THE CLOSE-TO-CONVEX ANALOGUE OF R. SINGH'S STARLIKE FUNCTIONS

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ABSTRACT. For f analytic in the unit disk $\mathbb{D}$, we consider the close-to-convex analogue of a class of starlike functions intodoced in 1968 by R. Singh. Coefficient and other results are obtained for this class of functions defined by $|zf'(z)/g(z) - 1| < 1$ for $z \in \mathbb{D}$, where g is starlike in $\mathbb{D}$.

1. PRELIMINARIES

Let $\mathcal{H}$ denote the class of functions f analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S} \subset \mathcal{A}$ be the class of functions univalent (i.e. one-to-one) in $\mathbb{D}$. Any function $f \in \mathcal{A}$ has the following series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Denote by $\mathcal{S}^*$ the subclass of $\mathcal{S}$ of starlike functions. It is well-known that $f \in \mathcal{S}^*$ if, and only if,

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \ z \in \mathbb{D}.$$

Denote by $\mathcal{C}$ the subclass of $\mathcal{S}^*$ of convex functions. It is well-known that $f \in \mathcal{S}^*$ if, and only if,

$$f(z) = zg'(z), \quad \text{for some} \quad g \in \mathcal{C}.$$

By $\mathcal{P}$ we denote the class of Carathéodory functions p which are analytic in $\mathbb{D}$, satisfying the condition $\mathfrak{Re}\{p(z)\} > 0$ for $z \in \mathbb{D}$, with

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{1.2}$$

Suppose now that f is analytic in $\mathbb{D}$, then f is close-to-convex if, and only if, there exists $\alpha \in (-\pi/2, \pi/2)$, and a function $g \in \mathcal{S}^*$ such that

$$\mathfrak{Re}\left\{e^{i\alpha}\frac{zf'(z)}{g(z)}\right\} > 0, \ z \in \mathbb{D}.$$

When $\alpha = 0$, we denote this class of close-to-convex functions by $\mathcal{K}$, and note that $\mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$.

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Suppose next that $f \in \mathcal{A}$, and is given by (1.1), and for $z \in \mathbb{D}$, satisfies

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1.$$

This class of functions was introduced in 1968 by Singh [6]. Denoting this class by $\mathcal{S}^*_u$, it is clear that $\mathcal{S}^*_u \subset \mathcal{S}^*$. In [6], Singh showed that if $f \in \mathcal{S}^*_u$, then $|a_n| \le 1/(n-1)$ for $n \ge 2$, and that this inequality is sharp. Other properties of functions in $\mathcal{S}^*_u$ were also given in [6].

We now define the close-to-convex analogue of the class $\mathcal{S}^*_u$ as follows.

Definition 1.1. *We say $f \in \mathcal{K}_u$, if for $f \in \mathcal{A}$, there exists $g \in \mathcal{S}^*$, such that*

$$\left| \frac{zf'(z)}{g(z)} - 1 \right| < 1, \ \ z \in \mathbb{D}.$$

Again it is clear that $\mathcal{S}^*_u \subset \mathcal{K}_u \subset \mathcal{K} \subset \mathcal{S}$.

Remark 1.

Although $\mathcal{K}_u$ represents the natural close-to-convex analogue of $\mathcal{S}^*_u$, we shall see that obtaining sharp estimates for the coefficients for example, represents a much more difficult problem. We note that this phenomena is often reflected in extending results from $\mathcal{S}^*$ to $\mathcal{K}$, and will see in this paper that the class $\mathcal{K}_u$ gives rise to some significant and interesting problems.

2. Lemmas

A function ω is called a Schwarz function if $\omega \in \mathcal{H}$, $\omega(0) = 0$, and $|\omega(z)| < 1$ for $z \in \mathbb{D}$. We denote the class of Schwarz functions by Ω.

Note that for $p \in \mathcal{P}$ given by (1.2), we can write $p(z) = (1 + \omega(z))/(1 - \omega(z))$, for some $\omega \in \Omega$. So writing

$$\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n, \tag{2.1}$$

and equating coefficients gives

$$p_1 = 2\omega_1, \quad p_2 = 2\omega_2 + 2\omega_1^2. \tag{2.2}$$

We will need the following lemmas.

