Large deviations for neutral stochastic functional differential equations

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Abstract

In this paper, under a one-sided Lipschitz condition on the drift coefficient we adopt (via contraction principle) an exponential approximation argument to investigate large deviations for neutral stochastic functional differential equations.

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1 Introduction

As is well known, large deviation principle (LDP for short) is a branch of probability theory that deals with the asymptotic behaviour of rare events, and it has a wide range of applications, such as in mathematical finance, statistic mechanics, biology and so on. So the LDP for SDEs has been investigated extensively; see, e.g., [1, 2, 16] and references therein.

From the literature, we know there are two main methods to investigate the LDPs, one method is based on contraction principle in LDPs, that is, it relies on approximation arguments and exponential-type probability estimates; see e.g.,[3, 9, 10, 11, 12, 13, 16, 17] and references therein. [9, 13, 17, 19] concerned about the LDP for SDEs driven by Brownian motion or Poisson measure, [10] investigated the LDP for invariant distributions of memory gradient diffusions. [11] investigated how rapid-switching behaviour of solution X_t^ϵ affects the small-noise asymptotics of X_t^ϵ-modulated diffusion processes on the certain interval.

The other one is weak convergence method, which has also been applied in establishing LDPs for a various stochastic dynamic systems; see e.g.,[1, 2, 4, 5, 6, 7]. According to the compactness argument in this method of the solution space of corresponding skeleton equation, the weak convergence is done for Borel measurable functions whose existence is based on Yamada-Watanabe theorem. In [4, 5, 7], the authors study an LDP for SDEs/SPDEs.

Compared with the weak convergence method, there are few literature about the LDP for SFDEs, [16] gave result about LDP for SDEs with point delay, and large deviations for perturbed reflected diffusion processes was investigated in [3]. The aim of this paper is to study the LDP for neutral stochastic functional differential equations (NSFDEs), which extends the result in [16].

The structure of this paper is as follows. In section 2, we introduce some preliminary results and notation. In section 3, we state the main results about LDP for NSFDEs and give the corresponding proofs.

Before giving the preliminaries, a few words about the notation are in order. Throughout this paper, $C > 0$ stipulates a generic constant, which might change from line to line and depend on the time parameters.

2 Preliminaries

Let $(\mathbb{R}^d, \langle \cdot, \cdot \rangle, |\cdot|)$ be the d-dimensional Euclidean space with the inner product $\langle \cdot, \cdot \rangle$ which induces the norm $|\cdot|$. Let $\mathbb{M}^{d\times d}$ denote the set of all $d \times d$ matrices, which is equipped with the Hilbert-Schimidt norm $\|\cdot\|_{HS}$. A^* stands for the transpose of the matrix A. For a sub-interval $\mathbb{U} \subseteq \mathbb{R}$, $C(\mathbb{U}; \mathbb{R}^d)$ means the family of all continuous functions $f : \mathbb{U} \to \mathbb{R}^d$. Let $\tau > 0$ be a fixed number and $\mathscr{C} = C([-\tau, 0]; \mathbb{R}^d)$, endowed with the uniform norm $\|f\|_\infty := \sup_{-\tau\leq\theta\leq 0} |f(\theta)|$. For fixed $t \geq 0$, let $f_t \in \mathscr{C}$ be defined by $f_t(\theta) = f(t + \theta), \theta \in [-\tau, 0]$. In terminology, $(f_t)_{t\geq 0}$ is called the segment (or window) process corresponding to $(f(t))_{t\geq -\tau}$.

In this paper, we are interested in the following NSFDE

$$\mathrm{d}\{X^\epsilon(t) - G(X_t^\epsilon)\} = b(X_t^\epsilon)\mathrm{d}t + \sqrt{\epsilon}\sigma(X_t^\epsilon)\mathrm{d}W(t), \quad t \in [0, T], \quad X_0^\epsilon = \xi \in \mathscr{C}, \tag{2.1}$$

where $G, b : \mathscr{C} \to \mathbb{R}^d$, $\sigma : \mathscr{C} \to \mathbb{R}^d \times \mathbb{R}^d$ and $\{W(t)\}_{t\geq 0}$ is a d-dimensional Brownian motion on some filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$.

The proofs of main results will be based on an extension of the contraction principle in [8, Theorem 4.2.23]. To make the content self-contained, we recall it as follows:

Lemma 2.1. *Let $\{\mu_\epsilon\}$ be a family of probability measures that satisfies the LDP with a good rate function I on a Hausdorff topological space $\mathcal{X}$, and for $m = 1, 2, \cdots$, let $f_m : \mathcal{X} \to \mathcal{Y}$ be continuous functions, with $(\mathcal{Y}, d)$ a metric space. Assume there exists a measurable map $f : \mathcal{X} \to \mathcal{Y}$ such that for every $\alpha < \infty$,*

$$\limsup_{m\to\infty} \sup_{\{x:I(x)\leq\alpha\}} d(f_m(x), f(x)) = 0. \tag{2.2}$$

Then any family of probability measures $\{\widetilde{\mu}_\epsilon\}$ for which $\{\mu_\epsilon \circ f_m^{-1}\}$ are exponentially good approximations satisfies the LDP in $\mathcal{Y}$ with the good rate function $I'(y) = \inf\{I(x) : y = f(x)\}$.

We now state the classical exponential inequality for stochastic integral, which is crucial in proving the exponential approximation. For more details, please refer to Stroock [18, lemma 4.7].

Lemma 2.2. *Let* $\alpha : [0,\infty)\times\Omega \to \mathbb{R}^d\times\mathbb{R}^d$ *and* $\beta : [0,\infty)\times\Omega \to \mathbb{R}^d$ *be* $(\mathscr{F}_t)_{t\geq 0}$*-progressively measurable processes. Assume that* $\|\alpha(\cdot)\|_{HS} \leq A$ *and* $|\beta| \leq B$*. Set* $\xi(t) := \int_0^t \alpha(s)\mathrm{d}W(s) + \int_0^t \beta(s)\mathrm{d}s$ *for* $t \geq 0$*. Let* $T > 0$ *and* $R > 0$ *satisfy* $d^{\frac{1}{2}}BT < R$*. Then*

$$P\Big(\sup_{0\leq t\leq T} |\xi(t)| \geq R\Big) \leq 2d\exp\Big(\frac{-(R-d^{\frac{1}{2}}BT)^2}{2A^2dT}\Big). \tag{2.3}$$

3 LDP for NSFDE

Let H denote the Cameron-Martin space, i.e.

$$H = \Big\{h(t) = \int_0^t \dot{h}(s)\mathrm{d}s : [0,T] \to \mathbb{R}^d; \int_0^T |\dot{h}(s)|^2\mathrm{d}s < +\infty\Big\},$$

which is a Hilbert space endowed with the inner product as follows:

$$\langle f, g\rangle_H = \int_0^T \dot{f}(s)\dot{g}(s)\mathrm{d}s.$$

We define

$$L_T(h) = \begin{cases} \frac{1}{2}\int_0^T |\dot{h}(t)|^2\mathrm{d}t, & \text{if } \ h \in H, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

The well-known Schilder theorem (see [8]) states that the laws μ_ϵ of $\{\sqrt{\epsilon}W(t)\}_{t\in[0,T]}$ satisfies the LDP on $C([0,T];\mathbb{R}^d)$ with the rate function $L_T(\cdot)$.

To investigate the LDP for the laws of $\{X^\epsilon(t)\}_{t\in[-\tau,T]}$, we give the following assumptions about coefficients.

(**H1**) There exists a constant $L > 0$ such that

$$2\langle \xi(0) - \eta(0) + G(\eta) - G(\xi), b(\xi) - b(\eta)\rangle \leq L\|\xi - \eta\|_\infty^2,$$

and

$$\|\sigma(\xi) - \sigma(\eta)\|_{HS}^2 \leq L\|\xi - \eta\|_\infty^2, \quad \xi, \eta \in \mathscr{C};$$

(H2) There exists a constant $\kappa \in (0,1)$ such that

$$|G(\xi) - G(\eta)| \leq \kappa\|\xi - \eta\|_\infty,\ \xi, \eta \in \mathscr{C}; \tag{3.2}$$

(**H3**) There exists a constant $M > 0$ such that

$$|b(\xi)| \vee \|\sigma(\xi)\|_{HS} \leq M, \forall \xi \in \mathscr{C}.$$

Remark 3.1. The one-sided Lipschitz condition on the drift coefficient in (**H1**) is different from the global Lipschitz condition in [2]. Moreover, our method below is different from that of [2].

