y_t|^2 + \frac{1}{2(1+\lambda\bar{\delta})}\left(|z|^{2(1+\lambda\bar{\delta})} - |\mathbb{E}y_t|^{2(1+\lambda\bar{\delta})}\right) \\ &\quad - \frac{|\mathbb{E}y_t|}{1+2\lambda\bar{\delta}}\left(\mathrm{sgn}(z\mathbb{E}y_t)|z|^{1+2\lambda\bar{\delta}} - |\mathbb{E}y_t|^{1+2\lambda\bar{\delta}}\right) =: \chi(z,\mathbb{E}y_t). \end{aligned} \tag{3.8}$$

Then, it yields the following

$$\rho_{y_t}(z) \geq \frac{\mathbb{E}|y_t-\mathbb{E}y_t|}{2C(\epsilon,\delta)\varrho_t\left(1+|z|^{\bar{\epsilon}\bar{\delta}}\right)\left(1+|z|^{\bar{\epsilon}\bar{\delta}}+\varrho_t^{\frac{\bar{\epsilon}}{2}}\right)}\exp\left(-\frac{\chi(z,\mathbb{E}y_t)}{\tilde{C}(\delta)\varrho_t}\right).$$

As for the upper bound in Theorem 3.1, we get the following result.

Corollary 3.3 *Under the assumptions in Theorem 3.1, there exists $z_0 > 0$ such that*

$$\rho_{y_t}(z) \leq \frac{\mathbb{E}|y_t-\mathbb{E}y_t|}{2\tilde{C}(\delta)\varrho_t}\left(1+|z|^{2\lambda\bar{\delta}}\right)\exp\left(-\frac{|z-\mathbb{E}y_t|^{2(1-\bar{\epsilon}\bar{\delta})} - |\mathrm{sgn}z\cdot z_0-\mathbb{E}y_t|^{2(1-\bar{\epsilon}\bar{\delta})}}{4(1-\bar{\epsilon}\bar{\delta})C(\epsilon,\delta)\varrho_t}\right)$$

holds for all $|z| > z_0$.

Proof. We first observe that by Theorem 3.1 our problem can be reduced to show that the following inequality

$$\begin{aligned}&\int_0^{z-\mathbb{E}y_t} \frac{u\mathrm{d}u}{\left(1+|u+\mathbb{E}y_t|^{\bar{\epsilon}\bar{\delta}}\right)\left(1+|u+\mathbb{E}y_t|^{\bar{\epsilon}\bar{\delta}}+\varrho_t^{\frac{\bar{\epsilon}}{2}}\right)} \\ \geq\ & \frac{1}{4(1-\bar{\epsilon}\bar{\delta})}\left(|z-\mathbb{E}y_t|^{2(1-\bar{\epsilon}\bar{\delta})} - |\mathrm{sgn}z\cdot z_0-\mathbb{E}y_t|^{2(1-\bar{\epsilon}\bar{\delta})}\right)\end{aligned} \tag{3.9}$$

holds for each $|z| > z_0$. In order to prove (3.9), we start by noticing that

$$\lim_{u\to+\infty} \frac{u}{\left(1+|u+\mathbb{E}y_t|^{\bar{\epsilon}\bar{\delta}}\right)\left(1+|u+\mathbb{E}y_t|^{\bar{\epsilon}\bar{\delta}}+\varrho_t^{\frac{\bar{\epsilon}}{2}}\right)}\frac{1}{\frac{u}{|u|^{2\bar{\epsilon}\bar{\delta}}}} = 1,$$

and then for some sufficiently large $u_0 > 0$, we have for $u \geq u_0$

$$\frac{u}{\left(1+|u+\mathbb{E}y_t|^{\bar{\epsilon}\bar{\delta}}\right)\left(1+|u+\mathbb{E}y_t|^{\bar{\epsilon}\bar{\delta}}+\varrho_t^{\frac{\bar{\epsilon}}{2}}\right)} \geq \frac{u}{2|u|^{2\bar{\epsilon}\bar{\delta}}}.$$

Consequently, there exists $z_0 > 0$ large enough such that (3.9) holds for any $|z| \geq z_0$. Indeed, when $z \leq -z_0$ (the choosing of z_0 depends on the above argument and moreover satisfies $z_0 - |\mathbb{E}y_t| > 0$), we get