Lemma 2.1. [1], [3, p.78]. *Let $\omega \in \Omega$ and be given by (2.1). Then for all $n =$ $2, 3, \ldots$,*

$$|\omega_{2n-1}| \le 1 - |\omega_1|^2 - |\omega_2|^2 - |\omega_3|^2 - \ldots - |\omega_n|^2,$$

and for all $n = 1, 2, 3, \ldots$,

$$|\omega_{2n}| \le 1 - |\omega_1|^2 - |\omega_2|^2 - |\omega_3|^2 - \ldots - |\omega_{n-1}|^2 - |\omega_n|^2.$$

Lemma 2.2. [4]. *Let $\omega \in \Omega$ and be given by (2.1). If $\mu \in \mathbb{C}$, then*

$$|\omega_2 - \mu\,\omega_1^2| \leq \max\{1, |\mu|\}. \tag{2.3}$$

Using (2.2) and (2.3) immediately gives the following.

Lemma 2.3. *Let $p \in \mathcal{P}$ and be given by (1.2). Then for $\mu \in \mathbb{C}$,*

$$|p_2 - \mu p_1^2| \leq 2\max\{1, |2\mu - 1|\}.$$

The inequality is sharp for each complex μ.

We shall also need the following (see e.g. [7]).

Lemma 2.4. *Let $p \in \mathcal{P}$ and be given by (1.2). Then for $n \geq 1$, $|p_n| \leq 2$, and*

$$\left|p_2 - \frac{1}{2}p_1^2\right| \leq 2 - \frac{1}{2}\left|p_1^2\right|.$$

The following Fekete-Szegö type inequalities of Keogh and Merkes [4] will be used extensively in Section 5.

Lemma 2.5. [4]. *Let $g \in \mathcal{S}^*$, and be given by*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n. \tag{2.4}$$

Then for any $\mu \in \mathbb{C}$,

$$|b_3 - \mu b_2^2| \leq \max\{1, |4\mu - 3|\},$$

and

$$|b_3 - \mu b_2^2| \leq 1 + (|4\mu - 3| - 1)\frac{|b_2|^2}{4}. \tag{2.5}$$

Both inequalities are sharp.

We will also use the following lemma concerning functions in $\mathcal{P}$, the proof of which follows easily from Lemma 2.5.

Lemma 2.6. *Let $p \in \mathcal{P}$. Then for any $t \in \mathbb{C}$,*

$$|p_2 - tp_1^2| \leq 2 + (|2t - 1| - 1)\frac{|p_1|^2}{2}. \tag{2.6}$$

The inequality is sharp.

Proof. Let $p \in \mathcal{P}$, then there exists a function $g \in \mathcal{S}^*$ given by (2.4) such that

$$p(z) = \frac{zg'(z)}{g(z)},\ z \in \mathbb{D}.$$

Thus

$$\begin{aligned} b_2 &= p_1, \\ b_3 &= \frac{1}{2}(p_2 + p_1^2). \end{aligned}$$

Substituting in (2.5) gives

$$|p_2-(2\mu-1)p_1^2|\le 2+(|4\mu-3|-1)\,\frac{|p_1|^2}{2},$$

for all complex μ. Writing $\mu=(t+1)/2$, gives (2.6) for all complex t.

The function $p(z)=(1+z)/(1-z)$ shows that the result is sharp for $|2t-1|\ge 1$, and $p(z)=(1+z^2)/(1-z^2)$ shows the sharpness for $|2t-1|\le 1$. □

Lemma 2.7. [5], [2, p.67]. *Suppose that $f\in\mathcal{S}$, and that $z=re^{i\theta}\in\mathbb{D}$. If*

$$m'(r)\le|f'(z)|\le M'(r),$$

where $m'(r)$ and $M'(r)$ are real-valued functions of r in $[0,1)$, then

$$\int_0^r m'(t)\,\mathrm{d}t\le|f(z)|\le\int_0^r M'(r)\,\mathrm{d}t.$$

We begin with some distortion theorems.

3. Distortion Theorems

Theorem 3.1. *If $f\in\mathcal{K}_u$ and $z=re^{i\theta}$, $0\le r<1$, then*

$$\frac{1-r}{(1+r)^2}\le|f'(z)|\le\frac{1+r}{(1-r)^2}, \tag{3.1}$$

and

$$\frac{2r}{1+r}-\log(1-r)\le|f(z)|\le\frac{2r}{1-r}+\log(1-r). \tag{3.2}$$

The inequalities are sharp.