Remark 3.2. From (**H1**), (**H2**), it is easy to see that

$$2\langle \xi(0) - G(\xi), b(\xi)\rangle \leq L_2(1 + \|\xi\|_\infty^2), \quad |G(\xi)|^2 \leq \kappa^2\|\xi\|_\infty^2, \ \xi \in \mathscr{C}. \tag{3.3}$$

Remark 3.3. Let $\mu(\mathrm{d}\theta) \in \mathscr{P}([-\tau, 0])$ and let

$$\begin{aligned} G(\xi) &= \alpha_1 \int_{-\tau}^0 \xi(\theta)\mu(\mathrm{d}\theta), \quad \sigma(\xi) = \alpha_2 \int_{-\tau}^0 \xi(\theta)\mu(\mathrm{d}\theta), \\ b(\xi) &= -\alpha_3\xi(0) - \alpha_4\Big(\xi(0) - \alpha_1 \int_{-\tau}^0 \xi(\theta)\mu(\mathrm{d}\theta)\Big)^{1/3} + \alpha_5 \int_{-\tau}^0 \xi(\theta)\mu(\mathrm{d}\theta), \end{aligned}$$

for some constants $\alpha_i, i = 1, \cdots, 5$ such that $\alpha_1 \leq \kappa$, $\Big(\alpha_3(\alpha_1 - 1) + \alpha_5(1 + \alpha_1)\Big) \vee \alpha_2^2 \leq L$, then the assumptions (**H1**) and (**H2**) hold true. In fact, by the Hölder inequality, one has

$$|G(\xi) - G(\eta)|^2 \leq \alpha_1^2 \int_{-\tau}^0 |\xi(\theta) - \eta(\theta)|^2\mu(\mathrm{d}\theta) \leq \alpha_1^2\|\xi - \eta\|_\infty^2 \int_{-\tau}^0 \mu(\mathrm{d}\theta) = \alpha_1^2\|\xi - \eta\|_\infty^2,$$

noting that

$$-\alpha_4\langle \xi(0) - \eta(0) - (G(\xi) - G(\eta)), (\xi(0) - G(\xi))^{1/3} - (\eta(0) - G(\eta))^{1/3}\rangle \leq 0,$$

so

$$\begin{aligned} &\langle \xi(0) - \eta(0) - (G(\xi) - G(\eta)), b(\xi) - b(\eta)\rangle \\ &\leq -\alpha_3|\xi(0) - \eta(0)|^2 + \alpha_3|\xi(0) - \eta(0)||G(\xi) - G(\eta)| \\ &\quad + \alpha_5|\xi(0) - \eta(0)| \int_{-\tau}^0 |\xi(\theta) - \eta(\theta)|\mu(\mathrm{d}\theta) - \alpha_5|G(\xi) - G(\eta)| \int_{-\tau}^0 |\xi(\theta) - \eta(\theta)|\mu(\mathrm{d}\theta) \\ &\leq \alpha_3(\alpha_1 - 1) + \alpha_5(1 + \alpha_1)\|\xi - \eta\|_\infty^2, \\ &\|\sigma(\xi) - \sigma(\eta)\|_{HS}^2 \leq \alpha_2^2 \int_{-\tau}^0 |\xi(\theta) - \eta(\theta)|^2\mu(\mathrm{d}\theta) \leq \alpha_2^2\|\xi - \eta\|_\infty^2. \end{aligned}$$

Therefore, the assumptions hold if the constants $\alpha_i, i = 1, \ldots, 5$ satisfy the conditions above.

Let $F(h)$ be the unique solution of the following deterministic equation:

$$\begin{cases} F(h)(t) - G(F_t(h)) = F(h)(0) - G(F_0(h)) + \int_0^t b\Big(F_s(h)\Big)\mathrm{d}s \\ \qquad\qquad + \int_0^t \sigma\Big(F_s(h)\Big)\dot{h}(s)\mathrm{d}s, \quad t \in [0, T], \\ F_0(h)(\theta) = \xi(\theta), \quad \theta \in [-\tau, 0]. \end{cases} \tag{3.4}$$

Herein, $F_t(h)(\theta) = F(h)(t + \theta)$, $\theta \in [-\tau, 0]$.

The first main result of this section is stated as follows.

Theorem 3.1. *Under the assumptions (***H1***)-(***H3***), it holds that* $\{\mu_\epsilon, \epsilon > 0\}$*, the law of* $X^\epsilon(\cdot)$ *on* $C([-\tau,T];\mathbb{R}^d)$*, satisfies the LDP with the rate function below*

$$I(f) := \inf\Big\{L_T(h); F(h) = f, h \in H\Big\}, \quad f \in C([-\tau,T];\mathbb{R}^d), \tag{3.5}$$

where $L_T(h)$ *is defined as in* (3.1)*. That is,*

(i) for any closed subset $C \subset C([-\tau,T];\mathbb{R}^d)$*,*

$$\limsup_{\epsilon\to 0} \log \mu_\epsilon(C) \le -\inf_{f\in C} I(f),$$

(ii) for any open subset $G \subset C([-\tau,T];\mathbb{R}^d)$*,*

$$\liminf_{\epsilon\to 0} \log \mu_\epsilon(G) \ge -\inf_{f\in G} I(f).$$

We can extend the result of Theorem 3.1 to the case that b, σ are not necessary to satisfy the bounded condition (**H3**). We only assume b is locally Lipschitz with polynomial growth, that is, there exist constants $L > 0, q \in \mathbb{N}$, such that $\forall \xi, \eta \in \mathscr{C}$, we have

$$|b(\xi) - b(\eta)| \le L(\|\xi\|_\infty^q + \|\eta\|_\infty^q)\|\xi - \eta\|_\infty. \tag{3.6}$$

Let $\mathbf{0}$ denote the function such that $\mathbf{0}(\theta) = 0, \theta \in [-\tau, 0]$. We can see that b is polynomial growth

$$|b(\xi)| \le |b(\xi) - b(\mathbf{0}) + b(\mathbf{0})| \le L\|\xi\|_\infty^{q+1} + |b(\mathbf{0})| \le \widehat{L}(\|\xi\|_\infty^{q+1} + 1),$$

where $\widehat{L} = \max\{L, |b(\mathbf{0})|\}$.

we state the second result as follows.

Theorem 3.2. *Under the assumptions (***H1***), (***H2***) and* (3.6)*, it holds that* $\{\mu_\epsilon, \epsilon > 0\}$*, the law of* $X^\epsilon(\cdot)$ *on* $C([-\tau,T];\mathbb{R}^d)$*, satisfies the large deviation principle with the rate function below*

$$I(f) := \inf\Big\{L_T(h); F(h) = f, h \in H\Big\}, \quad f \in C([-\tau,T];\mathbb{R}^d), \tag{3.7}$$

where $L_T(h)$ *is defined as in* (3.1)*.*

In the sequel, we first finish the proof of Theorem 3.1.

Before giving the proof of Theorem 3.1, we prepare some lemmas.

We construct $X^{\epsilon,n}(\cdot)$ by exploiting an approximate scheme, that is, for a real positive number s, let $[s] = \sup\{k \in \mathbb{Z} : k \le s\}$ be its integer part. For any $n \in N_0$, we consider the following NSFDE

$$\mathrm{d}\{X^{\epsilon,n}(t) - G(X_t^{\epsilon,n})\} = b(X_t^{\epsilon,n})\mathrm{d}t + \sqrt{\epsilon}\sigma(\widehat{X}_t^{\epsilon,n})\mathrm{d}W(t), \quad t \ge 0, \quad X_0^{\epsilon,n} = \xi, \tag{3.8}$$

where, for $t \ge 0$,

$$\widehat{X}_t^{\epsilon,n}(\theta) := X^{\epsilon,n}((t+\theta) \wedge t_n), \quad t_n := [nt]/n, n \ge 1, \theta \in [-\tau, 0].$$

According to [14, Theorem 2.2, p.204], (3.8) has a unique solution by solving piece-wisely with the time length $1/n$.

Next, we show that $\{X^{\epsilon,n}, \epsilon > 0\}$ defined by (3.8) approximates $\{X^\epsilon, \epsilon > 0\}$.

