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Doctoral Thesis

Aspects of Gauge/String dualities

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Declaration of authorship

I, Stefano Speziali, declare that this thesis titled *"Aspects of Gauge/String dualities"* and the work presented in it are based on my research projects in collaboration with Mohammad Akhond, Prof. Adi Armoni, Prof. Yolanda Lozano, Prof. Carlos Nunez, Anayeli Ramirez, Dr. Dibakar Roychowdhury and Dr. Salomon Zacarias. In particular, this thesis is based on

[1] C. Nunez, D. Roychowdhury, S. Speziali, S. Zacarias,
Holographic Aspects of Four Dimensional $\mathcal{N} = 2$ *SCFTs and their Marginal Deformations*,
Nuclear Physics B 943 (2019) 114617
arXiv:1901.0288 [hep-th].

[2] A. Armoni, M. Akhond, S. Speziali,
Phases of $\mathbf{U(N_c)}$ QCD_3 *from Type 0 Strings and Seiberg Duality*,
Journal of High Energy Physics, 09 (2019) 111,
arXiv:1908.04324 [hep-th].

[3] S. Speziali,
Spin 2 fluctuations in 1/4 BPS AdS_3/CFT_2,
Journal of High Energy Physics, 03 (2020) 079,
arXiv:1910.14390 [hep-th].

[4] Y. Lozano, C. Nunez, A. Ramirez, S. Speziali
M-strings and AdS_3 *solutions to M-theory with small* $\mathcal{N} = (0, 4)$ *supersymmetry*,
Journal of High Energy Physics, 08 (2020) 118,
arXiv:2005.06561 [hep-th].

[5] Y. Lozano, C. Nunez, A. Ramirez, S. Speziali
New AdS_2 *backgrounds and* $\mathcal{N} = 4$ *Conformal Quantum Mechanics*,
arXiv:2005.06561 [hep-th].

[6] Y. Lozano, C. Nunez, A. Ramirez, S. Speziali
AdS_2 *duals to ADHM quivers with Wilson lines*,
arXiv:2011.13932 [hep-th].

The first part of the thesis will use results from [1], [3], [4], [5] and [6]. The material presented in [2] will be the subject of the second part. This thesis was handed in for a viva voce before [5] and [6] were submitted to the arXiv. Hyperlinks and arXiv numbers for [5] and [6] were included only afterwards.

I confirm that:

This work has not previously been accepted in substance for any degree and is not being concurrently submitted for any degree.

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Date: ....11/10/2020..

Statement 1

This thesis is the result of my own investigations, except where otherwise stated. Where correction services have been used, the extent and nature of the correction is clearly marked in a footnote(s). Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended.

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Date: ....11/10/2020..

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Abstract

In this thesis we investigate on the role that *dualities* play nowadays in our understanding of string and Quantum Field Theories.

The first part is devoted to holographic dualities, where the case of the string dual of certain one-, two- and four dimensional field theories is explored in detail. We begin in Chapter 1 by discussing the holographic dual of a broad class of four-dimensional field theories. We specialise to the case of quiver field theories where a number of formulas computing charges, number of branes and Linking Numbers are given. We then introduce marginal deformations in the supergravity backgrounds to uncover infinite new families of solutions to type II supergravity and M-theory. In Chapter 2, we discuss spin 2 fluctuations around a class of warped AdS_3 backgrounds. We identify explicitly the dual operators of a given protected sector. Also, a formula for the central charge of the dual two-dimensional field theories is derived. In Chapter 3, we introduce two new classes of geometries with an AdS_2 factor. We outline the importance to black hole physics and in one case give a prescription for the dual field theories.

The second part is devoted to QFT/QFT dualities in Chern-Simons theories, and how dualities between Chern-Simons theories in three dimensions can be motivated in string theory. It is found that a known duality between three-dimensional Chern-Simons theories with unitary gauge symmetry can be motivated by a brane setup in Type 0B string theory with an orientifold. In particular, the phase diagram of unitary QCD_3 is inferred from a dual theory.

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Introduction

Introduction and summary

One of the most fascinating tools to study interacting Quantum Field Theories (QFTs) developed in the last years is provided by the concept of *duality*. By duality it is usually meant some kind of relationship between two quantum theories that are formulated in terms of different degrees of freedom but which describe the same physics. Over the years, different kind of dualities have been spelled out, and they were often crucial in mapping out problems in theoretical physics. In particular, dualities exist between QFTs, between string theories or even between QFTs and string theories, the most notable example being the Maldacena duality, often referred to as AdS/CFT duality.

Examples of QFT/QFT dualities include the Ising model, bosonization, and free electromagnetism, where the duality can be constructed explicitly. There are also more convoluted examples, like supersymmetric Yang-Mills theories, where often the duality can only be conjectured (and tested).

There are also remarkable examples of dualitites between string theories, like T- or S-duality, and their generalisations, proving somehow that distinct string theories are essentially different classical limits of a single quantum theory.

The holographic duality of Maldacena is somehow different from those just mentioned, as it offers a crossover between QFTs and string theories. It is quite remarkable because, among other things, string theory is considered a promising candidate for a consistent theory of quantum gravity, whereas quantum field theory on flat spacetime does not seem to be describing any theory of gravity. Also, according to Maldacena, strongly interacting quantum field theories are mapped to classical supergravities which, in some cases, might be tractable. Indeed, it is a general hope to be able to do explicit computations in strongly interacting systems.

All the dualities mentioned so far will play a role in the present thesis. Even though the first part – made of three chapters – strongly relies on the AdS/CFT duality – we discuss the examples of certain one-, two- and four-dimensional supersymmetric theories – other types of dualities, like T- or S-dualities, are crucial in some computations. The second part – made of one chapter – instead, is devoted to dualities between quantum field theories, and we will discuss a particular example of a duality between three-dimensional Chern-Simons theories with fundamental matter.

Dualities have played a prominent role in theoretical physics in the last (at least) fifty years. Providing a precise hystorical reconstruction is nearly impossible. However, there are nice reviews and books which aim at giving an account for the modern understanding of dualities in theoretical physics. See for instance [7, 8] for very insightful reviews of dualities for fields and strings, or the books [9, 10] for recent developments in the AdS/CFT correspondence. See also the comprehensive review on AdS/CFT [11] and references therein.

We now outline the content of single chapters by summarising the main results contained in them.

Part I

In the first part of the thesis, we deal with holographic dualities. We will be concerned mainly with the *weak form* of the duality, namely that regime of parameters for which the string theory is weakly coupled and formulated on a weakly curved space-time. In this regime, string theory is well approximated by classical supergravity, the dual field theory is instead strongly coupled and its number of degrees of freedom very large[1]. The structure of Part I is as follows.

Chapter 1 In Chapter 1, after a brief account for the supersymmetry algebra of four-dimensional field theories with 8 supercharges and their string theory origin, we discuss their holographic dual. The problem of formulating the gravity dual of $\mathcal{N}=2$ four-dimensional superconformal field theories was originally addressed in [12] in M-theory, where the holographic description is based on AdS_5 spaces preserving 16 supersymmetries. The general form of such solutions contains an S^2 and an S^1, which realise geometrically the $SU(2)_R \times U(1)_r$ R-symmetry, in addition to the AdS_5 factor, which realises geometrically the conformal group in four dimensions, $SO(4,2)$. The remaining three-dimensional space is not specified by symmetry and is parametrised by three coordinates, (x^1, x^2, y). It was shown in [13] that it is within this class of solutions that we can find the holographic duals of the class S theories found in [14].

[1] This statement is made more quantitative in the main text.

It turns out that a generic solution is completely specified by a function D, which satisfies the 3d Toda equation

$$(\partial_{x^1}^2 + \partial_{x^2}^2)D + \partial_y^2 e^D = 0\,, \tag{1}$$

and the boundary conditions imposed on it. Boundary conditions are needed in order for the eleven dimensional background to be regular everywhere (with the sole exception of the points where physical sources are localised).

Equation (1) is notoriously difficult to solve, because of its non linearity. Assuming an extra $U(1)$ symmetry in the (x^1, x^2) plane, it turns out that the problem can be reduced to type IIA and simplified considerably. After a change of variables, the Type IIA solutions are given in terms of a solution V of the three-dimensional cylindrical Laplace equation in σ, η

$$\frac{1}{\sigma}\partial_\sigma \sigma \partial_\sigma V + \partial_\eta^2 V = 0\,. \tag{2}$$

Now, of course, boundary conditions must be imposed on V.

Such solutions to type IIA supergravity are found to be holographically dual to four-dimensional field theories emerging from intersections of $D4$, $D6$ and $NS5$ branes which preserve 8 Poincaré supersymmetries – the number of supersymmetries gets enhanced to 16 in the near-horizon limit, when the Poincaré group is extended to the conformal group.

The correspondence between the supergravity backgrounds and the dual field theory is made precise once we consider the "line charge density" $\lambda(\eta)$ defined by

$$\lambda = \sigma \partial_\sigma V|_{\sigma=0}\,. \tag{3}$$

The interpretation of λ as a line charge density for the potential V solving the equation (1) is sharpened in the appendix of Chapter 1. In particular, λ turns out to be *piecewise linear and continuous* in order to have a well-defined quantisation of fluxes [13]. For the dual field theory there is a gauge group $U(\lambda_n)$ whenever λ reaches an integer value $\lambda_n = \lambda(n)$ for any $n \in \mathbb{Z}_{>0}$, while there is a flavour symmetry with unitary flavour group whenever λ has a kink at some integer η_i. The rank of the flavour group is determined, roughly speaking, by the change in the slope of λ and is shown to be always an integer.

There are interesting quantities characterising the quantum field theory which can be reproduced on the gravity side. Here we follow closely [1], work done in collaboration with Prof. Carlos Nunez, Dr. Dibakar Roychowdhury and Dr. Salomon Zacarias. In particular, there is a set of topological invariants – invariant under Hanany-Witten moves – called "linking numbers", associated with $NS5$ and $D6$ branes. For superconformal field theories the linking numbers are all equal, $K_1 = K_2 = \cdots = K_{N_5} = K$, and it is proposed [1] that they can be obtained from the gravity dual by computing[2] $K = \frac{\pi^4}{4}\lambda'(N_5)$, which in turn implies that $\sum_{i=1}^{N_5} K_i = \frac{\pi^4}{4}\lambda'(N_5)N_5$. After a simple algebra, we also show that the sum of the $NS5$-Linking Numbers is given by integrating a Page four-flux over a compact four-manifold at $\sigma = 0$. In a similar fashion, we show that the sum of the Linking numbers associated with the $D6$ branes is such that it satisfies $\sum_{i=1}^{N_5} K_i + \sum_{j=1}^{N_6} L_j = 0$ [15].

One more thing that can be computed from the supergravity solution is the leading order in N_5 and N_6, when both $N_5, N_6 \to \infty$, of the central charge of the dual field theory. This is shown to be given by

$$c = \frac{\pi^{13}}{2^6}\int_0^{N_5} \lambda^2(\eta)d\eta\,. \tag{4}$$

All these formulas have been tested in different examples of quiver four-dimensional SCFTs [1].

As a final application of holographic duality for four-dimensional quiver SCFTs, we show how to compute some marginal deformations of the Gaiotto-Maldacena geometries following [1]. The deformations in question are usually referred to as TsT deformations [16], and involve the introduction of a "marginal" parameter γ. Roughly speaking, when we have a background with two $U(1)$ isometries, say parametrised by two coordinates ξ and β, a TsT transformation consists in T-dualising the first variable ξ, shifting $\beta \to \beta + \gamma\xi$, and T-dualising back along ξ. We perform such an operation on the Gaiotto-Maldacena $\mathcal{N} = 2$ backgrounds in Type IIA and also on their IIB counterpart, obtained after computing the T-dual along the $U(1)_r$ isometry. The backgrounds obtained in this way – given in the main text – do not display an $SU(2)_R \times U(1)_r$ R-symmetry anymore, and they are argued to be dual to some $\mathcal{N} = 1$ SCFTs in four dimensions.

The parameter γ introduces a marginal deformation, and thus no RG flow is sup-

[2] Here N_5 corresponds to the number of $NS5$ branes, whereas N_6 is the number of $D6$ branes present in the Type IIA backgrounds.

posed to take place. However, supersymmetry is argued to be broken, $\mathcal{N}=2\to\mathcal{N}=1$. It was proposed in [1], following [16], that this happens because of interactions of the form

$$W=e^{i\pi\gamma\mathcal{R}}W_{\mathcal{N}=2}\,, \tag{5}$$

where $W_{\mathcal{N}=2}$ is the superpotential of $\mathcal{N}=2$ theories, $W_{\mathcal{N}=2}=\sum \mathrm{tr}\Phi Q\tilde{Q}$, and $\mathcal{R}$ a combination of the $U(1)\times U(1)$ charges of the fields. When $\gamma=0$, i.e. no TsT transformation, we recover the original $\mathcal{N}=2$ theories and dual backgrounds.

We conclude Chapter 1 with a summary of the main results found in [1]. Appendices at the end sharpen some of the computations both in supergravity and field theory done in the main text.

Chapter 2 A large family of warped $AdS_3\times S^2\times \mathrm{CY}_2$ solutions to type IIA supergravity was found in [17]. In [18, 19], a subsector of such a family of backgrounds was argued to be dual to a special class of two-dimensional quiver SCFTs with 4 Poincaré supercharges and small $\mathcal{N}=(0,4)$ superconformal algebra.

In two dimensions, the two-dimensional conformal algebra splits into two copies of the Virasoro algebra – the left-moving and right-moving sectors – and each copy can be extended to include $\mathcal{N}$ supersymmetries. The resulting superconformal algebra is referred to as the super Virasoro algebra and is usually given as an $\mathfrak{so}(\mathcal{N})$ Kac-Moody algebra. Because of the splitting, we can study multiplets of individual sectors, and then simply take the direct product of multiplets in each sector to form a full superconformal multiplet.

The case relevant to us is that of *small* $\mathcal{N}=4$ global algebra (one $\mathfrak{su}(2)$ R-symmetry subalgebra as opposed to two $\mathfrak{su}(2)$'s R-symmetry subalgebras for *large* $\mathcal{N}=(0,4)$). States (and operators) in the dual field theory are then classified according to this algebra. Using the holographic dictionary, such operators are mapped to fluctuations of the supergravity fields.

The aim of Chapter 2 is to study a portion of the spectrum of the fluctuations – spin 2 fluctuations – around the background $AdS_3\times S^2\times \mathrm{CY}_2$ mentioned above, and classify them according to the maximal bosonic subalgebra realised by the background.

Following [3], we derive an equation for spin 2 fluctuations around the warped $AdS_3\times S^2\times \mathrm{CY}_2$ solutions mentioned above. These are of the form $h_{\mu\nu}(x,z)=h^{[tt]}_{\mu\nu}(x)\psi(z)$, with $h^{[tt]}_{\mu\nu}(x)$ a transverse-traceless mode on the AdS_3 subspace and $\psi(z)$ an internal mode function of the internal coordinates, denoted collectively as z. $h^{[tt]}_{\mu\nu}(x)$

is found to solve a massive equation for rank-2 tensors in AdS_3 spacetimes

$$\Box^{(2)}_{AdS_3} h^{[tt]}_{\mu\nu}(x) = (M^2-2) h^{[tt]}_{\mu\nu}(x)\,, \tag{6}$$

with M the mass of the graviton, while $\psi(z)$ factorises as[3] $\psi = \sum_{lm} u^l \phi_{lm}(\rho) Y_{lm}(S^2)$. ϕ solves an equation which reads

$$\frac{d}{d\rho}\left(u^{2(l+1)}\frac{d\phi_{lm}}{d\rho}\right) = -(M^2 - 4l(l+1))\widehat{h}_4 h_8 u^{2l}\phi_{lm}\,, \tag{7}$$

with u, $\widehat{h}_4$ and h_8 warping factors characterising the warped $AdS_2 \times S^3 \times \mathrm{CY}_2$ depending on ρ only.

Such ordinary differential equation admits a class of *universal minimal* solutions which depends on the general structure of the backgrounds, but is otherwise insensitive to specific data characterising the geometries. For such solutions, we have $M^2 = 4l(l+1)$ which implies, via the holographic formula $\Delta(\Delta-2) = M^2$, that the dual gauge-invariant operators have scaling dimension $\Delta = 2l+2$, with $l = 0, 1, 2, \ldots$ the quantum number associated with the $SU(2)_R$ R-symmetry of the small $\mathcal{N} = (0,4)$ superconformal algebra.

When $l = 0$ (massless graviton), the dual operator has dimension 2 and is a singlet under the R-symmetry. We associate it with the (anti-)holomorphic stress-energy tensor of the dual SCFT.

It is likewise easy to show that operators with scaling dimension $\Delta = 2l+2$ belong to short multiplets of the small $\mathcal{N} = (0,4)$ algebra also for $l > 0$.

As a final application, we show that fluctuating the IIA action, we get an action for $h^{[tt]}_{\mu\nu}(x)$ which reads

$$S[h] = \sum_{lm} C_{lm} \int d^3x \sqrt{-g_{AdS_3}}\, (h^{[tt]}_{lm})^{\mu\nu} \left\{\Box^{(2)}_{AdS_3} + 2 - M^2\right\} (h^{[tt]}_{lm})_{\mu\nu}\,, \tag{8}$$

with C_{lm} some coefficients obtained after an integration over the internal space. For the massless graviton $l = m = 0$, C_0 is given by

$$C_0 = \frac{1}{4\kappa_{10}^2}\mathrm{vol}_{\mathrm{CY}_2}\int_{\mathcal{I}_\rho} d\rho\, \widehat{h}_4 h_8\,, \tag{9}$$

[3]In [3] the geometry of interest is that of a warped $AdS_2 \times S^3 \times \mathrm{T}^4$ and the factorisation is $\psi = \sum_{lmn} u^l \phi_{lmn}(\rho) Y_{lm}(S^2) e^{in\cdot\theta}$, with $\theta_i \cong \theta_i + 2\pi$ parametrising T^4. Here, we consider a small simplification which does not change the final outcome.

which is shown in [19] to capture the leading order of the central charge of the dual quiver field theories. This does not come as a surprise as the quadratic action for $h_{\mu\nu}$ computes the two-point function of the dual stress-energy tensor (Schwinger function) whose coefficient is known to be the central charge of the field theory.

In the appendix, we discuss in some detail one example on *non-universal* solutions ($M^2 > 4l(l+1)$). We also give details on representations of the small $\mathcal{N} = (0,4)$ superconformal algebra and review basic facts of $\mathcal{N} = (0,2)$ and $\mathcal{N} = (0,4)$ supersymmetric theories in two dimensions.

Chapter 3 In Chapter 3, we discuss two different classes of backgrounds with an AdS_2 factor. They are both obtained from the warped $AdS_3 \times S^2 \times \mathrm{CY}_2$ solutions to Type IIA briefly discussed in Chapter 2, after some manipulations in supergravity. Being new entries in the classification of AdS_2 spacetimes, we will give a thorough analysis of both geometries. Here we draw on results from [4], [5] and [6], work done in collaboration with Prof. Yolanda Lozano, Prof. Carlos Nunez and Anayeli Ramirez. The two distinct classes of backgrounds will make up the two main sections of Chapter 3. Let us give a brief overview of both.

The first class of backgrounds is given in terms of a warped $AdS_2 \times S^2 \times \mathrm{CY}_2$ in Type IIB supergravity. Using local coordinates, it is in general possible to write AdS_3 spacetime as a fibration over an AdS_2 space with fibre an S^1. We then T-dualise along the fibre direction to get $AdS_2 \times S^2 \times \mathrm{CY}_2$. In fact, the backgrounds inherits also an S^1 (the fibre). The S^1 is shown to be broken by the presence of probe $D1$ and $D5$ branes. Schematically, the background metric reads

$$ds^2 = f_1 ds^2_{AdS_2} + f_2 ds^2_{S^2} + f_3 ds^2_{\mathrm{CY}_2} + f_1^{-1}(d\rho^2 + d\psi^2)\,, \tag{10}$$

with the warping factors f_1, f_2 and f_3 functions of ρ only and suitably given explicitly in terms of three functions u, $\widehat{h}_4$ and h_8.

Remarkably, we are able to define a suitable "central charge" that coincides with that of the "parent" $AdS_3 \times S^2 \times \mathrm{CY}_2$ and given by the very same formula (9). We argue that, in Quantum Mechanics, such a formula captures the dynamics of the degenerate ground states, i.e. counts their number. It allows us also to define a dual QM in the form of a quiver Quantum Mechanics given by the dimensional reduction of the two-dimensional quiver field theory dual to the original $AdS_3 \times S^2 \times \mathrm{CY}_2$.

As last insight, we give a minimisation procedure in supergravity in order to compute the central charge. It turns out that a properly defined functional $\mathcal{C}$, given in terms of fields in the R-R sector, coincides with the central charge of the theory upon extremisation. We offer also another point of view on the matter by showing that the functional $\mathcal{C}$ is essentially captured by the product of suitably defined electric and magnetic charges associated with D branes in the background.

The second class of backgrounds is given instead in terms of a warped $AdS_2 \times S^3 \times \text{CY}_2$ in Type IIA supergravity. It is obtained from the class of $AdS_3 \times S^2 \times \text{CY}_2$ backgrounds already mentioned after a Wick rotation. The Wick rotation is described in great detail in the main text and essentially corresponds to performing the following operation

$$AdS_3 \to -S^3\,, \qquad S^2 \to -AdS_2\,. \tag{11}$$

In order to get a spacetime with the correct signature it is necessary to perform a further analytic continuation on the warping factors and spacetime coordinates. Schematically, the metric of the spacetime after the Wick rotation reads

$$ds^2 = g_1 ds^2_{AdS_2} + g_2 ds^2_{S^3} + g_3 ds^2_{\text{CY}_2} + g_4 d\rho^2\,, \tag{12}$$

with the warping factors g_1, g_2, g_3 and g_4 functions of ρ only and given explicitly again in terms of three functions u, $\widehat{h}_4$ and h_8.

Likewise, a suitable central charge is defined and shown to emerge from a minimisation principle in supergravity.

The construction of the dual Quantum Mechanics turns out to be more subtle and will be given in [6].

We conclude with a brief summary of the main results found in Chapter 3 and possible future directions.

Part II

In the second part of the thesis, we deal with dualities between Quantum Field Theories. More specifically, we focus on three-dimensional gauge Chern-Simons theories and Seiberg-like dualities. Part II is made of a single chapter whose structure is as follows.

Chapter 1 The main content of Part II, Chapter 1, is based on [2], work done in collaboration with Mohammad Akhond and Prof. Adi Armoni. We begin with a brief introduction to Chern-Simons theories in three dimensions. We introduce the Chern-Simons functional and discuss its degrees of freedom. We argue that, roughly speaking, a Chern-Simons theory is specified by a choice of a compact gauge group G, a level k and data necessary to define topological invariants such as orientation and framing.

