Gelfand-type problem for turbulent jets.

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Abstract

We consider the model of auto-ignition (thermal explosion) of a free round reactive turbulent jet introduced in [11]. This model falls into the general class of Gelfand-type problems and constitutes a boundary value problem for a certain semi-linear elliptic equation that depends on two parameters: α characterizing the flow rate and λ (Frank-Kamenetskii parameter) characterizing the strength of the reaction. Similarly to the classical Gelfand problem, this equation admits a solution when the Frank-Kamenetskii parameter λ does not exceed some critical value $\lambda^*(\alpha)$ and admits no solutions for larger values of λ. We obtain the sharp asymptotic behavior of the critical Frank-Kamenetskii parameter in the strong flow limit ($\alpha \gg 1$). We also provide a detailed description of the extremal solution (i.e., the solution corresponding to λ^*) in this regime.

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1 Introduction.

In this paper we are concerned with the existence and quantitative properties of solutions of the following problem:

$$\begin{cases} -\Delta u - \alpha r \varphi(r) \frac{\partial}{\partial r} u = \lambda \psi(r) f(u) & \text{in} \quad B, \\ u > 0 & \text{in} \quad B, \\ u = 0 & \text{on} \quad \partial B, \end{cases} \tag{1.1}$$

where B is the unit disk in $\mathbb{R}^2$ centered at the origin, $\lambda > 0,\ \alpha > 0$ are parameters, $f : [0, \infty) \to (0, \infty)$ is a C^1 convex non-decreasing function satisfying:

$$\int_0^\infty \frac{ds}{f(s)} < \infty, \tag{1.2}$$

$\varphi(r), \psi(r)$ are non-negative, non-increasing Lipshitz continuous functions on $[0, 1]$ satisfying $\varphi(0) = \psi(0) = 1$. Moreover, we assume that $\varphi(r)$ is positive in $[0, 1)$. In addition we assume that

$$\int_0^1 M(s)ds < \infty, \tag{1.3}$$

where

$$M(s) := \max_{r\in[0,s]} \frac{\psi(r)}{\varphi(r)}. \tag{1.4}$$

Equation (1.1) was recently introduced in [11] as a model of the auto-ignition event in free round reactive turbulent jets. In the context of this model, u, f, φ, and ψ are respectively the (appropriately normalized)

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temperature, reaction rate, flow velocity profile, and product of concentrations of the oxidizing and reactive components over cross-sections of the jet, while λ is the Frank-Kamenetskii parameter representing the ratio of the heat release of the reaction and the thermal conductivity and α is the ratio of the injection velocity and the thermal conductivity.

The typical examples considered in the physical literature (e.g. [1,16]) are $f(u) = \exp(u)$, $\varphi(r) = \exp(-4r^2)$ or $\varphi(r) = (1-r^{3/2})^2$, and $\psi = \varphi^{2\mathrm{Sc}}$, where Sc is a Schlichting number which for round turbulent jets is $\mathrm{Sc} \approx 0.75$, in which cases our assumptions (1.2) and (1.3) are easily seen to be satisfied.

Problem (1.1) falls into the general class of Gelfand-type problems. The classical Gelfand problem can be obtained from problem (1.1) by removing the advection term, making the right hand side of the equation independent of r, that is, by setting $\alpha = 0$ and $\psi = 1$, and replacing the unit disk B in $\mathbb{R}^2$ by a general bounded domain in $\mathbb{R}^n$. That problem was introduced in 1938 by Frank-Kamenetskii as a model of thermal explosion in a combustion vessel with ideally thermally conducting walls (see [8,17,19] for more details), but became known in the mathematical community due to the chapter written by Barenblatt in a famous review of Gelfand [9]. The general properties of solutions of the classical Gelfand problem were studied quite extensively in both mathematical and physical literature, see book [6] for a review of results. The problem considered in this paper inherits many nice features of the classical Gelfand problem. The following proposition summarizes the properties of solutions of problem (1.1) relevant to the present work.

Proposition 1.1. *For fixed $\alpha > 0$, there exists an extremal value of the Frank-Kamenetskii parameter $\lambda^* = \lambda^*(\alpha) \in (0, \infty)$ such that:*

i) Problem (1.1) admits a unique minimal (i.e., smallest) classical positive solution $u_{\lambda,\alpha}$ for $\lambda \in (0, \lambda^*)$;

ii) Problem (1.1) admits a unique extremal solution u^*_α defined as

$$u^*_\alpha(x) := \lim_{\lambda \nearrow \lambda^*} u_{\lambda,\alpha}(x), \tag{1.5}$$

which is also classical;

iii) Minimal solutions of (1.1) for $\lambda \in (0, \lambda^*]$ are radially symmetric, strictly decreasing and satisfy the semi-stability condition

$$\int_B |\nabla \eta|^2 \mu dx \geq \lambda \int_B \psi f'(u_{\lambda,\alpha}) \eta^2 \mu dx, \quad \forall \eta \in H^1_0(B), \tag{1.6}$$

where

$$\mu(r) = \exp\left(\alpha \int_0^r s\varphi(s) ds\right); \tag{1.7}$$

iv) There are no solutions for (1.1) when $\lambda > \lambda^*$.

The proposition above is quite standard. We will present a sketch of the proof in the next section for completeness.

In the context of the auto-ignition problem, the extremal value $\lambda^*(\alpha)$ and the extremal solution u^*_α play a very special role. Indeed, in the context of the theory developed in [11], as any theory based on Frank-Kamenetskii approach, the existence of a solution for (1.1) indicates auto-ignition failure. From physical standpoint, that means that the reactive component undergoes partial oxidation, which results in establishing a self-similar temperature profile given by the minimal solution of (1.1). In contrast, the absence of a solution for (1.1) indicates successful auto-ignition. Therefore, $\lambda^*(\alpha)$ determines the boundary between the successful auto-ignition and the absence thereof. The extremal value of the Frank-Kamenetskii parameter $\lambda^*(\alpha)$ indicates the maximal reaction intensity for a given flow rate for which auto-ignition does not take place. The extremal solution determines the maximal possible self-similar profile.

In practical applications $\alpha \gg 1$ and hence one needs to understand the behavior of $\lambda^*(\alpha)$ for large α. This observation raises the question of asymptotic behavior of $\lambda^*(\alpha)$ as $\alpha \to \infty$. Lower and upper bounds on $\lambda^*(\alpha)$ as $\alpha \to \infty$ were derived in [11, Theorem 3.1] in the special case $f(u) = \exp(u)$. In this paper we establish a sharp asymptotic of $\lambda^*(\alpha)$ and give quite precise description of an extremal solution in this limit for a general class of nonlinearities $f(u)$ and functions φ, ψ under very mild regularity assumptions. Our main results are given by the following theorems.

The first theorem gives sharp asymptotic for $\lambda^*(\alpha)$ for large values of α:

Theorem 1.1. *Let*

$$K := \int_0^\infty \frac{ds}{f(s)}. \tag{1.8}$$

Then,

$$\lim_{\alpha\to\infty} \lambda^*(\alpha)\left(\frac{2K\alpha}{\log\alpha}\right)^{-1} = 1. \tag{1.9}$$

The second theorem provides details of the behavior of the extremal solution when $\alpha \gg 1$:

Theorem 1.2. *Let u^*_α be the extremal solution of* (1.1)*. Then, as $\alpha \to \infty$, we have*

$$i)\ u^*_\alpha(x) \to 0 \quad \forall x \in B \setminus \{0\}, \qquad u^*_\alpha(0) \to \infty, \tag{1.10}$$

and

$$ii)\ \int_B u^*_\alpha(x)dx \to 0, \qquad \int_B \psi(x) f(u^*_\alpha(x))dx \to f(0)\int_B \psi(x)dx. \tag{1.11}$$

While Proposition 1.1 ensures that extremal solutions of problem (1.1) are bounded, the establishment of a reasonable uniform upper bound appeared to be a difficult task. However, in case of sufficiently regular non-linearities or non-linearities with sufficiently fast growth at infinity we have the following result.

Theorem 1.3. *Assume that there exist constants $0 < c_0 < 1$, $c_1 > 1$ and $t_0 > 0$ such that*

$$f(t_2) \geq c_1 f(t_1), \quad t_2 > t_1 > t_0, \tag{1.12}$$

implies

$$(1 - c_0)f'(t_2) \geq f'(t_1). \tag{1.13}$$

Then, assuming that α is sufficiently large,

$$u^*_\alpha(0) \leq cA, \tag{1.14}$$

where A is the solution of

$$f'(A) = c\log\alpha \tag{1.15}$$

and $c > 0$ is some constant independent of α.

Moreover,

$$\int_B (u^*_\alpha(x))^p dx \to 0 \quad as \quad \alpha \to \infty, \tag{1.16}$$

for any $1 \leq p < \infty$.

Remark 1.1. *It is easy to check that the assumptions of Theorem 1.3 are satisfied for most typical non-linearities such as $f(u) = \exp(u)$ and $f(u) = (1+u)^p$ for $p > 1$. Moreover, they are also satisfied provided f is a C^2 function such that $f'(s), f''(s) > 0$ and $f''(s)$ is strictly increasing on $(0,\infty)$ (see Lemma 5.2).*

Remark 1.2. *In the case of exponential non-linearity, we also have that*

$$\lambda^*(\alpha) = \frac{2\alpha}{\log\alpha}\left(1 + O\left(\frac{1}{\sqrt{\log\alpha}}\right)\right) \tag{1.17}$$

and

$$u^*_\alpha(0) = O(\log(\log\alpha)), \tag{1.18}$$

as $\alpha \to \infty$, see Lemma 5.3. These results are consistent with the formal asymptotic extremal solution of (1.1) *obtained in* [11].

The paper is organized as follows. In section 2 we are setting up the stage by giving necessary definitions, providing standard results and introducing rescaling which makes the analysis more convenient. In sections 3,4,5 we give proofs of theorems 1.1, 1.2, and 1.3 respectively.

2 Preliminaries.

In this section we outline a proof of Proposition 1.1 and introduce alternative forms of problem (1.1), which will be used in the later sections.

First we observe that problem (1.1) can be written in the divergence form. Indeed one can verify by direct computations that (1.1) can be rewritten as follows

$$\begin{cases} -\nabla\cdot(\mu(r)\nabla u) = \lambda\mu(r)\psi(r)f(u) & \text{in} \quad B, \\ u > 0 & \text{in} \quad B, \\ u = 0 & \text{on} \quad \partial B, \end{cases} \tag{2.1}$$

where μ is given by (1.7). We note that $\mu \in C^{1,\omega}$ $(0 < \omega < 1)$ as follows from definition and properties of φ, while ψ is Lipshitz continuous. The results presented in this section deal with the situation when $\alpha > 0$ is fixed. Therefore, we will omit subscript α when referring to minimal and extremal solutions of (2.1).

Proposition 1.1 is basically a compilation of well known results (or their minimal adaptations) presented in [2–5, 12–14, 18]. It follows from the sequence of Lemmas 2.1-2.5 presented below.

A proof of Proposition 1.1 is based on the construction of sub and super-solutions for problem (2.1). Following [7, 15], we define a classical positive super-solution of (2.1) as a function $\bar{u} \in C^2(B) \cap C(\bar{B})$ positive in B such that

$$\begin{cases} -\nabla\cdot(\mu(r)\nabla \bar{u}) \geq \lambda\mu(r)\psi(r)f(\bar{u}) & \text{in} \quad B, \\ \bar{u} \geq 0 & \text{on} \quad \partial B, \end{cases} \tag{2.2}$$

and a classical non-negative sub-solution of (2.1) as a function $\underline{u} \in C^2(B) \cap C(\bar{B})$ non-negative in $\bar{B}$ such that

$$\begin{cases} -\nabla\cdot(\mu(r)\nabla \underline{u}) \leq \lambda\mu(r)\psi(r)f(\underline{u}) & \text{in} \quad B, \\ \underline{u} = 0 & \text{on} \quad \partial B, \end{cases} \tag{2.3}$$

We note that under the assumptions of this paper $\underline{u} = 0$ is always a sub-solution.