Proof. Write

$$f'(z)=\frac{g(z)}{z}[1+\omega(z)], \tag{3.3}$$

for some $g\in\mathcal{S}^*$, and some $\omega\in\Omega$.

It is well-known that for $g\in\mathcal{S}^*$, with $z=re^{i\theta}$, $0\le r<1$, then

$$\frac{1}{(1+r)^2}\le\left|\frac{g(z)}{z}\right|\le\frac{1}{(1-r)^2}. \tag{3.4}$$

Thus using the Schwarz lemma, we have

$$1-r\le|1+\omega(z)|\le 1+r, \tag{3.5}$$

and so from (3.3), using (3.4) and (3.5), we immediately obtain (3.1).

The inequalities in (3.1) are sharp when $f_1\in\mathcal{K}_u$ is given by

$$f_1'(z)=\frac{g_0(z)}{z}(1+z),\quad\text{and}\quad g_0(z)=\frac{z}{(1-z)^2},$$

in which case

$$f_1'(-r)=\frac{1-r}{(1+r)^2},\quad\text{and}\quad f_1'(r)=\frac{1+r}{(1-r)^2}.$$

Clearly (3.2) follows from Lemma 2.7, since $\mathcal{K}_u \subset \mathcal{S}$.

The upper bound in (3.2) is sharp for $f_1 \in \mathcal{K}_u$ given by

$$f_1(z) = \frac{2z}{1-z} + \log(1-z),$$

and the lower bound for $f_2 \in \mathcal{K}_u$ given by

$$f_2(z) = \frac{2z}{1+z} - \log(1+z).$$

□

4. Coefficients

In [6], Singh was able to use the method of Clunie to obtain sharp coefficient estimates for functions in $\mathcal{S}_u^*$. Since this is not possible in $\mathcal{K}_u$, the problem of extending the coefficient inequalities in [6] to the class $\mathcal{K}_u$ appears not to be straightforward, with exact bound not easy to find. We give the following.

Theorem 4.1. *Let $f \in \mathcal{K}_u$, and be given by* (1.1)*. Then*

$$|a_2| \le \frac{3}{2}, \quad |a_3| \le \frac{5}{3}, \quad |a_4| \le \frac{7.3731\ldots}{4} = 1.8443\ldots, \quad |a_5| \le \frac{8}{5} + \frac{3}{5\sqrt[3]{4}} = 1.97\ldots.$$

The inequalities for $|a_2|$ and $|a_3|$ are sharp.

Proof. Write

$$zf'(z) = g(z)[1+\omega(z)], \tag{4.1}$$

for some $g \in \mathcal{S}^*$ and some $\omega \in \Omega$.

Equating coefficients in (4.1), and using (2.1) and (2.4) gives

$$2a_2 = b_2 + w_1, \tag{4.2}$$
$$3a_3 = b_3 + b_2 w_1 + w_2, \tag{4.3}$$
$$4a_4 = b_4 + b_3 w_1 + b_2 w_2 + w_3, \tag{4.4}$$

where for $n \ge 1$, $|b_n| \le n$ and $|w_n| \le 1$. Therefore (4.2) gives

$$2|a_2| \le |b_2| + |w_1| \Rightarrow 2|a_2| \le 3.$$

Now write $x_1 = |w_1|$, $x_2 = |w_2|$, and $x_3 = |w_3|$, and so from (4.3) we obtain

$$3|a_3| \le |b_3| + |b_2||w_1| + |w_2|,$$

so that Lemma 2.1 implies

$$3|a_3| \le 3 + 2|w_1| + (1 - |w_1|^2) \le 5,$$

since $0 \le 4 + 2x_1 - x_1^2 \le 5$ for $x_1 \in [0,1]$.