When fermions are present, special care must be given to the fermion path integral. The fermion path integral in general needs regularisation, and can be shown to be given by[4] $Z_\psi = |Z_\psi| \exp\left(-i\frac{\pi}{2}\eta\right)$, with η the regularised Atiyah-Patodi-Singer (APS) invariant. The APS theorem relates the η invariant to Chern-Simons actions. When fermions are given a mass m, at low energies they can be integrated out and the Chern-Simons level gets shifted. The combined effect of regularising the fermion path integral and integrating out massive fermions has the consequence of producing a shift of the Chern-Simons level by some integer quantity. This will be important to understand the IR regime of $U(N_c)$ QCD$_3$, our theory of interest, to be mentioned momentarily.

We then move on to adding some structure to Chern-Simons theories. In particular, we give the theory a supersymmetric structure and discuss $\mathcal{N}=2$ supersymmetric Chern-Simons theories. We discuss in some detail what supermultiplets we have at disposal, and how such theories can be realised on intersections of $D3$, $D5$, $NS5$ and bound states of $NS5$ and $D5$ branes, often denoted as (p,q) fivebranes – for p $NS5$ branes and q $D5$ branes – in Type IIB string theory.

Giveon and Kutasov [20] proposed a Seiberg-like duality for $\mathcal{N}=2$ Cherns-Simons theory with level k and N_f flavours of fundamental quarks Q^i and $\tilde{Q}_i$, $i=1,\ldots,N_f$ starting from string theory. The dual theory – the magnetic theory – is realised on an equivalent brane configuration after an "irrelevant" operation is performed on the original brane web so as to "swap" the $NS5$ and $(1,k)$ branes.

Following [20], we argue that also non-supersymmetric dualities for Chern-Simons theories can be embedded into string theory settings. The Chern-Simons duality we would like to realise in string theory is one of those proposed in [21],

$$U(N_c)_{K,K\pm N_c} \oplus N_f\,\psi \longleftrightarrow U\left(K+\frac{N_f}{2}\right)_{-N_c,-N_c\mp(K+N_f/2)} \oplus N_f\,\phi \tag{13}$$

[4]Strictly speaking this is true for fermions in complex representations. In the case of real representations, the path integral gets regularised as $Z_\psi = |Z_\psi| \exp\left(-i\frac{\pi}{4}\eta\right)$.

It turns out that [2] a suitable string theory embedding is that of Type 0B string theory. After an overview of Type 0B string theory, in order to realise $U(N_c)$ QCD3, we embed the gauge theory in a Hanany-Witten brane configuration. The brane configuration consists of N_c $D3$ branes suspended between an $NS5$ branes and a $(1,k)$ fivebrane. In addition, there exist N_f flavour branes and an O′3 orientifold plane.

The low energy theory arising from fluctuations on such a brane setup has the same "supermultiplet" structure of $\mathcal{N}=2$ Chern-Simons theory in three dimensions with a major difference. The gaugino λ in the "vector multiplet" $\mathcal{V}_{\mathcal{N}=2}=(A_\mu,\sigma,\lambda)$ is projected on to the two-index antisymmetric representation of the gauge group, thus breaking supersymmetry completely.

The resulting theory is referred to as Orientifold QCD (OQCD). When supersymmetry is broken, in general, there is no mechanism that prevents fields from getting a mass at one (or higher) loop(s). It is argued in the main text that, as we flow to the IR, some of the fields can be integrated out and the resulting electric theory is

$$\text{electric IR:} \qquad U(N_c)_{K,K-N_c} \oplus N_f \text{ fermions}\,, \tag{14}$$

with $K=k-(N_c-2)-N_f/2$ the shifted level.

In order to get the magnetic dual, we swap the $NS5$ and $(1,k)$ branes in the original electric brane web. The magnetic theory is then read off from the low energy excitations on the branes. The resulting low energy spectrum is that of a "vector multiplet" and a "matter multiplet" of quarks and squarks as before and a singlet meson "chiral multiplet" $M=(M,\chi)$ with the "mesino" χ projected on to the the two-index representation of the flavour symmetry $SU(N_f)$. The number of colours for the dual theory is found to be $\tilde{N}_c=N_f+k+2-N_c$.

The claim of [2] is that, upon following the RG flow of the magnetic theory, we are able to determine the IR phases of (O)QCD$_3$.

The phases of QCD$_3$ are then suitably described by a two-dimensional diagram, where the vertical axis represents the number of quark flavours, while the horizontal axis represents the combination $\kappa=k+2-N_c$. Then, various phases of QCD$_3$ corresponds to different regions of the phase diagram:

- $\kappa \geq N_f$ (bosonisation phase). Here it is where we recover the bosonisation duality given in equation (13) and proposed in [21]. The low-energy Chern-Simons level is $K=\kappa-N_f/2$. Thus, $\kappa \geq N_f$ simply means $K \geq N_f/2$, consistent with the original proposal in [21]. We argue that such a phase is captured by studying the dynamics of the magnetic theory after squark (ϕ) condensation.

- $-N_f < \kappa < 0$ (symmetring breaking phase I). In this region the number of colours of the magnetic theory, $\tilde{N}_c$, is smaller than the number of flavours, $\tilde{N}_c = N_f + \kappa < N_f$. Therefore, upon condensation of the dual squark ϕ, the theory is fully higgsed in the IR and described a Grassmanian corresponding to the following symmetry breaking pattern

$$SU(N_f) \to S\left[U\left(\frac{N_f}{2} + K\right) \times U\left(\frac{N_f}{2} - K\right)\right] . \tag{15}$$

  This is reminiscent of the symmetry breaking pattern proposed by Komargodski and Seiberg [22] in the case of QCD$_3$ with $SU(N_c)$ gauge symmetry.

- $0 < \kappa < N_f$ (symmetry breaking phase II). This is the most subtle phase to interpret. String theory seems to predict a new bosonisation phase. However, Komargodski and Seiberg [22] argued that, for the case of $SU(N_c)$ QCD$_3$, there are two ways of breaking the global symmetry as in formula (15), and that they rely on two distinct bosonic duals. In fact, after properly tuning the mass of the dual squarks, the dual magnetic theory in the regime $0 < \kappa < N_f$ precisely accounts for another way of breaking the flavour symmetry.

- $\kappa < -N_f$ (no Seiberg duality phase). This phase is trivial as there is no Seiberg duality. This is similar to the case of Seiberg duality in four dimensions when $N_f < N_c + 2$ where there is no Seiberg duality (the dual gauge group is trivial).

Appendices

Besides appendices for individual chapters, I have decided to set also an appendix at the end of the thesis where basic facts about type II supergravities are reviewed. In particular, the supergravity equations of motion and conventions which are used in the main thesis, especially in Part I, are given. This should help the reader follow and reproduce the main results of the various backgrounds spelled out along the thesis.

We begin with Type IIB and then move on to massive Type IIA supergravity. Massless Type IIA, which is the relevant case for Chapter 1, is obtained by setting $F_{(0)} = 0$ in all formulas.

Part I

Holographic dualities

Chapter 1

Holography for four-dimensional $\mathcal{N} = 2$ SCFTs

This chapter is made of three main sections. In the first section, we discuss a special class of four-dimensional quiver gauge theories and their gravity duals. We begin with a brief discussion of the four-dimensional $\mathcal{N} = 2$ superconformal algebra and then move on to introducing quiver gauge theories realised on four-, five-, and six-branes in Type IIA string theory. We review material from [12] on the construction of the gravity dual of a large class of four-dimensional $\mathcal{N} = 2$ superconformal theories in M and string theory. From Subsection 1.1.5 onwards we present material from [1], where we give some details on how to make precise the correspondence between conformal quiver gauge theories with $\mathcal{N} = 2$ supersymmetries and their gravity duals.

In Section 1.2, following [1], we test in great detail the results found in the first part of the chapter for the particular examples of two distinct superconformal quiver field theories. In particular, we show how to reproduce holographically the central charge and Linking Numbers of the special case of two different SCFTs.

In the third and last section (Section 1.3), we discuss a special class of marginal deformations of the backgrounds discussed in the previous two sections. How such marginal deformations are implemented on both the supergravity backgrounds and the dual superconformal field theories is discussed in detail.

Appendices at the end of the chapter give further details on both the supergravity backgrounds and quantum field theories discussed in the main text.

1.1 $\mathcal{N} = 2$ SCFTs and their dual backgrounds

1.1.1 $\mathcal{N} = 2$ superconformal symmetry in four dimension

The goal of this subsection is to give the reader some of the essential notions about four-dimensional superconformal field theories with sixteen supercharges – eight ordinary supersymmetries and eight superconformal symmetries. We mostly follow the notational conventions of Wess and Bagger [23] for $d = 4$, $\mathcal{N} = 1$ supersymmetry. In particular, tensors in irreducible representations of the four dimensional Lorentz algebra, $\mathfrak{so}(1,3)$, will carry indices[1] $\alpha, \dot{\alpha}$ running over $1,2$. Moreover, $SU(2)$ indices are raised and lowered with $\epsilon^{12} = \epsilon_{12} = 1$. Complex conjugation implies that $(\psi_\alpha)^\dagger = \bar{\psi}_{\dot{\alpha}}$.

The four dimensional $\mathcal{N} = 2$ supersymmetry algebra will be denoted by $\mathfrak{s}$. It has even and odd parts

$$\mathfrak{s} = \mathfrak{s}^0 \oplus \mathfrak{s}^1 \,, \tag{1.1}$$

where the even subalgebra is[2]

$$\mathfrak{s}^0 = \mathfrak{so}(1,3) \oplus \mathfrak{su}(2)_{\mathcal{R}} \oplus \mathfrak{u}(1)_r \tag{1.2}$$

while the odd subalgebra, understood as a representation of $\mathfrak{s}^0$, is

$$\mathfrak{s}^1 = [(\mathbf{2},\mathbf{1};\mathbf{2})_{+1} \oplus (\mathbf{1},\mathbf{2};\mathbf{2})_{-1}] \,, \tag{1.3}$$

where $(\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{2})$ are fundamental spinors of $SO(1,3)$ in conjugate representations, and ± 1 refers to the charge with respect $U(1)_r$. A common basis for the odd subalgebra is given by Q^i_α, $\bar{Q}^i_{\dot{\alpha}}$, where i is a fundamental index for $SU(2)_R$. There is a reality constraint between Q and $\bar{Q}$,

$$(Q^i_\alpha)^\dagger = \bar{Q}_{\dot{\alpha} i} \equiv \epsilon_{ij} \bar{Q}^j_{\dot{\alpha}} \,. \tag{1.4}$$

The commutators of the odd generators are[3]

$$\{Q^i_\alpha, \bar{Q}_{\dot{\alpha} j}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^i_j \,, \quad \{Q^i_\alpha, Q_{\beta j}\} = 0 \,, \quad \{\bar{Q}^i_{\dot{\alpha}}, \bar{Q}_{\dot{\beta} j}\} = 0 \,. \tag{1.5}$$

The commutators of the even generators with the odd generators are fixed by symmetry and (super) Jacobi identities. In particular, $SU(2)_R$ rotates the index i.

In supersymmetric theories, the $U(1)_r$ symmetry can be broken explicitly, spontaneously or it can be anomalous. It is not hard to argue that such a symmetry must

[1] This follows from the isomorphism $SO(1,3) \cong Sl(2,\mathbb{C})/\mathbb{Z}_2$. α is a fundamental index for $Sl(2,\mathbb{C})$ and complex conjugation defines an inequivalent representation. It is customary to use $\dot{\alpha}, \dot{\beta}$... to label indices in such conjugate representation.

[2] If we include translations, we should really consider the full Poincaré group in four dimensions given by $SO(1,3) \ltimes T_4$, sometimes denoted $ISO(1,3)$.

[3] Here we neglect central extensions as they are unimportant in what follows.

not be broken in conformal theories. Indeed, the conformal and R-symmetry anomaly belong to the same supermultiplet.

We might wonder how the $\mathcal{N}=2$ supersymmetry algebra changes if we add also a conformal structure. The Poincaré algebra gets extended to the full conformal algebra, $\mathfrak{so}(2,4)$, which, along with the Poincaré Q supercharges, forms a larger superconformal algebra, $\mathfrak{S}$. In order for $\mathfrak{S}$ to close, we must include also the superconformal symmetries S^i_α, $\bar{S}^i_{\dot\alpha}$, which arise from commutators of special conformal transformations with Poincaré supersymmetries, $[K,Q]\sim S$. They are as many as the Q's, doubling the number of supersymmetries, and satisfy the algebra

$$\{\bar{S}^{i\dot\alpha},S^\alpha_j\}=2(\bar\sigma^\mu)^{\dot\alpha\alpha}K_\mu\delta^i_j\,,\quad \{S^\alpha_i,S^\beta_j\}=0\,,\quad \{\bar{S}^{i\dot\alpha},\bar{S}^{j\dot\beta}\}=0\,. \tag{1.6}$$

The anticommutators of the Q's and the S's are

$$\{Q^i_\alpha,\bar{S}^{j\dot\alpha}\}=0\,,\quad \{S^\alpha_i,\bar{Q}_{j\dot\alpha}\}=0\,, \tag{1.7}$$

together with

$$\begin{aligned}\{Q^i_\alpha,S^\beta_j\}&=4(\delta^i{}_j(M_\alpha{}^\beta-\frac{i}{2}\delta_\alpha{}^\beta D)-\delta_\alpha{}^\beta R^i{}_j)\,,\\ \{\bar{S}^{i\dot\alpha},\bar{Q}_{j\dot\beta}\}&=4(\delta^i{}_j(\overline{M}^{\dot\alpha}{}_{\dot\beta}-\frac{i}{2}\delta^{\dot\alpha}{}_{\dot\beta}D)-\delta^{\dot\alpha}{}_{\dot\beta}R^i{}_j)\,,\end{aligned} \tag{1.8}$$

where

$$M_\alpha{}^\beta=-\frac{i}{4}(\sigma^\mu\bar\sigma^\nu)_\alpha{}^\beta M_{\mu\nu}\,,\qquad \overline{M}^{\dot\alpha}{}_{\dot\beta}=-\frac{i}{4}(\bar\sigma^\mu\sigma^\nu)^{\dot\alpha}{}_{\dot\beta}M_{\mu\nu}\,. \tag{1.9}$$

The commutators for $M_\alpha{}^\beta$ and $\overline{M}^{\dot\alpha}{}_{\dot\beta}$ can be deduced from those for $M_{\mu\nu}$. The commutators between even and odd generators can be inferred again by symmetry. The full superconformal algebra can be found for instance in [24] or in appendix A of [25]. $R^i{}_j$ are the generators for the $U(2)$ R-symmetry and obey the standard $\mathfrak{u}(2)$ Lie algebra[4]

$$[R^i{}_j,R^k{}_l]=\delta^k{}_jR^i{}_l-\delta^i{}_lR^k{}_j\,. \tag{1.10}$$

There are many known examples of four dimensional $\mathcal{N}=2$ theories, e.g. 4d $\mathcal{N}=2$ gauge theories or the class S theories found by Gaiotto [14]. In the following we will be mainly concerned with 4d $\mathcal{N}=2$ quiver gauge theories, for reasons that

[4] When $\mathcal{N}=4$, the R-symmetry group is actually $SU(4)$, as the $U(1)$ inside the $U(4)$ is central and decouples from the algebra. When $\mathcal{N}=2$, we can retain the full $U(2)$ R-symmetry.

will be clear in the coming sections. In the next subsection, we review some of the connections between four-dimensional gauge theories and branes in string theory.

1.1.2 Four-dimensional $\mathcal{N}=2$ SCFTs from branes

We now review the four-dimensional $\mathcal{N}=2$ theories that emerge as the low energy limit on $D4$ branes ending on $NS5$ branes and intersecting $D6$ branes. In particular, these theories can be seen to emerge from the fluctuation of open strings stretched between D branes and are represented by linear quivers. Standard references for the material covered in this section are [26] and [27].

The starting point of our discussion will be the brane web of Table 1.1.

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
$D4$	$-$	$-$	$-$	$-$			$-$			
$D6$	$-$	$-$	$-$	$-$				$-$	$-$	$-$
$NS5$	$-$	$-$	$-$	$-$	$-$	$-$				

Table 1.1: Brane setup, where $-$ mark the spacetime directions spanned by the various branes.

It is known that in string theory a Dp brane stretching along $(x^1,\dots,x^p)$ preserves supercharges of the form $\epsilon_L Q_L + \epsilon_R Q_R$ with

$$\epsilon_L = \Gamma^0\dots\Gamma^p\epsilon_R\,, \tag{1.11}$$

while an $NS5$ brane in type IIA string theory[5] stretched along $(x^1,\dots,x^5)$ preserves supercharges of the form $\epsilon_L Q_L + \epsilon_R Q_R$ with[6]

$$\begin{aligned}\epsilon_L &= \Gamma^0\dots\Gamma^5\epsilon_L\\ \epsilon_R &= \Gamma^0\dots\Gamma^5\epsilon_R\,.\end{aligned} \tag{1.12}$$

It is fairly easy to see that a solution to the equations (1.11), (1.12) for the system in Table 1.1 leaves a total of eight supersymmetries undetermined, thus preserving

[5] In type IIB the equation for ϵ_R picks up a minus sign, $\epsilon_R = -\Gamma^0\dots\Gamma^5\epsilon_R$.

[6] Here ϵ_L and ϵ_R are ten dimensional Majorana-Weyl spinors with given chirality. Indeed, it is only in 2 (mod 8) dimensions that we can impose a Weyl condition along with a reality condition. Thus, we can think of ϵ_L and ϵ_R as objects with sixteen real and independent components.

eight of the thirty two supersymmetries. This is the amount of supersymmetry of four-dimensional $\mathcal{N} = 2$ supersymmetric theories.

Notice that, for the brane web (1.1), the ten-dimensional Lorentz group is decomposed in the following fashion

$$SO(1,9) \to SO(1,3) \times SO(2) \times SO(3)\,, \tag{1.13}$$

where $SO(1,3)$ acts on $(x^0, \ldots, x^3)$, while $SO(2)$ and $SO(3)$ act on (x^4, x^5) and (x^7, x^8, x^9), respectively. We will interpret the $SO(1,3)$ as the Lorentz group in four dimensions, while $SO(2)$ and $SO(3)$ are global symmetries. Decomposing the ten-dimensional supersymmetries (spinors) as in (1.13), we find that they transform as a doublet of $SO(3)$ and carry charge under $SO(2) = U(1)$. Therefore, we can interpret $SO(3)$ and $SO(2)$ as R-symmetries. They are nothing but the $SU(2)_R \times U(1)_r$ R-symmetry of four-dimensional $\mathcal{N} = 2$ theories.

In the following, we will specialise to the case where the $D4$ branes have finite extension in the x^6 direction, as we are interested in macroscopic four-dimensional theories, and are stretched between parallel $NS5$ branes. Moreover, all the $D4$ and $NS5$ branes are at $x^7 = x^8 = x^9 = 0$, while the $D6$ branes are at $x^4 = x^5 = 0$.

Let us now point out a nice phenomenon that takes place when moving branes first noted by Hanany and Witten in type IIB string theory. They argued that, whenever a $D5$ branes moves past an $NS5$ brane, a $D3$ brane between them is created (or annihilated). The same applies to type IIA string theory: whenever a $D6$ brane moves past an $NS5$ brane we have that a $D4$ brane is created (or annihilated) between them. As a matter of fact, it turns out that brane webs with inequivalent low-energy physics are characterised by a set of topological invariants, called Linking Numbers. There are two natural definitions of Linking Numbers, K_i and L_j, for the i-th $NS5$ brane and the j-th $D6$ brane, respectively. These are given by

$$\begin{aligned} K_i &= N^{right}_{D4,i} - N^{left}_{D4,i} - N^{right}_{D6,i}\,, \\ L_j &= N^{right}_{D4,j} - N^{left}_{D4,j} + N^{left}_{NS,j}\,, \end{aligned} \tag{1.14}$$

where N^{right}_{D4} ($N^{left}_{D4,i}$) is the number of $D4$ branes ending on the i-th $NS5$ brane from the right (left) and $N^{right}_{D6,i}$ is the number of $D6$ branes placed on the right of the i-th $NS5$ brane in a generic brane web. Likewise, $N^{left}_{NS,j}$ is the number of $NS5$ branes on the left of the j-th $D6$ brane. As shown in [28], they must satisfy the following conservation equation

$$\sum_{i=1}^{N_5} K_i + \sum_{j=1}^{N_6} L_j = 0\,. \tag{1.15}$$

Later in this chapter, we will show how these Linking Numbers can be computed holographically from the gravity solution given by the backreaction of generic $D4$, $D6$ and $NS5$ branes on to the geometry.

Since the $D4$ branes have finite extension in the x^6 direction, the low-energy physics will be effectively four-dimensional. We will discuss the low-energy limit of brane dynamics in a somewhat quantitative way in a moment. For now, let us spell out the field content arising from the quantisation of open strings for the brane web in (1.1).

Open strings stretched between, say k_n, coincident $D4$ branes give rise to an $\mathcal{N} = 2$ super Yang-Mills theory with gauge group $U(k_n)$. Witten showed that the $U(1)$ factor inside each $U(k_n)$ decouples from the dynamics at low energies, leaving as a total gauge group $\prod SU(k_n)$ [26]. Strings stretched between adjacent $D4$ branes, say between k_{n-1} and k_n $D4$ branes, gives rise to $\mathcal{N} = 2$ hypermultiplets in the bifundamental representation $(\mathbf{k}_{n-1}, \bar{\mathbf{k}}_n)$ of adjacent gauge groups. Also, $D4 - D6$ strings stretched between k_n $D4$ branes and d_n $D6$ branes give rise fundamental hypermultiplets in the representation $\mathbf{k}_n$.

The field theory built in this way is then a quiver gauge field theory with $\mathcal{N} = 2$ supersymmetry and is represented in Figure 1.1.

The one-loop beta function for each gauge group $SU(k_n)$ is given by

$$b_{0,n} = -2k_n + k_{n-1} + k_{n+1} + d_n\,, \tag{1.16}$$

and the $\mathcal{N} = 2$ theory is conformal if the beta function vanishes[7] for each n. This can be seen as a condition on the number of flavours (d_n)

$$d_n = 2k_n - k_{n-1} - k_{n+1}\,. \tag{1.17}$$

The classical gauge coupling for each gauge node is read off from the DBI action for a $D4$ brane after reducing over the interval $[x_i^6, x_{i+1}^6]$ and reads

$$\frac{1}{g_n^2} = \frac{x_{i+1}^6 - x_i^6}{g_s l_s}\,, \tag{1.18}$$

[7] $\mathcal{N} = 2$ supersymmetry forbids any further correction beyond one loop (and in fact even non-perturbatively), as it happens instead in $\mathcal{N} = 1$ theories.