Lemma 2.1. *Assume* (2.1) *admits a positive classical super-solution $\bar{u}$. Then* (2.1) *admits a unique minimal, positive classical solution $u_\lambda \in C^{2,\omega}(\bar{B})$. This minimal solution is radially symmetric, strictly decreasing and bounded by $\bar{u}$ from above.*

Proof. The minimal solution u_λ of (2.1) is obtained by a construction using monotone iteration arguments. Namely, we consider a sequence of functions $\{u_n\}_{n=0}^{\infty}$ with $u_0 = 0$ and u_n defined by

$$\begin{cases} -\nabla\cdot(\mu(r)\nabla u_n) + \Omega u_n = \lambda\mu(r)\psi(r)f(u_{n-1}) + \Omega u_{n-1} & \text{in} \quad B, \\ u_n = 0 & \text{on} \quad \partial B, \end{cases} \tag{2.4}$$

for $n \geq 1$, where $\Omega > 0$ is an arbitrary constant.

As follows from [10, Theorem 6.14], for each n problem (2.4) admits a unique solution $u_n \in C^{2,\omega}(\bar{B})$. Each u_n is radial, as follows from the uniqueness. We now define the minimal solution of (2.1) as

$$u_\lambda(x) := \lim_{n\to\infty} u_n(x). \tag{2.5}$$

By [18, Theorem 2.1] we have that u_λ defined by (2.5) satisfies $\bar{u} \geq u_\lambda > 0$ in B, belongs to $C^{2,\omega}(\bar{B})$ and solves (2.1) classically. Moreover, since each u_n is radially symmetric we have that u_λ is also radially symmetric so that $u_\lambda(x) = u_\lambda(|x|) = u_\lambda(r)$. Consequently, any minimal solution constructed above satisfies

$$\begin{cases} -\frac{d}{dr}\left(r\mu(r)\frac{d}{dr}u_\lambda\right) = \lambda r\mu(r)\psi(r)f(u_\lambda) & 0 < r < 1, \\ \frac{d}{dr}u_\lambda(0) = 0, & u_\lambda(1) = 0. \end{cases} \tag{2.6}$$

Integrating (2.6) we also have that for $r \in (0, 1]$,

$$\frac{d}{dr}u_\lambda(r) = -\frac{\lambda}{r\mu(r)}\int_0^r \mu(s)\psi(s)f(u_\lambda(s))sds < 0, \tag{2.7}$$

and hence u_λ is strictly decreasing. □

The following lemma uses the notion of a weak solution. Similarly to [3], we define a weak solution of (2.1) as a non-negative function $u \in L^1(B)$ such that $\psi f(u)\mathrm{dist}_{\partial B} \in L^1(B)$, where $\mathrm{dist}_{\partial B}(x)$ is the distance from x to the boundary of B, and

$$-\int_B u\nabla\cdot(\mu\nabla\zeta)dx = \int_B \psi\mu f(u)\zeta dx, \tag{2.8}$$

for all $\zeta \in C^2(\bar{B})$ with $\zeta = 0$ on ∂B.

Lemma 2.2. *Problem* (2.1) *admits a minimal classical solution* u_λ *for* $0 < \lambda < \lambda^* < \infty$*. Moreover, the extremal solution* u^* *defined by* (1.5) *is a weak solution of* (2.1).

Proof. First observe that u_λ is a non-decreasing function of λ. This follows from the fact that $u_{\lambda'}$ is a super-solution for problem (2.1) with $\lambda < \lambda'$. Hence, if (2.1) with $\lambda = \lambda'$ admits a classical solution, then (2.1) admits a classical solution for $\lambda \in (0, \lambda']$.

Next, let τ be a solution of

$$\begin{cases} -\nabla\cdot(\mu\nabla\tau) = 1 & \text{in} \quad B, \\ \tau = 0 & \text{on} \quad \partial B. \end{cases} \tag{2.9}$$

It is easy to see that τ is a super-solution for (2.1) provided $\lambda \le \big(\mu(1)f(\tau(0))\big)^{-1}$. This establishes the existence of a minimal solution for small enough λ.

Now let us show that $\lambda^* < \infty$, which is done by a slight adaptation of [3, Lemma 5]. This adaptation is needed because ψ might be zero in some portion of B. By convexity of f we have that there is $\varepsilon > 0$ such that $f(s) > \varepsilon s$ for $s \ge 0$. Hence

$$-\nabla\cdot(\mu\nabla u) \ge \varepsilon\lambda\psi\mu u \quad \text{in} \quad B. \tag{2.10}$$

Let κ_1, ξ_1 be the principal eigenvalue and the corresponding eigenfunction of the generalized eigenvalue problem

$$\begin{cases} -\nabla\cdot(\mu\nabla\xi) = \kappa\psi\mu\xi & \text{in} \quad B, \\ \xi = 0 & \text{on} \quad \partial B. \end{cases} \tag{2.11}$$

Variational characterization of κ_1 and arguments identical to these of [10, Theorem 8.38] show that $\kappa_1 > 0$ and $\xi_1 > 0$ in B.

Multiplying (2.10) by ξ_1 and integrating by parts we obtain that

$$\kappa_1\int_B \psi\mu u\xi_1 dx \ge \varepsilon\lambda\int_B \psi\mu u\xi_1 dx. \tag{2.12}$$

Thus, $\varepsilon\lambda \le \kappa_1$ and hence $\lambda^* \le \kappa_1/\varepsilon$.

Finally, proceeding as in the proof of [3, Lemma 5] with the only modification that $-\Delta$ is replaced by $-\nabla\cdot(\mu\nabla(\cdot))$, we recover that u^* is a weak solution of (2.1). □

Lemma 2.3. *Problem* (2.1) *admits neither classical, nor weak solutions for* $\lambda > \lambda^*$.

Proof of this lemma is a line by line adaptation of [3, Theorem 3] with the only difference that $-\Delta$ is replaced by $-\nabla\cdot(\mu\nabla(\cdot))$ and is omitted here.

As a next step we give a proof of the semi-stability condition (1.6). The proof is similar to the one of [18, Theorem 4.2], with a slight modification, which is required due to the restriction on regularity of the nonlinear term in (2.1).

Lemma 2.4. *The semi-stability condition* (1.6) *holds for any minimal solution* u_λ *with* $\lambda \in (0, \lambda^*]$.

Proof. Let

$$\mathcal{L}\eta = -\nabla\cdot(\mu\nabla\eta) - \lambda\mu\psi f'(u_\lambda)\eta, \quad \tilde{\mathcal{L}}\tilde{\eta} = -\nabla\cdot(\mu\nabla\tilde{\eta}) - \lambda\mu\psi f'_\varepsilon(u_\lambda)\tilde{\eta}, \tag{2.13}$$

where f'_ε is C^ω function such that

$$|f'(s) - f'_\varepsilon(s)| < \varepsilon \quad \text{for} \quad s \in [0, u_\lambda(0)], \tag{2.14}$$

for some $\varepsilon > 0$, and set $\lambda_1, \tilde{\lambda}_1$ to be the principal eigenvalues of $\mathcal{L}$ and $\tilde{\mathcal{L}}$ respectively.

Assume first that $\lambda < \lambda^*$. We claim that $\lambda_1 \geq 0$. To show that, we note that the first eigenfunction of $\tilde{\mathcal{L}}$, $\tilde{\eta}_1$ is positive in B and $\tilde{\eta}_1 \in C^{2,\omega}(\bar{B})$ as follows from [10, Theorem 6.15, Theorem 8.38]. Choosing the normalization in such a way that $||\tilde{\eta}_1||_{C^1(\bar{B})} = 1$, we observe that $\tilde{u} = u_\lambda - \varepsilon\tilde{\eta}_1$ is in $C^{2,\omega}(\bar{B})$ and positive in B, provided that ε is sufficiently small. The latter is guaranteed by the fact that the normal derivative of u_λ on the boundary is strictly positive as follows from Hopf's lemma [15, Chapter 2, Theorem7].

We next observe that

$$-\nabla \cdot (\mu \nabla \tilde{u}) - \lambda \mu \psi f(\tilde{u}) = -\varepsilon \tilde{\lambda}_1 \tilde{\eta}_1 + \lambda \mu \psi R_\varepsilon, \tag{2.15}$$

with

$$R_\varepsilon = [f(u_\lambda) - \varepsilon f'(u_\lambda)\tilde{\eta}_1 - f(u_\lambda - \varepsilon\tilde{\eta}_1)] + \varepsilon \left[f'(u_\lambda) - f'_\varepsilon(u_\lambda)\right] \tilde{\eta}_1. \tag{2.16}$$

Clearly $|R_\varepsilon| = o(\varepsilon\eta_1)$ as $\varepsilon \to 0$. Thus, if $\tilde{\lambda}_1 < 0$, then $\tilde{u}$ is a classical positive super-solution of (2.1) strictly below u_λ in B, which contradicts the minimality of u_λ. Hence, $\tilde{\lambda}_1 \geq 0$.

Next observe that $\lambda_1, \tilde{\lambda}_1$ admit the variational characterization

$$\lambda_1 = \inf_{\eta \in \mathcal{A}} \int_B (|\nabla \eta|^2 - \lambda \psi f'(u_\lambda)\eta^2)\mu dx, \quad \tilde{\lambda}_1 = \inf_{\eta \in \mathcal{A}} \int_B (|\nabla \eta|^2 - \lambda \psi f'_\varepsilon(u_\lambda)\eta^2)\mu dx, \tag{2.17}$$

where $\mathcal{A} := \{\eta \in H_0^1(B) : ||\eta||_{L^2(B)} = 1\}$. This immediately implies that $\lambda_1 \geq \tilde{\lambda}_1 - \lambda\mu(1)\varepsilon$. Since ε is arbitrarily small, we have that $\lambda_1 \geq 0$ and hence (1.6) holds for $\lambda < \lambda^*$. Moreover, since u^* is an increasing point-wise limit of u_λ, it also holds for $\lambda = \lambda^*$.

□

Lemma 2.5. *An extremal solution of* (2.1) *is classical.*

Proof. Taking $\tilde{f}(s) = f(s) - f(0)$ and setting $\eta = \tilde{f}(u_\lambda)$ in (1.6) we have that

$$\int_B |\nabla u_\lambda| \left(f'(u_\lambda)\right)^2 \mu dx \geq \lambda \int_B f'(u_\lambda) \left(\tilde{f}(u_\lambda)\right)^2 \mu\psi dx. \tag{2.18}$$

Next taking $g(t) = \int_0^t (f'(s))^2 ds$, multiplying first equation in (2.1) by $g(u_\lambda)$ and integrating the result by parts we have

$$\int_B |\nabla u_\lambda|^2 \left(f'(u_\lambda)\right)^2 \mu dx = \lambda \int_B f(u_\lambda) g(u_\lambda) \psi \mu dx. \tag{2.19}$$

Combining (2.18), (2.19), we have

$$\int_B f'(u_\lambda) \left(\tilde{f}(u_\lambda)\right)^2 \psi \mu dx \leq \int_B f(u_\lambda) g(u_\lambda) \psi \mu dx. \tag{2.20}$$

Using this inequality instead of Eq.(4) in [14] and arguing exactly as in [14, Theorem 1] we conclude that the extremal solution is classical.

□

In this paper we are mostly concerned with minimal solutions of problem (1.1), which by Proposition 1.1 are classical and radially symmetric. Therefore, we will only consider solutions of (1.1) that are radially symmetric.

