Characterizing the cosmological gravitational wave background: Anisotropies and non-Gaussianity

Nicola Bartolo,[1,2,3] Daniele Bertacca,[1,2] Sabino Matarrese,[1,2,3,4] Marco Peloso,[1,2] Angelo Ricciardone,[2] Antonio Riotto,[5,6] and Gianmassimo Tasinato[7]

[1] *Dipartimento di Fisica e Astronomia "Galileo Galilei", Universitá di Padova, 35131 Padova, Italy*
[2] *INFN, Sezione di Padova, 35131 Padova, Italy*
[3] *INAF—Osservatorio Astronomico di Padova, I-35122 Padova, Italy*
[4] *Gran Sasso Science Institute, I-67100 L'Aquila, Italy*
[5] *Department of Theoretical Physics and Center for Astroparticle Physics (CAP), CH-1211 Geneva 4, Switzerland*
[6] *CERN, Theoretical Physics Department, Geneva, Switzerland*
[7] *Department of Physics, Swansea University, Swansea SA2 8PP, United Kingdom*

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A future detection of the stochastic gravitational wave background (SGWB) with gravitational wave (GW) experiments is expected to open a new window on early universe cosmology and on the astrophysics of compact objects. In this paper we study SGWB anisotropies, that can offer new tools to discriminate between different sources of GWs. In particular, the cosmological SGWB inherits its anisotropies both (i) at its production and (ii) during its propagation through our perturbed universe. Concerning (i), we show that it typically leads to anisotropies with order one dependence on frequency. We then compute the effect of (ii) through a Boltzmann approach, including contributions of both large-scale scalar and tensor linearized perturbations. We also compute for the first time the three-point function of the SGWB energy density, which can allow one to extract information on GW non-Gaussianity with interferometers. Finally, we include nonlinear effects associated with long wavelength scalar fluctuations, and compute the squeezed limit of the 3-point function for the SGWB density contrast. Such limit satisfies a consistency relation, conceptually similar to that found in the literature for the case of cosmic microwave background perturbations.

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I. INTRODUCTION

The current ground-based interferometers are close to reaching the expected sensitivity to detect the stochastic gravitational wave background (SGWB) from unresolved astrophysical sources [1]. Future space-based (such as LISA [2] and DECIGO [3]) and Earth-based (like the Einstein Telescope [4,5] and Cosmic Explorer [6]) interferometers have the potential to detect the SGWB of cosmological origin (see [7–10] for reviews of possible cosmological sources). It is likely that a detection of a cosmological SGWB background will require the ability to discriminate it against the astrophysical signal. Astrophysical GW background (AGWB) arises from the superposition of the signals emitted by a large population of unresolved sources that are mainly dominated by two types of events: (i) the periodic long-lived sources (e.g., the early inspiraling phase of binary systems) where the frequency is expected to evolve very slowly compared to the observation time; (ii) the short-lived burst sources, e.g., core collapse to neutron stars or black holes, oscillation modes, r-mode instabilities in rotating neutron stars, magnetars and super-radiant instabilities (for example, see [11,12]). Several techniques have been developed to distinguish among the various backgrounds. The most obvious tool for this component separation is the frequency dependence [13], as several cosmological mechanisms are peaked at some given characteristic scale. However, future detectors will allow for a better angular resolution of anisotropies of the astrophysical background. Therefore, another tool could be the directionality dependence of the SGWB [14–19] and, as we explore here, its statistics.

In this work, we discuss graviton propagation through a Boltzmann approach [15] as it is typically done for the cosmic microwave background (CMB). Specifically, we construct and evolve the equation for the distribution f of gravitons in a FLRW background, plus first-order scalar and tensor perturbations (we also consider how nonlinear effects for the specific case of squeezed non-Gaussianity, as we discuss at the end of this introduction). At the unperturbed level, following the isotropy and homogeneity of the background, the distribution depends only on time and on the GW frequency $p/2\pi$ (where $\vec{p}$ is the physical momentum of the gravitons) through the combination $q \equiv pa$, where a is the scale factor of the universe. Namely, the gravitons freely propagate, and their physical momentum redshifts during the propagation. This property is shared by any free massless particles, and, in particular, also by the CMB photons. On the other hand, differently from the photon distribution, the initial population of gravitons is not expected to be thermal (as we have in mind production mechanisms, such as inflation [20,21], phase transitions [22], or enhanced density perturbations leading to primordial black holes [23–25], which occur at energies well below the Planck scale) which leaves in the distribution a sort of "memory" of the initial state. As we show, the fact that the spectrum is nonthermal generically results in angular anisotropies that have an order one dependence on the GW frequency. This is in contrast with the CMB case, for which this dependence only arises at second order in perturbation theory.

This initial state will in general be anisotropic, as no mechanism of GW production can be perfectly homogeneous. Additional anisotropies are induced by the GW propagation in the perturbed universe. As we are interested in large scale, we work in a regime of a large hierarchy $q \gg k$ between the GW (comoving) momentum q and the (comoving) momentum k of the large scale perturbations. We confirm that in the angular power spectrum, the Sachs Wolfe (SW) effect is dominating on large scales also for gravitons, while the integrated Sachs-Wolfe contribution is subdominant.

We employ this approach to study the non-Gaussianity of the SGWB energy density. Although we are not aware of any dedicated analysis in this sense, it is reasonable to expect that the SGWB produced by incoherent astrophysical sources is Gaussian, due to the central limit theorem. An analysis of detection methods of non-Gaussianity in a GWB induced by short-duration signals has been done in [26], while in [27,28] it has been studied how to use higher-order cumulants to characterize properties of non-Gaussianity in the AGWB (see also [29] for a recent analysis). In light of this fact, a measurement of non-Gaussianity would be a signal of large scale coherency, that would likely point to a cosmological origin of the signal. Previous works showed that inflation can result in a sizeable and nonvanishing 3-point function $\langle h^3 \rangle$ for the graviton wave function, but that this is generically non-observable in interferometers [23,24], due to the decoherence of the phase the GW wave-function h induced by the GW propagation, and due to the finite duration of the measurement (see [30] for a possible exception to this conclusion, occurring for a very specific shape of the bispectrum). Since the phase does not affect the GW energy density, we argue that the energy density is a much better variable to study the statistics of the SGWB. Also in this case, the non-Gaussianity can be induced both by the production mechanism and the propagation. As an example of the former, in Ref. [25] we recently computed the 3-point function of the SGWB energy density that arises in the presence of non-Gaussianity of the scalar perturbations of the local shape (in the presence of this non-Gaussianity, a long-scale mode of momentum k can modulate the power of the short-scalr scalar perturbations that are responsible for the primordial black holes formation). Here we study the 3-point function induced by the GW propagation. This is also proportional to the non-Gaussianity of the scalar perturbations. In this sense, the SGWB can be used as a novel probe (beyond the CMB and the large scale structure) of the non-Gaussianity of the scalar perturbations.

Although in most of this work we limit our attention to linearized fluctuations, in Sec. VI we consider nonlinear effects induced by long-wavelength scalar perturbations, which modulate correlation functions involving short-wavelength modes. We make use of a powerful method first introduced by Weinberg in [31], which focuses on adiabatic systems, and identifies the effects of long modes with an appropriate coordinate transformation. Applying this method to our setup, we compute how nonlinearities induce a nonvanishing squeezed limit of the 3-point function for the SGWB density contrast. We determine how such squeezed limit depends on the scale dependence of the spectrum of primordial scalar fluctuations, on the momentum dependence of the background SGWB distribution, and on the time, scale, and direction dependence of the scalar transfer functions connecting primordial to late-time adiabatic scalar fluctuations.

The paper is organized as follows. In Sec. II we present the computation and the formal solution of the Boltzmann equation for GW propagation. In Sec. III we decompose the formal solution in spherical harmonics, paralleling a treatment that is familiar in the study of CMB perturbations. In Sec. IV we compute the angular power spectrum and bispectrum of the SGWB perturbations. In Sec. V we review one physical mechanism that can result in a sizeable cosmological SGWB with some degree of anisotropy. In Sec. VI we study nonlinear effects on the squeezed bispectrum. In Sec. VII we comment on the observability of such anisotropies. These results are discussed and summarized in Sec. VIII. The paper is concluded by three Appendixes. Appendix A contains the details of the computation of the anisotropies due to the large-scale tensor perturbations. Appendix B provides some intermediate steps on the computation of the GW bispectrum induced by tensor modes. Finally, Appendix C presents an immediate connection between our formal solutions and the CMB results obtained in the case of initial thermal state.

Part of the results contained in the present work were also summarized in [32].

II. BOLTZMANN EQUATION FOR GRAVITATIONAL WAVES

We consider first-order perturbations around a Friedmann-Lemaitre-Robertson-Walker (FLRW) background in the Poisson gauge

$$ds^2 = a^2(\eta)[-e^{2\Phi}d\eta^2 + (e^{-2\Psi}\delta_{ij} + \chi_{ij})dx^i dx^j], \quad (2.1)$$

where $a(\eta)$ is the scale factor as a function of the conformal time η. Φ and Ψ are scalar perturbations while χ_{ij} represent the transverse-traceless tensor perturbations. We neglect linear vector modes since they are not produced at first order in standard mechanisms for the generation of cosmological perturbations (as scalar field inflation), and we consider tensor modes at linearized order.

Given the statistical nature of the GW we can define a distribution function of gravitons as $f = f(x^\mu, p^\mu)$, which is a function of their position x^μ and momentum $p^\mu = dx^\mu/d\lambda$, where λ is an affine parameter along the GW trajectory. As we will see, observables as number density, spectral energy density, and flux (directions) can be derived from the distribution function. The graviton distribution function obeys the Boltzmann equation

$$\mathcal{L}[f] = \mathcal{C}[f(\lambda)] + \mathcal{I}[f(\lambda)], \quad (2.2)$$

where $\mathcal{L} \equiv d/d\lambda$ is the Liouville term, while $\mathcal{C}$ and $\mathcal{I}$ account, respectively, for the collision of GWs along their path, and for their emissivity from cosmological and astrophysical sources [15]. The collision among GWs affects the distribution at higher orders (in an expansion series in the gravitational strength $1/M_P$, where M_P is the Planck mass) with respect to the ones we are considering, and they can be disregarded in our analysis (see [33] and references therein for a discussion of collisional effects involving gravitons). The emissivity can be due to astrophysical processes (such as black hole merging) in the relatively late universe, as well as cosmological processes, such as inflation or phase transitions. In this work we are only interested in the stochastic GW background of cosmological origin, so we treat the emissivity term as an initial condition on the GW distribution. This leads us to study the free Boltzmann equation $df/d\eta = 0$ in the perturbed universe

$$\frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial x^i}\frac{dx^i}{d\eta} + \frac{\partial f}{\partial q}\frac{dq}{d\eta} + \frac{\partial f}{\partial n^i}\frac{dn^i}{d\eta} = 0, \quad (2.3)$$

where $\hat{n} \equiv \hat{p}$ is the GW direction of motion, and where we have used the comoving momentum $q \equiv |\vec{p}|a$ (as opposed to the physical one, used in [15,34]). This simplifies the equations by factorizing out the universe expansion. The first two terms in (2.3) encode free streaming, that is the propagation of perturbations on all scales. At higher order this term also includes gravitational time delay effects. The third term causes the red-shifting of gravitons, including the Sachs-Wolfe, integrated Sachs-Wolfe, and Rees-Sciama effects. The fourth term vanishes to first order, and describes the effect of gravitational lensing. We shall refer to these terms as the free-streaming, redshift, and lensing terms, respectively, as customarily done in CMB physics.

Keeping only the terms up to first order in the perturbations, Eq. (2.3) gives

$$\frac{\partial f}{\partial \eta} + n^i \frac{\partial f}{\partial x^i} + \left[\frac{\partial \Psi}{\partial \eta} - n^i \frac{\partial \Phi}{\partial x^i} + \frac{1}{2} n_i n_j \frac{\partial \chi_{ij}}{\partial \eta}\right] q \frac{\partial f}{\partial q} = 0, \quad (2.4)$$

where we have followed the standard procedure developed for the CMB in [34,35]. The distribution function f can be expanded as

$$f(\eta, x^i, q, n^i) = \bar{f}(q) + f^{(1)}(\eta, x^i, q, n^i) + \ldots.$$
$$\equiv \bar{f}(q) - q \frac{\partial \bar{f}}{\partial q} \Gamma(\eta, x^i, q, n^i) + \ldots, \quad (2.5)$$

where the dominant, homogeneous, and isotropic contribution $\bar{f}(q)$ solves the zeroth-order Boltzmann equation. The function $f^{(1)}(\eta, x^i, q, n^i)$ is the solution of the first-order equation, and the ellipses denote the higher-order solutions in a perturbative expansion. In this expression we have parametrized the first-order solution in terms of the function Γ, so to simplify the first-order Boltzmann equation [15]. For a thermal distribution with temperature T, one finds $\Gamma = \delta T/T$. This is particularly the case for the CMB, for which, due to the thermalization, the temperature anisotropies are frequency independent up to second order in the perturbations. For gravitons, as we already mentioned, the collisional term is extremely small, and, for a generic production mechanism, Γ generically retains an order one dependence on frequency (as we show below, also for the GW case the propagation effects induce frequency-independent perturbations at linear order).

The zeroth-order homogeneous Boltzmann equation simply reads $\partial \bar{f}/\partial \eta = 0$, and it is solved by any distribution that is a function only of the comoving momentum q, namely $f = \bar{f}(q)$. In our approach this solution is simply given as the homogeneous part of the initial condition. As a consequence, the physical momentum of the individual gravitons redshifts proportionally to $1/a$, and the physical graviton number density $n \propto \int d^3p \bar{f}(q)$ is diluted as a^{-3} as the universe expands. This is also the case for CMB photons, whose distribution function $\bar{f}_{\rm CMB} = (e^{p/T} - 1)^{-1}$ is only controlled by the ratio $p/T \propto ap = q$, where T is the temperature of the CMB bath. We see that these rescalings with a are a consequence of the free particle propagation in the expanding FLRW background, and they do not rely on the distribution being thermal.