The bound for $|a_4|$ is more complicated. Again from (4.4) and Lemma 2.1 we have

$$\begin{aligned} 4|a_4| &\le |b_4| + |b_3 w_1| + |b_2||w_2| + |w_3|, \\ &\le 4 + 3x_1 + 2x_2 + x_3, \end{aligned}$$

and so

$$0 \le x_1 \le 1, \quad x_2 \le 1 - x_1^2, \quad x_3 \le 1 - x_1^2 - x_2^2.$$

We therefore need to find

$$\max_H g(x_1, x_2, x_3), \tag{4.5}$$

where $g(x_1, x_2, x_3) = 4 + 3x_1 + 2x_2 + x_3$, and

$$H = \{(x_1, x_2, x_3) : \; x_1 \le 1, \quad x_2 \le 1 - x_1^2, \quad x_3 \le 1 - x_1^2 - x_2^2\}.$$

It is clear that

$$\max_H g(x_1, x_2, x_3) = \max_{\partial H} g(x_1, x_2, x_3).$$

Hence we consider (4.5) on the boundary ∂H. If $x_3 = 1 - x_1^2 - x_2^2$, and $x_2 = 1 - x_1^2$, then

$$g(x_1, x_2, x_3) = 6 + 3x_1 - x_1^2 - x_1^4, \; 0 \le x_1 \le 1.$$

Solving this equation (using Wolfram Alpha), we obtain

$$\max\{6 + 3x_1 - x_1^2 - x_1^4 : \; 0 \le x_1 \le 1\} = 7.3731\ldots \text{ at } x_1 = 0.72808\ldots,$$

where

$$7.3731\ldots = \frac{1}{24}\left\{148 - \frac{968}{\sqrt[3]{54181 + 2259\sqrt{753}}} + \sqrt[3]{54181 + 2259\sqrt{753}}\right\},$$

$$0.72808\ldots = \frac{\sqrt[3]{27 + \sqrt{753}}}{2\sqrt{39}} - \frac{1}{\sqrt[3]{3(27 + \sqrt{753})}}.$$

Hence

$$|a_4| \le \frac{7.3731\ldots}{4} = 1.8443\ldots.$$

Applying the same method for a_5 gives

$$5|a_5| \le 8 + \frac{3}{\sqrt[3]{4}} = 9.889\ldots, \text{and so} \quad |a_5| \le 1.97\ldots.$$

The inequalities for a_2 and a_3 are sharp when

$$f(z) = \frac{2z}{1-z} + \log(1-z) = z + \sum_{n=2}^{\infty} \frac{2n-1}{n} z^n.$$

□

Inequalities for the coefficients of close-to-convex functions can exhibit unpredictable behaviour (see e.g the solution to the Fekete-Szegö problem [4]). On the basis of the extremal function for the coefficients a_2 and a_3 above, the obvious conjecture is the following, which may prove not to be correct.

Conjecture.

Let $f \in \mathcal{K}_u$, and be given by (1.1). Then for $n \ge 2$,

$$|a_n| \le \frac{2n-1}{n}.$$

Remark 2.

It is clear that other non-sharp bounds for $|a_n|$ when $n \geq 5$ can be obtained using the same techniques used in the proof of Theorem 4.1. However the analysis becomes more involved as n increases, and requires computer aided numerical methods.

We also note that the coefficients a_n are bounded. To see this, it follows from (4.1) and the Schwarz Lemma that, with $z = re^{i\theta} \in \mathbb{D}$,

$$\begin{aligned} n|a_n| &\leq \frac{1}{2\pi r^n} \int_0^{2\pi} |zf'(z)| \, d\theta \\ &\leq \frac{1}{\pi r^n} \int_0^{2\pi} |g(z)| \, d\theta. \end{aligned}$$

Since $\int_0^{2\pi} |g(z)| \, d\theta = \mathcal{O}(1-r)^{-1}$, as $r \to 1$ for $g \in \mathcal{S}^*$, choosing $r = 1 - 1/n$ shows that $a_n = \mathcal{O}(1)$ as $n \to \infty$.

5. Fekete-Szegö Theorems

We first give the following bounds for Fekete-Szegö functonal, noting that not all the inequalities are shown to be sharp.

