FRAMED MOTIVES OF ALGEBRAIC VARIETIES (AFTER V. VOEVODSKY)

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In memory of Vladimir Voevodsky

ABSTRACT. A new approach to stable motivic homotopy theory is given. It is based on Voevodsky's theory of framed correspondences. Using the theory, framed motives of algebraic varieties are introduced and studied in the paper. They are the major computational tool for constructing an explicit quasi-fibrant motivic replacement of the suspension $\mathbb{P}^1$-spectrum of any smooth scheme $X \in Sm/k$. Moreover, it is shown that the bispectrum

$$(M_{fr}(X), M_{fr}(X)(1), M_{fr}(X)(2), \ldots),$$

each term of which is a twisted framed motive of X, has the motivic homotopy type of the suspension bispectrum of X. Furthermore, an explicit computation of infinite $\mathbb{P}^1$-loop motivic spaces is given in terms of spaces with framed correspondences. We also introduce big framed motives of bispectra and show that they convert the classical Morel–Voevodsky motivic stable homotopy theory into an equivalent local theory of framed bispectra. As a topological application, it is proved that the framed motive $M_{fr}(pt)(pt)$ of the point $pt = \operatorname{Spec} k$ evaluated at pt is a quasi-fibrant model of the classical sphere spectrum whenever the base field k is algebraically closed of characteristic zero.

CONTENTS

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1. INTRODUCTION

The main objective of the present paper is to suggest a new approach to the classical stable motivic homotopy theory of Morel–Voevodsky [18]. Let us give more details.

In [24] Voevodsky develops the theory of (pre-)sheaves with framed correspondences. One of its aims was to give another framework for $SH(k)$ more amenable to explicit calculations (see his Nordfjordeid Lectures [5, Remark 2.15] or his unpublished notes [24]).

Let Sm/k be the category of smooth separated schemes of finite type over a field k. Recall that the category of framed correspondences $Fr_*(k)$, invented by Voevodsky [24, Section 2], is defined as follows. Its objects are those of Sm/k and morphisms sets $Fr_*(X,Y) = \sqcup_{n\geqslant 0} Fr_n(X,Y)$ are defined by means of certain geometric data (see Section 2 below). The category $Fr_*(k)$ contains a subcategory $Fr_0(k)$ whose objects are those of Sm/k and the morphisms set between schemes X and Y is the pointed set $Fr_0(X,Y)$. The latter set coincides with the pointed set $\mathrm{Mor}_\bullet(X_+,Y_+)$ of pointed morphisms between pointed schemes X_+ and Y_+. Following Voevodsky [24], we put for every $Y \in Sm/k$ (see Section 2 below):

$$Fr(X,Y) := \operatorname{colim}(Fr_0(X,Y) \xrightarrow{\sigma_Y} Fr_1(X,Y) \xrightarrow{\sigma_Y} \dots \xrightarrow{\sigma_Y} Fr_n(X,Y) \xrightarrow{\sigma_Y} \dots)$$

and refer to it as the set of stable framed correspondences. The set $Fr(X,Y)$ is pointed, contravariantly functorial with respect to $X \in Fr_*(k)$ and covariantly functorial with respect to $Y \in Fr_0(k)$.

Replacing Y by a simplicial object $Y^\bullet$ in $Fr_0(k)$, we get a pointed simplicial set $Fr(X,Y^\bullet)$. Replacing X by the standard cosimplicial object $\Delta^\bullet_k$ in Sm/k and taking the diagonal, we get a pointed simplicial set $Fr(\Delta^\bullet_k,Y^\bullet)$. We put

$$\pi_r^{\mathrm{fr}}(Y^\bullet) := \pi_r(Fr(\Delta^\bullet_k,Y^\bullet))$$

and call $\pi_r^{\mathrm{fr}}(Y^\bullet)$ the rth singular algebraic stable homotopy group of $Y^\bullet$.

If we replace $Y^\bullet$ by the simplicial circle S^1 (canonically regarded as an object of $\Delta^{\mathrm{op}}Fr_0(k)$) and take the base field to be the field of complex numbers $\mathbb{C}$, we can state our first result:

Theorem 1.1. *The geometric realization of the simplicial set $Fr(\Delta^\bullet_{\mathbb{C}},S^1)$ has the homotopy type of the topological space $\Omega^\infty\Sigma^\infty(S^1_{top})$, where S^1_{top} stands for the usual topological circle.*

The key point of the statement is this: the classical topological space of the theorem is recovered as the simplicial set $Fr(\Delta^\bullet_{\mathbb{C}},S^1)$, which is described in terms of algebraic varieties only. This is one of the computational miracles of framed correspondences.

Our next result is in the spirit of the preceding theorem. It extends the celebrated theorem of Suslin and Voevodsky [21] on singular algebraic homology to the singular algebraic stable homotopy defined above.

Theorem 1.2. *The assignment $X \mapsto \pi_*^{\mathrm{fr}}(X\otimes S^1)$ is a generalized homology theory on $Sm/\mathbb{C}$. Moreover, passing to homotopy groups with finite coefficients, we get equalities*

$$\pi_s^{\mathrm{fr}}(X\otimes S^1;\mathbb{Z}/m) = \pi_s^{st}(X_+\wedge S^1_{top};\mathbb{Z}/m)$$

for all integers s,m with $m \neq 0$.

Also, the first part of this theorem is true over any infinite perfect field k. Namely, the assignment $X \mapsto \pi_*^{fr}(X\otimes S^1)$ is a generalized homology theory on the category Sm/k.

The reader will find the proofs of Theorems 1.1 and 1.2 in Section 11.

From now on we deal with the category Sm/k of smooth algebraic varieties over an infinite perfect field k unless otherwise specified. Following [18], recall that a pointed motivic space is a pointed simplicial Nisnevich sheaf on Sm/k. The category of pointed motivic spaces will be denoted by $sShv_\bullet(Sm/k)$. Voevodsky conjectured that if the motivic space $Fr(\Delta_k^\bullet\times -,Y^\bullet)$ is locally connected in the Nisnevich topology, then it is weakly equivalent, locally in the Nisnevich topology, to the motivic space $\Omega_{\mathbb{P}^1}^\infty\Sigma_{\mathbb{P}^1}^\infty(Y_+^\bullet)$.

The proof of Theorem 6.4 shows that the motivic space $Fr(\Delta_k^\bullet\times -,Y^\bullet)$ is a commutative monoid in $H_\bullet^{\mathrm{nis}}(k)$. In particular, the Nisnevich sheaf $\underline{\pi}_0(Fr(\Delta^\bullet\times -,Y^\bullet))$ is a sheaf of commutative monoids. If this sheaf is a sheaf of Abelian groups, then the space $Fr(\Delta^\bullet\times -,Y^\bullet)$ is called a (locally) group-like motivic space. The following result (see Theorem 10.7) answers Voevodsky's conjecture in the affirmative.

Theorem 1.3. *Let k be an infinite perfect field. Then for any simplicial object $Y^\bullet$ in $Fr_0(k)$ the canonical morphism*

$$Fr(\Delta_k^\bullet\times -,Y^\bullet)\to\Omega_{\mathbb{P}^1}^\infty\Sigma_{\mathbb{P}^1}^\infty(Y_+^\bullet)$$

is locally a group completion. Furthermore, if $Fr(\Delta_k^\bullet\times -,Y^\bullet)$ is locally a group-like motivic space, then the canonical morphism

$$Fr(\Delta_k^\bullet\times -,Y^\bullet)\to\Omega_{\mathbb{P}^1}^\infty\Sigma_{\mathbb{P}^1}^\infty(Y_+^\bullet)$$

is a local weak equivalence. In particular, the latter is true whenever $Fr(\Delta^\bullet\times -,Y^\bullet)$ is locally connected.

The preceding theorem shows that the theory of framed correspondences produces a machinery for computing motivic infinite loop spaces.

To prove Voevodsky's conjecture, we introduce and study framed motives as well as big framed motives in the present paper. The main goal of the machinery of framed motives is to find an explicit $\mathbb{A}^1$-local replacement of the functor

$$\Omega_{\mathbb{G}}^\infty\Sigma_{\mathbb{G}}^\infty\Sigma_{S^1}^\infty : H_{\mathbb{A}^1}(k)\to SH_{S^1}(k).$$

To this end, we firstly regard $SH_{S^1}(k)$ as the full subcategory of the ordinary stable homotopy category $SH_{S^1}^{\mathrm{nis}}(k)$ consisting of $\mathbb{A}^1$-local spectra. Then in Theorem 11.7 we construct an explicit functor $M_{fr}: H_{\mathbb{A}^1}(k)\to SH_{S^1}(k)$ together with a functor isomorphism

$$\alpha:\Omega_{\mathbb{G}}^\infty\Sigma_{\mathbb{G}}^\infty\Sigma_{S^1}^\infty\to M_{fr}.$$

In order to formulate the main results of the theory of big framed motives, consider the full subcategory $SH_{\mathrm{nis}}^{fr}(k)$ of $SH(k)$ consisting of framed bispectra E such that for any $i,j\geqslant 0$ the simplicial framed sheaf $E_{i,j}$ is $\mathbb{A}^1$-local regarded as an ordinary motivic space and $\{E_{i,j}^f\}$ regarded as an ordinary bispectrum is stably motivically fibrant in the stable motivic model structure. Here "f" refers to a fibrant replacement in the local model structure on $sShv_\bullet(Sm/k)$.

The main results of the theory of big framed motives are Theorems 12.4 and 12.5. Theorem 12.4 says that an explicitly constructed functor

$$\mathscr{M}_{fr}^b: SH(k)\to SH_{\mathrm{nis}}^{fr}(k)$$

converts classical Morel–Voevodsky stable motivic homotopy theory $SH(k)$ into an equivalent local homotopy theory of $\mathbb{A}^1$-local framed bispectra from $SH_{\mathrm{nis}}^{fr}(k)$, thus producing a new approach to stable motivic homotopy theory. The main ingredients of this equivalent local homotopy theory are framed motivic spaces of the form $C_*Fr(-,Y)$ with $Y\in\Delta^{\mathrm{op}}Fr_0(k)$ a simplicial scheme as well as their framed motives $M_{fr}(Y)$. Theorem 12.5 states that the morphisms set in $SH(k)$ between two bispectra E and E' is the set $\pi_0(E^c,\mathscr{M}_{fr}^b(E')^f)$ of ordinary morphisms between bispectra E^c and $\mathscr{M}_{fr}^b(E')_f$ modulo naive homotopy.

Let us indicate some applications of the theory of framed motives and big framed motives. The fact that α is a functor isomorphism yields the following statement: for any $X \in Sm/k$ and any simplicial object $Y^\bullet$ in $Fr_0(k)$ one has a canonical isomorphism

$$SH(k)(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}X_+, \Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}Y^\bullet_+[n]) = SH^{\mathrm{nis}}_{S^1}(k)(\Sigma^\infty_{S^1}X_+, M_{fr}(Y^\bullet)[n]), \quad n \geqslant 0, \tag{1}$$

(see Theorem 11.5). In particular, the isomorphism α yields that for any simplicial object $X^\bullet$ in $Fr_0(k)$ the projection $X^\bullet \times \mathbb{A}^1 \to X^\bullet$ induces an isomorphism $M_{fr}(X^\bullet \times \mathbb{A}^1) \cong M_{fr}(X^\bullet)$. Furthermore, the functor M_{fr} converts any elementary distinguished Nisnevich square of k-smooth varieties to a Mayer–Vietoris exact triangle (see Theorem 8.10). Another important property of framed motives is as follows: given a morphism $\varphi : Y^\bullet \to Z^\bullet$ of simplicial objects in $Fr_0(k)$ such that the morphism $\Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}(\varphi)$ is an isomorphism in $SH(k)$, the morphism $M_{fr}(\varphi)$ is a local equivalence.

By definition, the framed motive of a smooth scheme $X \in Sm/k$ over a field k is a motivic S^1-spectrum $M_{fr}(X)$ whose terms are certain explicit motivic spaces with framed correspondences (see Definition 5.2). We use framed motives to construct an explicit quasi-fibrant motivic replacement (i.e. an Ω-spectrum in positive degrees) of the suspension $\mathbb{P}^1$-spectrum $\Sigma^\infty_{\mathbb{P}^1}X_+$ in Theorem 4.1 (here $\mathbb{P}^1$ is pointed at ∞). Another application is to show in Theorem 11.1 that an explicitly constructed bispectrum

$$M^{\mathbb{G}}_{fr}(X) = (M_{fr}(X), M_{fr}(X)(1), M_{fr}(X)(2), \dots),$$

each term of which is a twisted framed motive of X, has the motivic homotopy type of the suspension bispectrum $\Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}X_+$ of X. Moreover, if we take a stable local fibrant replacement $M_{fr}(X)(n)_f$ of each twisted framed motive then the bispectrum

$$M^{\mathbb{G}}_{fr}(X)_f = (M_{fr}(X)_f, M_{fr}(X)(1)_f, M_{fr}(X)(2)_f, \dots)$$

is motivically fibrant by [1, Theorem A]. These definitions and results hold equally for simplicial objects in the category $Fr_0(k)$. We should point out that for any simplicial object $Y^\bullet$ in $Fr_0(k)$ there is a canonical morphism of bispectra

$$M^{\mathbb{G}}_{fr}(Y^\bullet) \to \mathscr{M}^b_{fr}(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}Y^\bullet_+),$$

which is an isomorphism in $SH(k)$. Thus the composite morphism $\Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}Y^\bullet_+ \to M^{\mathbb{G}}_{fr}(Y^\bullet) \to \mathscr{M}^b_{fr}(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}Y^\bullet_+)$ is an isomorphism in $SH(k)$. This yields an equality (see Theorem 12.5)

$$SH(k)(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}X_+, \Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}Y^\bullet_+) = \pi_0(\mathscr{M}^b_{fr}(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}Y^\bullet_+)^f_{0,0}(X)).$$

Here "f" refers to a fibrant replacement in the local model structure on $sShv_\bullet(Sm/k)$.

Let us also give some applications of the isomorphism (1). Since $M_{fr}(Y^\bullet)$ is a sheaf of Segal S^1-spectra, it follows that for any $n < 0$ the Nisnevich sheaf $\pi^{\mathbb{A}^1}_{n,0}(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}Y^\bullet_+)$ vanishes. By varying $Y^\bullet$, one gets a much stronger vanishing property. Namely, for any $n < r$ one has $\pi^{\mathbb{A}^1}_{n,r}(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}Y^\bullet_+) = 0$. We can also compute $\pi^{\mathbb{A}^1}_{n,r}(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}Y^\bullet_+)$ for $n = r$ with $r \leqslant 0$ as

$$\pi^{\mathbb{A}^1}_{-n,-n}(\Sigma^\infty_{S^1}\Sigma^\infty_{\mathbb{G}}Y^\bullet_+)(K) = H_0(\mathbb{Z}F(\Delta^\bullet_K, Y^\bullet \times \mathbb{G}_m^{\wedge n})), \quad n \geqslant 0.$$

Here K/k is any field extension and $\mathbb{Z}F(\Delta^\bullet_K, Y^\bullet \times \mathbb{G}_m^{\wedge n}))$ is an explicit chain complex of free Abelian groups. If $X = \mathrm{Spec}(k)$, $\mathrm{char}\, k = 0$, then using Neshitov's computation $H_0(\mathbb{Z}F(\Delta^\bullet_K, \mathbb{G}_m^{\wedge n})) = K^{MW}_n(K)$ [19] we recover the celebrated theorem of Morel [17] for Milnor–Witt K-theory for fields of characteristic zero.

We also give an explicit computation of the suspension functor (see Theorem 10.5)

$$\Sigma^\infty_{\mathbb{P}^1} : H_{\mathbb{A}^1}(k) \to SH(k).$$

It is isomorphic to an explicitly constructed functor

$$M_{\mathbb{P}^{\wedge 1}} : H_{\mathbb{A}^1}(k) \to SH(k)$$

that takes a motivic space to a spectrum consisting of spaces with framed correspondences.

As a topological application, using the machinery of framed motives together with a theorem of Levine [16], we show in Theorem 11.9 that $M_{fr}(pt)(pt)$, the framed motive of the point $pt = \operatorname{Spec} k$ evaluated at pt, is a quasi-fibrant model (i.e. an Ω-spectrum in positive degrees) of the classical sphere spectrum if the base field k is algebraically closed of characteristic zero.

Finally, we show that every $\mathbb{P}^1$-spectrum E is actually isomorphic in $SH(k)$ to a framed spectrum in a canonical way, functorially in E. Furthermore, there is an equivalence of categories

$$SH(k) \xrightarrow{\cong} SH^{fr}(k),$$

where $SH^{fr}(k)$ is a full subcategory of framed $\mathbb{P}^1$-spectra (see Theorem 13.4).

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2. Voevodsky's framed correspondences

In this section we collect basic facts about framed correspondences and framed functors in the sense of Voevodsky [24]. We start with preparations.

Let S be a scheme and let Z be a closed subscheme. Recall that an *étale neighborhood of Z in S* is a triple $(W', \pi' : W' \to S, s' : Z \to W')$ satisfying the conditions:

(i) π' is an étale morphism;

(ii) $\pi' \circ s'$ coincides with the inclusion $Z \hookrightarrow S$ (thus s' is a closed embedding);

(iii) $(\pi')^{-1}(Z) = s'(Z)$

A morphism between two étale neighborhoods $(W', \pi', s') \to (W'', \pi'', s'')$ of Z in S is a morphism $\rho : W' \to W''$ such that $\pi'' \circ \rho = \pi'$ and $\rho \circ s' = s''$. Note that such ρ is automatically étale by [11, VI.4.7].

Definition 2.1 (Voevodsky [24])**.** For k-smooth schemes X, Y and $n \geqslant 0$, an *explicit framed correspondence* Φ of level n consists of the following data:

1. a closed subset Z in $\mathbb{A}^n_X$ which is finite over X;
2. an etale neighborhood $p : U \to \mathbb{A}^n_X$ of Z in $\mathbb{A}^n_X$;
3. a collection of regular functions $\varphi = (\varphi_1, \ldots, \varphi_n)$ on U such that $\cap_{i=1}^n \{\varphi_i = 0\} = Z$;
4. a morphism $g : U \to Y$.

The subset Z will be referred to as the *support* of the correspondence. We shall also write triples $\Phi = (U, \varphi, g)$ or quadruples $\Phi = (Z, U, \varphi, g)$ to denote explicit framed correspondences.

Two explicit framed correspondences Φ and Φ' of level n are said to be *equivalent* if they have the same support and there exists an open neighborhood V of Z in $U \times_{\mathbb{A}^n_X} U'$ such that, on V, the morphism $g \circ pr$ agrees with $g' \circ pr'$ and $\varphi \circ pr$ agrees with $\varphi' \circ pr'$. A *framed correspondence of level n* is an equivalence class of explicit framed correspondences of level n.

Let $Fr_n(X,Y)$ denote the set of framed correspondences from X to Y. We consider it as a pointed set with the basepoint being the class 0_n of the explicit correspondence with $U = \emptyset$.

As an example, the set $Fr_0(X,Y)$ coincides with the set of pointed morphisms $X_+ \to Y_+$. In particular, for a connected scheme X one has

$$Fr_0(X,Y) = \mathrm{Hom}_{Sm/k}(X,Y) \sqcup \{0_0\}.$$

If $f : X' \to X$ is a morphism of schemes and $\Phi = (U, \varphi, g)$ is an explicit correspondence from X to Y then

$$f^*(\Phi) := (U' = U \times_X X', \varphi \circ pr, g \circ pr)$$

is an explicit correspondence from X' to Y.

Remark 2.2. Let $\Phi = (Z, \mathbb{A}^n_X \xleftarrow{p} U, \varphi : U \to \mathbb{A}^n_k, g : U \to Y) \in Fr_n(X,Y)$ be an explicit framed correspondence of level n. It can more precisely be written in the form

$$((\alpha_1, \alpha_2, \dots, \alpha_n), f, Z, U, (\varphi_1, \varphi_2, \dots, \varphi_n), g) \in Fr_n(X,Y)$$

consisting of

- $\diamond$ $Z \subset \mathbb{A}^n_X$ a closed subset finite over X,
- $\diamond$ an etale neighborhood $(\alpha_1, \alpha_2, \dots, \alpha_n), f) = p : U \to \mathbb{A}^n_k \times X$ of Z,
- $\diamond$ a collection of regular functions $\varphi = (\varphi_1, \dots, \varphi_n)$ on U such that $\cap_{i=1}^n \{\varphi_i = 0\} = Z$,
- $\diamond$ a morphism $g : U \to Y$.

We shall usually drop $((\alpha_1, \alpha_2, \dots, \alpha_n), f)$ from notation and just write

$$(Z, U, (\varphi_1, \varphi_2, \dots, \varphi_n), g) = ((\alpha_1, \alpha_2, \dots, \alpha_n), f, Z, U, (\varphi_1, \varphi_2, \dots, \varphi_n), g).$$

The following definition is to describe compositions of framed correspondences.

Definition 2.3. Let X, Y and S be k-smooth schemes, let

$$a = ((\alpha_1, \alpha_2, \dots, \alpha_n), f, Z, U, (\varphi_1, \varphi_2, \dots, \varphi_n), g)$$

be an explicit correspondence of level n from X to Y and let

$$b = ((\beta_1, \beta_2, \dots, \beta_m), f', Z', U', (\psi_1, \psi_2, \dots, \psi_m), g') \in Fr_m(Y,S)$$

be an explicit correspondence of level m from Y to S. We define their composition as an explicit correspondence of level $n+m$ from X to S by

$$((\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_m), f, Z \times_Y Z', U \times_Y U', (\varphi_1, \varphi_2, \dots, \varphi_n, \psi_1, \psi_2, \dots, \psi_m), g').$$

Clearly, the composition of explicit correspondences respects the equivalence relation on them and defines associative maps

$$Fr_n(X,Y) \times Fr_m(Y,S) \to Fr_{n+m}(X,S).$$

Given $X, Y \in Sm/k$, denote by $Fr_+(X,Y)$ the set $\bigvee_n Fr_n(X,Y)$. The composition of framed correspondences defined above gives a category $Fr_+(k)$. Its objects are those of Sm/k and the morphisms are given by the sets $Fr_+(X,Y), X, Y \in Sm/k$. Since the naive morphisms of schemes can be identified with certain framed correspondences of level zero, we get a canonical functor

$$Sm/k \to Fr_+(k).$$

The category $Fr_+(k)$ has the zero object. It is the empty scheme. One can easily see that for a framed correspondence $\Phi : X \to Y$ and a morphism $f : X' \to X$, one has $f^*(\Phi) = \Phi \circ f$.

Definition 2.4. Let X, Y, S and T be smooth schemes. There is an *external product*

$$Fr_n(X,Y) \times Fr_m(S,T) \xrightarrow{-\boxtimes-} Fr_{n+m}(X \times S, Y \times T)$$

given by

$$((\alpha_1, \alpha_2, \dots, \alpha_n), f, Z, U, (\varphi_1, \varphi_2, \dots, \varphi_n), g) \boxtimes ((\beta_1, \beta_2, \dots, \beta_m), f', Z', U', (\psi_1, \psi_2, \dots, \psi_m), g') =$$
$$((\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_m), f \times f', Z \times Z', U \times U', (\varphi_1, \varphi_2, \dots, \varphi_n, \psi_1, \psi_2, \dots, \psi_m), g \times g').$$

For the constant morphism $c\colon \mathbb{A}^1 \to pt$, we set (following Voevodsky [24])

$$\Sigma = - \boxtimes (t, c, \{0\}, \mathbb{A}^1, t, c)\colon Fr_n(X,Y) \to Fr_{n+1}(X,Y)$$

and refer to it as the *suspension*.

Also, following Voevodsky [24], one puts

$$Fr(X,Y) = \operatorname{colim}(Fr_0(X,Y) \xrightarrow{\Sigma} Fr_1(X,Y) \xrightarrow{\Sigma} \dots \xrightarrow{\Sigma} Fr_n(X,Y) \xrightarrow{\Sigma} \dots)$$

and refers to it as the *set of stable framed correspondences*. The above external product induces external products

$$Fr_n(X,Y) \times Fr(S,T) \xrightarrow{-\boxtimes-} Fr(X \times S, Y \times T),$$
$$Fr(X,Y) \times Fr_0(S,T) \xrightarrow{-\boxtimes-} Fr(X \times S, Y \times T).$$

Definition 2.5. (I) Let Y be a k-smooth scheme and $S \subset Y$ be a closed subset and let $U \in Sm/k$. An *explicit framed correspondence of level $m \geqslant 0$ from U to $Y/(Y-S)$* is a tuple:

$$(Z, W, \varphi_1, \dots, \varphi_m; g : W \to Y),$$

where Z is a closed subset of $U \times \mathbb{A}^m$, finite over U, W is an étale neighborhood of Z in $U \times \mathbb{A}^m$, $\varphi_1, \dots, \varphi_m$ are regular functions on W, and g is a regular map such that $Z = Z(\varphi_1, \dots, \varphi_m) \cap g^{-1}(S)$. The set Z is called the *support* of the explicit framed correspondence. We shall also write quadruples $\Phi = (Z, W, \varphi; g)$ to denote explicit framed correspondences.

(II) Two explicit framed correspondences $(Z, W, \varphi; g)$ and $(Z', W', \varphi'; g')$ of level m are said to be *equivalent* if $Z = Z'$ and there exists an étale neighborhood W'' of Z in $W \times_{\mathbb{A}^m_U} W'$ such that $\varphi \circ pr$ agrees with $\varphi' \circ pr'$ and the morphism $g \circ pr$ agrees with $g' \circ pr'$ on W''.

(III) A *framed correspondence of level m from U to $Y/(Y-S)$* is the equivalence class of an explicit framed correspondence of level m from U to $Y/(Y-S)$. We write $Fr_m(U, Y/(Y-S))$ to denote the set of framed correspondences of level m from U to $Y/(Y-S)$. We regard it as a pointed set whose distinguished point is the class $0_{Y/(Y-S),m}$ of the explicit correspondence $(Z, W, \varphi; g)$ with $W = \emptyset$.

(IV) If $S = Y$ then the pointed set $Fr_m(U, Y/(Y-S))$ coincides with the pointed set $Fr_m(U,Y)$ of framed correspondences of level m from U to Y.

Definition 2.6. A *framed presheaf* $\mathscr{F}$ on Sm/k is a contravariant functor from $Fr_+(k)$ to the category of sets. A *framed functor* $\mathscr{F}$ on Sm/k is a contravariant functor from $Fr_+(k)$ to the category of pointed sets such that $\mathscr{F}(\emptyset) = pt$ and $\mathscr{F}(X \sqcup Y) = \mathscr{F}(X) \times \mathscr{F}(Y)$.

A framed Nisnevich sheaf on Sm/k is a framed presheaf $\mathscr{F}$ such that its restriction to Sm/k is a Nisnevich sheaf.

Note that the representable presheaves on $Fr_+(k)$ are not framed functors.

Construction 2.7. We set $Fr_+(-,Y/(Y-S)) := \bigvee_{m\geqslant 0} Fr_m(-,Y/(Y-S))$ and define the structure of a framed presheaf on it as follows. Let X,Y and S be k-smooth schemes and let

$$\Psi = (Z', \mathbb{A}^k \times V \xleftarrow{(\alpha,\pi')} W', \psi_1, \psi_2, \dots, \psi_k; g: W' \to U) \in Fr_k(V,U)$$

be an explicit correspondence of level k from V to U. Suppose

$$\Phi = (Z, \mathbb{A}^m \times U \xleftarrow{(\beta,\pi)} W, \varphi_1, \varphi_2, \dots, \varphi_m; g': W \to Y) \in Fr_m(U, Y/(Y-S))$$

is an explicit correspondence of level m from U to $Y/(Y-S)$. We define $\Psi^*(\Phi)$ as an explicit correspondence of level $k+m$ from V to $Y/(Y-S)$ as

$$(Z\times_U Z', \mathbb{A}^{k+m} \times V \xleftarrow{(\alpha,\beta,\pi')} W' \times_U W, \psi_1, \psi_2, \dots, \psi_k, \varphi_1, \varphi_2, \dots, \varphi_m,, g' \circ pr_W) \in Fr_{k+m}(V, Y/(Y-S)).$$

Clearly, the pullback operation $(\Psi, \Phi) \mapsto \Psi^*(\Phi)$ of explicit correspondences respects the equivalence relation on them. We get a pairing

$$Fr_k(V,U) \times Fr_m(U,Y/(Y-S)) \to Fr_{k+m}(V,Y/(Y-S)) \tag{2}$$

making $Fr_+(-,Y/(Y-S))$ a $Fr_+(k)$-presheaf.

Denote by

$$\sigma_{Y/(Y-S)} : Fr_m(U,Y/(Y-S)) \to Fr_{m+1}(U,Y/(Y-S)) \tag{3}$$

a map, which takes $\Phi = (Z,W,\varphi;g)$ to $(Z\times\{0\}, W\times\mathbb{A}^1, \varphi\circ pr_W, pr_{\mathbb{A}^1}; g)$.

Following Voevodsky [24] we give the following

Definition 2.8. We shall refer to the set

$$Fr(U,Y/(Y-S)) :=$$
$$= \operatorname{colim}(Fr_0(U,Y/(Y-S)) \xrightarrow{\sigma_{Y/(Y-S)}} Fr_1(U,Y/(Y-S)) \xrightarrow{\sigma_{Y/(Y-S)}} Fr_2(U,Y/(Y-S)) \cdots)$$

as the *set of stable framed correspondences from U to $Y/(Y-S)$*.

Remark 2.9. It is straightforward to check that for any framed correspondences $\Psi \in Fr_n(U',U)$ and $\Phi \in Fr_m(U,Y/(Y-S))$ one has $\sigma_{Y/(Y-S)}(\Psi^*(\Phi)) = \Psi^*(\sigma_{Y/(Y-S)}(\Phi))$. This shows that the assignment $U \mapsto Fr(U,Y/(Y-S))$ from Definition 2.8 is a framed presheaf.

For a scheme X we let Et/X denote the category of schemes separated and étale over X.