To study radially symmetric solutions, it is convenient to introduce the following rescaling

$$u(r) := v(\sqrt{\alpha} r) = v(y). \tag{2.21}$$

Substituting (2.21) into (1.1) we get

$$\begin{cases} -v'' - \left(\frac{1}{y} + y\varphi_\alpha(y)\right) v' = \frac{\lambda}{\alpha} \psi_\alpha(y) f(v), & 0 < y < \sqrt{\alpha}, \\ v'(0) = 0, & v(\sqrt{\alpha}) = 0. \end{cases} \tag{2.22}$$

Here and below

$$\varphi_\alpha(y) := \varphi\left(\frac{y}{\sqrt{\alpha}}\right), \quad \psi_\alpha(y) := \psi\left(\frac{y}{\sqrt{\alpha}}\right). \tag{2.23}$$

Also from now on we reserve the notation $(\cdot)'$ for the derivative with respect to the y variable, that is, $(\cdot)' = \frac{d}{dy}(\cdot)$.
We now set

$$G(y) := \int_{v(y)}^{\infty} \frac{ds}{f(s)}, \tag{2.24}$$

and note that the mapping $[0,+\infty) \ni v \mapsto \int_v^\infty \frac{ds}{f(s)} \in (0,K]$ is a C^2 strictly monotone decreasing bijection and hence its inverse is also a strictly monotone decreasing C^2 function from $(0,K]$ to $[0,+\infty)$. Differentiating (2.24), we obtain

$$G' = -\frac{v'}{f(v)}, \quad G'' = -\frac{v''}{f(v)} + \left(\frac{v'}{f(v)}\right)^2 f_v(v) = -\frac{v''}{f(v)} + (G')^2 f_v(v), \tag{2.25}$$

where

$$f_v(v) = \frac{df(v)}{dv}. \tag{2.26}$$

In what follows we will use the notation (2.26) for the derivative of the nonlinearity f with respect to its argument to avoid the confusion with the derivative of the corresponding composition with respect to y.

Combining (2.22) and (2.25) yields the equation

$$\begin{cases} G'' + \left(\frac{1}{y} + y\varphi_\alpha(y)\right) G' = \frac{\lambda}{\alpha}\psi_\alpha(y) + (G')^2 f_v(v), & 0 < y < \sqrt{\alpha}, \\ G'(0) = 0, & G(\sqrt{\alpha}) = K, \end{cases} \tag{2.27}$$

where v is related to G via (2.24).

In what follows we will work with both (2.22) and (2.27) as alternative versions of (1.1).

We next define super-solution for problem (2.22) and sub-solution for (2.27).

A function $\bar{v} > 0$ on $[0,\sqrt{\alpha})$ belonging to $C^2((0,\sqrt{\alpha})) \cap C([0,\sqrt{\alpha}])$ is a positive super-solution of (2.22) provided it verifies

$$\begin{cases} -\bar{v}'' - \left(\frac{1}{y} + y\varphi_\alpha(y)\right) \bar{v}' \geq \frac{\lambda}{\alpha}\psi_\alpha(y) f(\bar{v}), & 0 < y < \sqrt{\alpha}, \\ \bar{v}'(0) = 0, & \bar{v}(\sqrt{\alpha}) \geq 0. \end{cases} \tag{2.28}$$

Similarly, a function $\underline{G} > 0$ on $[0,\sqrt{\alpha}]$ belonging to $C^2((0,\sqrt{\alpha})) \cap C([0,\sqrt{\alpha}])$ is a positive classical sub-solution of (2.27) if

$$\begin{cases} \underline{G}'' + \left(\frac{1}{y} + y\varphi_\alpha(y)\right) \underline{G}' \geq \frac{\lambda}{\alpha}\psi_\alpha(y) + \left(\underline{G}'\right)^2 f_v(v), & 0 < y < \sqrt{\alpha} \\ \underline{G}'(0) = 0, & \underline{G}(\sqrt{\alpha}) \leq K. \end{cases} \tag{2.29}$$

The following result follow from Lemma 2.1.

Corollary 2.1. *Assume* (2.22) *has a positive super-solution* $\bar{v}$*. Then* (2.22) *has a minimal positive classical solution* v_λ*. Moreover,* $v_\lambda(y) \leq \bar{v}(y)$ *in* $y \in [0,\sqrt{\alpha}]$.

Note that if $\underline{G}$ is a positive sub-solution of (2.27), then, the function $\bar{v}$ implicitly defined by (2.24) is a positive super-solution of (2.22). Thus, the above corollary can be also restated as

Corollary 2.2. *Assume that* (2.27) *has a positive sub-solution* $\underline{G}$*. Then* (2.22) *has a minimal positive classical solution* v_λ*.*

3 Asymptotic behavior of λ^* as $\alpha \to \infty$.

In this section we establish an asymptotic behavior of $\lambda^*(\alpha)$ for sufficiently large α and give a proof of Theorem 1.1. In what follows we will work with (2.27), which is an alternative form of (1.1).

We start with the upper bound for λ^*, which is given by the following lemma.

Lemma 3.1. *Assume (1.1) has a solution. Then, for α sufficiently large, $\lambda^*(\alpha)$ obeys the following upper bound:*

$$\lambda^*(\alpha) \le \frac{2K\alpha}{\log\alpha}\left(1+\frac{c}{\log\alpha}\right), \tag{3.1}$$

where $c>0$ is some constant independent of α.

Proof. First we observe that the first equation in (2.27) can be rewritten in the divergence form

$$\left(y\exp\left(\int_0^y s\varphi_\alpha(s)ds\right)G'(y)\right)' = \left[\frac{\lambda}{\alpha}\psi_\alpha(y) + \left(G'(y)\right)^2 f_v(v(y))\right] y\exp\left(\int_0^y s\varphi_\alpha(s)ds\right). \tag{3.2}$$

Therefore,

$$\left(y\exp\left(\int_0^y s\varphi_\alpha(s)ds\right)G'(y)\right)' \ge \frac{\lambda}{\alpha} y\psi_\alpha(y)\exp\left(\int_0^y s\varphi_\alpha(s)ds\right). \tag{3.3}$$

Integrating (3.3) from 0 to y and using the boundary condition at zero in (2.27), we get

$$G'(y) \ge \frac{\lambda}{\alpha}\frac{1}{y}\int_0^y z\psi_\alpha(z)\exp\left(-\int_z^y s\varphi_\alpha(s)ds\right)dz. \tag{3.4}$$

Next observe that due to the monotonicity of ψ and φ we have

$$G'(y) \ge \frac{\lambda}{\alpha}\frac{\psi_\alpha(y)}{y}\int_0^y z\exp\left(-\int_z^y sds\right)dz = \frac{\lambda}{\alpha}\frac{\psi_\alpha(y)}{y}\left(1-\exp\left(-\frac{y^2}{2}\right)\right). \tag{3.5}$$

Since $\psi(0)=1$ and ψ is Lipshitz continuous, we have that

$$\psi(x) \ge 1-\frac{x}{h}, \tag{3.6}$$

for some constant $0<h\le 1$ independent of α. Therefore,

$$\psi_\alpha(y) \ge 1-\frac{y}{h\sqrt{\alpha}}. \tag{3.7}$$

Consequently, (3.5) and (3.7) yield that for $y\in[0,h\sqrt{\alpha}]$

$$\begin{aligned} G'(y) &\ge \frac{\lambda}{\alpha}\frac{1}{y}\left(1-\exp\left(-\frac{y^2}{2}\right)\right) + \frac{\lambda}{\alpha}\frac{(\psi_\alpha(y)-1)}{y}\left(1-\exp\left(-\frac{y^2}{2}\right)\right) \\ &\ge \frac{\lambda}{\alpha}\frac{1}{y}\left(1-\exp\left(-\frac{y^2}{2}\right)\right) - \frac{\lambda}{h\alpha^{3/2}}\left(1-\exp\left(-\frac{y^2}{2}\right)\right). \end{aligned} \tag{3.8}$$

Integrating (3.8) from 0 to $h\sqrt{\alpha}$, we obtain

$$G(h\sqrt{\alpha}) - G(0) \ge \frac{\lambda}{\alpha}(J_1 - J_2), \tag{3.9}$$

where

$$J_1 = \int_0^{h\sqrt{\alpha}}\left(1-\exp\left(-\frac{y^2}{2}\right)\right)\frac{dy}{y}, \qquad J_2 = \frac{1}{h\sqrt{\alpha}}\int_0^{h\sqrt{\alpha}}\left(1-\exp\left(-\frac{y^2}{2}\right)\right)dy. \tag{3.10}$$

Since $G(h\sqrt{\alpha}) \le K$ and $G(0) > 0$ we have from (3.9) that

$$K\alpha \ge \lambda(J_1 - J_2). \tag{3.11}$$

Next observe that the following estimates for J_1 and J_2 hold:

$$\begin{aligned} J_1 &= \int_1^{h\sqrt{\alpha}} \frac{dy}{y} - \int_1^{h\sqrt{\alpha}} \exp\left(-\frac{y^2}{2}\right)\frac{dy}{y} + \int_0^1 \left(1 - \exp\left(-\frac{y^2}{2}\right)\right)\frac{dy}{y} \\ &\ge \int_1^{h\sqrt{\alpha}} \frac{dy}{y} - \int_1^{\infty} \exp\left(-\frac{y^2}{2}\right)\frac{dy}{y} = \frac{\log\alpha}{2} + \log h - c \end{aligned} \tag{3.12}$$

(here and below c stands for a positive constant independent of α that may change from line to line), and

$$J_2 \le \frac{1}{h\sqrt{\alpha}} \int_0^{h\sqrt{\alpha}} dy = 1. \tag{3.13}$$

As a result of (3.12) and (3.13) we have that

$$J_1 - J_2 \ge \frac{\log\alpha}{2} - c. \tag{3.14}$$

Combining (3.11) and (3.14), we conclude that for (2.27) to admit a solution, we necessarily need

$$\lambda \le \frac{2K\alpha}{\log\alpha - c} \le \frac{2K\alpha}{\log\alpha}\left(1 + \frac{c}{\log\alpha}\right), \tag{3.15}$$

for α sufficiently large.

This observation immediately implies (3.1), which establishes an upper bound for $\lambda^*(\alpha)$. □

To obtain a lower bound for $\lambda^*(\alpha)$ we need two technical lemmas, where the role of the parameter w will be clarified later.

Lemma 3.2. *Let $\tilde{G}$ be the solution of the following linear problem:*

$$\begin{cases} \tilde{G}'' + \left(\frac{1}{y} + y\varphi_\alpha(y)\right)\tilde{G}' = \beta_w\left(\psi_\alpha(y) + \frac{M_\alpha^2(y)}{\alpha}\right), & 0 < y < \sqrt{\alpha} \\ \tilde{G}'(0) = 0, & \tilde{G}(\sqrt{\alpha}) = K, \end{cases} \tag{3.16}$$

with sufficiently large α. Here

$$M_\alpha(t) := M\left(\frac{t}{\sqrt{\alpha}}\right), \tag{3.17}$$

with M given by (1.4), $\varphi_\alpha, \psi_\alpha$ given by (2.23),

$$\beta_w = \frac{2(K - \varepsilon_w)}{\log\alpha + c}, \qquad \varepsilon_w = \int_w^\infty \frac{ds}{f(s)}, \tag{3.18}$$

where $w > 0$ is an arbitrary number and c is a sufficiently large constant independent of α.

Then, $\tilde{G}(y)$ is an increasing function on $y \in [0, \sqrt{\alpha}]$ and

$$\tilde{G}(y) \ge \varepsilon_w > 0, \quad \text{on} \quad [0, \sqrt{\alpha}]. \tag{3.19}$$

Proof. Using computations identical to (3.2), we rewrite (3.16) in the divergence form. Integrating the result from 0 to y and using the boundary condition at zero in (3.16), we get

$$\tilde{G}'(y) = \frac{\beta_w}{y}\int_0^y z\psi_\alpha(z)\exp\left(-\int_z^y s\varphi_\alpha(s)ds\right)dz + \frac{\beta_w}{\alpha y}\int_0^y zM_\alpha^2(z)\exp\left(-\int_z^y s\varphi_\alpha(s)ds\right)dz = J_3 + J_4. \tag{3.20}$$

Clearly $J_3, J_4 > 0$ and hence $\tilde{G}$ is strictly increasing. Therefore, we only need to prove the positivity of $\tilde{G}(0)$.

