# Exponential Convergence in Entropy and Wasserstein for McKean-Vlasov SDEs [*]

**Panpan Ren**[c)], **Feng-Yu Wang**[a,b)]

[a)] Center for Applied Mathematics, Tianjin University, Tianjin 300072, China

[b)] Department of Mathematics, Swansea University, Bay Campus, SA1 8EN, United Kingdom

[c)] Department of Mathematics, City University of HongKong, HongKong, China

rppzoe@gmail.com, wangfy@tju.edu.cn

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## Abstract

The following type of exponential convergence is proved for (non-degenerate or degenerate) McKean-Vlasov SDEs:

$$\mathbb{W}_2(\mu_t, \mu_\infty)^2 + \mathrm{Ent}(\mu_t|\mu_\infty) \le c e^{-\lambda t} \min\left\{\mathbb{W}_2(\mu_0, \mu_\infty)^2, \mathrm{Ent}(\mu_0|\mu_\infty)\right\}, \quad t \ge 1,$$

where $c, \lambda > 0$ are constants, $\mu_t$ is the distribution of the solution at time $t$, $\mu_\infty$ is the unique invariant probability measure, Ent is the relative entropy and $\mathbb{W}_2$ is the $L^2$-Wasserstein distance. In particular, this type of exponential convergence holds for some (non-degenerate or degenerate) granular media type equations generalizing those studied in [8, 12] on the exponential convergence in a mean field entropy.

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# 1   Introduction

The convergence in entropy for stochastic systems is an important topic in both probability theory and mathematical physics, and has been well studied for Markov processes by using the log-Sobolev inequality, see for instance [6] and references therein. However, the existing results derived in the literature do not apply to McKean-Vlasov SDEs due to the nonlinearity

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of the associated Fokker-Planck equations. In 2003, Carrillo, McCann and Villani [8] proved the exponential convergence in a mean field entropy of the following granular media equation for probability density functions $(\rho_t)_{t\geq 0}$ on $\mathbb{R}^d$:

$$(1.1) \qquad \partial_t \rho_t = \Delta \rho_t + \mathrm{div}\big\{\rho_t \nabla(V + W * \rho_t)\big\},$$

where the internal potential $V \in C^2(\mathbb{R}^d)$ satisfies $\mathrm{Hess}_V \geq \lambda I_d$ for a constant $\lambda > 0$ and the $d \times d$-unit matrix $I_d$, and the interaction potential $W \in C^2(\mathbb{R}^d)$ satisfies $W(-x) = W(x)$ and $\mathrm{Hess}_W \geq -\delta I_d$ for some constant $\delta \in [0, \lambda/2)$. Recall that we write $M \geq \lambda I_d$ for a constant $\lambda$ and a $d \times d$-matrix $M$, if $\langle Mv, v\rangle \geq \lambda |v|^2$ holds for any $v \in \mathbb{R}^d$. To introduce the mean field entropy, let $\mu_V(\mathrm{d}x) := \frac{e^{-V(x)}\mathrm{d}x}{\int_{\mathbb{R}^d} e^{-V(x)}\mathrm{d}x}$, recall the classical relative entropy

$$\mathrm{Ent}(\nu|\mu) := \begin{cases} \mu(\rho \log \rho), & \text{if } \nu = \rho\mu, \\ \infty, & \text{otherwise} \end{cases}$$

for $\mu, \nu \in \mathscr{P}$, the space of all probability measures on $\mathbb{R}^d$, and consider the free energy functional

$$E^{V,W}(\mu) := \mathrm{Ent}(\mu|\mu_V) + \frac{1}{2}\int_{\mathbb{R}^d \times \mathbb{R}^d} W(x-y)\mu(\mathrm{d}x)\mu(\mathrm{d}y), \quad \mu \in \mathscr{P},$$

where we set $E^{V,W}(\mu) = \infty$ if either $\mathrm{Ent}(\mu|\mu_V) = \infty$ or the integral term is not well defined. Then the associated mean field entropy $\mathrm{Ent}^{V,W}$ is defined by

$$(1.2) \qquad \mathrm{Ent}^{V,W}(\mu) := E^{V,W}(\mu) - \inf_{\nu \in \mathscr{P}} E^{V,W}(\nu), \quad \mu \in \mathscr{P}.$$

According to [8], for $V$ and $W$ satisfying the above mentioned conditions, $E^{V,W}$ has a unique minimizer $\mu_\infty$, and $\mu_t(\mathrm{d}x) := \rho_t(x)\mathrm{d}x$ for probability density $\rho_t$ solving (1.1) converges to $\mu_\infty$ exponentially in the mean field entropy:

$$\mathrm{Ent}^{V,W}(\mu_t) \leq e^{-(\lambda-2\delta)t}\mathrm{Ent}^{V,W}(\mu_0), \quad t \geq 0.$$

Recently, this result was generalized in [12] by establishing the uniform log-Sobolev inequality for the associated mean field particle systems, such that $\mathrm{Ent}^{V,W}(\mu_t)$ decays exponentially for a class of non-convex $V \in C^2(\mathbb{R}^d)$ and $W \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$, where $W(x,y) = W(y,x)$ and $\mu_t(\mathrm{d}x) := \rho_t(x)\mathrm{d}x$ for $\rho_t$ solving the nonlinear PDE

$$(1.3) \qquad \partial_t \rho_t = \Delta \rho_t + \mathrm{div}\big\{\rho_t \nabla(V + W \circledast \rho_t)\big\},$$

where

$$(1.4) \qquad W \circledast \rho_t := \int_{\mathbb{R}^d} W(\cdot, y)\rho_t(y)\mathrm{d}y.$$

In this case, $\mathrm{Ent}^{V,W}$ is defined in (1.2) for the free energy functional

$$E^{V,W}(\mu) := \mathrm{Ent}(\mu|\mu_V) + \frac{1}{2}\int_{\mathbb{R}^d \times \mathbb{R}^d} W(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y), \quad \mu \in \mathscr{P}.$$

To study (1.3) using probability methods, we consider the following McKean-Vlasov SDE with initial distribution $\mu_0$:

$$\mathrm{d}X_t = \sqrt{2}\mathrm{d}B_t - \nabla\{V + W \circledast \mathscr{L}_{X_t}\}(X_t)\mathrm{d}t, \tag{1.5}$$

where $B_t$ is the $d$-dimensional Brownian motion, $\mathscr{L}_{X_t}$ is the distribution of $X_t$, and

$$(W \circledast \mu)(x) := \int_{\mathbb{R}^d} W(x,y)\mu(\mathrm{d}y), \quad x \in \mathbb{R}^d, \mu \in \mathscr{P} \tag{1.6}$$

provided the integral exists. Let $\rho_t(x) = \frac{(\mathscr{L}_{X_t})(\mathrm{d}x)}{\mathrm{d}x}$, $t \geq 0$. By Itô's formula and the integration by parts formula, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}(\rho_t f)(x)\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(X_t)] = \mathbb{E}\big[\big(\Delta - \nabla V - \nabla\{W \circledast \rho_t\}\big)f(X_t)\big]\\
&= \int_{\mathbb{R}^d}\rho_t(x)\big\{\Delta f - \langle\nabla V + \nabla\{W \circledast \rho_t\}, \nabla f\rangle\big\}(x)\mathrm{d}x\\
&= \int_{\mathbb{R}^d}f(x)\big\{\Delta\rho_t + \mathrm{div}[\rho_t\nabla V + \rho_t\nabla(W \circledast \rho_t)]\big\}(x)\mathrm{d}x, \quad t \geq 0, \ f \in C_0^\infty(\mathbb{R}^d).
\end{aligned}$$

Therefore, $\rho_t$ solves (1.3). On the other hand, by this fact and the uniqueness of (1.1) and (1.5), if $\rho_t$ solves (1.1) with $\mu_0(\mathrm{d}x) := \rho_0(x)\mathrm{d}x$, then $\rho_t(x)\mathrm{d}x = \mathscr{L}_{X_t}(\mathrm{d}x)$ for $X_t$ solving (1.5) with $\mathscr{L}_{X_0} = \mu_0$.

To extend the study of [8, 12], in this paper we investigate the exponential convergence in entropy for the following McKean-Vlasov SDE on $\mathbb{R}^d$:

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t + b(X_t, \mathscr{L}_{X_t})\mathrm{d}t, \tag{1.7}$$

where $W_t$ is the $m$-dimensional Brownian motion on a complete filtration probability space $(\Omega, \{\mathscr{F}_t\}_{t \geq 0}, \mathbb{P})$,

$$\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^m, \quad b : \mathbb{R}^d \times \mathscr{P}_2 \to \mathbb{R}^d$$

are measurable, and $\mathscr{P}_2$ is the class of probability measures on $\mathbb{R}^d$ with $\mu(|\cdot|^2) < \infty$.

Since the "mean field entropy" associated with the SDE (1.7) is not available, and it is less explicit even exists as in (1.2) for the special model (1.5), we intend to study the exponential convergence of $\mathscr{L}_{X_t}$ in the classical relative entropy Ent and the Wasserstein distance $\mathbb{W}_2$. Recall that for any $p \geq 1$, the $L^p$-Wasserstein distance is defined by

$$\mathbb{W}_p(\mu_1, \mu_2) := \inf_{\pi \in \mathscr{C}(\mu_1, \mu_2)}\left(\int_{\mathbb{R}^d \times \mathbb{R}^d}|x-y|^p\pi(\mathrm{d}x, \mathrm{d}y)\right)^{\frac{1}{p}}, \quad \mu_1, \mu_2 \in \mathscr{P}_p,$$

where $\mathscr{C}(\mu_1, \mu_2)$ is the set of all couplings of $\mu_1$ and $\mu_2$.

Unlike in [8, 12] where the mean field particle systems are used to estimate the mean field entropy, in this paper we use the log-Harnack inequality introduced in [22, 18] and the Talagrand inequality developed in [20, 6, 16], see Theorem 2.1 below. Thus, the key point of the present study is to establish these two types of inequalities for McKean-Vlasov SDEs. Since the log-Harnack inequality is not yet available when $\sigma$ depends on the distribution, in (1.7) we

only consider distribution-free $\sigma$. In particular, for a class of granular media type equations generalizing the framework of [8, 12], we prove

$$\mathbb{W}_2(\mu_t, \mu_\infty)^2 + \text{Ent}(\mu_t|\mu_\infty) \le ce^{-\lambda t} \min\left\{\mathbb{W}_2(\mu_0, \mu_\infty)^2, \text{Ent}(\mu_0|\mu_\infty)\right\}, \quad t \ge 1$$

for $\mu_t(\mathrm{d}x) := \rho_t(x)\mathrm{d}x$ and some constants $c, \lambda > 0$, see Theorem 2.2 below for details.

The remainder of the paper is organized as follows. In Section 2, we state our main results for non-degenerate and degenerate models respectively, where the first case includes the granular media type equations (1.3) or the corresponding Mckean-Vlasov SDE (1.5) as a special example, and the second case deals with the McKean-Vlasov stochastic Hamiltonian system referring to the degenerate granular media equation. The main results are proved in Sections 3-5 respectively, where Section 4 establishes the log-Harnack inequality for McKean-Vlasov stochastic Hamiltonian systems.

# 2 Main results and examples

We first present a criterion on the exponential convergence for McKean-Vlasov SDEs by using the log-Harnack and Talagrand inequalities, and prove (2.15) for the granular media type equations (2.10) below which generalizes the framework of [12]. Then we state our results for solutions of SDE (1.7) with non-degenerate and degenerate noises respectively.