Theorem 2.10 (Voevodsky [24])**.** *Let X be a k-smooth scheme. Then for any scheme Y the functor $U \mapsto Fr_n(U,Y)$ from Et/X to $Sets_\bullet$ is a sheaf in the etale topology.*

The proof of the latter theorem given in [24] yields the following

Corollary 2.11. *Given $Y \in Sm/k$ and any closed subset S in Y, the presheaf $Fr_n(-,Y/(Y-S))$ on Sm/k is a pointed Nisnevich sheaf. Also, the framed presheaf $Fr(-,Y/(Y-S))$ is a framed Nisnevich sheaf.*

3. The Voevodsky Lemma

In this section we discuss the Voevodsky lemma computing framed correspondences in terms of morphisms of associated Nisnevich sheaves. It is crucial in our analysis. Corollary 3.3 and a sketch of its proof was communicated to us by A. Suslin. It very much helped the authors in understanding Voevodsky's notes [24].

Construction 3.1. Given an explicit framed correspondence $\alpha = (Z, W, g : W \to Y)$ from X to $Y/(Y-S)$ of level zero, consider an elementary distinguished square of the form

$$\begin{tikzcd} W - Z \arrow[r, "in"] \arrow[d, "(\rho)|_{W-Z}"'] & W \arrow[d, "\rho"] \\ X - Z \arrow[r, "in"'] & X \end{tikzcd}$$

where $\rho : W \to X$ is an étale neighborhood of Z. Let $q : Y \to F := Y/(Y-S)$ be the canonical morphism of Nisnevich sheaves. Take a morphism of sheaves $q \circ g : W \to F$ and a morphism of sheaves $c : X - Z \to F$ sending $X - Z$ to the distinguished point of F. These two morphisms agree on $W - Z$. Thus there is a unique morphism of Nisnevich sheaves

$$s_{(Z,W,g)} : X \to F$$

such that $s_{(Z,W,g)} \circ in = c$ and $s_{(Z,W,g)} \circ \rho = q \circ g$. Clearly, the sheaf morphism $s_{(Z,W,g)}$ depends only on the equivalence class of (Z, W, g) in $Fr_0(U, Y/(Y-S))$. The assignment $(Z, W, g) \mapsto s_{(Z,W,g)}$ defines a map of pointed sets

$$a_{X,Y/(Y-S)} : Fr_0(X, Y/(Y-S)) \to \mathrm{Mor}_{Shv}(X, Y/(Y-S)).$$

The map $a_{X,Y/(Y-S)}$ is natural in X with respect to morphisms of smooth varieties. Hence $a_{Y/(Y-S)} : Fr_0(-, Y/(Y-S)) \to \mathrm{Mor}_{Shv}(-, Y/(Y-S))$ is a morphism of presheaves on the category Sm/k. Using Corollary 2.11, the morphism

$$a_{Y/(Y-S)} : Fr_0(-, Y/(Y-S)) \to \mathrm{Mor}_{Shv}(-, Y/(Y-S))$$

is a morphism of Nisnevich sheaves on Sm/k.

Lemma 3.2 (Voevodsky's Lemma)**.** *Let Y be a k-smooth scheme and let $S \subset Y$ be a closed subset. The morphism of pointed Nisnevich sheaves*

$$a_{Y/(Y-S)} : Fr_0(-, Y/(Y-S)) \to \mathrm{Mor}_{Shv}(-, Y/(Y-S)) \tag{4}$$

is an isomorphism.

Proof. Since $a_{Y/(Y-S)}$ is a morphism of Nisnevich sheaves, it suffices to check that for any essentially k-smooth local Henselian U the map $a_{X,Y/(Y-S)}$ is a bijection. Let U be local essentially smooth Henselian with the closed point $u \in U$. Since U is local Henselian the following holds: for any non-empty closed subset Z in U the Henselization U^h_Z of U at Z coincides with U itself. This shows that

$$Fr_0(U, Y/(Y-S)) - 0_0 = \{(Z, U, f : U \to Y) \mid Z = f^{-1}(S), Z \neq \emptyset\} = \{f : U \to Y | f(u) \in S\}.$$

Here 0_0 is the distinguished point in $Fr_0(U, Y/(Y-S))$. The map $a_{U,Y/(Y-S)}$ takes a triple (Z, U, f) to the morphism $q \circ f : U \to Y/(Y-S)$, where $q : Y \to Y/(Y-S)$ is the quotient map.

Let $Y_u(U) \subset Y(U)$ be the subset of U-points of Y consisting of $g \in Y(U)$ with $g(u) \in S$. Then the map $Y(U) = \mathrm{Mor}_{Shv}(U, Y) \to \mathrm{Mor}_{Shv_\bullet}(U, Y/(Y-S))$ taking $f \in Y(U)$ to $q \circ f$ identifies $Y_u(U)$ with the subset $\mathrm{Mor}_{Shv_\bullet}(U, Y/(Y-S)) - *$. In fact,

$$\mathrm{Mor}_{Shv}(U, Y/(Y-S)) = (Y_u(U) \sqcup (Y-S)(U))/(Y-S)(U)) = Y_u(U) \sqcup *, \tag{5}$$

where $*$ is a singleton. Hence the map

$$a_{U,Y/(Y-S)}|_{Fr_0(U,Y/(Y-S))-\{0_0\}} : Fr_0(U, Y/(Y-S)) - \{0_0\} \to \mathrm{Mor}_{Shv_\bullet}(U, Y/(Y-S)) \setminus *$$

is a bijection. Thus the map $a_{U,Y/(Y-S)}$ is a bijection, too. $\square$

Corollary 3.3. *Let Y be a k-smooth scheme and let $S \subset Y$ be a closed subset. Let X be a k-smooth variety and $B \subset X$ a closed subset. Suppose $Fr_0(X/B, Y/(Y-S))$ is the subset of $Fr_0(X, Y/(Y-S))$ consisting of framed correspondences (Z, W, g) with $Z \cap B = \emptyset$. Then the map of pointed sets*

$$a_{X/B,Y/(Y-S)} : Fr_0(X/B, Y/(Y-S)) \to \mathrm{Mor}_{Shv_\bullet}(X/B, Y/(Y-S))$$

is a bijection.

Proof. Consider a commutative diagram

$$\begin{tikzcd} Fr_0(X/B, Y/(Y-S)) \arrow[r, "a_{X/B,Y/(Y-S)}"] \arrow[d, "in"'] & \mathrm{Mor}_{Shv_\bullet}(X/B, Y/(Y-S)) \arrow[d, "r^*"] \\ Fr_0(X, Y/(Y-S)) \arrow[r, "a_{X,Y/(Y-S)}"'] & \mathrm{Mor}_{Shv}(X, Y/(Y-S)) \end{tikzcd},$$

where r^* is induced by the quotient map $r : X \to X/(X-S)$. Since r is an epimorphism the map r^* is injective. The map *in* is an inclusion by the definition of $Fr_0(X/B, Y/(Y-S))$. By Lemma 3.2 the map $a_{X,Y/(Y-S)}$ is bijective. Thus the map $a_{X/B,Y/(Y-S)}$ is injective. It remains to check its surjectivity.

Let $g : X \to Y/(Y-S)$ be a Nisnevich sheaf morphism. It is in $\mathrm{Mor}_{Shv_\bullet}(X/B, Y/(Y-S))$ if and only if $g(B)$ is the distinguished point $*$ in $Y/(Y-S)$.

Let $g : X \to Y/(Y-S)$ be a Nisnevich sheaf morphism such that $g(B) = *$. We claim that g is in the image of $a_{X/B,Y/(Y-S)}$. By Lemma 3.2 there is an explicit framed correspondence $(Z, W, \tilde{g} : W \to Y)$ from X to $Y/(Y-S)$ such that $g = a_{X,Y/(Y-S)}((Z, W, \tilde{g} : W \to Y))$. The latter equality means that the morphism g is unique such that $g(X-Z) = *$ and $q \circ \tilde{g} = g \circ \rho$. If $g(B) = *$, then $B \cap Z = \emptyset$. Indeed, if b is a closed point of the closed subset $B \cap Z$ then $* \neq \tilde{g}(b) \in S$. On the other hand, $\tilde{g}(b) = g(b) = *$. We see that $B \cap Z = \emptyset$, $(Z, W, \tilde{g} : W \to Y)$ is in $Fr_0(X/B, Y/(Y-S))$, and $g = a_{X/B,Y/(Y-S)}((Z, W, \tilde{g} : W \to Y))$ as claimed. □

Remark 3.4. Let $n > 0$ be an integer. Let $B_n \subset (\mathbb{P}^1)^n$ be a closed subset which is the union of all subsets of the form $\mathbb{P}^1 \times \ldots \times \{\infty\} \times \ldots \times \mathbb{P}^1$. Set $B_0 = \{\infty\}$. For any $X, Y \in Sm/k$ and any $n \geqslant 0$ the inclusion

$$Fr_n(X, Y) \subset Fr_0(X \times (\mathbb{P}^1)^n / X \times B_n, Y \times \mathbb{A}^n / Y \times (\mathbb{A}^n - \{0\}))$$

is an equality. To see this, it suffices to check that any element $(Z, W, f : W \to Y \times \mathbb{A}^n)$ from $Fr_0(X \times (\mathbb{P}^1)^n / X \times B_n, Y \times \mathbb{A}^n / Y \times (\mathbb{A}^n - \{0\}))$ is contained in $Fr_n(X, Y)$. Since $Z \cap (X \times B_n) = \emptyset$, it follows that $Z \subset X \times \mathbb{A}^n$. Since Z is closed in $X \times (\mathbb{P}^1)^n$, Z is projective over X. Since Z is also affine over X, it is finite over X. Giving a morphism $f : W \to Y \times \mathbb{A}^n$ is the same as giving n functions $\varphi_1, \ldots, \varphi_n$ and a morphism $g : W \to Y$. The condition $Z = f^{-1}(Y \times \{0\})$ is equivalent to that of $Z = \{\varphi_1 = \ldots = \varphi_n = 0\}$. The desired equality is checked.

The preceding remark and Corollary 3.3 imply the following

Proposition 3.5 (Voevodsky)**.** *For any $X, Y \in Sm/k$ and any $n \geqslant 0$ the map*

$$a_{n,X,Y} = a_{X \times (\mathbb{P}^1)^n / X \times B_n, Y \times \mathbb{A}^n / Y \times (\mathbb{A}^n - \{0\})} : Fr_n(X, Y) \to$$
$$\to \mathrm{Hom}_{Shv_\bullet^{\mathrm{nis}}(Sm/k)}(X_+ \wedge (\mathbb{P}^1, \infty)^{\wedge n}, Y_+ \wedge (\mathbb{A}^1/(\mathbb{A}^1 - 0))^n) =$$
$$= \mathrm{Hom}_{Shv_\bullet^{\mathrm{nis}}(Sm/k)}(X_+ \wedge (\mathbb{P}^1, \infty)^{\wedge n}, Y_+ \wedge T^n)$$

is a bijection.

In what follows we shall write $\mathbb{P}^{\wedge n}$ for $(\mathbb{P}^1,\infty)^{\wedge n}$ and $\mathrm{Hom}(X_+\wedge\mathbb{P}^{\wedge n},Y_+\wedge T^n)$ instead of $\mathrm{Hom}_{Shv_\bullet^{\mathrm{nis}}(Sm/k)}(X_+\wedge\mathbb{P}^{\wedge n},Y_+\wedge T^n)$. We shall also write $\mathscr{F}r_n(X,Y)$ to denote $\mathrm{Hom}(X_+\wedge\mathbb{P}^{\wedge n},Y_+\wedge T^n)$.

Consider two categories $Fr_+(k)$ and $\mathscr{F}r_+(k)$, where the objects in both categories are those of Sm/k. The category $Fr_+(k)$ is defined in 2.3. The morphisms between X and Y in $\mathscr{F}r_+(k)$ are defined as $\bigvee_{n\geqslant 0}\mathscr{F}r_n(X,Y)$. The composition is defined as follows. Given two morphisms $\alpha: X_+\wedge\mathbb{P}^{\wedge m}\to Y_+\wedge T^m$ and $\beta: Y_+\wedge\mathbb{P}^{\wedge n}\to S_+\wedge T^n$, define a morphism $\beta\circ\alpha\in\mathscr{F}r_{m+n}(X,S)$ as the composite

$$X_+\wedge\mathbb{P}^{\wedge m}\wedge\mathbb{P}^{\wedge n}\xrightarrow{\alpha\wedge\mathrm{id}} Y_+\wedge T^m\wedge\mathbb{P}^{\wedge n}\cong T^m\wedge Y_+\wedge\mathbb{P}^{\wedge n}\xrightarrow{\mathrm{id}\wedge\beta} T^m\wedge Y_+\wedge T^n\cong Y_+\wedge T^m\wedge T^n.$$

It is straightforward to check commutativity of the diagram

$$\begin{tikzcd} Fr_m(X,Y)\times Fr_m(Y,S) \arrow[r, "\circ"] \arrow[d, "a\times a"'] & Fr_{m+n}(X,S) \arrow[d, "a"] \\ \mathscr{F}r_m(X,Y)\times\mathscr{F}r_n(Y,S) \arrow[r, "\circ"] & \mathscr{F}r_{m+n}(X,S). \end{tikzcd}$$

These observations imply the following

Corollary 3.6. *The functor*

$$a: Fr_+(k)\to\mathscr{F}r_+(k)$$

is an isomorphism of categories.

It is also worthwhile to make the following

Remark 3.7. One has that

$$Fr_n(X,Y/(Y-S))\subset Fr_0(X\times(\mathbb{P}^1)^n/X\times B_n, Y\times\mathbb{A}^n/(Y\times\mathbb{A}^n-S\times\{0\}))$$

is an equality. To prove this it suffices to check that any element $(Z,W,f:W\to Y\times\mathbb{A}^n)$ from $Fr_0(X\times(\mathbb{P}^1)^n/X\times B_n, Y\times\mathbb{A}^n/(Y\times\mathbb{A}^n-S\times\{0\}))$ is contained in $Fr_n(X,Y(Y-S))$. Since $Z\cap(X\times B_n)=\emptyset$, $Z\subset X\times\mathbb{A}^n$. Since Z is closed in $X\times(\mathbb{P}^1)^n$, Z is projective over X. Since Z is affine over X, it is also finite over X. Giving a morphism $f:W\to Y\times\mathbb{A}^n=\mathbb{A}^n\times Y$ is the same as giving n functions $\varphi_1,\dots,\varphi_n$ and a morphism $g:W\to Y$. The condition $Z=f^{-1}(S\times\{0\})$ is equivalent to that of $Z=\{\varphi_1=\dots=\varphi_n=0\}\cap g^{-1}(S)$. The equality is checked.

The previous remark and Corollary 3.3 imply the following

Proposition 3.8 (Voevodsky)**.** *For any $X,Y\in Sm/k$ and any $n\geqslant 0$, the map*

$$a_{n,X,Y/(Y-S)}: Fr_n(X,Y/(Y-S))\to\mathrm{Hom}_{Shv_\bullet}(X_+\wedge\mathbb{P}^{\wedge n},Y\times\mathbb{A}^n/(Y\times\mathbb{A}^n-S\times\{0\}))=$$

$$=\mathrm{Hom}_{Shv_\bullet}(X_+\wedge\mathbb{P}^{\wedge n},Y/(Y-S)\wedge T^n)$$

is a bijection.

For brevity, we shall write $\mathscr{F}r_n(X,Y/(Y-S))$ to denote $\mathrm{Hom}_{Shv_\bullet}(X_+\wedge\mathbb{P}^{\wedge n},Y/(Y-S)\wedge T^n)$.

Remark 3.9. For any pointed Nisnevich sheaf $\mathscr{F}$, the presheaf $\mathscr{F}r_+(\mathscr{F}):=\bigvee_{n\geqslant 0}\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n},\mathscr{F}\wedge T^n)$ is a framed presheaf. Indeed, define for any $U,X\in Sm/k$ and any m,n a map of pointed sets

$$\mathrm{Hom}(U_+\wedge\mathbb{P}^{\wedge m},X_+\wedge T^m)\times\mathrm{Hom}(X_+\wedge\mathbb{P}^{\wedge n},\mathscr{F}\wedge T^n)\to\mathrm{Hom}(U_+\wedge\mathbb{P}^{\wedge(m+n)},\mathscr{F}\wedge T^{m+n}).$$

If $\alpha : U_+ \wedge \mathbb{P}^{\wedge m} \to X_+ \wedge T^m$ and $s : X_+ \wedge \mathbb{P}^{\wedge n} \to \mathscr{F} \wedge T^n$ are morphisms of pointed Nisnevich sheaves, then we define $\alpha^*(s)$ as the composite morphism

$$U_+ \wedge \mathbb{P}^{\wedge m} \wedge \mathbb{P}^{\wedge n} \xrightarrow{\alpha\wedge\mathrm{id}} X_+ \wedge T^m \wedge \mathbb{P}^{\wedge n} \cong T^m \wedge X_+ \wedge \mathbb{P}^{\wedge n} \xrightarrow{\mathrm{id}\wedge s} T^m \wedge \mathscr{F} \wedge T^n \cong \mathscr{F} \wedge T^m \wedge T^n.$$

By Corollary 3.6 the categories $Fr_+(k)$ and $\mathscr{F}r_+(k)$ are isomorphic. The category isomorphism makes $\mathscr{F}r_+(\mathscr{F})$ a framed presheaf. In the special case when $\mathscr{F} = Y/(Y-S)$ the bijections $a_{n,X,Y/(Y-S)}$ induce isomorphisms of framed presheaves $a_{Y/(Y-S)} : Fr_+(-,Y/(Y-S)) \to \mathscr{F}r_+(-,Y/(Y-S))$, where $Fr_+(-,Y/(Y-S)) := \bigvee_{n\geqslant 0} Fr_n(-,Y/(Y-S))$.

Definition 3.10. For a pointed Nisnevich sheaf $\mathscr{F}$ set

$$\mathscr{F}r(-,\mathscr{F}) = \mathrm{colim}(\mathscr{F} \xrightarrow{\sigma} \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, \mathscr{F}\wedge T) \xrightarrow{\sigma} \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 2}, \mathscr{F}\wedge T^2) \xrightarrow{\sigma} \cdots),$$

where $\sigma(\Phi : U_+ \wedge \mathbb{P}^{\wedge n} \to \mathscr{F} \wedge T^n) = (U_+ \wedge \mathbb{P}^{\wedge n+1} \xrightarrow{\Phi\wedge 1_{\mathbb{P}^{\wedge 1}}} \mathscr{F} \wedge T^n \wedge \mathbb{P}^{\wedge 1} \xrightarrow{1\wedge\sigma} \mathscr{F} \wedge T^{n+1})$. Observe that $\mathscr{F}r(-,\mathscr{F})$ is a framed Nisnevich sheaf.

In the special case $\mathscr{F} = Y/(Y-S)$ one has that

$$a_{Y/(Y-S)} = \mathrm{colim}_n\, a_{n,Y/(Y-S)} : Fr(-,Y/(Y-S)) \to \mathscr{F}r(-,Y/(Y-S)) \tag{6}$$

is an isomorphism of framed sheaves.

In what follows we shall identify the isomorphic framed sheaves $Fr(-,Y/(Y-S))$ and $\mathscr{F}r(-,Y/(Y-S))$. If we write $Fr(-,Y/(Y-S))$ then we use the geometric description of the sheaf. In turn, the use of $\mathscr{F}r(-,Y/(Y-S))$ will mostly refer to the equivalent categorical description. The reader should always keep in mind the equivalent descriptions of both framed sheaves thanks to Voevodsky's Lemma.

4. Motivic version of Segal's theorem

After collecting necessary facts about framed correspondences in previous sections, we can formulate the main computational result of the paper. It is reminiscent of Segal's theorem computing the suspension spectrum $\Sigma^\infty_{S^1} X$ of a topological space X as the Segal spectrum of an associated Γ-space $B\Sigma_X$ (see [20, Section 3] for more details). The motivic counterpart of the Segal theorem computes the suspension $\mathbb{P}^1$-spectrum $\Sigma^\infty_{\mathbb{P}^1} X_+$ of a smooth algebraic variety X in terms of associated motivic spaces with framed correspondences. In a certain sense the theory of framed correspondences gives rise to an infinite $\mathbb{P}^1$-loop space machine. In order to formulate the theorem, we need some preparations.

In Section 2 we introduced Nisnevich sheaves $Fr_s(-,Y/Y-S)$ and $Fr(-,Y/Y-S)$. In the special case when $Y = X\times\mathbb{A}^n$ and $S = X\times 0$ we shall write $Fr_s(-,X\times T^n)$ and $Fr(-,X\times T^n)$ to denote the Nisnevich sheaves $Fr_s(-,X\times\mathbb{A}^n/(X\times\mathbb{A}^n - X\times 0))$ and $Fr(-,X\times\mathbb{A}^n/(X\times\mathbb{A}^n - X\times 0))$, respectively. Recall that an element of $Fr_s(-,X\times T^n)$ can be written as a tuple $(Z,W,(\varphi_1,\dots,\varphi_s;\psi_1,\dots,\psi_n) : W\to\mathbb{A}^{s+n}; g : W\to X)$ such that the support $Z = Z(\varphi_1,\dots,\varphi_s)\cap Z(\psi_1,\dots,\psi_n) = Z(\varphi_1,\dots,\varphi_s,\psi_1,\dots,\psi_n)$.

For $X\in Sm/k$ and integers $s,n\geqslant 0$ consider a morphism of pointed Nisnevich sheaves

$$\sigma_{s,n} : Fr_s(-,X\times T^n) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, Fr_s(-,X\times T^{n+1})). \tag{7}$$

It takes $(Z,W,\varphi_1,\dots,\varphi_s;\psi_1,\dots,\psi_n;g)$ to $(Z\times 0, W\times\mathbb{A}^1,\varphi_1,\dots,\varphi_s;\psi_1,\dots,\psi_n,pr_{\mathbb{A}^1};g)$. On the other hand, we have canonical maps (3)

$$\sigma_{s,X\times T^n} : Fr_s(-,X\times T^n) \to Fr_{s+1}(-,X\times T^n) \tag{8}$$

taking $(Z, W, \varphi_1, ..., \varphi_s; \psi_1, ..., \psi_n; g)$ to $(Z \times 0, W \times \mathbb{A}^1, \varphi_1, ..., \varphi_s, pr_{\mathbb{A}^1}; \psi_1, ..., \psi_n; g)$.

We have a commutative diagram

$$\begin{tikzcd} Fr_s(-, X \times T^n) \arrow[r, "\sigma_{s,n}"] \arrow[d, "\sigma_{s,X\times T^n}"'] & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, Fr_s(-, X \times T^{n+1})) \arrow[d, "(\sigma_{s,X\times T^{n+1}})_*"] \\ Fr_{s+1}(-, X \times T^n) \arrow[r, "\sigma_{s+1,n}"] & \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, Fr_{s+1}(-, X \times T^{n+1})). \end{tikzcd}$$

Passing to colimits in the s-direction, we get a morphism of pointed sheaves

$$\sigma_n : Fr(-, X \times T^n) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, Fr(-, X \times T^{n+1})).$$

We can form a $\mathbb{P}^1$-spectrum

$$Fr_{\mathbb{P}^{\wedge 1}, T}(X) = (Fr(-, X), Fr(-, X \times T), Fr(-, X \times T^2), ...) \tag{9}$$

with structure morphisms given by the σ_n morphisms.

We can now take the Suslin simplicial construction of each motivic space of the spectrum $Fr_{\mathbb{P}^{\wedge 1}, T}(X)$ to form a $\mathbb{P}^1$-spectrum

$$M_{\mathbb{P}^{\wedge 1}}(X) = (C_*Fr(-, X), C_*Fr(-, X \times T), C_*Fr(-, X \times T^2), ...)$$

with structure maps defined by $C_*(\sigma_n)$-s. Recall that for every $n \geqslant 0$, $C_*Fr(-, X \times T^n) := Fr(\Delta^\bullet \times -, X \times T^n)$. Here $\Delta^\bullet$ is the cosimplicial object $k \mapsto \mathrm{Spec}(k[t_0, \ldots, t_k]/(t_0 + \cdots + t_k - 1))$ in Sm/k.

There is a canonical morphism of $\mathbb{P}^1$-spectra

$$\varkappa : \Sigma^\infty_{\mathbb{P}^1} X_+ \to M_{\mathbb{P}^{\wedge 1}}(X)$$

given by the section $\mathrm{id}_X \in Fr_0(X, X)$ (recall that a morphism from the suspension spectrum of a variety $X \in Sm/k$ to any other spectrum is fully determined by a section of the zeroth motivic space of the spectrum at X).

By [14, 2.7] the category of simplicial Nisnevich sheaves on Sm/k has the injective local model structure with cofibrations monomorphisms and local weak equivalences. Take a fibrant replacement $C_*Fr(-, X \times T^n)_f$ of every motivic space $C_*Fr(-, X \times T^n)$ within the injective local model structure. We then arrive at a $\mathbb{P}^1$-spectrum

$$M_{\mathbb{P}^{\wedge 1}}(X)_f = (C_*Fr(-, X)_f, C_*Fr(-, X \times T)_f, C_*Fr(-, X \times T^2)_f, ...).$$

Notice that $M_{\mathbb{P}^{\wedge 1}}(X)_f$ is a fibrant replacement of the $\mathbb{P}^1$-spectrum $M_{\mathbb{P}^{\wedge 1}}(X)$ within the level injective local model structure of $\mathbb{P}^1$-spectra. Let

$$\varkappa_f : \Sigma^\infty_{\mathbb{P}^1} X_+ \to M_{\mathbb{P}^{\wedge 1}}(X) \to M_{\mathbb{P}^{\wedge 1}}(X)_f$$

denote the composite morphism.

A motivic counterpart of the Segal theorem says:

Theorem 4.1. *Let k be an infinite perfect field. Then the following statements are true:*

(1) *The morphism $\varkappa_f : \Sigma^\infty_{\mathbb{P}^1} X_+ \to M_{\mathbb{P}^{\wedge 1}}(X)_f$ is a stable motivic equivalence of $\mathbb{P}^1$-spectra.*
(2) *The $\mathbb{P}^1$-spectrum $M_{\mathbb{P}^{\wedge 1}}(X)_f$ is a motivically fibrant Ω-spectrum in positive degrees. This means that for every positive integer $n > 0$, each motivic space $C_*(Fr(-, X \times T^n))_f$ is motivically fibrant in the Morel–Voevodsky* [18] *motivic model category of simplicial Nisnevich sheaves. Furthermore, the structure map*

$$C_*(Fr(-, X \times T^n))_f \to \Omega_{\mathbb{P}^1}(C_*(Fr(-, X \times T^{n+1}))_f)$$

is a weak equivalence schemewise.

We shall also extend Theorem 4.1 to directed colimits of simplicial schemes (see Theorem 10.1). The next two corollaries are immediate consequences of the preceding theorem.

Corollary 4.2. *Let k be an infinite perfect field. Then for any positive integer $m>0$ the natural morphism of $\mathbb{P}^1$-spectra*

$$\varkappa_f:\Sigma^\infty_{\mathbb{P}^1}(X_+\wedge\mathbb{P}^{\wedge m})\to M_{\mathbb{P}^{\wedge 1}}(X\times T^m)_f:=(C_*Fr(-,X\times T^m)_f,C_*Fr(-,X\times T^{m+1})_f,C_*Fr(-,X\times T^{m+2})_f,\dots)$$

is a fibrant replacement of the suspension $\mathbb{P}^1$-spectrum $\Sigma^\infty_{\mathbb{P}^1}(X_+\wedge\mathbb{P}^{\wedge m})$ in the stable motivic model structure of $\mathbb{P}^1$-spectra in the sense of Jardine [15].

Given a $\mathbb{P}^1$-spectrum E, let $\mathscr{E}$ be an Ω-spectrum stably equivalent to E. By $\Omega^\infty_{\mathbb{P}^1}(E)$ we mean the zeroth motivic space $\mathscr{E}_0$ of $\mathscr{E}$. If $E=\Sigma^\infty_{\mathbb{P}^1}\mathscr{X}$ is the suspension $\mathbb{P}^1$-spectrum of a pointed motivic space $\mathscr{X}$, we shall write $\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(\mathscr{X})$ to denote $\Omega^\infty_{\mathbb{P}^1}(E)$.

Corollary 4.3. *Let k be an infinite perfect field. Then for any positive integer $m>0$ the natural morphism of motivic spaces*

$$C_*(Fr(X\times T^m))\to\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(X_+\wedge\mathbb{P}^{\wedge m})$$

is a stalkwise weak equivalence for the Nisnevich topology. In particular, for any field extension K/k the natural morphism of simplicial sets

$$Fr(\Delta^\bullet_K,X\times T^m)\to\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(X_+\wedge\mathbb{P}^{\wedge m})(K)$$

is a weak equivalence.

The proof of Theorem 4.1 is lengthy and is postponed. Although it states something for motivic spaces, the main strategy to prove it is to use the machinery of framed motives introduced and studied in this paper. By definition, the framed motive of a variety or a sheaf is a S^1-spectrum of simplicial Nisnevich sheaves, and hence may have nothing to do with Theorem 4.1 at first glance. But this is not the case! It is the theory of framed motives that allows us to prove Theorem 4.1.

The proof also depends on a theorem of [8] (complemented by [3] in characteristic 2) about homotopy invariant presheaves with framed correspondences and further two papers [1, 9], in which the Cancellation Theorem for framed motives of algebraic varieties is proved and framed motives of relative motivic spheres are computed.

5. Framed motives

As we have mentioned above, framed motives give the main technical tool to prove Theorem 4.1. Before introducing them, we fix the following useful

General Framework. Let $(\mathscr{V},\otimes)$ be a closed symmetric monoidal category and let $\mathscr{C}$ be a bicomplete category which is tensored and cotensored over $\mathscr{V}$. Then for every $V\in\mathscr{V}$ and $X\in\mathscr{C}$ there are defined objects $V\otimes X$, $X\otimes V$, $\underline{\mathrm{Hom}}(V,X)$ of $\mathscr{C}$. They are all functorial in V and X. Moreover, for every morphism $u:V\to V'$ in $\mathscr{V}$ the square

$$\begin{tikzcd} X \arrow[r, "-\otimes V"] \arrow[d, "-\otimes V'"'] & \underline{\mathrm{Hom}}(V,X\otimes V) \arrow[d, "u_*"] \\ \underline{\mathrm{Hom}}(V',X\otimes V') \arrow[r, "u^*"'] & \underline{\mathrm{Hom}}(V,X\otimes V') \end{tikzcd} \tag{10}$$

is commutative.

As an important example, one can take $\mathscr{V}$ to be the category $(sShv_\bullet(Sm/k),\wedge)$ of Nisnevich sheaves of pointed simplicial sets and one can take $\mathscr{C}$ to be either $sShv_\bullet(Sm/k)$ or the category of S^1-spectra of simplicial Nisnevich sheaves.