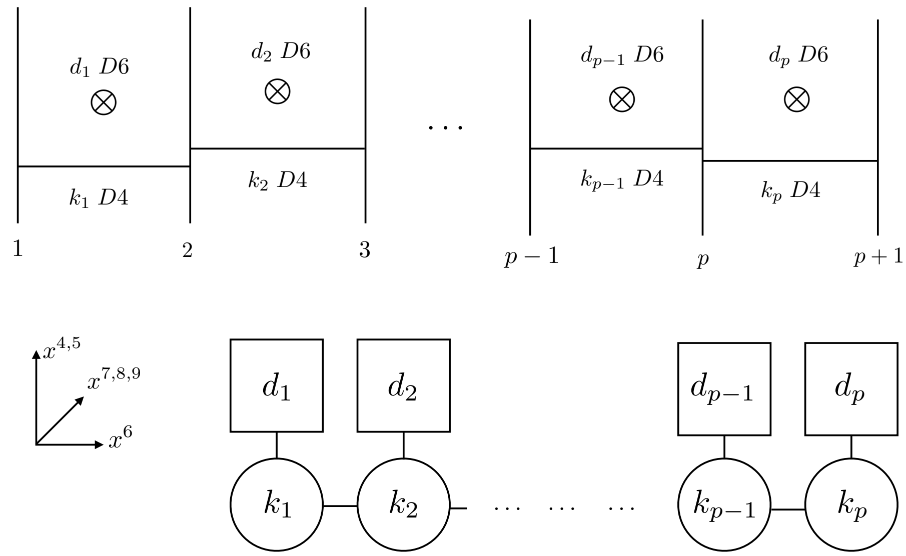

Figure 1.1: The quiver and Hanany-Witten setup for a generic situation. The vertical lines denote individual $NS5$ branes extended along the (x^4, x^5) space. The horizontal lines denote $D4$ branes, extended along x^6, in between fivebranes. The crossed-circles represent $D6$ branes, that extend along the (x^7, x^8, x^9) directions. All the branes share the Minkowski directions. Such a brane configuration realises the isometry $SO(1,3) \times SO(3) \times SO(2)$.

where g_s and l_s are the string coupling and length, respectively. In order to have a finite YM coupling in the low-energy limit ($l_s \to 0$), adjacent $NS5$ branes must be close to each other in string units. We have strong coupling when the distance between $NS5$ branes goes to zero in units of the string length. This particular limit, as we shall see later, is particularly useful when employing a holographic description for the four-dimensional quiver field theories.

As we are studying four-dimensional field theories, we might wonder how the theta angle for each gauge node θ_n is encoded in the brane description. It turns out that it is not immediately visible in type IIA tring theory, but becomes apparent when we uplift to M-theory. It is believed that M-theory on $\mathbb{R}^{1,9} \times S^1$ is equivalent to type IIA string theory on $\mathbb{R}^{1,9}$. If we denote the eleventh coordinate by x^{10}, and we take it to have period of $2\pi R_{10}$, with $R_{10} = g_s l_s$, the theta angle for each gauge group $SU(k_n)$

is given by the difference between the position of the $(n+1)$th and nth $M5$ branes

$$\theta_n = \frac{x^{10}_{n+1} - x^{10}_n}{g_s l_s} \,. \tag{1.19}$$

Let us make a few more remarks before ending this section. As the eleventh dimension depends on the string coupling, it turns out that the strong coupling limit of Type IIA string theory is better understood in M-theory, where one additional dimension shows up. This is somehow different from Type IIB string theory, which enjoys a strong/weak self-duality.

As we said before, M-theory on $\mathbb{R}^{1,9} \times S^1$ is equivalent to type IIA string theory on $\mathbb{R}^{1,9}$. Thus, it should be possible to reinterpret every object in Type IIA string theory in M-Theory. It turns out that, as showed by Witten in [26], reinterpreting $D4$ and $NS5$ branes as $M5$ branes in M-theory (and $D6$ branes as KK-monopoles), it is possible to find explicit "solutions" for a large family of four-dimensional $\mathcal{N}=2$ field theories with zero or negative beta function. In full generality, by "solving" a theory we usually mean that we want to understand as many things as possible from it. One large piece of information is to understand what the theory flows to in the IR. As a matter of fact, it turns out that there is a one-to-one correspondence between the quantum vacua of $\mathcal{N}=2$ SYM and supersymmetric configurations of an $M5$ brane with worldvolume $\mathbb{R}^{1,3} \times \Sigma$, with Σ a (complex) curve in a (real) four-dimensional space whose equation can be written down explicitly. This is quite a huge subject that has had a prolific continuation both in mathematics and physics. Some of the most significant follow-ups include [29, 30, 14, 31]. See also [32, 33] and references therein.

Let us now move on to the holographic description of the quantum field theories just introduced.

1.1.3 LLM solutions in M-theory

The aim of this subsection is to briefly review the M-theory solutions found by Lin, Lunin and Maldacena (LLM) [12] dual to some four-dimensional $\mathcal{N}=2$ theories constructed in [14]. We will see later that assuming an extra $U(1)$ symmetry in M-theory allows us to find in Type IIA the holographic duals to the quiver gauge theories discussed in the previous subsection.

In holography, global symmetries of a field theory are usually realised as isometries of the dual background. The most general solution in eleven dimension realising the four-dimensional $\mathcal{N}=2$ superalgebra, $\mathfrak{su}(2,2|2)$, preserving sixteen supercharges was found in [12]. In particular, in order to realise geometrically the $\mathfrak{so}(2,4)\oplus\mathfrak{su}(2)_R\oplus\mathfrak{u}(1)_r$ subalgebra, the authors of [12] considered an eleven-dimensional spacetime with AdS_5, S^2 and S^1 subspaces. The remaining three-dimensional space is not determined by symmetry and is parametrised by three coordinates, (x_1, x_2, y). In a convenient form, the full background reads

$$\begin{aligned} \frac{ds^2}{\kappa^{2/3}} &= 4f_1 ds^2_{AdS_5} + f_2 ds^2_{S^2} + f_3(d\beta + A_i dx^i)^2 + f_4 dy^2 + f_5 \delta_{ij} dx_i dx_j \\ G_{(4)} &= dA_{(3)} = \kappa F_{(2)} \wedge \widehat{\mathrm{vol}}_{S^2}\,, \end{aligned} \tag{1.20}$$

where the warping factors are

$$\begin{aligned} f_1 &= e^{2\tilde\lambda}\,, & f_2 &= y^2 e^{-4\tilde\lambda}\,, & f_3 &= 4e^{2\tilde\lambda}(1-y^2e^{-6\tilde\lambda})\,, \\ f_4 &= \frac{e^{-4\tilde\lambda}}{1-y^2e^{-6\tilde\lambda}}\,, & f_5 &= f_4 e^D\,, & A_i &= \frac{1}{2}\epsilon_{ij}\partial_j D\,, \end{aligned} \tag{1.21}$$

whereas $F_{(2)}$ is given by

$$F_{(2)} = 2\left[(d\beta + A_i dx^i)\wedge d(y^3 e^{-6\tilde\lambda}) + y(1-y^2e^{-6\tilde\lambda})dA_i\wedge dx^i - \frac{1}{2}\partial_y e^D dx^1\wedge dx^2\right]. \tag{1.22}$$

The function $\tilde\lambda$ is given in terms of the function D through

$$e^{-6\tilde\lambda} = -\frac{\partial_y D}{y(1-y\partial_y D)}\,. \tag{1.23}$$

It was shown in [12] that, in order for the background to preserve sixteen real supersymmetries and the equations of motion to be satisfied, the function D has to satisfy the following Toda equation

$$(\partial^2_{x_1} + \partial^2_{x_2})D + \partial^2_y e^D = 0\,. \tag{1.24}$$

Boundary conditions for the Toda equation

Solutions to the equation (1.24) must be supplemented with some boundary conditions in order for the background in (1.20) to be regular everywhere. We discuss them in turn.

Regularity of the metric at $y=0$, where the S^2 collapses to zero size, requires that

$$\partial_y D|_{y=0} = 0\,, \qquad e^D|_{y=0} = \text{finite}\,, \tag{1.25}$$

in such a way $\partial_y D/y$ has smooth finite limit when y approaches zero. Moreover, we should take into account that our theory is realised on $D4$ branes stretched between $NS5$ branes. The latter, when uplifted to M-theory, become $M5$ branes and therefore we need to have a non-trivial four-cycle that supports a non-trivial flux for $G_{(4)}$. This is achieved by demanding that the S^1 shrinks at some finite value $y=y_c$. A non-trivial four-cycle, call it $\mathcal{M}_4$, is then given by $S^2\times S^1$ warped over the interval $I_y=\{y|0\leq y\leq y_c\}$. In other words, the fact that the S^1 shrinks at $y=y_c$ makes the spacetime terminate in the y direction at $y=y_c$, and we can then define a compact four-cycle as just stated. From the metric (1.20), this is achieved by demanding that

$$\lim_{y\to y_c} \partial_y D = \infty\,, \tag{1.26}$$

which, in turn, implies that

$$\lim_{y\to y_c} e^{-6\tilde\lambda} = \frac{1}{y_c^2}\,. \tag{1.27}$$

Then regularity of the metric (we want the f_5 in (1.21) and the background with it to be finite when $y\sim y_c$) demands

$$e^D|_{y\sim y_c} \sim y-y_c\,. \tag{1.28}$$

The four-form flux on $\mathcal{M}_4$ then reads

$$\begin{aligned}\int_{\mathcal{M}_4} G_{(4)} &= \kappa\int_{S^2}\widehat{\text{vol}}_{S^2}\int_{S^1\times I_y} F_2|_{x_i=\text{const}}\\ &= \kappa(4\pi)^2\int_0^{y_c} dy\partial_y(y^3e^{-6\tilde\lambda})\\ &= \kappa(4\pi)^2 y_c\,.\end{aligned} \tag{1.29}$$

Notice that to get this result we used all the boundary conditions above. Setting $\kappa=\frac{\pi}{2}l_p^3$, we find that y_c must be a positive integer number.

Finally, let us just mention that if we want to add explicit $M5$ brane sources to the background we must modify the Toda equation (1.24) as to have singular sources located at $x = x_0^{(i)}$ and extended along y in the following fashion

$$(\partial_{x_1}^2 + \partial_{x_2}^2)D + \partial_y^2 e^D = -2\pi\delta^{(2)}(x - x^{(i)})\theta(2N_5 - y)\,. \tag{1.30}$$

1.1.4 Solution with an extra $U(1)$ symmetry

The equation (1.24) is notoriously difficult to solve and explicit eleven-dimensional background are difficult to find. Nonetheless, there is a way of simplifying the problem by assuming that the background has an additional isometry generated by the Killing vector ∂_{x_1}. In this way, we can reduce the eleven-dimensional M-theory background to the ten dimensions of type IIA supergravity. This is most conveniently (though not easily) done by considering an implicit change of coordinates where the function D and the coordinates (x_2, y) are traded for a new function V and coordinates (σ, η) in the following way

$$e^D = \sigma^2\,, \quad y = \sigma\partial_\sigma V \qquad x_2 = \partial_\eta V\,. \tag{1.31}$$

It turns out that [13], with the change of variables (1.31), the Toda equation (1.24) reduces to a linear partial differential equation for V,

$$\ddot{V} + \sigma^2 V'' = 0\,, \qquad \dot{V} \equiv \sigma\partial_\sigma V\,, \qquad V' = \partial_\eta V\,. \tag{1.32}$$

The string frame metric reads

$$\frac{ds^2}{\alpha'\mu^2} = 4f_1 ds^2_{AdS_5} + f_2(d\sigma^2 + d\eta^2) + f_3 ds^2_{S^2} + f_4 d\beta^2\,, \tag{1.33}$$

where, the quantity $\mu^2 = L^2/\alpha'$ indicates the size of the spacetime in units of α'. The remaining NS-NS fields read

$$e^{2\phi} = f_8\,, \qquad B_{(2)} = \mu^2\alpha' f_5 \widehat{\text{vol}}_{S^2}\,, \tag{1.34}$$

while the R-R fields are

$$C_{(1)} = \mu^4\alpha'^{1/2} f_6 d\beta\,, \qquad C_{(3)} = \mu^6\alpha'^{3/2} f_7 d\beta \wedge \widehat{\text{vol}}_{S^2}\,. \tag{1.35}$$

The warping factors are given by[8]

$$f_1=\left(\frac{2\dot V-\ddot V}{V''}\right)^{\frac12},\qquad f_2=f_1\frac{2V''}{\dot V},\qquad f_3=f_1\frac{2V''\dot V}{\Delta},\qquad f_4=f_1\frac{4V''}{2\dot V-\ddot V}\sigma^2$$
$$f_5=2\left(\frac{\dot V\dot V'}{\Delta}-\eta\right),\qquad f_6=\frac{2\dot V\dot V'}{2\dot V-\ddot V},\qquad f_7=-\frac{4\dot V^2V''}{\Delta},\qquad f_8=\left(\frac{4(2\dot V-\ddot V)^3}{\mu^{12}V''\dot V^2\Delta^2}\right)^{\frac12}, \tag{1.36}$$

while Δ is given by

$$\Delta=(2\dot V-\ddot V)V''+(\dot V')^2\,. \tag{1.37}$$

For future purposes, it is useful to uplift the backgrounds (1.33), (1.34) and (1.35) to M-theory. Such a procedure is explained in Appendix B and gives the result

$$\begin{aligned}\frac{ds^2}{\kappa^{2/3}}&=4F_1ds^2_{AdS_5}+F_2(d\sigma^2+d\eta^2)+F_3ds^2_{S^2}+F_4d\beta^2+F_5(dy+\tilde A d\beta)^2\\ A_{(3)}&=\kappa(F_6d\beta+F_7dy)\wedge\widehat{\mathrm{vol}}_{S^2}\,,\end{aligned} \tag{1.38}$$

where the warping factors are

$$F_1=\left(\frac{\dot V\Delta}{2V''}\right)^{1/3},\qquad F_2=F_1\frac{2V''}{\dot V},\qquad F_3=F_1\frac{2V''\dot V}{\Delta},\qquad F_4=F_1\frac{4V''}{2\dot V-\ddot V}\sigma^2,$$
$$F_5=F_1\frac{2(2\dot V-\ddot V)}{\dot V\Delta},\qquad F_6=-4\frac{\dot V^2V''}{\Delta},\qquad F_7=2\left(\frac{\dot V\dot V'}{\Delta}-\eta\right),\qquad \tilde A=\frac{2\dot V\dot V'}{2\dot V-\ddot V}, \tag{1.39}$$

where we have that $\kappa^{2/3}=\mu^4\alpha'$.

Boundary conditions for the Laplace equation

Just as for the case of the Toda equation, the solution to the Laplace equation (1.32) must be supplemented with boundary conditions in order for the background (1.33), (1.34) and (1.35) to be regular everywhere (again, except at points where brane sources are located). These can be found just by inspection of the ten dimensional background or by translating the boundary condition for the Toda equation to the ten-dimensional case. In either case we find that these are easily expressed in terms of $\dot V$ by demanding

[8] We use the same symbols f_i for the warping factors as in (1.21). Hopefully this will not cause any confusion.

that[9]

$$\dot{V}|_{\eta=0,\eta_c} = 0\,. \tag{1.40}$$

Moreover, it turns out to be useful to define the following quantity

$$\lambda = \dot{V}|_{\sigma=0}\,. \tag{1.41}$$

Quantisation of fluxes imposes non-trivial conditions on what forms on λ. In particular, it turns out that [13]

- λ must be piecewise linear and continuous, made of segments of the form $\lambda = a_i\eta + q_i$, with $a_i \in \mathbb{Z}$
- $\lambda(0) = \lambda(\eta_c) = 0$
- the change of the gradient of λ at a kink must be a non-positive integer, i.e. $a_i - a_{i-1} \in \mathbb{Z}_{\leq 0}$
- The positions of the kinks along the η axis must appear at integer values for η.

1.1.5 Generic solutions to the Laplace equation

In the following, we shall consider two different types of solutions to the Laplace equation (1.32). The first type of solution, call it $V_1(\sigma,\eta)$, is well defined for the whole range of the σ coordinate, and was first discussed in [34, 35]. The second type of solutions, call it $V_2(\sigma,\eta)$, should be thought of as series expansion close to $\sigma = 0$, and is a generalisation of a solution already presented in [36, 37]. The potentials in each case read

$$V_1(\sigma,\eta) = -\sum_{n=1}^{\infty} \frac{c_n}{w_n} K_0(w_n\sigma)\sin(w_n\eta), \quad w_n = \frac{n\pi}{N_5}. \tag{1.42}$$

$$V_2(\sigma,\eta) = F(\eta) + G(\eta)\log\sigma + \sum_{k=1}^{\infty} \sigma^{2k}\left(h_k(\eta) + \widehat{f}_k(\eta)\log\sigma\right). \tag{1.43}$$

The coefficients c_n in equation (1.42) can be thought of as the Fourier coefficients of the odd-extended function $\lambda(\eta)$ – see equation (1.41) – in the interval $[-N_5, N_5]$. More

[9] η_c determines the end of the space in the η direction. The fact that the space is compact in the η direction comes about in a similar fashion to the compactness of y for the eleven-dimensional spacetime. This will be further discussed later in this section.

in detail

$$c_n = \frac{n\pi}{N_5^2}\int_{-N_5}^{N_5}\lambda(\eta)\sin(w_n\eta)d\eta\,,\quad w_n=\frac{n\pi}{N_5}. \tag{1.44}$$

On the other hand, the functions $h_k, \widehat{f}_k$ in equation (1.43) can be given explicitly in terms of the input functions $F(\eta), G(\eta)$ according to the following recursive relations

$$\begin{aligned} h_1(\eta) &= \frac{1}{4}(G''(\eta)-F''(\eta))\,, & \widehat{f}_1(\eta) &= -\frac{1}{4}G''(\eta),\\ h_k(\eta) &= -\frac{1}{4k^2}\big(h''_{k-1}(\eta)-\frac{1}{k}\widehat{f}''_{k-1}(\eta)\big)\,, & \widehat{f}_k(\eta) &= -\frac{1}{4k^2}\widehat{f}''_{k-1}(\eta)\,, \end{aligned} \tag{1.45}$$

with $k=2,3,4\ldots$

Making use of the equation (1.42), we obtain the following nice expression for $\lambda(\eta)$

$$\lambda(\eta)=\sigma\partial_\sigma V_1(\sigma,\eta)|_{\sigma=0}=\sum_{n=1}^{\infty}\frac{c_n}{w_n}\sin(w_n\eta)\,, \tag{1.46}$$

whereas, from the equation (1.43), we find

$$\lambda(\eta)=\sigma\partial_\sigma V_2(\sigma,\eta)|_{\sigma=0}=G(\eta). \tag{1.47}$$

Because of the asymptotic behaviour of the modified Bessel function $K_0(\sigma)$ at infinity[10], we find that V_1 is vanishingly small at infinity. Actually, the fact that V vanishes sufficiently fast when $\sigma=\infty$ is a necessary condition, as in the following we will be computing integrals over the coordinates of the internal manifold (thus, over σ) in order to get the holographic central charge. Convergence of such integrals is in general achieved if V goes to zero fast enough at $\sigma=\infty$. We will see that this is always the case for the physical backgrounds we will consider.

Convergence properties of the solution V_2 are less clear. For this reason, in the rest of this chapter, we will be mainly concerned with solutions to the Laplace equation in the form given in equation (1.42).

In Appendix C we quote the series expansions for all the warping factors appearing in the background close to $\sigma=0$ and $\sigma=\infty$, computed using the profiles for V given in equations (1.42), (1.43).

Let us now move on to the discussion of the detailed correspondence between the backgrounds in equations (1.33), (1.34) and (1.35) and the conformal field theories of

[10] $K_0(z)\sim\frac{e^{-z}}{\sqrt{z}}$ close to $z=\infty$. Asymptotic expansions of this kind will be reviewd in Appendix C.

interest.

1.1.6 Correspondence with a conformal quiver field theory

The quantum field theories we are interested in are $\mathcal{N} = 2$ SCFTs with gauge symmetry given as the product of many gauge groups, $SU(k_1) \times SU(k_2) \times .... \times SU(k_n)$. Thus, the field theories possess n $\mathcal{N} = 2$ vector multiplets, $n-1$ hypermultiplets transforming in the bifundamental of each pair of consecutive gauge groups and a set of hypers (behaving as flavours) transforming in the fundamental of each gauge group. The condition of zero beta function, namely that for each gauge factor the number of colours equals twice the number of flavours, gets translated to

$$2k_i = d_i + k_{i+1} + k_{i-1}\,, \tag{1.48}$$

with $i = 1, \ldots, n$. We denote by d_i the number of fundamental hypers for the i-th group and with k_{i+1} and k_{i-1} the ranks of the two groups neighboring the i-th node. Following [38], we can define the "forward and backwards lattice derivatives"

$$\partial_+ k_i = k_{i+1} - k_i\,, \quad \partial_- k_i = k_i - k_{i-1}\,. \tag{1.49}$$

In terms of $\partial_\pm$, the condition that all the beta functions vanish reads

$$d_i = 2k_i - k_{i+1} - k_{i-1} = -\partial_+ k_i + \partial_- k_i = -\partial_+\partial_- k_i\,. \tag{1.50}$$

Since the number of fundamental hypers d_i is non-negative, we find that $\partial_+\partial_- k_i \leq 0$, for each $i = 1, \ldots, n$. We will refer to this property as the "convexity" of the rank function k_i. Let us also define the "slope function"

$$s_i = k_i - k_{i-1} = \partial_- k_i\,, \tag{1.51}$$

from which we read off that the number of flavours at each node is given by

$$d_i = -\partial_+ s_i\,. \tag{1.52}$$

Again, given that $d_i \geq 0$, we find $\partial_+ s_i \leq 0$. This simply means that the slope function, understood as a function of the discrete variable i, is monotonically decreasing.

It is in general possible to give a "continuum version" of the lattice derivatives and slope function just defined. Let us see how this works out. Define the "rank function" $R(\eta)$, where η parametrises the "field theory space", i.e. the direction along which gauge and flavour groups are distributed. The first derivative of $R(\eta)$ gives us the slope, $R'=s$, while the second derivative the number of fundamentals, $-R''=d$. Let us clarify this with a generic example.