## 2.1 A criterion with application to Granular media type equations

In general, we consider the following McKean-Vlasov SDE:

$$(2.1) \qquad\qquad \mathrm{d}X_t = \sigma(X_t, \mathscr{L}_{X_t})\mathrm{d}W_t + b(X_t, \mathscr{L}_{X_t})\mathrm{d}t,$$

where $W_t$ is the $m$-dimensional Brownian motion and

$$\sigma : \mathbb{R}^d \times \mathscr{P}_2 \to \mathbb{R}^d \otimes \mathbb{R}^m, \quad b : \mathbb{R}^d \times \mathscr{P}_2 \to \mathbb{R}^d$$

are measurable. We assume that this SDE is strongly and weakly well-posed for square integrable initial values. It is in particular the case if $b$ is continuous on $\mathbb{R}^d \times \mathscr{P}_2$ and there exists a constant $K > 0$ such that

$$(2.2) \quad \begin{aligned} &\langle b(x,\mu) - b(y,\nu), x - y\rangle^+ + \|\sigma(x,\mu) - \sigma(y,\nu)\|^2 \le K\{|x-y|^2 + \mathbb{W}_2(\mu,\nu)^2\}, \\ &|b(0,\mu)| \le c\left(1 + \sqrt{\mu(|\cdot|^2)}\right), \quad x, y \in \mathbb{R}^d, \mu, \nu \in \mathscr{P}_2, \end{aligned}$$

see for instance [28]. See also [14, 30] and references therein for the well-posedness of McKean-Vlasov SDEs with singular coefficients. For any $\mu \in \mathscr{P}_2$, let $P_t^*\mu = \mathscr{L}_{X_t}$ for the solution $X_t$ with initial distribution $\mathscr{L}_{X_0} = \mu$. Let

$$P_t f(\mu) = \mathbb{E}[f(X_t)] = \int_{\mathbb{R}^d} f \mathrm{d}P_t^*\mu, \quad t \ge 0, f \in \mathscr{B}_b(\mathbb{R}^d).$$

We have the following equivalence on the exponential convergence of $P_t^*\mu$ in Ent and $\mathbb{W}_2$.

**Theorem 2.1.** *Assume that $P_t^*$ has a unique invariant probability measure $\mu_\infty \in \mathscr{P}_2$ such that for some constants $t_0, c_0, C > 0$ we have the log-Harnack inequality*

$$(2.3) \qquad P_{t_0}(\log f)(\nu) \le \log P_{t_0} f(\mu) + c_0 \mathbb{W}_2(\mu, \nu)^2, \quad \mu, \nu \in \mathscr{P}_2,$$

*and the Talagrand inequality*

$$(2.4) \qquad \mathbb{W}_2(\mu, \mu_\infty)^2 \le C\mathrm{Ent}(\mu|\mu_\infty), \quad \mu \in \mathscr{P}_2 f \in \mathscr{B}_b(\mathbb{R}^d).$$

(1) *If there exist constants $c_1, \lambda, t_1 \ge 0$ such that*

$$(2.5) \qquad \mathbb{W}_2(P_t^*\mu, \mu_\infty)^2 \le c_1 \mathrm{e}^{-\lambda t} \mathbb{W}_2(\mu, \mu_\infty)^2, \quad t \ge t_1, \mu \in \mathscr{P}_2,$$

*then*

$$(2.6) \qquad \begin{aligned} &\max\left\{c_0^{-1}\mathrm{Ent}(P_t^*\mu|\mu_\infty), \mathbb{W}_2(P_t^*\mu, \mu_\infty)^2\right\} \\ &\le c_1 \mathrm{e}^{-\lambda(t-t_0)} \min\left\{\mathbb{W}_2(\mu, \mu_\infty)^2, C\mathrm{Ent}(\mu|\mu_\infty)\right\}, \quad t \ge t_0 + t_1, \mu \in \mathscr{P}_2. \end{aligned}$$

(2) *If for some constants $\lambda, c_2, t_2 > 0$*

$$(2.7) \qquad \mathrm{Ent}(P_t^*\mu|\mu_\infty) \le c_2 \mathrm{e}^{-\lambda t} \mathrm{Ent}(\mu|\mu_\infty), \quad t \ge t_2, \nu \in \mathscr{P}_2,$$

*then*

$$(2.8) \qquad \begin{aligned} &\max\left\{\mathrm{Ent}(P_t^*\mu, \mu_\infty), C^{-1}\mathbb{W}_2(P_t^*\mu, \mu_\infty)^2\right\} \\ &\le c_2 \mathrm{e}^{-\lambda(t-t_0)} \min\left\{c_0\mathbb{W}_2(\mu, \mu_\infty)^2, \mathrm{Ent}(\mu|\mu_\infty)\right\}, \quad t \ge t_0 + t_2, \mu \in \mathscr{P}_2. \end{aligned}$$

When $\sigma\sigma^*$ is invertible and does not depend on the distribution, the log-Harnack inequality (2.3) has been established in [28]. The Talagrand inequality was first found in [20] for $\mu_\infty$ being the Gaussian measure, and extended in [6] to $\mu_\infty$ satisfying the log-Sobolev inequality

$$(2.9) \qquad \mu_\infty(f^2 \log f^2) \le C\mu_\infty(|\nabla f|^2), \quad f \in C_b^1(\mathbb{R}^d), \mu_\infty(f^2) = 1,$$

see [16] for an earlier result under a curvature condition, and see [21] for further extensions.

To illustrate this result, we consider the granular media type equation for probability density functions $(\rho_t)_{t\ge 0}$ on $\mathbb{R}^d$:

$$(2.10) \qquad \partial_t \rho_t = \mathrm{div}\left\{a\nabla\rho_t + \rho_t a\nabla(V + W \circledast \rho_t)\right\},$$

where $W \circledast \rho_t$ is in (1.4), and the functions

$$a : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d, \quad V : \mathbb{R}^d \to \mathbb{R}, \quad W : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$$

satisfy the following assumptions.

$(H_1)$ $a := (a_{ij})_{1\le i,j\le d} \in C_b^2(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d)$, and $a \ge \lambda_a I_d$ for some constant $\lambda_a > 0$.

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$(H_2)$ $V \in C^2(\mathbb{R}^d), W \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$ with $W(x,y) = W(y,x)$, and there exist constants $\kappa_0 \in \mathbb{R}$ and $\kappa_1, \kappa_2, \kappa_0' > 0$ such that

$$(2.11) \qquad \text{Hess}_V \geq \kappa_0 I_d, \quad \kappa_0' I_{2d} \geq \text{Hess}_W \geq \kappa_0 I_{2d},$$

$$(2.12) \qquad \langle x, \nabla V(x) \rangle \geq \kappa_1 |x|^2 - \kappa_2, \quad x \in \mathbb{R}^d.$$

Moreover, for any $\lambda > 0$,

$$(2.13) \qquad \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-V(x) - V(y) - \lambda W(x,y)} dx dy < \infty.$$

$(H_3)$ There exists a function $b_0 \in L^1_{loc}([0, \infty))$ with

$$r_0 := \frac{\|\text{Hess}_W\|_\infty}{4} \int_0^\infty e^{\frac{1}{4} \int_0^t b_0(s) ds} dt < 1$$

such that for any $x, y, z \in \mathbb{R}^d$,

$$\langle y - x, \nabla V(x) - \nabla V(y) + \nabla W(\cdot, z)(x) - \nabla W(\cdot, z)(y) \rangle \leq |x - y| b_0(|x - y|).$$

For any $N \geq 2$, consider the Hamiltonian for the system of $N$ particles:

$$H_N(x_1, \cdots, x_N) = \sum_{i=1}^N V(x_i) + \frac{1}{N-1} \sum_{1 \leq i < j \leq N} W(x_i, x_j),$$

and the corresponding finite-dimensional Gibbs measure

$$\mu^{(N)}(dx_1, \cdots, x_N) = \frac{1}{Z_N} e^{-H_N(x_1, \cdots, x_N)} dx_1 \cdots dx_N,$$

where $Z_N := \int_{\mathbb{R}^{dN}} e^{-H_N(x)} dx < \infty$ due to (2.13) in $(H_2)$. For any $1 \leq i \leq N$, the conditional marginal of $\mu^{(N)}$ given $z \in \mathbb{R}^{d(N-1)}$ is given by

$$\mu_z^{(N)}(dx) := \frac{1}{Z_N(z)} e^{-H_N(x|z)} dx, \quad Z_N(z) := \int_{\mathbb{R}^d} e^{-H_N(x|z)} dx,$$

$$H_N(x|z) := V(x) - \log \int_{\mathbb{R}^{d(N-1)}} e^{-\sum_{i=1}^{N-1} \{V(z_i) + \frac{1}{N-1} W(x, z_i)\}} dz_1 \cdots dz_{N-1}.$$

We have the following result.

**Theorem 2.2.** *Assume $(H_1)$-$(H_3)$. If there is a constant $a > 0$ such that the uniform log-Sobolev inequality*

$$(2.14) \qquad \mu_z^{(N)}(f^2 \log f^2) \leq \frac{1}{\beta} \mu_z^{(N)}(|\nabla f|^2), \quad f \in C_b^1(\mathbb{R}^d), \mu_z^{(N)}(f^2) = 1, N \geq 2, z \in \mathbb{R}^{d(N-1)}$$

*holds, then there exists a unique $\mu_\infty \in \mathscr{P}_2$ and a constant $c > 0$ such that*

$$(2.15) \quad \mathbb{W}_2(\mu_t, \mu_\infty)^2 + \text{Ent}(\mu_t|\mu_\infty) \leq c e^{-\lambda_a \beta (1-r_0)^2 t} \min \left\{ \mathbb{W}_2(\mu_0, \mu_\infty)^2 + \text{Ent}(\mu_0|\mu_\infty) \right\}, \quad t \geq 1$$

*holds for any probability density functions $(\rho_t)_{t \geq 0}$ solving (2.10), where $\mu_t(dx) := \rho_t(x) dx, t \geq 0$.*

This result allows $V$ and W to be non-convex. For instance, let $V = V_1 + V_2 \in C^2(\mathbb{R}^d)$ such that $\|V_1\|_\infty \wedge \|\nabla V_1\|_\infty < \infty$, $\text{Hess}_{V_2} \geq \lambda I_d$ for some $\lambda > 0$, and $W \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$ with $\|W\|_\infty \wedge \|\nabla W\|_\infty < \infty$. Then the uniform log-Sobolev inequality (2.14) holds for some constant $\beta > 0$. Indeed, by the Bakry-Emery criterion, $\mu_2(dx) := \frac{1}{\int_{\mathbb{R}^d} e^{-V(x)dx}} e^{-V_2(x)} dx$ satisfies the log-Sobolev inequality

$$\mu_2(f^2 \log f^2) \leq \frac{2}{\lambda} \mu_2(|\nabla f|^2), \quad f \in C_b^1(\mathbb{R}^d), \mu_2(f^2) = 1.$$

then (2.14) with some constant $\beta > 0$ follows by the stability of the log-Sobolev inequality under bounded perturbations (see [10, 9]) as well as Lipschitz perturbations (see [1]) for the potential $V_2$. Moreover, assumptions $(H_1)$-$(H_3)$ hold provided $\|\text{Hess}_W\|_\infty$ is small enough such that $r_0 < 1$. So, Theorem 2.2 applies. See [12] for more concrete examples satisfying $(H_1)$-$(H_3)$ and (2.14).

## 2.2   The non-degenerate case

In this part, we make the following assumptions:

$(A_1)$  $b$ is continuous on $\mathbb{R}^d \times \mathscr{P}_2$ and there exists a constant $K > 0$ such (2.25) holds.

