On a generalized population dynamics equation with environmental noise

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Abstract We establish the existence and uniqueness of global (in time) positive strong solutions for a generalized population dynamics equation with environmental noise, while the global existence fails for the deterministic equation. Particularly, we prove the global existence of positive strong solutions for the following stochastic differential equation

$$dX_t = (\theta X_t^{m_0} + kX_t^m)dt + \varepsilon X_t^{\frac{m+1}{2}} \varphi(X_t) dW_t,\ t > 0,\ X_t > 0,\ m > m_0 \geqslant 1, \quad X_0 = x > 0,$$

with $\theta, k, \varepsilon \in \mathbb{R}$ being constants and $\varphi(r) = r^\vartheta$ or $|\log(r)|^\vartheta$ ($\vartheta > 0$), and we also show that the index $\vartheta > 0$ is sharp in the sense that if $\vartheta = 0$, one can choose certain proper constants θ, k and ε such that the solution X_t will explode in a finite time almost surely.

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1 Introduction and main results

Ordinary differential equation (ODE) is used in the study of the dynamical behaviour of entity which in the sense of the given context must remain nonnegative for all times. One typical equation is described by the following form [2]:

$$\begin{cases} \dot{X}_t = X_t b(X_t),\ t > 0, \\ X_0 = x > 0. \end{cases} \tag{1.1}$$

For example, when $b(r) = \theta + kr$ (θ and k are given real numbers), (1.1) describes the single-species population dynamics (see [16]), and for this equation, there is a unique solution which is positive. However, if $k > 0$, the unique solution will explode at the following defined finite time

$$T = \begin{cases} \frac{1}{\theta}\log(1+\frac{\theta}{kx}), & \text{if } \ \theta, k > 0, \\ \frac{1}{\theta}\log(1+\frac{\theta}{kx}), & \text{if } \ \theta < 0, k > 0 \text{ and } kx+\theta > 0, \\ \frac{1}{kx}, & \text{if } \ \theta = 0, k > 0. \end{cases}$$

Since the population dynamics is often subject to an environment noise ([7, 15]), this equation can be amended into the following stochastic differential equation (SDE) with small multiplicative noise

$$\begin{cases} dX_t = X_t[b(X_t)dt + \varepsilon X_t dW_t], \ t > 0, \\ X_0 = x > 0, \end{cases} \tag{1.2}$$

where $\{W_t\}_{t\geqslant 0}$ is a 1-dimensional standard Wiener process defined on a given stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$. For $b(X_t) = \theta + kX_t$ ($k, \theta \in \mathbb{R}$), and any $\varepsilon > 0$, Mao, Marion and Renshaw [12] proved that there is a unique global positive strong solution $X_t(x)$ for (1.2).

Inspired by Mao, Marion and Renshaw [12], one can extend the stochastic population dynamics (1.2) to the following more general equation (hereafter called a generalized population dynamics equation)

$$\begin{cases} dX_t = X_t(b(X_t) + kX_t^{m-1})dt + \varepsilon X_t^{\alpha} dW_t, \ t > 0, \ m > 1, \\ X_0 = x > 0, \end{cases} \tag{1.3}$$

for some $\alpha \geqslant 1$. A very natural question is whether one can have an analogue result of Mao, Marion and Renshaw [12] to (1.3). To be more precise, it is very interesting to investigate whether there exists an optimal index $\alpha_0 = \alpha(m)$ for each $m > 1$ such that for every $\alpha > \alpha_0$, there is a unique global positive strong solution $X_t(x)$ for (1.3)? In other words, **does such an optimal α_0 exist and can one further get α_0 explicitly if it exists?**

In this paper, we are concerned with the above question for an even more general SDE

$$\begin{cases} dX_t = X_t(b(X_t) + kX_t^{m-1})dt + \varepsilon X_t^{\frac{m+1}{2}} \varphi(X_t) dW_t, \ t > 0, \ m > 1, \\ X_0 = x > 0, \end{cases} \tag{1.4}$$

where k and ε are real numbers, φ is a continuously differentiable function on $\mathbb{R}_+ = [0, +\infty)$. Our first main result is the following