$$\begin{aligned}
&\int_0^{z-\mathbb{E}y_t} \frac{u\mathrm{d}u}{\left(1+|u+\mathbb{E}y_t|^{\bar\epsilon\bar\delta}\right)\left(1+|u+\mathbb{E}y_t|^{\bar\epsilon\bar\delta}+\varrho_t^{\frac{\bar\epsilon}{2}}\right)}\\
=&\int_{z-\mathbb{E}y_t}^0 \frac{-u\mathrm{d}u}{\left(1+|u+\mathbb{E}y_t|^{\bar\epsilon\bar\delta}\right)\left(1+|u+\mathbb{E}y_t|^{\bar\epsilon\bar\delta}+\varrho_t^{\frac{\bar\epsilon}{2}}\right)}\\
\geq&\int_{z-\mathbb{E}y_t}^{-z_0-\mathbb{E}y_t} \frac{-u\mathrm{d}u}{\left(1+|u+\mathbb{E}y_t|^{\bar\epsilon\bar\delta}\right)\left(1+|u+\mathbb{E}y_t|^{\bar\epsilon\bar\delta}+\varrho_t^{\frac{\bar\epsilon}{2}}\right)}\\
\geq&\int_{z-\mathbb{E}y_t}^{-z_0-\mathbb{E}y_t} \frac{-u\mathrm{d}u}{2|u|^{2\bar\epsilon\bar\delta}}\\
=&\frac{1}{4(1-\bar\epsilon\bar\delta)}\left((-z+\mathbb{E}y_t)^{2(1-\bar\epsilon\bar\delta)}-(z_0+\mathbb{E}y_t)^{2(1-\bar\epsilon\bar\delta)}\right)\\
=&\frac{1}{4(1-\bar\epsilon\bar\delta)}\left(|z-\mathbb{E}y_t|^{2(1-\bar\epsilon\bar\delta)}-|\mathrm{sgn}z\cdot z_0-\mathbb{E}y_t|^{2(1-\bar\epsilon\bar\delta)}\right).
\end{aligned}$$

Along the same lines as above, we can easily check the case $z \geq z_0$. This completes our proof. □

Example 3.4 *Let us consider the BSDE (3.1) with $T = 1, h(x) = x^3 + x, f(s, x, y, z) = x, b_s = 0, \sigma_s = 1$ and $\eta_0 = 0$, that is*

$$y_t = (B_1^H)^3 + B_1^H + \int_t^1 B_s^H \mathrm{d}s - \int_t^1 z_s \mathrm{d}B_s^H.$$

Then, by [22, Theorem 3] and [31, Theorem 4.3.1] we can show that the unique solution is given by

$$y_t = (B_t^H)^3 + (2-t)B_t^H, \quad z_t = 3(B_t^H)^2 + 2 - t, \quad t \in [0,1],$$

from which we derive that y_t admits a density ρ_{y_t} with respect to the Lebesgue measure. However, it is obvious that y_t does not have Gaussian tail. Since $u(t,x) = x^3 + (2-t)x, u_x(t,x) = 3x^2 + 2 - t$, we can easily verify that the conditions in Theorem 3.1 hold. Consequently, for each $t \in (0,1]$ we can provide non-Gaussian type tail estimates for the density ρ_{y_t} due to Theorem 3.1 and Corollary 3.3.

Notice that $z_t = \sigma_t u_x(t,\eta_t)$ and then $D_u^H z_t = \sigma_t u_{xx}(t,\eta_t)\sigma_\cdot$, following exactly the same line as the proof of Theorem 3.1 then yields a result for z_t, which we state as follows

Theorem 3.5 *Let $t \in (0,T]$. Suppose that $0 < \underline{u_x(t,\cdot)} < +\infty, \overline{u_{xx}(t,\cdot)} < +\infty$ and there exist positive constants L, λ satisfying $u_{xx}(t,\cdot) \geq \frac{1}{L(1+|\cdot|^\lambda)}$. Then the law of z_t has a density ρ_{z_t}, and for any $\epsilon, \delta > 0$ there exists positive constants C_1 and C_2 such that*

$$\begin{aligned}
&\frac{\mathbb{E}|z_t-\mathbb{E}z_t|}{C_1\sigma_t^2\varrho_t\left(1+|z|^{\bar\epsilon\bar\delta}\right)\left(1+|z|^{\bar\epsilon\bar\delta}+\varrho_t^{\frac{\bar\epsilon}{2}}\right)}\exp\left(-\frac{1}{C_2\sigma_t^2\varrho_t}\int_0^{z-\mathbb{E}z_t}u\left(1+|u+\mathbb{E}z_t|^{2\lambda\bar\delta}\right)\mathrm{d}u\right)\leq\rho_{z_t}(z)\\
\leq&\frac{\mathbb{E}|z_t-\mathbb{E}z_t|}{C_2\sigma_t^2\varrho_t}\left(1+|z|^{2\lambda\bar\delta}\right)\exp\left(-\frac{1}{C_1\sigma_t^2\varrho_t}\int_0^{z-\mathbb{E}z_t}\frac{u\mathrm{d}u}{\left(1+|u+\mathbb{E}z_t|^{\bar\epsilon\bar\delta}\right)\left(1+|u+\mathbb{E}z_t|^{\bar\epsilon\bar\delta}+\varrho_t^{\frac{\bar\epsilon}{2}}\right)}\right).
\end{aligned}$$

Similar to Remark 3.2 and Corollary 3.3, we have the following estimates.