Consider the quiver of Figure 1.2. For this quiver to represent an $\mathcal{N}=2$ SCFT,

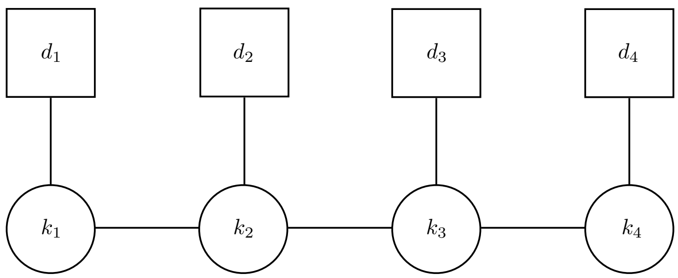

Figure 1.2: A generic quiver. The squares indicate flavour groups and the circles gauge groups.

the following conditions must be satisfied

$$\begin{aligned} d_1 &= 2k_1 - k_2\,, & d_2 &= 2k_2 - k_1 - k_3\,, \\ d_3 &= 2k_3 - k_2 - k_4\,, & d_4 &= 2k_4 - k_3\,. \end{aligned} \tag{1.53}$$

We then construct the rank function

$$R(\eta) = \begin{cases} k_1\eta & 0 \le \eta < 1 \\ (k_2-k_1)(\eta-1)+k_1 & 1 \le \eta < 2 \\ (k_3-k_2)(\eta-2)+k_2 & 2 \le \eta < 3 \\ (k_4-k_3)(\eta-3)+k_3 & 3 \le \eta < 4 \\ -k_4(\eta-4)+k_4 & 4 \le \eta \le 5. \end{cases}$$

Computing $R'(\eta)$ we find the piecewise discontinuous function

$$R'(\eta)=\begin{cases} k_1 & 0\leq\eta<1\\ (k_2-k_1) & 1\leq\eta<2\\ (k_3-k_2) & 2\leq\eta<3\\ (k_4-k_3) & 3\leq\eta<4\\ -k_4 & 4\leq\eta\leq 5. \end{cases}$$

Each entry in $R'(\eta)$, $(k_1, k_2-k_1, \dots)$, is nothing but the difference between the rank of two adjacent gauge groups, what we called "slope function" in (1.51). Moreover, if we calculate $-R''(\eta)$ we find precisely the function that gives us the number of fundamental hypermultiplets at each gauge group

$$\begin{aligned} d(\eta)=-R''(\eta)=&(2k_1-k_2)\delta(\eta-1)+(2k_2-k_1-k_3)\delta(\eta-2)\\ &+(2k_3-k_2-k_4)\delta(\eta-3)+(2k_4-k_3)\delta(\eta-4)\,. \end{aligned} \tag{1.54}$$

Indeed, according to equation (1.53), (1.54) simply tells us that we have $d_1=2k_1-k_2$ fundamental hypers connected to the first gauge group $SU(k_1)$, $d_2=2k_2-k_1-k_3$ connected to the second, and so on.

The connection between the gravitational picture and the field theory comes from the identification of the functions

$$\lambda(\eta)=R(\eta). \tag{1.55}$$

This is a non-trivial step as it relates the "field theory space" with the space coordinate η in IIA or M-theory background [13].

The logic to follow is then clear. First choose a conformal quiver field theory. Then write the rank function $R(\eta)$ and use this function as the boundary condition for the Laplace-like problem in equation (1.32) setting $\lambda(\eta)=R(\eta)$.

Trustability of the holographic description

The validity of the supergravity solutions in (1.33), (1.34) and (1.35) was carefully analysed in [35]. Such backgrounds have physical $D6$ and $NS5$ branes, and close to these branes both scalar curvature invariants (in units of α') and the string coupling become large. In general, we cannot trust holographic calculations in regions of the spacetime where $g_s = e^{\phi}$ and/or $\frac{\alpha'}{R_{eff}}$ are arbitrary large.

Another way to look at this is to consider our backgrounds as defining a two-dimensional manifold $V = V(\sigma, \eta)$ in a three-dimensional space parametrised by three coordinates (σ, η, V). The points in the σ, η space at which the $D6$ and $NS5$ branes are located are singular points on this manifold. The information obtained after holographic calculations close to those points is not reliable.

The idea then would be to "localise" the singluar regions in small patches on the manifold defined by $V = V(\sigma, \eta)$ close to the singular points. In order to do this, it was suggested in [35] that one has to take N_5 (the range of the η-coordinate) to be very large, hence dealing with long-linear quivers. We could also scale the function $\lambda(\eta) \to N_c\lambda(\eta)$. In this way we are changing the number of $D4$ and $D6$ branes (but keeping the number of $NS5$ branes fixed) having good control over string loop corrections (in a 't Hooft limit, with $g_s N_4$ fixed). Similarly, rescaling of the η coordinate increases the number of fivebranes reducing curvature divergences.

To sum up, we shall consider in all our comparisons between SCFTs and holographic results the range of the η coordinate N_5 to be very large. Also the function $\lambda(\eta)$ is considered to be scaled up by a large factor N_c. N_c will turn to be proportional to the number of $D4$ and $D6$ branes as we explain below.

1.1.7 Page Charges

In this section we work out the Page charges associated with the background above. As it is well known [39], it is this kind of charges that is conserved, localised and quantised, even though they are not invariant under (large) gauge transformations. As we will see in the coming sections, they imply the quantisation of some constants directly related to the number branes present in the supergravity background.

Let us briefly revisit the idea behind the Page charges. First of all, we aim at writing the modified Bianchi identities as the exterior derivative of some differential forms. When external sources are present, it is this exterior derivative that identifies a current and the corresponding charge. Let us see how this works for the case of $D4$ branes. The modified Bianchi identity for $F_{(4)}$ is[11]

$$dF_{(4)} - H_{(3)} \wedge F_{(2)} = 0\,. \tag{1.56}$$

[11] See Appendix IV at the end of the thesis where equations and conventions of Type IIA supergravity are reviewed.

An easy computation reveals that such an equation can be re-written as

$$d\big(F_{(4)} - F_{(2)} \wedge B_{(2)} + \frac{F_{(0)}}{2!} B_{(2)} \wedge B_{(2)}\big) = 0\,, \tag{1.57}$$

The term in parenthesis is the closed differential form we are looking for and, in presence of sources, this equation would read

$$d\big(F_{(4)} - F_{(2)} \wedge B_{(2)} + \frac{F_{(0)}}{2!} B_{(2)} \wedge B_{(2)}\big) = \star j^{\text{Page}}_{D4}\,. \tag{1.58}$$

This equation gives us the definition of Page form $\widehat{F}_{(4)}$ as the term in parenthesis, together with Page current $\star j^{\text{Page}}_{D4}$. Given such a definition, the Page current is automatically conserved and localised. The corresponding charge is given by

$$Q_{D4} = \frac{1}{(2\pi)^3}\int_{\mathcal{M}_5} d\widehat{F}_{(4)} = \frac{1}{(2\pi)^3}\int_{\partial\mathcal{M}_5} \widehat{F}_{(4)}\,, \tag{1.59}$$

for some compact five-manifold $\mathcal{M}_5$.

$\widehat{F}_{(4)}$ has a nice expression as a rank-four form in the polyform $\widehat{F}_{(k)} = (F \wedge e^{-B_{(2)}})_{(k)}$. It turns out that the fluxes $\widehat{F}_{(k)}$ are closed and provide a good definition of quantised charges associated with all the fluxes. They are usually referred to as Page fluxes or Page differential forms. The general expression for the Page charge of a Dp brane is therefore given by[12]

$$(2\pi)^{7-p} g_s \alpha'^{\frac{7-p}{2}} Q_{Dp} = \int_{\mathcal{M}_{8-p}} \widehat{F}_{(8-p)}\,, \tag{1.60}$$

where $\mathcal{M}_{8-p}$ is any $(8-p)$-dimensional compact manifold.

Let us now move on to computing the Page charges associated with a generic Gaiotto-Maldacena background. These charges are identified with the number of branes in the associated Hanany-Witten set-up.

Using the expressions for the fields in equation (1.34) and (1.35), we find for the NS-NS $H_{(3)}$ field

$$H_{(3)} = dB_{(2)} = \mu^2 \alpha' \left(\partial_\sigma f_5 d\sigma + \partial_\eta f_5 d\eta\right) \wedge \widehat{\text{vol}}_{S^2}\,, \tag{1.61}$$

[12]The ten-dimensional gravitational constant and the brane tension are related to the string coupling and length by $2\kappa_{10}^2 T_{Dp} = (2\pi)^{7-p} g_s \alpha'^{\frac{7-p}{2}}$.

while for the R-R sector

$$\begin{aligned}\widehat{F}_{(2)} &= \mu^4\alpha'^{\frac{1}{2}}\left(\partial_\sigma f_6 d\sigma + \partial_\eta f_6 d\eta\right)\wedge d\beta\,,\\ \widehat{F}_{(4)} &= \mu^6\alpha'^{\frac{3}{2}}\left[(\partial_\sigma f_7 - f_5\partial_\sigma f_6)d\sigma + (\partial_\eta f_7 - f_5\partial_\eta f_6)d\eta\right]\widehat{\mathrm{vol}}_{S^2}\wedge d\beta\end{aligned} \tag{1.62}$$

Let us specify the non-trivial cycles over which we are going to integrate the Page fluxes

$$\Sigma_2 = (\beta,\eta)|_{\sigma=0}\,,\quad \Sigma_3 = (\eta,\chi,\xi)|_{\sigma=0}\,,\quad \tilde{\Sigma}_3 = (\eta,\chi,\xi)|_{\sigma=\infty},\quad \Sigma_4 = (\beta,\eta,\chi,\xi)|_{\sigma=0}. \tag{1.63}$$

We then find for the $NS5$ branes

$$\begin{aligned}Q_{NS} &= \frac{1}{(2\pi)^2 g_s^2\alpha'}\mu^2\alpha'\int\widehat{\mathrm{vol}}_{S^2}\int_0^{\eta_f} d\eta\,\partial_\eta f_5(\sigma,\eta)|_{\sigma=0} = \frac{\mu^2}{g_s^2\pi}[f_5(0,\eta_f) - f_5(0,0)]\,,\\ \tilde{Q}_{NS} &= \frac{1}{(2\pi)^2 g_s^2\alpha'}\mu^2\alpha'\int\widehat{\mathrm{vol}}_{S^2}\int_0^{\eta_f} d\eta\,\partial_\eta f_5(\sigma,\eta)|_{\sigma=\infty} = \frac{\mu^2}{g_s^2\pi}[f_5(\infty,\eta_f) - f_5(\infty,0)]\,,\end{aligned} \tag{1.64}$$

while for $D6$ branes

$$Q_{D6} = \frac{1}{2\pi g_s\alpha'^{\frac{1}{2}}}\mu^4\alpha'^{\frac{1}{2}}\int_0^{2\pi} d\beta\int_0^{\eta_f} d\eta\,\partial_\eta f_6(\sigma,\eta)|_{\sigma=0} = \frac{\mu^4}{g_s}[f_6(0,\eta_f) - f_6(0,0)]\,. \tag{1.65}$$

In what follows, we will set $g_s = 1$ and use the expansion for the functions f_5, f_6, f_7 quoted in Appendix C. It is then easy to see that the number of $NS5$ branes is given by

$$Q_{NS5} = \tilde{Q}_{NS5} = \frac{2\mu^2}{\pi}\eta_f\,, \tag{1.66}$$

while the number of $D6$ reads

$$Q_{D6} = \mu^4(\lambda'(0) - \lambda'(\eta_f))\,. \tag{1.67}$$

If we choose $\mu^2 = \frac{\pi}{2}$, we find that η_f is quantised to be an integer. Thus, from now on we set $\eta_f = N_5$ with $N_5 \in \mathbb{Z}_{\geq 0}$. Of course, for physically sensible solutions $N_5 \geq 2$.

What about the quantisation of $D6$ branes? Let us again consider the example of Figure 1.2. The quantity $\lambda'(0) - \lambda'(\eta_f)$ is easily computed to be $k_1 + k_4$ (of course an integer number). In order to get a well-quantised number of $D6$ branes let us rescale λ such as $\lambda \to N_c\lambda$, with N_c any number. This is of course always possible as the Toda equation (1.24) is a linear equation. Defining $N_6 = \frac{\pi^2}{4}N_c$, an integer, it is a trivial matter to see that, for the example in Figure 1.2,

$$Q_{D6} = \left(\frac{\pi}{2}\right)^2 N_c(k_1 + k_4) = \sum_{i=1}^{4} d_i\,. \tag{1.68}$$

So not only Q_{D6} is a well-quantised integer, but it also reproduces the correct number of flavours in our theory. We will see in this chapter more examples of these formulas at work.

The calculation of the $D4$ brane charge is more subtle. Let us compute the Page charge associated with $D4$ branes

$$\begin{aligned} q_4 &= \frac{1}{(2\pi)^3\alpha'^{\frac{3}{2}}}\int_{\Sigma_4}\widehat{F}_4 \\ &= \frac{1}{(2\pi)^2}4\pi\mu^6\left[f_7(0,\eta_f) - f_7(0,0)) - \int_0^{\eta_f} f_5(0,\eta)\partial_\eta f_6(0,\eta)d\eta\right] \\ &= \frac{2}{\pi}\mu^6\eta_f\lambda'(\eta_f)\,. \end{aligned} \tag{1.69}$$

Again, referring to the example of Figure 1.2, it is fairly easy to see that q_4 does not reproduce the charge of $D4$ brane. The reason relies on the fact that upon Hanany-Witten transition the number of $D4$'s is not conserved in each interval. We will next what is the physical meaning of equation (1.69).

It is then necessary to find a new formula that counts the number of $D4$ branes properly. Fortunately, this is possible and the correct formula for counting $D4$ brane charge is given by

$$Q_{D4} = \frac{2}{\pi}\mu^6\int_0^{\eta_f}\lambda(\eta)d\eta. \tag{1.70}$$

The proof of such a formula is given in Appendix D. In this section we limit ourselves to see that it works well in some interesting examples.

Making use of the fact that $\mu^2 = \frac{\pi}{2}$ and $N_6 = \frac{\pi^2}{4}N_c$, it is not difficult to see that (1.70) reproduces the correct number of colour branes for the example of Figure 1.2.

In Section 1.2 we shall test equations (1.66), (1.67) and (1.70) in different examples. Now, let us move on to deriving some general expressions for the linking numbers.

1.1.8 Linking numbers

The linking numbers for general brane webs in string theory were defined by Hanany and Witten in [28]. In this work, we are dealing with brane webs constructed from the intersection of $D4$ and $NS5$ branes, together with transverse $D6$ branes. For the sake of readability, we present again the definition of the linking numbers for the i^{th} fivebrane (K_i) and for the j^{th} D6 brane (L_j)

$$\begin{aligned} K_i &= N_{D4}^{right} - N_{D4}^{left} - N_{D6}^{right}\,, \\ L_j &= N_{D4}^{right} - N_{D4}^{left} + N_{NS}^{left} \end{aligned} \tag{1.71}$$

As shown in [28], they must satisfy the following conservation equation

$$\sum_{i=1}^{N_5} K_i + \sum_{j=1}^{N_6} L_j = 0\,. \tag{1.72}$$

The linking numbers are topological invariants and they do not change under Hanany-Witten moves. They can be easily computed by simple counting of branes in a brane web. The goal of this subsection is to use the dual supergravity background to compute these invariants.

As a matter of fact, for the case of the $NS5$ branes, we find that in our generic conformal backgrounds the linking numbers are all equal $K_1 = K_2 = \cdots = K_{N5}$. This is easily seen by using the formula (1.71) applied to any $NS5$ brane in a generic brane web of the kind studied in this chapter. A simple proposal that can be pushed forward for the holographic computation of the linking numbers is given by the following formula

$$K_i = \frac{2}{\pi}\mu^6 \lambda'(\eta_f)\,. \tag{1.73}$$

The sum of all the linking numbers simply gives

$$\sum_{i=1}^{N_5} K_i = \frac{2}{\pi}\mu^6 \lambda'(\eta_f)\eta_f\,. \tag{1.74}$$

where we used that $\eta_f = N_5$. Going back to the equation (1.69), we see that $\sum_{i=1}^{N_5} K_i$ is given by

$$\sum_{i=1}^{N_5} K_i = \frac{1}{2\kappa_{10}^2 T_{D4}} \int_{\Sigma_4} \widehat{F}_4\,. \tag{1.75}$$

where, again, we take $\Sigma_4 = (\eta, \chi, \xi, \beta)|_{\sigma=0}$, as specified in equation (1.63).

It is possible to obtain also a nice expression for the linking number of the $D6$ branes using the supergravity background. In a general Hanany-Witten setup, stacks of $D6$ branes are placed at different points $\eta_1, \eta_2, ...\eta_l$ along the η direction. As explained previously in this chapter, the number of $D6$ branes for each stack, say the jth, is in general given by $\lambda'(\eta_j-\epsilon)-\lambda'(\eta_j+\epsilon)$. Also, all the branes in the j-th stack have the same linking number[13] $L_j=\eta_j$. If we sum over all the $D6$ branes (the total number of $D6$ branes is N_6) we get

$$\sum_j L_j = -\frac{2\mu^6}{\pi}\sum_{j=1}^{N_6}\lambda'(\eta_j)\eta_j = -\frac{2\mu^6}{\pi}\lambda'(\eta_f)\eta_f. \tag{1.76}$$

To calculate this explicitly in supergravity, we perform a large gauge transformation on the field $C_{(1)}$ at each point η_i where the stacks of D6 branes are placed,

$$C_{(1)} \to C_{(1)} + \mu^4\alpha'^{\frac{1}{2}}\left(\lambda'(\eta_j-\epsilon)-\lambda'(\eta_j+\epsilon)\right)d\beta. \tag{1.77}$$

We equate the $D6$ linking numbers with the flux that we calculate on the four manifold $\tilde{\Sigma}_4=(\eta,\chi,\xi,\beta)|_{\sigma=\infty}$. We propose the formula

$$\sum_{i=1}^{N_6} L_i = \frac{1}{2\kappa_{10}^2 T_{D4}}\int_{\tilde{\Sigma}_4} F_{(4)} + C_{(1)}\wedge H_{(3)} = -\frac{2}{\pi}\mu^6\lambda'(\eta_f)\eta_f. \tag{1.78}$$

In Section 1.2 and in Appendix E, we evaluate the expressions of equations (1.75), (1.78) in various examples and check them against the expressions derived from the Hanany-Witten set-up, finding a precise match.

Let us now discuss another observable characterising the CFT that has a nice holographic description, the central charge.

1.1.9 Central charge for Gaiotto-Maldacena backgrounds

The aim of this subsection is to find an holographic expression for the central charge of a generic $\mathcal{N}=2$ SCFT of the type we are describing in this chaper by using the solutions of (1.33), and (1.34) and (1.35). In order to achieve this goal we are going to make use of the formalism developed in [40, 41, 42].

[13] Remember that for us $\eta_j=1,2,3,\dots,N_5$, where N_5 is the number of $NS5$ branes. Applying the formula (1.71) to any of the $D6$ branes of the j-th stack simply amounts to computing $L_j=0-0+\eta_j$.

Consider the background metric given in equation (1.33) and rewrite it in the following fashion

$$ds^2 = a(R,y)(dx_{1,3}^2 + b(R)dR^2) + g_{ij}(R,y)dy^i dy^j\,, \tag{1.79}$$

where R stands for the AdS_5 radius while y^i denote collectively the coordinates of the internal space. $dx_{1,3}^2$ is simply the metric of four-dimensional Minkowski space, whereas g_{ij} is the metric of the internal space.

Using Poincaré coordinates for the AdS part of the ten-dimensional metric and comparing with equations (1.33), we identify

$$a(R,y^i) = 4\mu^2\alpha' R^2 f_1\,, \qquad b(R) = 1/R^4\,. \tag{1.80}$$

Following [40], we can define the quantities

$$\mathcal{V}_{\text{int}} = \int dy \sqrt{e^{-4\phi}\det(g_{ij})a(r,\theta)^d}\,, \qquad \widehat{H} = \mathcal{V}_{\text{int}}^2 \tag{1.81}$$

The holographic central charge, when the dual CFT is $(d+1)$-dimensional, is then defined as

$$c_{\text{holo}} = \frac{d^d}{G_N} b(R)^{d/2} \frac{\widehat{H}^{\frac{2d+1}{2}}}{(\widehat{H}')^d}\,. \tag{1.82}$$

Note that in our case we have

$$\sqrt{e^{-4\phi}\det g_{ij} a(R,y^i)^3} = 2^5\alpha'^4\mu^{14}R^3\sigma\sin\chi V''\dot{V}\,. \tag{1.83}$$

Thus, the "internal volume" $\mathcal{V}_{int}$ reduces to

$$\begin{aligned}\mathcal{V}_{int} &= 2^5\alpha'^4\mu^{14}R^3\int_0^\pi \sin\chi d\chi \int_0^{2\pi} d\beta \int_0^{2\pi} d\xi \int_0^\infty \int_0^{\eta_f} \sigma\dot{V}V'' d\sigma d\eta \\ &= \mathcal{N}R^3\,,\end{aligned} \tag{1.84}$$

where we have defined the quantity

$$\mathcal{N} = 2^7\pi^2\alpha'^4\mu^{14}\int_0^{\eta_f}\lambda^2(\eta)d\eta. \tag{1.85}$$

To get to the last expression we have used equation (1.32), the fact that $\dot{V}(\sigma,\eta)|_{\sigma=\infty} = 0$ and the definition of $\lambda(\eta)$ given in equation (1.41). The integral in (1.85) can be explicitly evaluated for the generic solution in equation (1.42) as

$$\int_0^{\eta_f} d\eta \dot{V}^2|_{\sigma=0} = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \frac{c_m c_l}{ml}\frac{N_5^2}{\pi^2}\int_0^{\eta_f} \sin\omega_m\eta \sin\omega_l\eta d\eta \,. \tag{1.86}$$

We finally obtain for $\mathcal{V}_{int}$

$$\begin{aligned} \mathcal{V}_{int} &= 2^7\pi^2\alpha'^4\mu^{14}R^3\int_0^{\eta_f}\lambda^2(\eta)d\eta \\ &= 2^6 N_5^3 R^3\alpha'^4\mu^{14}\sum_{m=1}^{\infty}\frac{c_m^2}{m^2}\,. \end{aligned} \tag{1.87}$$

Now, coming back to our original goal, we find for the central charge

$$c = \frac{2\mu^{14}}{\pi^4}\int_0^{N_5}\lambda^2(\eta)d\eta = \frac{N_5^3\mu^{14}}{\pi^6}\sum_{m=1}^{\infty}\frac{c_m^2}{m^2}. \tag{1.88}$$

The last expression tells us that the central charge is proportional to the area subtended by the function $\lambda^2(\eta)$. These formulas are similar to those derived in dual to six dimensional SCFTs with $\mathcal{N}=(1,0)$ SUSY, see equation (2.14) of the paper [43].

On the CFT side, it was shown by the authors of [44] that an expression for the two central charges[14] a and c characterising a four-dimensional superconformal $\mathcal{N}=2$ theory can be written in terms of the number of $\mathcal{N}=2$ vector multiplets (n_v) and hypermultiplets (n_h) in the quiver. The expressions read

$$a = \frac{5n_v+n_h}{24\pi}\,, \quad c = \frac{2n_v+n_h}{12\pi}\,. \tag{1.89}$$

The comparison with the holographic result in equation (1.88) holds only when the IIA/M-theory background is trustable, that is when $N_5\to\infty$ and $N_c\to\infty$, in which case we also have $a=c$. In Section 1.2 and in Appendix E, we shall compare the result of equation (1.88) with the explicit field theoretical counting of degrees of freedom in equation (1.89), for various examples.

To summarise, in this section we discussed some observables of generic $\mathcal{N}=2$ SCFTs (brane charges, Linking Numbers, central charges) and presented some expres-

[14] In $d=4$ the trace of the energy momentum tensor on a curved background is given by $T^\mu_\mu = -aE_4 - cI_4$, where E_4 is the four-dimensional Euler density while I_4 is a conformal invariant built from the Weyl tensor. Thus, in 4d it is possible to define two conformal charges. In $d=2$ there is no conformal invariant of the right dimension and we have only the Euler density.

sions computed using generic holographic dual backgrounds. In the next section, we will study some particular CFTs and check the matching of physical observables using the holographic and field theoretical description.