Lemma 3.3. *Assume* (**H1**), (**H2**), *and* (**H3**) *hold, then for any* $\delta > 0$, *one has*

$$\lim_{n\to\infty}\limsup_{\epsilon\to 0}\epsilon\log P\Big(\sup_{-\tau\le t\le T}|X^{\epsilon}(t)-X^{\epsilon,n}(t)|>\delta\Big)=-\infty. \tag{3.9}$$

Proof. For notation brevity, we set $Z^{\epsilon,n}(t):=X^{\epsilon}(t)-X^{\epsilon,n}(t)$ and $Y^{\epsilon,n}(t):=X^{\epsilon}(t)-X^{\epsilon,n}(t)-(G(X^{\epsilon}_t)-G(X^{\epsilon,n}_t)), t\ge 0$. Noting $X^{\epsilon,n}_0=X^{\epsilon}_0=\xi$, we write $Y^{\epsilon,n}(t)$ as follows:

$$Y^{\epsilon,n}(t)=\int_0^t(b(X^{\epsilon}_s)-b(X^{\epsilon,n}_s))\mathrm{d}s+\sqrt{\epsilon}\int_0^t(\sigma(X^{\epsilon}_s)-\sigma(\widehat{X}^{\epsilon,n}_s))\mathrm{d}W(s).$$

It is easy to see from (3.2) that

$$\begin{aligned}|Z^{\epsilon,n}(t)|&\le|Y^{\epsilon,n}(t)|+|G(X^{\epsilon}_t)-G(X^{\epsilon,n}_t)|\\&\le|Y^{\epsilon,n}(t)|+\kappa\|X^{\epsilon}_t-X^{\epsilon,n}_t\|_\infty,\end{aligned}$$

and

$$\sup_{0\le t\le T}|Z^{\epsilon,n}(t)|\le\frac{1}{1-\kappa}\sup_{0\le t\le T}|Y^{\epsilon,n}(t)|. \tag{3.10}$$

For $\rho>0$, we define $\tau^{\epsilon}_{n_\rho}=\inf\{t\ge 0:\|X^{\epsilon,n}_t-\widehat{X}^{\epsilon,n}_t\|_\infty>\rho\}$, $Z^{\epsilon,n_\rho}=Z^{\epsilon,n}(t\wedge\tau^{\epsilon}_{n_\rho})$, $\xi^{\epsilon}_{n_\rho}=\inf\{t\ge 0:|Z^{\epsilon,n_\rho}(t)|\ge\delta\}$, and compute

$$\begin{aligned}P\Big(\sup_{0\le t\le T}|Z^{\epsilon,n}(t)|>\delta\Big)&=P\Big(\sup_{0\le t\le T}|Z^{\epsilon,n}(t)|>\delta,\tau^{\epsilon}_{n_\rho}\le T\Big)+P\Big(\sup_{0\le t\le T}|Z^{\epsilon,n}(t)|>\delta,\tau^{\epsilon}_{n_\rho}>T\Big)\\&\le P(\tau^{\epsilon}_{n_\rho}\le T)+P\Big(\sup_{0\le t\le T}|Z^{\epsilon,n}(t)|>\delta,\tau^{\epsilon}_{n_\rho}>T\Big)\\&\le P(\tau^{\epsilon}_{n_\rho}\le T)+P(\xi^{\epsilon}_{n_\rho}\le T).\end{aligned} \tag{3.11}$$

Observe that

$$\begin{aligned}X^{\epsilon,n}_t(\theta)-\widehat{X}^{\epsilon,n}_t(\theta)&=X^{\epsilon,n}(t+\theta)-X^{\epsilon,n}((t+\theta)\wedge t_n)\\&=(X^{\epsilon,n}(t+\theta)-X^{\epsilon,n}(t+\theta))I_{\{(t+\theta)<t_n\}}+(X^{\epsilon,n}(t+\theta)-X^{\epsilon,n}(t_n))I_{\{t_n\le(t+\theta)\}}\\&=(X^{\epsilon,n}(t+\theta)-X^{\epsilon,n}(t_n))I_{\{t_n\le(t+\theta)\}}\\&=G(X^{\epsilon,n}_{t+\theta})-G(X^{\epsilon,n}_{t_n})+\Big(\int_{t_n}^{t+\theta}b(X^{\epsilon,n}_s)\mathrm{d}s+\int_{t_n}^{t+\theta}\sqrt{\epsilon}\sigma(\widehat{X}^{\epsilon,n}_s)\mathrm{d}W(s)\Big).\end{aligned}$$

This, together with (3.2), yields

$$\sup_{0\le t\le T}\|X^{\epsilon,n}_t-\widehat{X}^{\epsilon,n}_t\|_\infty\le\frac{1}{1-\kappa}\sup_{0\le t\le T}\sup_{t_n-t\le\theta\le 0}\Big|\int_{t_n}^{t+\theta}b(X^{\epsilon,n}_s)\mathrm{d}s+\int_{t_n}^{t+\theta}\sqrt{\epsilon}\sigma(\widehat{X}^{\epsilon,n}_s)\mathrm{d}W(s)\Big|. \tag{3.12}$$

Taking (**H3**) into consideration and utilizing Lemma 2.2, one gets that

$$P\Big(\sup_{0\le t\le T}\|X^{\epsilon,n}_t-\widehat{X}^{\epsilon,n}_t\|_\infty\ge\rho\Big)\le 2d\exp\Big(-\frac{(n\rho(1-\kappa)-\sqrt{\mathrm{d}}M)^2}{2nM^2(1-\kappa)^2d\epsilon}\Big),$$

provided that $\frac{\sqrt{d}M}{(1-\kappa)n} < \rho$. This, together with the definition of stopping time $\tau^{\epsilon}_{n_\rho}$, implies that

$$\lim_{n\to\infty}\limsup_{\epsilon\to 0}\epsilon\log P(\tau^{\epsilon}_{n_\rho}\le T) = -\infty. \tag{3.13}$$

For $\lambda > 0$, let $\phi_\lambda(y) = (\rho^2+|y|^2)^\lambda$, an application of Itô's formula yields

$$\phi_\lambda(Y^{\epsilon,n_\rho}(t)) = \rho^{2\lambda} + M^{\epsilon,n_\rho}(t) + \int_0^{t\wedge\tau^{\epsilon}_{n_\rho}}\gamma^{\epsilon}_\lambda(s)\mathrm{d}s, \tag{3.14}$$

where $M^{\epsilon,n_\rho}(t) := 2\lambda\int_0^{t\wedge\tau^{\epsilon}_{n_\rho}}(\rho^2+|Y^{\epsilon,n}(s)|^2)^{\lambda-1}\sqrt{\epsilon}\langle Y^{\epsilon,n}(s),\sigma(X^{\epsilon,n}_s)-\sigma(\widehat{X}^{\epsilon,n}_s)\mathrm{d}W(s)\rangle$ is a martingale. Moreover, by (**H1**), we see that

$$\begin{aligned}
\gamma^{\epsilon}_\lambda(s) :&= 2\lambda(\rho^2+|Y^{\epsilon,n}(s)|^2)^{\lambda-1}\langle Y^{\epsilon,n}(s), b(X^{\epsilon}_s)-b(X^{\epsilon,n}_s)\rangle\\
&\quad + 2\lambda(\lambda-1)\epsilon(\rho^2+|Y^{\epsilon,n}(s)|^2)^{\lambda-2}|(\sigma(X^{\epsilon}_s)-\sigma(\widehat{X}^{\epsilon,n}_s))^*Y^{\epsilon,n}(s)|^2\\
&\quad + \lambda\epsilon(\rho^2+|Y^{\epsilon,n}(s)|^2)^{\lambda-1}\|\sigma(X^{\epsilon}_s)-\sigma(\widehat{X}^{\epsilon,n}_s)\|^2_{HS}\\
&\le 2L\lambda(\rho^2+|Y^{\epsilon,n}(s)|^2)^{\lambda-1}\|Z^{\epsilon,n}_s\|^2_\infty + \lambda(2\lambda-1)\epsilon(\rho^2+|Y^{\epsilon,n}(s)|^2)^{\lambda-1}\|(\sigma(X^{\epsilon}_s)-\sigma(\widehat{X}^{\epsilon,n}_s))\|^2_{HS}\\
&\le C_1(\rho^2+|Y^{\epsilon,n}(s)|^2)^{\lambda-1}\|Z^{\epsilon,n}_s\|^2_\infty + C_2(\rho^2+|Y^{\epsilon,n}(s)|^2)^{\lambda-1}\|X^{\epsilon,n}_s-\widehat{X}^{\epsilon,n}_s\|^2_\infty,
\end{aligned} \tag{3.15}$$

where $C_1 = 2L\lambda[(2\lambda-1)\epsilon+1]$, $\quad C_2 = 2L\lambda\epsilon(2\lambda-1)$.