Corollary 3.6 *With the same preamble as in Theorem 3.5, then the density ρ_{z_t} fulfills the following bounds*

$$\rho_{z_t}(z) \geq \frac{\mathbb{E}|z_t - \mathbb{E}z_t|}{C_1\sigma_t^2\varrho_t\left(1+|z|^{\bar{\epsilon}\bar{\delta}}\right)\left(1+|z|^{\bar{\epsilon}\bar{\delta}}+\varrho_t^{\frac{\bar{\epsilon}}{2}}\right)}\exp\left(-\frac{\chi(z,\mathbb{E}z_t)}{C_2\sigma_t^2\varrho_t}\right), \quad z\in\mathbb{R}$$

and

$$\rho_{z_t}(z) \leq \frac{\mathbb{E}|z_t - \mathbb{E}z_t|}{C_2\sigma_t^2\varrho_t}\left(1+|z|^{2\lambda\bar{\delta}}\right)\exp\left(-\frac{|z-\mathbb{E}z_t|^{2(1-\bar{\epsilon}\bar{\delta})}-|\mathrm{sgn}z\cdot z_0-\mathbb{E}z_t|^{2(1-\bar{\epsilon}\bar{\delta})}}{4(1-\bar{\epsilon}\bar{\delta})C_1\sigma_t^2\varrho_t}\right), \quad |z|>z_0$$

with some positive constant z_0.

Example 3.7 *Let us consider the following BSDE*

$$y_t = h(B_1^H) + \int_t^1 (B_s^H)^2\mathrm{d}s - \int_t^1 z_s\mathrm{d}B_s^H,$$

where $h(x) = x^4 + 2x^2$. According to [22, Theorem 3] and [31, Theorem 4.3.1], we deduce that the unique solution is shown by

$$y_t = (B_t^H)^4 + (3-t)(B_t^H)^2, \quad z_t = 4(B_t^H)^3 + 2(3-t)B_t^H, \quad t\in[0,1].$$

In this case, $u(t,x) = x^4 + (3-t)x^2, u_x(t,x) = 4x^3 + 2(3-t)x$ and $u_{xx}(t,x) = 12x^2 + 2(3-t)$ for all $(t,x)\in[0,1]\times\mathbb{R}$. It can be verified that the assumptions in Theorem 3.5 are satisfied. So, we obtain that for all $t\in(0,1], z_t$ has a density ρ_{z_t} with respect to the Lebesgue measure with non-Gaussian type lower and upper bounds by Theorem 3.5 and Corollary 3.6.

We conclude this section with a remark.

Remark 3.8 *(i) Comparing to the relevant results on BSDE driven by the standard Brownian motion ($H=\frac{1}{2}$) proved in [21, Theorem 5.6], it is clear to see that our results apply to more general BSDEs since we replace $B_t^{\frac{1}{2}}$ with $\eta_t = \eta_0 + \int_0^t b_s\mathrm{d}s + \int_0^t \sigma_s\mathrm{d}B_s^H$ and treat the case of fractional Brownian motion with arbitrary $H\in(\frac{1}{2},1)$ as driving process. Furthermore, an explicit low bound for the density of the solution y_t without any restriction for the variable z is shown in our Remark 3.2.*

(ii) The advantage of the above method of estimating densities allows us to obtain non-Gaussian type lower and upper bounds. The drawback is that Theorem 3.1 and Theorem 3.5 must be completed by an analysis of the following PDE

$$\begin{cases} u_t(t,x) + \frac{1}{2}\varrho_t' u_{xx}(t,x) + b(t)u_x(t,x) + f(t,x,u(t,x),\sigma_t u_x(t,x)) = 0, \\ u(T,x) = h(x), \end{cases}$$

which is studied in [15]. In the next two sections, we will present the Gaussian type densities estimates results where the only assumptions are those put on the data of BSDEs.

4 Linear BSDEs driven by fractional Brownian motions

In the previous section, we have presented the non-Gaussian type densities estimates for the fractional BSDE (3.1), which are indeed obtained by using the relation between this type of equation and a quasilinear PDE of mixed type. While the BSDE (3.1) is linear, we are able to prove the Gaussian type bounds and the tail probabilities with a direct and simpler method based on the explicit expression of the solutions in terms of the quasi-conditional expectation.

Consider the following linear BSDE

$$\begin{cases} \mathrm{d}y_t = -\left(\alpha_t + \beta_t y_t + \gamma_t z_t\right)\mathrm{d}t - z_t \mathrm{d}B_t^H, \\ y_T = \xi, \end{cases} \tag{4.1}$$

where $\alpha_t, \beta_t, \gamma_t$ are given as continuous and adapted processes.