Another example is the category $(Fr_0(k),\times,pt)$ and the category $Fr_+(k)$. The functor

$$Fr_+(k)\times Fr_0(k)\to Fr_+(k)$$

takes (X,Y) to $X\times Y$.

Let Γ^{op} be the category of finite pointed sets and pointed maps. Its skeleton has objects $n^+=\{0,1,\dots,n\}$. We shall also regard each finite pointed set as a pointed smooth scheme. For example, we identify n^+ with the pointed scheme $(\sqcup_1^n \operatorname{Spec} k)_+$ with the distinguished point + corresponding to $0\in n^+$. Note that $0^+=\emptyset_+$. A Γ-*space* is a covariant functor from Γ^{op} to the category of simplicial sets taking 0^+ to a one point simplicial set. A morphism of Γ-spaces is a natural transformation of functors. A Γ-space X is called *special* if the map $X((k+l)^+)\to X(k^+)\times X(l^+)$ induced by the projections from $(k+l)^+\cong k^+\vee l^+$ to k^+ and l^+ is a weak equivalence for all k and l. X is called *very special* if it is special and the monoid $\pi_0(X(1^+))$ is a group.

In what follows we shall regard Γ^{op} as a full subcategory of $sShv_\bullet(Sm/k)$ by means of the identification $K\in\Gamma^{\mathrm{op}}$ with the pointed scheme $(\operatorname{Spec} k\sqcup\ldots\sqcup\operatorname{Spec} k)_+$, where the coproduct is indexed by the non-based elements in K.

By the General Framework above, for every $\mathscr{F},\mathscr{G}\in sShv_\bullet(Sm/k)$ the association

$$K\in\Gamma^{\mathrm{op}}\mapsto \operatorname{Hom}_{sShv_\bullet(Sm/k)}(\mathscr{F},\mathscr{G}\wedge K)$$

gives rise to a Γ-space, where the right hand side is regarded as a discrete simplicial set. In particular, if $\mathscr{F}=X_+\wedge\mathbb{P}^{\wedge n}$ and $\mathscr{G}=\mathscr{H}\wedge T^n$ with $X\in Sm/k$, $\mathscr{H}\in sShv_\bullet(Sm/k)$, we have that the association

$$K\in\Gamma^{\mathrm{op}}\mapsto \mathscr{F}r_n(X,\mathscr{H}\wedge K):=\operatorname{Hom}_{sShv_\bullet(Sm/k)}(X_+\wedge\mathbb{P}^{\wedge n},\mathscr{H}\wedge T^n\wedge K)$$

is a Γ-space. Taking the colimit over n, we get that

$$K\in\Gamma^{\mathrm{op}}\mapsto \mathscr{F}r(X,\mathscr{H}\wedge K)=\operatorname{colim}_n(\mathscr{F}r_n(X,\mathscr{H}\wedge K))$$

is a Γ-space as well.

Using the geometric description of framed correspondences for $\mathscr{H}=Y_+$, $Y\in Sm/k$, the Γ-spaces $K\in\Gamma^{\mathrm{op}}\mapsto\mathscr{F}r_n(X,\mathscr{H}\wedge K)$ and $K\in\Gamma^{\mathrm{op}}\mapsto\mathscr{F}r(X,\mathscr{H}\wedge K)$ can equivalently be defined as

$$K\in\Gamma^{\mathrm{op}}\mapsto Fr_n(X,Y\otimes K)\quad\text{and}\quad Fr(X,Y\otimes K)$$

respectively. Here $Y\otimes K:=Y\sqcup\ldots\sqcup Y$ with the coproduct indexed by the non-based elements in K. Observe that $\emptyset\otimes K=\emptyset$ and $X\otimes *=\emptyset$. These Γ-spaces are functorial in X and Y in framed correspondences of level zero. The second Γ-space is furthermore a framed functor in X.

Definition 5.1. We define a category $SmOp(Fr_0(k))$, which will often be used in our constructions. Its objects are pairs (X,U), where $X\in Sm/k$ and $U\subset X$ is an open subset. A morphism between (X,U) and (X',U') in $SmOp(Fr_0(k))$ is a morphism $f\in Fr_0(X,X')$ such that $f(U)\subset U'$. We shall also identify $X\in Sm/k$ with the pair $(X,\emptyset)\in SmOp(Fr_0(k))$.

The category $SmOp(Fr_0(k))$ is symmetric monoidal with the monoidal product $\wedge$ given by

$$(X,U)\wedge(Y,V):=(X\times Y,X\times V\cup U\times Y).$$

The point pt is its monoidal unit. Also, by $(X,U)\sqcup(Y,V)$ we shall mean $(X\sqcup Y,U\sqcup V)$.

Let $\Delta^{\mathrm{op}}SmOp(\mathrm{Fr}_0(k))$ be the category of simplicial objects in $SmOp(Fr_0(k))$. There is an obvious functor $spc : SmOp(Fr_0(k)) \to Shv_\bullet(Sm/k)$ sending an object $(X,U) \in SmOp(Fr_0(k))$ to the Nisnevich sheaf X/U. Observe that this functor is a strict symmetric monoidal functor. It induces a functor

$$spc : \Delta^{\mathrm{op}}SmOp(Fr_0(k)) \to sShv_\bullet(Sm/k),$$

taking an object $[n] \mapsto (Y_n, U_n)$ to the simplicial Nisnevich sheaf $[n] \mapsto (Y_n/U_n)$.

Given $Y \in SmOp(Fr_0(k))$ there is a Γ-space

$$K \in \Gamma^{\mathrm{op}} \mapsto Fr(X, Y \otimes K) := \mathscr{F}r(X, spc(Y) \otimes K).$$

Notice that the right hand side has an explicit geometric description thanks to Voevodsky's lemma.

Definition 5.2. (1) The *framed motive $\mathscr{M}_{fr}(\mathscr{G})$ of a pointed Nisnevich simplicial sheaf $\mathscr{G}$* is the Segal S^1-spectrum $(C_*\mathscr{F}r(-,\mathscr{G}), C_*\mathscr{F}r(-,\mathscr{G}\wedge S^1), C_*\mathscr{F}r(-,\mathscr{G}\wedge S^2), \ldots)$ associated with the Γ-space $K \in \Gamma^{\mathrm{op}} \mapsto C_*\mathscr{F}r(-,\mathscr{G}\wedge K) = \mathscr{F}r(\Delta^\bullet_+ \wedge -, \mathscr{G}\wedge K)$. More precisely, each structure map

$$C_*\mathscr{F}r(-,\mathscr{G}\wedge S^m)\wedge S^1 \to C_*\mathscr{F}r(-,\mathscr{G}\wedge S^{m+1})$$

is given as follows. For any r and m, it coincides termwise with the natural morphisms

$$\bigvee \mathscr{F}r(\Delta^r_+ \wedge -, \mathscr{G}\wedge S^m) \to \mathscr{F}r(\Delta^r_+ \wedge -, \bigvee(\mathscr{G}\wedge S^m)),$$

where coproducts are indexed by non-basepoint elements of $S^1_n = n^+$.

(2) The *framed motive $M_{fr}(Y)$ of $Y \in \Delta^{\mathrm{op}}SmOp(Fr_0(k))$* is the framed motive $\mathscr{M}_{fr}(spc(Y))$. It is the Segal S^1-spectrum $(C_*Fr(-,Y), C_*Fr(-,Y\otimes S^1), C_*Fr(-,Y\otimes S^2), \ldots)$ associated with the Γ-space $K \in \Gamma^{\mathrm{op}} \mapsto C_*Fr(-,Y\otimes K) = Fr(\Delta^\bullet \times -, Y \otimes K)$.

(3) In particular, by the *framed motive $M_{fr}(Y)$ of a smooth algebraic variety $Y \in Sm/k$* we mean the framed motive of $(Y,\emptyset) \in SmOp(Fr_0(k))$.

Remark 5.3. (1) We should point out that it is framed motives $M_{fr}(Y^\bullet)$ of simplicial k-varieties which are used to construct the functor $M_{fr} : H_{\mathbb{A}^1}(k) \to SH_{S^1}(k)$ in Section 11.

(2) Framed motives of pointed Nisnevich simplicial sheaves are not suitable for constructing such a functor. In particular, we do not expect that the functor $\mathscr{G} \mapsto \mathscr{M}_{fr}(\mathscr{G})$ from Definition 5.2(1) preserves motivic equivalences. Therefore we do not expect that for a general motivic space $\mathscr{G}$ the value of the functor M_{fr} at $\mathscr{G}$ constructed in Section 11 has the stable motivic homotopy type of $\mathscr{M}_{fr}(\mathscr{G})$ from Definition 5.2(1).

(3) It is for this reason that the main result of [9] stating that the natural motivic equivalence $X \times (\mathbb{A}^1//\mathbb{G}_m)^{\wedge n} \to X \times T^n$ of motivic spaces induces for any $n \geqslant 1$ a motivic equivalence (and even a level Nisnevich local weak equivalence) $M_{fr}(X \times (\mathbb{A}^1//\mathbb{G}_m)^{\wedge n}) \simeq \mathscr{M}_{fr}(X_+ \wedge T^n)$ of S^1-spectra is not obvious at all. Here $\mathbb{A}^1//\mathbb{G}_m$ stands for the simplicial object in $Fr_0(k)$ which is obtained by taking the pushout of the diagram $\mathbb{G}_m \hookleftarrow \mathbb{A}^1 \overset{i_0}{\hookrightarrow} \mathbb{G}_m \otimes I$ in $\Delta^{\mathrm{op}}Fr_0(k)$, where $(I,1)$ is the pointed simplicial set $\Delta[1]$ with basepoint 1 (we also refer the reader to Section 8).

In other words, the framed motive of the sheaf $X_+ \wedge T^n$ is computed as the framed motive of the associated simplicial scheme. This result is necessary to prove Theorem 4.1. It is also of independent interest.

(4) More generally, we can raise a problem asking for which motivic spaces $\mathscr{G}$ the framed motive $\mathscr{M}_{fr}(\mathscr{G})$ from Definition 5.2(1) is isomorphic in $SH_{S^1}(k)$ to its image under the functor $M_{fr} : H_{\mathbb{A}^1}(k) \to SH_{S^1}(k)$ constructed in Section 11.

(5) However, the functor $M_{fr} : \Delta^{\mathrm{op}}(Fr_0(k)) \to Sp_{S^1}(k)$ does preserve motivic equivalences (see Corollary 11.6).

(6) Framed motives of the form $M_{fr}(Y)$ with $Y \in \Delta^{\rm op} SmOp(Fr_0(k))$ are of great utility in proving Theorem 4.1.

(7) The framed motive $M_{fr}(Y)$ or $\mathscr{M}_{fr}(\mathscr{G})$ is a symmetric semistable S^1-spectrum, because it is the value of the Γ-space $C_*Fr(-,Y)$ or $C_*\mathscr{F}r(-,\mathscr{G})$ at the sphere spectrum $\mathbb{S} = (S^0, S^1, S^2, \dots)$.

Our next goal is to show that the framed motive $M_{fr}(Y)$ of $Y \in \Delta^{\rm op} SmOp(Fr_0(k))$ is a positively fibrant Ω-spectrum. To this end we need to prove the "Additivity Theorem".

6. Additivity Theorem

In this section we prove the Additivity Theorem. It is reminiscent of the Additivity Theorem in algebraic K-theory. We shall use it to produce special Γ-spaces in the sense of Segal [20] for associated motivic spaces with framed correspondences. In particular, Segal's machine then implies that the framed motive of a variety or, more generally, $Y \in \Delta^{\rm op} SmOp(Fr_0(k))$ is a positively fibrant S^1-spectrum. This means that it is sectionwise an Ω-spectrum in positive degrees.

Following [24] for any $X \in Sm/k$ denote by m the explicit correspondence from X to $X \sqcup X$ with $U = (\mathbb{A}^1 - \{0\} \sqcup \mathbb{A}^1 - \{1\})_X$, $\varphi = (t-1) \sqcup t$, where $t : \mathbb{A}^1_X \to X$ is the projection and $g : (\mathbb{A}^1 - \{0\} \sqcup \mathbb{A}^1 - \{1\})_X \to X \sqcup X$. For any framed functor $\mathscr{F}$ it defines a map

$$\mathscr{F}(X) \times \mathscr{F}(X) = \mathscr{F}(X \sqcup X) \xrightarrow{m^*} \mathscr{F}(X).$$

Definition 6.1. Let $\mathscr{F}$ and $\mathscr{G}$ be two presheaves of sets on the category of k-smooth schemes and let $\varphi_0, \varphi_1 : \mathscr{F} \rightrightarrows \mathscr{G}$ be two morphisms. An $\mathbb{A}^1$-*homotopy* between φ_0 and φ_1 is a morphism $H : \mathscr{F} \to \underline{\rm Hom}(\mathbb{A}^1, \mathscr{G})$ such that $H_0 = \varphi_0$ and $H_1 = \varphi_1$. We write $\varphi_0 \sim \varphi_1$ if there is an $\mathbb{A}^1$-homotopy between φ_0 and φ_1. We say that $\varphi_0, \varphi_n : \mathscr{F} \rightrightarrows \mathscr{G}$ are $\mathbb{A}^1$-*homotopic*, if there is a chain of morphisms $\varphi_1, ..., \varphi_n$ such that $\varphi_i \sim \varphi_{i+1}$ for $i = 0, 1, ..., n-1$.

We now want to discuss matrix actions on framed correspondences and $\mathbb{A}^1$-homotopies associated to them. Let Y be a k-smooth scheme and let $A \in GL_n(k)$ be a matrix. Then A defines an automorphism $\varphi_{A \perp Id_m} : Fr_{n+m}(-,Y) \to Fr_{n+m}(-,Y)$ of the presheaf $Fr_{n+m}(-,Y)$ in the following way. Given $W \in Sm/k$ and an explicit framed correspondence of level n

$$\Phi = (\mathbb{A}^{n+m}_X \xleftarrow{p} U, \varphi : U \to \mathbb{A}^{n+m}_k, g : U \to Y) \in Fr_{n+m}(X,Y),$$

set $\varphi_{A \perp Id_m}(\Phi) = ((A \perp Id_m) \circ p, (A \perp Id_m) \circ \varphi, g)$. Within the notation of Remark 2.2

$$\varphi_{A \perp Id_m}(Z, U, (\varphi_1, \varphi_2, \dots, \varphi_{n+m}), g)) := ((A \perp Id_m)(Z), U, (A \perp Id_m) \circ (\varphi_1, \varphi_2, \dots, \varphi_{n+m}), g),$$

where $A \perp Id_m$ is a linear automorphism of $\mathbb{A}^{n+m}_k$. In more detail, if $(U, \mathbb{A}^{n+m}_X \xleftarrow{p} U, s : Z \to U)$ is an étale neighborhood of Z in $\mathbb{A}^{n+m}_X$, then we take

$$(U, \mathbb{A}^{n+m}_X \xleftarrow{(A \perp Id_m) \circ p} U, s \circ ((A \perp Id_m)^{-1}|_{(A \perp Id_m)(Z)}) : (A \perp Id_m)(Z) \to U)$$

as an étale neighborhood of $(A \perp Id_m)(Z)$ in $\mathbb{A}^{n+m}_X$. Clearly, $\sigma \circ \varphi_{A \perp Id_m} = \varphi_{A \perp Id_{m+1}} \circ \sigma$. Hence the maps $\varphi_{A \perp Id_m}$ give rise to a unique automorphism of presheaves on Sm/k

$$\varphi_A : Fr(-,Y) \to Fr(-,Y) \tag{11}$$

such that for any $m \geqslant 0$ one has $\varphi_A|_{Fr_{n+m}(-,Y)} = \varphi_{A \perp Id_m}$.

Definition 6.2. Let $A \in SL_n(k)$. Choose a matrix $A_s \in SL_n(k[s])$ such that $A_0 = id$ and $A_1 = A$. The matrices $A_s \perp Id_m \in SL_{n+m}(k[s])$, regarded as morphisms $\mathbb{A}^{n+m} \times \mathbb{A}^1 \to \mathbb{A}^{n+m}$, give rise to an $\mathbb{A}^1$-homotopy h between the automorphisms id and φ_A of $Fr(-,Y)$ as follows. Given $a = (Z, U, (\varphi_1, \varphi_2, ..., \varphi_{n+m}), g) \in Fr_n(-,Y)$, one sets

$$h(a) = (Z \times \mathbb{A}^1, U \times \mathbb{A}^1, (A_s \perp Id_m) \circ (\varphi \times id_{\mathbb{A}^1}), g \circ pr_U) \in Fr_n(W \times \mathbb{A}^1, Y).$$

In this way we get a morphism $h : Fr(-,Y) \to Fr(- \times \mathbb{A}^1, Y)$ such that $h_0 = id$ and $h_1 = \varphi_A$. We see that h is an $\mathbb{A}^1$-homotopy between the identity and φ_A.

Definition 6.3. Let $\tau \in \Sigma_n$ be an even permutation regarded as a matrix in $SL_n(k)$. Let $A_s \in SL_n(k[s])$ be such that $A_0 = id$ and $A_1 = \tau$. Then the morphism h from Definition 6.2 defines an $\mathbb{A}^1$-homotopy between the automorphisms φ_{id} and φ_τ of $Fr(-,Y)$.

We are now in a position to prove the Additivity Theorem. The category $SmOp(Fr_0(k))$ was introduced in Definition 5.1. Given a simplicial object Y of $SmOp(Fr_0(k))$, by $C_*Fr(-,Y)$ we mean as usual the diagonal of the bisimplicial presheaf $(m,n) \mapsto Fr(\Delta^m \times -, Y_n)$.

Theorem 6.4 (Additivity)**.** *For any two simplicial objects Y_1, Y_2 in $SmOp(Fr_0(k))$, the natural map $\alpha : Fr(-, Y_1 \sqcup Y_2) \to Fr(-,Y_1) \times Fr(-,Y_2)$, given by the morphisms* $\mathrm{id} \sqcup \emptyset : Y_1 \sqcup Y_2 \to Y_1, \emptyset \sqcup \mathrm{id} : Y_1 \sqcup Y_2 \to Y_2$, *induces a map of simplicial framed presheaves*

$$C_*(\alpha) : C_*Fr(-, Y_1 \sqcup Y_2) \to C_*Fr(-,Y_1) \times C_*Fr(-,Y_2),$$

which is a schemewise weak equivalence.

Proof. Since the realization functor takes simplicial weak equivalences of simplicial sets to weak equivalences, it is enough to prove the theorem for any objects Y_1, Y_2 in $SmOp(Fr_0(k))$. Moreover, it is sufficient to prove that for any $X \in Sm/k$ and any finite unpointed simplicial set K, the map

$$C_*(\alpha)(X,K) : [K, C_*Fr(X, Y_1 \sqcup Y_2)] \to [K, C_*Fr(X,Y_1) \times C_*Fr(X,Y_2)]$$

between the Hom-sets in the homotopy category $\mathrm{Ho}(sSets)$ of unpointed simplicial sets is a bijection. Furthermore, for the simplicity of the exposition we shall assume that Y_1, Y_2 are just k-smooth varieties. Let $Y = Y_1 \sqcup Y_2$. Define a morphism

$$\beta : Fr(-,Y_1) \times Fr(-,Y_2) \to Fr(-,Y)$$

of presheaves on Sm/k by the following commutative diagram:

$$\begin{tikzcd}
Fr(X,Y_1) \times Fr(X,Y_2) \arrow[r, "\beta"] \arrow[d, "(i_1)_* \times (i_2)_*"'] & Fr(X,Y) \\
Fr(X,Y) \times Fr(X,Y) & \\
Fr(X \sqcup X, Y) \arrow[u, "(j_1)^* \times (j_2)^*", "\cong"'] \arrow[uur, bend right, "m^*"'] &
\end{tikzcd}$$

Here $i_\varepsilon : Y_\varepsilon \to Y$, $\varepsilon = 1,2$, is the corresponding embedding.

We claim that $(\beta\alpha)|_{Fr_{2n}(-,Y)}$ is $\mathbb{A}^1$-homotopic to the inclusion $in_{2n} : Fr_{2n}(-,Y) \to Fr(-,Y)$ and $(\alpha\beta)|_{Fr_{2n}(-,Y_1) \times Fr_{2n}(-,Y_2)}$ is $\mathbb{A}^1$-homotopic to the inclusion

$$inc^1_{2n} \times inc^2_{2n} : Fr_{2n}(-,Y_1) \times Fr_{2n}(-,Y_2) \to Fr(-,Y_1) \times Fr(-,Y_2).$$

The first of these $\mathbb{A}^1$-homotopies will imply that $C_*(\beta) \circ C_*(\alpha)|_{C_*(Fr_{2n}(-,Y))}$ is simplicially homotopic to the inclusion $C_*(in_{2n})$, because $C_*(-)$ converts $\mathbb{A}^1$-homotopies into simplicial ones.

The second of these $\mathbb{A}^1$-homotopies will imply that $(C_*(\alpha)\circ C_*(\beta))|_{C_*(Fr_{2n}(-,Y_1)\times C_*(Fr_{2n}(-,Y_1))}$ is simplicially homotopic to the inclusion $C_*(inc^1_{2n}\times inc^2_{2n})$. It will follow that for any $X\in Sm/k$ and any finite simplicial set K the map $C_*(\alpha)(X,K)$ is bijective. Indeed, one should use the fact that the functor $[K,-]:\mathrm{Ho}(sSets)\to\mathrm{Ho}(sSets)$ commutes with sequential colimits whenever K is finite. It therefore remains to prove the claim.

Firstly, let us focus on the morphism $\alpha\beta$. The map $\alpha\beta$ is of the form

$$\rho_1\times\rho_2: Fr(X,Y_1)\times Fr(X,Y_2)\to Fr(X,Y_1)\times Fr(X,Y_2).$$

Here ρ_1 takes a framed correspondence $(Z_1,W_1,\underline{\varphi}^{(1)};g_1)$ of level n to the framed correspondence $(0\times Z_1,\mathbb{A}^1\times W_1,t_1,\underline{\varphi}^{(1)};g_1)$ of level $n+1$, and ρ_2 takes a framed correspondence $(Z_2,W_2,\underline{\varphi}^{(2)};g_2)$ of level n to the framed correspondence $(1\times Z_2,\mathbb{A}^1\times W_2,t_0-1,\underline{\varphi}^{(2)};g_2)$ of level $n+1$. We first observe that the morphism ρ_2 is $\mathbb{A}^1$-homotopic to the morphism $\rho_2^0: Fr(-,Y_2)\to Fr(-,Y_2)$ taking a framed correspondence $(Z_2,W_2,\underline{\varphi}^{(2)};g_2)$ of level n to the framed correspondence $(0\times Z_2,\mathbb{A}^1\times W_2,t_0,\underline{\varphi}^{(2)};g_2)$ of level $n+1$. To see this, send a framed correspondence $(Z_2,W_2,\underline{\varphi}^{(2)};g_2)$ of level n to the framed correspondence

$$(\Delta\times Z_2,\mathbb{A}^1\times\mathbb{A}^1\times W_2,t_0-\lambda,\underline{\varphi}^{(2)};g_2)$$

of level $n+1$. Evaluating the latter framed correspondence at $\lambda=1$, we get $\rho_2(Z_2,W_2,\underline{\varphi}^{(2)};g_2)$. Evaluating the same framed correspondence at $\lambda=0$, we get $(0\times Z_2,\mathbb{A}^1\times W_2,t_0,\underline{\varphi}^{(2)};g_2)$.

Let $n>0$ be an even integer and let $\tau\in\Sigma_{n+1}$ be the even permutation $(n+1,1,2,...,n)$. Let h_1 denote the associated $\mathbb{A}^1$-homotopy from Definition 6.3 between the automorphisms φ_{id} and φ_τ of $Fr(-,Y_1)$. Then $h_1\circ inc^1_n$ is an $\mathbb{A}^1$-homotopy between inc^1_n and $\varphi_\tau\circ inc^1_n=\rho_1|_{Fr_n(-,Y_1)}$: $Fr_n(-,Y_1)\to Fr(-,Y_1)$. Let h_2 be the associated $\mathbb{A}^1$-homotopy from Definition 6.3 between the automorphisms φ_{id} and φ_τ of $Fr(-,Y_2)$. Then $h_2\circ inc^2_n$ is an $\mathbb{A}^1$-homotopy between inc^2_n and $\varphi_\tau\circ inc^2_n=\rho^0_2|_{Fr_n(-,Y_2)}: Fr_n(-,Y_2)\to Fr(-,Y_2)$. Thus $(\alpha\beta)|_{Fr_{2n}(-,Y_1)\times Fr_{2n}(-,Y_2)}$ is $\mathbb{A}^1$-homotopic to the inclusion $inc^1_{2n}\times inc^2_{2n}$.

Next, let us focus on the morphism $\beta\alpha$. Since $Y=Y_1\sqcup Y_2$ every framed correspondence of level n from X to Y is of the form $a=(Z_1\sqcup Z_2,W_1\sqcup W_2,\underline{\varphi}^{(1)}\sqcup\underline{\varphi}^{(2)};g_1\sqcup g_2)$. One has,

$$\beta\alpha(a)=(0\times Z_1\sqcup 1\times Z_2,\mathbb{A}^1\times W_1\sqcup\mathbb{A}^1\times W_2),(t_0,\underline{\varphi}^{(1)})\sqcup(t_0-1,\underline{\varphi}^{(2)});g_1\sqcup g_2).$$

Firstly, the morphism $\beta\alpha$ is $\mathbb{A}^1$-homotopic to the morphism $\rho^0: Fr(-,Y)\to Fr(-,Y)$ taking a framed correspondence $(Z,W,\underline{\varphi};g)$ to the framed correspondence $(0\times Z,\mathbb{A}^1\times W,t_0,\underline{\varphi};g_2)$ of level $n+1$. To see this, send a framed correspondence $a=(Z_1\sqcup Z_2,W_1\sqcup W_2,\underline{\varphi}^{(1)}\sqcup\underline{\varphi}^{(2)};g_1\sqcup g_2)$ of level n to the framed correspondence of level $n+1$

$$(\mathbb{A}^1\times Z_1\sqcup\Delta\times Z_2,\mathbb{A}^1\times\mathbb{A}^1\times(W_1\sqcup W_2),(t_0,\underline{\varphi}^{(1)})\sqcup((t_0-\lambda),\underline{\varphi}^{(2)});g_1\sqcup g_2).$$

Evaluating the latter framed correspondence of level $n+1$ at $\lambda=1$, we get $\beta\alpha(a)$. Evaluating the same framed correspondence $\lambda=0$, we get $\rho^0(a)$. Furthermore, using the associated homotopy from Definition 6.3, we see that $\rho^0|_{Fr_{2n}(-,Y)}$ is $\mathbb{A}^1$-homotopic to the inclusion in_{2n}. Hence $\beta\alpha|_{Fr_{2n}(-,Y)}$ is $\mathbb{A}^1$-homotopic to the inclusion in_{2n}, as was to be proved. □

Now the Additivity Theorem 6.4 together with the Segal machine [20] imply the following

Theorem 6.5. *Let $Y\in\Delta^{\mathrm{op}}SmOp(Fr_0(k))$. Then the Γ-space $K\in\Gamma^{\mathrm{op}}\mapsto C_*Fr(-,Y\otimes K)$ is sectionwise special. As a consequence, the framed motive $M_{fr}(Y)$ of Y is sectionwise a positively fibrant Ω-spectrum, which is sectionwise (respectively locally in the Nisnevich topology) an*

Ω*-spectrum whenever the motivic space* $C_*Fr(-,Y)$ *is sectionwise (respectively locally in the Nisnevich topology) connected.*

Remark 6.6. Whenever we say that $M_{fr}(Y)$ is a (positively) fibrant Ω-spectrum we tacitly assume that each of its spaces $C_*Fr(-,Y\otimes S^n)$ is replaced with $Ex^\infty(C_*Fr(-,Y\otimes S^n))$, where Ex^∞ refers to Kan's complex. The spaces $Ex^\infty(C_*Fr(-,Y\otimes S^n))$ are then spaces with framed correspondences and sectionwise fibrant simplicial sets. A detailed description of the spaces will be given in Section 12. We can equally take any naive sectionwise fibrant resolution functor in the category of spaces with framed correspondences (which exists by standard arguments) in place of Ex^∞.

It is worth mentioning that the latter theorem is a kind of the "Cancellation Theorem for framed motives in the S^1-direction" (the meaning of this will become clear in the proof of Theorem 4.1(2)). One should also stress that the motivic spaces $C_*Fr(-,X\times T^n)_f$ from Theorem 4.1 are zero spaces of sheaves of S^1-spectra $M_{fr}(X\times T^n)_f$ (these are level local Nisnevich replacements of the framed motives $M_{fr}(X\times T^n)$). So each space $C_*Fr(-,X\times T^n)_f$ is part of the S^1-spectrum $M_{fr}(X\times T^n)_f$.

If we can prove that each S^1-spectrum $M_{fr}(X\times T^n)$, $n>0$, is motivically fibrant, then each space $C_*Fr(-,X\times T^n)_f$ becomes motivically fibrant (what is claimed in Theorem 4.1). Therefore our next goal is to investigate these kinds of fibrant motivic spaces coming from relevant S^1-spectra in more detail.