1.2 Examples of $\mathcal{N}=2$ SCFTs

In this section we work out two particularly simple and interesting examples for the potential function V in the form given in equation (1.42). We will explicitly check that the field theory and the holographic calculation match in the limit in which the supergravity description is trustable. In Appendix E we will discuss more elaborated CFTs, again obtaining a precise matching.

Let us first present the two basic examples that we will consider in this section.

1.2.1 Two interesting solutions of the Laplace equation

The first solution we deal with was already considered in [45] in the study of the non-Abelian T-dual of $AdS_5 \times S^5$. The charge density, or λ-profile, reads[15]

$$\lambda(\eta) = N_c \begin{cases} \eta & 0 \leq \eta \leq N_5 - 1 \\ (N_5-1)(N_5-\eta) & N_5 - 1 < \eta \leq N_5. \end{cases} \tag{1.90}$$

In this case, the Fourier coefficients in equations (1.42) and (1.44) are computed to be

$$c_m = \frac{2N_c N_5}{m\pi} \sin\left(\frac{m\pi(N_5-1)}{N_5}\right). \tag{1.91}$$

The associated quiver and Hanany-Witten set-up are shown in Figure 1.3.

The second solution has a λ-profile given by

$$\lambda(\eta) = N_c \begin{cases} \eta & 0 \leq \eta \leq 1 \\ 1 & 1 < \eta \leq N_5 - 1 \\ N_5 - \eta, & N_5 - 1 < \eta \leq N_5 \end{cases} \tag{1.92}$$

The Fourier coefficients are

$$c_n = \frac{2N_c}{n\pi}\left[\sin\left(\frac{n\pi}{N_5}\right) + \sin\left(\frac{n\pi(N_5-1)}{N_5}\right)\right]. \tag{1.93}$$

[15] Here and in the rest of the chapter $\frac{\pi^2}{4} N_c = N_6$.

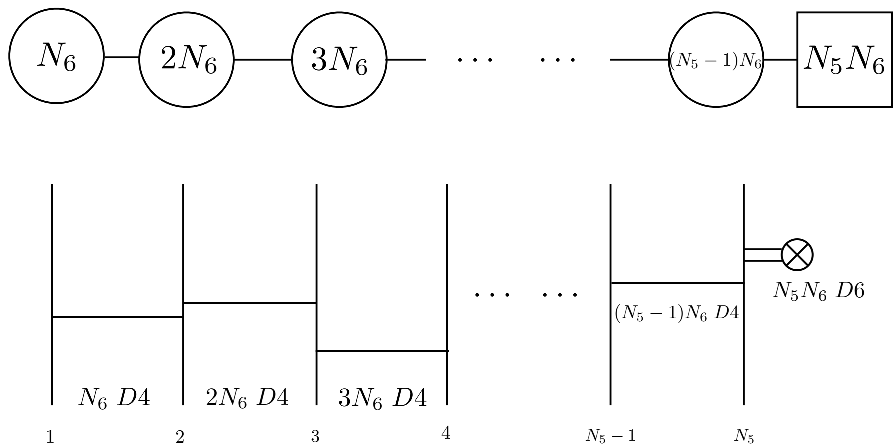

Figure 1.3: The quiver and Hanany-Witten set-up for the profile in equation (1.90). The vertical lines denote individual $NS5$ branes. The horizontal lines denote $D4$ branes and the crossed circles $D6$ branes.

The quiver and Hanany-Witten set up are shown in Figure 1.4.

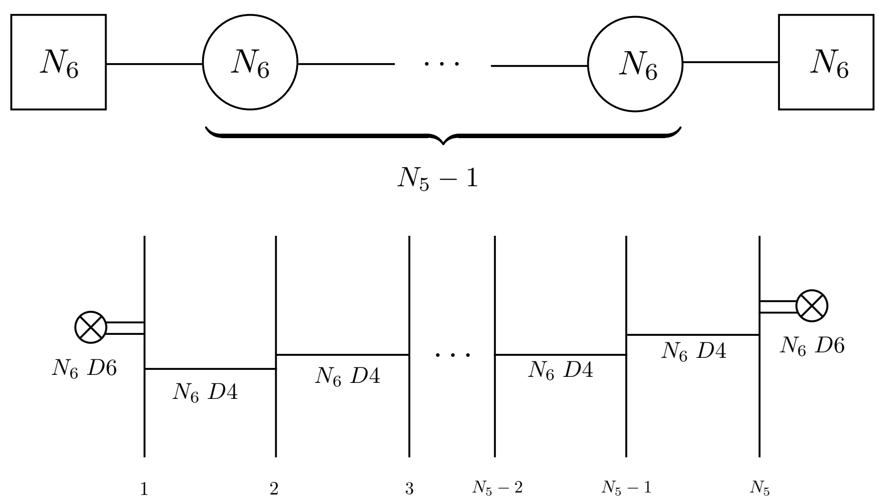

Figure 1.4: The quiver and Hanany-Witten set-up for the profile in equation (1.92).

For both examples, we will proceed as described above: given the function $\lambda(\eta)$, and the Fourier expansion of its odd-extension, we construct the potential in equation (1.42). We then construct the full background given in the equations (1.33), (1.34) and (1.35). In the following, we will show the details of the precise matching between field theoretical and holographic calculations of the observables in Section 1.1 for these cases.

1.2.2 Page charges and linking numbers

Let us start off by computing the expressions for Q_{NS5}, Q_{D6}, Q_{D4} in equations (1.66), (1.67) and (1.70) for the two backgrounds obtained using the equations (1.90), (1.92).

For the function $\lambda(\eta)$ in equation (1.90) we have that $\lambda'(\eta_f)=N_c(1-N_5)$, $\lambda'(0)=N_c$. This implies that

$$\begin{aligned} Q_{NS5} &= N_5\,, \\ Q_{D6} &= \mu^4 N_c N_5 = \left(\frac{\pi^2 N_c}{4}\right) N_5 = N_6 N_5\,. \end{aligned} \tag{1.94}$$

Finally, for the charge of $D4$ branes we find, using equation (1.70),

$$Q_{D4} = \frac{1}{2} N_6 N_5 (N_5-1). \tag{1.95}$$

These numbers coincide precisely with those we would obtain by simple inspection of Figure 1.3:

$$\begin{aligned} N_{NS5} &= N_5\,, \qquad N_{D6} = N_6 N_5\,, \\ N_{D4} &= N_6 \sum_{r=1}^{N_5-1} r = \frac{N_6}{2} N_5 (N_5-1)\,. \end{aligned} \tag{1.96}$$

Along the same lines, we find for the profile in equation (1.92)

$$\begin{aligned} Q_{NS5} &= N_5 \qquad Q_{D6} = 2\frac{\pi^2 N_c}{4} = 2N_6\,, \\ Q_{D4} &= N_6(N_5-1)\,. \end{aligned} \tag{1.97}$$

This results coincide with what we would obtain by simple inspection of the quiver and Hanany-Witten set-up displayed in Figure 1.4.

Concerning the linking numbers, we use the holographic expressions given in equations (1.75), (1.78). We find that the computation from the gravity side for the λ-profile given in equation (1.90) gives

$$-\sum_i K_i = \sum_j L_j = \frac{2}{\pi}\mu^6 N_5(N_5-1)N_c = N_6 N_5(N_5-1)\,. \tag{1.98}$$

This result is easily confirmed by studying the Hanany-Witten set-up in Figure 1.3. We find

$$\begin{aligned} K_i = N_6(1-N_5) \qquad &\Rightarrow \qquad \sum_i K_i = N_6 N_5(1-N_5), \\ L_i = N_5 - 1 \qquad &\Rightarrow \qquad \sum_j L_j = N_5 N_6(N_5-1)\,, \end{aligned} \tag{1.99}$$

which coincides exactly with (1.98).

The same matching is found for the quiver associated with the λ function given in equations (1.92) and Figure 1.4. Using equations (1.75) and (1.78), we find

$$\sum_i K_i = -\sum_j L_j = -\frac{2}{\pi}\mu^6 N_c N_5 = -N_5 N_6\,. \tag{1.100}$$

A simple inspection of the Hanany-Witten set-up of Figure 1.4, and using the equation (1.71), we find for the Linking Numbers of the $D6$ branes

$$\begin{aligned} &L_1 = \cdots = L_{N6} = 1\,, \quad \tilde{L}_1 = \cdots = \tilde{L}_{N6} = N_5 - 1 \\ &\Rightarrow \sum_j L_j = N_6 + N_6(N_5-1) = N_6 N_5 \end{aligned} \tag{1.101}$$

while for $NS5$ branes

$$K_1 = \cdots = K_{N5} = -N_6 \quad \Rightarrow \quad \sum_i K_i = -N_5 N_6\,, \tag{1.102}$$

where have denoted by L_j ($\tilde{L}_j$) the D6 branes to the left (right) of the Hanany-Witten set-up of Figure 1.4, finding exact agreement.

Let us now move on to the computation of the central charges for the two basic examples considered in this section, showing precise agreement between holographic and field theory calculation in the limit where the supergravity description is realiable.

1.2.3 Central charge

Using the holographic expression given in the equation (1.88), we compare the holographic central charge for each examples considered in this section (in the large N_c, N_5 limit) with its quantum field theory analogue, given in equations (1.89).

We start with the supergravity background obtained using the λ-profile in equation (1.90). Using the equations (1.90), (1.91) and (1.88), we find

$$c=\frac{2\mu^{14}}{\pi^4}\int_0^{N_5}\lambda^2 d\eta=\frac{2\mu^{14}}{3\pi^4}N_c^2N_5^3\left(1-\frac{1}{N_5}\right)^2\simeq\frac{2\mu^{14}}{3\pi^4}N_c^2N_5^3\,, \tag{1.103}$$

where we only kept the leading orderes as $N_5\to\infty$ and $N_c\to\infty$. As discussed previously, this is necessary in order to have a trustable holographic description. In a completely equivalent way, we could also work using the Fourier expansion of the λ function, employing equation (1.88), which implies

$$\begin{aligned} c&=\frac{4N_5^5N_c^2\mu^{14}}{\pi^8}\sum_{m=1}^{\infty}\frac{1}{m^4}\left[\sin\left(\frac{m\pi(N_5-1)}{N_5}\right)\right]^2\\ &=\frac{4N_5^5N_c^2\mu^{14}}{\pi^8}\left[\frac{\pi^4}{180}-45(\text{Polylog}[4,e^{i2\pi/N_5}]+\text{Polylog}[4,e^{-i2\pi/N_5}])\right]\\ &\simeq\frac{2\mu^{14}}{3\pi^4}N_c^2N_5^3\,. \end{aligned} \tag{1.104}$$

Using that $\mu^2=\frac{\pi}{2}$ and $N_6=\frac{\pi^2}{4}N_c$, we find the holographic result

$$c=\frac{N_5^3N_6^2}{12\pi}\,. \tag{1.105}$$

This is precisely the central charge we would obtain by means of a CFT calculation. Indeed, computing the number of vector multiplets and hypermultiplets for the quiver of Figure 1.3 as

$$\begin{aligned} n_v&=\sum_{r=1}^{N_5-1}r^2N_6^2-1=\frac{(N_5-1)}{6}(2N_5^2N_6-N_5N_6^2-6)\\ n_h&=\sum_{r=1}^{N_5-1}r(r+1)N_6^2=\frac{N_6^2}{3}N_5(N_5^2-1) \end{aligned} \tag{1.106}$$

and using the expression in equation (1.89), we obtain

$$c = \frac{(N_5 - 1)(N_5^2 N_6^2 - 2)}{12\pi} \simeq \frac{N_6^2 N_5^3}{12\pi}\,, \tag{1.107}$$

finding, in the large N_5 and large N_6 limit, a precise matching with the holographic calculation of equation (1.105).

It is easy to check also that equation (1.88) applied to the equations (1.92), (1.93) – our second example – for large N_c and N_5 leads to

$$c = \frac{2\mu^{14}}{\pi^4} N_c^2 N_5 = \frac{N_6^2 N_5}{4\pi}\,. \tag{1.108}$$

This expression is matched in the appropriate limit of the CFT calculation. For the quiver associated with the profile in equation (1.92), we have

$$\begin{aligned} n_v &= (N_6^2 - 1)(N_5 - 1)\,, \\ n_h &= (N_5 - 1)N_6^2\,, \end{aligned} \tag{1.109}$$

and

$$c = \frac{N_6^2 N_5}{4\pi}\left(1 - \frac{2}{3N_6^2} - \frac{1}{3N_5} + \frac{2}{N_5 N_6^2}\right) \simeq \frac{N_6^2 N_5}{4\pi}\,. \tag{1.110}$$

The reader can verify that the same expressions are obtained for the a central charge in the holographic limit (since $a = c$ in this case).

In Appendix E, we extend the precise matching of the Page charges, linking numbers and central charge to more general and elaborated SCFTs.

Let us now move on to the study of two solutions to the Laplace equation (1.32) that are qualitatively different from those discussed above in a way that will be soon clear.

1.2.4 The Sfetsos-Thompson solution

Let us discuss a particular solution obtained by Sfetsos and Thompson in [46], that received attention in the last few years. The potential V and the charge line density λ for the Sfetsos-Thompson solution are given by

$$V_{ST} = N_c\left(\eta \log\sigma - \frac{\eta\sigma^2}{2} + \frac{\eta^3}{3}\right)\,, \qquad \lambda(\eta) = N_c \eta\,, \tag{1.111}$$

respectively. In the language of equations (1.43) - (1.45) the defining functions are

$$F(\eta)=N_c\frac{\eta^3}{3}\,,\quad G(\eta)=N_c\eta\,,\quad h_1=-N_c\frac{\eta}{2}\,,\quad f_k=h_{k+1}=0\,,\tag{1.112}$$

with $k>1$. Notice that, in this particular solution, λ does not close off at some finite value η_f. Thus, the η-coordinate is not bounded from above. In other words, $\eta_f\to\infty$. This has unpleasant consequences. For example the associated quiver has gauge symmetry given as an infinite product of gauge groups, $\Pi_{k=1}^{\infty}SU(kN_6)$. As there are no kinks for the λ function (λ is only linear continuous and not piecewise linear continuous), there are no $D6$ brane sources. Similarly, the equations (1.66), (1.70) indicate the presence of an infinite number of five- and four-branes. The linking numbers do not satisfy equation (1.72) and the central charge in equation (1.88), diverges as $\eta_f\to\infty$. The bad behaviour of the field theory observables is mirrored by a singularity in the background at $\sigma=1$. Still, some quantities may have an acceptable behaviour[16].

These deficiencies might suggest that we should ignore the Sfetsos-Thompson solution as unphysical. However, the background generated by V_{ST} in equation (1.111) has a very interesting property: the string theory sigma model is integrable on this background, as was shown in [48]. In particular, it was shown in [37] that any other generic Gaiotto-Maldacena background as in equation (1.33) leads to a non-integrable (and chaotic) sigma model for the string theory.

These ideas were exploited in [49, 50] to show that the Sfetsos-Thompson solution is a member of a family of integrable backgrounds. Interestingly, the geometry and fluxes produced by the potential V_{ST}, together with the definitions in equation (1.33), were obtained in [46] by using non-Abelian T-duality. There are presently many new backgrounds that have been obtained using this powerful technique [51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68].

It is in this sense that the Sfetsos-Thompson solution stands out as a paradigmatic example of non-Abelian T-duality as generating technique. While the conformal field

[16] We could regulate physical quantities, like the central charges, using the Riemann ζ-function $\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s}$: for a strictly infinite conformal quiver with gauge group $\Pi_{k=1}^{\infty}SU(kN_6)$ joined by bifundamental hypers, we have that $n_v=\sum_{k=1}^{\infty}(k^2N_6^2-1)$ and $n_h=\sum_{k=1}^{\infty}(k^2+k)N_6^2$. We obtain that

$$\frac{a}{c}=\frac{5n_v+n_h}{4n_v+2n_h}=\frac{\sum_{k=1}^{\infty}6k^2N_6^2+kN_6^2-5}{\sum_{k=1}^{\infty}6k^2N_6^2+2kN_6^2-4}.$$

Using that $\zeta(-2)=0$, $\zeta(-1)=-\frac{1}{12}$ and $\zeta(0)\to\infty$, we find $\frac{a}{c}=\frac{5}{4}$, satisfying the Hofman-Maldacena bound [47].

theory obtained by following the prescription described in Section 1.1 is not well defined [17], it was proposed in [45] that the Sfetsos-Thompson solution should be embedded inside a "complete" Gaiotto-Maldacena geometry, that regulates the background and solves the above mentioned problems of the CFT. The authors of [45] suggested to consider the charge density in equation (1.90) as a regulator for λ_{ST}. Indeed, the solution in equation (1.42) with Fourier coefficients given in equation (1.91) is proposed to be the potential from which to obtain the "completed" background. This logic extended successfully [69, 70, 71, 72, 73] to other backgrounds generated by non-Abelian T-duality. Below we comment on other ways to think about the Sfetsos-Thompson background and its associated CFT.

A field theory view of the Sfetsos-Thompson background

Let us add a few comments about the field theoretical interpretation of the Sfetsos-Thompson background and non-Abelian T-duality (an operation on the string sigma model that generates a new background).

Consider $\mathcal{N}=4$ Super-Yang-Mills. The bosonic part of the global symmetries is $SO(2,4)\times SO(6)$. These symmetries are realised as isometries of the dual $AdS_5\times S^5$ background. Let us consider an $SO(4)\times SO(2)$ inside the $SO(6)$. The non-Abelian T-dual transformation proposed by Sfetsos and Thompson in [46] picks up an $SU(2)$ inside the $SO(6)$, say $SU(2)_L$ if[18] $SO(4)=SU(2)_L\times SU(2)_R$, and operates on it. This operation preserves the $SO(2,4)$ isometry, as the AdS_5 part of the space is inert. The same happens to the $SU(2)_R\times SO(2)$. Schematically the non-Abelian T-duality acts as

$$\begin{aligned}
&ds^2_{AdS_5}+d\alpha^2+\sin^2\alpha d\beta^2+\cos^2\alpha ds^2_{S^3}\to\\
&ds^2_{AdS_5}+d\alpha^2+\sin^2\alpha d\beta^2+\frac{d\rho^2}{\cos^2\alpha}+\frac{\rho^2\cos^2\alpha}{\rho^2+\cos^4\alpha}ds^2_{S^2}\to\\
&ds^2_{AdS_5}+\frac{1}{1-\sigma^2}(d\sigma^2+d\eta^2)+\eta^2 d\beta^2+\frac{\eta^2(1-\sigma^2)}{4\eta^2+(1-\sigma^2)^2}ds^2_{S^2}\,. \qquad (1.113)
\end{aligned}$$

In the last line we have changed variables $\sigma=\sin\alpha$ and $\rho=\eta$ in order to put the geometry in the "Gaiotto-Maldacena notation". The background is complemented by Ramond and Neveu-Scharz fields. For details see, for example, [45].

[17] In [36] the authors suggest that the system should be thought as a higher dimensional field theory with a conformal four-dimensional defect.

[18] We ignore the global $\mathbb{Z}_2$ in the isomorphism between $SO(4)$ and $SU(2)_L\times SU(2)_R$.

The result of the operation (1.113) is a background dual to an $\mathcal{N} = 2$ SCFT, with bosonic isometries $SO(2,4) \times SU(2)_R \times U(1)_r$. From a field theory point of view, one can point out two distinct operations on $\mathcal{N} = 4$ SYM that, acting on $SU(2)_L$, produce two different $\mathcal{N} = 2$ SCFTs. The first one is a modding of the internal space by a $\mathbb{Z}_k$ and is represented at the top of Figure 1.5. The second is a higssing operation represented in the lower part of Figure 1.5.

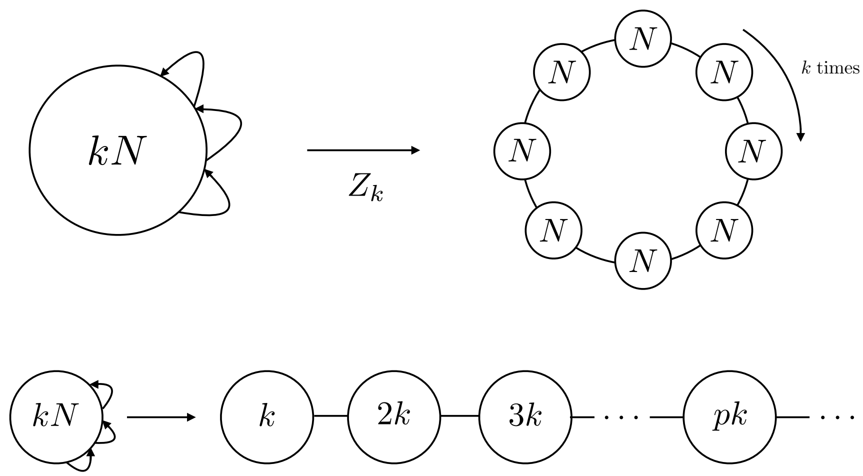

Figure 1.5: The two operations preserving conformality and $SU(2)_R \times U(1)_r$ as discussed in the text.

The ranks of the gauge groups are determined by conformality ($N_c = 2N_f$ at each node for $\mathcal{N} = 2$ SCFTs). While the option at the top of Figure 1.5 is well defined, the one at the bottom runs into a problem as the quiver should extend indefinetely. A way out in order to avoid an infinitely long quiver is to end such a linear quiver by the addition of a flavour group. This option is not available to the non-Abelian T-duality, as it would imply the creation of an isometry, an $SU(kp + k)$, and the presence of $D6$ sources to realise it. Also, if we do not end the quiver, we eventually "run-out" of degrees of freedom to create a new gauge group and conformality would be compromised. The Sfetsos-Thompson solution reflects this by generating a singularity.

Let us finally discuss a geometric aspect of the Sfetsos-Thompson background. Considering the σ-derivative of the generic potential V, $\dot{V}(\sigma, \eta) = \sigma\partial_\sigma V$. Using the equation (1.42), we have

$$\dot{V}(\sigma,\eta)=\sigma\partial_\sigma V(\sigma,\eta)=\sum_{k=1}^{\infty}c_k\sigma K_1\left(\frac{n\pi}{N_5}\right)\sin\left(\frac{n\pi}{N_5}\eta\right). \tag{1.114}$$

By Poisson summation, we rewrite this as [34]

$$\dot{V}(\sigma,\eta)=\frac{N_c}{2}\sum_{l=1}^{P}\sum_{m=-\infty}^{\infty}\int d\sigma\sigma\left[\frac{1}{\sqrt{\sigma^2+(\eta-\nu_l-m)^2}}-\frac{1}{\sqrt{\sigma^2+(\eta+\nu_l-m)^2}}\right]. \tag{1.115}$$

The values of the constants ν_l depend on the Fourier coefficients and can be found in [34]. Of all the terms in the sum of equation (1.115), we shall only keep for now the term $m=0$. We also consider the function $\dot{V}$ close to $\sigma=\eta=0$. To leading order in both σ and η we find

$$\dot{V}(\sigma,\eta)\simeq\dot{V}_{app}(\sigma,\eta)=\eta(c_1-c_2\sigma^2)=\dot{V}_{ST}\,. \tag{1.116}$$

This is somewhat reminiscent of what occurs when lifting $D2$ branes to eleven dimensions [74]. In that case, the correct solution is the one that contains the infinite number of "images" just like equation (1.114) does. The naive lifting of the $D2$ brane solution does not capture the full IR dynamics of $D2$ branes. By analogy this suggests that omitting the summation over the images in equation (1.115) misses the correct dynamics of the dual CFT, that the completion in [45] provides.