As a first step, we establish upper bounds on J_3 and J_4. To do so, let

$$F(t) := \int_0^t s\varphi_\alpha(s)ds, \tag{3.21}$$

and observe that

$$F(t) \leq \frac{t^2}{2}, \tag{3.22}$$

$$\begin{aligned} &\int_0^y z\varphi_\alpha(z)\exp\left(-\int_z^y s\varphi_\alpha(s)ds\right)dz = \int_0^y z\varphi_\alpha(z)\exp\left(F(z)-F(y)\right)dz \\ &= 1-\exp\left(-F(y)\right) \leq 1-\exp\left(-\frac{y^2}{2}\right), \end{aligned} \tag{3.23}$$

and

$$\int_0^y M_\alpha(z)dz = \int_0^y M\left(\frac{z}{\sqrt{\alpha}}\right)dz = \sqrt{\alpha}\int_0^{\frac{y}{\sqrt{\alpha}}} M(s)ds \leq \sqrt{\alpha}\int_0^1 M(s)ds = m\sqrt{\alpha}, \tag{3.24}$$

where

$$m := \int_0^1 M(s)ds. \tag{3.25}$$

Using (3.22) – (3.25) we obtain the following upper bounds on J_3, J_4 :

$$\begin{aligned} &J_3(y) = \frac{\beta_w}{y}\int_0^y \frac{\psi_\alpha(z)}{\varphi_\alpha(z)} z\varphi_\alpha(z)\exp\left(-\int_z^y s\varphi_\alpha(s)ds\right)dz \\ &\leq \frac{\beta_w}{y}M_\alpha(y)\int_0^y z\varphi_\alpha(z)\exp\left(-\int_z^y s\varphi_\alpha(s)ds\right)dz \leq \frac{\beta_w}{y}M_\alpha(y)\left(1-\exp\left(-\frac{y^2}{2}\right)\right), \end{aligned} \tag{3.26}$$

and

$$J_4 \leq \frac{\beta_w}{\alpha y}\int_0^y zM_\alpha^2(z)dz \leq \frac{\beta_w}{\alpha}\int_0^y M_\alpha^2(z)dz \leq \frac{\beta_w}{\alpha}M_\alpha(y)\int_0^y M_\alpha(z)dz \leq m\frac{\beta_w}{\sqrt{\alpha}}M_\alpha(y). \tag{3.27}$$

Let us also note that $M(t)$ is Lipshitz continuous on the interval $[0,k]$ for some $0 < k \leq 1$ independent of α as follows from the properties of φ and ψ and the definition of M. Therefore, when $y \in [0, k\sqrt{\alpha}]$,

$$M_\alpha(y) \leq 1 + l\frac{y}{\sqrt{\alpha}}, \tag{3.28}$$

for some constant $l \geq 0$ independent of α.

Bounds (3.26), (3.27), (3.28) and the observations presented below allow one to estimate the difference

$$\tilde{G}(\sqrt{\alpha}) - \tilde{G}(0) = \int_0^{\sqrt{\alpha}} \tilde{G}'(y)dy = \int_0^{\sqrt{\alpha}} J_3(y)dy + \int_0^{\sqrt{\alpha}} J_4(y)dy. \tag{3.29}$$

Indeed, we have

$$\int_0^1 \frac{M_\alpha(y)}{y}\left(1-\exp\left(-\frac{y^2}{2}\right)\right)dy \leq M_\alpha(1)\int_0^1\left(1-\exp\left(-\frac{y^2}{2}\right)\right)\frac{dy}{y} \leq \left(1+\frac{l}{\sqrt{\alpha}}\right)c \leq c, \tag{3.30}$$

and

$$\begin{aligned} &\int_1^{\sqrt{\alpha}} (M_\alpha(y)-1)\frac{dy}{y} = \int_1^{k\sqrt{\alpha}} (M_\alpha(y)-1)\frac{dy}{y} + \int_{k\sqrt{\alpha}}^{\sqrt{\alpha}} (M_\alpha(y)-1)\frac{dy}{y} \\ &\leq \frac{l}{\sqrt{\alpha}}\int_1^{k\sqrt{\alpha}} dy + \frac{1}{k\sqrt{\alpha}}\int_{k\sqrt{\alpha}}^{\sqrt{\alpha}} M_\alpha(y)dy \leq lk + \frac{1}{k}\int_k^1 M(s)ds \leq lk + \frac{m}{k} = c. \end{aligned} \tag{3.31}$$

Thanks to estimates (3.30) and (3.31) we have

$$\int_0^{\sqrt{\alpha}} J_3(y)dy \le \beta_w \int_0^1 \frac{M_\alpha(y)}{y}\left(1-\exp\left(-\frac{y^2}{2}\right)\right)dy + \beta_w \int_1^{\sqrt{\alpha}} \frac{dy}{y} + \beta_w \int_1^{\sqrt{\alpha}} \left(M_\alpha(y)-1\right)\frac{dy}{y}$$
$$\le \beta_w\left(\frac{\log\alpha}{2}+c\right), \tag{3.32}$$

and

$$\int_0^{\sqrt{\alpha}} J_4(y)dy \le m\frac{\beta_w}{\sqrt{\alpha}}\int_0^{\sqrt{\alpha}} M_\alpha(y)dy = m\beta_w \int_0^1 M(s)ds = m^2\beta_w. \tag{3.33}$$

Combining (3.29), (3.32) and (3.33) we obtain

$$\tilde{G}(\sqrt{\alpha}) - \tilde{G}(0) \le \beta_w\left(\frac{\log\alpha}{2}+c\right). \tag{3.34}$$

That yields

$$\tilde{G}(0) \ge K - \beta_w\left(\frac{\log\alpha}{2}+c\right). \tag{3.35}$$

Consequently,

$$\tilde{G}(0) \ge \varepsilon_w. \tag{3.36}$$

In view of the monotonicity of $\tilde{G}$ we then have that $\tilde{G} \ge \varepsilon_w$ on $[0,\sqrt{\alpha}]$.

□

Lemma 3.3. *Let α be sufficiently large and let $w > 0$ be an arbitrary number. Assume that*

$$\lambda \le \frac{2K\alpha}{\log\alpha}\left(1-\frac{\varepsilon_w}{K}-c\frac{f_v(w)+1}{\log\alpha}\right), \tag{3.37}$$

where ε_w is as in Lemma 3.2. Then problem (1.1) *admits a minimal positive strictly increasing solution satisfying $G(0) \ge \varepsilon_w$.*

Proof. We claim that $\tilde{G}$ constructed in Lemma 3.2 is a sub-solution for problem (2.27), provided λ satisfies (3.37). Indeed, using (3.20), (3.26) and (3.27) we observe that

$$\left(\tilde{G}'(y)\right)^2 \le c\beta_w^2 M_\alpha^2(y)\left(\frac{1}{\alpha}+\left[\frac{1-\exp\left(-\frac{y^2}{2}\right)}{y}\right]^2\right). \tag{3.38}$$

In particular, this observation and (3.28) imply that

$$\left(\tilde{G}'(y)\right)^2 \le c\beta_w^2 \le c\beta_w^2\left(\psi_\alpha(y)+\frac{M_\alpha^2(y)}{\alpha}\right) \quad \text{for} \quad y \in [0,\tilde{k}\sqrt{\alpha}], \tag{3.39}$$

and

$$\left(\tilde{G}'(y)\right)^2 \le c\beta_w^2\frac{M_\alpha^2(y)}{\alpha} \le c\beta_w^2\left(\psi_\alpha(y)+\frac{M_\alpha^2(y)}{\alpha}\right) \quad \text{for} \quad y \in [\tilde{k}\sqrt{\alpha},\sqrt{\alpha}], \tag{3.40}$$

where $\tilde{k} = \min[k, h/2]$ and k, h are as in Lemmas 3.2, 3.1 respectively. Therefore,

$$\left(\tilde{G}'(y)\right)^2 \le c\beta_w^2\left(\psi_\alpha(y)+\frac{M_\alpha^2(y)}{\alpha}\right) \quad \text{for} \quad y \in [0,\sqrt{\alpha}]. \tag{3.41}$$

Next we define implicitly $\tilde{v}(y)$ by the following formula

$$\tilde{G}(y) = \int_{\tilde{v}(y)}^{\infty} \frac{ds}{f(s)}. \tag{3.42}$$

Since $\tilde{G}(y)$ is an increasing function of y we have that $\tilde{v}(y)$ is a decreasing function of y. In view of this observation and the convexity of f we have

$$f_v(\tilde{v}(y)) \le f_v(w) \quad \text{for} \quad y \in [0, \sqrt{\alpha}]. \tag{3.43}$$

Therefore, $\tilde{G}$ will be a sub-solution for (2.27), provided that

$$\tilde{G}'' + \left(\frac{1}{y} + y\varphi_\alpha(y)\right)\tilde{G}' \ge \frac{\lambda}{\alpha}\psi_\alpha(y) + \left(\tilde{G}'\right)^2 f_v(w), \quad 0 < y < \sqrt{\alpha}. \tag{3.44}$$

Using (3.16) and (3.41), we observe that this condition is automatically satisfied if

$$\beta_w\left(\psi_\alpha(y) + \frac{M_\alpha^2(y)}{\alpha}\right) \ge \frac{\lambda}{\alpha}\psi_\alpha(y) + c\beta_w^2 f_v(w)\left(\psi_\alpha(y) + \frac{M_\alpha^2(y)}{\alpha}\right) \quad \text{for} \quad 0 < y < \sqrt{\alpha}, \tag{3.45}$$

which is in turn satisfied when

$$\beta_w(1 - c\beta_w f_v(w)) \ge \frac{\lambda}{\alpha}. \tag{3.46}$$

Straightforward computations show that (3.46) holds for all values of λ satisfying (3.37). Consequently, for λ satisfying (3.37), problem (2.27) admits a positive sub-solution and thus by Corollaries 2.1, 2.2 problem (2.22) and hence problem (1.1) admits a minimal solution. □

An immediate consequence of this lemma is the following corollary.

Corollary 3.1. *For α sufficiently large, we have*

$$\lambda^*(\alpha) \ge \sup_{w\in(0,\infty)} \frac{2K\alpha}{\log\alpha}\left(1 - \frac{\varepsilon_w}{K} - c\frac{f_v(w)+1}{\log\alpha}\right), \tag{3.47}$$

where w and ε_w are as in Lemma 3.2.

We now can proceed to the proof of Theorem 1.1.

Proof of Theorem 1.1. The statement of the theorem follows directly form Lemma 3.1 and Corollary 3.1. First we observe that by (3.1)

$$\limsup_{\alpha\to\infty} \frac{\lambda^*(\alpha)\log\alpha}{\alpha} \le 2K. \tag{3.48}$$

On the other hand given $\varepsilon > 0$ arbitrarily small, we can choose w sufficiently large so that $\varepsilon_w/K < \varepsilon/2$. This observation together with (3.47) gives

$$\liminf_{\alpha\to\infty} \frac{\lambda^*(\alpha)\log\alpha}{\alpha} \ge 2(K - \varepsilon). \tag{3.49}$$

In view of the fact that ε is arbitrarily small, (3.48) and (3.49) give (1.9). □

4 Asymptotic behavior of the extremal solution as $\alpha \to \infty$.

In this section we give a proof of Theorem 1.2. For convenience we split the section into two parts dealing with local and integral properties of the extremal solution respectively, that is, with parts one and two of Theorem 1.2. In this section we will use (2.22) as an alternative version of (1.1).

4.1 Local properties of the extremal solution.

In this subsection we give a proof of the first part of Theorem 1.2. The proof is based on the following two lemmas.

Lemma 4.1. *Assume that*

$$\vartheta < \frac{\lambda}{\lambda^*} \leq 1 \quad \text{with} \quad 0 < \vartheta \leq 1 \tag{4.1}$$

Then, for sufficiently large α, any radial solution of (1.1) satisfies for any $\frac{1}{4} > \delta > 0$ the following upper bound:

$$u(x) \leq c\left(\frac{1+\log(1/|x|)}{\log\alpha}\right), \qquad |x| \in \left[\frac{1}{\alpha^{\frac{1}{2}-2\delta}}, 1\right], \tag{4.2}$$

where the constant $c = c(\vartheta, \delta) > 0$ is independent of α.

Proof. First we observe that (2.22) can be rewritten in the divergence form

$$-\left(y \exp\left(\int_0^y s\varphi_\alpha(s)ds\right) v'(y)\right)' = \frac{\lambda}{\alpha} y \exp\left(\int_0^y s\varphi_\alpha(s)ds\right) \psi_\alpha(y) f(v(y)). \tag{4.3}$$

Integrating this equation and taking into account the boundary condition at zero in (2.22), we get

$$-yv'(y) = \frac{\lambda}{\alpha} I(y), \tag{4.4}$$

where

$$I(y) = \int_0^y f(v(z)) z\psi_\alpha(z) \exp\left(-\int_z^y s\varphi_\alpha(s)ds\right) dz = \exp(-F(y)) \int_0^y f(v(z)) z\psi_\alpha(z) \exp(F(z))\, dz, \tag{4.5}$$

and $F(t)$ is defined in (3.21).