7. Fibrant motivic spaces generated by S^1-spectra

Let $sShv_\bullet(Sm/k)$ denote the category of Nisnevich sheaves of pointed simplicial sets. It has the injective model structure [14] in which cofibrations are the monomorphisms and weak equivalences are stalkwise weak equivalences of simplicial sets. The category of S^1-spectra $Sp_{S^1}(sShv_\bullet(Sm/k))$ associated with $sShv_\bullet(Sm/k)$ will also be called the *category of ordinary S^1-spectra of simplicial Nisnevich sheaves*. It has level and stable model structures (the standard references here are [12, 15]). In this section we describe a class of motivic spaces coming from ordinary S^1-spectra of simplicial Nisnevich sheaves, which are fibrant in the motivic model category $sShv_\bullet(Sm/k)_{\mathrm{mot}}$ of Morel–Voevodsky [18]. This class occurs in our analysis. Recall that $sShv_\bullet(Sm/k)_{\mathrm{mot}}$ is obtained from $sShv_\bullet(Sm/k)$ by Bousfield localization with respect to the projections $p:X\times\mathbb{A}^1\to X$, $X\in Sm/k$. As above, the level/stable model category of S^1-spectra associated with $sShv_\bullet(Sm/k)_{\mathrm{mot}}$ will also be called the *level/stable injective model category of S^1-spectra.*

Proposition 7.1. *Let E be an S^1-spectrum in the category of simplicial Nisnevich sheaves such that each space $E_n\in sShv_\bullet(Sm/k)$ of the spectrum is fibrant in $sShv_\bullet(Sm/k)$. Suppose E is sectionwise an Ω-spectrum in the category of ordinary S^1-spectra of pointed simplicial sets. Suppose E is locally (-1)-connected in the Nisnevich topology. Finally suppose that for any integer n, the Nisnevich sheaf $\pi_n^{\mathrm{nis}}(E)$ is strictly homotopy invariant. Then the following statements are true:*

(1) *every motivic space E_n of E is motivically fibrant;*
(2) *E is fibrant in the stable injective motivic model structure of S^1-spectra.*

Proof. (1). Since E is sectionwise an Ω-spectrum, every E_n, $n\geqslant 0$, is sectionwise fibrant. Therefore it suffices to prove that E_n is $\mathbb{A}^1$-local. So we have to check that for any smooth variety

X the projection $p : X \times \mathbb{A}^1 \to X$ induces a weak equivalence of simplicial sets $p^* : E_n(X) \to E_n(X \times \mathbb{A}^1)$. Since E is sectionwise an Ω-spectrum it suffices to check that the pull-back map $p^* : E(X) \to E(X \times \mathbb{A}^1)$ is a stable equivalence of ordinary S^1-spectra. So it is sufficient to verify that for any integer r the map $p^* : \pi_r(E(X)) \to \pi_r(E(X \times \mathbb{A}^1))$ is an isomorphism. Consider two convergent spectral sequences

$$H^p_{Nis}(X, \pi^{\mathrm{nis}}_q(E)) \Rightarrow \pi_{q-p}(E(X)) \quad \text{and} \quad H^p_{Nis}(X \times \mathbb{A}^1, \pi^{\mathrm{nis}}_q(E)) \Rightarrow \pi_{q-p}(E(X \times \mathbb{A}^1)).$$

The projection p induces a pull-back morphism between these two spectral sequences. This morphism is an isomorphism on the second page, because each Nisnevich sheaf $\pi^{\mathrm{nis}}_n(E)$ is strictly homotopy invariant by assumption. Hence $p^* : \pi_r(E(X)) \to \pi_r(X \times \mathbb{A}^1)$ is an isomorphism. Assertion (2) easily follows from assertion (1). □

We refer the reader to [25] for the notion of radditive presheaves. Below we shall need the following

Lemma 7.2. *Let F be a radditive framed presheaf of Abelian groups. Then the associated sheaf in the Nisnevich topology has a unique structure of a framed presheaf such that the map $F \to F_{\mathrm{nis}}$ is a map of framed presheaves.*

Proof. This is proved in [24, 4.5]. □

Remark 7.3. We should stress that [24, 4.5] used in the proof of the preceding lemma is not true unless F is radditive.

Corollary 7.4. *Let k be an infinite perfect field and E be an S^1-spectrum of simplicial Nisnevich sheaves with framed correspondences. Suppose E is locally an Ω-spectrum in the Nisnevich topology. Suppose it is (-1)-connected locally in the Nisnevich topology. Finally, suppose that for any integer n the Nisnevich presheaf $\pi_n(E)$ is homotopy invariant, quasi-stable and radditive. Let $E \to E^f$ be a fibrant replacement of E in the level injective model structure of ordinary sheaves of S^1-spectra. Then the following statements are true:*

(1) *each motivic space E^f_n is motivically fibrant;*
(2) *the spectrum E^f is fibrant in the stable injective motivic model category of S^1-spectra.*

Proof. The Nisnevich presheaf $\pi_n(E)$ is a radditive framed presheaf. Hence the associated Nisnevich sheaf $\pi^{\mathrm{nis}}_n(E)$ is equipped with a unique structure of a framed presheaf such that the canonical morphism $\pi_n(E) \to \pi^{\mathrm{nis}}_n(E)$ is a morphism of framed presheaves by Lemma 7.2. By [8, 1.1] (complemented by [3] in characteristic 2) the Nisnevich sheaf $\pi^{\mathrm{nis}}_n(E)$ is strictly homotopy invariant, and hence so is the Nisnevich sheaf $\pi^{\mathrm{nis}}_n(E^f)$ of E^f. Our statement now follows from the previous proposition. □

Corollary 7.5. *Let k be an infinite perfect field and let Y be a simplicial object in $SmOp(Fr_0(k))$. Suppose the simplicial Nisnevich sheaf $C_*Fr(Y)$ is locally connected in the Nisnevich topology. Let $M_{fr}(Y) \to M_{fr}(Y)_f$ be a fibrant replacement in the level injective model structure of ordinary sheaves of S^1-spectra. Then:*

(1) *$M_{fr}(Y)_f$ is fibrant in the stable injective motivic model category of S^1-spectra;*
(2) *for any $n \geqslant 0$ and any fibrant replacement $C_*(Fr(-, Y \otimes S^n)) \to C_*(Fr(-, Y \otimes S^n))_f$ in $sShv_\bullet(Sm/k)$, the space $C_*(Fr(-, Y \otimes S^n))_f$ is fibrant in $sShv_\bullet(Sm/k)_{\mathrm{mot}}$.*

Proof. The zeroth space $C_*Fr(-, Y)$ of the framed spectrum $M_{fr}(Y)$ is locally connected. Hence the framed spectrum $M_{fr}(Y)$ is locally an Ω-spectrum by the Segal machine [20] and Theorem 6.5. The presheaves $\pi_n(M_{fr}(Y))$ are homotopy invariant, quasi-stable and radditive framed presheaves. Corollary 7.4 implies assertion (1). To prove the second one, note that any two fibrant replacements of $C_*Fr(Y)$ in $sShv_\bullet(Sm/k)$ are sectionwise weakly equivalent. Hence Corollary 7.4(1) implies the second assertion. $\square$

Under the notation of the preceding corollary we can now prove the following

Corollary 7.6. *Let k be an infinite perfect field. Then the following statements are true:*

(1) *For any integer $n > 0$, the S^1-spectrum $M_{fr}(X \times T^n)_f$ is motivically fibrant and the motivic space $C_*Fr(X \times T^n)_f$ is motivically fibrant.*
(2) *For any integer $n \geqslant 0$, the S^1-spectrum $M_{fr}(X \times T^n \times \mathbb{G}_m^{\wedge 1} \otimes S^1)_f$ is motivically fibrant and the motivic space $C_*(Fr(X \times T^n \times \mathbb{G}_m^{\wedge 1} \otimes S^1))_f$ is motivically fibrant.*
(3) *For any integer $n \geqslant 0$, the S^1-spectrum $M_{fr}(X \times T^n \times (\mathbb{A}^1//\mathbb{G}_m))_f$ is motivically fibrant and the motivic space $C_*Fr(X \times T^n \times (\mathbb{A}^1//\mathbb{G}_m))_f$ is motivically fibrant.*

Proof. By [9, A.1] the spaces $C_*Fr(X \times T^n)$, $C_*Fr(X \times T^n \times (\mathbb{A}^1//\mathbb{G}_m))$ of the corollary are locally connected in the Nisnevich topology. The space $C_*(Fr(X \times T^n \times \mathbb{G}_m^{\wedge 1} \otimes S^1))$ is, moreover, sectionwise connected. Now our assertions follow from Corollary 7.5. $\square$

We should stress that the previous corollary is of great utility in the proof of Theorem 4.1.

8. Comparing framed motives

One of the key properties of framed motives is that they convert motivic equivalences between certain motivic spaces to Nisnevich local weak equivalences. Some such motivic equivalences are discussed in this section. Its main result, Theorem 8.2, is an essential step in proving Theorem 4.1. We start with preparations.

Every category $\mathscr{A}$ with coproducts and zero object 0 has a natural action of finite pointed sets. For example, $\mathscr{A} = Fr_0(k)$ or, more generally, $\mathscr{A} = SmOp(Fr_0(k))$. Precisely, if $A \in \mathscr{A}$ and $(K,*)$ is a finite pointed set, then we set $A \otimes K := A \sqcup \ldots \sqcup A$, where the coproduct is taken over non-base elements of K. Clearly, $A \otimes K$ is functorial in A and K. Note that $A \otimes * = 0$ and $0 \otimes K = 0$.

This action is extended to an action of finite pointed simplicial sets on the category $\Delta^{\mathrm{op}}\mathscr{A}$ of simplicial objects in $\mathscr{A}$. Let $(I,1)$ denote the pointed simplicial set $\Delta[1]$ with basepoint 1. The *cone* of $A \in \mathscr{A}$ is the simplicial object $A \otimes I$ in $\mathscr{A}$. There is a natural morphism $i_0 : A \to A \otimes I$ in $\Delta^{\mathrm{op}}\mathscr{A}$. Given a morphism $f : A \to B$ in $\mathscr{A}$, denote by $B//_f A$ a simplicial object in $\mathscr{A}$ which is obtained from the pushout in $\Delta^{\mathrm{op}}\mathscr{A}$ of the diagram

$$B \xleftarrow{f} A \xhookrightarrow{i_0} A \otimes I$$

We can think of $B//_f A$ as a cone of f. In practice, if A is a subobject of B, we shall also write $B//A$ to denote the simplicial object $B//_\iota A$ in $\mathscr{A}$ with $\iota : A \to B$ the inclusion. We have a sequence of simplicial objects in $\mathscr{A}$

$$A \xrightarrow{f} B \to B//_f A \to A \otimes S^1.$$

In practice, this sequence is a typical "triangle" of an associated triangulated category (see, e.g., the proof of Theorem 8.2).

Notation 8.1. (1) In the particular example when $\mathscr{A} = Fr_0(k)$ and (X,x) is a pointed smooth variety, we shall write $X^{\wedge 1}$ to denote the cone $X//x$ of the inclusion $x \hookrightarrow X$. The most common example is $\mathbb{G}_m^{\wedge 1}$ given by the pointed scheme $(\mathbb{G}_m, 1)$. Regarding $\Delta^{\mathrm{op}}Fr_0(k)$ as a full subcategory of the symmetric monoidal category $\Delta^{\mathrm{op}} SmOp(Fr_0(k))$, we can take the nth monoidal power of $X//x$ for every $n > 0$, which we shall denote by $X^{\wedge n}$. The most common example is $\mathbb{G}_m^{\wedge n}$.

(2) If X is an open subset of $Y \in Sm/k$, we shall denote by $(Y//X)^{\wedge n}$ the nth monoidal power of $Y//X \in \Delta^{\mathrm{op}} Fr_0(k)$. The most common example will be $(\mathbb{A}^1//\mathbb{G}_m)^{\wedge n}$.

If $\mathscr{A} = SmOp(Fr_0(k))$ then the symmetric monoidal product on $SmOp(Fr_0(k))$ defined above gives rise to a natural pairing

$$SmOp(Fr_0(k)) \times \Delta^{\mathrm{op}} Fr_0(k) \to \Delta^{\mathrm{op}} SmOp(Fr_0(k)).$$

Composing it with the framed motive functor, we get a functor

$$M_{fr} : SmOp(Fr_0(k)) \times \Delta^{\mathrm{op}} Fr_0(k) \to Sp_{S^1}(sShv_\bullet(Sm/k)).$$

Taking pairings of $(X \times \mathbb{A}^n, X \times \mathbb{A}^n - X \times 0) \in SmOp(Fr_0(k))$ with $\mathbb{A}^1//\mathbb{G}_m \in \Delta^{\mathrm{op}} Fr_0(k)$ and $\mathbb{G}_m^{\wedge 1} \otimes S^1 \in \Delta^{\mathrm{op}} Fr_0(k)$, we get the framed motives $M_{fr}(X \times T^n \times (\mathbb{A}^1//\mathbb{G}_m))$ and $M_{fr}(X \times T^n \times \mathbb{G}_m^{\wedge 1} \otimes S^1)$, respectively.

Consider a commutative diagram in $\Delta^{\mathrm{op}} Fr_0(k)$

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[r] \arrow[d] & \mathbb{A}^1//\mathbb{G}_m \arrow[d, "\alpha"] \\
\mathbb{G}_m^{\wedge 1} \arrow[r] \arrow[d, equal] & \mathbb{A}^{\wedge 1} \arrow[r] \arrow[d] & \mathbb{A}^{\wedge 1}//\mathbb{G}_m^{\wedge 1} \arrow[d, "\beta"] \\
\mathbb{G}_m^{\wedge 1} \arrow[r] & \emptyset \arrow[r] & \mathbb{G}_m^{\wedge 1} \otimes S^1
\end{tikzcd} \tag{12}$$

It induces a morphism of framed motives

$$\beta_* \alpha_* : M_{fr}(X \times T^n \times (\mathbb{A}^1//\mathbb{G}_m)) \to M_{fr}(X \times T^n \times \mathbb{G}_m^{\wedge 1} \otimes S^1), \quad n \geqslant 0.$$

The main result of this section is as follows.

Theorem 8.2. *Let k be an infinite perfect field. Then the morphism $\beta_* \alpha_*$ is a stable Nisnevich local weak equivalence of S^1-spectra.*

We postpone the proof of the theorem. It requires the language of "linear framed motives".

Definition 8.3. Let X and Y be smooth schemes. Denote by

- $\diamond$ $\mathbb{Z}Fr_n(X,Y) := \widetilde{\mathbb{Z}}[Fr_n(X,Y)] = \mathbb{Z}[Fr_n(X,Y)]/\mathbb{Z} \cdot 0_n$, i.e the free Abelian group generated by the set $Fr_n(X,Y)$ modulo $\mathbb{Z} \cdot 0_n$;
- $\diamond$ $\mathbb{Z}F_n(X,Y) := \mathbb{Z}Fr_n(X,Y)/A$, where A is the subgroup generated by the elements

$$(Z \sqcup Z', U, (\varphi_1, \varphi_2, \ldots, \varphi_n), g) - \\ - (Z, U \setminus Z', (\varphi_1, \varphi_2, \ldots, \varphi_n)|_{U \setminus Z'}, g|_{U \setminus Z'}) - (Z', U \setminus Z, (\varphi_1, \varphi_2, \ldots, \varphi_n)|_{U \setminus Z}, g|_{U \setminus Z}).$$

We shall also refer to the latter relation as the *additivity property for supports*. In other words, it says that a framed correspondence in $\mathbb{Z}F_n(X,Y)$ whose support is a disjoint union $Z \sqcup Z'$ equals the sum of the framed correspondences with supports Z and Z' respectively. Note that $\mathbb{Z}F_n(X,Y)$ is $\mathbb{Z}[Fr_n(X,Y)]$ modulo the subgroup generated by the elements as above, because $0_n = 0_n + 0_n$ in this quotient group, and hence 0_n equals zero. Indeed, it is enough to observe that the support of 0_n equals $\emptyset \sqcup \emptyset$ and then apply the above relation to this support.

The elements of $\mathbb{Z}F_n(X,Y)$ are called *linear framed correspondences of level n* or just *linear framed correspondences*. It is worthwhile to mention that $\mathbb{Z}F_n(X,Y)$ is the free Abelian group generated by the elements of $Fr_n(X,Y)$ with connected support.

Denote by $\mathbb{Z}F_*(k)$ the additive category whose objects are those of Sm/k and with Hom-groups defined as

$$\mathrm{Hom}_{\mathbb{Z}F_*(k)}(X,Y) = \bigoplus_{n\geqslant 0} \mathbb{Z}F_n(X,Y).$$

The composition is induced by the composition in the category $Fr_*(k)$.

There is a functor $Sm/k \to \mathbb{Z}F_*(k)$ which is the identity on objects and which takes a regular morphism $f: X \to Y$ to the linear framed correspondence $1\cdot(X, X\times\mathbb{A}^0, pr_{\mathbb{A}^0}, f\circ pr_X) \in \mathbb{Z}F_0(k)$.

Definition 8.4. Let X,Y,S and T be schemes. The external product from Definition 2.4 induces a unique external product

$$\mathbb{Z}F_n(X,Y)\times\mathbb{Z}F_m(S,T) \xrightarrow{-\boxtimes -} \mathbb{Z}F_{n+m}(X\times S, Y\times T)$$

such that for any elements $a\in Fr_n(X,Y)$ and $b\in Fr_m(S,T)$ one has $1\cdot a\boxtimes 1\cdot b = 1\cdot(a\boxtimes b)\in \mathbb{Z}F_{n+m}(X\times S, Y\times T)$.

For the constant morphism $c\colon \mathbb{A}^1 \to pt$, we set

$$\Sigma := -\boxtimes 1\cdot(\{0\},\mathbb{A}^1,t,c)\colon \mathbb{Z}F_n(X,Y)\to\mathbb{Z}F_{n+1}(X,Y)$$

and refer to it as the *suspension*.

Definition 8.5. For any k-smooth variety Y there is a presheaf $\mathbb{Z}F_*(-,Y)$ on the category $\mathbb{Z}F_*(k)$ represented by Y. We also have a $\mathbb{Z}F_*(k)$-presheaf

$$\mathbb{Z}F(-,Y) := \mathrm{colim}(\mathbb{Z}F_0(-,Y)\xrightarrow{\Sigma}\mathbb{Z}F_1(-,Y)\xrightarrow{\Sigma}\cdots\xrightarrow{\Sigma}\mathbb{Z}F_n(-,Y)\xrightarrow{\Sigma}\cdots).$$

For a k-smooth variety X, the elements of $\mathbb{Z}F(X,Y)$ are also called *stable linear framed correspondences*. Stable linear framed correspondences do not form the morphisms of a category.

Remark 8.6. For any X,Y in Sm/k one has $\mathbb{Z}F_*(-,X\sqcup Y) = \mathbb{Z}F_*(-,X)\oplus\mathbb{Z}F_*(-,Y)$ and $\mathbb{Z}F(-,X\sqcup Y) = \mathbb{Z}F(-,X)\oplus\mathbb{Z}F(-,Y)$.

For every $(Y,Y-S)\in SmOp(Fr_0(k))$, $\mathbb{Z}F_*(k)$-presheaves $\mathbb{Z}F_*(-,Y/Y-S)$ and $\mathbb{Z}F(-,Y/Y-S)$ are defined in a similar fashion. Namely, each $\mathbb{Z}F_n(X,Y/Y-S)$ is the free Abelian group generated by the elements of $Fr_n(X,Y/Y-S)$ with connected support. Then we set $\mathbb{Z}F_*(X,Y/Y-S) = \oplus_{n\geqslant 0}\mathbb{Z}F_n(X,Y/Y-S)$. Finally, $\mathbb{Z}F(X,Y/Y-S)$ is obtained from $\mathbb{Z}F_*(X,Y/Y-S)$ by stabilization in the Σ-direction.

For every $Y\in\Delta^{\mathrm{op}}SmOp(Fr_0(k))$ there is a Γ-space

$$(K,*)\in\Gamma^{\mathrm{op}}\mapsto\mathbb{Z}F(-,(Y/Y-S)\otimes K).$$

Definition 8.7. The *linear framed motive* $LM_{fr}(Y)$ *of* $Y\in\Delta^{\mathrm{op}}SmOp(Fr_0(k))$ is the Segal S^1-spectrum $(C_*\mathbb{Z}F(-,Y), C_*\mathbb{Z}F(-,Y\otimes S^1), C_*\mathbb{Z}F(-,Y\otimes S^2),\ldots)$ of spaces in $sShv_\bullet(Sm/k)$ associated with the Γ-space $K\in\Gamma^{\mathrm{op}}\mapsto C_*\mathbb{Z}F(-,Y\otimes K) = \mathbb{Z}F(\Delta^\bullet\times -, Y\otimes K)$.

Note that $LM_{fr}(Y)$ is the Eilenberg–Mac Lane spectrum associated with the complex of Nisnevich sheaves $C_*\mathbb{Z}F(-,Y)$ (we often identify simplicial Abelian groups with their normalized complexes by the Dold–Kan correspondence). Therefore $\pi_*(LM_{fr}(Y)) = H_*(C_*\mathbb{Z}F(-,Y))$.

Proof of Theorem 8.2. Since the spectra $M_{fr}(X\times T^n\times(\mathbb{A}^1//\mathbb{G}_m))$ and $M_{fr}(X\times T^n\times\mathbb{G}_m^{\wedge 1}\otimes S^1)$ are sectionwise connected, $\beta_*\alpha_*$ is a stable Nisnevich local equivalence of spectra if and only if this is true of the induced map on homology

$$\beta_*\alpha_*:\mathbb{Z}M_{fr}(X\times T^n\times(\mathbb{A}^1//\mathbb{G}_m))\to\mathbb{Z}M_{fr}(X\times T^n\times\mathbb{G}_m^{\wedge 1}\otimes S^1).$$

Here both linear spectra are defined by taking free Abelian groups of every entry of $M_{fr}(X\times T^n\times(\mathbb{A}^1//\mathbb{G}_m))$ and $M_{fr}(X\times T^n\times\mathbb{G}_m^{\wedge 1}\otimes S^1)$, respectively. By [9, 1.2] the latter arrow is schemewise stably equivalent to the map of linear framed motives

$$\beta_*\alpha_*:LM_{fr}(X\times T^n\times(\mathbb{A}^1//\mathbb{G}_m))\to LM_{fr}(X\times T^n\times\mathbb{G}_m^{\wedge 1}\otimes S^1).$$

We see that the map of the theorem is a stable Nisnevich local equivalence of spectra if and only if the morphism of complexes of Nisnevich sheaves

$$\beta_*\alpha_*:C_*\mathbb{Z}F(-,X\times T^n\times(\mathbb{A}^1//\mathbb{G}_m))\to C_*\mathbb{Z}F(-,X\times T^n\times\mathbb{G}_m^{\wedge 1}\otimes S^1)$$

is a quasi-isomorphism.

Since $C_*\mathbb{Z}F(-,Y_1\sqcup Y_2)=C_*\mathbb{Z}F(-,Y_1)\oplus C_*\mathbb{Z}F(-,Y_2)$ for any $Y_1,Y_2\in\Delta^{\mathrm{op}}SmOp(Fr_0(k))$, the diagram (12) induces a commutative diagram of triangles of complexes of Nisnevich sheaves

$$\begin{tikzcd}
C_*\mathbb{Z}F(-,X\times T^n\times\mathbb{G}_m) \arrow[r]\arrow[d] & C_*\mathbb{Z}F(-,X\times T^n\times\mathbb{A}^1) \arrow[r]\arrow[d] & C_*\mathbb{Z}F(-,X\times T^n\times(\mathbb{A}^1//\mathbb{G}_m)) \arrow[r,"+"]\arrow[d,"\alpha"] & {} \\
C_*\mathbb{Z}F(-,X\times T^n\times\mathbb{G}_m^{\wedge 1}) \arrow[r]\arrow[d,equal] & C_*\mathbb{Z}F(-,X\times T^n\times\mathbb{A}^{\wedge 1}) \arrow[r]\arrow[d] & C_*\mathbb{Z}F(-,X\times T^n\times(\mathbb{A}^{\wedge 1}//\mathbb{G}_m^{\wedge 1})) \arrow[r,"+"]\arrow[d,"\beta"] & {} \\
C_*\mathbb{Z}F(-,X\times T^n\times\mathbb{G}_m^{\wedge 1}) \arrow[r] & 0 \arrow[r] & C_*\mathbb{Z}F(-,X\times T^n\times\mathbb{G}_m^{\wedge 1}\otimes S^1) \arrow[r,"+"] & {}
\end{tikzcd}\tag{13}$$

Firstly, we claim that α is a schemewise quasi-isomorphism of complexes of presheaves. Indeed, we have a map of two triangles of complexes of presheaves

$$\begin{tikzcd}
C_*\mathbb{Z}F(-,X\times T^n\times\mathbb{G}_m) \arrow[r]\arrow[d] & C_*\mathbb{Z}F(-,X\times T^n\times\mathbb{G}_m^{\wedge 1}) \arrow[r]\arrow[d] & C_*\mathbb{Z}F(-,X\times T^n\times(pt\otimes S^1)) \arrow[r,"+"]\arrow[d,equal] & {} \\
C_*\mathbb{Z}F(-,X\times T^n\times\mathbb{A}^1) \arrow[r] & C_*\mathbb{Z}F(-,X\times T^n\times\mathbb{A}^{\wedge 1}) \arrow[r] & C_*\mathbb{Z}F(-,X\times T^n\times(pt\otimes S^1)) \arrow[r,"+"] & {}
\end{tikzcd}$$

We see that the left square of the diagram is Mayer–Vietoris, hence α is a schemewise quasi-isomorphism of complexes of presheaves.

Secondly, we claim that β is a local quasi-isomorphism of complexes of Nisnevich sheaves. This is equivalent to showing that the complex $C_*\mathbb{Z}F(-,X\times T^n\times\mathbb{A}^{\wedge 1})$ of diagram (13) is locally quasi-isomorphic to zero. To prove the latter, consider a map of two triangles of complexes of sheaves

$$\begin{tikzcd}
C_*\mathbb{Z}F(-,X\times T^n\times pt) \arrow[r]\arrow[d,equal] & C_*\mathbb{Z}F(-,X\times T^n\times\mathbb{A}^1) \arrow[r]\arrow[d,equal] & C_*\mathbb{Z}F(-,X\times T^n\times\mathbb{A}^{\wedge 1}) \arrow[r,"+"]\arrow[d] & {} \\
C_*\mathbb{Z}F(-,X\times T^n\times pt) \arrow[r] & C_*\mathbb{Z}F(-,X\times T^n\times\mathbb{A}^1) \arrow[r] & C_*\mathbb{Z}F(-,X\times T^n\times\mathbb{A}^1)/C_*\mathbb{Z}F(-,X\times T^n\times pt) \arrow[r,"+"] & {}
\end{tikzcd}$$

The cohomology sheaves of the lower right complex are homotopy invariant and quasi-stable framed presheaves. By [8, 1.1] (complemented by [3] in characteristic 2) these cohomology sheaves are strictly homotopy invariant. The terms of this complex are contractible sheaves. Now the proof of [22, 1.10.2] yields the local acyclicity of the complex. It follows that the complex $C_*\mathbb{Z}F(-,X\times T^n\times\mathbb{A}^{\wedge 1})$ is locally acyclic, and hence β is locally a quasi-isomorphism. This completes the proof of the theorem. $\square$

In fact, the proof of Theorem 8.2 also shows the following fact.

Corollary 8.8. *Suppose k is an infinite perfect field. Then for every $n \geqslant 0$ and $X \in Sm/k$, the natural maps $M_{fr}(X \times T^n \times \mathbb{A}^1) \to M_{fr}(X \times T^n)$ and $LM_{fr}(X \times T^n \times \mathbb{A}^1) \to LM_{fr}(X \times T^n)$ are stable local weak equivalences of S^1-spectra.*

Let us take the nth power $(\beta\alpha)^{\wedge n} : (\mathbb{A}^1//\mathbb{G}_m)^{\wedge n} \to (\mathbb{G}_m^{\wedge 1} \otimes S^1)^{\wedge n}$ of the morphism $\beta\alpha$ in the symmetric monoidal category $\Delta^{\mathrm{op}} SmOp(Fr_0(k))$. Below we shall also need the following

Corollary 8.9. *Suppose k is an infinite perfect field. For every $n \geqslant 1$ and $X \in Sm/k$, the map $(\beta\alpha)_*^{\wedge n} : C_*Fr(-, X \times (\mathbb{A}^1//\mathbb{G}_m)^{\wedge n}) \to C_*Fr(-, X \times (\mathbb{G}_m^{\wedge 1} \otimes S^1)^{\wedge n})$ is a local Nisnevich weak equivalence of motivic spaces.*

Proof. The space $C_*Fr(-, X \times (\mathbb{G}_m^{\wedge 1} \otimes S^1)^{\wedge n})$ is plainly sectionwise connected. By [9, A.1] the space $C_*Fr(-, X \times (\mathbb{A}^1//\mathbb{G}_m)^{\wedge n})$ is locally connected. Therefore $M_{fr}(X \times (\mathbb{G}_m^{\wedge 1} \otimes S^1)^{\wedge n})$ is sectionwise an Ω-spectrum and $M_{fr}(X \times (\mathbb{A}^1//\mathbb{G}_m)^{\wedge n})$ is locally an Ω-spectrum by Theorem 6.5. Therefore our assertion would follow if we showed that the map $(\beta\alpha)_*^{\wedge n} : M_{fr}(X \times (\mathbb{A}^1//\mathbb{G}_m)^{\wedge n}) \to M_{fr}(X \times (\mathbb{G}_m^{\wedge 1} \otimes S^1)^{\wedge n})$ is a local Nisnevich weak equivalence of S^1-spectra. The latter follows by using induction in n, Theorem 8.2 and the fact that the realization of Nisnevich local weak equivalences is a local Nisnevich weak equivalence. ☐

We finish the section by proving the following useful result.

Theorem 8.10. *Suppose k is an infinite perfect field. For every $n \geqslant 0$ and every elementary Nisnevich square*

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & X' \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

the square of S^1-spectra

$$\begin{tikzcd} M_{fr}(U' \times T^n) \arrow[r] \arrow[d] & M_{fr}(X' \times T^n) \arrow[d] \\ M_{fr}(U \times T^n) \arrow[r] & M_{fr}(X \times T^n) \end{tikzcd}$$

is homotopy cartesian locally in the Nisnevich topology. The same is also true for linear framed motives.