Notice that BSDE (4.1) admits a unique solution under the condition (4.2) below. More specifically, set $\varsigma_t := \gamma_t + \int_0^t D_t^H \beta_s \mathrm{d}s$, $\bar{B}_t^H := B_t^H + \int_0^t \varsigma_s \mathrm{d}s$ and

$$R_t := \exp\left[-\int_0^t \left(K_H^{-1}\int_0^{\cdot} \varsigma_r \mathrm{d}r\right)(s)\mathrm{d}W_s - \frac{1}{2}\int_0^t \left(K_H^{-1}\int_0^{\cdot} \varsigma_r \mathrm{d}r\right)^2 (s)\mathrm{d}s\right].$$

It follows from the Novikov condition, i.e.

$$\mathbb{E}\exp\left[\frac{1}{2}\int_0^T \left(K_H^{-1}\int_0^{\cdot} \varsigma_r \mathrm{d}r\right)^2 (s)\mathrm{d}s\right] < \infty, \tag{4.2}$$

that $(R_t)_{t\in[0,T]}$ is an exponential martingale and then the Girsanov theorem implies that $(\bar{B}^H)_{t\in[0,T]}$ is a fractional Brownian motion under the probability measure $\mathbb{Q} := R_T \mathrm{d}\mathbb{P}$. Let $\rho_t := \exp\left\{\int_0^t \beta_s \mathrm{d}s\right\}$. Applying the fractional integration by parts formula to $\rho_t y_t$ yields that

$$\mathrm{d}(\rho_t y_t) = -\alpha_t \rho_t \mathrm{d}t - \rho_t z_t \mathrm{d}\bar{B}_t^H. \tag{4.3}$$

Then, there is a unique solution for BSDE (4.1), and moreover

$$y_t = \rho_t^{-1}\hat{\mathbb{E}}^{\mathbb{Q}}\left[\rho_T \xi + \int_t^T \alpha_s \rho_s \mathrm{d}s | \mathcal{F}_t\right], \tag{4.4}$$

where $\hat{\mathbb{E}}$ stands for the quasi-conditional expectation. More details can be found in [17, Theorem 5.1] or [32].

Next, we want to show the existence of densities for the marginal laws of the solution (y, z) to BSDE (4.2) and then to derive the Gaussian bounds for them via the relation (4.4) and Proposition 2.1. To this end, let $\alpha_t, \beta_t, \gamma_t$ be given continuous and deterministic functions and $\xi = h(\eta_T)$, in which η is defined in the previous section. Put

$$p_t(x) := \frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{x^2}{2t}},$$

and further denote

$$P_t g(x) := \int_{\mathbb{R}} p_t(x-y)g(y)\mathrm{d}y.$$

We first state the following useful lemma concerning the representation of the quasi-conditional expectation.

Lemma 4.1 *Assume that $g : \mathbb{R} \to \mathbb{R}$ is a measurable function of polynomial growth, then the following holds*

$$\hat{\mathbb{E}}\left(g(\eta_T)|\mathcal{F}_t\right) = \mathbb{P}_{\|\sigma\|_T^2-\|\sigma\|_t^2} g\left(\eta_0 + \int_0^T b_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s^H\right).$$

Proof. Though the proof is similar to the one proposed in [17, Theorem 3.8], yet we give a justification for the convenience of the reader. For $t \in [0, T]$, let $\tilde{\eta}_t := \eta_0 + \int_0^T b_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s^H$. Applying the Itô formula (see [17, Theorem 2.3] or [22, Corollary 35]) to $P_{\|\sigma\|_t^2-\|\sigma\|_s^2} g(\tilde{\eta}_s)$, we get

$$g(\tilde{\eta}_t) = P_{\|\sigma\|_t^2} g(\tilde{\eta}_0) + \int_0^t \frac{\partial}{\partial x} P_{\|\sigma\|_t^2-\|\sigma\|_s^2} g(\tilde{\eta}_s)\sigma_s \mathrm{d}B_s^H. \tag{4.5}$$

Choosing $t = T$ in the above equation and then taking the quasi-conditional expectation with respect to $\mathcal{F}_t$, we have

$$\hat{\mathbb{E}}\left(g(\tilde{\eta}_T)|\mathcal{F}_t\right) = P_{\|\sigma\|_T^2} g(\tilde{\eta}_0) + \int_0^t \frac{\partial}{\partial x} P_{\|\sigma\|_T^2-\|\sigma\|_s^2} g(\tilde{\eta}_s)\sigma_s \mathrm{d}B_s^H. \tag{4.6}$$

By the semigroup property of P_t, it is easy to verify that, for $0 \leq s \leq t \leq T$,

$$\frac{\partial}{\partial x} P_{\|\sigma\|_T^2-\|\sigma\|_s^2} g(x) = P_{\|\sigma\|_T^2-\|\sigma\|_t^2} \frac{\partial}{\partial x} P_{\|\sigma\|_t^2-\|\sigma\|_s^2} g(x).$$

Hence, this, together with (4.6) and (4.5), yields the desired result. □

Frow now on, let us suppose the following

(H2) $h : \mathbb{R} \to \mathbb{R}$ is twice differentiable and $0 < c \leq h' \leq C, 0 < \tilde{c} \leq h'' \leq \tilde{C}$, where $c, C, \tilde{c}$ and $\tilde{C}$ are constants.

Besides, we set

$$\vartheta_1(t) := \varrho_t \exp\left[2(T-t) \inf_{s\in[0,T]} \beta_s\right]$$

and

$$\vartheta_2(t) := \varrho_t \exp\left[2(T-t) \sup_{s\in[0,T]} \beta_s\right].$$

Recall that ϱ_t is defined in Section 3.

We are now in the position to state our main result of this section.