We claim that

$$I(y_2) \leq cM_\alpha(y_2) I(y_1) \quad \text{for} \quad y_2 > y_1, \quad y_1 \in [1, q\sqrt{\alpha}], \tag{4.6}$$

with some $0 < q \leq 1$ independent of α.

To prove this claim we observe that

$$I(y_2) = \exp(F(y_1) - F(y_2)) I(y_1) + \exp(-F(y_2)) \int_{y_1}^{y_2} f(v(z)) \frac{\psi_\alpha(z)}{\varphi_\alpha(z)} z\varphi_\alpha(z) \exp(F(z))\, dz. \tag{4.7}$$

Since $F(t)$ and $M_\alpha(t) \geq 1$ are increasing and $f(v(t))$ is decreasing on $[0, \sqrt{\alpha}]$, using calculations similar to (3.23), we obtain

$$\begin{aligned} I(y_2) &\leq I(y_1) + \exp(-F(y_2)) f(v(y_1)) M_\alpha(y_2) \int_{y_1}^{y_2} z\varphi_\alpha(z) \exp(F(z))\, dz \leq \\ &I(y_1) + f(v(y_1)) M_\alpha(y_2) (1 - \exp(F(y_2) - F(y_1))) \leq I(y_1) + f(v(y_1)) M_\alpha(y_2). \end{aligned} \tag{4.8}$$

Moreover, since φ, ψ are Lipshitz continuous and $\varphi(0) = \psi(0) = 1$ we can choose $q > 0$ independent of α such that

$$\frac{\psi_\alpha(y)}{\varphi_\alpha(y)} \geq \frac{1}{2}, \quad \varphi_\alpha(y) \geq \frac{1}{2} \quad \text{for} \quad y \leq q\sqrt{\alpha}. \tag{4.9}$$

Therefore, as follows from (4.9) and (4.5)

$$\begin{aligned} &I(y_1) \geq f((v(y_1)) \exp(-F(y_1)) \int_0^{y_1} z\psi_\alpha(z) \exp(F(z)) dz \geq \\ &\frac{1}{2} f((v(y_1)) \exp(-F(y_1)) \int_0^{y_1} z\varphi_\alpha(z) \exp(F(z)) dz = \frac{1}{2} f(v(y_1)) (1 - \exp(-F(y_1))) \geq cf(v(y_1)), \end{aligned} \tag{4.10}$$

provided $y_1 < q\sqrt{\alpha}$. Combining (4.8) and (4.10), we get (4.6).

Now we fix $\delta > 0$ sufficiently small. We claim that for large enough α,

$$v(\alpha^\delta) \le c, \tag{4.11}$$

where c depends only on ϑ and δ. To show that, we integrate (3.8) from zero to α^δ and arguing as in Lemma 3.1 obtain

$$G(\alpha^\delta) \ge \frac{\lambda}{\alpha}\left(\delta \log \alpha - c\right). \tag{4.12}$$

Since for α large enough,

$$\lambda^* \ge \frac{K\alpha}{\log \alpha}, \tag{4.13}$$

we have that for λ satisfying (4.1)

$$G(\alpha^\delta) \ge \frac{\vartheta\delta K}{2}. \tag{4.14}$$

Using the definition of G (see (2.24)), we have

$$\frac{1}{K}\int_{v(\alpha^\delta)}^{\infty} \frac{ds}{f(s)} \ge \frac{\vartheta\delta}{2} > 0. \tag{4.15}$$

Since the integral in (4.15) is bounded from below away from zero independently of α, we conclude that the lower limit in this integral is bounded from above, which gives (4.11).

Next, taking $y_2 = \alpha^{2\delta}$, we have from (4.6) that

$$I(\alpha^{2\delta}) \le cM_\alpha(\alpha^{2\delta})I(y), \qquad \alpha^\delta \le y \le \alpha^{2\delta}. \tag{4.16}$$

Since $1 \le M_\alpha(\alpha^{2\delta}) < c$, we have

$$I(y) \ge cI(\alpha^{2\delta}), \qquad \alpha^\delta \le y \le \alpha^{2\delta}, \tag{4.17}$$

and hence by (4.4)

$$-yv'(y) \ge -c\alpha^{2\delta}v'(\alpha^{2\delta}) = cD, \tag{4.18}$$

where

$$D = -\alpha^{2\delta}v'(\alpha^{2\delta}). \tag{4.19}$$

Integrating inequality (4.18), we obtain

$$v(\alpha^\delta) - v(\alpha^{2\delta}) = -\int_{\alpha^\delta}^{\alpha^{2\delta}} v'(y)dy \ge D\int_{\alpha^\delta}^{\alpha^{2\delta}} \frac{dy}{y} = c\delta D \log \alpha. \tag{4.20}$$

Therefore, from (4.11) and (4.20) we have

$$D \le \frac{c}{\log \alpha}. \tag{4.21}$$

On the other hand, from (4.6) with $y_1 = \alpha^{2\delta}$ and $y_2 = y$ we have

$$I(y) \le cM_\alpha(y)I(\alpha^{2\delta}), \quad y \ge \alpha^{2\delta}, \tag{4.22}$$

and, therefore, by (4.4), (4.19), (4.21) and (4.22)

$$-yv'(y) \le cM_\alpha(y)\left(-\alpha^{2\delta}v'(\alpha^{2\delta})\right) = cDM_\alpha(y) \le \frac{c}{\log \alpha}M_\alpha(y). \tag{4.23}$$

Thus, for $y \geq \alpha^{2\delta}$, we have

$$-v'(y) \leq \frac{c}{\log \alpha} \frac{M_\alpha(y)}{y}. \tag{4.24}$$

Integrating (4.24), we then obtain

$$v(y) = -\int_y^{\sqrt{\alpha}} v'(y)dy \leq \frac{c}{\log \alpha} \int_y^{\sqrt{\alpha}} M_\alpha(y) \frac{dy}{y}. \tag{4.25}$$

Now using (3.17),(3.24), (3.25) and (3.28) we estimate the right hand side of (4.25). First assume that $y \geq k\sqrt{\alpha}$, where k is as in Lemma 3.2. Then

$$\int_y^{\sqrt{\alpha}} M_\alpha(y) \frac{dy}{y} \leq \int_{k\sqrt{\alpha}}^{\alpha} M_\alpha(y) \frac{dy}{y} \leq \frac{1}{k\sqrt{\alpha}} \int_{k\sqrt{\alpha}}^{\sqrt{\alpha}} M_\alpha(y) dy = \frac{1}{k} \int_k^1 M(t)dt \leq \frac{m}{k} < c. \tag{4.26}$$

Next, when $y < k\sqrt{\alpha}$, we have

$$\begin{aligned} \int_y^{\sqrt{\alpha}} M_\alpha(y) \frac{dy}{y} = \int_{k\sqrt{\alpha}}^{\sqrt{\alpha}} M_\alpha(y) \frac{dy}{y} + \int_y^{k\sqrt{\alpha}} M_\alpha(y) \frac{dy}{y} \leq c + \int_y^{k\sqrt{\alpha}} \left(1 + l \frac{y}{\sqrt{\alpha}}\right) \frac{dy}{y} \leq \\ c + \int_y^{k\sqrt{\alpha}} \frac{dy}{y} + \frac{l}{\sqrt{\alpha}} \int_y^{k\sqrt{\alpha}} dy = c + \log(k\sqrt{\alpha}) - \log(y) + lk - l\frac{y}{\sqrt{\alpha}} \leq c + \log\left(\frac{\sqrt{\alpha}}{y}\right). \end{aligned} \tag{4.27}$$

Combining (4.25), (4.26) and (4.27), we get

$$v(y) \leq \frac{c}{\log \alpha} \left(1 + \log\left(\frac{\sqrt{\alpha}}{y}\right)\right), \quad y \geq \alpha^{2\delta}. \tag{4.28}$$

The latter inequality in terms of the original (unscaled) variables (see (2.21)) gives (4.2), which completes the proof. ∎

Remark 4.1. *We note that the statement of the lemma above concerns not only extremal but all radial solutions of problem* (1.1)*. That is, any radial solution of* (1.1) *with λ comparable with λ^* obeys* (4.2)*. This fact, in particular, implies that any radial solution of* (1.1) *with $\lambda \asymp \lambda^*$ tends to zero outside of the origin as $\alpha \to \infty$.*

Lemma 4.2. *The extremal solution u^*_α of* (1.1) *satisfies*

$$u^*_\alpha(0) \to \infty \quad \text{as} \quad \alpha \to \infty. \tag{4.29}$$

Proof. First we observe that due to the monotonicity of G', inequality (3.8), and arguments identical to those given in Lemma 3.1, we have

$$G(\sqrt{\alpha}) - G(\tilde{c}) = \int_{\tilde{c}}^{\sqrt{\alpha}} G'(y)dy \geq \int_{\tilde{c}}^{h\sqrt{\alpha}} G'(y)dy \geq \frac{\lambda}{\alpha} \frac{\log \alpha}{2} \left(1 - \frac{c}{\log \alpha}\right), \tag{4.30}$$

where $\tilde{c} \geq 0$ is an arbitrary constant independent of α. Also we observe that Theorem 1.1 implies

$$\lambda^*(\alpha) = \frac{2K\alpha}{\log \alpha} (1 - \sigma(\alpha)), \tag{4.31}$$

with

$$\sigma(\alpha) \to 0, \quad \text{as} \quad \alpha \to \infty. \tag{4.32}$$

Since $G(\sqrt{\alpha}) = K$, (4.30) and (4.31) give that for $\lambda = \lambda^*$ the following estimate holds:

$$G^*_\alpha(\tilde{c}) \leq K - \frac{\lambda^*(\alpha) \log \alpha}{2\alpha} \left(1 - \frac{c}{\log \alpha}\right) \leq c \left(\frac{1}{\log \alpha} + \sigma(\alpha)\right), \tag{4.33}$$

where $G^*_\alpha(y) = \int_{v^*_\alpha(y)}^\infty \frac{ds}{f(s)}$.

Thus, $G^*_\alpha(\tilde{c}) \to 0$ as $\alpha \to \infty$. Therefore,

$$\int_{v^*_\alpha(\tilde{c})}^\infty \frac{ds}{f(s)} \to 0. \tag{4.34}$$

Hence $v^*_\alpha(\tilde{c}) \to \infty$ as $\alpha \to \infty$ and, consequently, $v^*_\alpha(0) \to \infty$ as $\alpha \to \infty$.

Since $u^*_\alpha(0) = v^*_\alpha(0)$ we conclude that $u^*_\alpha(0) \to \infty$ as $\alpha \to \infty$. □

We now can give a proof of the first part of Theorem 1.2.

Proof of Theorem 1.2 part 1. By Lemma 4.1 we have that, for sufficiently large α,

$$u^*_\alpha(x) \le c\frac{\log(\log\alpha)}{\log\alpha}, \qquad |x| \ge \frac{1}{\log\alpha}. \tag{4.35}$$

Taking a limit as $\alpha \to \infty$ in (4.35), we obtain that $u^*_\alpha(x) \to 0$ for $x \ne 0$ as $\alpha \to \infty$. The fact that $u^*_\alpha(0) \to \infty$ as $\alpha \to \infty$ follows directly from Lemma 4.2. □

We now proceed to the proof of the second part of Theorem 1.2.

4.2 Integral properties of the extremal solution.

In this section we complete the proof of Theorem 1.2, which follows from the following two lemmas.

Lemma 4.3. *Let $\theta^*(\alpha)$ be the largest solution of the equation*

$$\frac{f(\theta)}{\theta} = c\log\alpha, \tag{4.36}$$

where $c > 0$ is a fixed constant.

Then, for arbitrarily small $\gamma > 0$, we have

$$\theta^*(\alpha) \le \alpha^\gamma, \tag{4.37}$$

provided α is sufficiently large.