Using the Burkholder-Davis-Gundy (BDG for short) inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T}M^{\epsilon,n_\rho}(t)\Big)\\
&\le 8\sqrt{2\epsilon}\lambda\Big(\mathbb{E}\int_0^{T\wedge\tau^{\epsilon}_{n_\rho}}(\rho^2+|Y^{\epsilon,n}(s)|^2)^{2\lambda-2}|Y^{\epsilon,n}(s)|^2\|\sigma(X^{\epsilon}_s)-\sigma(\widehat{X}^{\epsilon,n}_s)\|^2_{HS}\mathrm{d}s\Big)^{\frac{1}{2}}\\
&\le \frac{1}{2}\mathbb{E}\Big(\sup_{0\le t\le T\wedge\tau^{\epsilon}_{n_\rho}}(\rho^2+|Y^{\epsilon,n}(s)|^2)^\lambda\Big) + 64\lambda^2\epsilon\mathbb{E}\int_0^{T\wedge\tau^{\epsilon}_{n_\rho}}(\rho^2+|Y^{\epsilon,n}(s)|^2)^{\lambda-1}\|\sigma(X^{\epsilon}_s)-\sigma(\widehat{X}^{\epsilon,n}_s)\|^2_{HS}\mathrm{d}s\\
&\le \frac{1}{2}\mathbb{E}\Big(\sup_{0\le t\le T\wedge\tau^{\epsilon}_{n_\rho}}(\rho^2+|Y^{\epsilon,n}(s)|^2)^\lambda\Big) + 128L\lambda^2\epsilon\mathbb{E}\int_0^{T\wedge\tau^{\epsilon}_{n_\rho}}(\rho^2+|Y^{\epsilon,n}(s)|^2)^{\lambda-1}\|Z^{\epsilon,n}_s\|^2_\infty\mathrm{d}s\\
&\quad + 128\lambda^2\epsilon\mathbb{E}\int_0^{T\wedge\tau^{\epsilon}_{n_\rho}}(\rho^2+|Y^{\epsilon,n}(s)|^2)^{\lambda-1}\|\sigma(X^{\epsilon,n}_s)-\sigma(\widehat{X}^{\epsilon,n}_s)\|^2_{HS}\mathrm{d}s\\
&\le \frac{1}{2}\mathbb{E}\Big(\sup_{0\le t\le T\wedge\tau^{\epsilon}_{n_\rho}}(\rho^2+|Y^{\epsilon,n}(s)|^2)^\lambda\Big) + 128L\lambda^2\epsilon\mathbb{E}\int_0^{T\wedge\tau^{\epsilon}_{n_\rho}}(\rho^2+|Y^{\epsilon,n}(s)|^2)^{\lambda-1}\|Z^{\epsilon,n}_s\|^2_\infty\mathrm{d}s\\
&\quad + 128L\lambda^2\epsilon\mathbb{E}\int_0^{T\wedge\tau^{\epsilon}_{n_\rho}}(\rho^2+|Y^{\epsilon,n}(s)|^2)^{\lambda-1}\|X^{\epsilon,n}_s-\widehat{X}^{\epsilon,n}_s\|^2_\infty\mathrm{d}s.
\end{aligned} \tag{3.16}$$

Combining (3.15) and (3.16) and reformulating (3.14), one has

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T}\phi_\lambda(Y^{\epsilon,n_\rho}(t))\Big)\\
&\le 2\rho^{2\lambda}+4L\lambda(66\lambda\epsilon-\epsilon+1)\int_0^T\mathbb{E}(\rho^2+|Y^{\epsilon,n_\rho}(s)|^2)^{\lambda-1}\|Z_s^{\epsilon,n_\rho}\|_\infty^2\mathrm{d}s\\
&\quad+4L\lambda\epsilon(68\lambda-1)\int_0^T\mathbb{E}(\rho^2+|Y^{\epsilon,n_\rho}(s)|^2)^{\lambda-1}\|X_s^{\epsilon,n_\rho}-\widehat{X}_s^{\epsilon,n_\rho}\|_\infty^2\mathrm{d}s\\
&\le 2\rho^{2\lambda}+4L\lambda(66\lambda\epsilon-\epsilon+1)\int_0^T\mathbb{E}\Big(\sup_{0\le u\le s}(\rho^2+|Y^{\epsilon,n_\rho}(u)|^2)^{\lambda-1}\|Z_u^{\epsilon,n_\rho}\|_\infty^2\Big)\mathrm{d}s\\
&\quad+4L\lambda\epsilon(68\lambda-1)\int_0^T\mathbb{E}\Big(\sup_{0\le u\le s}(\rho^2+|Y^{\epsilon,n_\rho}(u)|^2)^{\lambda-1}\|X_u^{\epsilon,n_\rho}-\widehat{X}_u^{\epsilon,n_\rho}\|_\infty^2\Big)\mathrm{d}s\\
&\le 2\rho^{2\lambda}+(C_3+C_4)\int_0^T\mathbb{E}\Big(\sup_{0\le u\le s}(\rho^2+|Y^{\epsilon,n_\rho}(u)|^2)^{\lambda}\Big)\mathrm{d}s,
\end{aligned}
\tag{3.17}
$$

where $C_3=4\lambda(66\lambda\epsilon-\epsilon+1)\frac{L}{(1-\kappa)^2}$, $C_4=4L\lambda\epsilon(68\lambda-1)$. In the last step, we utilized the fact that $Y^{\epsilon,n_\rho}(t)=0, t\in[-\tau,0]$ and (3.10).

Choosing $\lambda=\frac{1}{\epsilon}$ and setting $\Phi^{\epsilon,n_\rho}(t):=(\rho^2+|Y^{\epsilon,n_\rho}(t\wedge\xi_{n_\rho}^\epsilon)|^2)^{1/\epsilon}$, by the Gronwall inequality, we obtain

$$
\mathbb{E}\Big(\sup_{0\le t\le T}\Phi^{\epsilon,n_\rho}(t)\Big)\le 2\rho^{2\lambda}\mathrm{e}^{(C_3+C_4)T}\le 2\rho^{2/\epsilon}\mathrm{e}^{C_5T/\epsilon},
$$

where $C_5=L\Big(\frac{268}{(1-\kappa)^2}+272\Big)$. Noting that

$$
\Phi^{\epsilon,n_\rho}(t)=(\rho^2+|Y^{\epsilon,n_\rho}(t)|^2)^{1/\epsilon}I_{\{t\le\xi_{n_\rho}^\epsilon\}}+(\rho^2+|Y^{\epsilon,n_\rho}(\xi_{n_\rho}^\epsilon)|^2)^{1/\epsilon}I_{\{\xi_{n_\rho}^\epsilon<t\}},
$$

so

$$
(\rho^2+(1-\kappa)^2\delta^2)^{1/\epsilon}P(\xi_{n_\rho}^\epsilon\le T)\le\mathbb{E}\Big(\sup_{0\le t\le T}\Phi^{\epsilon,n_\rho}(t)\Big),
$$

then we have

$$
P(\xi_{n_\rho}^\epsilon\le T)\le\Big(\frac{2^\epsilon\rho^2}{\rho^2+(1-\kappa)^2\delta^2}\Big)^{1/\epsilon}\mathrm{e}^{C_5T/\epsilon}.
$$

Thus,

$$
\limsup_{\epsilon\to0}\epsilon\log P(\xi_{n_\rho}^\epsilon\le T)\le\log\Big(\frac{\rho^2}{\rho^2+(1-\kappa)^2\delta^2}\Big)+C_5T.
$$

Finally, given $L>0$, choose ρ sufficiently small such that $\log\Big(\frac{\rho^2}{\rho^2+(1-\kappa)^2\delta^2}\Big)+C_5T\le-2L$. Next, utilizing (3.13), choose N such that $\limsup_{\epsilon\to0}\epsilon\log P(\tau_{n_\rho}^\epsilon\le T)\le-2L$ for $n\ge N$. Then, for $n\ge N$ there is an $0<\epsilon_n<1$ such that $P(\tau_{n_\rho}^\epsilon\le T)\le\mathrm{e}^{-L/\epsilon}$ and $P(\xi_{n_\rho}^\epsilon\le T)\le\mathrm{e}^{-L/\epsilon}$ for $0<\epsilon\le\epsilon_n$, so (3.11) leads to

$$
P\Big(\sup_{0\le t\le T}|Z^{\epsilon,n}(t)|\ge\delta\Big)\le2\mathrm{e}^{-L/\epsilon},\quad 0<\epsilon\le\epsilon_n.
$$

Thus,

$$\limsup_{\epsilon\to 0}\epsilon\log P\Big(\sup_{0\le t\le T}|Z^{\epsilon,n}(t)|>\delta\Big)\le -L,\qquad n\ge N.$$

The proof of the lemma is complete. □

For $n\ge 1$, define the map $F^n(\cdot):C_0([0,T],\mathbb{R}^d)\to C_\xi([-\tau,T],\mathbb{R}^d)$ by

$$\begin{cases} F^n(\omega)(t)-G(F^n_t(\omega))=F^n(\omega)(t_n)-G(F_{t_n}(\omega))+\int_{t_n}^t b(F^n_s(\omega))\mathrm{d}s \\ \qquad\qquad +\sigma(\widehat{F}^n_s(\omega))(\omega(t)-\omega(t_n)),\quad t_n\le t\le t_n+\frac{1}{n}, \\ F^n(\omega)(t)=\xi(t),\quad -\tau\le t\le 0, \end{cases}$$

where $F^n_s(\omega)(\theta)=F^n(\omega)(s+\theta)$ and $\widehat{F}^n_s(\omega)(\theta)=\widehat{F}^n(\omega)((s+\theta)\wedge s_n)$.