Proof. Since we deal with connected S^1-spectra, the proof of Theorem 8.2 shows that it suffices to verify our statement for linear framed motives. It follows from the proof of [24, 4.4] that the sequence of presheaves

$$0 \to \mathbb{Z}F_s(-, U' \times T^n) \to \mathbb{Z}F_s(-, U \times T^n) \oplus \mathbb{Z}F_s(-, X' \times T^n) \to \mathbb{Z}F_s(-, X \times T^n) \to 0$$

is locally exact for every $s \geqslant 0$. Passing to the colimit over s, the sequence of presheaves

$$0 \to \mathbb{Z}F(-, U' \times T^n) \to \mathbb{Z}F(-, U \times T^n) \oplus \mathbb{Z}F(-, X' \times T^n) \to \mathbb{Z}F(-, X \times T^n) \to 0$$

is locally exact as well. It follows that the sequence of Eilenberg–Mac Lane spectra

$$EM(\mathbb{Z}F(-, U' \times T^n)) \to EM(\mathbb{Z}F(-, U \times T^n)) \times EM(\mathbb{Z}F(-, X' \times T^n)) \to EM(\mathbb{Z}F(-, X \times T^n))$$

is locally a homotopy fibre sequence. Therefore the sequence

$$LM_{fr}(U' \times T^n) \to LM_{fr}(U \times T^n) \times LM_{fr}(X' \times T^n) \to LM_{fr}(X \times T^n) \tag{14}$$

is a homotopy fibre sequence in the motivic model structure of S^1-spectra, and hence so is the sequence

$$LM_{fr}(U' \times T^n)_f \to LM_{fr}(U \times T^n)_f \times LM_{fr}(X' \times T^n)_f \to LM_{fr}(X \times T^n)_f,$$

where "f" is as in Corollary 7.4. It follows from Corollary 7.4 that the latter sequence is a sequence of fibrant objects in the stable injective motivic model structure of S^1-spectra. Therefore this sequence is also locally a homotopy fibre sequence, and hence so is the sequence (14). $\square$

9. Proof of Theorem 4.1

In this section we prove Theorem 4.1. We first give a proof for the second statement of the theorem and then a proof for the first statement.

9.1. *Proof of Theorem 4.1(2)*

Recall that $A \in sShv_\bullet(Sm/k)$ is *finitely presentable* if the functor $\mathrm{Hom}_{sShv_\bullet(Sm/k)}(A,-)$ preserves directed colimits. Using the General Framework on p. 14, for every $A, L \in sShv_\bullet(Sm/k)$ with A finitely presentable there is a canonical morphism in $sShv_\bullet(Sm/k)$

$$C_*\mathscr{F}r(L) \to \underline{\mathrm{Hom}}(A, C_*\mathscr{F}r(L \wedge A))$$

as well as a canonical morphism of ordinary S^1-spectra of simplicial Nisnevich sheaves

$$\mathscr{M}_{fr}(L) \to \underline{\mathrm{Hom}}(A, \mathscr{M}_{fr}(L \wedge A)). \tag{15}$$

These induce morphisms of spaces and S^1-spectra respectively:

$$a_A : C_*\mathscr{F}r(L)_f \to \underline{\mathrm{Hom}}(A, C_*\mathscr{F}r(L \wedge A)_f)$$

and

$$\alpha_A : \mathscr{M}_{fr}(L)_f \to \underline{\mathrm{Hom}}(A, \mathscr{M}_{fr}(L \wedge A)_f).$$

Here $C_*\mathscr{F}r(L)_f$ (respectively $\mathscr{M}_{fr}(L)_f$) is a Nisnevich local fibrant replacement of $C_*\mathscr{F}r(L)$ (respectively a level Nisnevich local fibrant replacement of $\mathscr{M}_{fr}(L)$ in the category of ordinary S^1-spectra).

Lemma 9.1. *Suppose $u : A \to B$ is a motivic weak equivalence in $sShv_\bullet(Sm/k)$ between finitely presentable objects such that the induced map $u_* : \mathscr{M}_{fr}(L \wedge A) \to \mathscr{M}_{fr}(L \wedge B)$ is a stable Nisnevich local weak equivalence of spectra. Suppose $\mathscr{M}_{fr}(L)_f, \mathscr{M}_{fr}(L \wedge A)_f, \mathscr{M}_{fr}(L \wedge B)_f$ are all motivically fibrant S^1-spectra. Then $\alpha_A : \mathscr{M}_{fr}(L)_f \to \underline{\mathrm{Hom}}(A, \mathscr{M}_{fr}(L \wedge A)_f)$ is a sectionwise stable equivalence if and only if $\alpha_B : \mathscr{M}_{fr}(L)_f \to \underline{\mathrm{Hom}}(B, \mathscr{M}_{fr}(L \wedge B)_f)$ is.*

Proof. The commutative square (10) of the General Framework gives rise to a commutative square

$$\begin{tikzcd}
\mathscr{M}_{fr}(L)_f \arrow[r, "\alpha_A"] \arrow[d, "\alpha_B"'] & \underline{\mathrm{Hom}}(A, \mathscr{M}_{fr}(L \wedge A)_f) \arrow[d, "u_*"] \\
\underline{\mathrm{Hom}}(B, \mathscr{M}_{fr}(L \wedge B)_f) \arrow[r, "u^*"] & \underline{\mathrm{Hom}}(A, \mathscr{M}_{fr}(L \wedge B)_f)
\end{tikzcd}$$

By assumption, $\mathscr{M}_{fr}(L \wedge A)_f, \mathscr{M}_{fr}(L \wedge B)_f$ are motivically fibrant S^1-spectra, and hence u^* is a sectionwise stable equivalence. Since $\mathscr{M}_{fr}(L \wedge A) \to \mathscr{M}_{fr}(L \wedge B)$ is a stable Nisnevich local weak equivalence of spectra, it follows that $\mathscr{M}_{fr}(L \wedge A)_f \to \mathscr{M}_{fr}(L \wedge B)_f$ is a sectionwise stable weak equivalence of spectra. We see that the right vertical arrow u_* of the square is a sectionwise stable weak equivalence of spectra. Our statement now follows from the two-out-three property for weak equivalences. $\square$

Corollary 9.2. *Under the assumptions of Lemma 9.1 the map of spaces $a_A : C_*\mathscr{F}r(L)_f \to \underline{\mathrm{Hom}}(A, C_*\mathscr{F}r(L\wedge A)_f)$ is a sectionwise weak equivalence if and only if $a_B : C_*\mathscr{F}r(L)_f \to \underline{\mathrm{Hom}}(B, C_*\mathscr{F}r(L\wedge B)_f)$ is.*

Lemma 9.3. *Suppose $u : A \to B$ is a motivic weak equivalence in $sShv_\bullet(Sm/k)$ between finitely presentable objects. Suppose $\mathscr{M}_{fr}(L)_f, \mathscr{M}_{fr}(L\wedge B)_f$ are motivically fibrant S^1-spectra. Then the composite map of spaces $C_*\mathscr{F}r(L)_f \xrightarrow{a_B} \underline{\mathrm{Hom}}(B, C_*\mathscr{F}r(L\wedge B)_f) \xrightarrow{u^*} \underline{\mathrm{Hom}}(A, C_*\mathscr{F}r(L\wedge B)_f)$ is a sectionwise weak equivalence if and only if $\alpha_B : \mathscr{M}_{fr}(L)_f \to \underline{\mathrm{Hom}}(B, \mathscr{M}_{fr}(L\wedge B)_f)$ is a sectionwise stable equivalence of spectra.*

Proof. Our assumptions on spectra imply $C_*\mathscr{F}r(L)_f, C_*\mathscr{F}r(L\wedge B)_f$ are motivically fibrant spaces. It follows that $C_*\mathscr{F}r(L)_f \xrightarrow{a_B} \underline{\mathrm{Hom}}(B, C_*\mathscr{F}r(L\wedge B)_f) \xrightarrow{u^*} \underline{\mathrm{Hom}}(A, C_*\mathscr{F}r(L\wedge B)_f)$ is a sectionwise weak equivalence if and only if $C_*\mathscr{F}r(L)_f \xrightarrow{a_B} \underline{\mathrm{Hom}}(B, C_*\mathscr{F}r(L\wedge B)_f)$ is, because $u : A \to B$ is a motivic weak equivalence (recall that all spaces in the Morel–Voevodsky model category $sShv_\bullet(Sm/k)_{\mathrm{mot}}$ are cofibrant).

Again because of our assumptions on spectra we have that $C_*\mathscr{F}r(L)_f \xrightarrow{a_B} \underline{\mathrm{Hom}}(B, C_*\mathscr{F}r(L\wedge B)_f)$ is a sectionwise weak equivalence if and only if $\alpha_B : \mathscr{M}_{fr}(L)_f \to \underline{\mathrm{Hom}}(B, \mathscr{M}_{fr}(L\wedge B)_f)$ is a sectionwise stable equivalence of spectra. $\square$

We are now in a position to prove the second statement of Theorem 4.1.

Proof of Theorem 4.1(2). By Corollary 7.6 for any integer $n > 0$, the S^1-spectrum $M_{fr}(X\times T^n)_f$ is motivically fibrant and the motivic space $C_*Fr(X\times T^n)_f$ is motivically fibrant. Let $u : \mathbb{P}^{\wedge 1} = (\mathbb{P}^1, \infty) \to T$ be the canonical motivic weak equivalence in $sShv_\bullet(Sm/k)$. It is also given by the framed correspondence of level one $(\{0\}, \mathbb{A}^1, t) \in Fr_1(pt, pt)$. By Lemma 9.3 the map

$$C_*Fr(X\times T^n)_f \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, C_*Fr(X\times T^{n+1})_f)$$

is a sectionwise weak equivalence if and only if $\alpha_T : M_{fr}(X\times T^n)_f \to \underline{\mathrm{Hom}}(T, M_{fr}(X\times T^{n+1})_f)$ is a sectionwise stable equivalence of spectra.

Consider the zigzag of motivic weak equivalences

$$T \xleftarrow{\sim} (-, \mathbb{A}^1//\mathbb{G}_m)_+ \xrightarrow{\sim} (-, \mathbb{G}_m^{\wedge 1}\otimes S^1)_+,$$

where the right arrow is induced by $\beta\alpha$ of the diagram (12). By Corollary 7.6 for any integer $n \geqslant 0$, the S^1-spectra $M_{fr}(X\times T^n\times \mathbb{G}_m^{\wedge 1}\otimes S^1)_f, M_{fr}(X\times T^n\times(\mathbb{A}^1//\mathbb{G}_m))_f$ are motivically fibrant and $C_*(Fr(X\times T^n\times \mathbb{G}_m^{\wedge 1}\otimes S^1))_f, C_*Fr(X\times T^n\times(\mathbb{A}^1//\mathbb{G}_m))_f$ are motivically fibrant spaces.

By [9, 8.1] $M_{fr}(X\times T^n\times(\mathbb{A}^1//\mathbb{G}_m)) \to M_{fr}(X\times T^{n+1})$ is a stable Nisnevich local weak equivalence of spectra. By Theorem 8.2 $M_{fr}(X\times T^n\times(\mathbb{A}^1//\mathbb{G}_m)) \to M_{fr}(X\times T^n\times \mathbb{G}_m^{\wedge 1}\otimes S^1)$ is a stable Nisnevich local weak equivalence of spectra. By Lemma 9.1 $\alpha_T : M_{fr}(X\times T^n)_f \to \underline{\mathrm{Hom}}(T, M_{fr}(X\times T^{n+1})_f)$ is a sectionwise stable equivalence of spectra if and only if so is the map of spectra $\alpha_{\mathbb{G}_m^{\wedge 1}\otimes S^1} : M_{fr}(X\times T^n)_f \to \underline{\mathrm{Hom}}((\mathbb{G}_m^{\wedge 1}\otimes S^1)_+, M_{fr}(X\times T^n\times \mathbb{G}_m^{\wedge 1}\otimes S^1)_f)$.

Consider a commutative diagram

$$\begin{tikzcd} M_{fr}(X\times(\mathbb{A}^1//\mathbb{G}_m)^{\wedge n})_f \arrow[r, "\alpha_{\mathbb{G}_m^{\wedge 1}\otimes S^1}"] \arrow[d] & \underline{\mathrm{Hom}}((\mathbb{G}_m^{\wedge 1}\otimes S^1)_+, M_{fr}(X\times(\mathbb{A}^1//\mathbb{G}_m)^{\wedge n}\times \mathbb{G}_m^{\wedge 1}\otimes S^1)_f) \arrow[d] \\ M_{fr}(X\times T^n)_f \arrow[r, "\alpha_{\mathbb{G}_m^{\wedge 1}\otimes S^1}"] & \underline{\mathrm{Hom}}((\mathbb{G}_m^{\wedge 1}\otimes S^1)_+, M_{fr}(X\times T^n\times \mathbb{G}_m^{\wedge 1}\otimes S^1)_f) \end{tikzcd}$$

Here $M_{fr}(X \times (\mathbb{A}^1//\mathbb{G}_m)^n)_f$ is a stable Nisnevich local fibrant replacement of ordinary spectra and $(\mathbb{A}^1//\mathbb{G}_m)^{\wedge n}$ is from Notation 8.1. It follows from [9, 1.1] that the left vertical arrow is a sectionwise stable weak equivalence of spectra, hence so is the right vertical arrow. We see that the lower arrow is a sectionwise stable weak equivalence of spectra if and only if the upper arrow is. But the upper arrow is a sectionwise stable weak equivalence of spectra by the Cancellation Theorem for framed motives of algebraic varieties [1, Theorem A] and Theorem 6.5. $\square$

The proof of Theorem 4.1(2) and Corollary 8.9 also implies the following

Corollary 9.4. *Suppose k is an infinite perfect field. For any $n \geqslant 1$, the map $C_*Fr(-,X \times (\mathbb{A}^1//\mathbb{G}_m)^{\wedge n}) \to C_*Fr(-,X \times T^n)$ is a local Nisnevich weak equivalence of motivic spaces.*

9.2. *Proof of Theorem 4.1(1)*

In this section we finish the proof of Theorem 4.1. It remains to show part (1) of the theorem. Denote by $\mathbf{Spt}^{\mathbb{P}^1}(Sm/k)$ the category of $\mathbb{P}^1$-spectra, where $\mathbb{P}^1$ is pointed at ∞. We shall work with the injective stable motivic model structure on $\mathbf{Spt}^{\mathbb{P}^1}(Sm/k)$ (see [15] for details). The weak equivalences in this model category will be referred to as *stable equivalences*.

We define the *fake suspension functor* $\Sigma^{\ell}_{\mathbb{P}^1} : \mathbf{Spt}^{\mathbb{P}^1}(Sm/k) \to \mathbf{Spt}^{\mathbb{P}^1}(Sm/k)$ by $(\Sigma^{\ell}_{\mathbb{P}^1}\mathscr{Z})_n = \mathscr{Z}_n \wedge \mathbb{P}^1$ and structure maps

$$(\Sigma^{\ell}_{\mathbb{P}^1}\mathscr{Z}_n) \wedge \mathbb{P}^1 \xrightarrow{\sigma_n \wedge \mathbb{P}^1} \Sigma^{\ell}_{\mathbb{P}^1}\mathscr{Z}_{n+1},$$

where σ_n is a structure map of $\mathscr{Z}$. The fake suspension functor is left adjoint to the *fake loops functor* $\Omega^{\ell}_{\mathbb{P}^1} : \mathbf{Spt}^{\mathbb{P}^1}(Sm/k) \to \mathbf{Spt}^{\mathbb{P}^1}(Sm/k)$ defined by $(\Omega^{\ell}_{\mathbb{P}^1}\mathscr{Z})_n = \Omega_{\mathbb{P}^1}\mathscr{Z}_n = \underline{\mathrm{Hom}}(\mathbb{P}^1, \mathscr{Z}_n)$ and structure maps adjoint to

$$\Omega_{\mathbb{P}^1}\mathscr{Z}_n \xrightarrow{\Omega_{\mathbb{P}^1}\widetilde{\sigma}_n} \Omega_{\mathbb{P}^1}(\Omega_{\mathbb{P}^1}\mathscr{Z}_{n+1}),$$

where $\widetilde{\sigma}_n$ is adjoint to the structure map σ_n of $\mathscr{Z}$.

Define the *shift functors* $t : \mathbf{Spt}^{\mathbb{P}^1}(Sm/k) \to \mathbf{Spt}^{\mathbb{P}^1}(Sm/k)$ and $s : \mathbf{Spt}^{\mathbb{P}^1}(Sm/k) \to \mathbf{Spt}^{\mathbb{P}^1}(Sm/k)$ by $(s\mathscr{Z})_n = \mathscr{Z}_{n+1}$ and $(t\mathscr{Z})_n = \mathscr{Z}_{n-1}$, $(t\mathscr{Z})_0 = pt$, with the evident structure maps. Note that t is left adjoint to s.

Define $\Theta : \mathbf{Spt}^{\mathbb{P}^1}(Sm/k) \to \mathbf{Spt}^{\mathbb{P}^1}(Sm/k)$ to be the functor $s \circ \Omega^{\ell}_{\mathbb{P}^1}$, where s is the shift functor. Then we have a natural map $\iota_{\mathscr{Z}} : \mathscr{Z} \to \Theta\mathscr{Z}$, and we define

$$\Theta^{\infty}\mathscr{Z} = \mathrm{colim}(\mathscr{Z} \xrightarrow{\iota_{\mathscr{Z}}} \Theta\mathscr{Z} \xrightarrow{\Theta\iota_{\mathscr{Z}}} \Theta^2\mathscr{Z} \xrightarrow{\Theta^2\iota_{\mathscr{Z}}} \cdots \xrightarrow{\Theta^{n-1}\iota_{\mathscr{Z}}} \Theta^n\mathscr{Z} \xrightarrow{\Theta^n\iota_{\mathscr{Z}}} \cdots). \tag{16}$$

Set $\eta_{\mathscr{Z}} : \mathscr{Z} \to \Theta^{\infty}\mathscr{Z}$ to be the obvious natural transformation.

Lemma 9.5. *For every $\mathbb{P}^1$-spectrum $\mathscr{Z}$ the natural map $\eta_{\mathscr{Z}} : \mathscr{Z} \to \Theta^{\infty}\mathscr{Z}$ is a stable motivic weak equivalence.*

Proof. The assertion will follow from [12, 4.11] as soon as we find a weakly finitely generated model structure on pointed simplicial presheaves $sPre_\bullet(Sm/k)$ in the sense of [4] such that its model category of $\mathbb{P}^1$-spectra is Quillen equivalent to the injective stable motivic model structure of Jardine [15]. Such a model structure on $sPre_\bullet(Sm/k)$ is the flasque motivic model structure of Isaksen [13]. The fact that it is weakly finitely generated follows from [13, 3.10, 4.9, 5.1] and [6, 2.2]. $\square$

We are now in a position to prove Theorem 4.1(1).

Proof of Theorem 4.1(1). Let $\mathscr{X} \in sPre_\bullet(Sm/k)$ be a pointed motivic space. Consider its suspension spectrum

$$\Sigma^\infty_{\mathbb{P}^1}\mathscr{X} = (\mathscr{X}, \mathscr{X} \wedge \mathbb{P}^1, \mathscr{X} \wedge \mathbb{P}^{\wedge 2}, \dots).$$

We set

$$\Sigma^\infty_{\mathbb{P}^1,T}\mathscr{X} = (\mathscr{X}, \mathscr{X} \wedge T, \mathscr{X} \wedge T^2, \dots)$$

to be the $\mathbb{P}^1$-spectrum with structure maps defined by $(\mathscr{X} \wedge T^n) \wedge \mathbb{P}^1 \xrightarrow{\mathrm{id}\wedge\sigma} \mathscr{X} \wedge T^{n+1}$, where $\sigma : \mathbb{P}^1 \to T$ is the canonical motivic equivalence of sheaves.

Since the smash product of a motivic weak equivalence and a motivic space is again a motivic weak equivalence, we get a level motivic equivalence of spectra

$$\sigma : \Sigma^\infty_{\mathbb{P}^1}\mathscr{X} \to \Sigma^\infty_{\mathbb{P}^1,T}\mathscr{X}.$$

Let us take the Nisnevich sheaf $(\mathscr{X} \wedge T^n)_{\mathrm{nis}}$ of each space $\mathscr{X} \wedge T^n$ of $\Sigma^\infty_{\mathbb{P}^1,T}\mathscr{X}$. Then we get a $\mathbb{P}^1$-spectrum $\Sigma^\infty_{\mathbb{P}^1,T}\mathscr{X}_{\mathrm{nis}}$ and a level local weak equivalence of spectra

$$\nu : \Sigma^\infty_{\mathbb{P}^1,T}\mathscr{X} \to \Sigma^\infty_{\mathbb{P}^1,T}\mathscr{X}_{\mathrm{nis}}.$$

By Lemma 9.5 the natural map of spectra

$$\eta : \Sigma^\infty_{\mathbb{P}^1,T}\mathscr{X}_{\mathrm{nis}} \to \Theta^\infty\Sigma^\infty_{\mathbb{P}^1,T}\mathscr{X}_{\mathrm{nis}}$$

is a stable equivalence.

If we apply the Suslin complex C_* to the spectrum $\Theta^\infty\Sigma^\infty_{\mathbb{P}^1,T}\mathscr{X}_{\mathrm{nis}}$ levelwise, we get a spectrum $C_*(\Theta^\infty\Sigma^\infty_{\mathbb{P}^1,T}\mathscr{X}_{\mathrm{nis}})$. By [18, 2.3.8] the natural map of spectra

$$\delta : \Theta^\infty\Sigma^\infty_{\mathbb{P}^1,T}\mathscr{X}_{\mathrm{nis}} \to C_*(\Theta^\infty\Sigma^\infty_{\mathbb{P}^1,T}\mathscr{X}_{\mathrm{nis}})$$

is a level motivic weak equivalence. Likewise, the morphism δ can be defined for all spectra, which we will denote by the same letter below.

Denote by

$$\rho := \delta \circ \eta \circ \nu \circ \sigma : \Sigma^\infty_{\mathbb{P}^1}(\mathscr{X}) \to C_*(\Theta^\infty\Sigma^\infty_{\mathbb{P}^1,T}\mathscr{X}_{\mathrm{nis}}).$$

Then ρ is a stable motivic weak equivalence of $\mathbb{P}^1$-spectra.

Suppose $\mathscr{X}$ is represented by a smooth scheme X, i.e. $\mathscr{X} = X_+$. By construction, $M_{\mathbb{P}^{\wedge 1}}(X) = C_*Fr_{\mathbb{P}^{\wedge 1},T}(X)$ (see Section 4). As above, the natural map of spectra $\delta : Fr_{\mathbb{P}^{\wedge 1},T}(X) \to M_{\mathbb{P}^{\wedge 1}}(X)$ is a level motivic equivalence. There is a commutative diagram of $\mathbb{P}^1$-spectra

$$\begin{tikzcd}
\Sigma^\infty_{\mathbb{P}^1}(\mathscr{X}) \arrow[dr, "\nu\sigma"'] \arrow[drr, bend left, "\rho"] \arrow[dddr, bend right, "\varkappa"'] & & \\
& \Sigma^\infty_{\mathbb{P}^1,T}\mathscr{X}_{\mathrm{nis}} \arrow[r, "\delta\eta"] \arrow[d, "can"] & C_*\Theta^\infty(\Sigma^\infty_{\mathbb{P}^1,T}\mathscr{X}_{\mathrm{nis}}) \arrow[d, "C_*\Theta^\infty(can)"] \\
& Fr_{\mathbb{P}^{\wedge 1},T}(X) \arrow[r, "\delta\eta"] \arrow[d, "\delta"] & C_*\Theta^\infty(Fr_{\mathbb{P}^{\wedge 1},T}(X)) \\
& M_{\mathbb{P}^{\wedge 1}}(X) &
\end{tikzcd}$$

with *can* being the canonical map. We see that $\varkappa$ is a stable motivic weak equivalence if and only if *can* is. Since η is a stable motivic weak equivalence by Lemma 9.5, it follows that *can* is a stable motivic weak equivalence if and only if $C_*\Theta^\infty(can)$ is.

We claim that $C_*\Theta^\infty(can)$ is a level local weak equivalence. To show this, we first present the spectra $Fr_{\mathbb{P}^{\wedge 1},T}(X)$ and $M_{\mathbb{P}^{\wedge 1}}(X)$ in terms that use the language of symmetric spectra.

Given a motivic symmetric T-spectrum E, there is a canonical morphism of symmetric spectra $T\wedge E\to E[1]$, which is defined at each level by the composite map

$$T\wedge E_n \xrightarrow{tw} E_n\wedge T \xrightarrow{u} E_{n+1} \xrightarrow{\chi_{n,1}} E_{1+n},$$

where $\chi_{n,1}$ is the obvious shuffle permutation. Then by adjointness we have a morphism of symmetric T-spectra $E\to R^1(E):=\underline{\mathrm{Hom}}(T,E[1])$. It induces a morphism of $\mathbb{P}^1$-spectra $E\to R^1(E)\xrightarrow{\sigma^*} R^1_{mot}(E)$, where $\sigma:\mathbb{P}^{\wedge 1}\to T$ is the canonical motivic equivalence and $R^1_{mot}(E):=\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1},E[1])$. Set,

$$R^\infty_{mot}(E):=\mathrm{colim}(E\to R^1_{mot}(E)\to R^2_{mot}(E)\to\cdots).$$

If we apply Suslin's construction C_* levelwise, we get a $\mathbb{P}^1$-spectrum $C_*R^\infty_{mot}(E)$. If $E=\Sigma^\infty_T X_+$ with $X\in Sm/k$, then we have that $Fr_{\mathbb{P}^{\wedge 1},T}(X)=R^\infty_{mot}E$ and $M_{\mathbb{P}^{\wedge 1}}(X)=C_*R^\infty_{mot}E$.

Fix a number r and consider a two-dimensional sequence

$$A_{n,m}:=C_*\Theta^n(R^m_{mot}(\Sigma^\infty_T X_+))_r$$

(here the right hand side is the rth space of the spectrum $C_*\Theta^n(R^m_{mot}(\Sigma^\infty_T X_+)))$ with horizontal maps $A_{n,m}\to A_{n+1,m}$ induced by $\Theta^n\to\Theta^{n+1}$ and vertical maps $A_{n,m}\to A_{n,m+1}$ induced by $R^m_{mot}\to R^{m+1}_{mot}$. To prove the claim, it suffices to show that the map $\mathrm{colim}_n A_{n,0}\to \mathrm{colim}_{n,m}A_{n,m}$ is a local weak equivalence. Without loss of generality it is sufficient to prove that for every m,n the map

$$\mathrm{colim}_n A_{2n,2m}\to \mathrm{colim}_n A_{2n,2m+2} \tag{17}$$

is a local weak equivalence.

To prove that (17) is a local weak equivalence, let $f_n:A_{2n,2m}\to A_{2n,2m+2}$ be given by the map from the two dimensional sequence above. Define a map $g_n\colon A_{2n,2m+2}\to A_{2n+2,2m}$ as an identification via the associativity isomorphism

$$\begin{aligned}A_{2n,2m+2}&=C_*\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 2n},\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 2m+2},T^{2n+2m+2+r}))=\\&=C_*\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 2n+2},\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 2m},T^{2n+2m+2+r}))=A_{2n+2,2m}.\end{aligned}$$

Then g_nf_n differs from $i_n:A_{2n,2m}\to A_{2n+2,2m}$ by the action of an even permutation on $\mathbb{P}^{\wedge 2n+2m+2}$ and an even permutation on $T^{2n+2m+2+r}$. Thus g_nf_n and i_n are simplicially homotopic, because the action of an even permutation lifts to the action of a matrix from a special linear group (see, e.g., Definition 6.3) and using the fact that $\mathbb{A}^1$-homotopies become the usual ones because of Suslin's complex C_*. We also use here Voevodsky's Lemma from Section 3. Similarly, $f_{n+1}g_n$ differs from $j_n:A_{2n,2m+2}\to A_{2n+2,2m+2}$ by the action of an even permutation on $\mathbb{P}^{\wedge 2n+2m+4}$ and an even permutation on $T^{2n+2m+4+r}$. Therefore $f_{n+1}g_n$ is simplicially homotopic to j_n for the same reasons as above. Thus the map (17) on the colimits is a local weak equivalence, because it becomes an isomorphism on homotopy sheaves. This proves the claim.

We have shown that $C_*\Theta^\infty(can)$ is a level local weak equivalence. In particular, it is a stable motivic weak equivalence, and hence so is $\varkappa$. It remains to observe that the morphism $\varkappa_f$ is the composition of $\varkappa$ and a level local weak equivalence $M_{\mathbb{P}^{\wedge 1}}(X)\to M_{\mathbb{P}^{\wedge 1}}(X)_f$. Therefore $\varkappa_f$ is a stable motivic weak equivalence. This finishes the proof of part (1) of Theorem 4.1. □

10. Computing infinite $\mathbb{P}^1$-loop spaces

The purpose of this section is to produce a motivic infinite loop space machine for motivic spaces. This is one of the most impressive applications of Theorem 4.1.

Precisely, given a pointed motivic space $\mathscr{F}$, the main result here, Theorem 10.7, states that $C_*Fr(-,\mathscr{F}^c)^{\mathrm{gp}}$ is locally equivalent to the motivic space $\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(\mathscr{F})$, where $\mathscr{F}^c$ is a canonical replacement of $\mathscr{F}$ which is a directed colimit of simplicial smooth schemes and "gp" refers to a group completion of $C_*Fr(-,\mathscr{F}^c)$, locally in the Nisnevich topology.

Let $\Delta^{\mathrm{op}}\overrightarrow{Fr_0}(k)$ be the full subcategory of $sShv_\bullet(Sm/k)$ consisting of directed colimits of objects from $\Delta^{\mathrm{op}}Fr_0(k)$. Recall that $\Delta^{\mathrm{op}}Fr_0(k)$ can be regarded as a full subcategory of $sShv_\bullet(Sm/k)$ by the embedding sending $X \in Fr_0(k)$ to $X_+ \in Shv_\bullet(Sm/k)$.

In order to compute $\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(\mathscr{F})$ of any motivic space $\mathscr{F} \in sShv_\bullet(Sm/k)$ (see Theorem 10.7), we need the following extension of Theorem 4.1 to objects of $\Delta^{\mathrm{op}}\overrightarrow{Fr_0}(k)$:

Theorem 10.1. *Let k be an infinite perfect field and let Y be an object of $\Delta^{\mathrm{op}}\overrightarrow{Fr_0}(k)$. Then the following statements are true:*

(1) *The morphism $\varkappa_f : \Sigma^\infty_{\mathbb{P}^1}Y_+ \to M_{\mathbb{P}^{\wedge 1}}(Y)_f$ is a stable motivic equivalence of $\mathbb{P}^1$-spectra.*

(2) *The $\mathbb{P}^1$-spectrum $M_{\mathbb{P}^{\wedge 1}}(Y)_f$ is a motivically fibrant Ω-spectrum in positive degrees. This means that for every positive integer $n > 0$ each motivic space $C_*(Fr(-,Y \times T^n))_f$ is motivically fibrant in the Morel–Voevodsky* [18] *motivic model category of simplicial Nisnevich sheaves and the structure map*

$$C_*(Fr(-,Y \times T^n))_f \to \Omega_{\mathbb{P}^1}(C_*(Fr(-,Y \times T^{n+1}))_f)$$

is a weak equivalence schemewise.