$(A_2)$  $\sigma\sigma^*$ is invertible with $\lambda := \|(\sigma\sigma^*)^{-1}\|_\infty < \infty$, and there exist constants $K_2 > K_1 \geq 0$ such that for any $x, y \in \mathbb{R}^d$ and $\mu, \nu \in \mathscr{P}_2$,

$$\|\sigma(x) - \sigma(y)\|_{HS}^2 + 2\langle b(x,\mu) - b(y,\nu), x - y\rangle \leq K_1 \mathbb{W}_2(\mu,\nu)^2 - K_2 |x - y|^2.$$

According to [28, Theorem 2.1], if $(A_1)$ holds and $b(x,\mu)$ is continuous on $\mathbb{R}^d \times \mathscr{P}_2$, then for any initial value $X_0 \in L^2(\Omega \to \mathbb{R}^d, \mathscr{F}_0, \mathbb{P})$, (1.7) has a unique solution which satisfies

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^2\right] < \infty, \quad T \in (0, \infty).$$

Let $P_t^* \mu = \mathscr{L}_{X_t}$ for the solution with $\mathscr{L}_{X_0} = \mu$. We have the following result.

**Theorem 2.3.** *Assume $(A_1)$ and $(A_2)$. Then $P_t^*$ has a unique invariant probability measure $\mu_\infty$ such that*

$$(2.16) \quad \max\left\{\mathbb{W}_2(P_t^*\mu, \mu_\infty)^2, \text{Ent}(P_t^*\mu|\mu_\infty)\right\} \leq \frac{c_1}{t \wedge 1} e^{-(K_2-K_1)t} \mathbb{W}_2(\mu, \mu_\infty)^2, \quad t > 0, \mu \in \mathscr{P}_2$$

*holds for some constant $c_1 > 0$. If moreover $\sigma \in C_b^2(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^m)$, then there exists a constant $c_2 > 0$ such that for any $\mu \in \mathscr{P}_2, t \geq 1$,*

$$(2.17) \quad \max\left\{\mathbb{W}_2(P_t^*\mu, \mu_\infty)^2, \text{Ent}(P_t^*\mu|\mu_\infty)\right\} \leq c_2 e^{-(K_2-K_1)t} \min\left\{\mathbb{W}_2(\mu, \mu_\infty)^2, \text{Ent}(\mu|\mu_\infty)\right\}.$$

To illustrate this result, we consider the granular media equation (1.3), for which we take

$$(2.18) \quad \sigma = \sqrt{2}I_d, \qquad b(x,\mu) = -\nabla\{V + W \circledast \mu\}(x), \quad (x,\mu) \in \mathbb{R}^d \times \mathscr{P}_2.$$

The following example is not included by Theorem 2.2 since the function W may be non-symmetric.

**Example 2.1 (Granular media equation).** Consider (1.1) with $V \in C^2(\mathbb{R}^d)$ and $W \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying

$$(2.19) \qquad \text{Hess}_V \geq \lambda I_d, \quad \text{Hess}_W \geq \delta_1 I_d, \quad \|\text{Hess}_W\| \leq \delta_2$$

for some constants $\lambda_1, \delta_2 > 0$ and $\delta_1 \in \mathbb{R}$. If $\lambda + \delta_1 - \delta_2 > 0$, then there exists a unique $\mu_\infty \in \mathscr{P}_2$ and a constant $c > 0$ such that for any probability density functions $(\rho_t)_{t \geq 0}$ solving (1.3), $\mu_t(\mathrm{d}x) := \rho_t(x)\mathrm{d}x$ satisfies

$$(2.20) \quad \max\left\{ \mathbb{W}_2(\mu_t, \mu_\infty), \text{Ent}(\mu_t|\mu_\infty) \right\} \leq c e^{-(\lambda+\delta_1-\delta_2)t} \min\left\{ \mathbb{W}_2(\mu_0, \mu_\infty), \text{Ent}(\mu_0|\mu_\infty) \right\}, \quad t \geq 1.$$

*Proof.* Let $\sigma$ and $b$ be in (2.18). Then (2.19) implies $(A_1)$ and

$$\langle b(x, \mu) - b(y, \nu), x - y \rangle \leq -(\lambda_1 + \delta_1)|x-y|^2 + \delta_2|x-y|\mathbb{W}_1(\mu, \nu),$$

where we have used the formula

$$\mathbb{W}_1(\mu, \nu) = \sup\{\mu(f) - \nu(f) : \|\nabla f\|_\infty \leq 1\}.$$

So, by taking $\alpha = \frac{\delta_2}{2}$ and noting that $\mathbb{W}_1 \leq \mathbb{W}_2$, we obtain

$$\langle b(x, \mu) - b(y, \nu), x - y \rangle \leq -(\lambda + \delta_1 - \alpha)|x-y|^2 + \frac{\delta_2^2}{4\alpha}\mathbb{W}_1(\mu, \nu)^2$$

$$\leq -\left(\lambda + \delta_1 - \frac{\delta_2}{2}\right)|x-y|^2 + \frac{\delta_2}{2}\mathbb{W}_2(\mu, \nu)^2, \quad x, y \in \mathbb{R}^d, \mu, \nu \in \mathscr{P}_2.$$

Therefore, if (2.19) holds for $\lambda + \delta_1 - \delta_2 > 0$, Theorem 2.3 implies that $P_t^*$ has a unique invariant probability measure $\mu_\infty \in \mathscr{P}_2$, such that (2.20) holds for $\mu_0 \in \mathscr{P}_2$. When $\mu_0 \notin \mathscr{P}_2$, we have $\mathbb{W}_2(\mu_0, \mu_\infty)^2 = \infty$ since $\mu_\infty \in \mathscr{P}_2$. Combining this with the Talagrand inequality

$$\mathbb{W}_2(\mu_0, \mu_\infty)^2 \leq C\text{Ent}(\mu_0|\mu_\infty)$$

for some constant $C > 0$, see the proof of Theorem 2.3, we have $\text{Ent}(\mu_0|\mu_\infty) = \infty$ for $\mu_0 \notin \mathscr{P}_2$, so that (2.20) holds for all $\mu_0 \in \mathscr{P}$. $\qquad\square$

## 2.3 The degenerate case

When $\mathbb{R}^k$ with some $k \in \mathbb{N}$ is considered, to emphasize the space we use $\mathscr{P}(\mathbb{R}^k)$ $(\mathscr{P}_2(\mathbb{R}^k))$ to denote the class of probability measures (with finite second moment) on $\mathbb{R}^k$. Consider the following McKean-Vlasov stochastic Hamiltonian system for $(X_t, Y_t) \in \mathbb{R}^{d_1+d_2} := \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ :

$$(2.21) \qquad \begin{cases} \mathrm{d}X_t = BY_t\mathrm{d}t, \\ \mathrm{d}Y_t = \sqrt{2}\mathrm{d}W_t - \left\{ B^*\nabla V(\cdot, \mathscr{L}_{(X_t, Y_t)})(X_t) + \beta B^*(BB^*)^{-1}X_t + Y_t \right\}\mathrm{d}t, \end{cases}$$

where $\beta > 0$ is a constant, $B$ is a $d_1 \times d_2$-matrix such that $BB^*$ is invertible, and

$$V : \mathbb{R}^{d_1} \times \mathscr{P}_2(\mathbb{R}^{d_1+d_2}) \to \mathbb{R}^{d_2}$$

is measurable. Let

$$\psi_B((x,y),(\bar{x},\bar{y})) := \sqrt{|x-\bar{x}|^2 + |B(y-\bar{y})|^2}, \quad (x,y),(\bar{x},\bar{y}) \in \mathbb{R}^{d_1+d_2},$$

$$\mathbb{W}_2^{\psi_B}(\mu,\nu) := \inf_{\pi \in \mathscr{C}(\mu,\nu)} \left\{ \int_{\mathbb{R}^{d_1+d_2} \times \mathbb{R}^{d_1+d_2}} \psi_B{}^2 \mathrm{d}\pi \right\}^{\frac{1}{2}}, \quad \mu,\nu \in \mathscr{P}_2(\mathbb{R}^{d_1+d_2}).$$

We assume

**(C)** $V(x,\mu)$ is differentiable in $x$ such that $\nabla V(\cdot,\mu)(x)$ is Lipschitz continuous in $(x,\mu) \in \mathbb{R}^{d_1} \times \mathscr{P}_2(\mathbb{R}^{d_1+d_2})$. Moreover, there exist constants $\theta_1, \theta_2 \in \mathbb{R}$ with

(2.22) $$\theta_1 + \theta_2 < \beta,$$

such that for any $(x,y),(x',y') \in \mathbb{R}^{d_1+d_2}$ and $\mu,\mu' \in \mathscr{P}_2(\mathbb{R}^{d_1+d_2})$,

(2.23) $$\langle BB^*\{\nabla V(\cdot,\mu)(x) - \nabla V(\cdot,\mu')(x')\}, x - x' + (1+\beta)B(y-y') \rangle$$
$$\geq -\theta_1\psi_B((x,y),(x',y'))^2 - \theta_2\mathbb{W}_2^{\psi_B}(\mu,\mu')^2.$$

Obviously, **(C)** implies $(A_1)$ for $d = m = d_1 + d_2$, $\sigma = \mathrm{diag}\{0, \sqrt{2}I_{d_2}\}$, and

$$b((x,y),\mu) = \big(By, -B^*\nabla V(\cdot,\mu)(x) - \beta B^*(BB^*)^{-1}x - y\big).$$

So, according to [28], (2.21) is well-posed for any initial value in $L^2(\Omega \to \mathbb{R}^{d_1+d_2}, \mathscr{F}_0, \mathbb{P})$. Let $P_t^*\mu = \mathscr{L}_{(X_t,Y_t)}$ for the solution with initial distribution $\mu \in \mathscr{P}_2(\mathbb{R}^{d_1+d_2})$. In this case, (2.21) becomes

$$\begin{cases} \mathrm{d}X_t = BY_t\mathrm{d}t, \\ \mathrm{d}Y_t = \sqrt{2}\mathrm{d}W_t + Z_t(X_t,Y_t)\mathrm{d}t, \end{cases}$$

where $Z_t(x,y) := -B^*\{\nabla V(\cdot, P_t^*\mu)\}(x) + \beta B^*(BB^*)^{-1}x + y$. According to [26, Theorems 2.4 and 3.1], when $\mathrm{Hess}_V(\cdot, P_t^*\mu)$ is bounded,

$$\rho_t(z) := \frac{(P_t^*\mu)(\mathrm{d}z)}{\mathrm{d}z} = \frac{\mathrm{d}(\mathscr{L}_{(X_t,Y_t)})(\mathrm{d}z)}{\mathrm{d}z}$$

exists and is differentiable in $z \in \mathbb{R}^{d_1+d_2}$. Moreover, since **(C)** implies that the class

$$\{\partial_{y_j}, [\partial_{y_j}, (By)_i\partial_{x_i}] : 1 \leq i \leq d_1, 1 \leq j \leq d_2\}$$

spans the tangent space at any point (i.e. the Hörmander condition of rank 1 holds), according to the Hörmander theorem, $\rho_t \in C^\infty(\mathbb{R}^{d_1+d_2})$ for $t > 0$ provided $Z_t \in C^\infty(\mathbb{R}^{d_1+d_2})$ for $t \geq 0$.