Notice that, $X^{\epsilon,n}(t)=F^n(\sqrt{\epsilon}W)(t)$, which is a continuous map. Herein, W is a standard Brownian motion. For $h\in H$, we define

$$\begin{cases} F^n(h)(t)-G(F^n_t(h))=F^n(h)(0)-G(F^n_0(h))+\int_0^t b\Big(F^n_s(h)\Big)\mathrm{d}s \\ \qquad\qquad +\int_0^t\sigma\Big(\widehat{F}^n_s(h)\Big)\dot{h}(s)\mathrm{d}s,\, t\in[0,T], \\ F^n_0(h)(\theta)=\xi(\theta),\quad \theta\in[-\tau,0]. \end{cases}\tag{3.18}$$

The next lemma shows that the measurable map $F(h)(\cdot)$ can be approximated well by the continuous maps $F^n(h)(\cdot)$.

Lemma 3.4. *Under the assumptions of Theorem 3.1, we have*

$$\lim_{n\to\infty}\sup_{\{h:L_T(h)\le\alpha\}}\sup_{-\tau\le t\le T}\Big|F^n(h)(t)-F(h)(t)\Big|=0,\tag{3.19}$$

where $\alpha<\infty$ is a constant.

Proof. For simplicity, we first let $G(0)=0$. Set $M^n(t):=F^n(h)(t)-G(F^n_t(h))$, by fundamental inequality $(a+b)^2\le[1+\eta](a^2+\frac{b^2}{\eta})$ and (**H2**), we derive

$$\begin{aligned} |F^n(h)(t)|^2&=|F^n(h)(t)-G(F^n_t(h))+G(F^n_t(h))|^2 \\ &\le(1+\eta)\Big(\frac{|G(F^n_t(h))|^2}{\eta}+|F^n(h)(t)-G(F^n_t(h))|^2\Big) \\ &\le(1+\eta)\Big(\frac{\kappa^2\|F^n_t(h)\|^2_\infty}{\eta}+|F^n(h)(t)-G(F^n_t(h))|^2\Big). \end{aligned}$$

Letting $\eta=\frac{\kappa}{1-\kappa}$, we then have

$$\sup_{0\le t\le T}|F^n(h)(t)|^2\le\frac{\kappa}{1-\kappa}\|\xi\|^2_\infty+\frac{1}{(1-\kappa)^2}\sup_{0\le t\le T}|M^n(t)|^2.\tag{3.20}$$

On the other hand, it is easy to see that

$$|M^n(t)|^2\le(1+\kappa)^2\|F^n_t(h)\|^2_\infty.\tag{3.21}$$

By (**H1**), (**H2**), we obtain from (3.18) that

$$
\begin{aligned}
&|M^n(t)|^2 \le (1+\kappa)^2\|\xi\|_\infty^2 + \int_0^t 2\langle M^n(s), b(F_s^n(h)) + \sigma(\widehat{F}_t^n(h))\dot{h}(s)\rangle \mathrm{d}s \\
&\le (1+\kappa)^2\|\xi\|_\infty^2 + L_2\int_0^t (1+\|F_s^n(h)\|_\infty^2)\mathrm{d}s + \int_0^t |M^n(s)|^2\mathrm{d}s + \int_0^t |\sigma(\widehat{F}_t^n(h))\dot{h}(s)|^2\mathrm{d}s \\
&\le (1+\kappa)^2\|\xi\|_\infty^2 + L_2\int_0^t (1+\|F_s^n(h)\|_\infty^2)\mathrm{d}s + \int_0^t |M^n(s)|^2\mathrm{d}s \\
&+ L_2\int_0^t (1+\|\widehat{F}_t^n(h)\|_\infty^2)|\dot{h}(s)|^2\mathrm{d}s.
\end{aligned}
$$

Noting that $\|\widehat{F}_t^n(h)\|_\infty = \sup_{-\tau\le\theta\le 0} F^n(h)((t+\theta)\wedge t_n) \le \sup_{-\tau\le\theta\le 0} F^n(h)(t+\theta)$, which together with (3.20),(3.21), yields that

$$
\begin{aligned}
&\sup_{-\tau\le t\le T} |F^n(h)(t)|^2 \\
&\le \|\xi\|_\infty^2 + \sup_{0\le t\le T}|F^n(h)(t)|^2 \\
&\le \|\xi\|_\infty^2 + \frac{\kappa}{1-\kappa}\|\xi\|_\infty^2 + \frac{1}{(1-\kappa)^2}\sup_{0\le t\le T}|M^n(t)|^2 \\
&\le \frac{1-\kappa+(1+\kappa)^2}{(1-\kappa)^2}\|\xi\|_\infty^2 + \frac{1}{(1-\kappa)^2}\Big[(L_2+(1+\kappa)^2)\int_0^T \|F_s^n(h)\|_\infty^2\mathrm{d}s \\
&\quad + L_2\int_0^T \|F_s^n(h)\|_\infty^2|\dot{h}(s)|^2\mathrm{d}s + L_2\int_0^T |\dot{h}(s)|^2\mathrm{d}s\Big],
\end{aligned}
$$

by the Gronwall inequality, we get

$$
\begin{aligned}
&\sup_{n\ge 1}\sup_{-\tau\le t\le T}\Big|F^n(h)(t)\Big|^2 \\
&\le \Big(\frac{1-\kappa+(1+\kappa)^2}{(1-\kappa)^2}\|\xi\|_\infty^2 + \frac{2L_2L_T(h)}{(1-\kappa)^2}\Big)\exp\Big\{\frac{(L_2+(1+\kappa)^2)T+2L_2L_T(h)}{(1-\kappa)^2}\Big\} \\
&\le C_1(1+L_T(h))\exp\{C_2(1+L_T(h))\},
\end{aligned}
$$

where $C_1 = \Big(\frac{1-\kappa+(1+\kappa)^2}{(1-\kappa)^2}\|\xi\|_\infty^2\Big)\vee\Big(\frac{2L_2}{(1-\kappa)^2}\Big)$, $C_2 = \Big(\frac{(L_2+(1+\kappa)^2)T}{(1-\kappa)^2}\Big)\vee\Big(\frac{2L_2}{(1-\kappa)^2}\Big)$.

In particular,

$$
M_\alpha = \sup_{\{h;L_T(h)\le\alpha\}}\sup_{n\ge 1}\sup_{-\tau\le t\le T}\Big|F^n(h)(t)\Big|^2 \le C_1(1+\alpha)\exp\{C_2(1+\alpha)\} < \infty. \tag{3.22}
$$

Hence, in the same way as the argument of (3.12), we arrive at

(3.23)

$$
\begin{aligned}
\sup_{0\le t\le T}\|F_t^n(h)-\widehat{F}_t^n(h)\|_\infty &\le \frac{1}{1-\kappa}\sup_{0\le t\le T}\sup_{t_n-t\le\theta\le 0}\Big|\int_{t_n}^{t+\theta} b(F_s^n(h))\mathrm{d}s+\int_{t_n}^{t+\theta}\sigma(\widehat{F}_s^n(h))\dot h(s)\mathrm{d}s\Big|\\
&\le \frac{1}{1-\kappa}\sup_{0\le t\le T}\Big(\int_{t_n}^{t}|b(F_s^n(h))|\mathrm{d}s+\int_{t_n}^{t}|\sigma(\widehat{F}_s^n(h))\dot h(s)|\mathrm{d}s\Big)\\
&\le C_\alpha M_\alpha\Big(\frac{1}{n}\Big)^{1/2}\to 0,\quad \text{as } n\to\infty
\end{aligned}
$$

uniformly over the set $\{h; L_T(h)\le\alpha\}$.