1.2.5 An interesting particular solution

As extensively discussed, the general solution to (1.32) with boundary conditions as in (1.40) and (1.41), can be cast in the form given in (1.42). This solution is the infinite superposition of functions of the type $V\sim K_0(\frac{n\pi\sigma}{N_5})\sin\frac{n\pi\eta}{N_5}$, with suitable coefficients. A natural question is what is the physical content of each term in this superposition. To answer this question, we shall consider a solution to (1.32) that is simply given by

$$V(\sigma,\eta)=-K_0(\sigma)\sin\eta\,, \tag{1.117}$$

and study the background that this solution generates. Putting (1.117) into (1.33), (1.34) and (1.35) we find for the N-S sector

$$\begin{aligned}
\frac{ds_{10}^2}{L^2} &= 4\sigma\sqrt{\frac{K_2(\sigma)}{K_0(\sigma)}}ds^2_{AdS_5} + 2\frac{\sqrt{K_0(\sigma)\,K_2(\sigma)}}{K1(\sigma)}(d\sigma^2+d\eta^2)\\
&\quad + 2\frac{K_1(\sigma)\sqrt{K_0(\sigma)\,K_2(\sigma)}\sin^2\eta}{K_0(\sigma)\,K_2(\sigma)\sin^2\eta + K_1^2(\sigma)\cos^2\eta}ds^2_{S^2}(\chi,\xi) + 4\sigma\sqrt{\frac{K_0(\sigma)}{K_2(\sigma)}}d\beta^2\,,\\
B_{(2)} &= 2\alpha'\mu^2\left[-\eta + \frac{K_1^2(\sigma)\sin\eta\cos\eta}{K_1^2(\sigma)\cos^2\eta + K_0(\sigma)\,K_2(\sigma)\sin^2\eta}\right]\sin\chi\,d\xi\wedge d\chi\,,\\
e^{-2\phi} &= \frac{1}{2}\mu^6\sqrt{\frac{K_0(\sigma)}{K_2^3(\sigma)}}K_1(\sigma)\left[K_1^2(\sigma)\cos^2\eta + K_0(\sigma)\,K_2(\sigma)\sin^2\eta\right]\,,
\end{aligned}\tag{1.118}$$

while for the R-R sector

$$\begin{aligned}
C_{(1)} &= 2\mu^4\alpha'^{\frac{1}{2}}\frac{K_1^2(\sigma)\cos\eta}{K_2(\sigma)}d\beta\,,\\
C_{(3)} &= -4\alpha'^{\frac{3}{2}}\mu^6\frac{K_0(\sigma)\,K_1^2(\sigma)\sin^3\eta}{K_0(\sigma)\,K_2(\sigma)\sin^2\eta + K_1^2(\sigma)\cos^2\eta}\sin\chi\,d\xi\wedge d\chi\wedge d\beta\,.
\end{aligned}\tag{1.119}$$

To get some intuition about the physical meaning of this solution, we compare it with the background obtained in equations (2.44) - (2.47) of the paper [75] where Lin and Maldacena give the configuration corresponding to type IIA Neveu-Schwarz fivebranes on some $\mathbb{R}\times S^5$, with $\mathbb{R}$ discribing the time-direction. The solution to the equation (1.118) differs from the one in [75] by an "analytic continuation" (that as explained in Section 3.1 of [12] changes $ds^2_{S^5}\to -ds^2_{AdS_5}$ and $-dt^2\to d\beta^2$). Such analytic continuation should also be understood as exchanging the modified Bessel functions of the second kind, $K_0(\sigma), K_1(\sigma), K_2(\sigma)$, for modified Bessel functions of the first kind, $I_0(\sigma), I_1(\sigma), I_2(\sigma)$. See the equations (2.44) - (2.47) in [75].

This suggest that the solution in equation (1.118) represents NS fivebranes extended along $AdS_5\times S^1_\beta$. The function $\lambda(\eta)=\sin\eta$ associated with the potential in (1.117) does not have the property of being a piecewise linear and continuous function, as it is the case of our examples in equations (1.90), (1.92). We may think about the background in equation (1.118) as one where the position of the $D6$ branes has been smeared, and they are distributed along the whole η-direction.

Analysing the asymptotic behaviour close to the position of the fivebranes, we find that the metric, dilaton and B-field read as σ goes off to infinity

$$\begin{aligned} &ds^2 \simeq 4\sigma(ds^2_{AdS_5} + d\beta^2) + d\sigma^2 + d\eta^2 + \sin^2\eta ds^2_{S^2}\,, \\ &e^{4\Phi} \simeq e^{4\sigma}\sigma^2\,, \qquad B_{(2)} \simeq (\eta - \cos\eta\sin\eta)d\Omega_2\,. \end{aligned} \tag{1.120}$$

We see that the integral $\int H_3 = N_5$ and that the dilaton diverges close to the five branes.

Interestingly, these solutions can offer a connection with the proposal of the paper [76], according to which (see page 33 in [76]) any four-dimensional CFT of the type we are studying contains, in a suitable limit of parameters, a decoupled sector that is dual to the 6d (0,2) theory on $AdS_5 \times S^1$.

Let us study some of the observables previously calculated. We use the solution corresponding to the first harmonic $V(\sigma,\eta) = -N_c K_0(\frac{\pi\sigma}{N_5})\sin(\frac{\pi\eta}{N_5})$. Using the equations (1.66), (1.67),(1.70) and (1.88) we find for the number of $NS5$, $D6$ and $D4$ branes

$$\begin{aligned} Q_{NS} &= N_5\,, \\ Q_{D4} &= \frac{2N_5N_6}{\pi}\,, \\ Q_{D6} &= \frac{2\pi N_6}{N_5}\,, \end{aligned} \tag{1.121}$$

while for the central charge

$$c = \frac{N_6^2 N_5}{8\pi}\,. \tag{1.122}$$

The particular solution studied here should be thought of as representing a situation where the $D4$ and $D6$ branes are smeared over the Hanany-Witten set up. We cannot identify a localised gauge or flavour group.

Just like the solution of equation (1.117) could be thought of as a "smeared version" of the usual Gaiotto-Maldacena solutions with piecewise continuous $\lambda(\eta)$, it would also be interesting to study the potential and associated charge density,

$$V(\sigma,\eta) = e^{-\eta}\left[c_1 J_0(\sigma) - \frac{\pi}{2}Y_0(\sigma)\right] + \log\sigma, \quad \lambda(\eta) = 1 - e^{-\eta}.$$

as an approximation to the piecewise continuous solution of [77].

To complement this study, in Appendix F we present a new solution representing a black hole in a generic Gaiotto-Maldacena background and briefly discuss its thermodynamics.

Let us now move to the second part of this work, where we study holographically the marginal deformation of these $\mathcal{N} = 2$ SCFTs.

1.3 Marginal deformations of SCFTs and holography

The aim of this section is to discuss a particular marginal deformation of the $\mathcal{N} = 2$ SCFTs studied above. The method used to find the holographic dual to these marginal deformations are those developed by Lunin and Maldacena [16]. See also [78, 79]. We begin with a discussion of marginally deformed $\mathcal{N} = 2$ SCFTs using $\mathcal{N} = 1$ language.

1.3.1 Details about the deformation of the CFT

Consider a gauge field theory like the one represented as a quiver in Figure 1.6. Again, the gauge symmetry is given by a product of gauge groups, $SU(N_1) \times SU(N_2) \times \cdots \times SU(N_P)$. We have chiral fields transforming in the bifundamental of gauge and flavour groups. The flavour symmetry is indeed given by the finite product $U(F_1) \times ... \times U(F_P)$. We are using $\mathcal{N} = 1$ language, indicating an $\mathcal{N} = 2$ hypermultiplet as two arrows (two $\mathcal{N} = 1$ chiral multiplets). There are also $\mathcal{N} = 1$ adjoint chiral fields associated with each gauge group. Formulating a generic $\mathcal{N} = 2$ SCFT in terms of $\mathcal{N} = 1$ multiplets will turn out to be useful when studying marginal deformations.

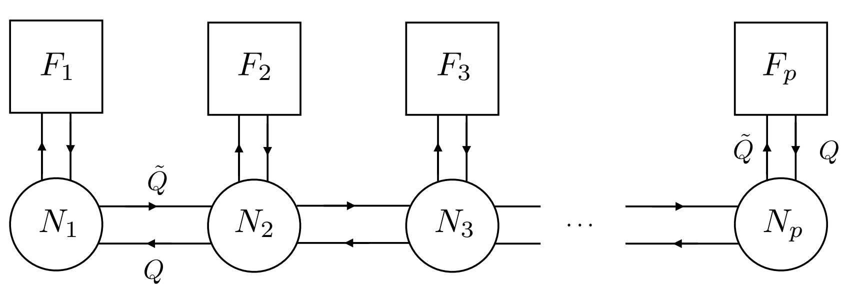

Figure 1.6: A generic $\mathcal{N} = 1$ CFT.

Following ideas spelled out in [80, 81], we use the fact that the R-symmetry can mix with flavour symmetries. In particular, associating a “flavour” charge to all the multiplets present in our theory in the following fashion[19]

$$F[Q] = F[\tilde{Q}] = 1\,, \qquad F[\Phi] = -2\,, \qquad F[\mathcal{W}] = 0\,, \tag{1.123}$$

we propose the following R-charge assignments

$$R_{\mathcal{N}=1} = R_0 + \frac{\epsilon}{2} F\,, \tag{1.124}$$

which, in turn, imply

$$R_{\mathcal{N}=1}[Q] = R_{\mathcal{N}=1}[\tilde{Q}] = \frac{1}{2} + \frac{\epsilon}{2}\,, \quad R_{\mathcal{N}=1}[\Phi] = 1 - \epsilon\,, \quad R_{\mathcal{N}=1}[\mathcal{W}_\alpha] = 1\,. \tag{1.125}$$

This is in line with the fact that marginal deformations do not change the number of degrees of freedom, but only the way in which the different fields interact.

To determine what value we should assign to ϵ, we use a-maximisation [82]. The a and c "central charges" in theories with at least $\mathcal{N} = 1$ supersymmetry are given by

$$a(\epsilon) = \frac{3}{32\pi}\left[3\mathrm{Tr}R^3_{\mathcal{N}=1} - \mathrm{Tr}R_{\mathcal{N}=1}\right]\,, \quad c(\epsilon) = \frac{1}{32\pi}\left[9\mathrm{Tr}R^3_{\mathcal{N}=1} - 5\mathrm{Tr}R_{\mathcal{N}=1}\right]\,. \tag{1.126}$$

For the quiver of Figure 1.6, we find that the contribution from the hypermultiplets, $H = (Q, \tilde{Q})$, and vectormultiplets $V = (\mathcal{W}_\alpha, \Phi)$ to the central charges a and c can be obtained by computing first the following quantities[20]

$$\begin{aligned}
\mathrm{Tr}R_H &= 2 \times \frac{\epsilon - 1}{2}\left(\sum_{j=1}^{P} N_j F_j + \sum_{j=1}^{P-1} N_j N_{j+1}\right) = n_H(\epsilon - 1)\,,\\
\mathrm{Tr}R^3_H &= 2 \times \frac{(\epsilon - 1)^3}{8}\left(\sum_{j=1}^{P} N_j F_j + \sum_{j=1}^{P-1} N_j N_{j+1}\right) = n_H \frac{(\epsilon - 1)^3}{4}\,,\\
\mathrm{Tr}R_V &= \sum_{j=1}^{P}(N_j^2 - 1)(1 - \epsilon) = n_V(1 - \epsilon)\,,\\
\mathrm{Tr}R^3_V &= \sum_{j=1}^{P}(N_j^2 - 1)(1 - \epsilon^3) = n_V(1 - \epsilon^3)
\end{aligned} \tag{1.127}$$

[19]Here Q and $\tilde{Q}$ stand for chiral and antichiral fields, respectively, while Φ and $\mathcal{W}$ represent the adjoint chiral field and vector multiplet associated with a generic gauge group.

[20]Here it should be remembered that only fermions contribute the computation. In particular, in superspace we have that the field strength $\mathcal{W}$ expands out as $\mathcal{W} = \lambda + \theta F + \ldots$, with λ the gluino field, whereas for chiral fields we have $Q = q + \theta\psi + \ldots$. Therefore, we find that $R_{\mathcal{N}=1}[\mathcal{W}] = R_{\mathcal{N}=1}[\lambda]$ and $R_{\mathcal{N}=1}[Q] = R_{\mathcal{N}=1}[\psi] + 1$.

where n_H, n_V is the total number of $\mathcal{N}=2$ hypermultiplets and vector multiplets in the quiver.

Using the equation (1.126), we find

$$\begin{aligned} a(\epsilon) &= \frac{3}{32\pi}\left[n_V\left(3-3\epsilon^3+\epsilon-1\right)+n_H\left(\frac{3(\epsilon-1)^3}{4}+1-\epsilon\right)\right]\,, \\ c(\epsilon) &= \frac{1}{32\pi}\left[9\left(n_V(1-\epsilon^3)+\frac{n_H}{4}(\epsilon-1)^3\right)-5\Big(n_V(1-\epsilon)+n_H(\epsilon-1)\Big)\right]. \end{aligned} \tag{1.128}$$

It is quite easy to see that $a(\epsilon)$ is maximised when $\epsilon=\frac{1}{3}$. Putting $\epsilon=\frac{1}{3}$ into (1.128), we find

$$a=\frac{5n_V+n_H}{24\pi}\,,\qquad c=\frac{2n_V+n_H}{12\pi}\,. \tag{1.129}$$

the very the same values as for the $\mathcal{N}=2$ central charges.

Thus, using equation (1.125), the R-charges are given by

$$R_{\mathcal{N}=1}[Q]=R_{\mathcal{N}=1}[\tilde{Q}]=R_{\mathcal{N}=1}[\Phi]=\frac{2}{3}\,,\quad R_{\mathcal{N}=1}[\mathcal{W}_\alpha]=1\,. \tag{1.130}$$

Note that a superpotential term of the form

$$W=h\sum_{j=1}^{P}\mathrm{Tr}\left[\Phi_jQ_j\tilde{Q}_j\right]\,, \tag{1.131}$$

has the correct R-charge, $R[W]=2$, and the correct mass dimension (h is dimensionless, i.e. marginal), satisfying $\dim[W]=3=\frac{3}{2}R[W]$. Other possible gauge invariant operators, like $\mathcal{O}_1=\mathrm{Tr}Q_j\tilde{Q}_j$ or $\mathcal{O}_2=\mathrm{Tr}\Phi_j^2$ satisfy the unitarity bound $1\leq\dim\mathcal{O}$.

In a generic $\mathcal{N}=1$ SCFT the dimension of a chiral operator $\mathcal{O}$ is given in terms of its R-charge as

$$\dim\mathcal{O}=\frac{3}{2}R_{\mathcal{O}}\,. \tag{1.132}$$

Also, quantum mechanically, we have

$$\dim\mathcal{O}=[\mathcal{O}]+\frac{1}{2}\widehat{\gamma}_{\mathcal{O}}\,, \tag{1.133}$$

with $[\mathcal{O}]$ the classical dimension of $\mathcal{O}$ and $\widehat{\gamma}_{\mathcal{O}}$ its anomalous dimension. It is then easy to see that $\widehat{\gamma}_Q=\widehat{\gamma}_{\tilde{Q}}=\widehat{\gamma}_\Phi=0$. This, in turn, implies that all the beta functions for the gauge couplings vanish $\beta_g\sim 2N_c-N_f(1-\widehat{\gamma})=0$, $\beta_h=0-\widehat{\gamma}_\Phi/2-\widehat{\gamma}_Q=0$.

We can check that the SCFTs we are dealing with, do satisfy the bound $\frac{1}{2}\leq\frac{a}{c}\leq\frac{3}{2}$, in agreement with [47].

A marginal deformation changes the superpotential by means of powers of e^{iR} (with R a combination of the R-charges of the fields participating to the interaction). There is not an RG-flow taking place, yet we are breaking SUSY $\mathcal{N} = 2 \to \mathcal{N} = 1$ via interaction terms. No degree of freedom is lost, as shown already by the calculation of the central charge, coincident with the $\mathcal{N} = 2$ values. We just have different interactions between fields, and different global symmetries.

Let us now discuss the holographic viewpoint of the above. We shall construct two different deformations of Gaiotto-Maldacena SCFTs. They will be described by a parameter γ (the marginal parameter of the deformation). We shall then calculate the central charge in each geometry, finding the same result as in the parent $\mathcal{N} = 2$ background. We will also compute the associated Page charges.

1.3.2 Backgrounds dual to marginal deformations

We now explore some backgrounds obtained as a result of duality transformations (T-duality, TsT and dimensional reduction from eleven dimensions) applied to the Gaiotto-Maldacena background in (1.33), (1.34) and (1.35). These backgrounds are proposed to be dual to some $\mathcal{N} = 1$ SCFTs, along the lines of what we have discussed in the previous subsection. To make this part more readable, we shall postpone the details of the computations in the appendix at the end of this chapter.

First, we present a new class of backgrounds in eleven dimensional supergravity and in Type IIA obtained using an $Sl(3, \mathbb{R})$ transformation, understood as a generalisation of the Lunin-Maldacena TsT [16]. Then, we present a different solution obtained first by moving a generic Gaiotto-Maldacena background to Type IIB (via T-duality) and then performing a TsT transformation.

The outcome is that of two new families of solutions, one in M-theory/IIA, the other in Type IIB. All of them will be described in terms of a potential function, $V(\sigma, \eta)$, which still satisfies the Laplace equation (1.32). Thus, for any solution to the Laplace equation with certain boundary conditions, we generate a new solution in IIA/M-theory or in Type IIB.

The γ-deformed backgrounds in eleven-dimensions and Type IIA

We shall now present one possible γ-deformation of the Gaiotto-Maldacena backgrounds. We follow the formalism of [79].

Consider the eleven dimensional background in (1.38) and (1.39) written in the following form

$$ds^2 = \mu^4\alpha'\left(\widehat{\Delta}^{-1/6}g_{\mu\nu}dx^\mu dx^\nu + \widehat{\Delta}^{1/3}M_{ab}\mathcal{D}\phi^a\mathcal{D}\phi^b\right)\,,$$
$$\kappa^{-1}A_{(3)} = C_{(0)}\mathcal{D}\phi^1\wedge\mathcal{D}\phi^2\wedge\mathcal{D}\phi^3 + \frac{1}{2}C_{(1)ab}\wedge\mathcal{D}\phi^a\wedge\mathcal{D}\phi^b + C_{(2)a}\wedge\mathcal{D}\phi^a + C_{(3)}\,, \tag{1.134}$$

where $\mathcal{D}\phi^a = d\phi^a + A^a_\mu dx^\mu$. Here, $a,b=1,2,3$ and $\phi^{1,2,3}=(\xi,\beta,y)$. We have chosen units such that all coordinates are dimensionless[21]. We have also made the following identifications

$$A^a_\mu = 0, \quad M_{ab} = \widehat{\Delta}^{-1/3}\begin{pmatrix} F_3\sin^2\chi & 0 & 0\\ 0 & F_4+F_5\tilde{A}^2 & \tilde{A}F_5\\ 0 & \tilde{A}F_5 & F_5\end{pmatrix}, \quad \widehat{\Delta} = F_3F_4F_5\sin^2\chi \tag{1.135}$$

as well as

$$C_{(1)\xi\beta} = F_6\sin\chi d\chi\,,\quad C_{(1)\xi y} = F_7\sin\chi d\chi\,,\quad C_{(0)} = C_{(2)} = C_{(3)} = 0\,, \tag{1.136}$$

and

$$\mu^4\alpha'\widehat{\Delta}^{-1/6}g_{\mu\nu}dx^\mu dx^\nu = \kappa^{2/3}\left[4F_1 ds^2_{AdS_5} + F_2(d\sigma^2+d\eta^2) + F_3d\chi^2\right]\,. \tag{1.137}$$

The functions F_i and $\tilde{A}$ have been defined in (1.39).

The background obtained after an $Sl(3,\mathbb{R})$ transformation, with parameter of the transformation γ, is constructed following the rules of [79].

We give the details of the construction applied to this particular case in Appendix G. The resulting eleven dimensional solution is given by

$$\frac{ds^2}{\kappa^{2/3}} = (1+\gamma^2\widehat{\Delta})^{1/3}\left(4F_1ds^2_{AdS_5} + F_2(d\sigma^2+d\eta^2) + F_3d\chi^2\right)$$
$$+ (1+\gamma^2\widehat{\Delta})^{-2/3}\left(F_3\sin^2\chi d\xi^2 + F_4\tilde{\mathcal{D}}\beta^2 + F_5(\tilde{\mathcal{D}}y + \tilde{A}\tilde{\mathcal{D}}\beta)^2\right)\,, \tag{1.138}$$
$$\kappa^{-1}A_{(3)} = (F_6\tilde{\mathcal{D}}\beta + F_7\tilde{\mathcal{D}}y)\wedge\widehat{\mathrm{vol}}_{S^2} - \frac{\gamma\widehat{\Delta}}{1+\gamma^2\widehat{\Delta}}d\xi\wedge\tilde{\mathcal{D}}\beta\wedge\tilde{\mathcal{D}}y\,,$$

[21] Here $\kappa^{2/3}=\mu^4\alpha'$. See Appendix B for details on the connection between eleven- and ten-dimensional backgrounds.

where

$$\tilde{\mathcal{D}}\beta = d\beta - \gamma F_7 \sin\chi d\chi\,, \quad \tilde{\mathcal{D}}y = dy + \gamma F_6 \sin\chi d\chi\,. \tag{1.139}$$

This is a solution of eleven-dimensional supergravity for any function $V(\sigma,\eta)$ solving the equation (1.32). Obviously, when $\gamma = 0$, this background reduces back to the one in (1.38) and (1.39).