Theorem 4.2 *Assume that (H2) holds. Then, for each $t \in (0, T], y_t$ and z_t possess densities p_{y_t} and p_{z_t}, respectively. Moreover, for almost all $x \in \mathbb{R}, p_{y_t}$ and p_{z_t} satisfy, respectively, the following bounds*

$$\frac{\mathbb{E}|y_t - \mathbb{E}y_t|}{2C^2\vartheta_2(t)} \exp\left(-\frac{(x-\mathbb{E}y_t)^2}{2c^2\vartheta_1(t)}\right) \leq p_{y_t}(x) \leq \frac{\mathbb{E}|y_t - \mathbb{E}y_t|}{2c^2\vartheta_1(t)} \exp\left(-\frac{(x-\mathbb{E}y_t)^2}{2C^2\vartheta_2(t)}\right).$$

and

$$\frac{\mathbb{E}|z_t - \mathbb{E}z_t|}{2\tilde{C}^2\vartheta_2(t)\sigma_t^2} \exp\left(-\frac{(x-\mathbb{E}z_t)^2}{2\tilde{c}^2\vartheta_1(t)\sigma_t^2}\right) \leq p_{z_t}(x) \leq \frac{\mathbb{E}|z_t - \mathbb{E}z_t|}{2\tilde{c}^2\vartheta_1(t)\sigma_t^2} \exp\left(-\frac{(x-\mathbb{E}z_t)^2}{2\tilde{C}^2\vartheta_2(t)\sigma_t^2}\right).$$

Proof. By (4.4), we obtain $y_t = \mathrm{e}^{\int_t^T \beta_s \mathrm{d}s} \hat{\mathbb{E}}^{\mathbb{Q}}[\xi|\mathcal{F}_t] + \rho_t^{-1}\int_t^T \alpha_s\rho_s \mathrm{d}s$. Hence, we have

$$D_u^H y_t = \mathrm{e}^{\int_t^T \beta_s \mathrm{d}s} D_u^H \left(\hat{\mathbb{E}}^{\mathbb{Q}}[\xi|\mathcal{F}_t]\right). \tag{4.7}$$

On the other hand, from Lemma 4.1, we get

$$\begin{aligned}
\hat{\mathbb{E}}^{\mathbb{Q}}[\xi|\mathcal{F}_t] = \hat{\mathbb{E}}^{\mathbb{Q}}[h(\eta_T)|\mathcal{F}_t] &= \hat{\mathbb{E}}^{\mathbb{Q}}\left[h\left(\eta_0 + \int_0^T b_s \mathrm{d}s - \int_0^T \sigma_s\gamma_s \mathrm{d}s + \int_0^T \sigma_s \mathrm{d}\bar{B}_s^H\right)|\mathcal{F}_t\right] \\
&= P_{\|\sigma\|_T^2 - \|\sigma\|_t^2} h\left(\eta_0 + \int_0^T b_s \mathrm{d}s - \int_0^T \sigma_s\gamma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}\bar{B}_s^H\right) \\
&= P_{\|\sigma\|_T^2 - \|\sigma\|_t^2} h\left(\eta_0 + \int_0^T b_s \mathrm{d}s - \int_t^T \sigma_s\gamma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s^H\right) \quad (4.8)
\end{aligned}$$

Then, for $u \in [0,t]$

$$D_u^H \left(\hat{\mathbb{E}}^{\mathbb{Q}}[\xi|\mathcal{F}_t]\right) = \sigma_u P_{\|\sigma\|_T^2 - \|\sigma\|_t^2} h'\left(\eta_0 + \int_0^T b_s \mathrm{d}s - \int_t^T \sigma_s\gamma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s^H\right). \tag{4.9}$$

This allows us to deduce from (4.7) that, for $u \in [0,t]$

$$D_u^H y_t = \sigma_u \mathrm{e}^{\int_t^T \beta_s \mathrm{d}s} P_{\|\sigma\|_T^2 - \|\sigma\|_t^2} h'\left(\eta_0 + \int_0^T b_s \mathrm{d}s - \int_t^T \sigma_s\gamma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s^H\right). \tag{4.10}$$

Notice that

$$\Phi_{y_t}(B^H) = D^H y_t = \sigma.\mathrm{e}^{\int_t^T \beta_s \mathrm{d}s} P_{\|\sigma\|_T^2 - \|\sigma\|_t^2} h'\left(\eta_0 + \int_0^T b_s \mathrm{d}s - \int_t^T \sigma_s\gamma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s^H\right),$$

then

$$\begin{aligned}
&\widetilde{\Phi_{y_t}^{\theta}}(B^H) \\
=\ & \Phi_{y_t}(\mathrm{e}^{-\theta}B^H + \sqrt{1-\mathrm{e}^{-2\theta}}B'^H) \\
=\ & \sigma.\mathrm{e}^{\int_t^T \beta_s \mathrm{d}s} P_{\|\sigma\|_T^2 - \|\sigma\|_t^2} h'\left(\eta_0 + \int_0^T b_s \mathrm{d}s - \int_t^T \sigma_s\gamma_s \mathrm{d}s + \mathrm{e}^{-\theta}\int_0^t \sigma_s \mathrm{d}B_s^H + \sqrt{1-\mathrm{e}^{-2\theta}}\int_0^t \sigma_s \mathrm{d}B_s'^H\right).
\end{aligned}$$