As anticipated, from the graviton distribution function, evaluated at the present time η_0, we can compute the SGWB energy density

$$\rho_{\mathrm{GW}}(\eta_0,\vec{x}) = \frac{1}{a_0^4}\int d^3q q f(\eta_0,\vec{x},q,\hat{n}) \equiv \rho_{\mathrm{crit},0}\int d\ln q\,\Omega_{\mathrm{GW}}(\vec{x},q), \quad (2.6)$$

where we have introduced the spectral energy density Ω_{GW} and the critical density $\rho_{\mathrm{crit}} = 3H^2M_p^2$. Here $H \equiv (1/a^2)da/d\eta$ is the Hubble rate. Following standard conventions, the suffix 0 denotes a quantity evaluated at the present time.

Contrary to most of the studies of the SGWB, that assume a homogeneous Ω_{GW}, in our case the GW energy density depends on space. We denote the homogeneous component of Ω_{GW} as

$$\bar{\Omega}_{\mathrm{GW}}(q) \equiv \frac{4\pi}{\rho_{\mathrm{crit},0}}\left(\frac{q}{a_0}\right)^4\bar{f}(q). \quad (2.7)$$

For the full spectral energy density, we define

$$\Omega_{\mathrm{GW}} \equiv \frac{1}{4\pi}\int d^2\hat{n}\omega_{\mathrm{GW}}(\vec{x},q,\hat{n}), \quad (2.8)$$

where $\omega_{\mathrm{GW}}(\vec{x},q,\hat{n}) \equiv q^4/(a^4\rho_{\mathrm{crit}})f(\vec{x},q,\hat{n})$, and we introduce the SGWB density contrast

$$\delta_{\mathrm{GW}} \equiv \frac{\omega_{\mathrm{GW}}(\vec{x},q,\hat{n}) - \bar{\Omega}_{\mathrm{GW}}(q)}{\bar{\Omega}_{\mathrm{GW}}(q)} = \left[4 - \frac{\partial\ln\bar{\Omega}_{\mathrm{GW}}(q)}{\partial\ln q}\right]\Gamma(\eta_0,\vec{x},q,\hat{n}). \quad (2.9)$$

In terms of the function Γ, the first-order Boltzmann equation reads [15]

$$\frac{\partial\Gamma}{\partial\eta} + n^i\frac{\partial\Gamma}{\partial x^i} = S(\eta,x^i,n^i), \quad (2.10)$$

where

$$S(\eta,x^i,n^i) = \frac{\partial\Psi}{\partial\eta} - n^i\frac{\partial\Phi}{\partial x^i} - \frac{1}{2}n^in^j\frac{\partial\chi_{ij}}{\partial\eta}$$

is the source function which includes the physical effects due to cosmological scalar and tensor inhomogeneities. We note that the source is q independent (thus showing that the anisotropies arising at first order from propagation effects are frequency independent, as we anticipated).

To solve this equation, it is convenient to Fourier transform with respect to spatial coordinates,

$$\Gamma \equiv \int\frac{d^3k}{(2\pi)^3}\mathrm{e}^{i\vec{k}\cdot\vec{x}}\Gamma(\eta,\vec{k},q,\hat{n}), \quad (2.11)$$

and analogously for the other variables (we use the same notation for a field and for its Fourier transform, as the context always clarifies which object we are referring to). This leads to

$$\Gamma' + ik\mu\Gamma = S(\eta,\vec{k},\hat{n}), \quad (2.12)$$

where from now on prime denotes a derivative with respect to conformal time, and where we denote by

$$\mu \equiv \hat{k}\cdot\hat{n}, \quad (2.13)$$

the cosine of the angle between the Fourier variable $\vec{k}$ and the direction of motion $\hat{n}$ of the GW. In Fourier space the source term reads

$$S = \Psi' - ik\mu\Phi - \frac{1}{2}n^in^j\chi'_{ij}. \quad (2.14)$$

With this information in mind, Eq. (2.12) is readily integrated to give

$$\Gamma(\eta,\vec{k},q,\hat{n}) = \mathrm{e}^{ik\mu(\eta_{\mathrm{in}}-\eta)}\Gamma(\eta_{\mathrm{in}},\vec{k},q,\hat{n}) + \int_{\eta_{\mathrm{in}}}^{\eta}d\eta'\mathrm{e}^{ik\mu(\eta'-\eta)}\left[\frac{d\Psi(\eta',\vec{k})}{d\eta'} - ik\mu\Phi(\eta',\vec{k}) - \frac{1}{2}n^in^j\frac{\partial\chi_{ij}(\eta',\vec{k})}{\partial\eta'}\right]. \quad (2.15)$$

We integrate the second term in the second line by parts, and obtain

$$\Gamma(\eta,\vec{k},q,\hat{n}) = \mathrm{e}^{ik\mu(\eta_{\mathrm{in}}-\eta)}[\Gamma(\eta_{\mathrm{in}},\vec{k},q,\hat{n}) + \Phi(\eta_{\mathrm{in}},\vec{k})] - \Phi(\eta,\vec{k}) + \int_{\eta_{\mathrm{in}}}^{\eta}d\eta'\mathrm{e}^{ik\mu(\eta'-\eta)}\left\{\frac{d[\Psi(\eta',\vec{k}) + \Phi(\eta',\vec{k})]}{d\eta'} - ik\mu\Phi(\eta',\vec{k}) - \frac{1}{2}n^in^j\frac{\partial\hat{\chi}_{ij}(\eta',\vec{k})}{\partial\eta'}\right\}, \quad (2.16)$$

with the last two terms in the first line being the boundary terms of this integration. In the following section, we decompose the $\hat{n}$ dependence of the solution (representing the arrival direction of the GW on our sky) in spherical harmonics. As we are not interested in the monopole term, we can disregard the $-\Phi(\eta,\vec{k})$ contribution to the solution, and write

$$\Gamma(\eta,\vec{k},q,\hat{n}) \equiv \int_{\eta_{\rm in}}^{\eta} d\eta' e^{ik\mu(\eta'-\eta)} \times \left\{ [\Gamma(\eta',\vec{k},q,\hat{n}) + \Phi(\eta',\vec{k})]\delta(\eta'-\eta_{\rm in}) + \frac{\partial[\Psi(\eta',\vec{k})+\Phi(\eta',\vec{k})]}{\partial\eta'} - \frac{1}{2} n^i n^j \frac{\partial\hat{\chi}_{ij}(\eta',\vec{k})}{\partial\eta'} \right\}. \tag{2.17}$$

The first term, which was disregarded in [15], carries the "memory" of the initial conditions. Due to this term, the GW energy density anisotropies are generically dependent on the frequency q. We discuss an example of this fact in Sec. V, where we study the SGWB produced in axion inflation.

Generally, this term has also a dependence on $\hat{n}$. This implies that the solution has a dependence on the direction $\hat{n}$, which is more general than the one arising from the projection of $\vec{k}$ on the line of sight $\hat{n}$. [Indeed, the remaining terms in Eq. (2.17) depend on $\hat{n}$ only through the $\mu \equiv \hat{k}\cdot\hat{n}$ combination. Thanks to this fact, they result in angular correlators that are statistically isotropic (as we show in the next two sections).] On the other hand, the angular dependence present in the first term of (2.17) could result in statistically anisotropic correlators (specifically, 2-point and 3-point correlators) that have a more general dependence on the multipoles coefficients ℓ_i and m_i than Eqs. (4.3). This would indicate an overall anisotropy of the mechanism responsible for the GW across the entire universe, and, ultimately, a departure from an exact FLRW geometry. While we believe that this can be an interesting topic for future exploration, the present work focuses on the statistically isotropic case, and we assume an initial condition of the form $\Gamma_{\rm in} = \Gamma(\eta_{\rm in},\vec{k},q)$, which guarantees such a condition.

In the case of gravitational wave backgrounds, what can be directly observed with laser interferometers is the (incoherent) superposition of the signal coming from directions over the whole sky under a certain projection through the response function [7]. However the quantity Γ is related to fluctuations of the flux or intensity of gravitons (in analogy with the flux of photons in CMB observations), and it is proportional to the square of the signal. Making use of the map-making algorithm, this is measurable from laser interferometers, but it is not a direct observable. Ways to understand how to measure this observable with interferometers are under development.

III. SPHERICAL HARMONICS DECOMPOSITION

We separate the solution (2.17) in three terms

$$\Gamma(\eta,\vec{k},q,\hat{n}) = \Gamma_I(\eta,\vec{k},q,\hat{n}) + \Gamma_S(\eta,\vec{k},\hat{n}) + \Gamma_T(\eta,\vec{k},\hat{n}), \tag{3.1}$$

where I, S, and T stand for initial, scalar, and tensor sourced terms respectively and they are given by

$$\begin{aligned} \Gamma_I(\eta,\vec{k},q,\hat{n}) &= e^{ik\mu(\eta_{\rm in}-\eta)}\Gamma(\eta_{\rm in},\vec{k},q), \\ \Gamma_S(\eta,\vec{k},\hat{n}) &= \int_{\eta_{\rm in}}^{\eta} d\eta' e^{ik\mu(\eta'-\eta)} \left[\Phi(\eta',\vec{k})\delta(\eta'-\eta_{\rm in}) + \frac{\partial[\Psi(\eta',\vec{k})+\Phi(\eta',\vec{k})]}{\partial\eta'}\right], \\ \Gamma_T(\eta,\vec{k},\hat{n}) &= -\frac{n^i n^j}{2}\int_{\eta_{\rm in}}^{\eta} d\eta' e^{ik\mu(\eta'-\eta)} \frac{\partial\hat{\chi}_{ij}(\eta',\vec{k})}{\partial\eta'}. \end{aligned} \tag{3.2}$$

Similarly to what is usually done for the CMB, in order to compute the angular power spectrum, in an all-sky analysis we decompose the fluctuations using spin-0 or spin-2 spherical harmonics. Since Γ is a scalar, we can express it as

$$\Gamma(\hat{n}) = \sum_\ell \sum_{m=-\ell}^{\ell} \Gamma_{\ell m} Y_{\ell m}(\hat{n}), \qquad \text{inverted by } \Gamma_{\ell m} = \int d^2 n \Gamma(\hat{n}) Y^*_{\ell m}(\hat{n}), \tag{3.3}$$

where we recall that $\hat{n}$ is the direction of motion of the GWs. More specifically,

$$\begin{aligned} \Gamma_{\ell m} &= \int d^2 n Y^*_{\ell m}(\hat{n}) \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} [\Gamma_I(\eta,\vec{k},q,\hat{n}) + \Gamma_S(\eta,\vec{k},\hat{n}) + \Gamma_T(\eta,\vec{k},\hat{n})] \\ &\equiv \Gamma_{\ell m,I} + \Gamma_{\ell m,S} + \Gamma_{\ell m,T}. \end{aligned} \tag{3.4}$$

A. Initial condition term and q-dependent anisotropies

Let us first evaluate the initial condition term

$$\Gamma_{\ell m,I} = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0}\Gamma(\eta_{\rm in},\vec{k},q)\int d^2n Y^*_{\ell m}(\hat{n})e^{-ik(\eta_0-\eta_{\rm in})\hat{k}\cdot\hat{n}}. \tag{3.5}$$

Following the standard treatment for CMB anisotropies [34], we make use of the identity

$$e^{-i\mathbf{k}\cdot\mathbf{y}} = \sum_\ell (-i)^\ell(2\ell+1)j_\ell(ky)P_\ell(\hat{k}\cdot\hat{y}) = 4\pi\sum_\ell\sum_{m=-\ell}^{\ell}(-i)^\ell j_\ell(ky)Y_{\ell m}(\hat{k})Y^*_{\ell m}(\hat{y}) \tag{3.6}$$

(where j_ℓ and P_ℓ are, respectively, spherical Bessel functions and Legendre polynomial) so to obtain

$$\Gamma_{\ell m,I} = 4\pi(-i)^l\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}_0} \times\Gamma(\eta_{\rm in},\vec{k},q)Y^*_{\ell m}(\hat{k})j_\ell(k(\eta_0-\eta_{\rm in})). \tag{3.7}$$

Here $\vec{x}_0$ denotes our location (that can be set to the origin), η_0 denotes the present time, and $\eta_{\rm in}$ the initial time. Once again we stress the peculiar property of the initial condition, namely its dependence on the frequency q. In Sec. IV we discuss how this imprints the SGWB angular spectrum.

B. Scalar sourced term

A second source of anisotropy is due to the GW propagation in the large-scale scalar perturbations of the universe (the wave number of these perturbations k is many orders of magnitude smaller than the GW frequency q, and the GW acts as a probe of this large-scale background). As long as the scalar perturbation is in the linear regime (which is the case for the large-scale modes that leave an impact on the large-scale anisotropies of our interest), we can express it [34] as a transfer function (a deterministic function that encodes the time dependence of the perturbations) times a stochastic variable ζ. This assumes the absence of isocurvature modes, and, in particular, of anisotropic stresses, as for example those due to the relic neutrinos. This also assumes that the statistical properties of ζ have been set well before the propagation stage that we are considering (for instance during inflation, or during some early phase transition). Therefore, the scalar perturbations are

$$\Phi(\eta,\vec{k}) = T_\Phi(\eta,k)\zeta(\vec{k}),\qquad \Psi(\eta,\vec{k}) = T_\Psi(\eta,k)\zeta(\vec{k}). \tag{3.8}$$

Under the above assumptions, $T_\Phi(\eta,k) = T_\Psi(\eta,k)$. However, we keep these two terms as distinct in our intermediate computations, so that the present analysis can be most easily generalized, if one wishes to introduce more general sources.