Theorem 5.1. *Let $f \in \mathcal{K}_u$, and be given by* (1.1)*, and let $\mu \in \mathbb{R}$.*

If $\mu \leq 0$, then

(5.1) $$|a_3 - \mu a_2^2| \leq \frac{5}{3} - \frac{9}{4}\mu.$$

If $0 \leq \mu \leq 2/3$, then

$$|a_3 - \mu a_2^2| \leq \frac{2(10 - 18\mu + 9\mu^2)}{3(4 - 3\mu)}.$$

If $2/3 \leq \mu \leq 1$, then

(5.2) $$|a_3 - \mu a_2^2| \leq \frac{2}{3}.$$

If $1 \leq \mu \leq 10/9$, then

(5.3) $$|a_3 - \mu a_2^2| \leq \frac{3\mu - 5}{3(3\mu - 4)}.$$

If $\mu \geq 10/9$, then

(5.4) $$|a_3 - \mu a_2^2| \leq \frac{9}{4}\mu - \frac{5}{3}.$$

Inequalities (5.1), (5.2) *and* (5.4) *are sharp.*

Proof. Since $f \in \mathcal{K}_u$, we can write

$$zf'(z) = g(z)\left[\frac{2p(z)}{1+p(z)}\right], \tag{5.5}$$

where $p \in \mathcal{P}$ and $g \in \mathcal{S}^*$. Thus equating coefficients in (5.5) we obtain from (1.2) and (2.4) the following two alternative expressions:

$$a_3 - \mu a_2^2 = \frac{1}{3}\left(b_3 - \frac{3b_2^2\mu}{4}\right) + \frac{b_2p_1}{12}(2-3\mu) + \frac{1}{6}\left(p_2 - \frac{p_1^2}{2}\right) - \frac{1}{16}p_1^2\mu, \tag{5.6}$$

$$a_3 - \mu a_2^2 = \frac{1}{3}\left(b_3 - \frac{3b_2^2\mu}{4}\right) + \frac{b_2p_1}{12}(2-3\mu) + \frac{1}{6}\Big(p_2 - \frac{4+3\mu}{8}p_1^2\Big). \tag{5.7}$$

We now treat the following cases.

Case 1. $\mu \le 0$.

We use (5.6) with $|p_1| = x$. Noting that $|b_2| \le 2$, and using Lemma 2.4 and Lemma 2.5, we obtain from (5.6)

$$\begin{aligned} |a_3 - \mu a_2^2| &= \left|\frac{1}{3}\left(b_3 - \frac{3b_2^2\mu}{4}\right) + \frac{b_2p_1}{12}(2-3\mu) + \frac{1}{6}\left(p_2 - \frac{p_1^2}{2}\right) - \frac{1}{16}p_1^2\mu\right| \\ &\le \frac{1}{3}|3\mu - 3| + \frac{1}{6}|2-3\mu|\,x + \frac{1}{6}\left|2 - \frac{x^2}{2}\right| - \frac{1}{16}x^2\mu \\ &= \frac{1}{3}(3-3\mu) + \frac{1}{6}(2-3\mu)x + \frac{1}{6}\left(2 - \frac{x^2}{2}\right) - \frac{1}{16}x^2\mu, \end{aligned} \tag{5.8}$$

where $x \in [0,2]$. Since the right hand side of (5.8) increases with respect to $x \in [0,2]$, we obtain

$$\begin{aligned} |a_3 - \mu a_2^2| &\le \left[\frac{1}{3}(3-3\mu) + \frac{1}{6}(2-3\mu)x + \frac{1}{6}\left(2 - \frac{x^2}{2}\right) - \frac{1}{16}x^2\mu\right]_{x=2} \\ &= \frac{5}{3} - \frac{9\mu}{4}. \end{aligned}$$

The result is sharp on choosing $b_3 = 3, b_2 = p_1 = p_2 = 2$ in (5.6), i.e. $g(z) = z/(1-z)^2$, $p(z) = (1+z)/(1-z)$.

Case 2. $0 \le \mu \le 2/3$.

We again use (5.6) with $x = |p_1|$ which gives

$$|a_3 - \mu a_2^2| \le \frac{1}{3}(3-3\mu) + \frac{1}{6}(2-3\mu)x + \frac{1}{6}\left(2 - \frac{x^2}{2}\right) + \frac{1}{16}x^2\mu.$$

Since the above expression has a maximum value at $x = 4(3\mu-2)/(3\mu-4)$ in $[0,2]$, the bound for $0 \le \mu \le 2/3$ follows.

Case 3. $2/3 \le \mu \le 1$.