**Theorem 2.4.** *Assume* **(C)**. *Then $P_t^*$ has a unique invariant probability measure $\mu_\infty$ such that for any $t > 0$ and $\mu \in \mathscr{P}_2(\mathbb{R}^{d_1+d_2})$,*

(2.24) $$\max\left\{\mathbb{W}_2(P_t^*\mu, \mu_\infty)^2, \mathrm{Ent}(P_t^*\mu|\mu_\infty)\right\} \leq \frac{ce^{-2\kappa t}}{(1 \wedge t)^3} \min\left\{\mathrm{Ent}(\mu|\mu_\infty), \mathbb{W}_2(\mu,\mu_\infty)^2\right\}$$

*holds for some constant $c > 0$ and*

(2.25) $$\kappa := \frac{2(\beta - \theta_1 - \theta_2)}{2 + 2\beta + \beta^2 + \sqrt{\beta^4 + 4}} > 0.$$

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**Example 2.2 (Degenerate granular media equation).** Let $m \in \mathbb{N}$ and $W \in C^\infty(\mathbb{R}^m \times \mathbb{R}^{2m})$. Consider the following PDE for probability density functions $(\rho_t)_{t \geq 0}$ on $\mathbb{R}^{2m}$:

$$(2.26) \qquad \partial_t \rho_t(x, y) = \Delta_y \rho_t(x, y) - \langle \nabla_x \rho_t(x, y), y \rangle + \langle \nabla_y \rho_t(x, y), \nabla_x(W \circledast \rho_t)(x) + \beta x + y \rangle,$$

where $\beta > 0$ is a constant, $\Delta_y, \nabla_x, \nabla_y$ stand for the Laplacian in $y$ and the gradient operators in $x, y$ respectively, and

$$(W \circledast \rho_t)(x) := \int_{\mathbb{R}^{2m}} W(x, z) \rho_t(z) \mathrm{d}z, \quad x \in \mathbb{R}^m.$$

If there exists a constant $\theta \in \left(0, \frac{2\beta}{1 + 3\sqrt{2 + 2\beta + \beta^2}}\right)$ such that

$$(2.27) \qquad |\nabla W(\cdot, z)(x) - \nabla W(\cdot, \bar{z})(\bar{x})| \leq \theta\big(|x - \bar{x}| + |z - \bar{z}|\big), \quad x, \bar{x} \in \mathbb{R}^m, z, \bar{z} \in \mathbb{R}^{2m},$$

then there exists a unique probability measure $\mu_\infty \in \mathscr{P}_2(\mathbb{R}^{2m})$ and a constant $c > 0$ such that for any probability density functions $(\rho_t)_{t \geq 0}$ solving (2.26), $\mu_t(\mathrm{d}x) := \rho_t(x) \mathrm{d}x$ satisfies

$$(2.28) \qquad \max\left\{\mathbb{W}_2(\mu_t, \mu_\infty)^2, \mathrm{Ent}(\mu_t | \mu_\infty)\right\} \leq ce^{-\kappa t} \min\left\{\mathbb{W}_2(\mu_0, \mu_\infty)^2, \mathrm{Ent}(\mu_0 | \mu_\infty)\right\}, \quad t \geq 1$$

holds for $\kappa = \frac{2\beta - \theta\left(1 + 3\sqrt{2 + 2\beta + \beta^2}\right)}{2 + 2\beta + \beta^2 + \sqrt{\beta^4 + 4}} > 0$.

*Proof.* Let $d_1 = d_2 = m$ and $(X_t, Y_t)$ solve (2.21) for

$$(2.29) \qquad\qquad B := I_m, \quad V(x, \mu) := \int_{\mathbb{R}^{2m}} W(x, z) \mu(\mathrm{d}z).$$

We first observe that $\rho_t$ solves (2.26) if and only if $\rho_t(z) = \frac{\mathrm{d}(P_t^* \mu)(\mathrm{d}z)}{\mathrm{d}z}$ for $\mu(\mathrm{d}z) = \rho_0(z)\mathrm{d}z$, where $P_t^* \mu := \mathscr{L}_{(X_t, Y_t)}$.

Firstly, let $\rho_t(z) = \frac{\mathscr{L}_{(X_t, Y_t)}(\mathrm{d}z)}{\mathrm{d}z}$ which exists and is smooth as explained before Theorem 2.4. By Itô's formula and the integration by parts formula, for any $f \in C_0^2(\mathbb{R}^{2m})$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^{2m}} (\rho_t f)(z)\mathrm{d}z = \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(X_t, Y_t)]$$

$$= \int_{\mathbb{R}^{2m}} \rho_t(x, y)\Big\{\Delta_y f(x, y) + \langle \nabla_x f(x, y), y \rangle - \langle \nabla_y f(x, y), \nabla_x V(x, \rho_t(z)\mathrm{d}z) + \beta x + y \rangle\Big\}\mathrm{d}x\mathrm{d}y$$

$$= \int_{\mathbb{R}^{2m}} f(x, y)\Big\{\Delta_y \rho_t(x, y) - \langle \nabla_x \rho_t(x, y), y \rangle + \langle \nabla_y \rho_t(x, y), \nabla_x \mu_t(W(x, \cdot)) + \beta x + y \rangle\Big\}\mathrm{d}x\mathrm{d}y.$$

Then $\rho_t$ solves (2.26).

On the other hand, let $\rho_t$ solve (2.26) with $\mu_0(\mathrm{d}z) := \rho_0(z)\mathrm{d}z \in \mathscr{P}_2(\mathbb{R}^{2m})$. By the integration by parts formula, $\mu_t(\mathrm{d}z) := \rho_t(z)\mathrm{d}z$ solves the non-linear Fokker-Planck equation

$$\partial_t \mu_t = L_{\mu_t}^* \mu_t$$

in the sense that for any $f \in C_0^\infty(\mathbb{R}^{d_1+d_2})$ we have

$$\mu_t(f) = \mu_0(f) + \int_0^t \mu_s(L_{\mu_s}f)\mathrm{d}s, \quad t \geq 0,$$

where $L_\mu := \Delta_y + y \cdot \nabla_x - \{\nabla_x \mu(W(x,\cdot)) + \beta x - y\} \cdot \nabla_y$. By the superposition principle, see [5, Section 2], we have $\mu_t = P_t^*\mu$.

Now, as explained in the proof of Example 2.1, by Theorem 2.4 we only need to verify (C) for $B, V$ in (2.29) and

$$(2.30) \qquad \theta_1 = \theta\Big(\frac{1}{2} + \sqrt{2 + 2\beta + \beta^2}\Big), \quad \theta_2 = \frac{\theta}{2}\sqrt{2 + 2\beta + \beta^2},$$

so that the desired assertion holds for

$$\kappa := \frac{2(\beta - \theta_1 - \theta_2)}{2 + 2\beta + \beta^2 + \sqrt{\beta^4 + 4}} = \frac{2\beta - \theta(1 + 3\sqrt{2 + 2\beta + \beta^2})}{2 + 2\beta + \beta^2 + \sqrt{\beta^4 + 4}}.$$

By (2.27) and $V(x,\mu) := \mu(W(x,\cdot))$, for any constants $\alpha_1, \alpha_2, \alpha_3 > 0$ we have

$$
\begin{aligned}
I &:= \big\langle \nabla V(\cdot,\mu)(x) - \nabla V(\cdot,\bar\mu)(\bar x), x - \bar x + (1+\beta)(y - \bar y) \big\rangle \\
&= \int_{\mathbb{R}^{2m}} \big\langle \nabla W(\cdot,z)(x) - \nabla W(\cdot,z)(\bar x), x - \bar x + (1+\beta)(y - \bar y) \big\rangle \mu(\mathrm{d}z) \\
&\quad + \big\langle \mu(\nabla_{\bar x}W(\bar x,\cdot)) - \bar\mu(\nabla_{\bar x}W(\bar x,\cdot)), x - \bar x + (1+\beta)(y - \bar y) \big\rangle \\
&\geq -\theta\big\{|x - \bar x| + \mathbb{W}_1(\mu,\bar\mu)\big\} \cdot \big(|x - \bar x| + (1+\beta)|y - \bar y|\big) \\
&\geq -\theta(\alpha_2 + \alpha_3)\mathbb{W}_2(\mu,\bar\mu)^2 - \theta\Big\{\Big(1 + \alpha_1 + \frac{1}{4\alpha_2}\Big)|x - \bar x|^2 + (1+\beta)^2\Big(\frac{1}{4\alpha_1} + \frac{1}{4\alpha_3}\Big)|y - \bar y|^2\Big\}.
\end{aligned}
$$

Take

$$\alpha_1 = \frac{\sqrt{2 + 2\beta + \beta^2} - 1}{2}, \quad \alpha_2 = \frac{1}{2\sqrt{2 + 2\beta + \beta^2}}, \quad \alpha_3 = \frac{(1+\beta)^2}{2\sqrt{2 + 2\beta + \beta^2}}.$$

We have

$$
\begin{aligned}
1 + \alpha_1 + \frac{1}{4\alpha_2} &= \frac{1}{2} + \sqrt{2 + 2\beta + \beta^2}, \\
(1+\beta)^2\Big(\frac{1}{4\alpha_1} + \frac{1}{4\alpha_3}\Big) &= \frac{1}{2} + \sqrt{2 + 2\beta + \beta^2}, \\
\alpha_2 + \alpha_3 &= \frac{1}{2}\sqrt{2 + 2\beta + \beta^2}.
\end{aligned}
$$

Therefore,

$$I \geq -\frac{\theta}{2}\sqrt{2 + 2\beta + \beta^2}\,\mathbb{W}_2(\mu,\bar\mu)^2 - \theta\Big(\frac{1}{2} + \sqrt{2 + 2\beta + \beta^2}\Big)|(x,y) - (\bar x, \bar y)|^2,$$

i.e. (C) holds for $B$ and $V$ in (2.29) where $B = I_m$ implies that $\psi_B$ is the Euclidean distance on $\mathbb{R}^{2m}$, and for $\theta_1, \theta_2$ in (2.30). $\qquad\square$

# 3 Proofs of Theorems 2.1 and 2.2

*Proof of Theorem 2.1.* (1) Since

$$\mathrm{Ent}(P_{t_0}^*\nu|P_{t_0}^*\mu) = \sup_{f\geq 0,(P_{t_0}f)(\mu)=1} P_{t_0}(\log f)(\nu),$$

(2.3) implies

$$\mathrm{Ent}(P_{t_0}^*\nu|P_{t_0}^*\mu) \leq c_0 \mathbb{W}_2(\mu,\nu)^2.$$

This together with $P_{t_0}^*\mu_\infty = \mu_\infty$ gives

(3.1) $$\mathrm{Ent}(P_{t_0}^*\mu|\mu_\infty) \leq c_0 \mathbb{W}_2(\mu,\mu_\infty)^2, \quad \mu \in \mathscr{P}_2.$$

Combining (2.5) with (2.4) and (3.1), we obtain

$$\mathbb{W}_2(P_t^*\mu,\mu_\infty)^2 \leq c_1 \mathrm{e}^{-\lambda t}\mathbb{W}_2(\mu,\mu_\infty)^2 \leq c_1 \mathrm{e}^{-\lambda t} \min\left\{\mathbb{W}_2(\mu,\mu_\infty)^2, C\mathrm{Ent}(\mu|\mu_\infty)\right\}, \quad t \geq t_1$$

and

$$\mathrm{Ent}(P_t^*\mu|\mu_\infty) = \mathrm{Ent}(P_{t_0}^*P_{t-t_0}^*\mu|\mu_\infty) \leq c_0\mathbb{W}_2(P_{t-t_0}^*\mu,\mu_\infty)^2 \leq c_0 c_1 \mathrm{e}^{-\lambda(t-t_0)}\mathbb{W}_2(\mu,\mu_\infty)^2$$
$$= \{c_0 c_1 \mathrm{e}^{\lambda t_0}\}\mathrm{e}^{-\lambda t}\min\left\{\mathbb{W}_2(\mu,\mu_\infty)^2, C\mathrm{Ent}(\mu|\mu_\infty)\right\}, \quad t \geq t_0 + t_1.$$

Therefore, (2.6) holds.