With this in mind, the scalar sourced term becomes

$$\Gamma_S(\eta_0,\vec{k},\hat{n}) = \int_{\eta_{\rm in}}^{\eta_0}d\eta' e^{ik\mu(\eta'-\eta_0)}\left[T_\Phi(\eta',k)\delta(\eta'-\eta_{\rm in}) + \frac{\partial[T_\Psi(\eta',k)+T_\Phi(\eta',k)]}{\partial\eta'}\right]\zeta(\vec{k}) \equiv \int_{\eta_{\rm in}}^{\eta_0}d\eta' e^{-ik\mu(\eta_0-\eta')}\mathcal{T}_S(\eta',k)\zeta(\vec{k}), \tag{3.9}$$

and we note that we are assuming a single adiabatic mode [i.e., $\zeta(\vec{k})$ is the operator associated with the conserved curvature perturbation at superhorizon scales]. Proceeding as above,

$$\Gamma_{\ell m,S} = 4\pi(-i)^l\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}_0}\zeta(\vec{k})Y^*_{\ell m}(\hat{k}) \times\left\{T_\Phi(\eta_{\rm in},k)j_\ell(k(\eta_0-\eta_{\rm in})) + \int_{\eta_{\rm in}}^{\eta_0}d\eta'\frac{\partial[T_\Psi(\eta',k)+T_\Phi(\eta',k)]}{\partial\eta'}j_\ell(k(\eta_0-\eta'))\right\}. \tag{3.10}$$

As we can see, also the SGWB, feels, similarly to the CMB, a Sachs-Wolfe and integrated Sachs-Wolfe effect, which are represented by the first and the second term in (3.10), respectively.

C. Tensor sourced term

Finally, the third contribution $\Gamma_{\ell m,T}$ is due to the GW propagation in the large-scale tensor modes

$$\Gamma_{\ell m,T} = -\int d^2nY^*_{\ell m}(\hat{n})\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}_0}\frac{n^in^j}{2} \times\int_{\eta_{\rm in}}^{\eta}d\eta' e^{ik\mu(\eta'-\eta_0)}\frac{\partial\chi_{ij}(\eta',\vec{k})}{\partial\eta'}. \tag{3.11}$$

To evaluate such term we decompose the tensor modes in right- and left-handed (respectively $\lambda = \pm 2$) circular polarizations (see e.g., [36]),

$$\chi_{ij} \equiv \sum_{\lambda=\pm2}e_{ij,\lambda}(\hat{k})\chi(\eta,k)\xi_\lambda(k^i). \tag{3.12}$$

The three factors involved in each term are, respectively, the tensor circular polarization operator, the tensor mode function (equal for the two polarizations), and the stochastic variable for that tensor polarization (that is the analog of ζ we discussed in the scalar case).

Inserting this decomposition in Eq. (3.11), a lengthy algebra, that we report in Appendix A, leads to

$$\Gamma_{\ell m,T} = \pi(-i)^\ell \sqrt{\frac{(\ell+2)!}{(\ell-2)!}} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0} \sum_{\lambda=\pm 2} {}_{-\lambda}Y^*_{\ell m}(\Omega_k)\xi_\lambda(\vec{k}) \int_{\eta_{\rm in}}^{\eta_0} d\eta \chi'(\eta,k) \frac{j_\ell(k(\eta_0-\eta))}{k^2(\eta_0-\eta)^2}, \tag{3.13}$$

which is formally analogous to Eq. (3.10), with the product $\hat{\zeta} Y^*_{\ell m}$ replaced by the combination $\sum_{\lambda=\pm 2} \hat{\xi}_\lambda(\vec{k})\,{}_{-\lambda}Y^*_{\ell m}(\Omega_k)$, involving the spin-2 spherical harmonics, and with the scalar transfer function replaced by the tensor one.

D. Summary of the three contributions

The results derived in the three previous subsections can be written in the (slightly) more compact form

$$\Gamma_{\ell m,I}(q) = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0} \Gamma(\eta_{\rm in},\vec{k},q) Y^*_{\ell m}(\hat{k}) j_\ell(k(\eta_0-\eta_{\rm in})),$$

$$\Gamma_{\ell m,S} = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0} \zeta(\vec{k}) Y^*_{\ell m}(\hat{k}) \mathcal{T}^S_\ell(k,\eta_0,\eta_{\rm in}),$$

$$\Gamma_{\ell m,T} = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0} \sum_{\lambda=\pm 2} {}_{-\lambda}Y^*_{\ell m}(\Omega_k)\xi_\lambda(\vec{k}) \mathcal{T}^T_\ell(k,\eta_0,\eta_{\rm in}), \tag{3.14}$$

where we have introduced the linear transfer function $\mathcal{T}^{X(z)}_\ell$, with $X = S, T$ which represents the time evolution of the graviton fluctuations originated from the primordial perturbation

$$\mathcal{T}^S_\ell(k,\eta_0,\eta_{\rm in}) \equiv T_\Phi(\eta_{\rm in},k) j_\ell(k(\eta_0-\eta_{\rm in})) + \int_{\eta_{\rm in}}^{\eta_0} d\eta' \frac{\partial[T_\Psi(\eta,k)+T_\Phi(\eta,k)]}{\partial\eta} j_\ell(k(\eta-\eta_{\rm in})),$$

$$\mathcal{T}^T_\ell(k,\eta_0,\eta_{\rm in}) \equiv \sqrt{\frac{(\ell+2)!}{(\ell-2)!}} \frac{1}{4} \int_{\eta_{\rm in}}^{\eta_0} d\eta \frac{\partial\chi(\eta,k)}{\partial\eta} \frac{j_\ell(k(\eta_0-\eta))}{k^2(\eta_0-\eta)^2}. \tag{3.15}$$

IV. CORRELATORS OF GW ANISOTROPIES AND SGWB NON-GAUSSIANITY

We now compute the 2-point $\langle \Gamma_{\ell m}\Gamma^*_{\ell' m'}\rangle$ and the 3-point $\langle \Gamma_{\ell_1 m_1}\Gamma_{\ell_2 m_2}\Gamma_{\ell_3 m_3}\rangle$ angular correlators of the solutions (3.14). The statistical operators entering in these solutions are the four momentum-dependent quantities $\Gamma(\eta_{\rm in},\vec{k},q)$, $\zeta(\vec{k})$, $\xi_R(\vec{k})$, and $\xi_L(\vec{k})$, while the other terms encode deterministic effects such has the time evolution of the large-scale modes (in the linearized theory of the cosmological perturbations) and the projection of the GW anisotropies in the harmonic space. In this study, we assume that the stochastic variables are nearly Gaussian, with the 2-point functions

$$\langle \Gamma(\eta_{\rm in},\vec{k},q)\Gamma^*(\eta_{\rm in},\vec{k}',q)\rangle = \frac{2\pi^2}{k^3} P_I(q,k)(2\pi)^3\delta(\vec{k}-\vec{k}'),$$

$$\langle \zeta(\vec{k})\zeta^*(\vec{k}')\rangle = \frac{2\pi^2}{k^3} P_\zeta(k)(2\pi)^3\delta(\vec{k}-\vec{k}'),$$

$$\langle \xi_\lambda(\vec{k})\xi^*_{\lambda'}(\vec{k}')\rangle = \frac{2\pi^2}{k^3} P_\lambda(k)\delta_{\lambda,\lambda'}(2\pi)^3\delta(\vec{k}-\vec{k}'), \tag{4.1}$$

and a subdominant 3-point component

$$\langle \Gamma(\eta_{\rm in},\vec{k},q)\Gamma^*(\eta_{\rm in},\vec{k}',q)\Gamma^*(\eta_{\rm in},\vec{k}'',q)\rangle = B_I(q,k,k',k'')(2\pi)^3\delta(\vec{k}+\vec{k}'+\vec{k}'')$$

$$\langle \zeta(\vec{k})\zeta(\vec{k}')\zeta(\vec{k}'')\rangle = B_\zeta(k,k',k'')(2\pi)^3\delta(\vec{k}+\vec{k}'+\vec{k}'')$$

$$\langle \xi_\lambda(\vec{k})\xi_{\lambda'}(\vec{k}')\xi_{\lambda''}(\vec{k}'')\rangle = B_\lambda(\vec{k},\vec{k}',\vec{k}'')\delta_{\lambda,\lambda'}\delta_{\lambda,\lambda''}(2\pi)^3\delta(\vec{k}+\vec{k}'+\vec{k}''). \tag{4.2}$$

The assumption of nearly Gaussian modes is experimentally verified for the large-scale perturbations of ζ and of ξ_λ, as obtained from the CMB data [37]. We assume that this is the case also for the initial condition term.

The expressions (4.1) and (4.2) can be readily used to compute the angular correlators of the solutions in (3.14). Moreover, for simplicity of exposition, we have here assumed that the various terms are not cross-correlated. These result in separate sets of correlators for the three terms in (3.14). This assumption can be easily relaxed, and in fact, we did so in [25] where we studied the anisotropic distribution of the GW originated in models with primordial black holes, as we review in Sec. V.

The computations performed so far assume statistical isotropy (recall the discussion at the end of Sec. II). Correspondingly, when we combine (4.1) and (4.2) with (3.14) we obtain angular correlators with well specific dependence on the multipole indices. Specifically, the two point correlators have the dependence

$$\langle \Gamma_{\ell m} \Gamma^*_{\ell' m'} \rangle \equiv \delta_{\ell\ell'} \delta_{mm'} \tilde{C}_\ell, \qquad \langle \Gamma_{\ell_1 m_1} \Gamma_{\ell_2 m_2} \Gamma_{\ell_3 m_3} \rangle \equiv \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m+2 & m_3 \end{pmatrix} \tilde{b}_{\ell\ell'\ell''}, \tag{4.3}$$

while, under the above assumption, the angular power spectrum and the reduced bispectrum consists of the three separate contributions

$$\tilde{C}_\ell = \tilde{C}_{\ell,I}(q) + \tilde{C}_{\ell,S} + \tilde{C}_{\ell,T}, \qquad \tilde{b}_{\ell_1\ell_2\ell_3} = \tilde{b}_{\ell_1\ell_2\ell_3,I}(q) + \tilde{b}_{\ell_1\ell_2\ell_3,S} + \tilde{b}_{\ell_1\ell_2\ell_3,T}. \tag{4.4}$$

We recall that the form of the bispectrum factorizes the Wigner-3j symbols [38], which are nonvanishing only provided that $\sum_i m_i = 0$ and that the three ℓ_i satisfy the triangular inequalities.

In the following we provide the explicit expression for the various contributions to the power spectrum and the reduced bispectrum.

A. Angular power spectrum of GW energy density

We start with the computation of the two-point function of the initial condition term. From the first of (3.14) we can write

$$\langle \Gamma_{\ell m,I}(q) \Gamma^*_{\ell' m',I}(q) \rangle = (4\pi)^2 (-i)^{\ell-\ell'} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0} \int \frac{d^3k'}{(2\pi)^3} e^{-i\vec{k}'\cdot\vec{x}_0} \langle \Gamma(\eta_{\rm in}, \vec{k}, q) \Gamma^*(\eta_{\rm in}, \vec{k}', q) \rangle \\ \times Y^*_{\ell m}(\hat{k}) Y_{\ell' m'}(\hat{k}') j_\ell(k(\eta_0 - \eta_{\rm in})) j_{\ell'}(k'(\eta_0 - \eta_{\rm in})). \tag{4.5}$$

The correlator of the initial condition term is then given by the first of (4.1). Using this, and the orthonormality condition of the spherical harmonics, $\int d^2\hat{n}_s Y_{\ell m s} Y^*_{\ell' m'} = \delta_{\ell\ell'} \delta_{mm'}$, leads to

$$\langle \Gamma_{\ell m,I}(q) \Gamma^*_{\ell' m',I}(q) \rangle = \delta_{\ell\ell'} \delta_{mm'} 4\pi \int \frac{dk}{k} [j_\ell(k(\eta_0 - \eta_{\rm in}))]^2 P_I(q,k), \tag{4.6}$$

which indeed is of the form dictated by statistical isotropy. The other two terms are obtained analogously. Altogether, we find

$$\tilde{C}_{\ell,I}(q) = 4\pi \int \frac{dk}{k} [j_\ell(k(\eta_0 - \eta_{\rm in}))]^2 P_I(q,k), \\ \tilde{C}_{\ell,S} = 4\pi \int \frac{dk}{k} \mathcal{T}_\ell^{(S)2}(k, \eta_0, \eta_{\rm in}) P_\zeta(k), \\ \tilde{C}_{\ell,T} = 4\pi \int \frac{dk}{k} \mathcal{T}_\ell^{(T)2}(k, \eta_0, \eta_{\rm in}) \sum_{\lambda=\pm 2} P_\lambda(k). \tag{4.7}$$

We know from the CMB that the large-scale tensor modes have a power smaller than the scalar ones. At large scale, the scalar contribution is dominated by the term proportional to the initial value of Φ in $\mathcal{T}_\ell^{(0)}$, which is the analog of the SW contribution for the CMB. The large-scale modes that we are considering reentered the horizon during matter domination. For these modes, ignoring the late time dark energy domination, $T_\Phi = T_\Psi = 3/5$ [34]. So, for scale invariant power spectra,

$$\tilde{C}_\ell \simeq \tilde{C}_{\ell,I}(q) + \tilde{C}_{\ell,S} \simeq \frac{2\pi}{\ell(\ell+1)} \left[P_I(q) + \left(\frac{3}{5}\right)^2 P_\zeta \right]. \tag{4.8}$$

The second term can be compared to the SW contribution to the CMB anisotropies. In that case, the final temperature anisotropy is 1/3 times the scalar perturbation at the last scattering surface, while Φ at that moment decreased by a factor 9/10 in the transition from radiation to matter domination [34]. With this in mind, the second term in (4.8) leads to $C_\ell^{\rm SW} = (3/10)^2 \tilde{C}_{\ell,S}$, in agreement with the CMB literature. On the other hand, if the two contributions are correlated, as it would be the case for adiabatic initial condition for Γ_I, then both terms in (4.8) contribute to the SW effect for the SGWB.