For notation brevity, we set $D^n(h)(t):=F^n(h)(t)-F(h)(t)-(G(F_t^n(h))-G(F_t(h)))$, similarly, it is easy to see from (**H1**),(**H2**) that

$$
\sup_{0\le t\le T}|F^n(h)(t)-F(h)(t)|^2\le\frac{1}{(1-\kappa)^2}\sup_{0\le t\le T}|D^n(h)(t)|^2, \tag{3.24}
$$

and

$$
|D^n(h)(t)|^2\le(1+\kappa)^2\|F_t^n(h)-F_t(h)\|_\infty^2. \tag{3.25}
$$

Using (3.4) and (3.18), we deduce

(3.26)

$$
\begin{aligned}
|D^n(h)(t)|^2 \le& \int_0^t 2|\langle D^n(h)(s), b(F_s^n(h))-b(F_s(h))\rangle|\mathrm{d}s\\
&+\int_0^t 2|\langle D^n(h)(s),[\sigma(\widehat{F}_s^n(h)-\sigma(F_s^n(h))+\sigma(F_s^n(h))-\sigma(F_s(h))]\dot h(s)\rangle|\mathrm{d}s\\
\le& L\int_0^t\|F_s^n(h)-F_s(h)\|_\infty^2\mathrm{d}s+\int_0^t|D^n(h)(s)|^2\mathrm{d}s\\
&+L\int_0^t|F^n(h)(s)-F(h)(s)|^2|\dot h(s)|^2\mathrm{d}s+L\int_0^t\|\widehat{F}_s^n(h))-F_s^n(h))\|_\infty^2|\dot h(s)|^2\mathrm{d}s,
\end{aligned}
$$

which, together with (3.23), (3.24) and (3.25), yields that

$$
\begin{aligned}
\sup_{-\tau\le t\le T}|F^n(h)(t)-F(h)(t)|^2\le&\frac{1}{(1-\kappa)^2}\Big\{(L+(1+\kappa)^2)\int_0^T\|F_s^n(h)-F_s(h)\|_\infty^2\mathrm{d}s\\
&+L\int_0^T\|F_s^n(h)-F_s(h)\|_\infty^2|\dot h(s)|^2\mathrm{d}s+2L\alpha C_\alpha M_\alpha\Big(\frac{1}{n}\Big)^{1/4}\Big\},
\end{aligned}
$$

it follows from the Gronwall inequality that,

$$
\sup_{-\tau\le t\le T}|F^n(h)(t)-F(h)(s)|^2\le\frac{2L\alpha C_\alpha M_\alpha\big(\frac{1}{n}\big)^{1/4}}{(1-\kappa)^2}\exp\Big\{\frac{(L+(1+\kappa)^2)T+2L\alpha}{(1-\kappa)^2}\Big\}.
$$

Hence, the desired assertion is followed by taking $n \to \infty$.

If $G(\mathbf{0}) \neq 0$, by (**H2**) and the fundamental inequality, for any $\xi \in \mathscr{C}$ and $\epsilon > 0$, we have

$$\begin{aligned}|G(\xi)|^2 &\leq |G(\xi) - G(\mathbf{0}) + G(\mathbf{0})|^2 \leq (1+\epsilon)|G(\xi) - G(\mathbf{0})|^2 + (1 + 1/\epsilon)|G(\mathbf{0})|^2 \\ &\leq \kappa^2(1+\epsilon)\|\xi\|_\infty^2 + (1+1/\epsilon)|G(\mathbf{0})|^2.\end{aligned}$$

Taking ϵ sufficiently small such that $\kappa(1+\epsilon) < 1$, the proof can be complete by repeating the one above. □

We now complete the

Proof of Theorem 3.1. Notice that $X^{\epsilon,n}(s) = F^n(\sqrt{\epsilon}W)(s)$, where W is the d-dimentional Brownian motion. Then by the contraction principle in large deviations theory, we get that the law of $X^{\epsilon,n}(s)$ satisfies an LDP. Then Lemma 3.3 states that $X^{\epsilon,n}(s)$ approximates exponentially $X^{\epsilon}(s)$. Furthermore, Lemma 3.4 shows that the extension of contraction principle to measurable maps $F(h)(\cdot)$ can be approximated well by continuous maps $F^n(h)(\cdot)$, i.e. Lemma 3.3, so the proof of Theorem 3.1 follows from Lemma 2.1. □

In the sequel, we will finish the proof of Theorem 3.2.

Lemma 3.5. *Under (**H1**) and (**H2**), then for $R > 0$ we have*

$$\lim_{R\to\infty} \limsup_{\epsilon\to 0} \epsilon \log P\Big(\sup_{-\tau\leq t\leq T} |X^\epsilon(t)| > R\Big) = -\infty. \tag{3.27}$$

Proof. For notation brevity, we set $Y^\epsilon(t) := X^\epsilon(t) - G(X_t^\epsilon)$, from (**H2**) and fundamental inequality, it yields that

$$|Y^\epsilon(t)|^2 \leq (1+\kappa)^2 \|X_t^\epsilon\|_\infty^2, \tag{3.28}$$

and

$$\sup_{-\tau\leq t\leq T} |X^\epsilon(t)|^2 \leq \frac{1}{1-\kappa}\|\xi\|_\infty^2 + \frac{1}{(1-\kappa)^2} \sup_{0\leq t\leq T} |Y^\epsilon(t)|^2. \tag{3.29}$$

For $\lambda > 0$, applying the Itô formula, (**H1**), (**H2**) and (3.3) yield

$$\begin{aligned}(1+|Y^\epsilon(t)|^2)^\lambda &\leq (1 + (1+\kappa)^2\|\xi\|_\infty^2)^\lambda + \lambda \int_0^t (1+|Y^\epsilon(s)|^2)^{\lambda-1} 2\langle Y^\epsilon(s), b(X_s^\epsilon)\rangle \mathrm{d}s \\ &\quad + 2\lambda(\lambda-1)\epsilon \int_0^t (1+|Y^\epsilon(s)|^2)^{\lambda-2} |\sigma(X_s^\epsilon) Y^\epsilon(s)|^2 \mathrm{d}s \\ &\quad + \lambda\epsilon \int_0^t (1+|Y^\epsilon(s)|^2)^{\lambda-1} \|\sigma(X_s^\epsilon)\|_{HS}^2 \mathrm{d}s + M^{\epsilon,\lambda}(t) \\ &\leq (1 + (1+\kappa)^2\|\xi\|_\infty^2)^\lambda + M^{\epsilon,\lambda}(t) \\ &\quad + \lambda L_2(1+2\lambda\epsilon-\epsilon) \int_0^t (1+|Y^\epsilon(s)|^2)^{\lambda-1}(1+\|X_s^\epsilon\|_\infty^2)\mathrm{d}s \\ &\leq (1 + (1+\kappa)^2\|\xi\|_\infty^2)^\lambda + M^{\epsilon,\lambda}(t) \\ &\quad + \lambda L_2 C_1(1+2\lambda\epsilon-\epsilon) \int_0^t \Big(\sup_{0\leq u\leq s} (1+|Y^\epsilon(u)|^2)^\lambda\Big)\mathrm{d}s,\end{aligned} \tag{3.30}$$

where $C_1 = (1+\frac{\|\xi\|_\infty^2}{(1-\kappa)}) \vee (\frac{1}{(1-\kappa)^2})$, $M^{\epsilon,\lambda}(t) = 2\lambda\epsilon \int_0^t (1+|Y^\epsilon(s)|^2)^{\lambda-1}\langle Y^\epsilon(s), \sigma(X_s^\epsilon)\mathrm{d}W(s)\rangle$, and in the last step, we used (3.29).

Noting that $\|X_s^\epsilon\|_\infty^2 \le \|\xi\|_\infty^2 + \Big(\sup_{0\le u\le s}|X^\epsilon(u)|^2\Big)$, by (**H1**), (3.29) and the BDG inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T} M^{\epsilon,\lambda}(t)\Big)\\
&\le 8\sqrt{2\epsilon}\lambda\bigg(\mathbb{E}\int_0^T (1+|Y^\epsilon(s)|^2)^{2\lambda-1}\|\sigma(X_s^\epsilon)\|_{HS}^2\mathrm{d}s\bigg)^{1/2}\\
&\le \frac{1}{2}\mathbb{E}\Big(\sup_{0\le t\le T}(1+|Y^\epsilon(s)|^2)^\lambda\Big) + 64L_2\lambda^2\epsilon\mathbb{E}\int_0^T (1+|Y^\epsilon(s)|^2)^{\lambda-1}(1+\|X_s^\epsilon\|_\infty^2)\mathrm{d}s\\
&\le \frac{1}{2}\mathbb{E}\Big(\sup_{0\le t\le T}(1+|Y^\epsilon(s)|^2)^\lambda\Big) + 64L_2\lambda^2C_1\epsilon\mathbb{E}\int_0^T \Big(\sup_{0\le u\le s}(1+|Y^\epsilon(u)|^2)^\lambda\Big)\mathrm{d}s.
\end{aligned}\tag{3.31}$$

Substituting (3.31) into (3.30), and reformulating (3.30), we arrive at

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T}(1+|Y^\epsilon(t)|^2)^\lambda\Big)\\
&\le 2(1+(1+\kappa)^2\|\xi\|_\infty^2)^\lambda + 2L_2C_1\lambda[66\lambda\epsilon+1-\epsilon]\int_0^T \mathbb{E}\Big(\sup_{0\le u\le s}(1+|Y^\epsilon(u)|^2)^\lambda\Big)\mathrm{d}s.
\end{aligned}\tag{3.32}$$