We can bring this family of solutions down to Type IIA supergravity performing a reduction along the direction y – the details of this reduction are discussed in Appendix B – and write all the functions in terms of those defined in (1.36). The N-S sector in Type IIA reads

$$\begin{aligned}
\frac{ds^2_{10}}{\alpha'\mu^2} &= 4f_1 ds^2_{AdS_5} + f_2(d\sigma^2 + d\eta^2) + f_3 d\chi^2 + \frac{f_3\sin^2\chi}{(1+\gamma^2 f_3 f_4 \sin^2\chi)} d\xi^2 \\
&\quad + \frac{f_4}{(1+\gamma^2 f_3 f_4\sin^2\chi)}(d\beta - \gamma f_5 \sin\chi d\chi)^2 \\
e^{2\phi} &= \frac{f_8}{(1+\gamma^2 f_3 f_4\sin^2\chi)}\,, \\
B_{(2)} &= \frac{\mu^2\alpha'}{(1+\gamma^2 f_3 f_4 \sin^2\chi)}\left(f_5 \widehat{\mathrm{vol}}_{S^2} - \gamma f_3 f_4 \sin^2\chi d\xi\wedge d\beta\right)\,,
\end{aligned} \tag{1.140}$$

while the R-R sector is

$$\begin{aligned}
C_{(1)} &= \mu^4 \alpha'^{\frac{1}{2}} \left(f_6 d\beta + \gamma(f_7 - f_5 f_6)\sin\chi d\chi\right)\,, \\
C_{(3)} &= \frac{\mu^6\alpha'^{3/2}}{(1+\gamma^2 f_3 f_4\sin^2\chi)} f_7 d\beta\wedge\widehat{\mathrm{vol}}_{S^2}\,.
\end{aligned} \tag{1.141}$$

As expected, when $\gamma = 0$, we are back to the Gaiotto-Maldacena backgrounds in (1.33), (1.34) and (1.35).

To sum up, we have constructed a new family of backgrounds with $SO(2,4)\times U(1)_\beta \times U(1)_\xi$ bosonic isometries. For any solution to the Laplace equation (1.32), we have valid new backgrounds. The isometries of the background suggest that the it preserves $\mathcal{N}=1$ supersymmetry instead of $\mathcal{N}=2$ (which would need an $SU(2)$ R-symmetry). One possible strategy to prove SUSY would be to put this background to the coordinates of [83, 84], but finding such a change of coordinates is not immediate. Nevertheless, given the arguments explained in [85], it seems likely that some amount of supersymmetry is preserved.

We suggest that the integrability of the $\mathcal{N}=2$ Sfetsos-Thompson solution [46] should translate into the integrability of the string sigma model in the background of equation (1.140) for the case in which the functions f_i are derived from the Sfetsos-Thompson potential in (1.111). It would be interesting to find the Lax pair along the lines of [48].

The gamma-deformed Type IIB backgrounds

The goal now is to write down the backgrounds obtained by moving the Gaiotto-Maldacena solutions to Type IIB supergravity via a T-duality and then performing a Lunin-Maldacena TsT transformation.

Let us apply a T-duality along the β direction of the background in (1.33), (1.34) and (1.35). Using the usual Buscher rules we find the following T-dual N-S sector

$$\begin{aligned} \frac{ds^2}{\alpha'\mu^2} &= 4f_1 ds^2_{AdS_5} + f_2(d\sigma^2 + d\eta^2) + f_3 ds^2_{S^2} + f_4^{-1}\frac{d\beta^2}{\mu^4}\,, \\ B_{(2)} &= \alpha'\mu^2 f_5 \widehat{\text{vol}}_{S^2}\,, \qquad e^{2\phi} = \frac{f_8}{\mu^2 f_4}\,, \end{aligned} \tag{1.142}$$

whilst the R-R potentials and corresponding field strengths are

$$\begin{aligned} C_{(0)} &= \mu^4 f_6 \qquad C_{(2)} = \alpha'\mu^6 f_7\, \widehat{\text{vol}}_{S^2}\,, \\ F_{(1)} &= dC_{(0)} \qquad F_{(3)} = dC_{(2)} - H_{(3)}C_{(0)}\,, \end{aligned} \tag{1.143}$$

Let us apply now the TsT transformation to this solution. Following the rules of the papers [16, 78] (the details are given in Appendix G.2) we find the TsT transformed background

$$\begin{aligned} \frac{ds^2}{\alpha'\mu^2} &= 4f_1 ds^2_{AdS_5} + f_2(d\sigma^2 + d\eta^2) + f_3 d\chi^2 \\ &\quad + \frac{1}{f_4 + \gamma^2 f_3 \sin^2\chi}\left(f_3 f_4 \sin^2\chi d\xi^2 + (d\beta - \gamma f_5 \sin\chi d\chi)^2\right)\,, \\ e^{2\phi} &= \frac{f_8}{\mu^2(f_4 + \gamma^2 f_3 \sin^2\chi)}\,, \\ B_{(2)} &= \alpha'\mu^2\left(\frac{\gamma f_3 \sin^2\chi}{f_4 + \gamma^2 f_3\sin^2\chi}(d\beta - \gamma f_5 \sin\chi d\chi)\wedge d\xi + f_5\widehat{\text{vol}}_{S^2}\right)\,, \\ C_{(0)} &= \mu^4 f_6\,, \\ C_{(2)} &= \alpha'\mu^6\left(\frac{\gamma f_6 f_3 \sin^2\chi}{f_4 + \gamma^2 f_3\sin^2\chi}(d\beta - \gamma f_5 \sin\chi d\chi)\wedge d\xi + f_7\widehat{\text{vol}}_{S^2}\right)\,, \end{aligned} \tag{1.144}$$

where γ is the deformation parameter. In addition, it is easily seen that after turning off the deformation parameter γ the above background reduces to that in equations (1.142) and (1.143).

The same comments as those written below equation (1.140) apply here. For any potential function satisfying (1.32), the background of equation (1.144) is a solution to the Type IIB equations of motion. The $SO(2,4)\times U(1)_\xi \times U(1)_\beta$ isometries suggest that some SUSY is preserved. The construction of a Lax pair for the string sigma model on equation (1.144), for the f_i evaluated with the potential V_{ST} in equation (1.111) should be related to that in [48] via dualities.

Let us now compute some observables of these backgrounds.

Page charges and central charge

We follow Subsection 1.1.7 and compute the Page charges of the backgrounds in equations (1.140), (1.141) and (1.142) - (1.144). For the Type IIA solutions in (1.140), (1.141) let us define the following cycles

$$\Sigma_2 = (\eta,\beta)|_{\sigma=0}\,, \quad \widehat{\Sigma}_2 = (\eta,\chi)|_{\sigma=0}\,, \quad \Sigma_3 = (\eta,\chi,\xi)|_{\sigma=\infty}\,, \quad \widehat{\Sigma}_3 = (\sigma,\beta,\xi)\,. \tag{1.145}$$

For the $NS5$ and $D6$ branes we find the following associated Page charges

$$\begin{aligned} Q_{NS5} &= \frac{1}{2\kappa_{10}^2 T_{NS5}}\int_{\Sigma_3} H_{(3)}\,, \qquad \widehat{Q}_{NS5} = \frac{1}{2\kappa_{10}^2 T_{NS5}}\int_{\widehat{\Sigma}_3} H_{(3)}\,,\\ Q_{D6} &= \frac{1}{2\kappa_{10}^2 T_{D6}}\int_{\Sigma_2} F_{(2)}\,, \qquad \widehat{Q}_{D6} = \frac{1}{2\kappa_{10}^2 T_{D6}}\int_{\widehat{\Sigma}_2} F_{(2)}\,. \end{aligned} \tag{1.146}$$

For Q_{NS5} and Q_{D6} we find same results as in Subsection 1.1.7, namely

$$Q_{NS5} = -\frac{2}{\pi}\mu^2 N_5\,, \qquad Q_{D6} = \mu^4(\lambda'(\eta_f) - \lambda'(0)). \tag{1.147}$$

As before, in order to have a well-quantised number of branes, this implies that $\mu^2 = \frac{\pi}{2}$. Hence $Q_{NS5} = N_5$ and, as before the definition $N_6 = \frac{\pi^2}{4}N_c$ should be used.

The integral defining $\widehat{Q}_{NS5}$ can be performed to give

$$\begin{aligned} \widehat{Q}_{NS5} &= \frac{1}{4\pi^2\alpha'}\mu^2\alpha'\gamma\int d\xi d\beta\int_0^\infty d\sigma\partial_\sigma\left(\frac{f_3 f_4\sin^2\chi}{1+\gamma^2 f_3 f_4\sin^2\chi}\right)\\ &= -\frac{\mu^2}{\gamma} = \widehat{N}_5\,. \end{aligned} \tag{1.148}$$

The last equation implies a new quantisation condition, $2\gamma\widehat{N}_5=\pi$. In particular, this should be thought of as a condition on the parameter γ.

It may be confusing that in the limit of $\gamma\to 0$ the new charge of five branes diverges. However, it should be observed that $\widehat{Q}_{NS5}$ vanishes in the first place when $\gamma\to 0$ from its very definition.

Similarly, one can compute $\widehat{Q}_{D6}$ to get

$$\begin{aligned}\widehat{Q}_{D6}&=\frac{1}{2\pi\alpha'^{\frac{1}{2}}}\gamma\mu^4\alpha'^{\frac{1}{2}}\int_0^{\pi}d\chi\sin\chi\int_0^{\eta_f}d\eta\partial_\eta[f_7(0,\eta)-f_5(0,\eta)f_6(0,\eta)]\\&=-\frac{\gamma\mu^4}{\pi}\left[f_7(0,\eta)-f_5f_6(0,\eta)\right]_0^{\eta_f}=\gamma\frac{\pi}{2}N_5\lambda'(N_5).\end{aligned}\tag{1.149}$$

For the solutions of Type IIB in equation (1.142) - (1.144), we define the cycles

$$\Sigma_1=(\eta)|_{\sigma=0}\,,\quad \Sigma_3=(\eta,\chi,\xi)|_{\sigma=\infty}\,,\quad \widehat{\Sigma}_3=(\sigma,\beta,\xi)|_{\eta=\eta_0}\,.\tag{1.150}$$

Using this, we compute the following $NS5$ brane charges

$$\begin{aligned}Q_{NS5}&=\frac{1}{2\kappa_{10}^2T_{NS5}}\int_{\Sigma_3}H_{(3)}=\frac{\mu^2\alpha'}{4\pi^2\alpha'}\int\widehat{\text{vol}}_{S^2}\int_0^{N_5}d\eta\,\partial_\eta\frac{f_5f_4}{f_4+\gamma^2f_3\sin^2\chi}\\&=\frac{2\mu^2}{\pi^2}N_5\,,\\\widehat{Q}_{NS5}&=\frac{1}{2\kappa_{10}^2T_{NS5}}\int_{\widehat{\Sigma}_3}H_{(3)}=\frac{\mu^2}{4\pi^2}\int d\xi d\beta\int_0^{\infty}d\sigma\,\partial_\sigma\frac{\gamma f_3\sin^2\chi}{f_4+\gamma^2f_3\sin^2\chi}\\&=\frac{\mu^2}{\gamma}=\widehat{N}_5\end{aligned}\tag{1.151}$$

and $D7$ brane charges

$$Q_{D7}=\frac{1}{2\kappa_{10}^2T_{D7}}\int_{\Sigma_1}F_{(1)}=\mu^4(\lambda'(N_5)-\lambda'(0))\,.\tag{1.152}$$

As in the Type IIA case, we see that a new set of $NS5$ branes appear and we need to impose that $\gamma=\frac{\pi}{2\widehat{N}_5}$ in order to have a well-defined number of them.

Let us now study the *central charges* for the type IIA solution above. We follow the procedure outlined in Subsection 1.1.9. For the Type IIA solutions, we identify, from the equation (1.140),

$$\det(g_{int}) = (\alpha'\mu^2)^5 \frac{f_2^2 f_3^2 f_4 \sin^2\chi}{(1+\gamma^2 f_3 f_4 \sin^2\chi)^2}\,, \quad a(R) = 4\alpha'\mu^2 f_1 R^2\,,$$
$$e^{-4\phi} = \frac{(1+\gamma^2 f_3 f_4 \sin^2\chi)^2}{f_8^2}\,. \tag{1.153}$$

A straightforward computation shows that the internal volume $\mathcal{V}_{int}$ is

$$\begin{aligned}\mathcal{V}_{int} &= \int d\eta d\sigma d\chi d\xi d\beta \sqrt{e^{-4\phi}\det[g_{int}]a(R)^3} \\ &= 64\pi^2\alpha'^4\mu^8 \int_0^{\eta_f} d\eta \int_0^\infty d\sigma \frac{f_1^{3/2} f_4^{1/2} f_2 f_3}{f_8}\,.\end{aligned} \tag{1.154}$$

Using, as above, that $\lim_{\sigma\to\infty} \dot{V}(\sigma,\eta) = 0$, after some straightforward algebra we find that the internal volume in equation (1.154) is precisely equal to that in (1.84). This implies, following the steps outlined in equations (1.84) – (1.88), that the central charge for the γ-deformed background in (1.140) and (1.141) is equal to the central charge of the original Gaiotto-Maldacena background given in (1.33), (1.34) and (1.35), and given by equation (1.88). The same happens in Type IIB. This is in line with the fact that these solutions represent SCFTs which have the same number of degrees of freedom, with only different interaction potentials.

These solutions, to be compared with those found in [86, 87, 88], are realising what we explained in Subsection 1.3.1, namely they behave as $\mathcal{N} = 1$ SCFTs with vanishing anomalous dimensions. They have the same number of degrees of freedom of the parent $\mathcal{N} = 2$ SCFTs.

1.4 Summary

In this chapter, after a detailed introduction to four-dimensional $\mathcal{N} = 2$ SCFTs and their gravity duals, we have presented several formulas for the dictionary between the SCFTs and the dual supergravity backgrounds. Formulas calculating charges, number of branes and Linking Number associated with Hanany-Witten setups were spelled out. All these were given in terms of λ, the function defining boundary conditions for the Laplace-like equation that encodes all the information of the supergravity background. We have tested these expressions in various examples of increasing level of complexity and presented a proof for them, when available.

We have built the holographic description of marginal deformations of some $\mathcal{N} = 2$ four-dimensional SCFTs. Some infinite families of solutions were constructed, again with all the information being encoded by a Laplace equation and its boundary conditions. New solutions were explored, observables calculated and CFT interpretation presented.

A Physical Interpretation of $\lambda(\eta)$

The equation (1.32) does not look like a typical Laplace problem in two dimensions, but in fact like a Laplace problem in three dimensions, with a cyclic coordinate[22]. Below, we show that the interpretation of the quantity $\lambda(\eta)$ in equation (1.46) is precisely that of a line charge density. In order to prove this, we consider the solutions for the potential V in the form given in equation(1.42) and use an integral representation for the Bessel function $K_0(w_n\sigma)$,

$$K_0(w_n\sigma) = \int_0^\infty \frac{\cos(w_n\sigma t)}{\sqrt{t^2+1}}dt\,. \tag{1.155}$$

Using that $2\cos x \sin y = \sin(x+y) - \sin(x-y)$, the potential in (1.42) can be rewritten as

$$\begin{aligned} V(\sigma,\eta) &= -\sum_{n=1}^{\infty}\frac{c_n}{2w_n}\left[\int_0^\infty \frac{\sin\left(w_n(\eta+\sigma t)\right)}{\sqrt{t^2+1}}dt - \int_0^\infty \frac{\sin\left(w_n(-\eta+\sigma t)\right)}{\sqrt{t^2+1}}dt\right] \\ &= -\sum_{n=1}^{\infty}\frac{c_n}{2w_n}\left[\int_{-\infty}^\infty \frac{\sin\left(w_n u\right)}{\sqrt{(u-\eta)^2+\sigma^2}}du\right]\,. \end{aligned} \tag{1.156}$$

Now, swapping the sum and the integral, and using equation (1.46), we find

$$\begin{aligned} V(\sigma,\eta) &= -\int_{-\infty}^\infty \frac{\lambda(u)}{2\sqrt{(u-\eta)^2+\sigma^2}}du \\ &= -\int_0^\infty \frac{\lambda(u)}{\sqrt{(u-\eta)^2+\sigma^2}}du\,. \end{aligned} \tag{1.157}$$

This precisely the electric potential produced by an odd-extended density of charge λ

[22] The Laplace equation in a three-dimensional space, with metric given in cylindrical coordinates as $ds^2 = d\sigma^2 + \sigma^2 d\varphi^2 + d\eta^2$, where σ represents the radial distance, φ the azimuth angle and η the height, is simply $\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\left(\sigma\frac{\partial V}{\partial\sigma}\right) + \frac{\partial^2 V}{\partial\varphi^2} + \frac{\partial^2 V}{\partial\eta^2} = 0$. If we take V to be independent of φ, the three-dimensional Laplace equation reduces to the (1.32)

along the η-axis, at some generic point (σ, η). This makes clear the interpretation as an electrostatic problem.

B The 11d Supergravity-Type IIA connection

In this appendix we give some detail on the uplift of the ten-dimensional background in equations (1.33), (1.34) and (1.35). We will pay special attention to how the ten-dimensional constants, μ and α', g are related to κ, the sole eleven-dimensional parameter that appears in M-theory.

Consider the usual Ansatz for the uplift to eleven dimensions

$$\begin{aligned} ds_{11}^2 &= e^{-\frac{2}{3}\phi} ds_{10}^2 + e^{\frac{4}{3}\phi}(dx^{10} + C_{(1)})^2 \,, \\ A_{(3)} &= C_{(3)} + B_{(2)} \wedge dx^{10} \,, \end{aligned} \tag{1.158}$$

where $A_{(3)}$ the three-form in eleven dimensions and x^{10} the eleventh coordinate, $x^{10} \cong x^{10} + 2\pi g_s l_s$. Using that

$$e^{-\frac{2}{3}\phi} = f_8^{-1/3} = \mu^2 \left(\frac{4(2\dot{V} - \ddot{V})^3}{V''\dot{V}^2\Delta^2} \right)^{-1/6} \,, \tag{1.159}$$

we find that the eleven dimensional metric is given by

$$\begin{aligned} ds_{11}^2 =& \alpha'\mu^4 \left(\frac{4(2\dot{V} - \ddot{V})^3}{V''\dot{V}^2\Delta^2} \right)^{-1/6} \left[4f_1 ds_{AdS_5}^2 + f_2(d\sigma^2 + d\eta^2) + f_3 ds_{S^2}^2(\chi, \xi) + f_4 d\beta^2 \right] \\ &+ \frac{1}{\mu^4} \left(\frac{4(2\dot{V} - \ddot{V})^3}{V''\dot{V}^2\Delta^2} \right)^{1/3} (dx^{10} + \mu^4 \sqrt{\alpha'} f_6 d\beta)^2. \end{aligned} \tag{1.160}$$

Notice that we are using conventions where coordinates are dimensionless, with the sole exception of x^{10}, which has dimensions of length. Let us then rescale x^{10} as $x^{10} = \alpha'^{\frac{1}{2}} \mu^4 y$. We find

$$\begin{aligned} ds_{11}^2 =& \alpha'\mu^4 \left(\frac{4(2\dot{V} - \ddot{V})^3}{V''\dot{V}^2\Delta^2} \right)^{-1/6} \left[4f_1 ds_{AdS_5}^2 + f_2(d\sigma^2 + d\eta^2) + f_3 ds_{S^2}^2(\chi, \xi) + f_4 d\beta^2 \right] \\ &+ \alpha'\mu^4 \left(\frac{4(2\dot{V} - \ddot{V})^3}{V''\dot{V}^2\Delta^2} \right)^{1/3} (dy + f_6 d\beta)^2 \,. \end{aligned} \tag{1.161}$$

Identifying $\mu^4\alpha' = \kappa^{2/3}$, and after a simple algebra, we find the background metric given in (1.38), (1.39).

We can proceed in a similar way to find the eleven-dimensional three-form

$$\begin{aligned} A_{(3)} &= \mu^6\alpha'^{3/2} f_7 d\beta \wedge \widehat{\mathrm{vol}}_{S^2} + \mu^2\alpha' f_5 \widehat{\mathrm{vol}}_{S^2} \wedge dx^{10} \\ &= \kappa\left(f_7 d\beta + f_5 dy\right) \wedge \widehat{\mathrm{vol}}_{S^2}\,, \end{aligned} \tag{1.162}$$

upon using $x^{10} = \alpha'^{\frac{1}{2}}\mu^4 y$ and $\mu^4\alpha' = \kappa^{2/3}$.

Following a similar procedure, we can connect the eleven dimensional background in equation (1.138) with the corresponing background in type IIA given in equation (1.140).

C Expansion of the various background functions

In this appendix we quote some of the asymptotic expansions for the warping factors appearing in the background close to $\sigma = 0$ and $\sigma = \infty$. When studying the behaviour of the brackground close to $\sigma = 0$, we will use the expressions for the potential V given in (1.42) and (1.43), while when studying the asymptotic behaviour at infinity we will use (1.42) only.

C.1 Expansion of the various background functions using the solution in equation (1.42)

Consider first the expression given in (1.42)

$$\begin{aligned} \dot{V}(\sigma,\eta) &= \sum_{n=1}^{\infty} \frac{c_n}{w_n}(w_n\sigma)K_1(w_n\sigma)\sin(w_n\eta)\,, \\ \dot{V}'(\sigma,\eta) &= \sum_{n=1}^{\infty} c_n(w_n\sigma)K_1(w_n\sigma)\cos(w_n\eta)\,, \\ \ddot{V}(\sigma,\eta) &= -\sum_{n=1}^{\infty} \frac{c_n}{w_n}(w_n\sigma)^2 K_0(w_n\sigma)\sin(w_n\eta)\,, \\ V''(\sigma,\eta) &= \sum_{n=1}^{\infty} c_n w_n(w_n)K_0(w_n\sigma)\sin(w_n\eta)\,. \end{aligned} \tag{1.163}$$

Now, take $\dot{V}$, $\dot{V}'$, $\ddot{V}$, V'' as just given to compute

$$\begin{aligned} 2\dot{V} - \ddot{V} &= \sum_{n=1}^{\infty} \frac{c_n}{w_n} (w_n\sigma)^2 K_2(w_n\sigma)\sin(w_n\eta)\,, \\ 2\dot{V}\dot{V}' &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{c_n}{w_n} c_k (w_n\sigma)(w_k\sigma) K_1(w_n\sigma) K_1(w_k\sigma)\sin(w_n\eta)\cos(w_k\eta) \\ \Delta &= \sum_{n=1}^{\infty} \frac{c_n}{w_n} (w_n\sigma)^2 K_2(w_n\sigma)\sin(w_n\eta) \sum_{k=1}^{\infty} c_k (w_k) K_0(w_k\sigma)\sin(w_k\eta) \\ &\quad + \left[\sum_{n=1}^{\infty} c_n (w_n\sigma) K_1(w_n\sigma)\cos(w_n\eta)\right]^2 . \end{aligned} \tag{1.164}$$

To discuss the behaviour close to $\sigma = 0$, we use that

$$\begin{aligned} K_0(z) &\simeq \log 2 - \gamma - \log z + \frac{z^2}{4}(1 + \log 2 - \gamma - \log z)\,, \\ zK_1(z) &\simeq 1 + \frac{z^2}{4}(2\gamma - 1 - \log 4 + 2\log z)\,, \\ z^2 K_2(z) &\simeq 2 - \frac{z^2}{2}(3 - 4\gamma + \log 16 - 4\log z)\,, \end{aligned} \tag{1.165}$$

as $z \simeq 0$. It is then easy to see that

$$\begin{aligned} 2\dot{V} - \ddot{V} &\simeq 2\sum_{n=1}^{\infty} \frac{c_n}{w_n}\sin(w_n\eta) = 2\lambda(\eta)\,, \\ 2\dot{V}\dot{V}' &\simeq 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{c_n}{w_n} c_k \sin(w_n\eta)\cos(w_k\eta) = 2\lambda(\eta)\lambda'(\eta)\,, \\ \Delta &\simeq \log\sigma\,, \end{aligned} \tag{1.166}$$

as $\sigma \simeq 0$.