Proof. Let $\theta > 1$. Then, by the convexity of f, we have

$$f(s) \le g(s) = f(\sqrt{\theta}) + \frac{f(\theta) - f(\sqrt{\theta})}{\theta - \sqrt{\theta}}(s - \sqrt{\theta}), \quad s \in [\sqrt{\theta}, \theta], \tag{4.38}$$

and

$$f(t\theta) \le tf(\theta) + (1-t)f(0), \quad t \in [0,1]. \tag{4.39}$$

In particular, setting $t = 1/\sqrt{\theta}$, the latter inequality gives

$$f(\sqrt{\theta}) \le \frac{f(\theta)}{\sqrt{\theta}} + c. \tag{4.40}$$

Next let

$$\rho(\theta) = \int_{\sqrt{\theta}}^\theta \frac{ds}{f(s)}. \tag{4.41}$$

By (4.38) we have

$$\rho(\theta) \ge \int_{\sqrt{\theta}}^\theta \frac{ds}{g(s)} = \frac{\theta - \sqrt{\theta}}{f(\theta) - f(\sqrt{\theta})}\log\left(\frac{f(\theta)}{f(\sqrt{\theta})}\right). \tag{4.42}$$

This observation together with (4.40) implies that for θ sufficiently large we have

$$\rho(\theta) > c\frac{\theta\log(\theta)}{f(\theta)}. \tag{4.43}$$

By (1.2), $\rho(\theta) \to 0$ as $\theta \to \infty$ and therefore,

$$\frac{f(\theta)}{\theta} > \log(\theta)\chi(\theta), \tag{4.44}$$

with some function $\chi(\theta)$ having the property that $\chi(\theta) \to \infty$ as $\theta \to \infty$. In view of this observation we have

$$\log(\theta^*)\chi(\theta^*) \le c\log\alpha. \tag{4.45}$$

The statement of the lemma then follows immediately. □

Lemma 4.4. *Let $\delta, \gamma > 0$ be arbitrary fixed small numbers such that $\gamma + 4\delta < 1$. If α is large enough, then there exists a point $a < \alpha^{2\delta}$ such that*

$$\int_0^a v_\alpha^*(y)ydy \le c\alpha^{\gamma+4\delta}, \qquad \int_0^a \psi_\alpha(y)f(v_\alpha^*(y))ydy \le c\alpha^{\gamma+4\delta}\log\alpha, \tag{4.46}$$

and

$$v_\alpha^*(y) \le c\alpha^\gamma, \quad f(v_\alpha^*(y)) \le c\alpha^\gamma\log\alpha \quad \text{when} \quad y \ge a, \tag{4.47}$$

where $c = c(\delta,\gamma) > 0$ is a constant independent of α.

Proof. First we claim that

$$v_\alpha^*(\alpha^\delta) > c. \tag{4.48}$$

Indeed, arguing as in Lemma 4.1 (see Eqs. (4.3), (4.4), (4.5)), we have that

$$-v'(y) = \frac{\lambda}{\alpha}\frac{\exp(-F(y))}{y}\int_0^y \frac{\psi_\alpha(z)}{\varphi_\alpha(z)}z\varphi_\alpha(z)f(v(z))\exp(F(z))dz, \tag{4.49}$$

where F is defined by (3.21).

Next, using (4.9), (3.23), and the fact that $f(v(y)) \ge f(0) > 0$, we have that for $\alpha^{2\delta} \le y \le q\sqrt{\alpha}$

$$\begin{aligned} -v'(y) &\ge c\frac{\lambda}{\alpha}\frac{\exp(-F(y))}{y}\int_0^y z\varphi_\alpha(z)\exp(F(z))dz = \\ &c\frac{\lambda}{\alpha}\frac{1}{y}(1-\exp(-F(y)) \ge c\frac{\lambda}{\alpha}\frac{1}{y}(1-\exp(-c\alpha^{4\delta}) \ge c\frac{\lambda}{\alpha}\frac{1}{y}. \end{aligned} \tag{4.50}$$

Integrating this expression from $\alpha^{2\delta}$ to $q\sqrt{\alpha}$ we get

$$v(\alpha^{2\delta}) \ge v(q\sqrt{\alpha}) + c\frac{\lambda}{\alpha}(\frac{1}{2} - 2\delta)(\log\alpha - c). \tag{4.51}$$

For $\delta < \frac{1}{4}$, the latter inequality implies that for sufficiently large α

$$v(\alpha^{2\delta}) \ge c\frac{\lambda}{\alpha}\log\alpha. \tag{4.52}$$

In particular, we have

$$v_\alpha^*(\alpha^{2\delta}) \ge c\frac{\lambda^*\log\alpha}{\alpha} \ge c, \tag{4.53}$$

which proves our claim.

Next let

$$\Gamma(y) = y[v_\alpha^*(y)]' + y^2\varphi_\alpha(y)v_\alpha^*(y). \tag{4.54}$$

We note that since φ is Lipschitz continuous, $\frac{d}{dr}\varphi(r)$ is defined almost everywhere and $\left|\frac{d}{dr}\varphi(r)\right| \le c$. This fact and the definition of φ_α (2.23) imply that $|\varphi'_\alpha(y)| \le \frac{c}{\sqrt{\alpha}}$. Direct computations and (2.22) give

$$\Gamma'(y) = \left[\left(2\varphi_\alpha(y) + y\varphi'_\alpha(y)\right)v_\alpha^*(y) - \frac{\lambda^*}{\alpha}\psi_\alpha(y)f(v_\alpha^*(y))\right]y \tag{4.55}$$

for almost every y.

Clearly, we have $\Gamma(0) = \Gamma'(0) = 0$. Assume first that $v_\alpha^*(0) \ge v_0$ where v_0 is the largest solution of

$$4v_0 = \frac{\lambda^*}{\alpha}f(v_0). \tag{4.56}$$

Then $\Gamma(y)$ is negative in some small neighborhood of $y = 0$. On the other hand $\Gamma(\alpha^{2\delta}) > 0$ as follows from (4.24) and (4.53). Consequently, there exists a point $y = a \in (0, \alpha^{2\delta})$ such that $\Gamma(a) = 0$. This in particular implies that there exists a point $0 < a_0 < a$ where Γ attains its minimum. At that point we have

$$\left(2\varphi_\alpha(a_0) + O\left(\frac{a_0}{\sqrt{\alpha}}\right)\right)v_\alpha^*(a_0) = \frac{\lambda^*}{\alpha}\psi_\alpha(a_0)f(v_\alpha^*(a_0)). \tag{4.57}$$

This implies that

$$cv_\alpha^*(a_0) = \frac{\lambda^*}{\alpha}f(v_\alpha^*(a_0)). \tag{4.58}$$

Therefore, as follows from Lemma 4.3 and the monotonicity of $v_\alpha^*(y)$, for $y \ge a_0$ we have

$$v_\alpha^*(y) < c\alpha^\gamma, \qquad f(v_\alpha^*(y)) \le c\alpha^\gamma \log\alpha. \tag{4.59}$$

Taking into account that $a_0 < a$ and the monotonicity of $v_\alpha^*(y)$, the latter inequalities give (4.47).

Next integrating (4.55) we have

$$\int_0^a \Gamma'(y)dy = \int_0^a \left[\left(2\varphi_\alpha(y) + y\varphi'_\alpha(y)\right)v_\alpha^*(y) - \frac{\lambda^*}{\alpha}\psi_\alpha(y)f(v_\alpha^*(y))\right]ydy = 0. \tag{4.60}$$

Therefore,

$$\int_0^a \left(2\varphi_\alpha(y) + y\varphi'_\alpha(y)\right)v_\alpha^*(y)ydy = \int_0^a \frac{\lambda^*}{\alpha}\psi_\alpha(y)f(v_\alpha^*(y))ydy. \tag{4.61}$$

This observation and Jensen's inequality imply that

$$\langle v\rangle \ge c\frac{\lambda^*}{\alpha}f(\langle v\rangle), \tag{4.62}$$

where

$$\langle v\rangle = \frac{2}{a^2}\int_0^a v_\alpha^*(y)ydy \tag{4.63}$$

is an average of v over $[0, a]$. Consequently, by Lemma 4.3 we have

$$\langle v\rangle \le c\alpha^\gamma, \tag{4.64}$$

and thus

$$\int_0^a v_\alpha^*(y)ydy < c\alpha^{\gamma+4\delta}. \tag{4.65}$$

The latter inequality and (4.61) imply that

$$\int_0^a \psi_\alpha(y) f(v_\alpha^*(y)) y dy < c\alpha^{\gamma+4\delta}\log\alpha. \tag{4.66}$$

This proves (4.46).

Assume now that $v_\alpha^*(0) \le v_0$. In this case by Lemma 4.3

$$v_\alpha^*(y) \le \alpha^\gamma, \quad f(v_\alpha^*(y)) \le c\alpha^\gamma \log\alpha \quad \text{on} \quad [0, \sqrt{\alpha}]. \tag{4.67}$$

which proves (4.47). Using these inequalities and taking $a = \alpha^{2\delta}$ we also have (4.46) for this case. □

We now can proceed to the proof of the second part of Theorem 1.2.

Proof of Theorem 1.2 part 2. We first observe that

$$\int_0^{\sqrt{\alpha}} v_\alpha^*(y) y dy = \left\{ \int_0^a + \int_a^{\alpha^{2\delta}} + \int_{\alpha^{2\delta}}^{\frac{\sqrt{\alpha}}{\log\alpha}} + \int_{\frac{\sqrt{\alpha}}{\log\alpha}}^{\sqrt{\alpha}} \right\} v_\alpha^*(y) y dy = L_1 + L_2 + L_3 + L_4, \tag{4.68}$$

where a is as in Lemma 4.4.

By (4.46), (4.47) (see Lemma 4.4) we have that

$$L_1, L_2 \le c\alpha^{\gamma+4\delta}. \tag{4.69}$$

Moreover by (4.28) (see Lemma 4.1) we also have that

$$L_3 \le c\frac{\alpha}{(\log\alpha)^2}, \quad L_4 \le c\frac{\log(\log\alpha)}{\log\alpha}\alpha. \tag{4.70}$$

Hence,

$$\int_0^{\sqrt{\alpha}} v_\alpha^*(y) y dy \le c\frac{\log(\log\alpha)}{\log\alpha}\alpha. \tag{4.71}$$

Observing that

$$\int_B u_\alpha^*(x)dx = \frac{2\pi}{\alpha}\int_0^{\sqrt{\alpha}} v_\alpha^*(y) y dy \le c\frac{\log(\log\alpha)}{\log\alpha}, \tag{4.72}$$

taking a limit as $\alpha \to \infty$ in the right hand side of (4.72), and using the positivity of u_α^*, we obtain the first part of (1.11).

Next, we perform computations similar to those above:

$$\int_0^{\sqrt{\alpha}} \psi_\alpha(y) f(v_\alpha^*(y)) y dy = \left\{ \int_0^a + \int_a^{\alpha^{2\delta}} + \int_{\alpha^{2\delta}}^{\frac{\sqrt{\alpha}}{\log\alpha}} + \int_{\frac{\sqrt{\alpha}}{\log\alpha}}^{\sqrt{\alpha}} \right\} \psi_\alpha(y) f(v_\alpha^*(y)) y dy =$$
$$P_1 + P_2 + P_3 + P_4. \tag{4.73}$$

By (4.46), (4.47) (see Lemma 4.4) we have that

$$P_1, P_2 \le c\alpha^{\gamma+4\delta}\log\alpha. \tag{4.74}$$

By (4.11) we have

$$P_3 \le c\frac{\alpha}{(\log\alpha)^2}, \tag{4.75}$$

and by (4.28)

$$P_4 \le f\left(c\frac{\log(\log\alpha)}{\log\alpha}\right)\int_{\frac{\sqrt{\alpha}}{\log\alpha}}^{\sqrt{\alpha}} \psi_\alpha(y) y dy, \tag{4.76}$$

Arguing as above we then have

$$\int_B \psi(x) f(u_\alpha^*(x))dx = \frac{2\pi}{\alpha}\int_0^{\sqrt{\alpha}} \psi_\alpha(y) f(v_\alpha^*(y)) y dy \le \frac{c}{(\log\alpha)^2} + \frac{2\pi}{\alpha} f\left(c\frac{\log(\log\alpha)}{\log\alpha}\right)\int_{\frac{\sqrt{\alpha}}{\log\alpha}}^{\sqrt{\alpha}} \psi_\alpha(y) y dy$$
$$= \frac{c}{(\log\alpha)^2} + f\left(c\frac{\log(\log\alpha)}{\log\alpha}\right)\int_{B\setminus B\left(0,\frac{1}{\log(\alpha)}\right)} \psi(x)dx. \tag{4.77}$$

Therefore,

$$\int_B \psi(x) f(u_\alpha^*(x))dx \le f(0)\int_B \psi(x)dx + \tilde{\sigma}(\alpha), \tag{4.78}$$

for some $\tilde{\sigma}(\alpha)$ having the property that $\tilde{\sigma}(\alpha)\to 0$ as $\alpha\to\infty$. In view of this observation and the fact that

$$\int_B \psi(x) f(u_\alpha^*(x))dx > f(0)\int_B \psi(x)dx, \tag{4.79}$$

which follows from the positivity of u_α^*, we have the second part of (1.11), which completes the proof.