B. Angular bispectrum of GW energy density

The characterization of the non-Gaussian properties of the SGWB is a potential tool to discriminate whether a SGWB has a primordial or astrophysical origin. The primordial

3-point function of the GW field, $\langle h^3 \rangle$, is unobservable, due to the decoherence of the associated phase (because of the propagation, and the finite duration of the measurement [23,24]), with, possibly, the exception of very specific shapes [30,39]. It is more convenient to consider the non-Gaussianity associated with the GW energy density angular distribution, which is not affected by this problem [25]. This gives rise to the bispectra in (4.4), which we evaluate now.

As we did for the power spectrum, also in this case we start from the initial condition term. Combining the first of (3.14) and the first of (4.2) leads to

$$\left\langle \prod_{i=1}^{3} \Gamma_{\ell_i m_i, I}(q) \right\rangle = \prod_{i=1}^{3} \left[4\pi (-i)^{\ell_i} \int \frac{d^3 k_i}{(2\pi)^3} Y^*_{\ell_i m_i}(\hat{k}_i) j_{\ell_i}(k_i(\eta_0 - \eta_{\rm in})) \right] B_I(q, k_1, k_2, k_3)(2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3). \tag{4.9}$$

We then use the representation of the Dirac δ function,

$$\delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) = \int \frac{d^3 y}{(2\pi)^3} e^{i(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)\cdot \vec{y}} = \int_0^\infty dy y^2 \int d\Omega_y \prod_{i=1}^{3} \left[2 \sum_{L_i M_i} i^{L_i} j_{L_i}(k_i y) Y^*_{L_i M_i}(\Omega_y) Y_{L_i M_i}(\hat{k}_i) \right], \tag{4.10}$$

and the orthonormality of the spherical harmonics, to arrive at

$$\left\langle \prod_{i=1}^{3} \Gamma_{\ell_i m_i, I}(q) \right\rangle = \mathcal{G}^{m_1 m_2 m_3}_{\ell_1 \ell_2 \ell_3} \int_0^\infty dr\, r^2 \prod_{i=1}^{3} \left[\frac{2}{\pi} \int dk_i k_i^2 j_{\ell_i}(k_i(\eta_0 - \eta_{\rm in})) j_{\ell_i}(k_i r) \right] B_I(q, k, k', k''), \tag{4.11}$$

where we have introduced the Gaunt integrals

$$\mathcal{G}^{m_1 m_2 m_3}_{\ell_1 \ell_2 \ell_3} \equiv \int d^2 \hat{n} Y_{\ell_1 m_1}(\hat{n}) Y_{\ell_2 m_2}(\hat{n}) Y_{\ell_3 m_3}(\hat{n}) = \sqrt{\frac{(2\ell_1 + 1)(2\ell_2 + 1)(2\ell_3 + 1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \tag{4.12}$$

We remark that also the bispectrum from the initial condition is also generally an $\mathcal{O}(1)$ dependence on the GW frequency.

An analogous computation leads to the contribution from the scalar modes

$$\left\langle \prod_{i=1}^{3} \Gamma_{\ell_i m_i, S} \right\rangle = \mathcal{G}^{m_1 m_2 m_3}_{\ell_1 \ell_2 \ell_3} \int_0^\infty dr r^2 \prod_{i=1}^{3} \left[\frac{2}{\pi} \int dk_i k_i^2 \mathcal{T}^S_{\ell_i}(k_i, \eta_0, \eta_{\rm in}) j_{\ell_i}(k_i r) \right] B_\zeta(k, k', k''). \tag{4.13}$$

For the tensor sourced contribution we have

$$\left\langle \prod_{i=1}^{3} \Gamma_{\ell_i m_i, T} \right\rangle = \sum_{\lambda = \pm 2} \prod_{i=1}^{3} \left[4\pi (-i)^{\ell_i} \int \frac{k_i^2 dk_i}{(2\pi)^3} \mathcal{T}^T_{\ell, i}(k_i, \eta_0, \eta_{\rm in}) \int d\Omega_{k_i}\, {}_{-\lambda} Y^*_{\ell_i m_i}(\Omega_{k_i}) \right] \left\langle \prod_{i=1}^{3} \xi_\lambda(\vec{k}_i) \right\rangle. \tag{4.14}$$

Following [40], in Appendix B we show that also this contribution can be cast in a similar form to the previous two terms:

$$\left\langle \prod_{i=1}^{3} \Gamma_{\ell_i m_i, T} \right\rangle = \mathcal{G}^{m_1 m_2 m_3}_{\ell_1 \ell_2 \ell_3} \left[\prod_{i=1}^{3} 4\pi (-i)^{\ell_i} \int \frac{k_i^2 dk_i}{(2\pi)^3} \mathcal{T}^T_{\ell, i}(k_i, \eta_0, \eta_{\rm in}) \right] \sum_{\lambda = \pm 2} \tilde{\mathcal{F}}^\lambda_{\ell_1 \ell_2 \ell_3}(k_1, k_2, k_3), \tag{4.15}$$

where

$$\tilde{\mathcal{F}}^\lambda_{\ell_1 \ell_2 \ell_3}(k_1, k_2, k_3) \equiv \sqrt{4\pi} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^{-1} \sum_{m_1, m_2, m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \left[\prod_{i=1}^{3} \int d\Omega_{k_i} \frac{{}_{-\lambda} Y^*_{\ell_i m_i}(\Omega_{k_i})}{\sqrt{2\ell_i + 1}} \right] \langle \xi_\lambda(\vec{k}_1) \xi_\lambda(\vec{k}_2) \xi_\lambda(\vec{k}_3) \rangle. \tag{4.16}$$

We remark once again that we have neglected for simplicity all the mixed scalar-tensor correlators.

C. Reduced bispectrum and estimation

The three contributions to the bispectrum found above have the correct form (4.3) as dictated by statistical isotropy. For convenience, we collect here the explicit form of the reduced bispectra contributing to (4.4)

$$\tilde{b}_{\ell_1\ell_2\ell_3,I} = \int_0^\infty dr\, r^2 \prod_{i=1}^3 \left[\frac{2}{\pi}\int dk_i k_i^2 j_{\ell_i}[k_i(\eta_0-\eta_{\rm in})] j_{\ell_i}(k_i r)\right] B_I(q,k_1,k_2,k_3),$$

$$\tilde{b}_{\ell_1\ell_2\ell_3,S} = \int_0^\infty dr\, r^2 \prod_{i=1}^3 \left[\frac{2}{\pi}\int dk_i k_i^2 \mathcal{T}^S_{\ell_i}(k_i,\eta_0,\eta_{\rm in}) j_{\ell_i}(k_i r)\right] B_\zeta(k,k',k''),$$

$$\tilde{b}_{\ell_1\ell_2\ell_3,T} = \frac{4}{\pi^2}\sum_{\lambda=\pm2}\sum_{m_i}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0 & 0\end{pmatrix}^{-2}\mathcal{G}^{m_1m_2m_3}_{\ell_1\ell_2\ell_3}\left[\prod_{i=1}^3 \frac{(-i)^{\ell_i}}{2\ell_i+1}\int d^3k_i \mathcal{T}^T_{\ell,i}(k_i)_{-\lambda} Y^*_{\ell_i m_i}(\Omega_{k_i})\right]$$
$$\times\,\delta(\vec{k}+\vec{k}'+\vec{k}'')B_\lambda(\vec{k},\vec{k}',\vec{k}''). \tag{4.17}$$

To estimate the SGWB bispectrum, we consider only the scalar source contribution $\tilde{B}_{\ell_1\ell_2\ell_3,S}$ and we assume the simplest small nonlinear coupling *local* ansatz for the curvature perturbation

$$\zeta(\vec{x}) = \zeta_g(\vec{x}) + \frac{3}{5} f_{\rm NL}\zeta_g^2(\vec{x}), \tag{4.18}$$

where $\zeta_g(\vec{x})$ denotes the linear Gaussian part of the perturbation. With the local ansatz, the bispectrum of the scalar perturbations assumes the form [41,42]

$$B_\zeta(k_1,k_2,k_3) = \frac{6}{5} f_{\rm NL}\left[\frac{2\pi^2}{k_1^3}P_\zeta(k_1)\frac{2\pi^2}{k_2^3}P_\zeta(k_2) + 2\ \text{permutations}\right]. \tag{4.19}$$

We insert this in the second line of (4.17) and we assume a matter transfer function $T_\Phi(\eta,k) = T_\Psi(\eta,k) = 3/5g(\eta)$ with the growth factor $g(\eta) = 1$ and a scale invariant spectrum for the primordial curvature fluctuations. We can then integrate over one of the internal momenta k_i,

$$\frac{2}{\pi}\int dk\, k^2 j_\ell(k\eta_0) j_\ell(kr)|_{\ell\gg1} = \frac{\delta(\eta_0 - r)}{\eta_0^2}. \tag{4.20}$$

The relation (4.20) is exact if k ranges up to infinity, which is not the case for the innermost momentum [as the integral (4.20) is performed first, this will necessarily be the momentum that we order to be the innermost one], due to the triangular inequalities associated with the bispectrum. The condition $\ell \gg 1$ ensures that the support of the integration occurs at sufficiently small k, so that the relation (4.20) becomes exact at large ℓ. The result then allows us to immediately perform the integral over r. We then find that the reduced bispectrum from the scalar contribution, assuming that the SW is the dominant contribution, is

$$\tilde{b}_{\ell_1\ell_2\ell_3,S} = \frac{162}{625} f_{\rm NL}\left(4\pi\int\frac{dk_1}{k_1} j^2_{\ell_1}(k_1\eta_0)\mathcal{P}_\zeta(k_1)\right) \times\left(4\pi\int\frac{dk_2}{k_2} j^2_{\ell_2}(k_2\eta_0)\mathcal{P}_\zeta(k_2)\right) + 2\ \text{permutations}. \tag{4.21}$$

This result can also be written in terms of the 2-point functions found in Eq. (4.7):

$$\tilde{b}_{\ell_1\ell_2\ell_3,S} \simeq 2 f_{\rm NL}[\tilde{C}_{\ell_1,S}\tilde{C}_{\ell_2,S} + \tilde{C}_{\ell_1,S}\tilde{C}_{\ell_3,S} + \tilde{C}_{\ell_2,S}\tilde{C}_{\ell_3,S}], \tag{4.22}$$

which resembles the one for the CMB angular bispectrum in the Sachs-Wolfe regime [41]. So, the SGWB bispectrum is specified by the $f_{\rm NL}$ parameter and the angular spectrum. Also in this estimate we neglected a possible correlation between the initial and scalar source contributions that should be taken into account when, for instance, Γ_I is controlled by the adiabatic scalar perturbation (see [25] for an example).

V. AN EXAMPLE: THE AXION-INFLATION CASE

The goal of this section is to understand under which conditions the initial term $\Gamma_I(q)$ has a nontrivial q dependence that distinguishes it from the other contributions to the anisotropy. There are several mechanisms for the generation of a cosmological GW signal visible at interferometer scales (see [8–10] for recent review). In this section we focus on a specific mechanism: we consider the case where an axion inflaton ϕ sources gauge fields, which in turn generates a large GW background. In particular we consider the specific evolution shown in Fig. 4 of [43], where the inflaton potential is chosen so to lead to a peak in the GW signal at LISA frequencies, without overproducing scalar perturbations and primordial black holes. The amount of GWs sourced in this mechanism is controlled by the parameter $\xi \equiv (\dot{\phi}/2f_\phi H)$, where f_ϕ is the decay constant of the axion inflaton. The present fractional energy in GW, $\Omega_{\rm GW}(\eta_0, q)$, is related to the primordial GW power spectrum $P_\lambda(\eta_{\rm in}, q)$ by

$$\Omega_{\rm GW}(\eta_0, q) = \frac{3}{128}\Omega_{\rm rad}\sum_\lambda P_\lambda(\eta_{\rm in}, q) \times \left[\frac{1}{2}\left(\frac{q_{\rm eq}}{q}\right)^2 + \frac{4}{9}(\sqrt{2}-1)\right]. \quad (5.1)$$

This relation, taken from [10], interpolates between large and small scales. Since we are interested in the modes with $q \gg q_{\rm eq}$, that entered the horizon during radiation domination, we consider only the second term in the square bracket, and we find

$$\Omega_{\rm GW}(\eta_0, q) = \text{constant} \times \sum_\lambda P_\lambda(\eta_{\rm in}, q), \quad (5.2)$$

and, as we will see, the constant term is not relevant for our computation.

We are interested in the contribution from the initial condition $\Gamma_{\rm in}$. So we can set the long modes $\zeta(\vec{k}) = h_\lambda(\hat{k}) = 0$ in this discussion. We therefore assume that the value of the energy density that arrives to the location $\vec{x}$ from the direction $\hat{n}$ is controlled by the parameter

$$\xi = \bar{\xi} + \delta\xi(\vec{x} + d\hat{n}). \quad (5.3)$$

In this relation, ξ is the value that this parameter had during inflation at the location $\vec{x} + d\hat{n}$, where d is the distance covered by the gravitons between the initial and the present time (equal for all directions, since we are disregarding the effect of the long scale modes ζ; we note that these modes will contribute to the term Γ_S, that we are not discussing in this section). In writing this relation, we have assumed that the parameter ξ is in turn controlled by a dynamical field (the rolling axion, in the example of [43]), which results in the background value $\bar{\xi}$, and in the perturbation $\delta\xi$.