(2) Similarly, if (2.7) holds, then (2.4) and (3.1) imply

$$\mathrm{Ent}(P_t^*\mu|\mu_\infty) \leq c_2 \mathrm{e}^{-\lambda(t-t_0)}\min\left\{\mathrm{Ent}(P_{t_0}^*\mu|\mu_\infty), \mathrm{Ent}(\mu|\mu_\infty)\right\}$$
$$\leq c_2 \mathrm{e}^{-\lambda(t-t_0)}\min\left\{c_0\mathbb{W}_2(\mu,\mu_\infty)^2, \mathrm{Ent}(\mu|\mu_\infty)\right\}, \quad t \geq t_0 + t_2$$

and

$$C^{-1}\mathbb{W}_2(P_t^*\mu,\mu_\infty)^2 \leq \mathrm{Ent}(P_{t-t_0}^*P_{t_0}^*\mu|\mu_\infty)$$
$$\leq c_2 \min\left\{\mathrm{e}^{-\lambda t}\mathrm{Ent}(\mu|\mu_\infty), \mathrm{e}^{-\lambda(t-t_0)}\mathrm{Ent}(P_{t_0}^*\mu|\mu_\infty)\right\}$$
$$\leq c_2 \mathrm{e}^{-\lambda(t-t_0)}\min\left\{\mathrm{Ent}(\mu|\mu_\infty), c_0\mathbb{W}_2(\mu,\mu_\infty)^2\right\}, \quad t \geq t_0 + t_2.$$

Then (2.8) holds, and the proof is finished. $\qquad\square$

*Proof of Theorem 2.2.* By [12, Theorem 10], there exits a unique $\mu_\infty \in \mathscr{P}_2$ such that

(3.2) $$\mathrm{Ent}^{V,W}(\mu_\infty) = 0.$$

Let $\mu_0 = \rho_0 \mathrm{d}x \in \mathscr{P}_2$. We first note that $\mu_t = P_t^*\mu_0 := \mathscr{L}_{X_t}$ for $X_t$ solving the distribution dependent SDE (2.1) with

(3.3) $$\sigma(x,\mu) = \sqrt{2a(x)}, \quad b(x,\mu) = \sum_{j=1}^{d}\partial_j a_{\cdot,j}(x) - a\nabla\{V + W \circledast \mu\}(x), \quad x \in \mathbb{R}^d, \mu \in \mathscr{P}_2.$$

Obviously, for this choice of $(\sigma, b)$, assumptions $(H_1)$ and $(H_2)$ imply condition (2.25) for some constant $K > 0$, so that the McKean-Vlasov (2.1) SDE is weakly and strongly well-posed. For any $N \geq 2$, let $\mu_t^{(N)} = \mathscr{L}_{X_t^{(N)}}$ for the mean field particle system $X_t^{(N)} = (X_t^{N,k})_{1 \leq i \leq N}$:

$$(3.4) \quad \mathrm{d}X_t^{N,k} = \sqrt{2}\sigma(X_t^{N,k})\mathrm{d}B_t^k + \Big\{ \sum_{j=1}^{d} \partial_j a_{\cdot j}(X_t^{N,k}) - a(X_t^{N,k})\nabla_k H_N(X_t^{(N)}) \Big\}\mathrm{d}t, \quad t \geq 0,$$

where $\nabla_k$ denotes the gradient in the $k$-th component, and $\{X_0^{N,k}\}_{1 \leq i \leq N}$ are i.i.d. with distribution $\mu_0 \in \mathscr{P}_2$. According to the propagation of chaos, see [19], $(H_1)$-$(H_3)$ imply

$$(3.5) \qquad\qquad \lim_{N \to \infty} \mathbb{W}_2(\mathscr{L}_{X_t^{N,1}}, P_t^*\mu_0) = 0.$$

Next, our conditions imply (25) and (26) in [12] for $\rho_{LS} = \beta(1 - r_0)^2$. So, by [12, Theorem 8(2)], we have the log-Sobolev inequality

$$(3.6) \qquad \mu^{(N)}(f^2 \log f^2) \leq \frac{2}{\beta(1 - r_0)^2}\mu^{(N)}(|\nabla f|^2), \quad f \in C_b^1(\mathbb{R}^{dN}), \mu^{(N)}(f^2) = 1.$$

By [6], this implies the Talagrand inequality

$$(3.7) \qquad \mathbb{W}_2(\nu^{(N)}, \mu^{(N)})^2 \leq \frac{2}{\beta(1 - r_0)^2}\mathrm{Ent}(\nu^{(N)}|\mu^{(N)}), \quad t \geq 0, N \geq 2, \nu^{(N)} \in \mathscr{P}(\mathbb{R}^{dN}).$$

On the other hand, by Itô's formula we see that the generator of the diffusion process $X_t^{(N)}$ is

$$L^{(N)}(x^{(N)}) = \sum_{i,j,k=1}^{d} \Big\{ a_{ij}(x^{N,k})\partial_{x_i^{N,k}}\partial_{x_j^{N,k}} + \partial_j a_{ij}(x^{N,k})\partial_{x_i^{N,k}} - a_{ij}(x^{N,k})\big[\partial_{x_i^{N,k}}H_N(x^{(N)})\big]\partial_{x_i^{N,k}} \Big\},$$

for $x^{(N)} = (x^{N,1}, \cdots, x^{N,N}) \in \mathbb{R}^{dN}$, where $x_i^{N,k}$ is the $i$-th component of $x^{N,k} \in \mathbb{R}^d$. Using the integration by parts formula, we see that this operator is symmetric in $L^2(\mu^{(N)})$:

$$\mathscr{E}^{(N)}(f, g) := \int_{\mathbb{R}^{dN}} \langle a^{(N)}\nabla f, \nabla g \rangle \mathrm{d}\mu^{(N)} = -\int_{\mathbb{R}^{dN}} (fL^{(N)}g)\mathrm{d}\mu^{(N)}, \quad f, g \in C_0^{\infty}(\mathbb{R}^{dN}),$$

where $a^{(N)}(x^{(N)}) := \mathrm{diag}\{a(x^{N,1}), \cdots, a(x^{N,N})\}, x^{(N)} = (x^{N,1}, \cdots, x^{N,N}) \in \mathbb{R}^{dN}$. So, the closure of the pre-Dirichelt form $(\mathscr{E}^{(N)}, C_0^{\infty}(\mathbb{R}^{dN}))$ in $L^2(\mu^{(N)})$ is the Dieichlet form for the Markov semigroup $P_t^{(N)}$ of $X_t^{(N)}$. By $(H_1)$ we have $a^{(N)} \geq \lambda_a I_{dN}$, so that (3.6) implies

$$\mu^{(N)}(f^2 \log f^2) \leq \frac{2}{\beta\lambda_a(1 - r_0)^2}\mathscr{E}^{(N)}(f, f), \quad f \in C_b^1(\mathbb{R}^{dN}), \mu^{(N)}(f^2) = 1.$$

It is well known that this log-Sobolev inequality implies the exponential convergence

$$(3.8) \qquad \begin{aligned} \mathrm{Ent}(\mu_t^{(N)}|\mu^{(N)}) &\leq \mathrm{e}^{-\lambda_a\beta(1-r_0)^2 t}\mathrm{Ent}(\mu_0^{(N)}|\mu^{(N)}) \\ &= \mathrm{e}^{-\lambda_a\beta(1-r_0)^2 t}\mathrm{Ent}(\mu^{\otimes N}|\mu^{(N)}), \quad t \geq 0, N \geq 2, \end{aligned}$$

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see for instance [3, Theorem 5.2.1]. Moreover, since $\mathrm{Hess}_V$ and $\mathrm{Hess}_W$ are bounded from below, $(H_1)$ implies that the Bakry-Emery curvature of the generator of $X_t^{(N)}$ is bounded by a constant. Then according to [22], there exists a constant $K \geq 0$ such that the Markov semigroup $P_t^{(N)}$ of $X_t^{(N)}$ satisfies the log-Harnack inequality

$$(3.9) \qquad P_t^{(N)} \log f(x) \leq \log P_t^{(N)} f(y) + \frac{K \rho^{(N)}(x,y)^2}{2(1 - \mathrm{e}^{-2Kt})}, \quad 0 < f \in \mathscr{B}_b(\mathbb{R}^{dN}), t > 0, x, y \in \mathbb{R}^{dN},$$

where $\rho^{(N)}$ is the intrinsic distance induced by the Dirichlet form $\mathscr{E}^{(N)}$. Since $a^{(N)} \geq \lambda_a I_{dN}$, we have $\rho^{(N)}(x,y)^2 \leq \lambda_a^{-1}|x - y|^2$. So, (3.9) implies (2.3) for $P_t^{(N)}$ replacing $P_{t_0}$ and $c_0 = \frac{K}{2\lambda_a(1-\mathrm{e}^{-2Kt})}$ :

$$P_t^{(N)}(\log f)(\nu) \leq \log P_t^{(N)} f(\mu) + \frac{K \, \mathbb{W}_2(\mu,\nu)^2}{2\lambda_a(1 - \mathrm{e}^{-2Kt})}, \quad 0 < f \in \mathscr{B}_b(\mathbb{R}^{dN}), t > 0, \mu, \nu \in \mathscr{P}_2(\mathbb{R}^{dN}).$$

Thus, by Theorem 2.1, (3.8) implies

$$(3.10) \qquad \mathbb{W}_2(\mu_t^{(N)}|\mu^{(N)})^2 \leq \frac{c_1 \mathrm{e}^{-\lambda_a \beta(1-r_0)^2 t}}{1 \wedge t} \mathbb{W}_2(\mu^{\otimes N}, \mu^{(N)})^2, \quad t > 0, N \geq 2$$

for some constant $c_1 > 0$. Moreover, (3.7), (3.2) and [12, Lemma 17] yield

$$(3.11) \qquad \begin{aligned} \lim_{N \to \infty} \frac{1}{N} \mathbb{W}_2(\mu_\infty^{\otimes N}, \mu^{(N)})^2 &\leq \limsup_{N \to \infty} \frac{2}{\beta(1-r_0)^2 N} \mathrm{Ent}(\mu_\infty^{\otimes N}|\mu^{(N)})^2 \\ &= \frac{2}{\beta(1-r_0)^2} \mathrm{Ent}^{V,W}(\mu_\infty) = 0. \end{aligned}$$

Combining this with (3.10) we derive

$$(3.12) \qquad \begin{aligned} \limsup_{N \to \infty} \frac{1}{N} \mathbb{W}_2(\mu_t^{(N)}, \mu_\infty^{\otimes N})^2 &= \limsup_{N \to \infty} \frac{1}{N} \mathbb{W}_2(\mu_t^{(N)}, \mu^{(N)})^2 \\ &\leq \frac{c_1 \mathrm{e}^{-\lambda_a \beta(1-r_0)^2 t}}{1 \wedge t} \limsup_{N \to \infty} \frac{1}{N} \mathbb{W}_2(\mu_0^{\otimes N}, \mu^{(N)})^2 \\ &= \frac{c_1 \mathrm{e}^{-\lambda_a \beta(1-r_0)^2 t}}{1 \wedge t} \limsup_{N \to \infty} \frac{1}{N} \mathbb{W}_2(\mu_0^{\otimes N}, \mu_\infty^{\otimes N})^2 = \frac{c_1 \mathrm{e}^{-\lambda_a \beta(1-r_0)^2 t}}{1 \wedge t} \mathbb{W}_2(\mu_0, \mu_\infty)^2, \quad t > 0. \end{aligned}$$