Theorem 1.1 *Let k be a real number which is not zero. Assume $rb(r) \in \mathcal{C}^1(\mathbb{R}_+)$ and there exist two positive real numbers c_0 and m_0 ($< m$) such that*

$$|b(r)| \leqslant c_0(1 + r^{m_0 - 1}), \quad r \in \mathbb{R}_+. \tag{1.5}$$

Assume in addition that $r^{\frac{m+1}{2}}\varphi(r) \in \mathcal{C}^1(\mathbb{R}_+)$. Let $\beta \in (0, 1)$ and suppose there is a $r_0 > 0$ such that

$$\inf_{r \geqslant r_0} \varphi(r) > \sqrt{\frac{2|k|}{(1-\beta)\varepsilon^2}}. \tag{1.6}$$

There is a unique global positive strong solution $X_\cdot(x)$ to (1.4), namely, for every $t > 0$, $X_t(x)$ satisfies

$$X_t(x) = x + \int_0^t X_s(b(X_s) + kX_s^{m-1})ds + \varepsilon \int_0^t X_s^{\frac{m+1}{2}} \varphi(X_s)dW_s, \quad \mathbb{P} - a.s.,$$

and for all $t > 0$, $X_t(x)$ is positive valued almost surely. Moreover, for every $T > 0$

$$\sup_{0 \leqslant t \leqslant T} \mathbb{E}X_t^\beta(x) < +\infty. \tag{1.7}$$

Remark 1.1 *(i) From the above result, one can claim that a multiplicative noise suppresses explosion for the solution. Besides, there exist a number of research works devoted to understanding the effect of noise on solutions of ODEs. We just mention a few here, for instance, noise gave rise to stochastic resonance [5, 14], noise enhanced stability [1, 11], noise delayed extinction [17], noise stimulated explosion [10], and so on.*

(ii) When $\varepsilon > 0$, from [8, Theorem 2.3 (i) and Theorem 2.8 (i)], there is a unique global positive strong solution for (1.4). There are two advantages of our present result: (a) the constant ε can take negative real number; (b) one can get the moment estimate for the strong solution. Moreover, our proof method here is different from the proofs of Theorem 2.3 (i) and Theorem 2.8 (i) in [8].

(iii) Let b be in (1.3) which satisfies (1.5) and let $\alpha = (m + 1 + \vartheta)/2$ ($\vartheta > 0$). For every $\beta \in (0, 1)$, if we let $r_0 = \left(\sqrt{\frac{3|k|}{(1-\beta)\varepsilon^2}}\right)^{\frac{2}{\vartheta}}$, then

$$\inf_{r \geqslant r_0} \varphi(r) = \inf_{r \geqslant r_0} |r|^{\frac{\vartheta}{2}} = \sqrt{\frac{3|k|}{(1-\beta)\varepsilon^2}} > \sqrt{\frac{2|k|}{(1-\beta)\varepsilon^2}}.$$

By Theorem 1.1, there is a unique global positive strong solution $X_t(x)$ for (1.3). However, if $\vartheta = 0$, then this conclusion no longer holds true.

To illustrate the last point in Remark 1.1, let us present a counterexample. For simplicity we assume $b = 0$.

Example 1.1 *Let $b = \vartheta = 0$, $\varepsilon = k = 1$, $m = 3$ and $\alpha = 2$. Then (1.3) degenerates into:*

$$\begin{cases} dX_t = X_t^3 dt + X_t^2 dW_t, \ t > 0, \\ X_0 = x > 0. \end{cases} \tag{1.8}$$

The unique strong solution for (1.8) can be represented by $X_t(x) = (x^{-1} - W_t)^{-1}$. Defining the lifetime by

$$\tau = \inf\{t > 0, X_t(x) = 0 \ or +\infty\},$$

then

$$\tau = \inf\{t > 0, X_t(x) = +\infty\} = \inf\{t > 0, W_t = 1/x\}$$

for $\mathbb{P}(W_t = +\infty) = 0$*. Since all paths of Winer process* W_t *are continuous and* $W_0 = 0$*, for any given real number* $x_0 > 0$*, we derive*

$$\begin{aligned}\mathbb{P}\{\tau \leqslant x_0^2\} = 1 - \mathbb{P}\{\tau > x_0^2\} &= 1 - \mathbb{P}\{\sup_{0\leqslant t\leqslant x_0^2} W_t < 1/x\} \\ &= 1 - \frac{2}{\sqrt{2\pi}x_0}\int_0^{x^{-1}} e^{-\frac{r^2}{2x_0^2}}dr \\ &= 1 - \sqrt{\frac{2}{\pi}}\int_0^{x^{-1}x_0^{-1}} e^{-\frac{r^2}{2}}dr.\end{aligned}$$