□

We now turn to the proof of Theorem 1.3.

5 Proof of Theorem 1.3.

The proof of Theorem 1.3 requires the following lemma. This lemma is based on a rescaled version of inequality (2.20), which was first introduced in [5].

Lemma 5.1. *Let v_α^* be an extremal solution of* (2.22) *and set*

$$f^\sharp(t) = f(v_\alpha^*(1)+t), \quad f_v^\sharp(t) = f_v(v_\alpha^*(1)+t), \quad \tilde{f}^\sharp(t) = f^\sharp(t) - f^\sharp(0), \quad g^\sharp(t) = \int_0^t (f_v^\sharp(s))^2 ds. \tag{5.1}$$

Assume that there exist constants $0<\tilde{c}_0<1$ and $\tilde{c}_1>1$ such that for sufficiently large α

$$f^\sharp(t) \ge \tilde{c}_1 f^\sharp(0), \quad t\ge 0, \tag{5.2}$$

implies

$$(1-\tilde{c}_0)\tilde{f}^\sharp(t) f_v^\sharp(t) \ge g^\sharp(t). \tag{5.3}$$

Then, for sufficiently large α,

$$v_\alpha^*(0) \le v_\alpha^*(1) + c\frac{f(v_\alpha^*(1))}{\log\alpha}, \tag{5.4}$$

where $c>0$ is some constant independent of α.

Proof. As a first step we establish an inequality similar to (2.20).
Let

$$\phi(y) := v_\alpha^*(y) - v_\alpha^*(1). \tag{5.5}$$

By (2.22), this function verifies

$$\begin{cases} -(y\mu_\alpha(y)\phi')' = \frac{\lambda^*}{\alpha} y\psi_\alpha(y)\mu_\alpha(y) f^\sharp(\phi), & 0<y<1, \\ \phi'(0)=0, & \phi(1)=0, \end{cases} \tag{5.6}$$

where

$$\mu_\alpha(y) = \exp\left(\int_0^y s\varphi_\alpha(s)ds\right). \tag{5.7}$$

Multiplying the first equation in (5.6) by $g^\sharp(\phi)$, integrating by parts and taking into account the boundary conditions in (5.6), we get

$$\int_0^1 \left(f_v^\sharp(\phi)\ \phi'\right)^2 d\tilde{\nu} = \frac{\lambda^*}{\alpha}\int_0^1 f^\sharp(\phi)g^\sharp(\phi)d\nu, \tag{5.8}$$

where

$$d\tilde{\nu}(y) = y\mu_\alpha(y)dy, \qquad d\nu(y) = \psi_\alpha(y)d\tilde{\nu}(y). \tag{5.9}$$

We also note that the semi-stability condition (1.6) implies that

$$\int_0^{\sqrt{\alpha}} (\eta')^2 d\tilde{\nu} \geq \frac{\lambda^*}{\alpha}\int_0^{\sqrt{\alpha}} f_v^\sharp(\phi)\eta^2 d\nu, \quad \forall \eta \in H^1([0,\sqrt{\alpha}]) \quad \text{such that} \quad \eta(\sqrt{\alpha}) = 0. \tag{5.10}$$

Taking (in the spirit of the arguments in [14] and [6, Section 4.3])

$$\eta(y) = \begin{cases} \tilde{f}^\sharp(\phi(y)) & 0 \leq y \leq 1, \\ 0 & 1 < y \leq \sqrt{\alpha}, \end{cases} \tag{5.11}$$

and substituting this test function into (5.10), we obtain

$$\int_0^1 \left(f_v^\sharp(\phi)\phi'\right)^2 d\tilde{\nu} \geq \frac{\lambda^*}{\alpha}\int_0^1 f_v^\sharp(\phi)(\tilde{f}^\sharp(\phi))^2 d\nu. \tag{5.12}$$

Combining (5.8) and (5.12), we obtain

$$\int_0^1 f_v^\sharp(\phi)(\tilde{f}^\sharp(\phi))^2 d\nu \leq \int_0^1 f^\sharp(\phi)g^\sharp(\phi)d\nu. \tag{5.13}$$

Next let $\tilde{c}_2 > \tilde{c}_1$. Then,

$$\int_0^1 f^\sharp(\phi)g^\sharp(\phi)d\nu = \left\{\int_{X_1} + \int_{X_2}\right\} f^\sharp(\phi)g^\sharp(\phi)d\nu = I_1 + I_2, \tag{5.14}$$

where

$$X_1 = \{f^\sharp(\phi) \leq \tilde{c}_2 f^\sharp(0)\}, \quad X_2 = \{f^\sharp(\phi) > \tilde{c}_2 f^\sharp(0)\}. \tag{5.15}$$

By the assumption of the lemma, we have

$$I_2 \leq (1-\tilde{c}_0)\int_{X_2} f^\sharp(\phi)\tilde{f}^\sharp(\phi)f_v^\sharp(\phi)d\nu. \tag{5.16}$$

Moreover,

$$f^\sharp(\phi) \leq \frac{\tilde{c}_2}{\tilde{c}_2 - 1}\tilde{f}^\sharp(\phi) \quad \text{on} \quad X_2. \tag{5.17}$$

Combining these two observations, we conclude that

$$I_2 \leq \left(1 - \frac{\tilde{c}_0}{2}\right)\int_{X_2} f_v^\sharp(\phi)\left(\tilde{f}^\sharp(\phi)\right)^2 d\nu \tag{5.18}$$

provided $\tilde{c}_2$ is sufficiently large.

Next we observe that (5.13), (5.14) and (5.18) imply that

$$\frac{\tilde{c}_0}{2}\int_{X_2} f_v^\sharp(\phi)\left(\tilde{f}^\sharp(\phi)\right)^2 d\nu \leq I_1. \tag{5.19}$$

Noting that $f_v^\sharp$ is non-decreasing, we have

$$g^\sharp(t) = \int_0^t f_v^\sharp(s) f_v^\sharp(s) ds \le f_v^\sharp(t) \int_0^t f_v^\sharp(s) ds = f_v^\sharp(t) \tilde{f}^\sharp(t). \tag{5.20}$$

Consequently,

$$I_1 \le \int_{X_1} f_v^\sharp(\phi) f^\sharp(\phi) \tilde{f}^\sharp(\phi) d\nu. \tag{5.21}$$

Hence, from (5.19) and (5.21) we get

$$\frac{\tilde{c}_0}{2} \int_{X_2} f_v^\sharp(\phi) \left(\tilde{f}^\sharp(\phi)\right)^2 d\nu \le \int_{X_1} f_v^\sharp(\phi) f^\sharp(\phi) \tilde{f}^\sharp(\phi) d\nu. \tag{5.22}$$

Since

$$\min_{X_2} f_v^\sharp(\phi) \ge \max_{X_1} f_v^\sharp(\phi), \tag{5.23}$$

we have from (5.22) that

$$\frac{\tilde{c}_0}{2} \int_{X_2} \left(\tilde{f}^\sharp(\phi)\right)^2 d\nu \le \int_{X_1} f^\sharp(\phi) \tilde{f}^\sharp(\phi) d\nu, \tag{5.24}$$

and thus

$$\int_{X_2} \left(f^\sharp(\phi)\right)^2 d\nu \le c(f^\sharp(0))^2. \tag{5.25}$$

On the other hand, as follows from the definition of X_1, we have

$$\int_{X_1} \left(f^\sharp(\phi)\right)^2 d\nu \le c(f^\sharp(0))^2. \tag{5.26}$$

Consequently,

$$\int_0^1 \left(f^\sharp(\phi)\right)^2 d\nu \le c(f^\sharp(0))^2. \tag{5.27}$$

This estimate and the standard elliptic L^p–estimates [10, Theorem 8.16] imply that

$$\phi(0) \le c\frac{\lambda^*}{\alpha} f^\sharp(0), \tag{5.28}$$

and therefore

$$v_\alpha^*(0) \le v_\alpha^*(1) + c\frac{f(v_\alpha^*(1))}{\log \alpha}. \tag{5.29}$$

□

We now turn to the proof of Theorem 1.3.

Proof of Theorem 1.3. First let us show that the assumptions of Lemma 5.1 hold under the assumptions of Theorem 1.3. Observe that, as follows from the proof of Lemma 4.2, $v_\alpha^*(1) \to \infty$ as $\alpha \to \infty$. In view of this fact, the assumptions of Lemma 5.1 can be restated in the following form. There exist constants $0 < \tilde{c}_0 < 1$, $\tilde{c}_1 > 1$ and $t_0 > 0$ such that

$$f(t_2) > \tilde{c}_1 f(t_1) \quad t_2 > t_1 > t_0, \tag{5.30}$$

implies

$$(1 - \tilde{c}_0) f_v(t_2)(f(t_2) - f(t_1)) \ge \int_{t_1}^{t_2} \left(f_v(s)\right)^2 ds. \tag{5.31}$$

Assume that the assumptions of Theorem 1.3 hold. If $\tilde{c}_1 > c_1$ (which we can alway ensure) we can choose $\tilde{t} \in (t_1, t_2)$ such that

$$f(\tilde{t}) = \frac{1}{c_1} f(t_2). \tag{5.32}$$

Then, by (1.13)

$$(1 - c_0) f_v(t_2) \geq f_v(\tilde{t}). \tag{5.33}$$

Using the monotonicity of f_v, (5.32) and (5.33), we obtain

$$\begin{aligned} &\int_{t_1}^{t_2} (f_v(s))^2 \, ds = \left\{ \int_{t_1}^{\tilde{t}} + \int_{\tilde{t}}^{t_2} \right\} (f_v(s))^2 \, ds \leq f_v(\tilde{t}) \left(f(\tilde{t}) - f(t_1) \right) + f_v(t_2) \left(f(t_2) - f(\tilde{t}) \right) \leq \\ &f_v(t_2)(f(t_2) - f(t_1)) \left\{ \left(\frac{f(t_2) - f(\tilde{t})}{f(t_2) - f(t_1)} \right) + (1 - c_0) \left(\frac{f(\tilde{t}) - f(t_1)}{f(t_2) - f(t_1)} \right) \right\} = \\ &f_v(t_2)(f(t_2) - f(t_1)) \left\{ 1 - c_0 \left(\frac{f(\tilde{t}) - f(t_1)}{f(t_2) - f(t_1)} \right) \right\} \leq f_v(t_2)(f(t_2) - f(t_1)) \left\{ 1 - \frac{c_0}{c_1} \left(\frac{f(t_2) - c_1 f(t_1)}{f(t_2)} \right) \right\}. \end{aligned} \tag{5.34}$$

Now taking $\tilde{c}_1 > 2c_1$ in (5.30) we get

$$\left(\frac{f(t_2) - c_1 f(t_1)}{f(t_2)} \right) \geq \frac{1}{2}. \tag{5.35}$$

Hence,

$$\int_{t_1}^{t_2} (f_v(s))^2 \, ds \leq \left(1 - \frac{c_0}{2c_1} \right) f_v(t_2)(f(t_2) - f(t_1)), \tag{5.36}$$

which gives (5.31) with $\tilde{c}_0 = c_0/2c_1$.