Proof. The first statement of the theorem can be proved similarly to Theorem 4.1(1) for any $Y \in \Delta^{\mathrm{op}}\overrightarrow{Fr_0}(k)$. Without loss of generality it is enough to prove the second statement of the theorem for $Y \in \Delta^{\mathrm{op}}Fr_0(k)$. Indeed, we use the facts that the functor $M_{\mathbb{P}^{\wedge 1}}(-)$ respects directed colimits and directed colimits of locally fibrant objects are Nisnevich excisive (even more: they are fibrant in the local flasque model structure of sheaves in the sense of [13, 4.6]).

We first observe that each space $C_*(Fr(-,Y \times T^n))$, $n > 0$, is locally connected, because it is the geometric realization of a simplicial locally connected H-space $[k \in \Delta^{\mathrm{op}} \mapsto C_*(Fr(-,Y_k \times T^n))]$ and $\pi_0^{\mathrm{nis}}(C_*(Fr(-,Y \times T^n))) = 0$ by [10, 7.1]. Now Corollary 7.6 is true if we replace X by Y in it. Indeed, its proof relies on connectedness of the corresponding spaces, which we have just verified, and on Corollary 7.5. As a result, for every positive integer $n > 0$ each motivic space $C_*(Fr(-,Y \times T^n))_f$ is motivically fibrant in the Morel–Voevodsky [18] motivic model category of simplicial Nisnevich sheaves.

In order to show that the structure map

$$C_*(Fr(-,Y \times T^n))_f \to \Omega_{\mathbb{P}^1}(C_*(Fr(-,Y \times T^{n+1}))_f), \quad n > 0,$$

is a weak equivalence schemewise, we use Corollary 7.6 (replacing X by Y in it) and the proof of Theorem 4.1(2) (in that we also use the fact that the geometric realization of a simplicial stable local equivalence of S^1-spectra is a stable local equivalence) to say that this is equivalent to showing that the map

$$M_{fr}(Y \times (\mathbb{A}^1//\mathbb{G}_m)^n)_f \to \Omega_{\mathbb{G}_m^{\wedge 1}}\Omega_{S^1}(M_{fr}(Y \times (\mathbb{A}^1//\mathbb{G}_m)^n \otimes \mathbb{G}_m^{\wedge 1} \otimes S^1)_f)$$

is a schemewise stable weak equivalence of spectra. But the latter follows from the Cancellation Theorem for framed motives [1]. $\square$

Corollary 10.2. *Let k be an infinite perfect field. If $n > 0$ and $\varphi : \mathscr{X} \to \mathscr{Y}$ is a map between spaces in $\Delta^{\mathrm{op}}\overrightarrow{Fr_0}(k)$ such that $\Sigma^\infty_{\mathbb{P}^1}(\varphi)$ is an isomorphism in $SH(k)$ then the induced map $\varphi_* : C_*Fr(-,\mathscr{X}\times T^n) \to C_*Fr(-,\mathscr{Y}\times T^n)$ is a local weak equivalence of motivic spaces.*

In the situation when $\mathscr{X} \in \Delta^{\mathrm{op}}\overrightarrow{Fr_0}(k)$ is such that the space $C_*Fr(-,\mathscr{X})$ is locally connected we arrive at the following result:

Theorem 10.3. *Let k be an infinite perfect field. Suppose $\mathscr{X} \in \Delta^{\mathrm{op}}\overrightarrow{Fr_0}(k)$ is such that the space $C_*Fr(-,\mathscr{X})$ is locally connected. Then $C_*Fr(-,\mathscr{X})$ is an $\mathbb{A}^1$-local space and there is a local weak equivalence of motivic spaces*

$$C_*Fr(-,\mathscr{X}) \simeq \Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(\mathscr{X}_+).$$

Proof. The fact that $C_*Fr(-,\mathscr{X})$ is an $\mathbb{A}^1$-local space is proved similarly to Corollary 7.5(2). The theorem will follow if we show that the $\mathbb{P}^1$-spectrum $M_{\mathbb{P}^{\wedge 1}}(\mathscr{X})_f$ is motivically fibrant, because the suspension spectrum $\Sigma^\infty_{\mathbb{P}^1}\mathscr{X}_+$ is stably equivalent to $M_{\mathbb{P}^{\wedge 1}}(\mathscr{X})_f$ by Theorem 10.1(1). Now the fact that $M_{\mathbb{P}^{\wedge 1}}(\mathscr{X})_f$ is motivically fibrant repeats the proof of Theorem 10.1(2) word for word. $\square$

The proof of the preceding theorem shows the following

Corollary 10.4. *Under the assumptions of Theorem 10.3 the $\mathbb{P}^1$-spectrum $M_{\mathbb{P}^{\wedge 1}}(\mathscr{X})_f$ is motivically fibrant.*

By [2, 3.1] $sShv_\bullet(Sm/k)$ has the projective motivic model structure in which generating cofibrations are given by

$$X_+ \wedge \partial\Delta^n_+ \to X_+ \wedge \Delta^n_+, \quad X \in Sm/k, \quad n \geqslant 0.$$

Equivalently, this family can be regarded as a family in $\Delta^{\mathrm{op}}Fr_0(k)$ of the arrows $X \otimes \partial\Delta^n \to X \otimes \Delta^n$, $n \geqslant 0$.

Let $\mathscr{X} \mapsto \mathscr{X}^c$ be the cofibrant replacement functor in $sShv_\bullet(Sm/k)$ with respect to the projective model structure. Then $\mathscr{X}^c \in \Delta^{\mathrm{op}}\overrightarrow{Fr_0}(k)$, and hence Theorem 10.1 is applicable to it. It also follows from Corollary 10.2 that each functor

$$C_*Fr(-,(-)^c \times T^n) : \mathscr{X} \in sShv_\bullet(Sm/k) \mapsto C_*Fr(-,\mathscr{X}^c \times T^n) \in sShv_\bullet(Sm/k), \quad n \geqslant 1,$$

takes motivic weak equivalences to local weak equivalences. Thus we get a functor

$$C_*Fr(-,(-)^c \times T^n) : H_{\mathbb{A}^1}(k) \to H_{\mathrm{nis}}(k), \quad n \geqslant 1,$$

where $H_{\mathrm{nis}}(k)$ stands for the homotopy category of $sShv_\bullet(Sm/k)$ equipped with the local injective model structure.

Denote by $\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(H_{\mathrm{nis}}(k))$ the full subcategory of $H_{\mathrm{nis}}(k)$ consisting of the infinite $\mathbb{P}^1$-loop spaces. The above arguments together with Theorem 10.1 imply the following result:

Theorem 10.5. *Let k be an infinite perfect field. Then the following statements are true:*

(1) *The functor $C_*Fr(-,(-)^c \times T^n)_f : H_{\mathbb{A}^1}(k) \to H_{\mathrm{nis}}(k)$, $n \geqslant 1$, lands in $\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(H_{\mathrm{nis}}(k))$.*
(2) *For every $n \geqslant 1$ and $\mathscr{X} \in sShv_\bullet(Sm/k)$ the space $C_*Fr(-,\mathscr{X}^c \times T^n)_f$ has the motivic homotopy type of $\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(\mathscr{X} \wedge T^n)$. In particular, the functor $\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} \circ (- \wedge T^n) : H_{\mathbb{A}^1}(k) \to \Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(H_{\mathrm{nis}}(k))$ is isomorphic to the functor $\mathscr{X} \mapsto C_*Fr(-,\mathscr{X}^c \times T^n)_f$.*

(3) *For every* $\mathscr{X} \in sShv_\bullet(Sm/k)$ *the space* $\Omega_{\mathbb{P}^1}(C_*Fr(-,\mathscr{X}^c \times T)_f)$ *has the motivic homotopy type of* $\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(\mathscr{X})$. *In particular, the functor* $\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} : H_{\mathbb{A}^1}(k) \to \Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(H_{\mathrm{nis}}(k))$ *is isomorphic to the functor* $\mathscr{X} \mapsto \Omega_{\mathbb{P}^1}(C_*Fr(-,\mathscr{X}^c \times T)_f)$.

(4) *The functor* $\Sigma^\infty_{\mathbb{P}^1} : H_{\mathbb{A}^1}(k) \to SH(k)$ *is isomorphic to the functor* $\mathscr{X} \mapsto M_{\mathbb{P}^{\wedge 1}}(\mathscr{X}^c)_f$.

Corollary 10.6. *Let k be an infinite perfect field. If* $n > 0$ *then for any map of motivic spaces* $\varphi : \mathscr{Y} \to \mathscr{Z}$ *such that* $\Sigma^\infty_{\mathbb{P}^1}(\varphi)$ *is an isomorphism in* $SH(k)$ *the induced map* $\varphi_* : C_*Fr(-,\mathscr{Y}^c \times T^n) \to C_*Fr(-,\mathscr{Z}^c \times T^n)$ *is a local weak equivalence.*

Let $\mathscr{X} \in \Delta^{\mathrm{op}}\overrightarrow{Fr}_0(k)$. It follows from the Additivity Theorem that the π_0-sheaf of $C_*Fr(-,\mathscr{X})$ is a sheaf of Abelian monoids. In order to state the main result of this section, Theorem 10.7, we fix any local group completion $C_*Fr(-,\mathscr{X})^{\mathrm{gp}}$ of the motivic space $C_*Fr(-,\mathscr{X})$, which is functorial in $\mathscr{X}$. This means that there is a map of pointed motivic spaces

$$\alpha : C_*Fr(-,\mathscr{X}) \to C_*Fr(-,\mathscr{X})^{\mathrm{gp}}$$

which is a group completion locally in the Nisnevich topology. One such local group completion functor is given in assertion (3) of the preceding theorem. Another local group completion is given by $\Omega_{S^1}(C_*Fr(-,\mathscr{X} \otimes S^1))$.

We are now in a position to prove the main result of the section. It gives an explicit computation of motivic infinite loop spaces in terms of framed correspondences.

Theorem 10.7. *Let k be an infinite perfect field. Then the following statements are true:*

(1) *If* $\mathscr{X} \in \Delta^{\mathrm{op}}\overrightarrow{Fr}_0(k)$ *then* $C_*Fr(-,\mathscr{X})^{\mathrm{gp}}$ *is an* $\mathbb{A}^1$*-local space and there is a local equivalence of motivic spaces*

$$C_*Fr(-,\mathscr{X})^{\mathrm{gp}} \simeq \Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(\mathscr{X}).$$

(2) *If* $\mathscr{F} \in sShv_\bullet(Sm/k)$ *is a pointed motivic space then* $C_*Fr(-,\mathscr{F}^c)^{\mathrm{gp}}$ *is an* $\mathbb{A}^1$*-local space and there is a local equivalence of motivic spaces*

$$C_*Fr(-,\mathscr{F}^c)^{\mathrm{gp}} \simeq \Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(\mathscr{F}).$$

Proof. (1). Without loss of generality we may replace $C_*Fr(-,\mathscr{X})^{\mathrm{gp}}$ with $\Omega_{S^1}(C_*Fr(-,\mathscr{X} \otimes S^1))$, because there is a zigzag of local weak equivalences of motivic spaces $\Omega_{S^1}(C_*Fr(-,\mathscr{X} \otimes S^1)) \xrightarrow{\alpha} \Omega_{S^1}(C_*Fr(-,\mathscr{X} \otimes S^1)^{\mathrm{gp}}) \leftarrow C_*Fr(-,\mathscr{X}))^{\mathrm{gp}}$.

The space $C_*Fr(-,\mathscr{X} \otimes S^1)$ is sectionwise connected, and hence it is $\mathbb{A}^1$-local and there is a local weak equivalence of motivic spaces

$$C_*Fr(-,\mathscr{X} \otimes S^1) \simeq \Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(\mathscr{X}_+ \wedge S^1)$$

by Theorem 10.3. It follows that there is a local weak equivalence of motivic spaces

$$\Omega_{S^1}C_*Fr(-,\mathscr{X} \otimes S^1) \simeq \Omega_{S^1}\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(\mathscr{X}_+ \wedge S^1).$$

It remains to observe that there is a sectionwise weak equivalence of motivic spaces $\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(\mathscr{X}) \simeq \Omega_{S^1}\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(\mathscr{X}_+ \wedge S^1)$.

(2). This immediately follows from the first assertion of the theorem. $\square$

In turn, if $\mathscr{X} \in sShv_\bullet(Sm/k)$ is such that the space $C_*Fr(-,\mathscr{X}^c)$ is locally connected we get the following result:

Theorem 10.8. *Let k be an infinite perfect field and let $\mathcal{X} \in sShv_\bullet(Sm/k)$ be such that the space $C_*Fr(-,\mathcal{X}^c)$ is locally connected. Then $C_*Fr(-,\mathcal{X}^c)$ is an $\mathbb{A}^1$-local space and there is a local equivalence of motivic spaces*

$$\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(\mathcal{X}) \simeq C_*Fr(-,\mathcal{X}^c).$$

Proof. Since $C_*Fr(-,\mathcal{X}^c)$ is locally connected by assumption, it follows that $C_*Fr(-,\mathcal{X}^c)$ is locally equivalent to $C_*Fr(-,\mathcal{X}^c)^{\mathrm{gp}}$. The theorem now follows from Theorem 10.7. $\square$

The proof of the preceding theorem shows the following

Corollary 10.9. *Under the assumptions of Theorem 10.8 the $\mathbb{P}^1$-spectrum $M_{\mathbb{P}^{\wedge 1}}(\mathcal{X}^c)_f$ is motivically fibrant. Moreover, $\Sigma^\infty_{\mathbb{P}^1}\mathcal{X}$ is isomorphic to $M_{\mathbb{P}^{\wedge 1}}(\mathcal{X}^c)_f$ in $SH(k)$.*

11. FURTHER APPLICATIONS OF FRAMED MOTIVES

Having applied the machinery of framed motives to prove Theorem 4.1, we want to give further applications. One of the applications computes the suspension bispectrum $\Sigma^\infty_{S^1}\Sigma^\infty_{\mathbb{G}}X_+$ of a smooth algebraic variety X in terms of twisted framed motives of X. Another important application will be purely topological. It will compute the classical sphere spectrum $\Sigma^\infty_{S^1}S^0$ as the framed motive $M_{fr}(pt)(pt)$ of the point $pt = \operatorname{Spec} k$ evaluated at the point whenever the base field k is algebraically closed of characteristic zero.

Denote by $\mathbb{G}$ the cone $(\mathbb{G}_m)_+//pt_+$ of the embedding $pt_+ \overset{1}{\hookrightarrow} (\mathbb{G}_m)_+$ in the category of pointed simplicial presheaves $sPre_\bullet(Sm/k)$. It is termwise equal to

$$(-,\mathbb{G}_m)_+,(-,\mathbb{G}_m)_+\vee(-,pt)_+,(-,\mathbb{G}_m)_+\vee(-,pt)_+\vee(-,pt)_+,\ldots$$

Moreover, its sheafification equals $(\mathbb{G}_m^{\wedge 1})_+$, which is termwise equal to

$$(-,\mathbb{G}_m)_+,(-,\mathbb{G}_m\sqcup pt)_+,(-,\mathbb{G}_m\sqcup pt\sqcup pt)_+,\ldots$$

The sheafification is represented in the category $\Delta^{\mathrm{op}}(Fr_0(k))$ by the object $\mathbb{G}_m^{\wedge 1}$ (see Notation 8.1), which is termwise equal to

$$\mathbb{G}_m,\mathbb{G}_m\sqcup pt,\mathbb{G}_m\sqcup pt\sqcup pt,\ldots$$

One of the models for Morel–Voevodsky's $SH(k)$ can be defined in terms of $(S^1,\mathbb{G})$-bispectra (see, e.g., [15]). The main $(S^1,\mathbb{G})$-bispectrum we work with is given by the sequence of framed motives $M^{\mathbb{G}}_{fr}(X) = (M_{fr}(X), M_{fr}(X\times\mathbb{G}_m^{\wedge 1}), M_{fr}(X\times\mathbb{G}_m^{\wedge 2}),\ldots)$, $\quad X \in Sm/k$, where the simplicial objects $\mathbb{G}_m^{\wedge n} \in \Delta^{\mathrm{op}}(Fr_0(k))$ are those defined in Notation 8.1. Each structure map is defined as the composition

$$M_{fr}(X\times\mathbb{G}_m^{\wedge n}) \to \underline{\mathrm{Hom}}((\mathbb{G}_m^{\wedge 1})_+, M_{fr}(X\times\mathbb{G}_m^{\wedge n+1})) \to \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X\times\mathbb{G}_m^{\wedge n+1})),$$

where the left map is defined as (15). We shall also call $M_{fr}(X\times\mathbb{G}_m^{\wedge n})$ the *n-twisted framed motive of X*, and write $M_{fr}(X)(n)$ to denote $M_{fr}(X\times\mathbb{G}_m^{\wedge n})$. So we can write for brevity

$$M^{\mathbb{G}}_{fr}(X) = (M_{fr}(X), M_{fr}(X)(1), M_{fr}(X)(2),\ldots),$$

to denote the $(S^1,\mathbb{G})$-bispectrum $M^{\mathbb{G}}_{fr}(X)$.

It is worthwhile to mention that the bispectrum $M^{\mathbb{G}}_{fr}(X)$ is constructed in the same way as the bispectrum

$$M^{\mathbb{G}}_{\mathbb{K}}(X) = (M_{\mathbb{K}}(X), M_{\mathbb{K}}(X)(1), M_{\mathbb{K}}(X)(2),\ldots),\quad X\in Sm/k,$$

where each S^1-spectrum $M_{\mathbb{K}}(X)(n)=M_{\mathbb{K}}(X\times\mathbb{G}_m^{\wedge n})$ is the n-twisted K-motive of X in the sense of [7]. It was shown in [7] that $M_{\mathbb{K}}^{\mathbb{G}}(pt)$ represents the bispectrum $f_0(KGL)$.

The main result of this section is as follows.

Theorem 11.1. *Let k be an infinite perfect field. Then for any $X\in Sm/k$, the canonical map of bispectra $\gamma:\Sigma_{S^1}^\infty\Sigma_{\mathbb{G}}^\infty X_+\to M_{fr}^{\mathbb{G}}(X)$ is a stable motivic weak equivalence.*

Proof. It is enough to prove that on bigraded presheaves $\pi_{*,*}^{\mathbb{A}^1}(\gamma)$ is an isomorphism. Every bispectrum yields a $S^1\wedge\mathbb{G}$-spectrum by taking the diagonal. In order to avoid massive notation, we prove the theorem for the case $X=pt$. The same proof works for any $X\in Sm/k$. Let $r\geqslant 0$, s, $n\geqslant 1$ be integers with $s+n\geqslant 0$. There is a commutative diagram in the homotopy category $H_{\mathbb{A}^1}(k)$ of pointed motivic spaces

$$\begin{tikzcd}
{[S^r\wedge U_+\wedge(S^1\wedge\mathbb{G})^{s+n},(S^1\wedge\mathbb{G})^n]} \arrow[r, "(1_n)"] & {[S^r\wedge U_+\wedge(S^1\wedge\mathbb{G})^{s+n},C_*Fr((S^1\otimes\mathbb{G}_m^{\wedge 1})^{\wedge n})]} \\
{[S^r\wedge U_+\wedge(S^1\wedge\mathbb{G})^{s+n},(\mathbb{A}^1_+//(\mathbb{G}_m)_+)^n]} \arrow[r, "(2_n)"] \arrow[u, "(u^{\wedge n})_*"] \arrow[d, "(v^{\wedge n})_*"] & {[S^r\wedge U_+\wedge(S^1\wedge\mathbb{G})^{s+n},C_*Fr((\mathbb{A}^1//\mathbb{G}_m)^{\wedge n})]} \arrow[u, "u^n_*"'] \arrow[d, "v^n_*"'] \\
{[S^r\wedge U_+\wedge(S^1\wedge\mathbb{G})^{s+n},T^n]} \arrow[r, "(3_n)"] \arrow[d, "(u^{\wedge(s+n)})^*"] & {[S^r\wedge U_+\wedge(S^1\wedge\mathbb{G})^{s+n},C_*Fr(T^n)]} \arrow[d, "(u^{\wedge(s+n)})^*"'] \\
{[S^r\wedge U_+\wedge(\mathbb{A}^1_+//(\mathbb{G}_m)_+)^{s+n},T^n]} \arrow[r, "(4_n)"] & {[S^r\wedge U_+\wedge(\mathbb{A}^1_+//(\mathbb{G}_m)_+)^{s+n},C_*Fr(T^n)]} \\
{[S^r\wedge U_+\wedge T^{s+n},T^n]} \arrow[r, "(5_n)"] \arrow[u, "(v^{\wedge(s+n)})^*"] \arrow[d, "(\sigma^{\wedge(s+n)})^*"] & {[S^r\wedge U_+\wedge T^{s+n},C_*Fr(T^n)]} \arrow[u, "(v^{\wedge(s+n)})^*"'] \arrow[d, "(\sigma^{\wedge(s+n)})^*"'] \\
{[S^r\wedge U_+\wedge\mathbb{P}^{\wedge s+n},T^n]} \arrow[r, "(6_n)"] & {[S^r\wedge U_+\wedge\mathbb{P}^{\wedge s+n},C_*Fr(T^n)]}
\end{tikzcd}$$

In this diagram all left vertical arrows are bijections, because the natural maps $u:\mathbb{A}^1_+//(\mathbb{G}_m)_+\to S^1\wedge\mathbb{G}$, $v:\mathbb{A}^1_+//(\mathbb{G}_m)_+\to T$ and $\sigma:\mathbb{P}^{\wedge 1}\to T$ are motivic equivalences. All right vertical arrows are bijections, because the morphisms

$$C_*Fr((\mathbb{A}^1//\mathbb{G}_m)^{\wedge n})\xrightarrow{u^n_*}C_*Fr((S^1\otimes\mathbb{G}_m^{\wedge 1})^{\wedge n}),\ \ C_*Fr((\mathbb{A}^1//\mathbb{G}_m)^{\wedge n})\xrightarrow{v^n_*}C_*Fr(T^n)$$

are local equivalences by Corollaries 8.9 and 9.4.

Fit now each of the twelve vertices of the diagram into a direct colimit over n as follows. For vertices on the right hand side we form direct colimits with respect to the T-spectrum with entries $\{C_*Fr(T^n)\}$, with respect to the $\mathbb{A}^1_+//(\mathbb{G}_m)_+$-spectrum with entries $\{C_*Fr((\mathbb{A}^1//\mathbb{G}_m)^{\wedge n})\}$ as well as with respect to the $S^1\wedge\mathbb{G}$-spectrum with entries $\{C_*Fr((S^1\otimes\mathbb{G}_m^{\wedge 1})^{\wedge n})\}$. Likewise, for vertices on the left hand side we form direct colimits with respect to the T-spectrum with entries $\{T^n\}$, with respect to the $\mathbb{A}^1_+//(\mathbb{G}_m)_+$-spectrum with entries $\{(\mathbb{A}^1//\mathbb{G}_m)^{\wedge n}\}$, with respect to the $S^1\wedge\mathbb{G}$-spectrum with entries $\{(S^1\wedge\mathbb{G})^{\wedge n}\}$.

The family of morphisms $\{(6_n)\}$ forms a morphism on the direct colimit, since it corresponds to the T-spectra morphism $\{T^n\}\to\{C_*Fr(T^n)\}$. For $i=5,4,3$ the families of morphisms $\{(i_n)\}$ form morphisms on the direct colimits by the same reason. The family of morphisms $\{(1_n)\}$ forms a morphism on the direct colimit, since it corresponds to the $S^1\wedge\mathbb{G}$-spectra morphism $\{(S^1\wedge\mathbb{G})^{\wedge n}\}\to\{C_*Fr((S^1\otimes\mathbb{G}_m^{\wedge 1})^{\wedge n})\}$. Finally, the family of morphisms $\{(2_n)\}$ forms a morphism on the direct colimit, because it corresponds to the $\mathbb{A}^1_+//(\mathbb{G}_m)_+$-spectra morphism

$\{(\mathbb{A}^1//\mathbb{G}_m)^{\wedge n}\} \to \{C_*Fr((\mathbb{A}^1//\mathbb{G}_m)^{\wedge n})\}$. In a similar fashion for each vertical map the corresponding family of arrows forms a morphism on the direct colimits.

In this way we get a commutative diagram consisting of twelve direct colimits and morphisms between them. We also get a commutative diagram consisting of twelve groups and homomorphisms between them. In that diagram of groups all the vertical arrows are isomorphisms as mentioned above. The bottom arrow is an isomorphism by Theorem 4.1, and hence so is the top arrow. We conclude that for $r \geqslant 0$ the map of presheaves $\pi^{\mathbb{A}^1}_{r+2s,s}(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}(S^0)) \xrightarrow{\gamma_*} \pi^{\mathbb{A}^1}_{r+2s,s}(M^{\mathbb{G}}_{fr}(pt))$ is an isomorphism for any integer s. In other words, the map γ_* is an isomorphism on presheaves $\pi^{\mathbb{A}^1}_{a,b}$ with $2a-b \geqslant 0$. In particular, for any $U \in Sm/k$ and any $t > 0$ the map

$$\pi^{\mathbb{A}^1}_{2a,a}(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}(S^0))(U \times \mathbb{G}_m^{\times t}) \xrightarrow{\gamma_*} \pi^{\mathbb{A}^1}_{2a,a}(M^{\mathbb{G}}_{fr}(pt))(U \times \mathbb{G}_m^{\times t})$$

is an isomorphism. Note that $\pi^{\mathbb{A}^1}_{2a,a}(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}(S^0))(U_+ \wedge \mathbb{G}_m^{\wedge t})$ is a canonical direct summand of the group $\pi^{\mathbb{A}^1}_{2a,a}(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}(S^0))(U \times \mathbb{G}_m^{\times t})$, and the group $\pi^{\mathbb{A}^1}_{2a,a}(M^{\mathbb{G}}_{fr}(pt))(U \wedge \mathbb{G}_m^{\wedge t})$ is a canonical direct summand of the group $\pi^{\mathbb{A}^1}_{2a,a}(M^{\mathbb{G}}_{fr}(pt))(U \times \mathbb{G}_m^{\times t})$. Hence the map

$$\pi^{\mathbb{A}^1}_{2a-t,a}(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}(S^0))(U) = \pi^{\mathbb{A}^1}_{2a,a}(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}(S^0))(U_+ \wedge \mathbb{G}_m^{\wedge t}) \xrightarrow{\gamma_*} \pi^{\mathbb{A}^1}_{2a,a}(M^{\mathbb{G}}_{fr}(pt))(U_+ \wedge \mathbb{G}_m^{\wedge t}) = \pi^{\mathbb{A}^1}_{2a-t,a}(M^{\mathbb{G}}_{fr}(pt))(U)$$

is an isomorphism, too. Thus the map γ_* is an isomorphism on presheaves $\pi^{\mathbb{A}^1}_{a,b}$ with $2a-b<0$ and the theorem follows. □

Corollary 11.2. *Let k be an infinite perfect field and let X be smooth. Then $\pi^{\mathbb{A}^1}_{-n,-n}((\Sigma^\infty_{S^1}\Sigma^\infty_{\mathbb{G}}X_+)(pt))$, $n \geqslant 0$, is the Grothendieck group of the commutative monoid $\pi_0(C_*Fr(pt, X \times \mathbb{G}_m^{\wedge n}))$.*

Corollary 11.3. *Let k be an infinite perfect field and let X be smooth. Then*

$$\pi^{\mathbb{A}^1}_{-n,-n}(\Sigma^\infty_{S^1}\Sigma^\infty_{\mathbb{G}}X_+)(pt) = H_0(\mathbb{Z}F(\Delta^\bullet_k, X \times \mathbb{G}_m^{\wedge n})), \quad n \geqslant 0.$$

In particular, $\pi^{\mathbb{A}^1}_{-n,-n}(\Sigma^\infty_{S^1}\Sigma^\infty_{\mathbb{G}}S^0)(pt) = H_0(\mathbb{Z}F(\Delta^\bullet_k, \mathbb{G}_m^{\wedge n})) = K^{MW}_n(k)$ if $n \geqslant 0$ and $\operatorname{char} k = 0$.

Proof. The fact that $H_0(\mathbb{Z}F(\Delta^\bullet_k, \mathbb{G}_m^{\wedge n})) = K^{MW}_n(k)$, $n \geqslant 0$, was proved by Neshitov in [19] for fields of characteristic zero. Thus the statement recovers the celebrated theorem of Morel [17] for Milnor–Witt K-theory. □

It is also useful to have Theorem 11.1 for simplicial schemes or, more generally, for objects in $\Delta^{\mathrm{op}}\overrightarrow{Fr_0}(k)$ (cf. Theorem 10.1).

Theorem 11.4. *Let k be an infinite perfect field. Then for any $Y \in \Delta^{\mathrm{op}}\overrightarrow{Fr_0}(k)$, the canonical map of bispectra $\Sigma^\infty_{S^1}\Sigma^\infty_{\mathbb{G}}Y_+ \to M^{\mathbb{G}}_{fr}(Y)$ is a stable motivic weak equivalence.*

Proof. If we use Theorem 10.1, our proof repeats that of Theorem 11.1 word for word. □

Here is an application of the preceding theorem.

Theorem 11.5. *Suppose the base field k is infinite perfect. For any $Y \in \Delta^{\mathrm{op}}\overrightarrow{Fr_0}(k)$ one has a canonical isomorphism*

$$SH(k)(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}X_+, \Sigma^\infty_{\mathbb{G}}\Sigma^\infty_{S^1}Y_+[n]) = SH^{\mathrm{nis}}_{S^1}(k)(\Sigma^\infty_{S^1}X_+, M_{fr}(Y)[n]), \quad n \geqslant 0,$$

where $SH^{\mathrm{nis}}_{S^1}(k)$ is the stable homotopy category of Nisnevich sheaves of S^1-spectra.

Proof. Consider a bispectrum

$$M_{fr}^{\mathbb{G}}(Y)_f = (M_{fr}(Y)_f, M_{fr}(Y \times \mathbb{G}_m^{\wedge 1})_f, M_{fr}(Y \times \mathbb{G}_m^{\wedge 2})_f, \ldots)$$

obtained from $M_{fr}^{\mathbb{G}}(Y)$ by taking Nisnevich local stable fibrant replacements at each level. It is shown similarly to [1, Theorem B] that $M_{fr}^{\mathbb{G}}(Y)_f$ is a motivically fibrant bispectrum.