Therefore, $X_t(x)$ *will clearly explode in a finite time almost surely.*

Remark 1.2 *(i) Combining Theorem 1.1 and Example 1.1, one can see that the index* $\alpha_0 = (m+1)/2$ *is optimal and thus, we give a positive answer for the question we posed.*

(ii) If the noise in (1.8) vanishes, then the unique solution is given by $X_t(x) = (x^{-2} - 2t)^{-\frac{1}{2}}$*. In this case, the lifetime is* $T = x^{-2}/2$*. Let* τ *be given in* **Example** *1.1. Then*

$$\begin{aligned}\mathbb{P}\{\tau > x^{-2}/2\} &= \mathbb{P}\{\sup_{0\leqslant t\leqslant x^{-2}/2} W_t < 1/x\} \\ &= \frac{2}{\sqrt{\pi x^{-2}}}\int_0^{x^{-1}} e^{-\frac{r^2}{x^{-2}}}dr \\ &= \frac{2}{\sqrt{\pi}}\int_0^1 e^{-r^2}dr \approx 0.842 > 0.5. \qquad (1.9)\end{aligned}$$

From (1.9), with a 'fairly large probability', the lifetime for (1.8) will be prolonged.

Our second main result reads as follows

Theorem 1.2 *Let* k, b, c_0, m_0, φ *and* β *be given as in Theorem 1.1. We assume further that* $k > 0$ *and* $b(r) \geqslant 0$ *for* $r \in \mathbb{R}_+$*.*

(i) Then there is a real number $T_0 > 0$ *such that*

$$\sup_{0\leqslant t<T_0} \mathbb{E}X_t(x) = +\infty. \qquad (1.10)$$

(ii) If there is some $\gamma \in (\beta, 1)$ *such that*

$$\sup_{r>0}\varphi(r) < \sqrt{\frac{2k}{(1-\gamma)\varepsilon^2}} \ \Big(> \sqrt{\frac{2k}{(1-\beta)\varepsilon^2}}\Big), \qquad (1.11)$$

then there is a real number $T_0 > 0$ *such that*

$$\sup_{0\leqslant t<T_0} \mathbb{E}X_t^{\gamma}(x) = +\infty. \qquad (1.12)$$

Remark 1.3 *The above result can be used to discuss the blow up problems for stochastic parabolic equations on bounded domains, and for this topic, we refer the interested reader to [9, 13]. For more details, one also sees [3, 4].*

Let us give some examples to further illustrate our main results.

Example 1.2 *Let $m > 1$ be a given real number. Suppose $1 \leqslant m_0 < m$ and $\theta \in \mathbb{R}$, we set $b(r) = \theta r^{m_0 - 1}$ for $r \in \mathbb{R}_+$. Choose $\varphi(r) = \log(r)$, and consider the following SDE*

$$\begin{cases} dX_t = (\theta X_t^{m_0} + kX_t^m)dt + \varepsilon X_t^{\frac{m+1}{2}} \log(X_t)dW_t, \ t > 0, \ m > 1, \\ X_0 = x > 0, \end{cases} \tag{1.13}$$

where $k, \varepsilon \in \mathbb{R}$ are given real numbers. Then $rb(r) = \theta r^{m_0} \in \mathcal{C}^1(\mathbb{R}_+)$. For every $\beta \in (0,1)$, if one takes $r_0 = e^{\sqrt{\frac{3|k|}{1-\beta}}}$, then

$$\inf_{r \geqslant r_0} \log(r) = \sqrt{\frac{3|k|}{(1-\beta)\varepsilon^2}} > \sqrt{\frac{2|k|}{(1-\beta)\varepsilon^2}}.$$

By our Theorem 1.1, there is a unique global positive strong solution $X_t(x)$ to (1.13). Furthermore, for every $\beta \in (0,1)$, and every $T > 0$, $\sup_{0 \leqslant t \leqslant T} \mathbb{E}X_t^\beta(x) < +\infty$. However, by our Theorem 1.2, if $\theta \geqslant 0$ and $k > 0$, then there is a real number $T_0 > 0$ such that $\sup_{0 \leqslant t < T_0} \mathbb{E}X_t(x) = +\infty$.