For $R>0$, we define $\xi_R^\epsilon = \inf\{t\ge 0 : |X^\epsilon(t)| > R\}$, utilising BDG's inequality yields that

$$\mathbb{E}\Big(\sup_{0\le t\le T}(1+|Y^\epsilon(t\wedge\xi_R^\epsilon)|^2)^\lambda\Big) \le 2(1+(1+\kappa)^2\|\xi\|_\infty^2)^\lambda \exp\{2L_2C_1\lambda[66\lambda\epsilon+1-\epsilon]T\},$$

which implies that

$$\mathbb{E}\Big\{\Big(\sup_{0\le t\le T}(1+|Y^\epsilon(t\wedge\xi_R^\epsilon)|^2)^\lambda\Big)I_{\{\xi_R^\epsilon\le T\}}\Big\} \le 2(1+(1+\kappa)^2\|\xi\|_\infty^2)^\lambda \exp\{2L_2C_1\lambda[66\lambda\epsilon+1-\epsilon]T\},$$

$$\mathbb{P}\Big(\sup_{-\tau\le t\le T}|X^\epsilon(t)|>R\Big) \le \mathbb{P}(\xi_R^\epsilon\le T) \le \frac{2(1+(1+\kappa)^2\|\xi\|_\infty^2)^\lambda \exp\{2L_2C_1\lambda[66\lambda\epsilon+1-\epsilon]T\}}{\Big(1+[R-\frac{\kappa}{1-\kappa}\|\xi\|_\infty^2](1-\kappa)^2\Big)^\lambda},$$

choosing $\lambda=\frac{1}{\epsilon}$ yields that

$$\begin{aligned}
\epsilon\log\mathbb{P}\Big(\sup_{-\tau\le t\le T}|X^\epsilon(t)|>R\Big) &\le \log\frac{2(1+(1+\kappa)^2\|\xi\|_\infty^2)}{\Big(1+[R-\frac{\kappa}{1-\kappa}\|\xi\|_\infty^2](1-\kappa)^2\Big)} + \epsilon 2L_2C_1\lambda[66\lambda\epsilon+1-\epsilon]T\\
&\le \log\frac{2(1+(1+\kappa)^2\|\xi\|_\infty^2)}{\Big(1+[R-\frac{\kappa}{1-\kappa}\|\xi\|_\infty^2](1-\kappa)^2\Big)} + 2L_2C_1(67-\epsilon)T,
\end{aligned}$$

$$\lim_{R\to\infty}\limsup_{\epsilon\to 0}\epsilon\log P\Big(\sup_{-\tau\le t\le T}|X^\epsilon(t)|>R\Big) = -\infty.$$

The proof is therefore complete. □

In order to prove our theorem, we shall use the truncated method. For $R > 0$, set

$$b_R(\xi) = \begin{cases} b(\xi), & \text{if } \|\xi\|_\infty \le R; \\ b\big(\frac{R\xi}{\|\xi\|_\infty}\big), & \text{if } \|\xi\|_\infty > R. \end{cases}$$

Similarly, we can define σ_R.

In the following, we prove that b_R and σ_R satisfy the Lipschitz condition under the condition (3.6). We only give the proof for b_R.

Case (i), if $\|\xi\|_\infty \vee \|\eta\|_\infty \le R$, then

$$|b_R(\xi) - b_R(\eta)| = |b(\xi) - b(\eta)| \le 2LR^q\|\xi - \eta\|_\infty.$$

Case (ii), if $\|\xi\|_\infty \le R, \|\eta\|_\infty > R$, then

$$\begin{aligned} |b_R(\xi) - b_R(\eta)| &= \left| b(\xi) - b\left(\frac{R\eta}{\|\eta\|_\infty}\right)\right| \le 2LR^q \left|\xi - \frac{R\eta}{\|\eta\|_\infty}\right| \\ &\le 2LR^q \Big(\|\xi - \eta\|_\infty + \|\eta\|_\infty - R\Big) \le 4LR^q\|\xi - \eta\|_\infty. \end{aligned}$$

Similarly, we can show that b_R satisfies the Lipschitz condition if $\|\eta\|_\infty \le R, \|\xi\|_\infty > R$.

Case (iii), if $\|\xi\|_\infty \wedge \|\eta\|_\infty > R$, then

$$\begin{aligned} |b_R(\xi) - b_R(\eta)| &= |b(\frac{R\xi}{\|\xi\|_\infty}) - b(\frac{R\eta}{\|\eta\|_\infty})| \\ &\le 2LR^q\|\frac{R\xi}{\|\xi\|_\infty} - \frac{R\eta}{\|\xi\|_\infty} + \frac{R\eta}{\|\xi\|_\infty} - \frac{R\eta}{\|\eta\|_\infty}\| \\ &\le 4LR^q\frac{R}{\|\xi\|_\infty}\|\xi - \eta\|_\infty \le 4LR^q\|\xi - \eta\|_\infty. \end{aligned}$$

Since b_R and σ_R satisfy the Lipschitz condition, it is easy to verify that b_R and σ_R satisfy the assumptions (**H1**) and (**H3**).

Let $X^{\epsilon,R}(\cdot)$ be the solution to the NSFDE

$$\mathrm{d}\{X^{\epsilon,R}(t) - G(X_t^{\epsilon,R})\} = b_R(X_t^{\epsilon,R})\mathrm{d}t + \sqrt{\epsilon}\sigma_R(X_t^{\epsilon,R})\mathrm{d}W(t),\ t > 0,$$

with the initial datum $X_0^{\epsilon,R} = \xi(\theta), \quad \theta \in [-\tau, 0]$.

We recall a Lemma in [8], which is a key point in the proofs of following Lemmas.

Lemma 3.6. *Let N be a fixed integer. Then, for any $a_\epsilon^i \ge 0$,*

$$\limsup_{\epsilon\to 0} \epsilon \log\Big(\sum_{i=1}^N a_\epsilon^i\Big) = \max_{i=1}^N \limsup_{\epsilon\to 0} \epsilon \log a_\epsilon^i. \tag{3.33}$$

The lemma below states that $X^{\epsilon,R}(\cdot)$ is the uniformly exponential approximation of $X^\epsilon(\cdot)$ on the interval $[-\tau, T]$.

Lemma 3.7. *Assume (***H1***) and (***H2***) hold, then for any $T>0, \delta>0$, one has that:*

$$\lim_{R\to\infty}\limsup_{\epsilon\to 0}\epsilon\log P\Big(\sup_{-\tau\le t\le T}|X^\epsilon(t)-X^{\epsilon,R}(t)|>\delta\Big)=-\infty. \tag{3.34}$$

Proof. For notation simplicity, we set $Z^{\epsilon,R}(t):=X^\epsilon(t)-X^{\epsilon,R}(t)$ and $Y^{\epsilon,R}(t):=X^\epsilon(t)-X^{\epsilon,R}(t)-(G(X^\epsilon_t)-G(X^{\epsilon,R}_t))$.

From (**H2**), it is easy to see that

$$\sup_{0\le t\le T}|Z^{\epsilon,R}(t)|\le\sup_{0\le t\le T}\Big(\frac{1}{1-\kappa}|Y^{\epsilon,R}(t)|\Big).$$

Define $\xi^\epsilon_{R_1}:=\inf\{t\ge 0:|X^\epsilon(t)|\ge R_1\}$. For any $R\ge R_1$, we have

$$Y^{\epsilon,R}(t\wedge\xi^\epsilon_{R_1})=\int_0^{t\wedge\xi^\epsilon_{R_1}}(b_R(X^\epsilon_s)-b_R(X^{\epsilon,R}_s))\mathrm{d}s+\sqrt{\epsilon}\int_0^{t\wedge\xi^\epsilon_{R_1}}(\sigma_R(X^\epsilon_s)-\sigma_R(X^{\epsilon,R}_s))\mathrm{d}W(s). \tag{3.35}$$