Thus, according to (2.1), we have

$$\langle \Phi_{y_t}(B^H), \widetilde{\Phi_{y_t}^{\theta}}(B^H)\rangle_{\mathcal{H}} = \varrho_t \kappa(t,\theta),$$

where

$$\begin{aligned}
&\kappa(t,\theta) \\
=\ & \mathrm{e}^{2\int_t^T \beta_s \mathrm{d}s} P_{\|\sigma\|_T^2 - \|\sigma\|_t^2} h'\left(\eta_0 + \int_0^T b_s \mathrm{d}s - \int_t^T \sigma_s\gamma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s^H\right) \\
& \times P_{\|\sigma\|_T^2 - \|\sigma\|_t^2} h'\left(\eta_0 + \int_0^T b_s \mathrm{d}s - \int_t^T \sigma_s\gamma_s \mathrm{d}s + \mathrm{e}^{-\theta}\int_0^t \sigma_s \mathrm{d}B_s^H + \sqrt{1-\mathrm{e}^{-2\theta}}\int_0^t \sigma_s \mathrm{d}B_s'^H\right)
\end{aligned}$$

with $c^2\mathrm{e}^{2(T-t)\inf_{s\in[0,T]}\beta_s} \leq \kappa(t,\theta) \leq C^2\mathrm{e}^{2(T-t)\sup_{s\in[0,T]}\beta_s}$ due to (H2). Consequently, we arrive at the following bound

$$c^2\vartheta_1(t) \leq g_{y_t} \leq C^2\vartheta_2(t). \tag{4.11}$$

Next, we devote to computing $D_{\cdot}^{H} z_t$ and then estimating g_{z_t}. From (4.3), we know

$$y_0 - \rho_T \xi - \int_0^T \alpha_t \rho_t \mathrm{d}t = \int_0^T \rho_t z_t \mathrm{d}\bar{B}_t^H. \tag{4.12}$$

On the other hand, by the fractional Clark-Ocone formula (see [14] and [16]) on $(\bar{B}^H, \mathbb{Q})$, we can write

$$\begin{aligned} y_0 - \rho_T \xi - \int_0^T \alpha_t \rho_t \mathrm{d}t &= \int_0^T \hat{\mathbb{E}}^{\mathbb{Q}} \left[D_t^H \left(y_0 - \rho_T \xi - \int_0^T \alpha_t \rho_t \mathrm{d}t \right) |\mathcal{F}_t \right] \mathrm{d}\bar{B}_t^H \\ &= -\int_0^T \rho_T \hat{\mathbb{E}}^{\mathbb{Q}} \left[D_t^H \xi |\mathcal{F}_t \right] \mathrm{d}\bar{B}_t^H. \end{aligned} \tag{4.13}$$

Then, combining (4.12) with (4.13) yields the following

$$z_t = -\frac{\rho_T}{\rho_t} \hat{\mathbb{E}}^{\mathbb{Q}} \left[D_t^H \xi |\mathcal{F}_t \right] = -\sigma_t \mathrm{e}^{\int_t^T \beta_s \mathrm{d}s} \hat{\mathbb{E}}^{\mathbb{Q}} \left[h'(\eta_T) |\mathcal{F}_t \right].$$

Similar to (4.10) and (4.11), we obtain

$$D_u^H z_t = -\sigma_u \sigma_t \mathrm{e}^{\int_t^T \beta_s \mathrm{d}s} P_{\|\sigma\|_T^2 - \|\sigma\|_t^2} h'' \left(\eta_0 + \int_0^T b_s \mathrm{d}s - \int_t^T \sigma_s \gamma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s^H \right), \tag{4.14}$$

and

$$\tilde{c}^2 \vartheta_1(t) \sigma_t^2 \leq g_{z_t} \leq \tilde{C}^2 \vartheta_2(t) \sigma_t^2. \tag{4.15}$$

Finally, applying Proposition 2.1 to (4.11) and (4.15) respectively, we end up with the desired results and the proof is complete. □

Remark 4.3 *We can alternatively derive* (4.9) *in the above proof, by the following*

$$\begin{aligned} D_u^H \left(\hat{\mathbb{E}}^{\mathbb{Q}} [h(\eta_T) |\mathcal{F}_t] \right) &= \hat{\mathbb{E}}^{\mathbb{Q}} [D_u^H h(\eta_T) |\mathcal{F}_t] = \sigma_u \mathrm{I}_{[0,t]}(u) \hat{\mathbb{E}}^{\mathbb{Q}} [h'(\eta_T) |\mathcal{F}_t] \\ &= \sigma_u \mathrm{I}_{[0,t]}(u) P_{\|\sigma\|_T^2 - \|\sigma\|_t^2} h' \left(\eta_0 + \int_0^T b_s \mathrm{d}s - \int_t^T \sigma_s \gamma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s^H \right). \end{aligned}$$

where the last equality is similar to (4.8).