We apply (2.5) and (2.6) in (5.7) to obtain

$$
\begin{aligned}
|a_3-\mu a_2^2| &\leq \frac{1}{3}\left(1+(|3\mu-3|-1)\frac{|b_2|^2}{4}\right)+\frac{|b_2p_1|}{12}\,|2-3\mu| \\
&\quad +\frac{1}{6}\left(2+\left(\left|\frac{4+3\mu}{4}-1\right|-1\right)\frac{|p_1|^2}{2}\right) \\
&\leq \frac{1}{3}\left(1-\frac{3\mu-2}{4}|b_2|^2\right)+\frac{3\mu-2}{12}|p_1||b_2|+\frac{1}{6}\left(2-\frac{4-3\mu}{4}\frac{|p_1|^2}{2}\right) \\
&= -\frac{3\mu-2}{12}|b_2|^2+\frac{3\mu-2}{12}|p_1||b_2|-\frac{4-3\mu}{48}|p_1|^2+\frac{2}{3} \qquad (5.9)\\
&= \frac{3\mu-2}{12}\left(-y^2+xy-\frac{4-3\mu}{4(3\mu-2)}x^2\right)+\frac{2}{3},
\end{aligned}
$$

where $y=|b_2|\in[0,2]$, $x=|p_1|\in[0,2]$.

If $\mu=2/3$ then (5.2) follows at once from (5.9).

If $\mu\neq 2/3$, we dividing by $3\mu-2$, so that it suffices to show that

$$F(x,y)=-y^2+xy-\frac{4-3\mu}{4(3\mu-2)}x^2\leq 0$$

for all $2/3<\mu\leq 1$, $y\in[0,2]$ and $x\in[0,2]$.

Noting that $F(x,y)$ has no critical points in $(0,2)\times(0,2)$, we need only to check that $F(x,y)\leq 0$ when $x=0$ or $y=0$, which is trivial, and when $x=2$ or $y=2$.

If $x=2$, we have

$$F(2,y)=-y^2+2y-\frac{4-3\mu}{3\mu-2}=-(y-1)^2-\frac{6(1-\mu)}{3\mu-2}\leq 0,\quad \text{when}\quad 2/3<\mu\leq 1,$$

and if $y=2$, then

$$F(x,2)=-2(2-x)-\frac{4-3\mu}{4(3\mu-2)}x^2\leq 0,\quad \text{when}\quad 2/3<\mu\leq 1,$$

which establishes (5.2).

To show the result is sharp we choose $b_2=0$, $b_3=1$, $p_1=0$ and $p_2=2$ in (5.7), i.e. $g(z)=z/(1-z^2)$, $p(z)=(1+z^2)/(1-z^2)$.

Case 4. $1\leq\mu\leq 10/9$.

Applying (2.5) and (2.6) in (5.7) gives for all $\mu \geq 1$,

$$
\begin{aligned}
|a_3 - \mu a_2^2| &\leq \frac{1}{3}\left(1 + (|3\mu - 3| - 1)\frac{|b_2|^2}{4}\right) + \frac{|b_2 p_1|}{12}\,|2 - 3\mu| \\
&\qquad + \frac{1}{6}\left(2 + \left(\left|\frac{4+3\mu}{4} - 1\right| - 1\right)\frac{|p_1|^2}{2}\right) \\
&\leq \frac{1}{3}\left(1 - \frac{4-3\mu}{4}|b_2|^2\right) + \frac{3\mu - 2}{12}|p_1||b_2| + \frac{1}{6}\left(2 - \frac{4-3}{4}\frac{|p_1|^2}{2}\right) \\
&= -\frac{4-3\mu}{12}|b_2|^2 + \frac{3\mu-2}{12}|p_1||b_2| - \frac{4-3\mu}{48}|p_1|^2 + \frac{2}{3} \\
&= \frac{4-3\mu}{48}\left(-4y^2 + \frac{4(3\mu-2)}{4-3\mu}xy - x^2\right) + \frac{2}{3} := F(x,y),
\end{aligned}
$$

where $y = |b_2| \in [0,2]$, $x = |p_1| \in [0,2]$.