We then generalize the relation (5.1) to

$$\omega_{\rm GW}(\eta_0, \vec{x}, q, \hat{n}) = \text{constant} \times \sum_\lambda P_\lambda(q, \xi(\eta_0, \vec{x}, \hat{n})), \quad (5.4)$$

which has the background value $\bar{\Omega}_{\rm GW}(\eta_0, q) = \text{constant} \times \sum_\lambda P_\lambda(q, \bar{\xi})$. The constant factor drops in the ratio

$$4 - \frac{\partial \ln \bar{\Omega}_{\rm GW}(\eta_0, q)}{\partial \ln q} = 4 - \frac{\partial \ln [\sum_\lambda P_\lambda(q, \bar{\xi})]}{\partial \ln q}, \quad (5.5)$$

as well as in

$$\delta_{\rm GW}(\eta_0, \vec{x}, q, \hat{n}) = \frac{\sum P_\lambda(q, \xi(\eta_0, \vec{x}, \hat{n})) - \sum P_\lambda(q, \bar{\xi})}{\sum P_\lambda(q, \bar{\xi})} = \frac{\partial \ln [\sum_\lambda \ln P_\lambda(q, \bar{\xi})]}{\partial \bar{\xi}}\delta\xi(\vec{x} + d\hat{n}), \quad (5.6)$$

where we have expanded the GW primordial power spectrum to linear order in $\delta\xi$. In this way, the relation (2.9) can be recast in the form

$$\Gamma_I(\eta_0, \vec{x}_0, q, \hat{n}) \equiv \mathcal{F}(q, \bar{\xi})\delta\xi(\vec{x}_0 + d\hat{n}), \quad (5.7)$$

with

$$\mathcal{F}(q, \bar{\xi}) \equiv \frac{1}{4 - n_T}\frac{\partial \sum_\lambda [\ln P_\lambda(q, \bar{\xi})]}{\partial \bar{\xi}}, \qquad n_T \equiv \frac{\partial \ln [\sum_\lambda P_\lambda(q, \bar{\xi})]}{\partial \ln q}, \quad (5.8)$$

where we have also made use of the standard definition of the tensor spectral tilt n_T.

The question of whether we have or have not spectral distortion depends on whether the quantity $\mathcal{F}(q, \bar{\xi})$ is or is not q dependent. This provides an immediate criterion for evaluating whether and how much the GW anisotropies depend on frequency (as, in principle, one could imagine a GW power spectrum for which the dependence on q of $\mathcal{F}$ vanishes, or is extremely suppressed). This conclusion only assumes that the primordial GW signal is a function of some additional parameter ξ which has small spatial inhomogeneities, and therefore it likely applies to several other mechanisms.

We show in Fig. 1 the evolution of the function $\mathcal{F}$ corresponding to the GW production shown in Fig. 4 of [43]. We see that indeed this quantity presents a nontrivial

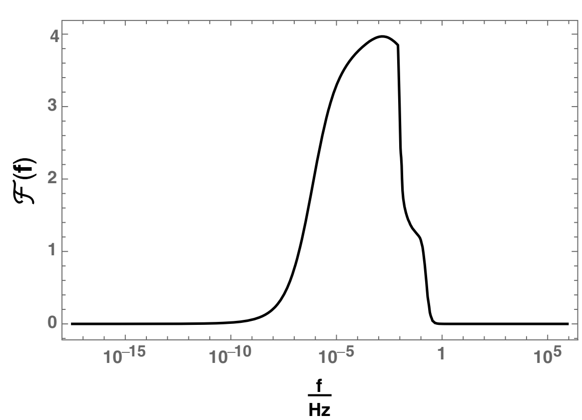

FIG. 1. Quantity $\mathcal{F}$ as a function of the frequency $f = q/2\pi$ of the GW signal for the model of axion inflation described in the text.

scale dependence, and therefore the correlators of the anisotropies will be different at different frequencies.

VI. SQUEEZED LIMIT AND CONSISTENCY RELATIONS OF THE SGWB

Nonlinear effects associated with the propagation of interacting GWs in a nonlinear universe lead to nonvanishing connected n-point functions even in absence of intrinsic, primordial non-Gaussianity. In particular, the squeezed limit of bispectra associated with GW observables should acquire a nonvanishing value, and satisfy consistency relations that resemble Maldacena's consistency relations [44]. This is analogous to what happens for CMB [45–47].

In this section we compute the squeezed limit of the bispectrum for the graviton distribution function in the case of adiabatic fluctuations. As in Sec. II, we write in momentum space

$$\omega_{\rm GW}(\eta, k^i, q, n^j) = \bar{\omega}_{\rm GW}(\eta, q)[1 + \delta_{\rm GW}(\eta, k^i, q, n^i)], \tag{6.1}$$

where $\bar{\omega}_{\rm GW}(\eta, q)$ is associated with the energy density of the isotropic SGWB. This quantity depends on time η and on the GW momentum q. Small anisotropies of the SGWB are controlled by the quantity $\delta_{\rm GW}$ given in Eq. (2.9). We rewrite it here, expressing it in terms of the function $\bar{f}(q)$:

$$\delta_{\rm GW}(\eta, \vec{k}, q, \vec{n}) = -\frac{\partial \ln \bar{f}(q)}{\partial \ln q} \Gamma_S(\eta, \vec{k}, q, \vec{n}), \tag{6.2}$$

where recall that Γ_S controls the fluctuations in the distribution function (see the definitions in Sec. II). In this section we focus on the contribution due to scalar fluctuations. We assume there is no anisotropic stress, and that scalar perturbations in Newtonian gauge satisfy the adiabaticity condition:

$$\Phi(\eta, \vec{k}) = \Psi(\eta, \vec{k}) = \frac{3}{5} g(\eta) \zeta(\vec{k}), \tag{6.3}$$

where $g(\eta)$ is a function mapping the superhorizon seed [controlled by $\zeta(\vec{k})$] to the scalar fluctuations at small scales (see e.g., [48,49]). It is generally time dependent although it is equal to unity in pure matter domination. Then the contribution $\Gamma_{\rm S}$ reads [see Eq. (3.9)]

$$\begin{aligned} \Gamma_{\rm S}(\eta, \vec{k}, \hat{n}) &= \frac{3}{5} \int_{\eta_{\rm in}}^{\eta} d\eta' e^{-ik\mu(\eta-\eta')} \\ &\quad \times \left[\delta(\eta' - \eta_{\rm in}) g(\eta') + \frac{1}{2} \partial_{\eta'} g(\eta') \right] \zeta(\vec{k}), \\ &\equiv T_S(\eta, k, \mu) \zeta(\vec{k}), \end{aligned} \tag{6.4}$$

where $\mu = \hat{n} \cdot \hat{k}$ and T_S is the definition for the scalar transfer function we adopt here. In matter domination this becomes

$$T_S = \frac{3}{5} e^{-ik\mu(\eta-\eta_{\rm in})}. \tag{6.5}$$

Notice that $\Gamma_{\rm S}$ does not depend on q. Assembling the definitions above, we can then write

$$\delta_{\rm GW}(\eta, \vec{k}, q, \vec{n}) = -\frac{\partial \ln \bar{f}(q)}{\partial \ln q} T_S(\eta, k, \mu) \zeta(\vec{k}). \tag{6.6}$$

Indicating with P_Γ the power spectrum, we can write the 2-point correlators in momentum space as

$$\begin{aligned} \langle \Gamma(\eta, \vec{k}_1, q, \hat{n}) \Gamma(\eta, \vec{k}_2, q, \hat{n}) \rangle' &= \frac{2\pi^2}{k_1^3} P_\Gamma(\eta, k_1, q, \hat{n}) \\ &= \frac{2\pi^2}{k_1^3} |T_S(\eta, k_1, \mu_1)|^2 P_\zeta(k_1), \end{aligned} \tag{6.7}$$

where a prime $'$ corresponds to correlators understanding the $(2\pi)^3 \delta(\sum \vec{k}_i)$ factor. Then,

$$\begin{aligned} P_\Gamma(\eta, k, \mu) &= \frac{\langle \Gamma_S(\eta, k, \mu) \Gamma_S(\eta, k', \mu) \rangle'}{2\pi^2/k^3} \\ &= |T_S(\eta, k, \mu)|^2 P_\zeta(k), \end{aligned} \tag{6.8}$$

$$P_{\delta_{\rm GW}}(\eta, k, q, \mu) = \left| \frac{\partial \ln \bar{f}(q)}{\partial \ln q} T_S(\eta, k, \mu) \right|^2 P_\zeta(k). \tag{6.9}$$

In matter domination, as we learned above, $|T_S|^2 = 9/25$, but in general $|T_S|^2$ can depend on η, k, $\hat{n}$.

In what follows, we study how the 2-point correlation functions of SGWB anisotropies, when evaluated at small scales k, are modulated by the presence of a long-scale mode $\zeta_L \equiv \zeta(\vec{k}_L)$, with $|\vec{k}_L| \ll |\vec{k}|$. Such modulation induces a nonvanishing squeezed limit for the 3-point function of $\delta_{\rm GW}$. The anisotropies $\delta_{\rm GW}$ depend on various quantities, (η, k, q, μ), which can be sensitive in a different way to the long mode. We use the systematic approach pioneered by Weinberg [31] that unambiguously associates the effects of a long mode with an appropriate coordinate transformation. We shall closely follow the treatment of [46], which develops the arguments of [31] for the case of CMB, applying it to the SGWB (for similar approaches see also [45,47]).

A. Long wavelength modes as coordinate transformations

We discuss how to identify the effects of a long mode with an appropriate coordinate transformation. We limit our attention to effects due to scalar fluctuations. The metric including long-wavelength scalars in Poisson gauge is

$$ds^2 = a^2(\eta)[-(1+2\Phi_L)d\eta^2 + (1-2\Psi_L)\delta_{ij}dx^i dx^j]. \tag{6.10}$$

We assume that the long-scale mode depends on a momentum $\vec{k}_L$, with magnitude much smaller than that of the momentum of the short-scale modes introduced in Eq. (6.17), but with a certain direction, and we discuss how the quantities (η, k, q, μ), transform under a coordinate redefinition adsorbing the long modes. We start by noticing that the following coordinate transformation preserves the Poisson gauge structure (ζ_L indicates the long mode of curvature fluctuations at large scales, responsible for the modulation effects):

$$\hat{\eta} = \eta + \epsilon(\eta)\zeta_L, \tag{6.11}$$

$$\hat{x}^i = x^i(1-\lambda\zeta_L), \tag{6.12}$$

with λ constant. After performing such gauge transformation,

$$\begin{aligned} \hat{\Phi}_L &= \Phi_L - \epsilon'\zeta_L - \mathcal{H}\epsilon\zeta_L, \\ \hat{\Psi}_L &= \Psi_L - \lambda\zeta_L + \mathcal{H}\epsilon\zeta_L, \end{aligned} \tag{6.13}$$

we can "gauge away" the long wavelength scalar modes making the gauge choice

$$\begin{aligned} \Phi_L &= (\epsilon' + \mathcal{H}\epsilon)\zeta_L, \\ \Psi_L &= (\lambda - \mathcal{H}\epsilon)\zeta_L, \end{aligned} \tag{6.14}$$

so that in the hat coordinates the metric is purely FRW with no long-wavelength perturbations. As explained in [31,46], in order to be consistent with the small k limit of Einstein equations, we need to impose the conditions (in absence of anisotropic stress)

$$\begin{aligned} \lambda &= 1, \\ \epsilon(\eta) &= \frac{1}{a^2(\eta)}\int_{\eta_*}^{\eta} d\eta' a^2(\eta'), \end{aligned} \tag{6.15}$$

where η_* is some initial reference time. Equation (6.15) immediately leads to the equality

$$\epsilon' = -2\mathcal{H}\epsilon + 1. \tag{6.16}$$

After performing the coordinate redefinition (6.11), (6.12), we can write a metric containing short-wavelength scalar fluctuations in Poisson gauge "on top" of long fluctuations:

$$ds^2 = a^2(\hat{\eta})[-(1+2\hat{\Phi}_S)d\hat{\eta}^2 + (1-2\hat{\Psi}_S)\delta_{ij}d\hat{x}^i d\hat{x}^j]. \tag{6.17}$$

In fact, such metric contains the long-scale modes within the definition of the hat coordinates. We can then express the perturbations in terms of the original coordinates (η, x^i) using again relations (6.11), (6.12). Such operation teach us how the short wavelength modes are modulated by the long wavelength ones:

$$\hat{\Phi}_S = \Phi_S + \Phi_L + 2\Phi_S\Phi_L + \epsilon\zeta_L\frac{\partial\Phi_S}{\partial\eta} - \lambda\zeta_L x^i\frac{\partial\Phi_S}{\partial x^i}, \tag{6.18}$$

$$\hat{\Psi}_S = \Psi_S + \Psi_L - 2\Psi_S\Psi_L + \epsilon\zeta_L\frac{\partial\Psi_S}{\partial\eta} - \lambda\zeta_L x^i\frac{\partial\Psi_S}{\partial x^i}. \tag{6.19}$$

Importantly, the short modes acquire a second-order correction due to long modes. As we shall discuss in what comes next, these nonlinear, higher-order corrections modulate the 2-point function for short modes, and lead to a nonvanishing squeezed limit for the 3-point function.

As a concrete example, that we shall use in what follows, we can consider the case of constant proportionality between pressure and energy density, $p = w\rho$. Being in this case $a(\eta) \propto \eta^{2/(1+3w)}$, $\mathcal{H} = 2/[\eta(1+3w)]$ we get

$$\epsilon(\eta)\zeta_L = \frac{1+3w}{5+3w}\eta\zeta_L, \tag{6.20}$$

and

$$\mathcal{H}\epsilon = \frac{2}{5+3w}, \tag{6.21}$$

which, for matter domination, gives $\mathcal{H}\epsilon = 2/5$.