Remark 1.4 *(i) In (1.13), $r^{\frac{m+1}{2}}\varphi(r) = r^{\frac{m+1}{2}} \log(r) \in \mathcal{C}^1(0, +\infty)$. Since*

$$\lim_{r \to 0} [r^{\frac{m+1}{2}} \log(r)] = \lim_{r \to 0} [r^{\frac{m+1}{2}} \log(r)]' = 0,$$

if one defines $r^{\frac{m+1}{2}} \log(r)|_{r=0} = 0$, then the function is continuously differentiable on $[0, +\infty)$. All assumptions in Theorems 1.1 and 1.2 are validated.

(ii) More generally, one can use $|\log(X_t)|^\vartheta$ $(\vartheta > 0)$ instead of $\log(X_t)$ in (1.13).

Example 1.3 *Let m, θ, k, m_0, θ and ε be given as in Example 1.2. Suppose $\vartheta > 0$ and $\varphi(r) = r^{\frac{\vartheta}{2}}$. Consider the following SDE*

$$\begin{cases} dX_t = (\theta X_t^{m_0} + kX_t^m)dt + \varepsilon X_t^{\frac{m+1+\vartheta}{2}} dW_t, \ t > 0, \ m > 1, \\ X_0 = x > 0. \end{cases} \tag{1.14}$$

Let β be in $(0,1)$. If we fetch $r_0 = \left(\sqrt{\frac{3|k|}{(1-\beta)\varepsilon^2}}\right)^{\frac{2}{\vartheta}}$, then

$$\inf_{r \geqslant r_0} |r|^{\frac{\vartheta}{2}} = \sqrt{\frac{3|k|}{(1-\beta)\varepsilon^2}} > \sqrt{\frac{2|k|}{(1-\beta)\varepsilon^2}}.$$

By our Theorem 1.1, there is a unique global positive strong solution $X_t(x)$, and for every small enough $\kappa > 0$, the $(1-\kappa)$-order moment of $X_t(x)$ is finite. But, by our Theorem 1.2, we conclude that the mean value of $X_t(x)$ explodes at a finite time.

2 Proof of Theorem 1.1

Since the coefficients are locally Lipschitz continuous, for every given initial value $x > 0$, there exists a unique local solution $X_t(x)$ on $[0, \tau)$ (see [6]), where τ is the lifetime, i.e.

$$\lim_{t \to \tau} X_t \in \{0, +\infty\}.$$

To show that for almost all $\omega \in \Omega$, the solution exists globally, it only suffices to prove that $\tau = +\infty$ a.s..

For $n > 0$, we define a stopping time by

$$\tau_n = \inf\{t \geqslant 0,\ X_t(x) \overline{\in} (\frac{1}{n}, n)\}. \tag{2.1}$$

Then $\tau_n < \tau$ and $\tau_\infty \leqslant \tau$. If we prove that $\tau_\infty = +\infty$ a.s., the desired result is then followed. We prove this statement by contradiction.

Assume that there is a constant $T > 0$ such that there is a positive real number δ and

$$\mathbb{P}\{\tau_\infty \leqslant T\} > \delta. \tag{2.2}$$

Then there a positive real number $N > 0$ such that for every $n \geqslant N$,

$$\mathbb{P}\{\tau_n \leqslant T\} \geqslant \delta. \tag{2.3}$$

Let $f(r) = r^\beta - 1 - \beta \log(r)$. Then $0 \leqslant f \in \mathcal{C}^2(0, +\infty)$. For every $t \in [0, \tau_n]$, $X_t > 0$ and thus we can use the Itô formula to $f(X_{t \wedge \tau_n})$,

$$\begin{aligned} f(X_{t\wedge\tau_n}) &= f(x) + \int_0^{t\wedge\tau_n} \Big[f'(X_s)X_s(b(X_s) + kX_s^{m-1}) + \frac{\varepsilon^2}{2} f''(X_s)X_s^{m+1}\varphi^2(X_s)\Big] ds \\ &\quad + \varepsilon \int_0^{t\wedge\tau_n} f'(X_s)X_s^{\frac{m+1}{2}} \varphi(X_s) dW_s. \end{aligned} \tag{2.4}$$