Thus to show (5.3) it suffices to establish that

$$
F(x,y) = \frac{2}{3} + \frac{4-3\mu}{48}\left(-4y^2 + \frac{4(3\mu-2)}{4-3\mu}xy - x^2\right) \leq \frac{2}{3} + \frac{\mu - 1}{4 - 3\mu} \tag{5.10}
$$

for all $1 \leq \mu \leq 10/9$, $y \in [0,2]$ and $x \in [0,2]$.

Again we notice that $F(x,y)$ has no critical points in $(0,2)\times(0,2)$. Hence we need only to check $F(x,y) \leq 0$ when $x = 0$ or $y = 0$, and when $x = 2$ or $y = 2$. It is clear from (5.10) that in these four cases $F(x,y)$ attains the greatest value when $x = 2$. Then

$$
F(2,y) = \frac{2}{3} + \frac{4-3\mu}{48}\left(-4y^2 + \frac{8(3\mu-2)}{4-3\mu}y - 4\right),
$$

and

$$
\begin{aligned}
\max_{0\leq y\leq 2} F(2,y) &= \frac{2}{3} + \left[\frac{4-3\mu}{48}\left(-4y^2 + \frac{8(3\mu-2)}{4-3\mu}y - 4\right)\right]_{y=(3\mu-2)/(4-3\mu)} \\
&= \frac{2}{3} + \frac{\mu-1}{4-3\mu} = \frac{3\mu-5}{3(3\mu-4)}.
\end{aligned}
$$

This gives (5.3).

Case 5. $\mu \geq 10/9$.

From (5.7) we obtain with $x = |p_1|$ and $y = |b_2|$,

$$
\begin{aligned}
|a_3 - \mu a_2^2| &\leq \frac{1}{3}\left(1 + (3\mu - 4)\frac{y^2}{4}\right) + \frac{xy}{12}(3\mu - 2) + \frac{1}{6}\left(2 - \frac{x^2}{2}\right) + \frac{\mu x^2}{16} \\
&:= H(x,y).
\end{aligned}
$$

Since the only critical point of $H(x,y)$ is when $x = y = 0$, and $H(0,0) = 2/3$, we need only to check the end points of $H(x,y)$ on $[0,2]\times[0,2]$. First $H(0,y) = 1/3 + 1/3(1 + 1/4(3\mu-4)y^2 \leq (3\mu-2)/3 \leq 9\mu/4 - 5/3$ when $\mu \geq 10/9$ and $0 \leq y \leq 2$.

Next $H(2,y) = \mu/4 + (3\mu-2)y/6 + 1/3(1+(3\mu-4)y^2/4)$, which increases on $y \in [0,2]$, and so $H(2,y) \le 9\mu/4 - 5/3$ again.

Next $H(x,0) = 2/3 + (3m-4)x^2/48$. Then $H'(x,0) = 0$ when either $x = 0$ or $\mu = 4/3$. Since $H(4/3,0) = 2/3 \le \mu/4 - 5/3$, we need only consider the cases $x = 0$, and $x = 2$, However since $H(0,0)$ is again $2/3$, and $H(2,0) = 1/3 + \mu/4 \le 9\mu/4 - 5/3$ the result follows in this case.

Finally $H(x,2) = (3\mu-3)/3 + (3\mu-2)x/6 + \mu x^2/16 + (2-x^2/2)/6$, which increases for $x \in [0,2]$ when $\mu \ge 10/9$. Since $H(2,2) = 9\mu/4 - 5/3$, the proof is complete.

The result is sharp on choosing $b_3 = 3, b_2 = p_1 = p_2 = 2$ in (5.8), i.e. $g(z) = z/(1-z)^2$, $p(z) = (1+z)/(1-z)$. □

The following Fekete-Szegö theorem for complex μ is probably not sharp.

Theorem 5.2. *Let $f \in \mathcal{K}_u$ and be given by* (1.1). *Then if $\mu \in \mathbb{C}$*

$$|a_3 - \mu a_2^2| \le \frac{1}{3}\left[\max\{1, |4\mu_1 - 3|\} + \max\{1, |2\mu_2 - 1|\} + |2 - 3\mu|\right], \tag{5.11}$$

where

$$\mu_1 = \frac{3\mu}{4}, \quad \mu_2 = \frac{4+3\mu}{8}.$$

Proof. From (5.7), we obtain

$$\begin{aligned} |a_3 - \mu a_2^2| &\le \frac{1}{3}\left|b_3 - \frac{3\mu}{4}b_2^2\right| + \frac{1}{12}|b_2 p_1|\,|2-3\mu| + \frac{1}{6}\left|p_2 - \frac{4+3\mu}{8}p_1^2\right| \\ &\le \frac{1}{3}\left|b_3 - \frac{3\mu}{4}b_2^2\right| + \frac{1}{3}|2-3\mu| + \frac{1}{6}\left|p_2 - \frac{4+3\mu}{8}p_1^2\right|. \end{aligned}$$