We also need to evaluate how the Fourier transform of a function $f(x^i)$ changes under a rescaling of spatial coordinates, as in Eq. (6.12). We find that if we apply a constant rescaling of spatial coordinates

$$f(x^i) \to f(x^i(1-\lambda\zeta_L)) \tag{6.22}$$

to a function f, then its Fourier transform, given by

$$f(x^i) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}}\tilde{f}(k^j),$$

transforms as (at first order in a ζ_L expansion)

$$\begin{aligned} f(x^i(1-\lambda\zeta_L)) &= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}(1-\lambda\zeta_L)}\tilde{f}(k^j) \\ &= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}}[(1+3\lambda\zeta_L)\tilde{f}(k^j(1+\lambda\zeta_L))]. \end{aligned} \tag{6.23}$$

This implies that up to first order in ζ_L, under the coordinate transformation we are interested in, we have

$$\tilde{f}(k^j) \to (1+3\lambda\zeta_L)\tilde{f}(k^j(1+\lambda\zeta_L)) \Rightarrow \tilde{f}(k^j) \to \tilde{f}(k^j) + 3\lambda\zeta_L\tilde{f}(k^j) + \lambda\zeta_L k^m \frac{\partial \tilde{f}(k^j)}{\partial k^m}. \tag{6.24}$$

As a last step, we now investigate how to transform the coordinates $(q, \hat{n}^i)$ that control the GW four-momentum. At first order, neglecting tensors, the GW four-momentum components are given by

$$P^0 = \frac{q}{a^2(\eta)} e^{-\Phi}, \qquad P^i = \frac{q}{a^2(\eta)} n^i e^{\Psi}. \tag{6.25}$$

We wish to express the previous quantities in terms of hat coordinates, including the effects of the long modes. In particular, we are interested in determining the quantities $\hat{q}$ and $\hat{n}^i$ that are contained into the GW four-momentum, when it is expressed in terms of hat coordinates. We use the fact that P^μ is a vector, transforming in the usual way under coordinate transformations [in particular transformations (6.11), (6.12)]. Using this fact, we find

$$\frac{\hat{q}}{a^2(\hat{\eta})} = (1+\epsilon'\zeta_L)\frac{q}{a^2(\eta)} e^{-\Phi_L}, \tag{6.26}$$

$$\frac{\hat{q}}{a^2(\hat{\eta})}\hat{n}^i = (1-\lambda\zeta_L)\frac{q}{a^2(\eta)} n^i e^{\Psi_L}. \tag{6.27}$$

Condition (6.26) gives, at first order in the long-scale modes,

$$\begin{aligned}\hat{q} &= \frac{a^2(\hat{\eta})}{a^2(\eta)}(1+\epsilon'\zeta_L)(1-\Phi_L)q \\ &= (1+2\mathcal{H}\epsilon\zeta_L+\epsilon'\zeta_L-\Phi_L)q \\ &= \left[1+\left(1-\frac{3}{5}g(\eta)\right)\zeta_L\right]q.\end{aligned} \tag{6.28}$$

On the other hand, condition (6.27) gives

$$\begin{aligned}\hat{n}^i &= \frac{a^2(\hat{\eta})}{a^2(\eta)}\frac{q}{\hat{q}}(1-\lambda\zeta_L)(1+\Psi_L)n^i \\ &= (1-\epsilon'\zeta_L-\lambda\zeta_L+2\Phi_L)n^i \\ &= \left[1-2\left(1-\mathcal{H}\epsilon(\eta)-\frac{3}{5}g(\eta)\right)\zeta_L\right]n^i.\end{aligned} \tag{6.29}$$

These are the results that we need. It is convenient to write more compact expressions as

$$\begin{aligned}\hat{q} &= (1+\beta_q(\eta)\zeta_L)q, \\ \hat{n}^i &= (1+\beta_n(\eta)\zeta_L)n^i,\end{aligned} \tag{6.30}$$

with $\beta_{q,n}$ functions of time

$$\begin{aligned}\beta_q(\eta) &= 1-\frac{3}{5}g(\eta), \\ \beta_n(\eta) &= -2\left(1-\mathcal{H}\epsilon(\eta)-\frac{3}{5}g(\eta)\right).\end{aligned} \tag{6.31}$$

In matter domination we find $\beta_q = 2/5$ and $\beta_n = 0$.

B. Coordinate transformations and the GW distribution function

We now apply the previous results to the problem at hand. We start by rewriting the GW energy density

$$\omega_{\rm GW}(\eta, k^i, q, n^i) = \bar{\omega}_{\rm GW}(\eta, q)[1+\delta_{\rm GW}(\eta, k^i, q, n^i)], \tag{6.32}$$

where

$$\bar{\omega}_{\rm GW}(\eta, q) = \frac{q^4}{a^4(\eta)\rho_{\rm crit}}\bar{f}(q), \tag{6.33}$$

and

$$\delta_{\rm GW}(\eta, \vec{k}, q, n^i) = -\frac{\partial \ln \bar{f}(q)}{\partial \ln q}\Gamma_S(\eta, \vec{k}, q, n^i). \tag{6.34}$$

We now transform each contribution in the previous formulas under the coordinate transformation discussed in Sec. VI A. The background quantities $\bar{\omega}_{\rm GW}$ and $\bar{f}(q)$ transform as

$$\bar{\omega}_{\rm GW}(\eta, q) \Rightarrow \bar{\omega}_{\rm GW}(\hat{\eta}, \hat{q}) = \bar{\omega}_{\rm GW}(\eta, q) \times \left[1+4(\beta_q-\mathcal{H}\epsilon)\zeta_L+\beta_q\frac{\partial \ln \bar{f}(q)}{\partial \ln q}\zeta_L\right], \tag{6.35}$$

where

$$\frac{\partial \ln \bar{f}(q)}{\partial \ln q} \Rightarrow \frac{\partial \ln \bar{f}(\hat{q})}{\partial \ln \hat{q}} = \frac{\partial \ln \bar{f}(q)}{\partial \ln q} + \beta_q(\eta)\frac{\partial^2 \ln \bar{f}(q)}{\partial (\ln q)^2}\zeta_L. \tag{6.36}$$

The quantity Γ_S is mapped to

$$\Gamma_S(\hat{\eta}, \hat{k}^i, \hat{q}, \hat{n}^i) = (1+3\zeta_L)\Gamma_S(\eta+\epsilon(\eta)\zeta_L, \vec{k}(1+\zeta_L), (1+\beta_q(\eta)\zeta_L)q, (1+\beta_n(\eta)\zeta_L)n^i), \tag{6.37}$$

that, expanded at linear order in ζ_L, becomes

$$\Gamma_S(\eta, k^i, q, n^i) \Rightarrow \Gamma_S(\hat{\eta}, \hat{k}^i, \hat{q}, \hat{n}^i) = (1+3\zeta_L)\Gamma_S(\eta, k^i, q, n^i) + \frac{\partial \Gamma_S}{\partial \eta}\epsilon(\eta)\zeta_L + k^i \frac{\partial \Gamma_S}{\partial k^i}\zeta_L + \beta_q(\eta)\frac{\partial \Gamma_S}{\partial \ln q}\zeta_L + \beta_n(\eta) n^j \frac{\partial \Gamma_S}{\partial n^j}\zeta_L. \tag{6.38}$$

We now assemble the results obtained. The SGWB energy density, including anisotropies, is modulated by the long mode ζ_L as

$$\omega_{\rm GW}(\hat{\eta}, \hat{k}^i, \hat{q}, \hat{n}^i) = \bar{\omega}_{GW}(\eta, q)\left[1 + \delta_{\rm GW} + \left(4\beta_q - 4\mathcal{H}\epsilon + \beta_q \frac{\partial \ln \bar{f}(q)}{\partial \ln q}\right)\zeta_L + \left(3 + \beta_q(\eta)\frac{\partial \ln q}{\partial \ln \bar{f}(q)}\frac{\partial^2 \ln \bar{f}(q)}{\partial (\ln q)^2}\right)\zeta_L \delta_{\rm GW} + \left(\frac{\partial \ln \Gamma}{\partial \eta}\epsilon(\eta) + k^i \frac{\partial \ln \Gamma}{\partial k^i} + \beta_n(\eta)\frac{\partial \ln \Gamma}{\partial \ln \mu}\right)\zeta_L \delta_{\rm GW}\right]. \tag{6.39}$$

Equation (6.39) is the basic expression that we need: all quantities on the right-hand side are evaluated in terms of the original coordinates without the hat. Notice that *even in absence of intrinsic small-scale anisotropies*, the GW energy density is modulated by the long mode: a dependence on ζ_L is indeed still present by setting $\delta_{\rm GW} = 0$ in Eq. (6.39). This is the effect studied by Alba and Maldacena [14]. For example, in pure matted domination, we have $\beta_q = \mathcal{H}\epsilon = 2/5$. Setting $\delta_{\rm GW} = 0$, Eq. (6.39) simply becomes

$$\hat{\omega}_{\rm GW}(\hat{\eta}, \hat{k}^i, \hat{q}, \hat{n}^i) = \bar{\omega}_{GW}(\eta, q)\left(1 + \frac{2}{5}\frac{\partial \ln \bar{f}(q)}{\partial \ln q}\zeta_L\right). \tag{6.40}$$

In this case, the modulation of $\omega_{\rm GW}$ is then controlled by the momentum dependence of the function $\bar{f}(q)$, associated with the isotropic distribution function of the SGWB energy density [14].

C. The squeezed limit of 3-point correlation functions

We now apply the general result of (6.39) to study how correlation functions of small-scale GW anisotropies $\delta_{\rm GW}$ are influenced by the long mode. We start by studying how 2-point correlation functions are modulated by ζ_L; we continue investigating the squeezed limit of the 3-point correlation functions.

Using Eq. (6.39), we find the result[1]

$$\langle \hat{\delta}_{\rm GW}(\vec{k}_1)\hat{\delta}_{\rm GW}(\vec{k}_2)\rangle' = (1 + \mathcal{M}\zeta_L)\langle \delta_{\rm GW}(\vec{k}_1)\delta_{\rm GW}(\vec{k}_2)\rangle', \tag{6.41}$$

where the modulating factor $\mathcal{M}$ reads

$$\mathcal{M} = 6 + 2\beta_q(\eta)\frac{\partial \ln q}{\partial \ln \bar{f}(q)}\frac{\partial^2 \ln \bar{f}(q)}{\partial (\ln q)^2} + \epsilon(\eta)\frac{\partial \ln \langle \Gamma_S(\vec{k}_1)\Gamma_S(\vec{k}_2)\rangle'}{\partial \eta} + k_1^i \frac{\partial \ln \langle \Gamma_S(\vec{k}_1)\Gamma_S(\vec{k}_2)\rangle'}{\partial k_1^i} + k_2^i \frac{\partial \ln \langle \Gamma_S(\vec{k}_1)\Gamma_S(\vec{k}_2)\rangle'}{\partial k_2^i} + \beta_q(\eta)\frac{\partial \ln \langle \Gamma_S(\vec{k}_1)\Gamma_S(\vec{k}_2)\rangle'}{\partial \ln q} + \beta_n(\eta) n^j \frac{\partial \ln \langle \Gamma_S(\vec{k}_1)\Gamma_S(\vec{k}_2)\rangle'}{\partial n^j}. \tag{6.42}$$

Notice that the contributions in the first line of Eq. (6.39) that depend only on the long mode (without being weighted by $\delta_{\rm GW}$) do not contribute to $\mathcal{M}$. Therefore they do not modulate the short-mode 2-point function.

We now apply to the results derived above the definitions of $\delta_{\rm GW}$ and Γ power spectra, Eqs. (6.8), (6.9). We find the following expression for the modulation of the power spectrum due to a long mode:

$$P_{\hat{\delta}_{\rm GW}}(\eta, k, q, \hat{n}, \vec{k}_L) = \left[1 + 2\frac{\partial \ln P_\zeta}{\partial \ln k}\zeta_{\vec{k}_L} + 2\beta_q(\eta)\frac{\partial \ln q}{\partial \ln \bar{f}(q)}\frac{\partial^2 \ln \bar{f}(q)}{\partial (\ln q)^2}\zeta(\vec{k}_L) + \left(\epsilon(\eta)\frac{\partial \ln |T_S|^2}{\partial \eta} + \frac{\partial \ln |T_S|^2}{\partial \ln k} + \beta_n(\eta)\frac{\partial \ln |T_S|^2}{\partial \ln \mu}\right)\zeta(\vec{k}_L)\right] P_{\delta_{\rm GW}}(\eta, k, q, \hat{n}). \tag{6.43}$$

[1]Each quantity is evaluated at the same value of η, q, n^i; hence, we understand such dependence. Here we indicate with $\hat{\delta}_{\rm GW}$ the quantity that receives the long-mode modulation.

All quantities inside the square brackets in the right-hand side are again evaluated at the same values of η, $\hat{n}$, k; hence we understand this dependence. We find that the power spectrum of $\delta_{\rm GW}$ is modulated by the long mode $\zeta(\vec{k}_L)$ through three (physically distinct) effects:

(1) A modulation due to the scale dependence of the primordial curvature spectrum, as in Maldacena's consistency relation. This is contained in the first line of Eq. (6.43), second term in the right-hand side. (Notice that the contributions coming from derivatives of the $1/k^3$ factor cancel out, as expected.)

(2) A contribution due to the momentum dependence of the background distribution $\bar{f}(q)$. This is contained in the first line of Eq. (6.43), third term in the right-hand side. This is a close relative of the effect pointed out by Alba and Maldacena [14], although it is not exactly the same result because we find contributions depending on second derivatives of the function $\bar{f}(q)$.