Observing that, for every $r > 0$, we have

$$\begin{aligned} & f'(r)r(b(r) + kr^{m-1}) + \frac{\varepsilon^2}{2} f''(r) r^{m+1} \varphi^2(r) \\ = \ & \beta(r^\beta - 1)b(r) + k\beta(r^{m+\beta-1} - r^{m-1}) + \frac{\varepsilon^2}{2}\beta(\beta-1)r^{m+\beta-1}\varphi^2(r) + \frac{\varepsilon^2\beta}{2} r^{m-1}\varphi^2(r) \\ \leqslant \ & c_0\beta(r^\beta + 1)(1 + r^{m_0-1}) + |k|\beta(r^{m+\beta-1} + r^{m-1}) \\ & + \frac{\varepsilon^2}{2}\beta(\beta-1)r^{m+\beta-1}\varphi^2(r) + \frac{\varepsilon^2\beta}{2}r^{m-1}\varphi^2(r) \\ = \ & \frac{\beta}{2} r^{m+\beta-1}\Big[2|k| - (1-\beta)\varepsilon^2\varphi^2(r)\Big] + c_0\beta(1 + r^\beta + r^{m_0-1} + r^{m_0+\beta-1}) \\ & + \frac{\beta}{2} r^{m-1}\Big[2|k| + \varepsilon^2\varphi^2(r)\Big]. \end{aligned} \tag{2.5}$$

In view of (1.6), there is a real number $\epsilon > 0$, such that for every $r \geqslant r_0$,

$$\varphi^2(r) \geqslant \frac{2|k| + \epsilon}{(1-\beta)\varepsilon^2}.$$

Therefore,

$$2|k| - (1-\beta)\varepsilon^2\varphi^2(r) = 2|k| - \frac{2|k|(1-\beta)}{2|k|+\epsilon}\varepsilon^2\varphi^2(r) - \frac{(1-\beta)\epsilon}{2|k|+\epsilon}\varepsilon^2\varphi^2(r) \leqslant -\frac{(1-\beta)\epsilon}{2|k|+\epsilon}\varepsilon^2\varphi^2(r). \quad (2.6)$$

By (2.5) and (2.6), for every $r \geqslant r_0$, we arrive at

$$\begin{aligned}
& f'(r)r(b(r)+kr^{m-1}) + \frac{\varepsilon^2}{2}f''(r)r^{m+1}\varphi^2(r) \\
\leqslant\ & -\frac{\beta(1-\beta)\epsilon\varepsilon^2}{2[2|k|+\epsilon]}r^{m+\beta-1}\varphi^2(r) + c_0\beta(1+r^\beta+r^{m_0-1}+r^{m_0+\beta-1}) + \frac{\beta}{2}r^{m-1}\Big[2|k|+\varepsilon^2\varphi^2(r)\Big] \\
=\ & \Big[-\frac{\beta(1-\beta)\epsilon\varepsilon^2}{4(2|k|+\epsilon)}r^{m+\beta-1}\varphi^2(r) + \frac{\beta\varepsilon^2}{2}r^{m-1}\varphi^2(r)\Big] \\
& + \Big[-\frac{\beta(1-\beta)\epsilon\varepsilon^2}{4(2|k|+\epsilon)}r^{m+\beta-1}\varphi^2(r) + c_0\beta(1+r^\beta+r^{m_0-1}+r^{m_0+\beta-1}) + \beta|k|r^{m-1}\Big] \\
\leqslant\ & \Big[-\frac{\beta(1-\beta)\epsilon\varepsilon^2}{4(2|k|+\epsilon)}r^{m+\beta-1}\varphi^2(r) + \frac{\beta\varepsilon^2}{2}r^{m-1}\varphi^2(r)\Big] \\
& + \Big[-\frac{\beta\epsilon}{4}r^{m+\beta-1} + c_0\beta(1+r^\beta+r^{m_0-1}+r^{m_0+\beta-1}) + \beta|k|r^{m-1}\Big] \\
=\ & \frac{\beta\varepsilon^2}{2}r^{m-1}\varphi^2(r)\Big[1-\frac{(1-\beta)\epsilon}{2(2|k|+\epsilon)}r^\beta\Big] \\
& + \beta\Big[c_0r^\beta + c_0r^{m_0-1} + c_0r^{m_0+\beta-1} + |k|r^{m-1} - \frac{\epsilon}{4}r^{m+\beta-1}\Big] + \beta c_0,
\end{aligned}$$