Now, let $\xi = (\xi_i)_{1 \leq i \leq N}$ and $\eta = (\eta_i)_{1 \leq i \leq N}$ be random variables on $\mathbb{R}^{dN}$ such that $\mathscr{L}_\xi = \mu_t^{(N)}, \mathscr{L}_\eta = \mu_\infty^{\otimes N}$ and

$$\sum_{i=1}^N \mathbb{E}|\xi_i - \eta_i|^2 = \mathbb{E}|\xi - \eta|^2 = \mathbb{W}_2(\mu_t^{(N)}, \mu_\infty^{\otimes N})^2.$$

We have $\mathscr{L}_{\xi_i} = \mathscr{L}_{X_t^{N,1}}, \mathscr{L}_{\eta_i} = \mu_\infty$ for any $1 \leq i \leq N$, so that

$$(3.13) \qquad N\mathbb{W}_2(\mathscr{L}_{X_t^{N,1}}, \mu_\infty)^2 \leq \sum_{i=1}^N \mathbb{E}|\xi_i - \eta_i|^2 = \mathbb{W}_2(\mu_t^{(N)}, \mu_\infty^{\otimes N})^2.$$

Substituting this into (3.12), we arrive at

$$\limsup_{N\to\infty} \mathbb{W}_2(\mathscr{L}_{X_t^{N,1}}, \mu_\infty)^2 \leq \frac{c_1 e^{-\lambda_a \beta(1-r_0)^2 t}}{1 \wedge t} \mathbb{W}_2(\mu, \mu_\infty)^2 \ , \quad t > 0.$$

This and (3.5) imply

(3.14) $$\mathbb{W}_2(P_t^*\mu, \mu_\infty)^2 \leq \frac{c_1 e^{-\lambda_a \beta(1-r_0)^2 t}}{1 \wedge t} \mathbb{W}_2(\mu, \mu_\infty)^2 \ , \quad t > 0.$$

By [28, Theorem 4.1], $(H_1)$-$(H_3)$ imply the log-Harnack inequality

(3.15) $$P_t(\log f)(\nu) \leq \log P_t f(\mu) + \frac{c_2}{1 \wedge t} \mathbb{W}_2(\mu, \nu)^2, \quad \mu, \nu \in \mathscr{P}_2, t > 0$$

for some constant $c_2 > 0$. Similarly to the proof of (3.13) we have

$$N \mathbb{W}_2(\mu_\infty, \mu^{(N,1)})^2 \leq \mathbb{W}_2(\mu_\infty^{\otimes N}, \mu^{(N)})^2,$$

where $\mu^{(N,1)} := \mu^{(N)}(\cdot \times \mathbb{R}^{d(N-1)})$ is the first marginal distribution of $\mu^{(N)}$. This together with (3.11) implies

$$\lim_{N\to\infty} \mathbb{W}_2(\mu^{(N,1)}, \mu_\infty)^2 = 0.$$

Therefore, applying (3.6) to $f(x)$ depending only on the firs component $x_1$, and letting $N \to \infty$, we derive the log-Sobolev inequality

$$\mu_\infty(f^2 \log f^2) \leq \frac{2}{\beta(1-r_0)^2} \mu_\infty(|\nabla f|^2), \quad f \in C_b^1(\mathbb{R}^d), \mu_\infty(f^2) = 1.$$

By [6], this implies (2.4) for $C = \frac{2}{\beta(1-r_0)^2}$. Combining this with the log-Harnack inequality and (3.14), by Theorem 2.1 we prove (2.15) for some constant $c > 0$ and $\mu_t = \mathscr{L}_{X_t} = P_t^*\mu_0$ for solutions to (2.1) with $b, \sigma$ in (3.3).

Similarly to the link of (1.3) and (1.5) shown in Introduction, for any probability density functions $\rho_t$ solving (2.10), we have $\rho_t dx = P_t^*\mu_0$ for $\mu_0 = \rho_0 dx \in \mathscr{P}_2$. So, we have proved (2.15) for $\rho_t$ solving (2.10) with $\mu_0 \in \mathscr{P}_2$. As explained in the proof of Example 2.1 that $\mathrm{Ent}(\mu_0, \mu_\infty) = \mathbb{W}_2(\mu, \mu_\infty) = \infty$ for $\mu_0 \notin \mathscr{P}_2$, so that the desired inequality (2.15) tribally true. Then the proof is finished. □

# 4 Proof of Theorems 2.3

According to [28, Theorem 3.1], $(A_1)$ and $(A_2)$ imply that $P_t^*$ has a unique invariant probability measure $\mu_\infty$ and

(4.1) $$\mathbb{W}_2(P_t^*\mu, \mu_\infty) \leq e^{-\frac{1}{2}(K_2-K_1)t} \mathbb{W}_2(\mu, \mu_\infty), \quad t \geq 0, \mu \in \mathscr{P}_2,$$

while [28, Corollary 4.3] implies

$$\mathrm{Ent}(P_t^*\mu | \mu_\infty) \leq \frac{c_0}{1 \wedge t} \mathbb{W}_2(\mu, \mu_\infty)^2, \quad t > 0, \mu \in \mathscr{P}_2$$

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for some constant $c_0 > 0$. Then for any $p > 1$, combining these with $P_t^* = P_{1 \wedge t}^* P_{(t-1)^+}^*$, we obtain

$$\operatorname{Ent}(P_t^* \mu | \mu_\infty) = \operatorname{Ent}(P_{1 \wedge t}^* P_{(t-1)^+}^* \mu | \mu_\infty) \leq \frac{c_0}{1 \wedge t} \mathbb{W}_2(P_{(t-1)^+}^* \mu, \mu_\infty)^2$$

$$\leq \frac{c_0 e^{-(K_2 - K_1)(t-1)^+}}{1 \wedge t} \mathbb{W}_2(\mu, \mu_\infty)^2 = \frac{c_0 e^{K_2 - K_1}}{1 \wedge t} e^{-(K_2 - K_1)t} \mathbb{W}_2(\mu, \mu_\infty)^2.$$

This together with (4.1) implies (2.16) for some constant $c_1 > 0$.

Now, let $\sigma \in C_b^2(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^m)$. To deduce (2.17) from (2.16), it remains to find a constant $c > 0$ such that the following Talagrand inequality holds:

$$\mathbb{W}_2(\mu, \mu_\infty)^2 \leq c \operatorname{Ent}(\mu | \mu_\infty), \quad \mu \in \mathscr{P}_2.$$

According to [6], this inequality follows from the log-Sobolev inequality

(4.2) $$\mu_\infty(f^2 \log f^2) \leq c \mu_\infty(|\nabla f|^2), \quad f \in C_b^1(\mathbb{R}^d), \mu_\infty(f^2) = 1.$$

To prove this inequality, we consider the diffusion process $\bar{X}_t$ on $\mathbb{R}^d$ generated by

$$\bar{L} := \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^*)_{ij} \partial_i \partial_j + \sum_{i=1}^\infty b_i(\cdot, \mu_\infty) \partial_i,$$

which can be constructed by solving the SDE

(4.3) $$d\bar{X}_t = \sigma(\bar{X}_t) dW_t + b(\bar{X}_t, \mu_\infty) dt.$$

Let $\bar{P}_t$ be the associated Markov semigroup. Since $P_t^* \mu_\infty = \mu_\infty$, when $\mathscr{L}_{X_0} = \mu_\infty$ the SDE (4.3) coincides with (1.7) so that by the uniqueness, we see that $\mu_\infty$ is an invariant probability measure of $\bar{P}_t$. Combining this with $(A_2)$ and Itô's formula, we obtain

(4.4) $$\mathbb{W}_2(\mathscr{L}_{\bar{X}_t}, \mu_\infty)^2 \leq e^{-K_2 t} \mathbb{W}_2(\mathscr{L}_{\bar{X}_0}, \mu_\infty)^2, \quad t > 0.$$

To prove the log-Sobolev inequality (4.2), we first verify the hyperboundedness of $\bar{P}_t$, i.e. for large $t > 0$ we have

(4.5) $$\|\bar{P}_t\|_{L^2(\mu_\infty) \to L^4(\mu_\infty)} < \infty.$$

It is easy to see that conditions $(A_1)$ and $(A_2)$ in Theorem 2.3 imply that $\sigma$ and $b(\cdot, \mu_\infty)$ satisfy conditions $(A1)$-$(A3)$ in [23] for $K = -(K_2 - K_1), \lambda_t^2 = \lambda$ and $\delta_t = \|\sigma\|_\infty$. So, by [23, Theorem 1.1(3)], we find a constant $C > 0$ such that the following Harnack inequality holds:

$$(\bar{P}_t f(x))^2 \leq \bar{P}_t f^2(y) \exp\left[\frac{C|x-y|^2}{e^{(K_2 - K_1)t} - 1}\right], \quad t > 0.$$

Then for any $f$ with $\mu_\infty(f^2) \leq 1$, we have

$$(\bar{P}_t f(x))^2 \int_{\mathbb{R}^d} \exp\left[-\frac{C|x-y|^2}{e^{(K_2 - K_1)t} - 1}\right] \mu_\infty(dy)$$

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$$\le \mu_\infty(\bar{P}_t f^2) = \mu_\infty(f^2) \le 1.$$

So,

(4.6)
$$\sup_{\mu_\infty(f^2)\le 1} |\bar{P}_t f(x)|^4 \le \frac{1}{\left(\int_{\mathbb{R}^d} e^{-\frac{C|x-y|^2}{e^{(K_2-K_1)t}-1}} \mu_\infty(\mathrm{d}y)\right)^2}$$

$$\le \frac{1}{\left(\int_{B(0,1)} e^{-\frac{C|x-y|^2}{e^{(K_2-K_1)t}-1}} \mu_\infty(\mathrm{d}y)\right)^2} \le C_1 \exp\left[C_1 e^{-(K_2-K_1)t}|x|^2\right], \quad t \ge 1, x \in \mathbb{R}^d.$$

Next, by $\|\sigma\|_\infty < \infty$, $(A_2)$ and Itô's formula, for any $k \in (0, K_2)$ there exists a constant $c_k > 0$ such that

$$\mathrm{d}|\bar{X}_t|^2 \le 2\langle \bar{X}_t, \sigma(\bar{X}_t)\mathrm{d}W_t\rangle + \left\{c_k - k|\bar{X}_t|^2\right\}\mathrm{d}t.$$

Then for any $\varepsilon > 0$,

(4.7)
$$\mathrm{d}e^{\varepsilon|\bar{X}_t|^2} \le 2\varepsilon e^{\varepsilon|\bar{X}_t|^2}\langle \bar{X}_t, \sigma(\bar{X}_t)\mathrm{d}W_t\rangle + \varepsilon e^{\varepsilon|\bar{X}_t|^2}\left\{c_k + 2\varepsilon\|\sigma\|_\infty^2|\bar{X}_t|^2 - k|\bar{X}_t|^2\right\}\mathrm{d}t.$$

When $\varepsilon > 0$ is small enough such that $2\varepsilon\|\sigma\|_\infty^2 < K_2$, there exist constants $c_1(\varepsilon), c_2(\varepsilon) > 0$ such that

$$\varepsilon e^{\varepsilon|\bar{X}_t|^2}\left\{c_k + 2\varepsilon\|\sigma\|_\infty^2|\bar{X}_t|^2 - k|\bar{X}_t|^2\right\} \le c_1(\varepsilon) - c_2(\varepsilon)e^{\varepsilon|\bar{X}_t|^2}.$$

Combining this with (4.7) we obtain

$$\mathrm{d}e^{\varepsilon|\bar{X}_t|^2} \le c_1(\varepsilon) - c_2(\varepsilon)e^{\varepsilon|\bar{X}_t|^2}\mathrm{d}t + 2\varepsilon e^{\varepsilon|\bar{X}_t|^2}\langle \bar{X}_t, \sigma(\bar{X}_t)\mathrm{d}W_t\rangle.$$