(3) A contribution due to the time, scale, and direction dependence of the transfer function of scalar modes. This is contained in the second line of Eq. (6.43).

In the previous discussion we learned how the long mode modulates the 2-point function. This effect is expected to lead to a nonvanishing squeezed limit for the 3-point function involving the anisotropies $\delta_{\rm GW}$. Indeed, expressing a large scale limit of $\delta_{\rm GW}$ in terms of ζ as

$$\hat{\delta}_{\rm GW}(\eta, k_3^i, q, n^i) = -\frac{\partial \ln \bar{f}(q)}{\partial \ln q} T_S(\eta, k_3^i, \mu_3)\zeta(\vec{k}_3), \tag{6.44}$$

for a small $|\vec{k}_3|$, we can write the schematic relation (all $\delta_{\rm GW}$'s are evaluated at the same values of η, n^i, q so we understand their dependence)

$$\begin{aligned}\lim_{\vec{k}_3\to 0}\langle\hat{\delta}_{\rm GW}(\vec{k}_1)\hat{\delta}_{\rm GW}(\vec{k}_2)\hat{\delta}_{\rm GW}(\vec{k}_3)\rangle &= -\frac{\partial \ln \bar{f}(q)}{\partial \ln q} T_S(\eta, k_3^i, \mu_3)\langle\langle\hat{\delta}_{\rm GW}(\vec{k}_1)\hat{\delta}_{\rm GW}(\vec{k}_2)\rangle\zeta(\vec{k}_3)\rangle \\ &= -\frac{\partial \ln \bar{f}(q)}{\partial \ln q} T_S(\eta, k_3^i, \mu_3)\langle\langle\delta_{\rm GW}(\vec{k}_1)\delta_{\rm GW}(\vec{k}_2)\rangle(1+\mathcal{M}\zeta_L)\zeta(\vec{k}_3)\rangle \\ &= -\frac{\partial \ln \bar{f}(q)}{\partial \ln q} T_S(\eta, k_3^i, \mu_3)\mathcal{M}\langle\langle\delta_{\rm GW}(\vec{k}_1)\delta_{\rm GW}(\vec{k}_2)\rangle\langle\zeta_L\zeta(\vec{k}_3)\rangle,\end{aligned} \tag{6.45}$$

where in the second line we used Eq. (6.41). This nonvanishing result gives the squeezed limit of the 3-point function for $\delta_{\rm GW}$. We adopt the following definition[2] for the nonlinear parameter $f_{\rm NL}^{\delta_{\rm GW}}$:

$$\lim_{\vec{k}_3\to 0}\langle\delta_{\rm GW}(\vec{k}_1)\delta_{\rm GW}(\vec{k}_2)\delta_{\rm GW}(\vec{k}_3)\rangle = f_{\rm NL}^{\delta_{\rm GW}}\left(\frac{4\pi^4}{k_1^3 k_3^3}\right)P_{\delta_{\rm GW}}(k_1)P_\zeta(k_3). \tag{6.46}$$

In our case, using the previous results, we find

$$\begin{aligned} f_{\rm NL}^{\delta_{\rm GW}} = -\frac{\partial \ln \bar{f}(q)}{\partial \ln q} T_S(\eta, k_3, \mu_3)\bigg[&2\frac{\partial \ln P_\zeta}{\partial \ln k_1} + 2\beta_q(\eta)\frac{\partial \ln q}{\partial \ln \bar{f}(q)}\frac{\partial^2 \ln \bar{f}(q)}{\partial(\ln q)^2} \\ &+ \epsilon(\eta)\frac{\partial \ln|T_S|^2}{\partial\eta} + \frac{\partial \ln|T_S|^2}{\partial \ln k_1} + \beta_n(\eta)\frac{\partial \ln|T_S|^2}{\partial \ln \mu_1}\bigg],\end{aligned} \tag{6.47}$$

and we can apply to this result the very same considerations made after Eq. (6.43).

The formula simplifies considerably in the case of pure matter domination. In this case, $T_S = 3/5$, $\beta_q = \mathcal{H}\epsilon = 2/5$. Then,

$$f_{\rm NL}^{\delta_{\rm GW}} = -\frac{6}{5}\frac{\partial \ln \bar{f}(q)}{\partial \ln q}\frac{\partial \ln P_\zeta}{\partial \ln k_1} - \frac{12}{25}\frac{\partial^2 \ln \bar{f}(q)}{\partial(\ln q)^2}. \tag{6.48}$$

Recalling that $\bar{f}(q)$ is related with the GW isotropic energy density $\Omega_{\rm GW}$ by the relation

$$\frac{\partial \ln \bar{f}}{\partial \ln q} = \frac{\partial \ln \Omega_{GW}}{\partial \ln q} - 4, \tag{6.49}$$

the nonlinearity parameter $f_{\rm NL}^{\delta_{\rm GW}}$ can then be enhanced in proximity to large values of second derivatives of $\Omega_{\rm GW}$ as a function of the scale q.

As an illustrative toy model which demonstrates this effect, we can consider a GW spectral density with the shape of a broken power law. The following parametrization for the spectral energy changes slope at a scale $q = q_\star$:

[2] We use $P_\zeta(k_3)$ instead of $P_{\delta_{\rm GW}}(k_3)$ in the next equation, in order to simplify the overall coefficients in the equations that come next. Recall that the definitions of P_ζ and $P_{\delta_{\rm GW}}$ are related by Eq. (6.9).

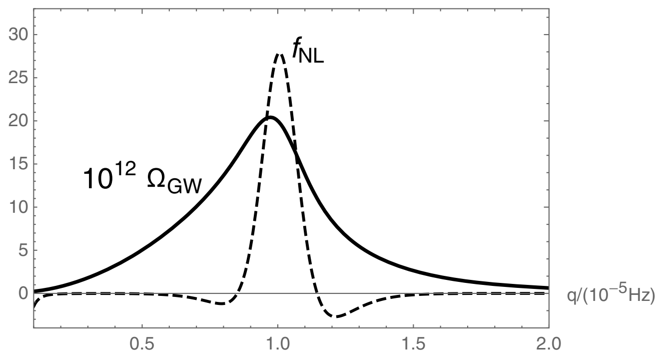

FIG. 2. Representation of the GW spectral density $\Omega_{\rm GW}$ and of $f_{\rm NL}^{\delta_{\rm GW}}$ for the model given in Eqs. (6.50), (6.51), choosing a scale invariant P_ζ. Notice that the magnitude of $f_{\rm NL}^{\delta_{\rm GW}}$ is amplified around the position where the spectral density changes slope. We have chosen the parameters $\alpha = 2$, $\beta = 5$, $\kappa_0 = 1/10$, $\Omega_0 = 10^{-12}$, $q_\star = 10^{-5}\ {\rm Hz}^{-1}$.

$$\Omega_{\rm GW}(q) = \frac{\Omega_0}{2}\left\{\left(\frac{q}{q_\star}\right)^{\alpha}\left[\tanh\left[\frac{(1-q/q_\star)}{\kappa_0}\right]+1\right] + \left(\frac{q}{q_\star}\right)^{-\beta}\left[\tanh\left[\frac{(q/q_\star-1)}{\kappa_0}\right]+1\right]\right\} \qquad (6.50)$$

with α, β positive numbers, while the functions inside the square brackets represent a regularization of twice the Heaviside function (that is approached when sending $\kappa_0 \to 0$). The function $\Omega_{\rm GW}$ has a large second derivative in proximity of the scale $q_\star$ where the change of slope occurs. The value of $f_{\rm NL}^{\delta_{\rm GW}}$ at $q_\star$ results (for a scale invariant spectrum of ζ)

$$f_{\rm NL}^{\delta_{\rm GW}} = \frac{3}{25}\frac{\alpha+\beta}{\kappa_0}(4-(\alpha+\beta)\kappa_0). \qquad (6.51)$$

Hence it can be enhanced taking small values of κ_0. See Fig. 2 for an illustration of this phenomenon, for a representative choice of parameters.

VII. DETECTABILITY OF ANISOTROPIC GWB

Prospects for direct measurement of the isotropic and anisotropic components of GWBs, both astrophysical and cosmological, by ongoing interferometers like LIGO-Virgo, and future space missions like LISA, DECIGO, and BBO have been performed in the literature (see e.g., [50–55]). While present GW interferometers are quite far from detecting the anisotropies of the SGWB since they are characterized by a poor angular resolution ($\ell \sim 4$–5) (see e.g., [1] for an analysis by the LIGO-Virgo Collaboration, and [56–62] using a map-making approach), future ground-based interferometers like ET and Cosmic Explorer could be sensitive to such a signal especially if it is characterized by a large monopole amplitude. To our knowledge, a detailed study for future ground-based interferometers is still missing, but it is plausible that a better angular resolution can be reached, just considering the longer baseline and better sensitivity of such detectors.

On the other hand, for the next-generation of space interferometers an analysis to quantify to which extent anisotropies of the GWB can be probed has been carried out in [55]. In these papers the strain sensitivity has been quantified for each detectable multipole moment and maps of the anisotropies have been produced. Fixing a SNR threshold, the minimum effective strain sensitivity, called $h^{\ell}_{\rm eff}(f)$, has been computed, taking into account the noise of detectors, the observation time, and some frequency interval. They find that future space interferometers may reach angular scales corresponding to $\ell \sim 8$–10 being also more sensitive to even multipoles than odd ones. Such results are ascribed to the properties of the response functions and to the geometric properties of the detectors. As a consequence, also the angular resolution of space interferometers is rather poor and the detectable multipole moments are very restrictive. However, some of the effects computed in this paper, that affect the SGWB of cosmological origin, like the Sachs-Wolfe and integrated Sachs-Wolfe, are effective on a very large scale, so a careful analysis is mandatory to really assess the possibility of their detection. Another relevant aspect to take into account is the amplitude of the monopole, which, as shown in Eq. (2.9), is a multiplicative factor in front of the observable spectrum. This means that, if we have primordial GW models which have a large (monopole) amplitude at scales probed by interferometers, this can increase the amplitude of the anisotropies at a level that can be probed by future GW detectors.

VIII. CONCLUSION

The amount of information extracted from the detection of GW signals by the LIGO-Virgo Collaboration has shown the power of gravitational waves to study an astrophysical compact object and to give relevant cosmological information on the late time universe. At the same level, the improving angular resolution of future GW detectors will allow one to extract precious information from the detection of the stochastic background of GWs generated both from the superposition of unresolved astrophysical sources and from cosmological sources, like inflation, phase transition, or topological defects. However, high sensitivity alone will not be sufficient for discriminating among different contributions. So it becomes necessary to characterize such backgrounds using observables that can give a clear hint about the origin of the signals. As recently studied, a parity-violating SGWB, which represents a smoking gun for some cosmological signals, can be probed using ground- and space-based interferometers [63]. Another important tool is the directionality dependence of the SGWB. As shown for astrophysical GW, the distribution of sources implies that the energy density is characterized by an anisotropic contribution beyond the isotropic one.

In the same way we expect that, analogously to CMB photons, also primordial GW are charaterized by anisotropies that can be generated both at the moment of production and during their propagation. In this paper we focused on the stochastic background of cosmological origin, and we studied the anisotropies due to the production mechanism (that we encode in an initial condition term) plus those generated from the propagation of GW on the perturbed universe, using a Boltzmann approach. We solved the Boltzmann equation for the graviton distribution function considering a FLRW metric with both scalar and tensor inhomogeneities. We showed that, contrary to CMB photons, at the moment of production, GWs, which are characterized by a nonthermal spectrum, generically result in angular anisotropies that have an order one dependence on the GW frequency. We provide a criterion to evaluate whether and how much the GW anisotropies depend on frequency. As an example, we evaluate this criterion in the case where an axion inflaton ϕ sources gauge fields, which in turn generates a large GW background.

Additional anisotropies are induced by the GW propagation in the large-scale scalar and tensor perturbations of the universe. We compute the angular power spectrum of the SGWB energy density, and, analogously to CMB photons, also the gravitons distribution function gets mainly affected by the Sachs-Wolfe effect on large scales, while the integrated Sachs-Wolfe effect is subdominant.

We then focused on a second observable that can be a crucial tool in discriminating an astrophysical from a cosmological background, namely its departure from Gaussian statistics. While we expect that the astrophysical background is Gaussian, due to central limit theorem, (some) cosmological backgrounds should show non-Gaussian statistics. We computed the 3-point function (bispectrum) of the SGWB energy density, which is not affected by de-correlation issues, both considering the effects at generation and due to propagation. We have shown that also the SGWB bispectrum carries a memory of the initial condition and that it is proportional to the non-Gaussianity of the scalar perturbations. In this sense, the SGWB can be used as a novel probe (beyond the CMB and the large scale structure) of the non-Gaussianity of the scalar perturbations.

Finally we considered nonlinear effects induced by long-wavelength scalar perturbations, which generate a modulation effect on the correlation functions of the short-wavelength modes. We identified the effects of long modes with an appropriate coordinate transformation and we computed the effect of nonlinearities in inducing a nonvanishing squeezed limit of the SGWB 3-point correlation function. We quantified the dependence of the squeezed bispectrum on the scale dependence of the spectrum of primordial scalar fluctuations similar to Maldacena consistency relation, on the momentum dependence of the background SGWB distribution function, and on the time, scale, and direction dependence of the scalar transfer function.

In summary, in this paper we have approached the possibility to use CMB techniques to describe the cosmological SGWB trying to characterize it using peculiar features that we do not expect to have in the astrophysical background. Of course the detectability with interferometers of such effects is one crucial step to address and we plan to work on it in a future paper. At the same time we also plan to analyze several additional physical effects that we have neglected in this first paper, like the effects of neutrinos on the GW amplitude or a possible direct dependence of Γ_I on $\hat{n}$, which would give distinctive signatures useful for the characterization.