We next turn to the proof of (1.14). First note that taking an arbitrary smooth test function η with support on $[1/2, 1]$ in (5.10) and using the monotonicity of f_v and v_α^* we have that

$$f_v(v_\alpha^*(1)) \leq c \log \alpha. \tag{5.37}$$

Next, by convexity,

$$f_v(t) \geq \frac{f(t) - f(0)}{t}, \quad t > 0. \tag{5.38}$$

Hence,

$$v_\alpha^*(1) f_v(v_\alpha^*(1)) + c \geq f(v_\alpha^*(1)). \tag{5.39}$$

In view of (5.37) and (5.39) we have that for sufficiently large α,

$$\frac{f(v_\alpha^*(1))}{\log \alpha} \leq c v_\alpha^*(1). \tag{5.40}$$

Combining this result with (5.4) we have

$$v_\alpha^*(0) \leq c v_\alpha^*(1), \tag{5.41}$$

which implies the result.

Finally, let us prove (1.16). Observe that (5.37), (5.38) and Lemma 4.3 imply that

$$v_\alpha^*(0) \leq c \alpha^\gamma, \tag{5.42}$$

for arbitrarily small $\gamma > 0$. This observation and Lemma 4.1 imply that

$$v_\alpha^*(y) < \begin{cases} c\alpha^\gamma & y \in [0, \alpha^{2\delta}], \\ c & y \in [\alpha^{2\delta}, \frac{\sqrt{\alpha}}{\log \alpha}], \\ c\frac{\log(\log \alpha)}{\log \alpha} & y \in [\frac{\sqrt{\alpha}}{\log \alpha}, \sqrt{\alpha}], \end{cases} \tag{5.43}$$

where $\delta > 0$ is arbitrarily small. Hence,

$$\int_0^{\sqrt{\alpha}} (v_\alpha^*(y))^p y dy \le c\left\{\frac{1}{\alpha^{1-p\gamma-4\delta}} + \frac{1}{(\log \alpha)^2} + \left(\frac{\log(\log \alpha)}{\log \alpha}\right)^p\right\}\alpha. \tag{5.44}$$

We thus have that for any $1 \le p < \infty$,

$$\int_B (u_\alpha^*(x))^p dx = \frac{2\pi}{\alpha}\int_0^{\sqrt{\alpha}} (v_\alpha^*(y))^p y dy \le \left\{\frac{1}{\alpha^{1-p\gamma-4\delta}} + \frac{1}{(\log \alpha)^2} + \left(\frac{\log(\log \alpha)}{\log \alpha}\right)^p\right\}. \tag{5.45}$$

In view that δ and γ can be chosen arbitrarily small we conclude that the expression in the braces in (5.45) goes to zero as $\alpha \to \infty$. This observation concludes the proof of the theorem. □

The following lemma gives an example of rather general nonlinearities that satisfy the assumptions of Theorem 1.3.

Lemma 5.2. *Assume that* $f \in C^2$, $\frac{df(s)}{ds}, \frac{d^2f(s)}{ds^2} > 0$, *and* $\frac{d^2f(s)}{ds^2}$ *is strictly increasing on* $(0, \infty)$. *Then,* f *satisfies the assumptions of Theorem 1.3.*

Proof. Let us show that under the assumptions of this lemma the assumptions of Theorem 1.3 are satisfied with $c_1 = 4$ and $c_0 = \frac{1}{3}$.

Take $t_2 > t_1$ such that

$$f(t_2) \ge 4f(t_1) \tag{5.46}$$

and consider two cases: case I in which $t_2 \le 2t_1$, and case II in which $t_2 \ge 2t_1$.

We start with case I. Assume that

$$f_v(t_1) \ge \frac{2}{3}f_v(t_2). \tag{5.47}$$

Then,

$$\int_{t_1}^{t_2} \frac{d^2}{ds^2}f(s) = f_v(t_2) - f_v(t_1) \le \frac{1}{3}f_v(t_2). \tag{5.48}$$

Let $t_3 = 2t_1 - t_2$ and note that under the assumption of case I we have that $0 \le t_3 \le t_1$. Since $t_1 - t_3 = t_2 - t_1$ and $\frac{d^2f(s)}{ds^2}$ is strictly increasing, we have from (5.48) that

$$\int_{t_3}^{t_1} \frac{d^2}{ds^2}f(s) = f_v(t_1) - f_v(t_3) \le \frac{1}{3}f_v(t_2). \tag{5.49}$$

Therefore,

$$f_v(t_3) \ge f_v(t_1) - \frac{1}{3}f_v(t_2), \tag{5.50}$$

and hence, by (5.47),

$$f_v(t_3) \ge \frac{1}{3}f_v(t_2). \tag{5.51}$$

Since f_v is an increasing function we also have

$$f_v(s) \ge \frac{1}{3}f_v(t_2), \quad s \in [t_3, t_1]. \tag{5.52}$$

Integrating (5.52) from t_3 to t_1 we have

$$f(t_1) - f(t_3) \geq \frac{1}{3} f_v(t_2)[t_2 - t_1]. \tag{5.53}$$

By convexity,

$$f_v(t_2)[t_2 - t_1] \geq f(t_2) - f(t_1). \tag{5.54}$$

Thus, from (5.53), (5.54) we obtain

$$f(t_3) \leq f(t_1) - \frac{1}{3}(f(t_2) - f(t_1)) = \frac{1}{3}(4f(t_1) - f(t_2)). \tag{5.55}$$

Therefore, by (5.46) we have

$$f(t_3) \leq 0, \tag{5.56}$$

which contradicts the strict positivity of f. Hence, we must have

$$f_v(t_1) \leq \frac{2}{3} f_v(t_2), \tag{5.57}$$

which completes the proof in case I.

We now turn to case II. In this case by the monotonicity of $\frac{d^2 f(s)}{ds^2}$ we have

$$f_v(t_2) - f_v(0) = \int_0^{t_2} \frac{d^2 f(s)}{ds^2} ds \geq \int_0^{2t_1} \frac{d^2 f(s)}{ds^2} ds \geq 2 \int_0^{t_1} \frac{d^2 f(s)}{ds^2} ds = 2(f_v(t_1) - f_v(0)). \tag{5.58}$$

Thus,

$$f_v(t_2) \geq 2 f_v(t_1) - f_v(0). \tag{5.59}$$

Choosing t_0 large enough, so that $f_v(0) \leq \frac{1}{2} f_v(t_0)$ we obtain (5.57), which completes the proof.

□

Finally, we summarize the properties of extremal solutions for the exponential nonlinearity.

Lemma 5.3. *Let $u_\alpha(x)$ be an extremal solution for problem (1.1) with $f(u) = e^u$. Then, as $\alpha \to \infty$, we have*

$$\lambda^*(\alpha) = \frac{2\alpha}{\log \alpha} \left(1 + O\left(\frac{1}{\sqrt{\log \alpha}}\right)\right), \tag{5.60}$$

$$u^*_\alpha(0) = O(\log \log \alpha), \tag{5.61}$$

$$\int_B (u^*_\alpha(x))^p dx \to 0, \quad 1 \leq p < \infty, \tag{5.62}$$

$$\int_B \psi(x) \exp(u^*_\alpha(x)) dx \to \int_B \psi(x) dx. \tag{5.63}$$

Proof. We first prove (5.60). Observe that estimate (3.47) for $f(u) = \exp(u)$ implies

$$\lambda^*(\alpha) \geq \frac{2\alpha}{\log \alpha} \left(1 - \frac{c}{\log \alpha} - c\tilde{R}(w)\right), \qquad \tilde{R}(w) = \exp(-w) + \frac{\exp(w)}{\log \alpha} \tag{5.64}$$

with an arbitrary w. It is easy to verify that $\tilde{R}(w)$ attains its minimum at

$$w = \frac{1}{2} \log \log \alpha, \tag{5.65}$$

hence

$$\min_{w\in(0,\infty)} \tilde{R}(w) = \frac{2}{\sqrt{\log\alpha}}. \tag{5.66}$$

Since (3.47) holds for an arbitrary w we have

$$\lambda^*(\alpha) \geq \frac{2\alpha}{\log\alpha}\left(1 - c\left(\frac{1}{\sqrt{\log\alpha}}\right)\right). \tag{5.67}$$

On the other hand, in the case $f(u) = \exp(u)$ estimate (3.1) takes the form

$$\lambda^*(\alpha) \leq \frac{2\alpha}{\log\alpha}\left(1 + \frac{c}{\log\alpha}\right). \tag{5.68}$$

Combining estimates (5.67) and (5.68) we get (5.60).

Next let us show that (5.61) holds. By (5.37)

$$\exp(v^*_\alpha(1)) \leq c\log\alpha, \tag{5.69}$$

and hence

$$v^*_\alpha(1) < c\log\log\alpha. \tag{5.70}$$

Using this observation, (5.41), and the fact that $u^*_\alpha(0) = v^*_\alpha(0)$, we have that

$$u^*_\alpha(0) \leq c\log\log\alpha. \tag{5.71}$$

On the other hand, we have from (5.67) and (4.33)

$$\exp(-u^*_\alpha(0)) \leq c\frac{1}{\sqrt{\log\alpha}}. \tag{5.72}$$

Hence,

$$u^*_\alpha(0) \geq \frac{1}{2}\log\log\alpha - c. \tag{5.73}$$

Combining (5.71) and (5.73), we get (5.61).

Finally (5.62) and (5.63) follow from Theorem 1.3 and Theorem 1.2 respectively. □

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References

[1] G. N. Abramovich, *The theory of turbulent jets*, The MIT Press, 2003.

[2] H. Berestycki, A. Kiselev, A. Novikov, and L. Ryzhik, *The explosion problem in a flow*, J. Anal. Math. **110** (2010), 31–65.

[3] H. Brezis, T. Cazenave, Y. Martel, and A. Ramiandrisoa, *Blow up for $u_t - \Delta u = g(u)$ revisited*, Adv. Differential Equations **1** (1996), no. 1, 73–90.

[4] X. Cabré and A. Capella, *Regularity of radial minimizers and extremal solutions of semilinear elliptic equations*, J. Funct. Anal. **238** (2006), no. 2, 709–733.

[5] M. G. Crandall and P. H. Rabinowitz, *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems*, Arch. Rational Mech. Anal. **58** (1975), no. 3, 207–218.

[6] L. Dupaigne, *Stable solutions of elliptic partial differential equations*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 143, Chapman & Hall/CRC, Boca Raton, FL, 2011.

[7] L. C. Evans, *Partial differential equations*, 2nd ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010.

[8] D. A. Frank-Kamenetskii, *Diffusion and heat transfer in chemical kinetics*, Plenum Press, New York, 1969.

[9] I. M. Gelfand, *Some problems in the theory of quasilinear equations*, Amer. Math. Soc. Transl. (2) **29** (1963), 295–381.

[10] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[11] P. V. Gordon, U. G. Hegde, and M. C. Hicks, *An elementary model for autoignition of free round turbulent jets*, SIAM J. Appl. Math. **78** (2018), no. 2, 705–718.

[12] J. P. Keener and H. B. Keller, *Positive solutions of convex nonlinear eigenvalue problems*, J. Differential Equations **16** (1974), 103–125.

[13] H. B. Keller and D. S. Cohen, *Some positone problems suggested by nonlinear heat generation*, J. Math. Mech. **16** (1967), 1361–1376.

[14] G. Nedev, *Regularity of the extremal solution of semilinear elliptic equations*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), no. 11, 997–1002.

[15] M. H. Protter and H. F. Weinberger, *Maximum principles in differential equations*, Springer-Verlag, New York, 1984.

[16] H. Schlichting, *Boundary Layer Theory*, McGraw-Hill; 6 Rev Ed edition, 1968.

[17] N. N. Semenov, *Thermal theory of combustion and explosion*, Physics-Uspekhi **23** (1940), 251-292.

[18] D. H. Sattinger, *Monotone methods in nonlinear elliptic and parabolic boundary value problems*, Indiana Univ. Math. J. **21** (1971/72), 979–1000.

[19] Ya. B. Zeldovich, G. I. Barenblatt, V. B. Librovich, and G. M. Makhviladze, *The mathematical theory of combustion and explosions*, Consultants Bureau [Plenum], New York, 1985.