Theorem 11.4 implies an isomorphism

$$SH(k)(\Sigma_{\mathbb{G}}^\infty \Sigma_{S^1}^\infty X_+, \Sigma_{\mathbb{G}}^\infty \Sigma_{S^1}^\infty Y_+[n]) = SH(k)(\Sigma_{\mathbb{G}}^\infty \Sigma_{S^1}^\infty X_+, M_{fr}^{\mathbb{G}}(Y)_f[n]), \quad n \geqslant 0.$$

But

$$SH(k)(\Sigma_{\mathbb{G}}^\infty \Sigma_{S^1}^\infty X_+, M_{fr}^{\mathbb{G}}(Y)_f[n]) \cong SH_{S^1}(k)(\Sigma_{S^1}^\infty X_+, M_{fr}(Y)_f[n]) \cong$$
$$SH_{S^1}^{\mathrm{nis}}(k)(\Sigma_{S^1}^\infty X_+, M_{fr}(Y)_f[n]) \cong SH_{S^1}^{\mathrm{nis}}(k)(\Sigma_{S^1}^\infty X_+, M_{fr}(Y)[n]),$$

as was to be shown. □

Corollary 11.6. *Suppose the base field k is perfect infinite. For any morphism $\varphi : Y \to Z$ in $\Delta^{\mathrm{op}}\overrightarrow{Fr_0}(k)$ such that $\Sigma_{\mathbb{G}}^\infty \Sigma_{S^1}^\infty(\varphi)$ is an isomorphism in $SH(k)$, the morphism of framed motives $M_{fr}(\varphi) : M_{fr}(Y) \to M_{fr}(Z)$ is a local stable equivalence of S^1-spectra.*

Let $\mathscr{X} \mapsto \mathscr{X}^c$ be the cofibrant replacement functor in $sShv_\bullet(Sm/k)$ (see p. 33). Then $\mathscr{X}^c$ is in $\Delta^{\mathrm{op}}\overrightarrow{Fr_0}(k)$, and hence we can apply Theorems 11.4-11.5 to it. It also follows from Corollary 11.6 that each functor

$$M_{fr}((-)^c) : \mathscr{X} \in sShv_\bullet(Sm/k) \mapsto M_{fr}(\mathscr{X}^c) \in Sp_{S^1}(sShv_\bullet(Sm/k))$$

takes motivic weak equivalences to stable local weak equivalences. Thus we get a functor

$$M_{fr}((-)^c) : H_{\mathbb{A}^1}(k) \to SH_{S^1}^{\mathrm{nis}}(k),$$

where $SH_{S^1}^{\mathrm{nis}}(k)$ stands for the homotopy category of $Sp_{S^1}(sShv_\bullet(Sm/k))$ equipped with the stable local injective model structure.

In a similar fashion, we define a functor

$$M_{fr}^{\mathbb{G}}((-)^c) : H_{\mathbb{A}^1}(k) \to SH(k).$$

Explicitly, it takes a motivic space $\mathscr{X}$ to the bispectrum $M_{fr}^{\mathbb{G}}(\mathscr{X}^c)$. In fact, we will extend the functor to $SH(k)$ in Section 12.

Denote by $\Omega_{\mathbb{G}}^\infty(SH_{S^1}^{\mathrm{nis}}(k))$ the full subcategory of $SH_{S^1}^{\mathrm{nis}}(k)$ consisting of the infinite $\mathbb{G}$-loop spectra. The above arguments together with Theorems 11.4-11.5 and Corollary 11.6 imply the following

Theorem 11.7. *Let k be an infinite perfect field. Then the following statements are true:*

(1) *The functor $M_{fr}((-)^c)_f : H_{\mathbb{A}^1}(k) \to SH_{S^1}^{\mathrm{nis}}(k)$ lands in $\Omega_{\mathbb{G}}^\infty(SH_{S^1}^{\mathrm{nis}}(k))$, where "$f$" refers to the stable local fibrant replacement of S^1-spectra.*

(2) *For every $\mathscr{X} \in sShv_\bullet(Sm/k)$ the spectrum $M_{fr}(\mathscr{X}^c)_f$ has the stable motivic homotopy type of $\Omega_{\mathbb{G}}^\infty \Sigma_{\mathbb{G}}^\infty \Sigma_{S^1}^\infty(\mathscr{X})$. In particular, the functor $\Omega_{\mathbb{G}}^\infty \Sigma_{\mathbb{G}}^\infty \Sigma_{S^1}^\infty : H_{\mathbb{A}^1}(k) \to \Omega_{\mathbb{G}}^\infty(SH_{S^1}^{\mathrm{nis}}(k))$ is isomorphic to the functor $\mathscr{X} \mapsto M_{fr}(\mathscr{X}^c)_f$.*

(3) *The functor $\Sigma_{\mathbb{G}}^\infty \Sigma_{S^1}^\infty : H_{\mathbb{A}^1}(k) \to SH(k)$ is isomorphic to the functor $\mathscr{X} \mapsto M_{fr}^{\mathbb{G}}(\mathscr{X}^c)$.*

Corollary 11.8. *Let k be an infinite perfect field. Then for any map of motivic spaces $\varphi : \mathscr{Y} \to \mathscr{X}$ such that $\Sigma_{S^1}^\infty \Sigma_{\mathbb{G}}^\infty(\varphi)$ is an isomorphism in $SH(k)$, the induced map $\varphi_* : M_{fr}(\mathscr{Y}^c) \to M_{fr}(\mathscr{X}^c)$ is a local stable equivalence.*

We finish the section with topological applications of framed motives. The first result gives an explicit model for the classical sphere spectrum $\Sigma^\infty_{S^1} S^0$.

Theorem 11.9. *Let k be an algebraically closed field of characteristic zero, with embedding $k \hookrightarrow \mathbb{C}$. Then the framed motive $M_{fr}(pt)(pt)$ of the point $pt = \operatorname{Spec} k$ evaluated at pt has the stable homotopy type of the classical sphere spectrum $\Sigma^\infty_{S^1} S^0$.*

Proof. By a theorem of Levine [16] the functor $c : SH \to SH(k)$, induced by the functor

$$sSets_\bullet \to sPre_\bullet(Sm/k)$$

sending a pointed simplicial set to the constant presheaf on Sm/k, is fully faithful. The functor c comes from a left Quillen functor (see the proof of [16, 6.5]). Its right Quillen functor from bispectra to ordinary S^1-spectra takes a bispectrum $E = (E_0, E_1, \dots)$ to $E_0(pt)$. Moreover, c induces an isomorphism $\pi_n(E) \to \pi^{\mathbb{A}^1}_{n,0}(c(E))$ for all spectra E.

Consider $M^{\mathbb{G}}_{fr}(pt)$. By Theorem 11.1 the canonical morphism

$$\Sigma^\infty_{S^1} \Sigma^\infty_{\mathbb{G}} S^0 = c(\Sigma^\infty_{S^1} S^0) \to M^{\mathbb{G}}_{fr}(pt)$$

is a motivic stable equivalence of bispectra. Consider a bispectrum

$$M^{\mathbb{G}}_{fr}(pt)_f = (M_{fr}(pt)_f, M_{fr}(\mathbb{G}_m^{\wedge 1})_f, M_{fr}(\mathbb{G}_m^{\wedge 2})_f, \dots)$$

obtained from $M^{\mathbb{G}}_{fr}(pt)$ by taking Nisnevich local stable fibrant replacements at each level. By [1, Theorem B] $M^{\mathbb{G}}_{fr}(pt)_f$ is a motivically fibrant bispectrum, and hence a fibrant replacement of $\Sigma^\infty_{S^1} \Sigma^\infty_{\mathbb{G}} S^0$.

It follows that $M_{fr}(pt)_f(pt)$ is a fibrant replacement of $\Sigma^\infty_{S^1} S^0$, because each homomorphism $\pi_n(\Sigma^\infty_{S^1} S^0) \to \pi^{\mathbb{A}^1}_{n,0}(c(\Sigma^\infty_{S^1} S^0)) \xrightarrow{\cong} \pi^{\mathbb{A}^1}_{n,0}(M^{\mathbb{G}}_{fr}(pt)_f) = \pi_n(M_{fr}(pt)_f(pt))$ is an isomorphism. It remains to observe that $M_{fr}(pt)_f(pt)$ is stably equivalent to $M_{fr}(pt)(pt)$. $\square$

The previous theorem immediately implies the following

Theorem 11.10. *Let k be an algebraically closed field of characteristic zero, with embedding $k \hookrightarrow \mathbb{C}$. Then the following statements are true:*

(1) *For every $n > 0$, the geometric realization of the simplicial set $Fr(\Delta^\bullet_k, S^n)$ has the homotopy type of the topological space $\Omega^\infty \Sigma^\infty(S^n_{top})$, where S^n_{top} stands for the usual topological n-sphere.*

(2) *The geometric realization of the simplicial set $Fr(\Delta^\bullet_k, pt)^{\mathrm{gp}}$ has the homotopy type of the topological space $\Omega^\infty \Sigma^\infty(S^0_{top})$, where "gp" refers to group completion.*

If we pass to homotopy groups with finite coefficients, then the preceding theorem has the following extension:

Theorem 11.11. *Let k be an algebraically closed field of characteristic zero, with embedding $k \hookrightarrow \mathbb{C}$. Then for any integers $m, N > 0$, $r \geqslant 0$ and any $X \in Sm/k$, there are canonical isomorphisms of Abelian groups*

$$\pi_r(Fr(\Delta^\bullet_k, X \otimes S^N); \mathbb{Z}/m) = \pi^{st}_r(X(\mathbb{C})_+ \wedge S^N_{top}; \mathbb{Z}/m)$$

and

$$\pi_r(Fr(\Delta^\bullet_k, X)^{\mathrm{gp}}; \mathbb{Z}/m) = \pi^{st}_r(X(\mathbb{C})_+; \mathbb{Z}/m).$$

Also, if k is any infinite perfect field, then the assignments $X \mapsto \pi_*(Fr(\Delta^\bullet_k, X)^{\mathrm{gp}})$ and $X \mapsto \pi_*(Fr(\Delta^\bullet_k, X \otimes S^N))$ are generalized homology theories on the category Sm/k.

Proof. The natural functor $X \in Sm/k \mapsto X(\mathbb{C}) \in \mathrm{Top}$ can be extended to a functor $Re : SH(k) \to SH$ (see, e.g., [16]). By [16, 7.2] Re induces an isomorphism

$$\pi^{\mathbb{A}^1}_{r,0}(\Sigma^\infty_{S^1}\Sigma^\infty_{\mathbb{G}}X_+;\mathbb{Z}/m)(pt) \xrightarrow{\cong} \pi^{st}_r(X(\mathbb{C})_+;\mathbb{Z}/m)$$

for any $r \in \mathbb{Z}$. Theorem 11.5 implies that $\pi^{\mathbb{A}^1}_{r,0}(\Sigma^\infty_{S^1}\Sigma^\infty_{\mathbb{G}}X_+;\mathbb{Z}/m)$ is computed as the Nisnevich sheaf $\pi^{Nis}_r(M_{fr}(X);\mathbb{Z}/m)$. We have that

$$\pi_r(M_{fr}(X);\mathbb{Z}/m)(pt) = \pi_r(M_{fr}(X)(pt);\mathbb{Z}/m) = \pi_r(Fr(\Delta^\bullet_k, X)^{\mathrm{gp}};\mathbb{Z}/m).$$

Thus we have verified the second isomorphism of the theorem. The first isomorphism is checked in a similar fashion.

Now the fact that the assignments $X \mapsto \pi_*(Fr(\Delta^\bullet_k, X)^{\mathrm{gp}})$ and $X \mapsto \pi_*(Fr(\Delta^\bullet_k, X \otimes S^N))$ are generalized homology theories on Sm/k with k infinite perfect immediately follow from Theorem 8.10 (verifying the excision property for homology theories) and Corollary 8.8 (verifying the homotopy invariance property for homology theories). $\square$

The preceding theorem is also an extension of the celebrated theorem of Suslin and Voevodsky [21] on singular algebraic homology.

12. The big framed motive functor $\mathscr{M}^b_{fr}$

In Section 11 we computed the functor $\Sigma^\infty_{S^1}\Sigma^\infty_{\mathbb{G}} : H_{\mathbb{A}^1}(k) \to SH(k)$ to be isomorphic to the functor $M^{\mathbb{G}}_{fr} : H_{\mathbb{A}^1}(k) \to SH(k)$. We extend the latter functor to bispectra below, but first we start with preparations. We assume in this section that the base field k is infinite perfect.

We first give an explicit description of $Ex^\infty(C_*\mathscr{F}r(\mathscr{X}))$ for every space $\mathscr{X} \in sShv_\bullet(Sm/k)$, where Ex^∞ is the Kan complex. Voevodsky [23, Section 3] defined a realization functor from simplicial sets to Nisnevich sheaves $|-| : sSets \to Shv_{\mathrm{nis}}(Sm/k)$ such that $|\Delta[n]| = \Delta^n$, where $\Delta[n]$ is the standard n-simplex. Under this notation the cosimplicial scheme $\Delta^\bullet$ equals $|\Delta[\bullet]|$. For every $\ell \geqslant 0$ denote by $sd^\ell\Delta^\bullet$ the cosimplicial Nisnevich sheaf $|sd^\ell\Delta[\bullet]|$. Under this notation we then have a canonical isomorphism of motivic spaces

$$Ex^\ell(C_*\mathscr{F}r(\mathscr{X})) = \mathscr{F}r(|sd^\ell\Delta[\bullet]|_+ \wedge -, \mathscr{X}).$$

It follows that $Ex^\ell(C_*\mathscr{F}r(\mathscr{X}))$ is a space with framed correspondences, as is the space

$$Ex^\infty(C_*\mathscr{F}r(\mathscr{X})) = \mathrm{colim}_\ell Ex^\ell(C_*\mathscr{F}r(\mathscr{X})).$$

Note that $Ex^\infty(C_*\mathscr{F}r(\mathscr{X}))$ is a sectionwise fibrant pointed simplicial set and the canonical map $C_*\mathscr{F}r(\mathscr{X}) \to Ex^\infty(C_*\mathscr{F}r(\mathscr{X}))$ is a sectionwise weak equivalence.

For brevity we denote by $\mathbf{G}$ the sheafification of $\mathbb{G}$. It equals the simplicial sheaf $(\mathbb{G}_m^{\wedge 1})_+ \in sShv_\bullet(Sm/k)$ (see Section 11). Given a $(S^1, \mathbf{G})$-bispectrum E of simplicial Nisnevich sheaves, replace it by a stably cofibrant bispectrum E^c in the stable projective motivic model structure of bispectra associated with the projective motivic structure on $sShv_\bullet(Sm/k)$ in the sense of [2] (we shall deal with this model structure throughout the section). Then each (i,j)-entry $E^c_{i,j}$ of E^c belongs to $\Delta^{\mathrm{op}}\overrightarrow{Fr}_0(k)$. We set

$$M^{\mathbb{G}}_{fr}(E)_{i,j} := Ex^\infty(C_*Fr(E^c_{i,j})), \quad i,j \geqslant 0.$$

The structure maps in the S^1- and $\mathbf{G}$-direction

$$Ex^\infty(C_*Fr(E^c_{i,j})) \to \underline{\mathrm{Hom}}(S^1, Ex^\infty(C_*Fr(E^c_{i+1,j}))),\ Ex^\infty(C_*Fr(E^c_{i,j})) \to \underline{\mathrm{Hom}}(\mathbf{G}, Ex^\infty(C_*Fr(E^c_{i,j+1})))$$

are obviously induced by the structure maps u_v, u_h of E^c (see [15, p. 488] for the relevant definitions on bispectra). Precisely, they are compositions

$$Ex^\infty(C_*Fr(E^c_{i,j})) \to \underline{\mathrm{Hom}}(S^1, Ex^\infty(C_*Fr(E^c_{i,j} \otimes S^1))) \xrightarrow{u_v} \underline{\mathrm{Hom}}(S^1, Ex^\infty(C_*Fr(E^c_{i+1,j})))$$

and

$$Ex^\infty(C_*Fr(E^c_{i,j})) \to \underline{\mathrm{Hom}}(\mathbf{G}, Ex^\infty(C_*Fr(E^c_{i,j} \otimes \mathbb{G}_m^{\wedge 1}))) \xrightarrow{u_h} \underline{\mathrm{Hom}}(\mathbf{G}, Ex^\infty(C_*Fr(E^c_{i,j+1})))$$

respectively.

For brevity we drop Ex^∞ from notation and tacitly assume below that all spaces like $C_*Fr(E^c_{i,j})$ are sectionwise fibrant with framed correspondences. We then have a canonical morphism of bispectra

$$\zeta : E^c \to M^{\mathbb{G}}_{fr}(E).$$

Clearly, ζ is functorial in E.

Denote by $SH^{fr}(k)$ the full subcategory of $SH(k)$ consisting of framed bispectra, i.e. those bispectra $\mathscr{E}$ such that each space $\mathscr{E}_{i,j}$ is a space with framed correspondences and the structure maps $\mathscr{E}_{i,j} \to \underline{\mathrm{Hom}}(S^1, \mathscr{E}_{i+1,j})$, $\mathscr{E}_{i,j} \to \underline{\mathrm{Hom}}(\mathbf{G}, \mathscr{E}_{i,j+1})$ preserve framed correspondences.

Theorem 12.1. *For every $(S^1, \mathbf{G})$-bispectrum of simplicial Nisnevich sheaves E the morphism of bispectra $\zeta : E^c \to M^{\mathbb{G}}_{fr}(E)$ is a stable motivic equivalence. In particular, E is isomorphic to $M^{\mathbb{G}}_{fr}(E)$ in $SH(k)$ by the zigzag of equivalences $E \leftarrow E^c \to M^{\mathbb{G}}_{fr}(E)$ and $M^{\mathbb{G}}_{fr}$ induces an equivalence of categories $M^{\mathbb{G}}_{fr} : SH(k) \xrightarrow{\simeq} SH^{fr}(k)$.*

Proof. Let $diag(E^c), diag(M^{\mathbb{G}}_{fr}(E))$ be the diagonal $S^1 \wedge \mathbf{G}$-spectra. They consist of motivic spaces $(E^c_{0,0}, E^c_{1,1}, \dots)$ and $(C_*Fr(E^c_{0,0}), C_*Fr(E^c_{1,1}), \dots)$ respectively. It suffices to show that $diag(\zeta) : diag(E^c) \to diag(M^{\mathbb{G}}_{fr}(E))$ is a stable motivic equivalence of $S^1 \wedge \mathbf{G}$-spectra.

By [15, p. 496] $diag(E^c)$ has a natural filtration

$$diag(E^c) = \mathrm{colim}_n L_n(diag(E^c)),$$

where $L_n(diag(E^c))$ is the spectrum

$$E^c_{0,0}, E^c_{1,1}, \dots, E^c_{n,n}, E^c_{n,n} \wedge (S^1 \wedge \mathbf{G}), E^c_{n,n} \wedge (S^1 \wedge \mathbf{G})^2, \dots$$

In turn, $diag(M^{\mathbb{G}}_{fr}(E))$ has a natural filtration

$$diag(M^{\mathbb{G}}_{fr}(E)) = \mathrm{colim}_n M^{S^1 \wedge \mathbb{G}}_{fr}(L_n(diag(E^c))),$$

where $M^{S^1 \wedge \mathbb{G}}_{fr}(L_n(diag(E^c)))$ is the spectrum

$$C_*Fr(E^c_{0,0}), C_*Fr(E^c_{1,1}), \dots, C_*Fr(E^c_{n,n}), C_*Fr(E^c_{n,n} \otimes (S^1 \otimes \mathbb{G}_m^{\wedge 1})), C_*Fr(E^c_{n,n} \otimes (S^1 \otimes \mathbb{G}_m^{\wedge 1})^2), \dots$$

It follows from Theorem 11.4 that each morphism $L_n(diag(E^c)) \to M^{S^1 \wedge \mathbb{G}}_{fr}(L_n(diag(E^c)))$ is a stable motivic equivalence, and hence so is $diag(\zeta)$. $\square$

Observe that the composite functor

$$H_{\mathbb{A}^1}(k) \xrightarrow{\Sigma^\infty_{\mathbb{G}} \Sigma^\infty_{S^1}} SH(k) \xrightarrow{M^{\mathbb{G}}_{fr}} SH(k)$$

is isomorphic to the functor $M^{\mathbb{G}}_{fr}$ of Theorem 11.7(3). Thus the functor of Theorem 12.1 extends the functor of Theorem 11.7(3).

Next, denote by $\mathscr{M}^b_{fr}$ the functor taking a bispectrum E to $(\Theta^\infty_{\mathbb{G}} \circ \Theta^\infty_{S^1} \circ M^{\mathbb{G}}_{fr})(E)$ and refer to it as the *big framed motive functor*. Here $\Theta^\infty_{S^1}$ applies to each S^1-spectrum of the bispectrum

$M_{fr}^{\mathbb{G}}(E)$ as in the formula (16), and $\Theta_{\mathbb{G}}^{\infty}$ similarly applies to the bispectrum $(\Theta_{S^1}^{\infty} \circ M_{fr}^{\mathbb{G}})(E)$ in $\mathbf{G}$-direction.

Theorem 12.2. *The following statements are true for every bispectrum E:*

(1) *the natural map $\mu : M_{fr}^{\mathbb{G}}(E) \to \mathscr{M}_{fr}^{b}(E)$ is a stable motivic equivalence of bispectra;*
(2) *for any $i, j \geqslant 0$ the space with framed correspondences $\mathscr{M}_{fr}^{b}(E)_{i,j}$ is $\mathbb{A}^1$-local as an ordinary motivic space;*
(3) *the bispectrum $\mathscr{M}_{fr}^{b}(E)^f$, obtained from $\mathscr{M}_{fr}^{b}(E)$ by taking Nisnevich local replacements $\mathscr{M}_{fr}^{b}(E)_{i,j}^f$ in all entries, is motivically fibrant.*

Proof. (1). Since the projective motivic model structure on $sShv_\bullet(Sm/k)$ is weakly finitely generated in the sense of [4], our assertion is proved similarly to that of Lemma 9.5.

(2). Write $E^c = (E_{*,0}^c, E_{*,1}^c, \ldots)$ as a collection of S^1-spectra. Then $M_{fr}^{\mathbb{G}}(E)$ is a collection of S^1-spectra $(C_*Fr(E_{*,0}^c), C_*Fr(E_{*,1}^c), \ldots)$ and

$$\Theta_{S^1}^{\infty} M_{fr}^{\mathbb{G}}(E) = (\Theta_{S^1}^{\infty}(C_*Fr(E_{*,0}^c)), \Theta_{S^1}^{\infty}(C_*Fr(E_{*,1}^c)), \ldots).$$

By construction, the (i, j)-entry equals

$$\Theta_{S^1}^{\infty} M_{fr}^{\mathbb{G}}(E)_{i,j} := \operatorname{colim}(C_*Fr(E_{i,j}^c) \to \underline{\mathrm{Hom}}(S^1, C_*Fr(E_{i+1,j}^c)) \to \underline{\mathrm{Hom}}(S^2, C_*Fr(E_{i+2,j}^c)) \to \cdots).$$

In each weight j, the S^1-spectrum $E_{*,j}^c$ has a natural filtration $E_{*,j}^c = \operatorname{colim}_n L_n(E_{*,j}^c)$, where $L_n(E_{*,j}^c)$ is the spectrum

$$E_{0,j}^c, E_{1,j}^c, \ldots, E_{n,j}^c, E_{n,j}^c \wedge S^1, E_{n,j}^c \wedge S^2, \ldots$$

Then $C_*Fr(E_{*,j}^c) = C_*Fr(\operatorname{colim}_n L_n(E_{*,j}^c)) = \operatorname{colim}_n C_*Fr(L_n(E_{*,j}^c))$, where $C_*Fr(L_n(E_{*,j}^c)$ is the spectrum

$$C_*Fr(E_{0,j}^c), C_*Fr(E_{1,j}^c), \ldots, C_*Fr(E_{n,j}^c), C_*Fr(E_{n,j}^c \otimes S^1), C_*Fr(E_{n,j}^c \otimes S^2), \ldots$$

Since each space $C_*Fr(E_{n,j}^c \otimes S^\ell)$, $\ell > 0$, is $\mathbb{A}^1$-local, then so is the space $\underline{\mathrm{Hom}}(S^m, C_*Fr(E_{n,j}^c \otimes S^\ell))$, where $m \geqslant 0$. Therefore each space of $\Theta_{S^1}^{\infty}(C_*Fr(E_{*,j}^c)) = \Theta_{S^1}^{\infty}(\operatorname{colim}_n C_*Fr(L_n(E_{*,j}^c))) = \operatorname{colim}_n \Theta_{S^1}^{\infty}(C_*Fr(L_n(E_{*,j}^c)))$ is $\mathbb{A}^1$-local, because so is each $\Theta_{S^1}^{\infty}(C_*Fr(L_n(E_{*,j}^c)))$. We use here the fact that a directed colimit of Nisnevich excisive spaces is Nisnevich excisive to conclude that a directed colimit of $\mathbb{A}^1$-local spaces is $\mathbb{A}^1$-local. We see that each $\Theta_{S^1}^{\infty} M_{fr}^{\mathbb{G}}(E)_{i,j}$ is $\mathbb{A}^1$-local. If we take a Nisnevich local fibrant resolution $\Theta_{S^1}^{\infty} M_{fr}^{\mathbb{G}}(E)_{i,j}^f$ in each (i, j)-entry, we obtain a bispectrum $\Theta_{S^1}^{\infty} M_{fr}^{\mathbb{G}}(E)^f$. Then each S^1-spectrum $\Theta_{S^1}^{\infty} M_{fr}^{\mathbb{G}}(E)_{*,j}^f$ of the bispectrum $\Theta_{S^1}^{\infty} M_{fr}^{\mathbb{G}}(E)^f$ is motivically fibrant in the local stable model structure of S^1-spectra. Indeed, this follows from the fact that the structure maps of the S^1-spectrum $\Theta_{S^1}^{\infty}(C_*Fr(E_{*,j}^c))$ are isomorphisms and that the functor $\underline{\mathrm{Hom}}(S^1, -)$ preserves local equivalences.

Next, the S^1-spectrum in each weight j of $\mathscr{M}_{fr}^{b}(E)$ equals by definition

$$\operatorname{colim}(\Theta_{S^1}^{\infty}(C_*Fr(E_{*,j}^c)) \to \underline{\mathrm{Hom}}(\mathbf{G}, \Theta_{S^1}^{\infty}(C_*Fr(E_{*,j+1}^c))) \to \underline{\mathrm{Hom}}(\mathbf{G}^{\wedge 2}, \Theta_{S^1}^{\infty}(C_*Fr(E_{*,j+2}^c))) \to \cdots).$$

We have shown above that $(\Theta_{S^1}^{\infty}(C_*Fr(E_{*,j}^c)))_f$ is a motivically fibrant S^1-spectrum. We claim that

$$\underline{\mathrm{Hom}}(\mathbf{G}, \Theta_{S^1}^{\infty}(C_*Fr(E_{*,j+1}^c))) \to \underline{\mathrm{Hom}}(\mathbf{G}, (\Theta_{S^1}^{\infty}(C_*Fr(E_{*,j+1}^c)))_f)$$

is a levelwise local equivalence. In this case it will follow that each space $\mathscr{M}_{fr}^{b}(E)_{i,j}$ is $\mathbb{A}^1$-local.

Since both spectra are sectionwise Ω-spectra, it suffices to prove that this arrow is a stable local equivalence. The presheaves of stable homotopy groups of the left spectrum are $\mathbb{A}^1$-invariant quasi-stable radditive with framed correspondences (see [7, Introduction]. Therefore our claim follows from the following

Sublemma. *Let $\mathscr{X}$ be an $\mathbb{A}^1$-local motivic S^1-spectrum whose presheaves of stable homotopy groups are homotopy invariant quasi-stable radditive presheaves with framed correspondences (see* [8] *for the definition of such presheaves). Suppose $\mathscr{X}^f$ is a local stable fibrant replacement of $\mathscr{X}$. Then the map of spectra $\underline{\mathrm{Hom}}(\mathbf{G},\mathscr{X}) \to \underline{\mathrm{Hom}}(\mathbf{G},\mathscr{X}^f)$ is a local stable equivalence.*

Proof. First let us compute $\pi_*^{\mathrm{nis}}(\underline{\mathrm{Hom}}((\mathbb{G}_m)_+,\mathscr{X}^f))$. We have $\mathscr{X}^f = \mathrm{hocolim}_{n\to-\infty}\mathscr{X}^f_{\geqslant n}$, where $\mathscr{X}_{\geqslant n}$ is the naive nth truncation of $\mathscr{X}$ in $Sp_{S^1}(sShv_\bullet(Sm/k))$. $\mathscr{X}$ has homotopy invariant, quasi-stable radditive presheaves with framed correspondences of stable homotopy groups $\pi_*(\mathscr{X})$. By [8, 1.1] (complemented by [3] in characteristic 2) the Nisnevich sheaves $\pi_*^{\mathrm{nis}}(\mathscr{X}_{\geqslant n})$ are strictly homotopy invariant. If $\mathscr{X}^f_{\geqslant n}$ is a stable local replacement of $\mathscr{X}_{\geqslant n}$, it follows from Proposition 7.1 that $\mathscr{X}^f_{\geqslant n}$ is motivically fibrant, hence $\mathscr{X}_{\geqslant n}$ is $\mathbb{A}^1$-local in $SH_{S^1}(k)$. Since

$$\underline{\mathrm{Hom}}((\mathbb{G}_m)_+,\mathscr{X}^f) = \mathrm{hocolim}_{n\to-\infty}\underline{\mathrm{Hom}}((\mathbb{G}_m)_+,\mathscr{X}^f_{\geqslant n})$$

one has $\pi_*^{\mathrm{nis}}(\underline{\mathrm{Hom}}((\mathbb{G}_m)_+,\mathscr{X}^f)) = \mathrm{colim}_{n\to-\infty}\pi_*^{\mathrm{nis}}(\underline{\mathrm{Hom}}((\mathbb{G}_m)_+,\mathscr{X}^f_{\geqslant n}))$.