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APPENDIX A: COMPUTATION OF THE TENSOR SOURCED TERM

In this Appendix we present the steps from Eq. (3.12) to Eq. (3.13) of the main text. The first goal is to obtain an explicit expression for the integrand in Eq. (3.11), when the integration variable $\vec{k}$ is oriented along the z axis. In the $\{+\times\}$ basis, related to the circular basis by

$$e_{ij,\lambda} \equiv \frac{e_{ij,+} + i\lambda e_{ij,\times}}{\sqrt{2}},$$

this orientation of $\vec{k}$ leads to

$$e_{ij,+}(\hat{k}_z) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$e_{ij,\times}(\hat{k}_z) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A1}$$

so that

$$\chi_{11}(\hat{k}_z) = -\chi_{22}(\hat{k}_z) = \chi(\eta, k)\frac{\xi_{-2}(\vec{k}) + \xi_2(\vec{k})}{2},$$
$$\chi_{12}(\hat{k}_z) = \chi_{21}(\hat{k}_z) = \chi(\eta, k)\frac{\xi_{-2}(\vec{k}) - \xi_2(\vec{k})}{2i}, \tag{A2}$$

while the other entries vanish.

We decompose the GW direction $\hat{n}$ in a basis having $\hat{k}$ as the z axis

$$\hat{n} = \left(\sqrt{1-\mu_{k,n}^2}\cos\phi_{k,n}, \sqrt{1-\mu_{k,n}^2}\sin\phi_{k,n}, \mu_{k,n}\right). \tag{A3}$$

In this basis

$$-\frac{n^i n^j}{2}\chi'_{ij}(\vec{k}=k\hat{k}_z) = -\frac{1-\mu_{k,n}^2}{4}\chi'(\eta,k) \times [\mathrm{e}^{2i\phi_{k,n}}\xi_2(\vec{k}) + \mathrm{e}^{-2i\phi_{k,n}}\xi_{-2}(\vec{k})]. \tag{A4}$$

Our goal is to compute

$$\Gamma_{\ell m,T} = \int\frac{d^3k}{(2\pi)^3}\mathrm{e}^{i\vec{k}\cdot\vec{x}_0}\int d^2\Omega_n \Gamma_T(\eta_0,\vec{k},\Omega_n)Y^*_{\ell m}(\Omega_n), \tag{A5}$$

with the knowledge that, when $\vec{k}$ is decomposed according to (A3) (namely, with $\hat{k}$ directed along thee z axis),

$$\Gamma_T(\eta_0,\vec{k},\Omega_{k,n}) = -\frac{1-\mu_{k,n}^2}{4}\sum_{\lambda=\pm2}\mathrm{e}^{i\lambda\phi_{k,n}}\xi_\lambda(\vec{k}) \times\int_{\eta_{\rm in}}^{\eta_0}d\eta\chi'(\eta,k)\mathrm{e}^{-i\mu_k(\eta_0-\eta)k}. \tag{A6}$$

We need to evaluate the integral (A5) for a generic orientation of $\vec{k}$. On the other hand, the explicit expression of the integrand (A6) holds only when $\vec{k}$ is oriented along the z axis. We cope with this by rotating the integrand of the $\int d^2\Omega_n$ integration into a basis in which the direction $\hat{n}$ is decomposed according to Eq. (A3).

To achieve this, we introduce the rotation matrix

$$S(\Omega_k) \equiv \begin{pmatrix} \cos\theta_k\cos\phi_k & -\sin\phi_k & \sin\theta_k\cos\phi_k \\ \cos\theta_k\sin\phi_k & \cos\phi_k & \sin\theta_k\sin\phi_k \\ -\sin\theta_k & 0 & \cos\theta_k \end{pmatrix}, \tag{A7}$$

in terms of which

$$\hat{k} = S(\Omega_k)\begin{pmatrix}0\\0\\1\end{pmatrix}, \qquad \begin{pmatrix}\sin\theta_n\cos\phi_n\\ \sin\theta_n\sin\phi_n\\ \cos\phi_n\end{pmatrix} = S(\Omega_k)\begin{pmatrix}\sin\theta_{k,n}\cos\phi_{k,n}\\ \sin\theta_{k,n}\sin\phi_{k,n}\\ \cos\phi_{k,n}\end{pmatrix}. \tag{A8}$$

Under this rotation

$$Y^*_{\ell m}(\Omega_n) = \sum_{m'=-\ell}^{\ell}D^{(\ell)}_{mm'}(S(\Omega_k))Y^*_{\ell m'}(\Omega_{k,n}), \qquad d\Omega_n = d\Omega_{k,n}, \tag{A9}$$

where the Wigner rotation matrix are given by

$$D^{(\ell)}_{ms}(S(\Omega_k)) \equiv \sqrt{\frac{4\pi}{2\ell+1}}(-1)^s {}_{-s}Y^*_{\ell m}(\Omega_k), \tag{A10}$$

in terms of the spin-weighted spherical harmonics

$${}_{-s}Y^*_{\ell m}(\Omega_k) \equiv (-1)^m\sqrt{\frac{(\ell+m)!(\ell-m)!(2\ell+1)}{4\pi(\ell+s)!(\ell-s)!}}\sin^{2\ell}\left(\frac{\theta_k}{2}\right) \times\sum_{r=0}^{\ell-s}\binom{\ell-s}{r}\binom{\ell+s}{r+s-m}(-1)^{\ell-r-s}\mathrm{e}^{im\phi_k}\cot^{2r+s-m}\left(\frac{\theta_k}{2}\right). \tag{A11}$$

With this relation, Eq. (A5) can be then rewritten as

$$\Gamma_{\ell m,T} = \int\frac{d^3k}{(2\pi)^3}\mathrm{e}^{i\vec{k}\cdot\vec{x}_0}\sum_{m'=-\ell}^{\ell}D^{(\ell)}_{mm'}(S(\Omega_k))\int d^2\Omega_{k,n}Y^*_{\ell m'}(\Omega_{k,n})\Gamma_T(\eta_0,\vec{k},\Omega_{k,n}), \tag{A12}$$

where now the innermost integrand is performed in a basis in which the $\hat{n}$ vector is decomposed according to (A3), so that the explicit expression (A6) can be used.

The inner integral evaluates to

$$\begin{aligned}\int d^2\Omega_{k,n} Y^*_{\ell m'}(\Omega_{k,n})\Gamma_T(\eta_0,\vec{k},\Omega_{k,n}) &= \int d^2\Omega_{k,n}\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m')!}{(\ell+m')!}}P_\ell^{m'}(\mu_{k,n})\mathrm{e}^{-im'\phi_{k,n}} \\ &\quad\times(-1)\frac{1-\mu_{k,n}^2}{4}\sum_{\lambda=\pm2}\mathrm{e}^{i\lambda\phi_{k,n}}\xi_\lambda(\vec{k})\int_{\eta_{\rm in}}^{\eta_0}d\eta\chi'(\eta,k)\mathrm{e}^{-i\mu_k(\eta_0-\eta)k} \\ &= -\int_{\eta_{\rm in}}^{\eta_0}d\eta\chi'(\eta,k)\int_{-1}^{1}d\mu_{k,n}\frac{1-\mu_{k,n}^2}{4}\mathrm{e}^{-i\mu_k(\eta_0-\eta)k}P_\ell^2(\mu_{k,n})2\pi \\ &\quad\times\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-2)!}{(\ell+2)!}}\sum_{\lambda=\pm2}\delta_{m'\lambda}\xi_\lambda(\vec{k}) \\ &= \int_{\eta_{\rm in}}^{\eta_0}d\eta\chi'(\eta,k)(-i)^\ell\frac{j_\ell(k(\eta_0-\eta))}{k^2(\eta_0-\eta)^2}\sqrt{4\pi(2\ell+1)}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\frac{1}{4}\sum_{\lambda=\pm2}\delta_{m'\lambda}\xi_\lambda(\vec{k}).\end{aligned} \tag{A13}$$

Inserting this into Eq. (A12), and using the relation (A10) for the Wigner elements we finally arrive to Eq. (3.13) of the main text.

APPENDIX B: TENSOR CONTRIBUTION TO THE GW BISPECTRUM

In this Appendix we present the steps from Eq. (4.14) to Eq. (4.15) of the main text. We start by introducing the quantity $\mathcal{F}^\lambda_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3)$ from Eq. (2.6) of [40]:

$$\left\langle\prod_{i=1}^{3}\int d\Omega_{k_i}\xi_\lambda(\vec{k}_i)_{-\lambda}Y^*_{\ell_i m_i}(\Omega_{k_i})\right\rangle \equiv (2\pi)^3\mathcal{F}^\lambda_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3)\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix} \tag{B1}$$

[where we have also used Eq. (2.6) of [40] at the left-hand side]. This relation is inverted by Eq. (2.7) of [40]:

$$\begin{aligned}\mathcal{F}^\lambda_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3) &= \sum_{m_1,m_2,m_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}\int d\Omega_{k_1}\int d\Omega_{k_2}\int d\Omega_{k_3} \\ &\quad\times{}_{-\lambda}Y^*_{\ell_1 m_1}(\Omega_{k_1})_{-\lambda}Y^*_{\ell_2 m_2}(\Omega_{k_2})_{-\lambda}Y^*_{\ell_3 m_3}(\Omega_{k_3})\frac{1}{(2\pi)^3}\langle\xi_\lambda(\vec{k}_1)\xi_\lambda(\vec{k}_2)\xi_\lambda(\vec{k}_3)\rangle.\end{aligned} \tag{B2}$$

We insert Eq. (B1) in Eq. (4.14) to obtain

$$\begin{aligned}\left\langle\prod_{i=1}^{3}\Gamma_{\ell_i m_i,T}\right\rangle &= \mathcal{G}^{m_1m_2m_3}_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0 & 0\end{pmatrix}^{-1}\sqrt{\frac{4\pi}{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}} \\ &\quad\times\left[\prod_{i=1}^{3}4\pi(-i)^{\ell_i}\int\frac{k_i^2dk_i}{(2\pi)^3}\mathcal{T}^T_{\ell,i}(k_i,\eta_0,\eta_{\rm in})\right](2\pi)^3\sum_{\lambda=\pm2}\mathcal{F}^\lambda_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3),\end{aligned} \tag{B3}$$

where the relation (4.12) has also been used. We collect some of the factors in this expression into the combination

$$\tilde{\mathcal{F}}^\lambda_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3) \equiv \begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0 & 0\end{pmatrix}^{-1}\sqrt{\frac{4\pi}{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}}(2\pi)^3\mathcal{F}^\lambda_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3), \tag{B4}$$

which then evaluates to the relation (4.16) in the main text. In terms of $\tilde{\mathcal{F}}$ we then recover Eq. (4.15) of the main text.

APPENDIX C: COMPARISON WITH THE CMB

In the CMB case for a temperature $T(\hat{n}) = \bar{T} + \delta T(\hat{n})$, we have

$$\bar{f}(p) = \frac{1}{e^{\frac{p}{\bar{T}}} - 1},$$

$$f(p,\hat{n}) = \frac{1}{e^{\frac{p}{T(\hat{n})}} - 1} = \frac{1}{e^{\frac{p}{\bar{T}}} - 1} + \frac{e^{\frac{p}{\bar{T}}}}{(e^{\frac{p}{\bar{T}}} - 1)^2}\frac{p}{\bar{T}}\frac{\delta T(\hat{n})}{\bar{T}}$$

$$= \bar{f}(p) - p\frac{\partial \bar{f}(p)}{\partial p}\frac{\delta T(\hat{n})}{\bar{T}}, \tag{C1}$$

from which it follows

$$\Gamma(\hat{n}) = \frac{\delta T(\hat{n})}{\bar{T}}, \quad p \text{ independent.} \tag{C2}$$

To connect with the description of the SGWB, we also define

$$w_{\rm CMB}(p,\hat{n}) = \frac{p^4 f(p,\hat{n})}{\rho_{\rm crit}}, \qquad \bar{w}_{\rm CMB}(p) = \frac{p^4 \bar{f}(p)}{\rho_{\rm crit}} \tag{C3}$$

so that we have the p-dependent quantity

$$\delta_{\rm CMB}(p,\hat{n}) \equiv \frac{w_{\rm CMB}(p,\hat{n}) - \bar{w}_{\rm CMB}(p)}{\bar{w}_{\rm CMB}(p)} = \frac{e^{\frac{p}{\bar{T}}}}{e^{\frac{p}{\bar{T}}} - 1}\frac{p}{\bar{T}}\frac{\delta T(\hat{n})}{\bar{T}} \tag{C4}$$

as well as the p-dependent quantity

$$4 - \frac{\partial \ln \bar{\omega}_{\rm CMB}(\eta_0,p)}{\partial \ln p} = 4 - \frac{\rho_{\rm crit}}{p^3 \bar{f}(p)}\left[\frac{4p^3\bar{f}(p)}{\rho_{\rm crit}} + \frac{p^4}{\rho_{\rm crit}}\frac{\partial \bar{f}(p)}{\partial p}\right]$$

$$= -p(e^{\frac{p}{\bar{T}}} - 1)\frac{-\frac{1}{\bar{T}}e^{\frac{p}{\bar{T}}}}{(e^{\frac{p}{\bar{T}}} - 1)^2} = \frac{\frac{p}{\bar{T}}e^{\frac{p}{\bar{T}}}}{e^{\frac{p}{\bar{T}}} - 1}. \tag{C5}$$

So the ratio

$$\frac{\delta_{\rm CMB}(p,\hat{n})}{4 - \frac{\partial \ln \bar{\omega}_{\rm CMB}(\eta_0,p)}{\partial \ln p}} = \frac{\delta T(\hat{n})}{\bar{T}} \tag{C6}$$

is indeed p independent.

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