which implies that there is a positive real number C such that

$$f'(r)r(b(r)+kr^{m-1}) + \frac{\varepsilon^2}{2}f''(r)r^{m+1}\varphi^2(r) \leqslant C, \quad \forall\, r \geqslant r_0. \quad (2.7)$$

Combining (2.7) and (2.5) and noting that the functions $r^{m+\beta-1}, r^{m-1}\varphi^2(r), r^\beta$ and r^{m_0-1} are continuous in r, we conclude that

$$f'(r)r(b(r)+kr^{m-1}) + \frac{\varepsilon^2}{2}f''(r)r^{m+1}\varphi^2(r) \leqslant C, \quad \forall\, r > 0. \quad (2.8)$$

By (2.8), from (2.4), it yields that

$$f(X_{t\wedge\tau_n}) \leqslant f(x) + Ct\wedge\tau_n + \int_0^{t\wedge\tau_n} f'(X_s)X_s^{\frac{m+1}{2}}\varphi(X_s)dW_s. \quad (2.9)$$

By taking the expectation in (2.9), we gain

$$\mathbb{E}f(X_{t\wedge\tau_n}) \leqslant f(x) + C\mathbb{E}t\wedge\tau_n.$$

In particular, we choose $t = T$, then

$$\mathbb{E}f(X_{T\wedge\tau_n}) \leqslant f(x) + CT, \quad (2.10)$$

which also implies that

$$\delta f(X_{\tau_n}) \leqslant \mathbb{P}\{\tau_n \leqslant T\}f(X_{\tau_n}) \leqslant \mathbb{E}f(X_{T\wedge\tau_n}) \leqslant f(x) + CT. \quad (2.11)$$

Observing that $X_{\tau_n} = \frac{1}{n}$ or n, and

$$f(n) = n^\beta - 1 - \beta \log(n), \quad f(\frac{1}{n}) = n^{-\beta} - 1 + \beta \log(n),$$

thus for every $n \geqslant N$, we conclude from (2.11) that

$$\delta(n^\beta - 1 - \beta \log(n)) \wedge (n^{-\beta} - 1 + \beta \log(n)) \leqslant f(x) + CT. \tag{2.12}$$

When n tends to infinity, the left hand side in (2.12) is infinity but the righthand side is finite, this contraction implies that (2.2) is not true and so the solution is positive for every $t > 0$ almost surely.

If we use the function $g(r) = r^\beta$ instead of $f(r) = r^\beta - 1 - \beta \log(r)$ and repeat all calculations from (2.4) to (2.10) to get

$$\mathbb{E} X^\beta_{t \wedge \tau_n}(x) \leqslant f(x) + Ct, \quad \forall \;\; t > 0.$$

Therefore,

$$\mathbb{E} X^\beta_t(x) \leqslant \liminf_{n \to \infty} \mathbb{E} X^\beta_{t \wedge \tau_n}(x) \leqslant f(x) + Ct, \quad \forall \;\; t > 0. \tag{2.13}$$

From (2.13), we get the estimate (1.7) and the proof is then completed. □

3 Proof of Theorem 1.2

(i) Let τ_n be given by (2.1). By taking the expectation in (1.4) for $X_{t \wedge \tau_n}$, we arrive at

$$\mathbb{E} X_{t \wedge \tau_n} = x + \mathbb{E} \int_0^{t \wedge \tau_n} X_s [b(X_s) + k X_s^{m-1}] ds \geqslant x + k \int_0^t \mathbb{E} X^m_{s \wedge \tau_n} ds. \tag{3.1}$$