Consider the Brown–Gersten convergent spectral sequence

$$H^p_{\mathrm{nis}}(U\times\mathbb{G}_m,\pi_q^{\mathrm{nis}}(\mathscr{X}_{\geqslant n})) \Longrightarrow \pi_{q-p}(\mathscr{X}^f_{\geqslant n})(U\times\mathbb{G}_m), \quad U\in Sm/k.$$

By [8, Section 17] each presheaf $U\mapsto H^p_{\mathrm{nis}}(U\times\mathbb{G}_m,\pi_q^{\mathrm{nis}}(\mathscr{X}_{\geqslant n}))$ is $\mathbb{A}^1$-invariant quasi-stable radditive with framed correspondences. If U is a smooth local Henselian scheme then by [8, 3.15(3')] there is an embedding

$$H^p_{\mathrm{nis}}(U\times\mathbb{G}_m,\pi_q^{\mathrm{nis}}(\mathscr{X}_{\geqslant n})) \hookrightarrow H^p_{\mathrm{nis}}(\mathbb{G}_{m,k(U)},\pi_q^{\mathrm{nis}}(\mathscr{X}_{\geqslant n})),$$

where $k(U)$ is the function field of U. By the Sublemma in [7, Appendix A] we have that $H^p_{\mathrm{nis}}(\mathbb{G}_{m,k(U)},\pi_q^{\mathrm{nis}}(\mathscr{X}_{\geqslant n}))=0$ for $p>0$, and hence $H^p_{\mathrm{nis}}(U\times\mathbb{G}_m,\pi_q^{\mathrm{nis}}(\mathscr{X}^f_{\geqslant n}))=0$ for $p>0$. We can conclude that

$$\pi_n^{\mathrm{nis}}(\underline{\mathrm{Hom}}((\mathbb{G}_m)_+,\mathscr{X}^f)) = \pi_n^{\mathrm{nis}}(\mathscr{X}^f)(\mathbb{G}_m\times -).$$

It also follows that

$$\pi_n^{\mathrm{nis}}(\underline{\mathrm{Hom}}(\mathbf{G},\mathscr{X}^f)) = (\pi_n^{\mathrm{nis}}(\mathscr{X}^f))_{-1} = (\pi_n^{\mathrm{nis}}(\mathscr{X}^f))_{-1}.$$

It remains to show that the morphism of $\mathbb{A}^1$-invariant radditive quasi-stable framed sheaves

$$(\pi_n(\underline{\mathrm{Hom}}(\mathbf{G},\mathscr{X})))^{\mathrm{nis}} = (\pi_n(\mathscr{X}))^{\mathrm{nis}}_{-1} \to (\pi_n^{\mathrm{nis}}(\mathscr{X}))_{-1}$$

is an isomorphism. Using [8, 3.15(3')] it suffices to check that it is an isomorphism for every field extension K/k. The homomorphism of Abelian groups

$$(\pi_n(\mathscr{X}))^{\mathrm{nis}}_{-1}(K) = (\pi_n(\mathscr{X}))_{-1}(K) \to (\pi_n^{\mathrm{nis}}(\mathscr{X}))_{-1}(K)$$

is an isomorphism, because for any $\mathbb{A}^1$-invariant radditive quasi-stable framed presheaf of Abelian groups $\mathscr{F}$ and every open $V\subset\mathbb{A}^1_K$, one has $\mathscr{F}(V)=\mathscr{F}^{\mathrm{nis}}(V)$ (see the proof of [8, 3.1]). $\square$

(3). By the previous assertion each space $\mathscr{M}^b_{fr}(E)_{i,j}$ is $\mathbb{A}^1$-local, and hence $\mathscr{M}^b_{fr}(E)^f_{i,j}$ is a motivically fibrant space. The proof of the assertion shows that $\mathscr{M}^b_{fr}(E)^f_{i,j}$ can be computed as the ith space of the motivically fibrant S^1-spectrum

$$\mathrm{colim}(\Theta^\infty_{S^1}(C_*Fr(E^c_{*,j}))^f \to \Omega_{\mathbf{G}}(\Theta^\infty_{S^1}(C_*Fr(E^c_{*,j+1}))^f) \to \Omega_{\mathbf{G}^{\wedge 2}}(\Theta^\infty_{S^1}(C_*Fr(E^c_{*,j+2}))^f) \to \cdots).$$

Moreover, this colimit is the S^1-spectrum in weight j of the bispectrum $\mathcal{M}_{fr}^b(E)^f$. By [12, 4.12] such a bispectrum must be motivically fibrant. This completes the proof. $\square$

Definition 12.3. (1) Given a $(S^1,\mathbf{G})$-bispectrum E and $i,j\geqslant 0$, denote by $\underline{\pi}_{i,j}^{fr}(E)$ the graded Nisnevich sheaf $\bigoplus_{n\geqslant 0}\pi_n^{\mathrm{nis}}(\mathcal{M}_{fr}^b(E)_{i,j})$ and refer to such sheaves as *framed sheaves of E*.

(2) Denote by $SH_{\mathrm{nis}}^{fr}(k)$ the full subcategory of $SH^{fr}(k)$ consisting of framed $(S^1,\mathbf{G})$-bispectra whose motivic spaces are $\mathbb{A}^1$-local as ordinary motivic spaces.

Theorem 12.2 implies the following

Theorem 12.4. *The following statements are true:*

1. *Framed sheaves detect stable weak equivalences of bispectra. Namely, a map of bispectra $f:E\to E'$ is a stable motivic equivalence if and only if $\underline{\pi}_{i,j}^{fr}(E)\xrightarrow{f_*}\underline{\pi}_{i,j}^{fr}(E)$ is an isomorphism of sheaves for all $i,j\geqslant 0$.*
2. *For every bispectrum E the bispectrum $\mathcal{M}_{fr}^b(E)$ belongs to $SH_{\mathrm{nis}}^{fr}(k)$. Moreover, the functor $\mathcal{M}_{fr}^b:SH(k)\to SH_{\mathrm{nis}}^{fr}(k)$ is an equivalence of categories and there is a natural isomorphism of endofunctors on $SH(k)$:*

$$\mathrm{id}\to\iota\circ\mathcal{M}_{fr}^b,$$

where $\iota:SH_{\mathrm{nis}}^{fr}(k)\to SH(k)$ is the natural inclusion.

In other words, the preceding theorem says that the functor $\mathcal{M}_{fr}^b$ converts the classical Morel–Voevodsky's stable motivic homotopy theory $SH(k)$ into an equivalent local homotopy theory of $\mathbb{A}^1$-local framed bispectra from $SH_{\mathrm{nis}}^{fr}(k)$. The main ingredients of this equivalent local homotopy theory are framed motivic spaces of the form $C_*Fr(-,Y)$, where $Y\in\Delta^{\mathrm{op}}Fr_0(k)$ is a simplicial scheme, as well as their framed motives $M_{fr}(Y)$.

We document this as follows.

Theorem 12.5. *The Morel–Voevodsky stable motivic homotopy category $SH(k)$ can be defined as follows. Its objects are $(S^1,\mathbf{G})$-bispectra and morphisms between two bispectra E,E' are given by the set $\pi_0(E^c,\mathcal{M}_{fr}^b(E')^f)$ of ordinary morphisms between bispectra E^c and $\mathcal{M}_{fr}^b(E')^f$ modulo naive homotopy. In particular, $SH(k)(\Sigma_{S^1}^\infty\Sigma_{\mathbb{G}}^\infty X_+,E')=\pi_0(\mathcal{M}_{fr}^b(E')_{0,0}^f(X))$ for any $X\in Sm/k$.*

13. Framed $\mathbb{P}^1$-spectra

In Theorem 10.5 we have shown that the suspension spectrum $\Sigma_{\mathbb{P}^1}^\infty\mathcal{X}$ of a motivic space $\mathcal{X}\in sShv_\bullet(Sm/k)$ is naturally equivalent to the spectrum

$$M_{\mathbb{P}^{\wedge 1}}(\mathcal{X}^c)=(C_*\mathcal{F}r(-,\mathcal{X}^c),C_*\mathcal{F}r(-,\mathcal{X}^c\wedge T),C_*\mathcal{F}r(-,\mathcal{X}^c\wedge T^2),\ldots).$$

There is an induced functor of homotopy categories

$$M_{\mathbb{P}^{\wedge 1}}:H_{\mathbb{A}^1}(k)\to SH(k),$$

which is equivalent to $\Sigma_{\mathbb{P}^1}^\infty$ (see Theorem 10.5). Observe that $M_{\mathbb{P}^{\wedge 1}}$ lands in the full subcategory of $\mathbb{P}^1$-spectra, denote it by $SH^{fr}(k)$, whose motivic spaces are $\mathbb{A}^1$-invariant with framed correspondences. This section will show the reader how to get naturally framed $\mathbb{P}^1$-spectra. Another approach for bispectra was illustrated in the previous section.

The purpose of this final section is to construct an equivalence of categories

$$SH(k) \xrightarrow{\cong} SH^{fr}(k)$$

(see Theorem 13.4). Moreover, we shall prove that every spectrum is isomorphic in $SH(k)$ to a framed Ω-spectrum from $SH^{fr}(k)$.

Throughout the section k is any field. By $\mathbf{Spt}^{\mathbb{P}^1}(Sm/k)$ (respectively $\mathbf{Spt}^{T}(Sm/k)$) we mean the category of $\mathbb{P}^1$-spectra (respectively T-spectra) associated with simplicial Nisnevich sheaves.

We shall consider the level and stable flasque model structures on $\mathbf{Spt}^{\mathbb{P}^1}(Sm/k)$ or $\mathbf{Spt}^{T}(Sm/k)$ in the sense of [13]. An advantage of these model structures is that a filtered colimit of fibrant objects is again fibrant [13, 5.3, 6.7] (in fact, both model structures are weakly finitely generated in the sense of [4]). A fibrant $\mathbb{P}^1$-spectrum with respect to the stable motivic model structure will also be referred to as an Ω-spectrum.

The motivic equivalence $\sigma : \mathbb{P}^{\wedge 1} \to T$ induces an adjoint pair

$$f : \mathbf{Spt}^{\mathbb{P}^1}(Sm/k) \rightleftarrows \mathbf{Spt}^{T}(Sm/k) : g,$$

where g is the forgetful functor. When proving Theorem 4.1(1) the reader may have observed that we first replaced the suspension spectrum $\Sigma^\infty_{\mathbb{P}^1}\mathcal{X}$ by the suspension T-spectrum $\Sigma^\infty_T\mathcal{X} = f(\Sigma^\infty_{\mathbb{P}^1}\mathcal{X})$ and then applied Θ^∞ to the $\mathbb{P}^1$-spectrum $\Sigma^\infty_{\mathbb{P}^1,T}\mathcal{X} := gf(\Sigma^\infty_{\mathbb{P}^1}\mathcal{X}) = g(\Sigma^\infty_T\mathcal{X})$ in order to get framed correspondences. We see that framed correspondences are obtained from the T-spectrum $\Sigma^\infty_T\mathcal{X}$.

We want to extend this construction to spectra. Suppose $\mathcal{E} \in \mathbf{Spt}^{T}(Sm/k)$. By [15, p. 496] $\mathcal{E} = (\mathcal{E}_0, \mathcal{E}_1, \ldots)$ has a natural filtration

$$\mathcal{E} = \operatorname{colim}_n L_n\mathcal{E},$$

where $L_n\mathcal{E}$ is the spectrum

$$\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_n, \mathcal{E}_n \wedge T, \mathcal{E}_n \wedge T^2, \ldots$$

Denote by $L_n^{\mathbb{P}^1}\mathcal{E}$ the spectrum $g(L_n\mathcal{E})$. By Lemma 9.5 the natural map of spectra

$$\eta_n : L_n^{\mathbb{P}^1}\mathcal{E} \to \Theta^\infty L_n^{\mathbb{P}^1}\mathcal{E}$$

is a stable equivalence. There is an isomorphism of spectra

$$\iota_n : \Theta^\infty L_n^{\mathbb{P}^1}\mathcal{E} \xrightarrow{\cong} \Theta^\infty L_{n,fr}^{\mathbb{P}^1}\mathcal{E} := (\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, \mathcal{F}r(\mathcal{E}_n)), \ldots, \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, \mathcal{F}r(\mathcal{E}_n)), \mathcal{F}r(\mathcal{E}_n), \mathcal{F}r(\mathcal{E}_n \wedge T), \ldots).$$

If we apply the Suslin complex functor C_* levelwise, we get a spectrum

$$C_*\Theta^\infty L_{n,fr}^{\mathbb{P}^1}\mathcal{E} := (\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, C_*\mathcal{F}r(\mathcal{E}_n)), \ldots, \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, C_*\mathcal{F}r(\mathcal{E}_n)), C_*\mathcal{F}r(\mathcal{E}_n), C_*\mathcal{F}r(\mathcal{E}_n \wedge T), \ldots).$$

By [18, 2.3.8] the natural map of spectra

$$\delta_n : \Theta^\infty L_{n,fr}^{\mathbb{P}^1}\mathcal{E} \to C_*\Theta^\infty L_{n,fr}^{\mathbb{P}^1}\mathcal{E}$$

is a level motivic weak equivalence.

Passing to the colimit over n, we get that the composite map of spectra

$$\operatorname{colim}_n(\delta_n\iota_n\eta_n) : g(\mathcal{E}) = \operatorname{colim}_n L_n^{\mathbb{P}^1}\mathcal{E} \to \operatorname{colim}_n C_*\Theta^\infty L_{n,fr}^{\mathbb{P}^1}\mathcal{E}$$

is a stable motivic weak equivalence. We use here the fact that a sequential colimit of stable motivic weak equivalences is a stable motivic weak equivalence by [15, 3.12].

Denote by $\Theta^\infty C_*\mathcal{F}r(\mathcal{E})$ the spectrum

$$\operatorname{colim}_n \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, C_*\mathcal{F}r(\mathcal{E}_n)), \operatorname{colim}_n \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, C_*\mathcal{F}r(\mathcal{E}_{1+n})), \operatorname{colim}_n \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, C_*\mathcal{F}r(\mathcal{E}_{2+n})), \ldots$$

Also, denote by $C_*\mathscr{F}r(\mathscr{E})$ the spectrum $(C_*\mathscr{F}r(\mathscr{E}_0), C_*\mathscr{F}r(\mathscr{E}_1), C_*\mathscr{F}r(\mathscr{E}_2), \dots)$. Each structure map $C_*\mathscr{F}r(\mathscr{E}_k) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, C_*\mathscr{F}r(\mathscr{E}_{k+1}))$, $k \geqslant 0$, is given by the natural composition

$$C_*\mathscr{F}r(\mathscr{E}_k) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, C_*\mathscr{F}r(\mathscr{E}_k \wedge \mathbb{P}^{\wedge 1})) \xrightarrow{\sigma_*} \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, C_*\mathscr{F}r(\mathscr{E}_k \wedge T)) \xrightarrow{(\varepsilon_k)_*} \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, C_*\mathscr{F}r(\mathscr{E}_{k+1})),$$

where $\varepsilon_k : \mathscr{E}_k \wedge T \to \mathscr{E}_{k+1}$ is the structure map of $\mathscr{E}$. Observe that for any $\mathscr{X} \in \Delta^{\mathrm{op}}\, \overrightarrow{Fr_0}\,(k)$ the spectrum $M_{\mathbb{P}^{\wedge 1}}(\mathscr{X})$ is isomorphic to $C_*\mathscr{F}r(\Sigma^\infty_{\mathbb{P}^1,T}(\mathscr{X}))$ in $SH(k)$.

The spectrum $\mathrm{colim}_n C_*\Theta^\infty L^{\mathbb{P}^1}_{n,fr}\mathscr{E}$ is naturally isomorphic to the spectrum $\Theta^\infty C_*\mathscr{F}r(\mathscr{E})$ and the stable motivic equivalence

$$\alpha : g(\mathscr{E}) \to \Theta^\infty C_*\mathscr{F}r(\mathscr{E})$$

factors as $g(\mathscr{E}) \xrightarrow{\beta} C_*\mathscr{F}r(\mathscr{E}) \xrightarrow{\gamma} \Theta^\infty C_*\mathscr{F}r(\mathscr{E})$. Here γ equals the stable motivic equivalence of Lemma 9.5 and each $\beta_n : \mathscr{E}_n \to C_*\mathscr{F}r(\mathscr{E}_n)$ is the obvious map of motivic spaces. The two-out-of-three property implies β is a stable motivic equivalence. It is plainly functorial in $\mathscr{E} \in \mathbf{Spt}^T(Sm/k)$.

Theorem 13.1. *For every T-spectrum $\mathscr{E}$ there is a natural stable motivic equivalence of spectra $\beta : g(\mathscr{E}) \to C_*\mathscr{F}r(\mathscr{E})$, functorial in $\mathscr{E}$. Moreover, a morphism of spectra $u : \mathscr{E} \to \mathscr{E}'$ is a stable motivic equivalence if and only if $C_*\mathscr{F}r(u) : C_*\mathscr{F}r(\mathscr{E}) \to C_*\mathscr{F}r(\mathscr{E}')$ is. In particular, we have a functor*

$$C_*\mathscr{F}r : \mathrm{Ho}(\mathbf{Spt}^T(Sm/k)) \to \mathrm{Ho}(\mathbf{Spt}^{\mathbb{P}^1}(Sm/k)),$$

which is an equivalence of categories.

Proof. The first statement has already been verified above. The second statement follows from the first statement and the fact that $u : \mathscr{E} \to \mathscr{E}'$ is a stable motivic equivalence if and only if so is $g(u) : g(\mathscr{E}) \to g(\mathscr{E}')$ (see [15, p. 477]). The functor

$$C_*\mathscr{F}r : \mathrm{Ho}(\mathbf{Spt}^T(Sm/k)) \to \mathrm{Ho}(\mathbf{Spt}^{\mathbb{P}^1}(Sm/k)),$$

is an equivalence of categories, because g is and $\beta : g(\mathscr{E}) \to C_*\mathscr{F}r(\mathscr{E})$ is a stable motivic equivalence, functorial in $\mathscr{E}$. □

Lemma 13.2. *For every T-spectrum $\mathscr{E}$, each space of the $\mathbb{P}^1$-spectrum $\Theta^\infty\mathscr{F}r(\mathscr{E})$ (respectively $\Theta^\infty C_*\mathscr{F}r(\mathscr{E})$) is a motivic space with framed correspondences.*

Proof. $\Theta^\infty\mathscr{F}r(\mathscr{E})$ is the spectrum

$$\mathrm{colim}_n \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, \mathscr{F}r(\mathscr{E}_n)), \mathrm{colim}_n \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, \mathscr{F}r(\mathscr{E}_{1+n})), \mathrm{colim}_n \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, \mathscr{F}r(\mathscr{E}_{2+n})), \dots$$

Given a sheaf F and $s > 0$, we claim that $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge s}, \mathscr{F}r(F))$ is a sheaf with framed correspondences (the internal Hom is taken in the category of pointed Nisnevich sheaves). To see this, define for any $U, X \in Sm/k$ and any m, n a map of pointed sets

$$\mathrm{Hom}(U_+ \wedge \mathbb{P}^{\wedge m}, X_+ \wedge T^m) \wedge \mathrm{Hom}(X_+ \wedge \mathbb{P}^{\wedge s+n}, F \wedge T^n) \to \mathrm{Hom}(U_+ \wedge \mathbb{P}^{\wedge (s+m+n)}, F \wedge T^{m+n}).$$

If $\alpha : U_+ \wedge \mathbb{P}^{\wedge m} \to X_+ \wedge T^m$ and $v : X_+ \wedge \mathbb{P}^{\wedge s+n} \to F \wedge T^n$ are morphisms of pointed Nisnevich sheaves, then we define $\alpha^*(v)$ as the composite morphism

$$U_+ \wedge \mathbb{P}^{\wedge s} \wedge \mathbb{P}^{\wedge m} \wedge \mathbb{P}^{\wedge n} \cong U_+ \wedge \mathbb{P}^{\wedge m} \wedge \mathbb{P}^{\wedge s+n} \xrightarrow{\alpha \wedge \mathrm{id}} X_+ \wedge T^m \wedge \mathbb{P}^{\wedge s+n} \cong$$

$$\cong T^m \wedge X_+ \wedge \mathbb{P}^{\wedge s+n} \xrightarrow{\mathrm{id} \wedge v} T^m \wedge F \wedge T^n \cong F \wedge T^m \wedge T^n.$$

Passing to the colimit over n, we get that $\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge s}, \mathscr{F}r(F))$ is a sheaf with framed correspondences as claimed.

Next, the composite morphism

$$\underline{\mathrm{Hom}}(\mathbb{P}^{\wedge s}, \mathscr{F}r(\mathscr{E}_k)) \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge s+1}, \mathscr{F}r(\mathscr{E}_k \wedge T)) \xrightarrow{(\varepsilon_k)_*} \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge s+1}, \mathscr{F}r(\mathscr{E}_{k+1})), \quad s,k \geqslant 0,$$

is plainly a morphism of framed Nisnevich sheaves. Recall that the left arrow is induced by the map taking $(v : X_+ \wedge \mathbb{P}^{\wedge s+n} \to \mathscr{E}_k \wedge T^n) \in \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge s}, \mathscr{F}r_n(\mathscr{E}_k))$ to the composition

$$(X_+ \wedge \mathbb{P}^{\wedge s+1+n} \cong X_+ \wedge \mathbb{P}^{\wedge s+n} \wedge \mathbb{P}^{\wedge 1} \xrightarrow{v\wedge\sigma} \mathscr{E}_k \wedge T^n \wedge T \cong \mathscr{E}_k \wedge T \wedge T^n) \in \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge s+1}, \mathscr{F}r_n(\mathscr{E}_k \wedge T)).$$

It follows that each motivic space $(\Theta^\infty \mathscr{F}r(\mathscr{E}))_k = \mathrm{colim}_n \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge n}, \mathscr{F}r(\mathscr{E}_{k+n}))$ of the spectrum $\Theta^\infty \mathscr{F}r(\mathscr{E})$ has framed correspondences. Obviously, the same is true for the spectrum $\Theta^\infty C_* \mathscr{F}r(\mathscr{E})$. □

Suppose $\mathscr{E} \in \mathbf{Spt}^T(Sm/k)$. Then there are isomorphisms of $\mathbb{P}^1$-spectra

$$\Theta^\infty(g(\mathscr{E})) = \Theta^\infty(\mathrm{colim}_n L_n^{\mathbb{P}^1} \mathscr{E}) \cong \mathrm{colim}_n \Theta^\infty(L_n^{\mathbb{P}^1} \mathscr{E})) \cong \Theta^\infty \mathscr{F}r(\mathscr{E}).$$

If $\mathscr{E}$ is levelwise motivically fibrant it follows that $\Theta^\infty \mathscr{F}r(\mathscr{E})$ is an Ω-spectrum, because so is $\Theta^\infty(g(\mathscr{E}))$ by [12, 4.6]. Using Lemma 13.2, we have shown therefore the following

Lemma 13.3. *For every levelwise fibrant T-spectrum $\mathscr{E}$, the spectrum $\Theta^\infty \mathscr{F}r(\mathscr{E}) \in \mathbf{Spt}^{\mathbb{P}^1}(Sm/k)$ is an Ω-spectrum with framed correspondences.*

We are now in a position to prove that $SH(k)$ is canonically equivalent to its full subcategory $SH^{fr}(k)$ of framed spectra, i.e. those spectra whose motivic spaces are $\mathbb{A}^1$-invariant with framed correspondences.

Theorem 13.4. *The functor $C_*\mathscr{F}r : \mathbf{Spt}^T(Sm/k) \to \mathbf{Spt}^{\mathbb{P}^1}(Sm/k)$ takes a spectrum $\mathscr{E}$ to a framed spectrum $C_*\mathscr{F}r(\mathscr{E})$. The composite functor*

$$SH(k) = \mathrm{Ho}(\mathbf{Spt}^{\mathbb{P}^1}(Sm/k)) \xrightarrow{f} \mathrm{Ho}(\mathbf{Spt}^T(Sm/k)) \xrightarrow{C_*\mathscr{F}r} \mathrm{Ho}(\mathbf{Spt}^{\mathbb{P}^1}(Sm/k)),$$

induces an equivalence of categories $SH(k) \xrightarrow{\sim} SH^{fr}(k)$. Moreover, every $\mathbb{P}^1$-spectrum is isomorphic in $SH(k)$ to a framed Ω-spectrum.

Proof. By construction, $C_*\mathscr{F}r(\mathscr{E})$ is an $\mathbb{A}^1$-invariant with framed correspondences. The next statement follows from Theorem 13.1 and the fact that f is an equivalence of categories (see [15, 2.13]). To show that every $\mathbb{P}^1$-spectrum E is isomorphic in $SH(k)$ to a framed Ω-spectrum, let E^c be a cofibrant replacement of E and $f(E^c)_{lf} \in \mathbf{Spt}^T(Sm/k)$ be a level fibrant replacement of the T-spectrum $f(E^c)$. Let Θ_T^∞ denote the stabilization functor in the category of T-spectra. Since the flasque motivic model structure on spaces is weakly finitely generated (this follows from [13, 3.10]), the T-spectrum $\Theta_T^\infty(f(E^c)_{lf})$ is fibrant by [12, 4.6] and the morphism $f(E^c)_{lf} \to \Theta_T^\infty(f(E^c)_{lf})$ is a stable equivalence by [12, 4.11]. The canonical arrow $g(\Theta_T^\infty(f(E^c)_{lf})) \to \Theta^\infty(gf(E^c)_{lf})$ is a level weak equivalence by [15, p. 477], and hence the composite morphism

$$E^c \to gf(E^c) \to g(\Theta_T^\infty(f(E^c)_{lf})) \to \Theta^\infty(gf(E^c)_{lf})$$

is a stable weak equivalence, where the left arrow is the adjunction unit morphism. The $\mathbb{P}^1$-spectrum $\Theta^\infty(gf(E^c)_{lf})$ is a framed Ω-spectrum by Lemma 13.3, so the zigzag

$$E \leftarrow E^c \to \Theta^\infty(gf(E^c)_{lf})$$

gives an isomorphism in $SH(k)$. □

The preceding theorem says that we can define the stable motivic homotopy theory as framed $\mathbb{P}^1$-spectra or even framed Ω-spectra. However, such framed Ω-spectra are hardly amenable for explicit calculations in general, because they require levelwise motivically fibrant replacements by construction. Instead, the main point of this paper is to show that whenever the base field k is infinite perfect, one can nevertheless construct explicit fibrant Ω-spectra using the framed correspondences of Voevodsky and the machinery of framed motives introduced and developed in this paper.

Namely, suppose $\mathscr{E} = (\mathscr{E}_0, \mathscr{E}_1, \ldots) \in \mathbf{Spt}^T(Sm/k)$ is such that its framed spectrum $C_*\mathscr{F}r(\mathscr{E})$ satisfies the following conditions:

(1) each space $C_*\mathscr{F}r(\mathscr{E}_n)$, $n \geqslant 0$, is locally connected;
(2) each space $C_*\mathscr{F}r(\mathscr{E}_n)$, $n \geqslant 0$, is σ-invariant (i.e. it takes the framed correspondence $\sigma_X = (X \times 0, X \times \mathbb{A}^1, pr_{\mathbb{A}^1}, pr_X)$ of level one to a weak equivalence of simplicial sets for every $X \in Sm/k$);
(3) each structure morphism induces a motivic equivalence

$$C_*\mathscr{F}r(\mathscr{E}_n)_f \to \underline{\mathrm{Hom}}(\mathbb{P}^{\wedge 1}, C_*\mathscr{F}r(\mathscr{E}_{n+1})_f), \quad n \geqslant 0,$$

where the subscript "f" refers to a local fibrant replacement.

Then the spectrum $C_*\mathscr{F}r(\mathscr{E})_f := (C_*\mathscr{F}r(\mathscr{E}_0)_f, C_*\mathscr{F}r(\mathscr{E}_1)_f, \ldots)$ is an Ω-spectrum stably equivalent to $\mathscr{E}$. In particular, if a motivic space $\mathscr{X}$ is such that its suspension T-spectrum satisfies $(1)-(3)$, then $C_*\mathscr{F}r(\mathscr{X})$ is locally equivalent to the space $\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}\mathscr{X}$ (see p. 14 for the definition of the latter space). The hardest condition in practice is condition (3), where the machinery of framed motives works in its full capacity.

14. Concluding remarks

In [24] Voevodsky defined the category of rational framed correspondences $Fr_*^{rat}(k)$ together with an obvious functor $Fr_*(k) \to Fr_*^{rat}(k)$. The definition of $Fr_*^{rat}(U,X)$ replaces regular functions on etale neighborhoods of supports by rational functions. It follows from [24] that $C_*Fr^{rat}(U,X)$ is a group-like space.

Conjecture 1 (Voevodsky). *Let k be an infinite perfect field. Then for any $X \in Sm/k$ the morphism of motivic spaces*

$$Fr(\Delta^\bullet_k \times -, X) \to Fr^{rat}(\Delta^\bullet_k \times -, X)$$

is Nisnevich locally a group completion map of simplicial sets.

To illustrate the importance of this conjecture, we state some of its consequences below. To be precise, we state that if Voevodsky's conjecture is true then so are Corollaries 14.1, 14.2 and Theorem 14.3.

Corollary 14.1. *If Voevodsky's conjecture is true then the geometric realization of the simplicial set $Fr^{rat}(\Delta^\bullet_{\mathbb{C}}, pt)$ has the homotopy type of the topological space $\Omega^\infty\Sigma^\infty(S^0)$.*

Given a field k and $X \in Sm/k$, put $\pi_s^{rfr}(X) := \pi_s(Fr^{rat}(\Delta^\bullet_k, X))$.

Corollary 14.2. *Suppose Voevodsky's conjecture is true. Let $k = \mathbb{C}$. Then the assignment $X \mapsto \pi_*^{rfr}(X)$ is a generalized homology theory on the category $Sm/\mathbb{C}$. Moreover, for any nonzero integer m, one has*

$$\pi_s^{rfr}(X; \mathbb{Z}/m) = \pi_s^{st}(X_+; \mathbb{Z}/m).$$

Also, the first part of this corollary is true for any infinite perfect field k. Namely, the assignment $X \mapsto \pi_*^{rfr}(X)$ is a generalized homology theory on the category Sm/k.

Theorem 14.3. *Suppose Voevodsky's conjecture is true. Let k be an infinite perfect field. Then for any $X \in Sm/k$ the canonical morphism*

$$Fr^{rat}(\Delta_k^\bullet \times -, X) \to \Omega_{\mathbb{P}^1}^\infty \Sigma_{\mathbb{P}^1}^\infty (X_+)$$

is a Nisnevich local weak equivalence of motivic spaces.

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