Setting $Z^\epsilon_{R_1}(t):=Z^{\epsilon,R}(t\wedge\xi^\epsilon_{R_1})$, $Y^\epsilon_{R_1}(t):=Y^{\epsilon,R}(t\wedge\xi^\epsilon_{R_1})$ and $\xi^\epsilon_{R,\delta}:=\inf\{t\ge 0:|Z^\epsilon_{R_1}(t)|\ge\delta\}$. Then, we have

$$\begin{aligned}
&P\Big(\sup_{-\tau\le t\le T}|Z^{\epsilon,R}(t)|>\delta\Big)\\
&=P\Big(\sup_{-\tau\le t\le T}|Z^{\epsilon,R}(t\wedge\xi^\epsilon_{R_1})|>\delta,I_{\{\xi^\epsilon_{R_1}\ge T\}}\Big)+P\Big(\sup_{-\tau\le t\le T}|Z^{\epsilon,R}(t\wedge\xi^\epsilon_{R_1})|>\delta,I_{\{\xi^\epsilon_{R_1}\le T\}}\Big)\\
&\le P(\xi^\epsilon_{R_1}\le T)+P(\xi^\epsilon_{R,\delta}\le T)\le P\Big(\sup_{-\tau\le t\le T}|X^\epsilon(t)|>R_1\Big)+P(\xi^\epsilon_{R,\delta}\le T).
\end{aligned} \tag{3.36}$$

By mimicking the argument in Lemma 3.3 for $t\le T\wedge\xi^\epsilon_{R_1}$, one gets

$$\mathbb{E}\Big(\sup_{0\le t\le T}\big(\rho^2+|Y^\epsilon_{R_1}(t)|^2\big)^{1/\epsilon}\Big)\le 2\rho^{2/\epsilon}\mathrm{e}^{CT/\epsilon}.$$

This implies that

$$P(\xi^\epsilon_{R,\delta}\le T)\le\Big(\frac{2^\epsilon\rho^2}{\rho^2+(1-\kappa)^2\delta^2}\Big)^{1/\epsilon}\mathrm{e}^{CT/\epsilon}.$$

Taking Logarithmic function into consideration, we have

$$\limsup_{\epsilon\to 0}\epsilon\log P(\xi^\epsilon_{R,\delta}\le T)\le\log\Big(\frac{\rho^2}{\rho^2+(1-\kappa)^2\delta^2}\Big)+CT.$$

This, together with (3.27),(3.33) and (3.36), implies

$$\begin{aligned}
&\lim_{R\to\infty}\limsup_{\epsilon\to 0}\epsilon\log P\Big(\sup_{-\tau\le t\le T}|Z^{\epsilon,R}(t)|>\delta\Big)\\
&\le\lim_{R\to\infty}\limsup_{\epsilon\to 0}\epsilon\log\Big(P\Big(\sup_{-\tau\le t\le T}|X^\epsilon(t)|>R_1\Big)+\lim_{R\to\infty}\limsup_{\epsilon\to 0}P(\xi^\epsilon_{R,\delta}\le T)\Big)\\
&\le\limsup_{\epsilon\to 0}\epsilon\log P\Big(\sup_{-\tau\le t\le T}|X^\epsilon(t)|>R_1\Big)\vee\Big\{\log\Big(\frac{\rho^2}{\rho^2+(1-\kappa)^2\delta^2}\Big)+CT\Big\}.
\end{aligned}$$

The conclusion follows from letting first $\rho\to 0$ and then $R_1\to\infty$ by Lemma 3.5. □

Proof of Theorem 3.2

For h with $L_T(h) < \infty$, let $F^R(h)$ be the solution of the equation below

$$F^R(h)(t) - G(F_t^R(h)) = F^R(h)(0) - G(F_0^R(h)) + \int_0^t b_R(F_s^R(h))\mathrm{d}s + \int_0^t \sigma_R(F_s^R(h))\dot{h}(s)\mathrm{d}s$$

with the initial datum $F_0^R(h)(\theta) = \xi(\theta), \quad \theta \in [-\tau, 0]$. Define

$$I_R(f) = \inf\Big\{\frac{1}{2}\int_0^T |\dot{h}(t)|^2\mathrm{d}t; \quad F^R(h) = f\Big\},$$

for each $f \in C([-\tau, T]; \mathbb{R}^d)$. If $\Big(\sup_{-\tau\le t\le T} |F(h)(t)|\Big) \le R$, then $F(h) = F^R(h)$.

$$I(f) = I_R(f), \quad \text{for all } f \text{ with } \Big(\sup_{-\tau\le t\le T} |f(t)|\Big) \le R.$$

Proof. For $R > 0$, and a closed subset $C \subset C([-\tau, T]; \mathbb{R}^d)$, set $C_R := C \cap \{f; \|f\|_\infty \le R\}$. C_R^δ denotes the δ-neighborhood of C_R. Denote by $\mu^{\epsilon,R}$ the law of X_R^ϵ. Then we have

$$\begin{aligned}
\mu_\epsilon(C) &= \mu_\epsilon(C_{R_1}) + \mu_\epsilon\Big(C, \sup_{-\tau\le t\le T}|X^\epsilon(t)| > R_1\Big)\\
&\le \mu_\epsilon(C_{R_1}) + P\Big(\sup_{-\tau\le t\le T}|X^\epsilon(t)| > R_1\Big)\\
&\le P\Big(\sup_{-\tau\le t\le T}|X^\epsilon(t) - X^{\epsilon,R}(t)| > \delta\Big) + \mu_\epsilon^R\Big(C_{R_1}^\delta\Big) + P\Big(\sup_{-\tau\le t\le T}|X^\epsilon(t)| > R_1\Big).
\end{aligned}$$

Taking the large deviation principle for $\{\mu_\epsilon^R, \epsilon > 0\}$ yields from 3.6 that

$$\begin{aligned}
&\limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(C)\\
&\le \limsup_{\epsilon\to 0} \epsilon \log\Big\{P\Big(\sup_{-\tau\le t\le T}|X^\epsilon(t) - X^{\epsilon,R}(t)| > \delta\Big) + \Big(-\inf_{f\in C_{R_1}^\delta} I_R(f)\Big)\\
&\quad + P\Big(\sup_{-\tau\le t\le T}|X^\epsilon(t)| > R_1\Big)\Big\}\\
&\le \Big(-\inf_{f\in C_{R_1}^\delta} I_R(f)\Big) \vee \Big(\limsup_{\epsilon\to 0} \epsilon \log P\big(\sup_{-\tau\le t\le T}|X^\epsilon(t)| > R_1\big)\Big)\\
&\quad \vee \Big(\limsup_{\epsilon\to 0} \epsilon \log P\big(\sup_{-\tau\le t\le T}|X^\epsilon(t) - X^{\epsilon,R}(t)| > \delta\big)\Big).
\end{aligned}$$

Then we obtain the upper bound (i) in Theorem 3.1, that is

$$\limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(C) \le -\inf_{f\in C} I(f),$$

by taking first $R \to \infty$, and $\delta \to 0$, then $R_1 \to \infty$. Let G be an open subset of $C([-\tau, T]; \mathbb{R}^d)$. Then for any $\phi_0 \in G$, and taking $\delta > 0$, we define $B(\phi_0, \delta) = \{f; \|f - \phi_0\|_\infty \le \delta\} \subset G$. Then using the large deviation principle for $\{\mu_\epsilon^R; \epsilon > 0\}$, one gets

$$\begin{aligned}
-I_R(\phi_0) &\le \liminf_{\epsilon\to 0} \epsilon \log \mu_\epsilon^R\Big(B(\phi_0, \frac{\delta}{2})\Big) \\
&\le \liminf_{\epsilon\to 0} \epsilon \log \Big\{ P\Big(\sup_{-\tau\le t\le T} |X^{\epsilon,R}(t) - \phi_0| \le \frac{\delta}{2}, \sup_{-\tau\le t\le T} |X^{\epsilon}(t) - X^{\epsilon,R}(t)| \le \frac{\delta}{2}\Big) \\
&\qquad + P\Big(\sup_{-\tau\le t\le T} |X^{\epsilon,R}(t) - \phi_0| \le \frac{\delta}{2}, \sup_{-\tau\le t\le T} |X^{\epsilon}(t) - X^{\epsilon,R}(t)| \ge \frac{\delta}{2}\Big)\Big\} \\
&\le \Big(\liminf_{\epsilon\to 0} \epsilon \log \mu_\epsilon(G)\Big) \vee \Big(\liminf_{\epsilon\to 0} \epsilon \log P\big(\sup_{-\tau\le t\le T} |X^{\epsilon}(t) - X^{\epsilon,R}(t)| \ge \frac{\delta}{2}\big)\Big).
\end{aligned}$$

Noting that $I_R(\phi_0) = I(\phi_0)$ provided that $\|\phi_0\|_\infty \le R$. Then we have

$$-I(\phi_0) \le \liminf_{\epsilon\to 0} \epsilon \log \mu_\epsilon(G), \text{ as } R \to \infty.$$

Owing to the arbitrary of ϕ_0, it follows that

$$-\inf_{f\in G} I(f) \le \liminf_{\epsilon\to 0} \epsilon \log \mu_\epsilon(G),$$

which is the lower bound (i) in Theorem 3.1, thus, the proof of Theorem 3.2 is complete. □

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