Set $Y_{t,n} = \mathbb{E} X_{t \wedge \tau_n}$, and let $Y_t = \mathbb{E} X_t$. Let $T > 0$, if $Y_t < +\infty$ for every $0 < t < T$, by using the dominated convergence theorem (since $0 < X_{t \wedge \tau_n} \leqslant 1 + X_t$) and Fatou's lemma, it leads from (3.1) to

$$Y_t = \lim_{n \to \infty} Y_{t,n} \geqslant x + k \int_0^t \liminf_{n \to \infty} Y^m_{s,n} ds = x + k \int_0^t Y^m_s ds. \tag{3.2}$$

Consider the following ODE

$$\begin{cases} \frac{dZ_t}{dt} = k Z_t^m, & t > 0, \;\; m > 1, \;\; k > 0, \\ Z_0 = x > 0. \end{cases}$$

Then the explicit solution of Z_t is given by

$$Z_t = \Big[x^{1-m} - k(m-1)t \Big]^{-\frac{1}{m-1}}.$$

Applying the comparison principle for ODE, we conclude that

$$Y_t \geqslant \Big[x^{1-m} - k(m-1)t \Big]^{-\frac{1}{m-1}}. \tag{3.3}$$

From (3.3), there is a $T_0 \leqslant x^{1-m}/(km-k)$ such that $\lim_{t\to T_0} Y_t = +\infty$.

(ii) By Theorem 1.1 the solution $X_t(x)$ is positive for all $t \geqslant 0$ almost surely, thus we can use the Itô formula to $X^{\gamma}_{t\wedge\tau_n}$, and then get

$$
\begin{aligned}
X^{\gamma}_{t\wedge\tau_n} &= x^{\gamma} + \frac{\gamma}{2}\int_0^{t\wedge\tau_n} \Big[2X_s^{\gamma}(b(X_s)+kX_s^{m-1}) - (1-\gamma)\varepsilon^2 X_s^{m+\gamma-1}\varphi^2(X_s)\Big]ds \\
&\quad + \gamma\varepsilon\int_0^{t\wedge\tau_n} X_s^{\frac{m+1}{2}+\gamma-1}\varphi(X_s)dW_s \\
&\geqslant x^{\gamma} + \frac{\gamma}{2}\int_0^{t\wedge\tau_n} \Big[2kX_s^{m+\gamma-1} - (1-\gamma)\varepsilon^2 X_s^{m+\gamma-1}\varphi^2(X_s)\Big]ds \\
&\quad + \gamma\varepsilon\int_0^{t\wedge\tau_n} X_s^{\frac{m+1}{2}+\gamma-1}\varphi(X_s)dW_s. \qquad (3.4)
\end{aligned}
$$

From (3.4), then

$$
\mathbb{E}X^{\gamma}_{t\wedge\tau_n} \geqslant x^{\gamma} + \frac{\gamma}{2}\mathbb{E}\int_0^{t\wedge\tau_n} \Big[2k - (1-\gamma)\varepsilon^2\varphi^2(X_s)\Big] X_s^{m+\gamma-1}ds, \qquad (3.5)
$$

With the help of (1.11), from (3.5), there is a real number $c > 0$ such that

$$
\mathbb{E}X^{\gamma}_{t\wedge\tau_n} \geqslant x^{\gamma} + c\mathbb{E}\int_0^{t\wedge\tau_n} X_s^{m+\gamma-1}ds = x^{\gamma} + c\int_0^t \mathbb{E}X^{m+\gamma-1}_{s\wedge\tau_n}ds,
$$

The arguments from (3.2) to (3.3) apply here again, we conclude that

$$
\mathbb{E}X_t^{\gamma} \geqslant \Big[x^{1-m} - \frac{c(m-1)}{\gamma}t\Big]^{-\frac{\gamma}{m-1}}. \qquad (3.6)
$$

From (3.6), there is a $T_0 \leqslant \gamma x^{1-m}/(cm-c)$ such that $\lim_{t\to T_0}\mathbb{E}X_t^{\gamma} = +\infty$. We are done. □

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