In view of the proof of Theorem 4.2, one can derive the following tail estimates for the probability laws of y_t and z_t.

Corollary 4.4 *Suppose (H2). Then there hold, for all $x > 0$,*

$$\mathbb{P}(y_t - \mathbb{E} y_t \geq x) \leq \exp\left(-\frac{x^2}{2C^2 \vartheta_2(t)} \right), \quad \mathbb{P}(y_t - \mathbb{E} y_t \leq -x) \leq \exp\left(-\frac{x^2}{2C^2 \vartheta_2(t)} \right), \tag{4.16}$$

and

$$\mathbb{P}(z_t - \mathbb{E} z_t \geq x) \leq \exp\left(-\frac{x^2}{2\tilde{C}^2 \vartheta_2(t) \sigma_t^2} \right), \quad \mathbb{P}(z_t - \mathbb{E} z_t \leq -x) \leq \exp\left(-\frac{x^2}{2\tilde{C}^2 \vartheta_2(t) \sigma_t^2} \right). \tag{4.17}$$

Proof. Noticing first that $g_F(x) = g_{F - \mathbb{E}F}(x)$. Then the relation (4.16) follows by (4.11) and Proposition 2.2 with $a_1 = 0$ and $a_2 = C^2 \vartheta_2(t)$. (4.17) can be verified similarly. □

5 BSDEs driven by Gaussian processes

In this section, we consider the following BSDE driven by a centered Gaussian process $\{X_t\}_{t\in[0,T]}$

$$\begin{cases} \mathrm{d}y_t = -f(t, X_t, y_t, z_t)\mathrm{d}V(t) + z_t \mathrm{d}^{\diamond} X_t, \\ y_T = h(X_T), \end{cases} \tag{5.1}$$

where $\mathrm{Var}X_t = V(t), t \in [0,T]$, is a strictly increasing, continuous function with $V(0) = 0$, the stochastic integral is the Wick-Itô integral defined by the S-transformation and the Wick product.

As stated in [5, Section 4.4], BSDE (5.1) covers a wide class of Gaussian processes as driving processes including fractional Brownian motion with $H \in (0,1)$ and fractional Wiener integral and so on, wherein a comparison with BSDE (3.1) is also given. Moreover, we notice that Bender [5] shows the existence and uniqueness results for BSDE (5.1) under the Lipschitz or even superquadratic growth conditions via the transfer theorem which can transfer the concerned problems to an auxiliary BSDE driven by a Brownian motion. Recall that the auxiliary BSDE is of the following form

$$\begin{cases} \mathrm{d}\bar{y}_t = -f(U(t), \bar{W}_t, \bar{y}_t, \bar{z}_t)\mathrm{d}t + \bar{z}_t \mathrm{d}\bar{W}_t, \\ \bar{y}_{V(T)} = h(\bar{W}_{V(T)}), \end{cases} \tag{5.2}$$

where $\{\bar{W}_t\}_{t\in[0,V(T)]}$ is a standard Brownian motion, $U(t), t \in [0, V(T)]$, is the inverse of V defined as

$$U(t) := \inf\{s \geq 0 : V(s) \geq t\},\ t \in [0, V(T)].$$

In this part, we aim to investigate the existence of densities and then derive their Gaussian estimates for the marginal laws of the solution (y, z) to BSDE (5.1). For this, we start by adopting the following set of conditions from [1, 5]: (H3)

(i) $h \in C_b^2(\mathbb{R})$ and $\inf\limits_{x\in\mathbb{R}} h''(x) > 0$;

(ii) $\mathbb{E}\int_0^T |f(t, X_t, 0, 0)|^2 \mathrm{d}V(t) < \infty$, and for each $t \in [0,T]$, $f(t,\cdot,\cdot,\cdot) \in C_b^2(\mathbb{R}^3)$ with all positive derivatives.

We state our final main result as follows

Theorem 5.1 *Assume (H3) holds true. Then, for all $t \in [0,T]$,*

(1) y_t has a density ρ_{y_t}, and there exist two strictly positive constants $c_1 < c_2$ such that for any $z \in \mathbb{R}$,

$$\begin{aligned} \frac{\mathbb{E}|y_t - \mathbb{E}y_t|}{c_2 V(t)} \exp\left(-\frac{(z - \mathbb{E}y_t)^2}{c_1 V(t)}\right) &\leq \rho_{y_t}(z) \\ &\leq \frac{\mathbb{E}|y_t - \mathbb{E}y_t|}{c_1 V(t)} \exp\left(-\frac{(z - \mathbb{E}y_t)^2}{c_2 V(t)}\right), \end{aligned} \tag{5.3}$$