Taking for instance $\bar{X}_0 = 0$, we get

$$\frac{c_2(\varepsilon)}{t}\int_0^t \mathbb{E}e^{\varepsilon|\bar{X}_s|^2}\mathrm{d}s \le \frac{1 + c_1(\varepsilon)t}{t}, \quad t > 0.$$

This together with (4.4) yields

$$\mu_\infty(e^{\varepsilon(|\cdot|^2\wedge N)}) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}e^{\varepsilon(|\bar{X}_s|^2\wedge N)}\mathrm{d}s \le \frac{c_1(\varepsilon)}{c_2(\varepsilon)}, \quad N > 0.$$

By letting $N \to \infty$ we derive $\mu_\infty(e^{\varepsilon|\cdot|^2}) < \infty$. Obviously, this and (4.6) imply (4.5) for large $t > 0$. Moreover, since $\|(\sigma\sigma^*)^{-1}\|_\infty < \infty$, $\sigma \in C_b^2(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^m)$ and noting that $(A_2)$ gives

$$\langle v, \nabla_v b(\cdot, \mu_\infty)\rangle \le -K_2|v|^2, \quad v \in \mathbb{R}^d,$$

we find a constant $K_0 \in \mathbb{R}$ such that for any $f \in C^\infty(\mathbb{R}^d)$,

$$\Gamma_2(f) := \frac{1}{2}\bar{L}|\sigma^*\nabla f|^2 - \langle \sigma^*\nabla f, \sigma^*\nabla\bar{L}f\rangle \ge K_0|\sigma^*\nabla f|^2,$$

i.e. the Bakry-Emery curvature of $\bar{L}$ is bounded below by a constant $K_0$. According to [17, Theorem 2.1], this and the hyperboundedness (4.5) imply the defective log-Sobolev inequality

(4.8)
$$\mu_\infty(f^2 \log f^2) \le C_1\mu_\infty(|\sigma^*\nabla f|^2) + C_2$$
$$\le C_1\|\sigma\|_\infty^2\mu_\infty(|\nabla f|^2) + C_2, \quad f \in C_b^1(\mathbb{R}^d), \mu_\infty(f^2) = 1$$

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for some constants $c_1, c_2 > 0$. Since $\bar{L}$ is elliptic, the invariant probability measure $\mu_\infty$ is equivalent to the Lebesgue measure, see for instance [7, Theorem 1.1(ii)], so that the Dirichlet form

$$\mathscr{E}(f,g) := \mu_\infty(\langle \nabla f, \nabla g \rangle), \quad f, g \in W^{1,2}(\mu)$$

is irreducible, i.e. $f \in W^{1,2}(\mu)$ and $\mathscr{E}(f,f) = 0$ imply that $f$ is constant. Therefore, by [25, Corollary 1.3], see also [15], the defective log-Sobolev inequality (4.8) implies the desired log-Sobolev inequality (4.2) for some constant $c > 0$. Hence, the proof is finished.

# 5 Proof of Theorem 2.4

We first establish the log-Harnack inequality for a more general model, which extends existing results derived in [13, 4] to the distribution dependent setting.

## 5.1 Log-Harnack inequality

Consider the following McKean-Vlasov stochastic Hamiltonian system for $(X_t, Y_t) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ :

$$(5.1) \qquad \begin{cases} \mathrm{d}X_t = (AX_t + BY_t)\mathrm{d}t, \\ \mathrm{d}Y_t = Z((X_t, Y_t), \mathscr{L}_{(X_t,Y_t)})\mathrm{d}t + \sigma \mathrm{d}W_t, \end{cases}$$

where $A$ is a $d_1 \times d_1$-matrix, $B$ is a $d_1 \times d_2$-matrix, $\sigma$ is a $d_2 \times d_2$-matrix, $W_t$ is the $d_2$-dimensional Brownian motion on a complete filtration probability space $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, and

$$Z : \mathbb{R}^{d_1+d_2} \times \mathscr{P}_2(\mathbb{R}^{d_1+d_2}) \to \mathbb{R}^{d_2}$$

is measurable. We assume

**(C)** $\sigma$ is invertible, $Z$ is Lipschitz continuous, and $A, B$ satisfy the following Kalman's rank condition for some $k \geq 1$:

$$\mathrm{Rank}[A^0 B, \cdots, A^{k-1}B] = d_1, \quad A^0 := I_{d_1}.$$

Obviously, this assumption implies $(A_1)$, so that (5.1) has a unique solution $(X_t, Y_t)$ for any initial value $(X_0, Y_0)$ with $\mu := \mathscr{L}_{(X_0,Y_0)} \in \mathscr{P}_2(\mathbb{R}^{d_1+d_2})$. Let $P_t^*\mu := \mathscr{L}_{(X_t,Y_t)}$ and

$$(P_t f)(\mu) := \int_{\mathbb{R}^{d_1+d_2}} f \mathrm{d}P_t^*\mu, \quad t \geq 0, f \in \mathscr{B}_b(\mathbb{R}^{d_1+d_2}).$$

By [28, Theorem 3.1], the Lipschitz continuity of $Z$ implies

$$(5.2) \qquad \mathbb{W}_2(P_t^*\mu, P_t^*\nu) \leq \mathrm{e}^{Kt}\mathbb{W}_2(\mu, \nu), \quad t \geq 0, \mu, \nu \in \mathscr{P}_2(\mathbb{R}^{d_1+d_2})$$

for some constant $K > 0$. We have the following result.

**Proposition 5.1.** *Assume* **(C)**. *Then there exists a constant $c > 0$ such that*

$$(5.3) \qquad (P_T \log f)(\nu) \leq \log(P_T f)(\mu) + \frac{c e^{cT}}{T^{4k-1} \wedge 1} \mathbb{W}_2(\mu, \nu)^2, \quad T > 0, \mu, \nu \in \mathscr{P}_2(\mathbb{R}^{d_1+d_2}).$$

*Consequently,*

$$(5.4) \qquad \operatorname{Ent}(P_T^* \nu | P_T^* \mu) \leq \frac{c e^{cT}}{T^{4k-1} \wedge 1} \mathbb{W}_2(\mu, \nu)^2, \quad T > 0, \mu, \nu \in \mathscr{P}_2(\mathbb{R}^{d_1+d_2}).$$

*Proof.* According to [28, Corollary 4.3], (5.3) implies (5.4). Below we prove (5.3) by using the coupling by change of measures summarized in [24, Section 1.1]. By the Kalman rank condition in **(C)**,

$$Q_T := \int_0^T t(T-t) e^{(T-t)A} BB^* e^{(T-t)A^*} dt$$

is invertible and there exists a constant $c_1 > 0$ such that

$$(5.5) \qquad \|Q_T^{-1}\| \leq \frac{c_1 e^{c_1 T}}{(T \wedge 1)^{2k+1}}, \quad T > 0,$$

see for instance [29, Theorem 4.2(1)].

Let $(X_0, Y_0), (\bar{X}_0, \bar{Y}_0) \in L^2(\Omega \to \mathbb{R}^{d_1+d_2}, \mathscr{F}_0, \mathbb{P})$ such that $\mathscr{L}_{(X_0, Y_0)} = \mu, \mathscr{L}_{(\bar{X}_0, \bar{Y}_0)} = \nu$ and

$$(5.6) \qquad \mathbb{E}(|X_0 - \bar{X}_0|^2 + |Y_0 - \bar{Y}_0|^2) = \mathbb{W}_2(\mu, \nu)^2.$$

Next, let $(X_t, Y_t)$ solve (5.1). Then $\mathscr{L}_{(X_t, Y_t)} = P_t^* \mu$. Consider the the modified equation with initial value $(\bar{X}_0, \bar{Y}_0)$:

$$(5.7) \qquad \begin{cases} d\bar{X}_t = (A\bar{X}_t + B\bar{Y}_t) dt, \\ d\bar{Y}_t = \left\{ Z((X_t, Y_t), P_t^* \mu) + \frac{Y_0 - \bar{Y}_0}{T} + \frac{d}{dt}\left[t(T-t) B^* e^{(T-t)A^*} v\right] \right\} dt + \sigma dW_t, \end{cases}$$

where

$$(5.8) \qquad v := Q_T^{-1} \left\{ e^{TA}(X_0 - \bar{X}_0) + \int_0^T \frac{t-T}{T} e^{(T-t)A} B(\bar{Y}_0 - Y_0) dt \right\}.$$

Then

$$(5.9) \qquad \begin{aligned} \bar{Y}_t - Y_t &= \bar{Y}_0 - Y_0 + \int_0^t \left\{ \frac{Y_0 - \bar{Y}_0}{T} + \frac{d}{dr}\left[r(T-r) B^* e^{(T-r)A^*} v\right] \right\} dr \\ &= \frac{T-t}{T}(\bar{Y}_0 - Y_0) + t(T-t) B^* e^{(T-t)A^*} v, \quad t \in [0, T]. \end{aligned}$$

Consequently, $Y_T = \bar{Y}_T$, and combining with Duhamel's formula, we obtain

$$(5.10) \qquad \bar{X}_t - X_t = e^{tA}(\bar{X}_0 - X_0) + \int_0^t e^{(t-r)A} B\left\{ \frac{T-r}{T}(\bar{Y}_0 - Y_0) + r(T-r) B^* e^{(T-r)A^*} v \right\} dr$$

for $t \in [0, T]$. This and (5.8) imply

$$\bar{X}_T - X_T = e^{TA}(\bar{X}_0 - X_0) + \int_0^T \frac{T - r}{T} e^{(T-r)A} B(\bar{Y}_0 - Y_0) \mathrm{d}r + Q_T v = 0,$$

which together with $Y_T = \bar{Y}_T$ observed above yields

(5.11) $$(X_T, Y_T) = (\bar{X}_T, \bar{Y}_T).$$

On the other hand, let

$$\xi_t = \sigma^{-1} \Big\{ \frac{1}{T}(Y_0 - \bar{Y}_0) + \frac{\mathrm{d}}{\mathrm{d}t}\Big[ t(T-t)B^* e^{(T-t)A^*} v \Big] + Z\big((X_t, Y_t), P_t^* \mu\big) - Z\big((\bar{X}_t, \bar{Y}_t), P_t^* \nu\big) \Big\}, \quad t \in [0, T].$$

By **(C)**, (5.2), (5.5), (5.8), (5.9), and (5.10), we find a constant $c_2 > 0$ such that

(5.12) $$|\xi_t|^2 \leq \frac{c_2}{(T \wedge 1)^{4k}} e^{c_2 T} \big\{ |X_0 - \bar{X}_0|^2 + |Y_0 - \bar{Y}_0|^2 + \mathbb{W}_2(\mu, \nu)^2 \big\}, \quad t \in [0, T].$$

So, the Girsanov theorem implies that

$$\tilde{W}_t := W_t + \int_0^t \xi_s \mathrm{d}s, \quad t \in [0, T]$$

is a $d_2$-dimensional Brownian motion under the probability measure $\mathbb{Q} := R\mathbb{P}$, where

(5.13) $$R := e^{-\int_0^T \langle \xi_t, \mathrm{d}W_t \rangle - \frac{1}{2} \int_0^T |\xi_t|^2 \mathrm{d}t}.$$