Applying Lemma 2.3 and Lemma 2.5 gives (5.11). □

6. The radius of convexity

We first recall the well-known condition that f maps $\mathbb{D}$ onto a convex domain if, and only if, $f \in \mathcal{A}$ and

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in \mathbb{D}.$$

A number $r_0 \in [0,1]$, is called the radius of convexity for a particular subclass of $\mathcal{A}$, if r_0 is the largest number such that

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$$

for all f in the subclass, and $|z| < r_0$. It was shown in [6] that the radius of convexity for functions in $\mathcal{S}_u^*$ is $(\sqrt{13}-3)/2$. We now show that when $f \in \mathcal{K}_u$, the radius of convexity is $(3-\sqrt{5})/2$.

Theorem 6.1. *The radius of convexity for $\mathcal{K}_u$ is*

$$r_0 = \frac{3-\sqrt{5}}{2} = 0.381966\ldots.$$

Proof. Since $f \in \mathcal{K}_u$, we write

$$zf'(z) = g(z)[1+\omega(z)],$$

for some $g \in \mathcal{S}^*$, and some $\omega \in \Omega$. Thus

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zg'(z)}{g(z)} + \frac{z\omega'(z)}{1+\omega(z)}. \tag{6.1}$$

It is well-known (see e.g. [7]), that for $g \in \mathcal{S}^*$, with $z = re^{i\theta}$, $0 \le r < 1$, then

$$\mathfrak{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \ge \frac{1-r}{1+r}.$$

Also from the Schwarz Lemma, $|w(z)| \le |z| = r$, and from [3, p.77],

$$|\omega'(z)| \le \frac{1-|\omega(z)|^2}{1-|z|^2} = \frac{1-|\omega(z)|^2}{1-r^2}.$$

Thus from (6.1), for $z = re^{i\theta}$, $0 \le r < 1$, we obtain

$$\begin{aligned}
\mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} &\ge \mathfrak{Re}\left\{\frac{zg'(z)}{g(z)}\right\} - \left|\frac{z\omega'(z)}{1+\omega(z)}\right| \\
&\ge \frac{1-r}{1+r} - \frac{r}{1-|\omega(z)|}|\omega'(z)| \\
&\ge \frac{1-r}{1+r} - \frac{r}{1-|\omega(z)|}\frac{1-|\omega(z)|^2}{1-r^2} \\
&= \frac{1-r}{1+r} - \frac{r(1+|\omega(z)|)}{1-r^2} \\
&\ge \frac{1-r}{1+r} - \frac{r(1+r)}{1-r^2} \\
&= \frac{1-3r+r^2}{1-r^2} > 0,
\end{aligned}$$

when $r \in [0, (3-\sqrt{5})/2)$. Thus the radius of convexity for the class $\mathcal{K}_u$ is at least $(3-\sqrt{5})/2$.

To see that this is the largest such radius, consider the function $f_0 \in \mathcal{K}_u$ defined by

$$f_0'(z) = g_0'(z)[1+\omega_0(z)], \quad g_0(z) = \frac{z}{(1-z)^2}, \quad \omega_0(z) = z.$$

Then

$$\begin{aligned}\left\{1+\frac{zf_0''(z)}{f_0'(z)}\right\}_{z=-r_0} &= \left\{\frac{zg_0'(z)}{g_0(z)}+\frac{z\omega_0'(z)}{1+\omega_0(z)}\right\}_{z=-r_0}\\ &= \left\{\frac{1+z}{1-z}+\frac{z}{1+z}\right\}_{z=-r_0}\\ &= \frac{1-3r_0+r_0^2}{1-r_0^2}\\ &= 0,\end{aligned}$$

which shows that the radius of convexity in the class $\mathcal{K}_u$ cannot be larger than r_0. □

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