(2) *if the generator $f(t,x,y,z)$ has a linear dependence on the z component, z_t possesses a density ρ_{z_t}, and furthermore if f only depends on the components (t,y), there exist two strictly positive constants $c_3 < c_4$ such that for any $z \in \mathbb{R}$,*

$$\begin{aligned} \frac{\mathbb{E}|z_t - \mathbb{E}z_t|}{c_4 V(t)} \exp\left(-\frac{(z-\mathbb{E}z_t)^2}{c_3 V(t)}\right) &\le \rho_{z_t}(z) \\ &\le \frac{\mathbb{E}|z_t - \mathbb{E}z_t|}{c_3 V(t)} \exp\left(-\frac{(z-\mathbb{E}z_t)^2}{c_4 V(t)}\right). \end{aligned} \tag{5.4}$$

Proof. By (H3), it follows by [12, Theorem 4.1] that the auxiliary BSDE (5.2) admits a unique solution $(\bar{y}, \bar{z})$ which has the following representation

$$\bar{y}_t = \phi(t, \bar{W}_t), \quad \bar{z}_t = \psi(t, \bar{W}_t), \quad t \in [0, V(T)], \tag{5.5}$$

where both $\phi, \psi : [0, V(T)] \times \mathbb{R} \to \mathbb{R}$ are deterministic functions.
Then, in light of [5, Theorem 4.2 and Theorem 4.4] or the transfer theorem asserted in [5, Theorem 3.1], we conclude that BSDE (5.2) has a unique solution (y, z) which can be written as

$$y_t = \phi(V(t), X_t), \quad z_t = \psi(V(t), X_t), \quad t \in [0, T]. \tag{5.6}$$

Therefore, taking into account the fact that $\mathrm{Var} X_t = V(t)$ we deduce that the law of y_t (resp. z_t) is the same as that of $\bar{y}_{V(t)}$ (resp. $\bar{z}_{V(t)}$).

On the other hand, by [1, Theorem 3.3] it follows that $\bar{y}_{V(t)}$ possesses a density $\rho_{\bar{y}_{V(t)}}$, and moreover there exist some strictly positive constants $c_1 < c_2$ such that for all $z \in \mathbb{R}$,

$$\begin{aligned} &\frac{\mathbb{E}|\bar{y}_{V(t)} - \mathbb{E}\bar{y}_{V(t)}|}{c_2 V(t)} \exp\left(-\frac{(z-\mathbb{E}\bar{y}_{V(t)})^2}{c_1 V(t)}\right) \le \rho_{\bar{y}_{V(t)}}(z) \\ \le\ &\frac{\mathbb{E}|\bar{y}_{V(t)} - \mathbb{E}\bar{y}_{V(t)}|}{c_1 V(t)} \exp\left(-\frac{(z-\mathbb{E}\bar{y}_{V(t)})^2}{c_2 V(t)}\right), \end{aligned} \tag{5.7}$$

which then yields our first claim.
As for z_t, provided that the generator f has a linear dependence on the z component, owing to [1, Theorem 4.3], we conclude that $\bar{z}_{V(t)}$ has a law which is absolutely continuous with respect to the Lebesgue measure. Moreover, if f depends only on (t,y) components, then applying [1, Theorem 4.6], we obtain the following Gaussian bounds for the density of $\bar{z}_{V(t)}$

$$\begin{aligned} \frac{\mathbb{E}|\bar{z}_{V(t)} - \mathbb{E}\bar{z}_{V(t)}|}{c_4 V(t)} \exp\left(-\frac{(z-\mathbb{E}\bar{z}_{V(t)})^2}{c_3 V(t)}\right) &\le \rho_{\bar{z}_{V(t)}}(z) \\ &\le \frac{\mathbb{E}|\bar{z}_{V(t)} - \mathbb{E}\bar{z}_{V(t)}|}{c_3 V(t)} \exp\left(-\frac{(z-\mathbb{E}\bar{z}_{V(t)})^2}{c_4 V(t)}\right), \end{aligned} \tag{5.8}$$

where $c_3 < c_4$ are two strictly positive constants.
We therefore obtain the other assertion. $\square$

Remark 5.2 *(i) If we choose Brownian motion as the driving Gaussian process, namely $X_t = B^{\frac{1}{2}}$, then our estimates (5.3) and (5.4) coincide with the estimates of [1, Theorem 3.3 and*

Theorem 4.6]. Note that the equation (5.1) *that we considered is more general than that of [1] since we allow X to be a wide class of Gaussian processes which includes fractional Brownian motion with arbitrary $H \in (0, 1)$ as special case. Therefore, the results stated in Theorem 5.1 cover that of [1].*

(ii) If we take $X_t = \eta_t$, then the equation (5.1) *is of the same form as* (3.1) *and* (4.1)*, which we considered in Section 3 and 4, respectively. By simple calculus we know that the result of Theorem 3.1 combined with Remark 3.2 and Corollary 3.3 is more elaborate than that of Theorem 5.1. Similarly, we also note that the estimates in Theorem 4.2 are better than the estimates* (5.3) *and* (5.4) *in Theorem 5.1. Indeed, the derivation of Theorem 5.1 is mainly based on the transfer theorem which is a time change type transformation allowing us to represent the equation* (5.1) *in terms of a class of BSDE driven by Brownian motion, while the arguments used in Section 3 and 4 focus on the structures of the original equations.*

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