Reformulating (5.7) as

$$\begin{cases} \mathrm{d}\bar{X}_t = (A\bar{X}_t + B\bar{Y}_t)\mathrm{d}t, \\ \mathrm{d}\bar{Y}_t = Z((\bar{X}_t, \bar{Y}_t), P_t^* \nu)\mathrm{d}t + \sigma \mathrm{d}\tilde{W}_t, \quad t \in [0, T], \end{cases}$$

by the weak uniqueness of (5.1) and that the distribution of $(\bar{X}_0, \bar{Y}_0)$ under $\mathbb{Q}$ coincides with $\mathscr{L}_{(\bar{X}_0, \bar{Y}_0)} = \nu$, we obtain $\mathscr{L}_{(\bar{X}_t, \bar{Y}_t)|\mathbb{Q}} = P_t^* \nu$ for $t \in [0, T]$. Combining this with (5.11) and using the Young inequality, for any $f \in \mathscr{B}_b^+(\mathbb{R}^{d_1 + d_2})$ we have

(5.14)
$$\begin{aligned} (P_T \log f)(\nu) &= \mathbb{E}[R \log f(\bar{X}_T, \bar{Y}_T)] = \mathbb{E}[R \log f(X_T, Y_T)] \\ &\leq \log \mathbb{E}[f(X_T, Y_T)] + \mathbb{E}[R \log R] = \log(P_T f)(\mu) + \mathbb{E}_{\mathbb{Q}}[\log R]. \end{aligned}$$

By (5.12), and (5.13), $\tilde{W}_t$ is a Brownian motion under $\mathbb{Q}$, and noting that $\mathbb{Q}|_{\mathscr{F}_0} = \mathbb{P}|_{\mathscr{F}_0}$ and (5.6) imply

$$\mathbb{E}_{\mathbb{Q}}\big(|X_0 - \bar{X}_0|^2 + |Y_0 - \bar{Y}_0|^2\big) = \mathbb{W}_2(\mu, \nu)^2,$$

we find a constant $c > 0$ such that

$$\mathbb{E}_{\mathbb{Q}}[\log R] = \frac{1}{2}\mathbb{E}_{\mathbb{Q}} \int_0^T |\xi_t|^2 \mathrm{d}t \leq \frac{c e^{cT}}{(T \wedge 1)^{4k-1}} \mathbb{W}_2(\mu, \nu)^2.$$

Therefore, (5.3) follows from (5.14). $\quad\square$

## 5.2 Proof of Theorem 2.4

We first prove the exponential convergence of $P_t^*$ in $\mathbb{W}_2$.

**Lemma 5.2.** *Assume* **(C)**. *Then there exists a constant* $c_1 > 0$ *such that*

$$(5.15) \qquad \mathbb{W}_2(P_t^*\mu, P_t^*\nu)^2 \le c_1 e^{-\kappa t}\mathbb{W}_2(\mu, \nu)^2, \quad t \ge 0, \mu, \nu \in \mathscr{P}_2(\mathbb{R}^{d_1+d_2}).$$

*Consequently,* $P_t^*$ *has a unique invariant probability measure* $\mu_\infty \in \mathscr{P}_2(\mathbb{R}^{d_1+d_2})$.

*Proof.* As shown in the proof of [28, Theorem 3.1(2)] that the second assertion follows from the first. So, it suffices to prove (5.15). For

$$(5.16) \qquad a := \left(\frac{1+\beta+\beta^2}{1+\beta}\right)^{\frac{1}{2}}, \quad r := a - \frac{\beta}{a} = \frac{1}{\sqrt{(1+\beta)(1+\beta+\beta^2)}} \in (0,1),$$

we define the distance

$$(5.17) \qquad \bar{\psi}_B((x,y), (\bar{x}, \bar{y})) := \sqrt{a^2|x-\bar{x}|^2 + |B(y-\bar{y})|^2 + 2ra\langle x-\bar{x}, B(y-\bar{y})\rangle}$$

for $(x,y), (\bar{x}, \bar{y}) \in \mathbb{R}^{d_1+d_2}$. Then there exists a constant $C > 1$ such that

$$(5.18) \qquad C^{-1}|(x-\bar{x}, y-\bar{y})| \le \bar{\psi}_B((x,y), (\bar{x}, \bar{y})) \le C|(x-\bar{x}, y-\bar{y})|.$$

Moreover, we claim that

$$(5.19) \qquad \bar{\psi}_B((x,y), (\bar{x}, \bar{y}))^2 \le \frac{2+2\beta+\beta^2+\sqrt{\beta^4+4}}{2(1+\beta)}\psi_B((x,y), (\bar{x}, \bar{y}))^2.$$

Indeed, by (5.16) and (5.17), for any $\varepsilon > 0$ we have

$$(5.20) \qquad \bar{\psi}_B((x,y), (\bar{x}, \bar{y}))^2 \le a^2(1+\varepsilon)|x-\bar{x}|^2 + \left(1 + \frac{1}{\varepsilon(1+\beta)(1+\beta+\beta^2)}\right)|B(y-\bar{y})|^2.$$

Obviously, by (5.16),

$$\varepsilon := \frac{1-a^2+\sqrt{(a^2-1)^2+4a^2(1+\beta)^{-1}(1+\beta+\beta^2)^{-1}}}{2a^2} = \frac{\sqrt{\beta^4+4}-\beta^2}{2(1+\beta+\beta^2)}$$

satisfies

$$a^2(1+\varepsilon) = 1 + \frac{1}{\varepsilon(1+\beta)(1+\beta+\beta^2)} = \frac{2+2\beta+\beta^2+\sqrt{\beta^4+4}}{2(1+\beta)}.$$

Thus, (5.19) follows from (5.20).

Now, let $(X_t, Y_t)$ and $(\bar{X}_t, \bar{Y}_t)$ solve (2.21) with $\mathscr{L}_{(X_0,Y_0)} = \mu, \mathscr{L}_{(\bar{X}_0,\bar{Y}_0)} = \nu$ such that

$$(5.21) \qquad \mathbb{W}_2(\mu, \nu)^2 = \mathbb{E}|(X_0 - \bar{X}_0, Y_0 - \bar{Y}_0)|^2.$$

Simply denote $\mu_t = \mathscr{L}_{(X_t,Y_t)}, \bar{\mu}_t = \mathscr{L}_{(\bar{X}_t,\bar{Y}_t)}$. By **(C)** and Itô's formula, and noting that (5.16) implies

$$a^2 - \beta - ra = 0, \quad 1 - ra = ra\beta = \frac{\beta}{1+\beta},$$

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we obtain

$$
\begin{aligned}
\frac{1}{2}\mathrm{d}\big\{\bar{\psi}_B((X_t,Y_t),(\bar{X}_t,\bar{Y}_t))^2\big\} &= \big\langle a^2(X_t-\bar{X}_t)+raB(Y_t-\bar{Y}_t),B(Y_t-\bar{Y}_t)\big\rangle\mathrm{d}t \\
&\quad + \big\langle B^*B(Y_t-\bar{Y}_t)+raB^*(X_t-\bar{X}_t),\ \beta B^*(BB^*)^{-1}(\bar{X}_t-X_t)+\bar{Y}_t-Y_t\big\rangle\mathrm{d}t \\
&\quad + \big\langle B^*B(Y_t-\bar{Y}_t)+raB^*(X_t-\bar{X}_t),\ B^*\{\nabla V(\bar{X}_t,\bar{\mu}_t)-\nabla V(X_t,\mu_t)\}\big\rangle\mathrm{d}t \\
&\leq \Big\{ -(1-ra)|B(Y_t-\bar{Y}_t)|^2+(a^2-\beta-ra)\langle X_t-\bar{X}_t,B(Y_t-\bar{Y}_t)\rangle-ra\beta|X_t-\bar{X}_t|^2 \\
&\qquad + \big\langle B^*B(Y_t-\bar{Y}_t)+(1+\beta)^{-1}B^*(X_t-\bar{X}_t),\ B^*\{\nabla V(\bar{X}_t,\bar{\mu}_t)-\nabla V(X_t,\mu_t)\}\big\rangle\Big\}\mathrm{d}t \\
&\leq \Big\{ \frac{\theta_2}{1+\beta}\mathbb{W}_2^{\psi_B}(\mu_t,\bar{\mu}_t)^2-\frac{\beta-\theta_1}{1+\beta}\psi_B((X_t,Y_t),(\bar{X}_t,\bar{Y}_t))^2\Big\}\mathrm{d}t.
\end{aligned}
$$

By (5.19) and the fact that

$$
\mathbb{W}_2^{\psi_B}(\mu_t,\bar{\mu}_t)^2 \leq \mathbb{E}[\psi_B((X_t,Y_t),(\bar{X}_t,\bar{Y}_t))^2],
$$

for $\kappa > 0$ in (2.25), we obtain

$$
\begin{aligned}
\frac{1}{2}\big\{&\mathbb{E}[\bar{\psi}_B((X_t,Y_t),(\bar{X}_t,\bar{Y}_t))^2]-\mathbb{E}[\bar{\psi}_B((X_s,Y_s),(\bar{X}_s,\bar{Y}_s))^2]\big\} \\
&\leq -\frac{\beta-\theta_1-\theta_2}{1+\beta}\int_s^t \mathbb{E}[\psi_B((X_r,Y_r),(\bar{X}_r,\bar{Y}_r))^2]\mathrm{d}r \\
&\leq -\kappa\int_s^t \mathbb{E}[\bar{\psi}_B((X_r,Y_r),(\bar{X}_r,\bar{Y}_r))^2]\mathrm{d}r,\quad t\geq s\geq 0.
\end{aligned}
$$

Therefore, Gronwall's inequality implies

$$
\mathbb{E}[\bar{\psi}_B((X_t,Y_t),(\bar{X}_t,\bar{Y}_t))^2] \leq \mathrm{e}^{-2\kappa t}\mathbb{E}[\bar{\psi}_B((X_0,Y_0),(\bar{X}_0,\bar{Y}_0))^2],\quad t\geq 0.
$$

Combining this with (5.18) and (5.21), we prove (5.15) for some constant $c > 0$. $\qquad\square$

*Proof of Theorem 2.4.* By Proposition 5.1 with $k=1$, Lemma 5.2 and Theorem 2.1, we only need to verify the Talagrand inequality. As shown in the beginning of [11, Section 3] that $\mu_\infty$ has the representation

$$
\mu_\infty(\mathrm{d}x,\mathrm{d}y)=Z^{-1}\mathrm{e}^{\bar{V}(x,y)}\mathrm{d}x\mathrm{d}y,\quad \bar{V}(x,y):=V(x,\mu_\infty)+\frac{\beta}{2}|(BB^*)^{-\frac{1}{2}}x|^2+\frac{1}{2}|y|^2,
$$

where $Z:=\int_{\mathbb{R}^{d_1+d_2}}\mathrm{e}^{-\bar{V}(x,y)}\mathrm{d}x\mathrm{d}y$ is the normalization constant. Since (2.23) implies

$$
BB^*\mathrm{Hess}_{V(\cdot,\mu_\infty)} \geq -\theta_1 I_{d_1},
$$

we deduce from (2.22) that

$$
\mathrm{Hess}_{\bar{V}} \geq \gamma I_{d_1+d_2},\quad \gamma:=1\wedge\frac{\beta-\theta_1}{\|B\|^2}>0.
$$

So, by the Bakry-Emery criterion [2], we have the log-Sobolev inequality

$$\mu_\infty(f^2 \log f^2) \le \frac{2}{\gamma} \mu_\infty(|\nabla f|^2), \quad f \in C_b^1(\mathbb{R}^{d_1+d_2}), \mu_\infty(f^2) = 1.$$

According to [6], this implies the Talagrand inequality

$$\mathbb{W}_2(\mu, \mu_\infty)^2 \le \frac{2}{\gamma} \mathrm{Ent}(\mu|\mu_\infty).$$

Then the proof if finished. $\qquad\square$

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