Consider also the following combinations and their asymptotic behaviour close to $\sigma = 0$

$$\begin{aligned} g_1 &= \frac{2\dot{V}\dot{V}'}{2\dot{V} - \ddot{V}} \simeq \lambda'(\eta)\,, \\ g_2 &= 2\left(\frac{\dot{V}\dot{V}'}{\Delta} - \eta\right) \simeq -2\eta\,, \\ g_3 &= -4\frac{\dot{V}^2 V''}{\Delta} \simeq -2\lambda(\eta)\,. \end{aligned} \tag{1.167}$$

Finally, turning to the asymptotic region at $\sigma = \infty$, we quote the following asymptotic expansion

$$
\begin{aligned}
K_0(z) &\simeq e^{-z}\sqrt{\frac{\pi}{2z}}\,, & z^2K_0(z) &\simeq e^{-z}\sqrt{\frac{\pi z^3}{2}}\,,\\
zK_1(z) &\simeq e^{-z}\sqrt{\frac{\pi z}{2}}\,, & z^2\partial_z K_1(z) &\simeq e^{-z}\sqrt{\frac{\pi z^3}{2}}\,,\\
z^2K_2(z) &\simeq e^{-z}\sqrt{\frac{\pi z^3}{2}}\,,
\end{aligned}
\tag{1.168}
$$

from which we find

$$
\begin{aligned}
2\dot{V}-\ddot{V} &\simeq c_1 e^{-w_1\sigma}\sqrt{\frac{\pi w_1\sigma^3}{2}}\sin(w_1\eta)\,,\\
2\dot{V}\dot{V}' &\simeq \frac{\pi c_1^2}{2w_1}\sin(w_1\eta)\cos(w_1\eta)e^{-2w_1\sigma}(w_1\sigma)\,,\\
\Delta &\simeq c_1^2\pi^2 w_1\sigma e^{-2w_1\sigma}\,.
\end{aligned}
\tag{1.169}
$$

D How to count D4 branes

In equation (1.70), we gave a formula that counts the number of $D4$ branes in different Hanany-Witten setups. This expression works nicely in the examples of the equations (1.90), (1.92) and in the more elaborated examples that we will study in Appendix E.

Here, we give a reasoning of why the equation (1.70) works for the generic profile $\lambda(\eta)$ given by

$$
\lambda(\eta)=N_c\begin{cases}
\frac{\lambda_1}{\eta_1}\eta & 0\le\eta\le\eta_1\\
\lambda_1+\left(\frac{\lambda_2-\lambda_1}{\eta_2-\eta_1}\right)(\eta-\eta_1) & \eta_1<\eta\le\eta_2\\
\lambda_2 & \eta_2<\eta\le\eta_3\\
\lambda_2+\left(\frac{\lambda_3-\lambda_2}{\eta_4-\eta_3}\right)(\eta-\eta_3) & \eta_3<\eta\le\eta_4\\
\lambda_3-\left(\frac{\lambda_3}{N_5-\eta_4}\right)(\eta-\eta_4) & \eta_4<\eta\le N_5\,,
\end{cases}
\tag{1.170}
$$

postponing the actual proof for any profile λ to the next subsection in this appendix. As explained in the main text, we set $N_6=\frac{\pi^2}{4}N_c$. The charge profile is drawn in Figure 1.7.

We shall count explicitly the number of $D4$ branes present in each interval $[\eta_i,\eta_{i+1}]$ and check that this is coincident with the result of equation (1.70).

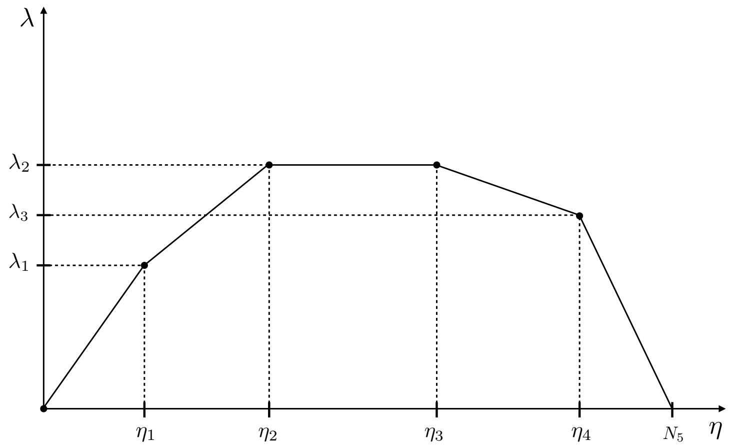

Figure 1.7: The charge density $\lambda(\eta)$ for the profile in equation (1.170).

Consider the portion of the Hanany-Witten setup shown[23] in Figure 1.8. This corresponds to the first interval ($\eta \in [0, \eta_1]$) for the piecewise continuous function $\lambda(\eta)$ in equation (1.170). We see that the number of D4 branes is

$$\begin{aligned} N_{D4} &= N_6 \frac{\lambda_1}{\eta_1}(1 + 2 + 3 + 4 + ... + \eta_1) \\ &= \frac{\lambda_1 N_6}{2}(\eta_1 + 1)\,. \end{aligned} \tag{1.171}$$

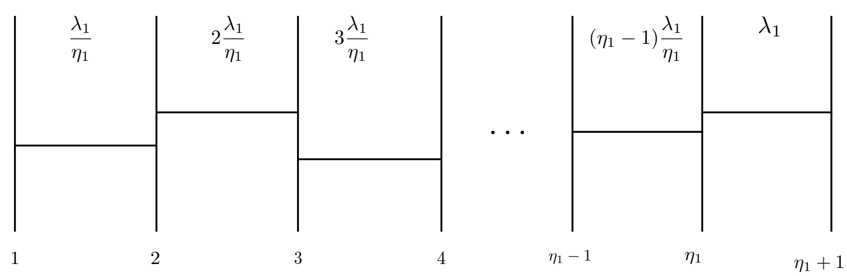

Figure 1.8: The Hanany-Witten set-up for the first interval $[0, \eta_1]$ of the profile in equation (1.170). The number of branes should be multiplied by N_6.

[23]In what follows, we will not draw the $D6$-flavour branes, to avoid clutter figures.

Let us now move on to studying the second interval, $\eta_1 < \eta \leq \eta_2$. In this case, the relevant part of the quiver and Hanany-Witten setup are shown in Figure 1.9. We count explicitly the number of $D4$ branes to find

$$\begin{aligned} N_{D4} &= N_6 \sum_{r=1}^{\eta_2-\eta_1-1} \left[\lambda_1 + \frac{\lambda_2-\lambda_1}{\eta_2-\eta_1} r\right] + N_6\lambda_2 \\ &= N_6 \frac{(\eta_2-\eta_1)(\lambda_1+\lambda_2)}{2} + N_6 \frac{\lambda_2-\lambda_1}{2}\,. \end{aligned} \tag{1.172}$$

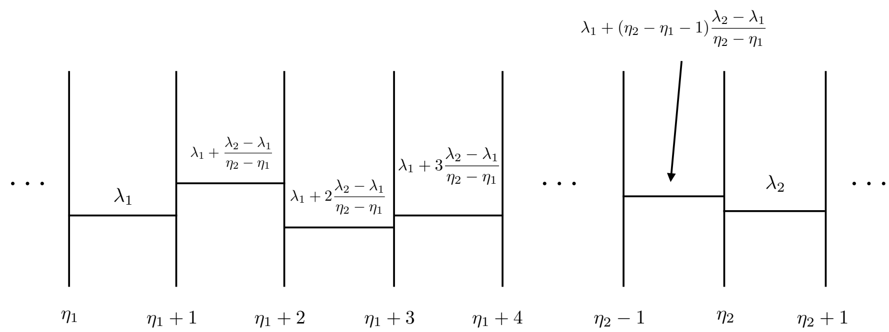

Figure 1.9: The Hanany-Witten set up corresponding to the second interval for the profile in equation (1.170). The number of branes should be multiplied by N_6.

In the $[\eta_2, \eta_3]$ interval, whose Hanany-Witten setup is drawn in Figure 1.10, we find

$$N_{D4} = N_6\lambda_2 \sum_{r=1}^{\eta_3-\eta_2} 1 = N_6\lambda_2(\eta_3-\eta_2). \tag{1.173}$$

The rest of the intervals will work similarly. Indeed, in the interval $[\eta_3, \eta_4]$, whose brane setup is depicted in Figure 1.11, we find

$$\begin{aligned} N_{D4} &= N_6 \sum_{r=1}^{\eta_4-\eta_3-1} \left[\lambda_2 + r\frac{(\lambda_3-\lambda_2)}{(\eta_4-\eta_3)}\right] + N_6\lambda_3 \\ &= N_6 \frac{(\lambda_2+\lambda_3)(\eta_4-\eta_3)}{2} + \frac{(\lambda_3-\lambda_2)}{2} N_6\,. \end{aligned} \tag{1.174}$$

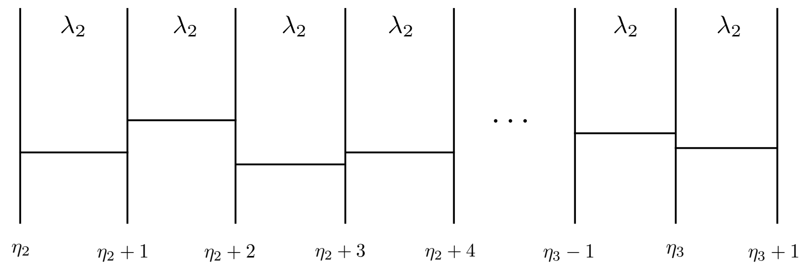

Figure 1.10: The Hanany-Witten set up corresponding to the third interval for the profile in equation (1.170). The number of branes should be multiplied by N_6.

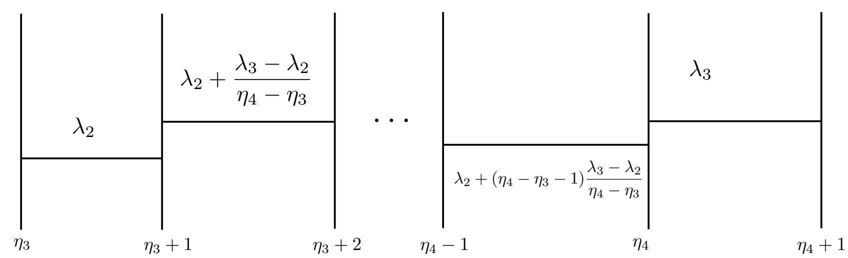

Figure 1.11: The Hanany-Witten set up corresponding to the fourth interval for the profile in equation (1.170). The number of branes should be multiplied by N_6.

For the $[\eta_4, N_5]$ interval, corresponding to the brane set-up of Figure 1.12, we have

$$\begin{aligned} N_{D4} &= N_6 \sum_{r=1}^{N_5-\eta_4-1} \left[\lambda_3 - \frac{\lambda_3}{N_5-\eta_4} r\right] \\ &= \frac{N_6\lambda_3}{2}(N_5-\eta_4) - \frac{N_6\lambda_3}{2}\,. \end{aligned} \tag{1.175}$$

Summing the results for the five intervals in equations (1.171) - (1.175), we find

$$\begin{aligned} N_{D4} &= \frac{N_6}{2}\left[\lambda_1\eta_1 + (\lambda_2+\lambda_1)(\eta_2-\eta_1) + 2\lambda_2(\eta_3-\eta_2) + (\lambda_2+\lambda_3)(\eta_4-\eta_3) + \lambda_3(N_5-\eta_4)\right] \\ &= \frac{2\mu^6}{\pi}\int_0^{N_5} \lambda(\eta)d\eta\,, \end{aligned} \tag{1.176}$$

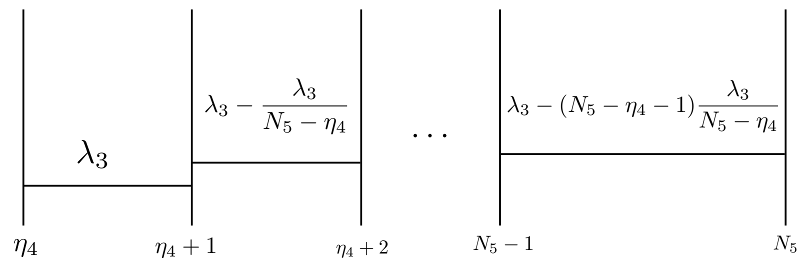

Figure 1.12: The Hanany-Witten set up corresponding to the last interval for the profile in equation (1.170). The number of branes should be multiplied by N_6.

where the last equality can be seen by direct integration of λ. Even though this result is obtained for a (generic) Gaiotto-Maldacena charge profile with four kinks only, it should be clear that (1.70) will work for any (acceptable) λ profile. Let us prove it.

D.1 A derivation for the formula (1.70)

In this section we will provide a derivation for the formula counting the number of D4 branes, see equation (1.70). To this end consider a non-trivial profile for the function $\lambda(\eta)$ respecting the boundary conditions stated around (1.41)

$$\lambda(\eta) = N_6 \begin{cases} \frac{\lambda_1}{\eta_1}\eta & 0 \le \eta \le \eta_1 \\ \lambda_1 + \left(\frac{\lambda_2 - \lambda_1}{\eta_2 - \eta_1}\right)(\eta - \eta_1) & \eta_1 < \eta \le \eta_2 \\ \vdots & \\ \lambda_{n-1} - \left(\frac{\lambda_n - \lambda_{n-1}}{\eta_n - \eta_{n-1}}\right)(\eta - \eta_{n-1}) & \eta_{n-1} < \eta \le \eta_n . \end{cases} \tag{1.177}$$

Notice that in order to satisfy the boundary conditions in equation (1.40) we must choose $\lambda_n = \lambda_0 = 0$. Following the previous section, it is not difficult to see that the counting of $D4$ branes of the Hanany-Witten set up can be done in the following way[24]

$$Q_{D4} = N_6 \sum_{s=1}^{n} \sum_{r=1}^{\eta_s - \eta_{s-1}} \left(\lambda_{s-1} + \frac{\lambda_s - \lambda_{s-1}}{\eta_s - \eta_{s-1}} r\right) . \tag{1.178}$$

Performing the inner sum (sum over r) explicitly leads to the following result

$$Q_{D4} = N_6 \sum_{s=1}^{n} \left(\frac{\lambda_{s-1} - \lambda_s}{2}\right) + N_6 \sum_{s=1}^{n} \frac{\lambda_s + \lambda_{s-1}}{2} (\eta_s - \eta_{s-1}) . \tag{1.179}$$

[24]Notice that this formula acquires a precise meaning only after the sum over r is carried out.

It is easy to see that the first of the two sums vanishes identically. Thus, we end up with the following result

$$Q_{D4} = N_6 \sum_{s=1}^{n} \frac{\lambda_s + \lambda_{s-1}}{2} (\eta_s - \eta_{s-1})\,. \tag{1.180}$$

Taking the continuous limit (i.e. sending n to infinity and taking infinitesimal the distance $\eta_s - \eta_{s-1}$) the approximation becomes exact and we get the formula in equation (1.70),

$$Q_{D4} = N_6 \int_0^{N_5} \lambda(\eta) d\eta\,, \tag{1.181}$$

where we have made the identification $\eta_n \equiv N_5$.

D.2 Counting D6 branes

$D6$ branes appear every time λ has a kink, i.e. every time the derivative $\lambda'(\eta)$ shows a discontinuity. The number of $D6$ branes at each kink is precisely that needed to have conformality at each gauge node, i.e. every gauge group $SU(\lambda_i)$ has $2\lambda_i$ flavours. We can compute the change in slope in each interval for the profile given in equation (1.170). We find in each interval

$$\begin{aligned} Q_{D6}^{(1)} &= N_6 \left(\frac{\lambda_2 - \lambda_1}{\eta_2 - \eta_1} - \frac{\lambda_1}{\eta_1} \right), \quad Q_{D6}^{(2)} = N_6 \left(0 - \frac{\lambda_2 - \lambda_1}{\eta_2 - \eta_1} \right), \\ Q_{D6}^{(3)} &= N_6 \left(\frac{\lambda_3 - \lambda_2}{\eta_4 - \eta_3} - 0 \right), \quad Q_{D6}^{(4)} = N_6 \left(- \frac{\lambda_3}{N_5 - \eta_4} - \frac{\lambda_3 - \lambda_2}{\eta_4 - \eta_3} \right), \end{aligned} \tag{1.182}$$

This, in turn, implies that

$$Q_{D6}^{total} = \sum_i Q_{D6}^{(i)} = N_6 \left[\frac{\lambda_3}{n_5 - \eta_4} + \frac{\lambda_1}{\eta_1} \right] = -\mu^4 N_c (\lambda'(N_5) - \lambda'(0))\,. \tag{1.183}$$

This is again consistent with (1.67).

E General $\mathcal{N}=2$ quivers and matching of observables

In this appendix we consider three different examples of Gaiotto-Maldacena backgrounds for $\mathcal{N}=2$ SCFTs. We show that the holographic formulas we have found in the main text for the number of $D4$ and $D6$ branes and the central charge precisely match with their field theory analogues.

E.1 First example

Let us begin with a λ-profile of the following form

$$\lambda(\eta) = N_c \begin{cases} \eta & 0 \leq \eta \leq \frac{N_5}{2} \\ (N_5-\eta) & \frac{N_5}{2} < \eta \leq N_5 \end{cases} \tag{1.184}$$

The associated quiver and the Hanany-Witten setup are depicted in Figure 1.13. Simply by inspection of Figure 1.13, it is easy to see that the number of $D4$ and $D6$ branes

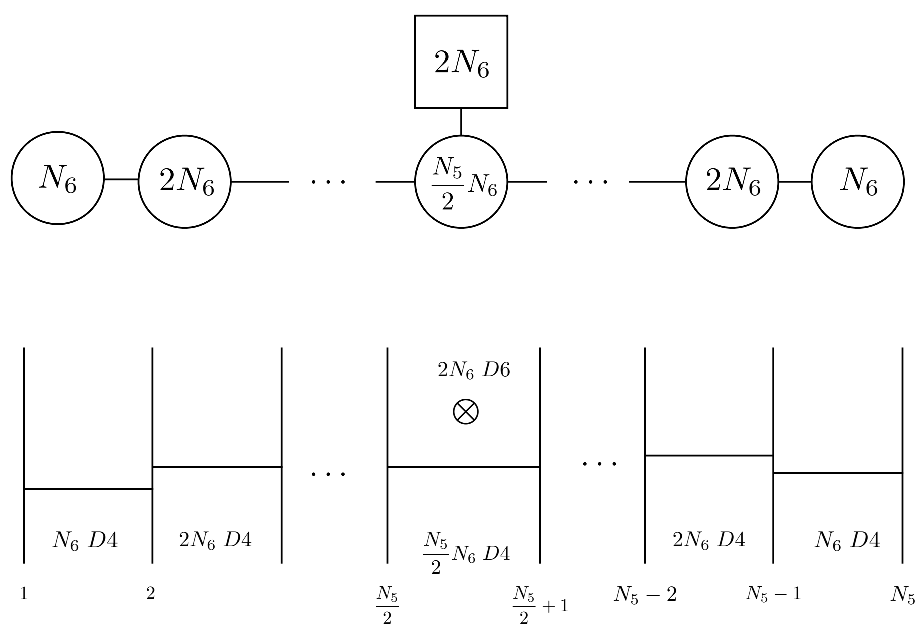

Figure 1.13: The quiver and Hanany-Witten set-up for the profile in equation (1.184).

is given by

$$\begin{aligned} N_{D4} &= \sum_{r=1}^{\frac{N_5}{2}} r N_6 + \sum_{r=1}^{\frac{N_5}{2}-1} N_6 \left(\frac{N_5}{2}-r\right) = \frac{N_6 N_5^2}{4}\,, \\ N_{D6} &= 2N_6\,. \end{aligned} \tag{1.185}$$

Let us count the number of vectormultiplets and hypermultiplets for the quiver of Figure 1.13,

$$\begin{aligned} n_v &= \sum_{r=1}^{\frac{N_5}{2}} r^2 N_6^2 - 1 + \sum_{r=1}^{\frac{N_5}{2}-1} N_6^2 \left(\frac{N_5}{2}-r\right)^2 - 1 = \frac{N_6^2 N_5^3}{12} + \frac{N_5}{6}(N_6^2-6)+1 \\ n_h &= \sum_{r=1}^{\frac{N_5}{2}-1} r(r+1)N_6^2 + N_5 N_6^2 + \sum_{r=0}^{\frac{N_5}{2}-1} N_6^2 \left(\frac{N_5}{2}-r\right)\left(\frac{N_5}{2}-r-1\right) = \frac{N_6^2 N_5}{12}(N_5^2+8)\,. \end{aligned} \tag{1.186}$$

The associated central charge is then given by

$$c = \frac{1}{48\pi}(N_6^2 N_5^3 + 4N_5(N_6^2-2)+8)\,. \tag{1.187}$$

which at leading order leads to

$$c = \frac{N_6^2 N_5^3}{48\pi}\,. \tag{1.188}$$

We can check that all these quantities are reproduced holographically by employing the equations (1.67), (1.70), and (1.88). We find for the number of $D4$ and $D6$ branes

$$\begin{aligned} N_{D4} &= \frac{2}{\pi}\mu^6 \int_0^{\eta_f} \lambda(\eta) d\eta = \frac{N_6 N_5^2}{4}\,, \\ N_{D6} &= -\mu^4(\lambda'(\eta_f)-\lambda'(0)) = 2N_6\,, \end{aligned} \tag{1.189}$$

while for the central charge

$$c = \frac{2}{\pi^4}\mu^{14} \int_0^{\eta_f} \lambda^2(\eta) d\eta = \frac{N_6^2 N_5^3}{48\pi}\,, \tag{1.190}$$

in agreement with the CFT computation.

Let us now compute the linking numbers for the Hanany-Witten setup in Figure

1.13. Using the definition already given in (1.71), we find

$$K_i = -N_6\,, \qquad L_j = \frac{N_5}{2}\,. \tag{1.191}$$

where $i = 1, \ldots N_5$ and $j = 1, \ldots, 2N_6$. We can easily see that the equation (1.72) is satisfied. On the supergravity side, we can compute the linking numbers for the $NS5$ and $D6$ branes using the equations (1.75) and (1.78) and the λ profile in (1.184). We find

$$\sum_{i=1}^{N_5} K_i = \frac{2}{\pi}\mu^6 \lambda'(\eta_f)\eta_f = -\frac{2}{\pi}\mu^6 N_c N_5 \equiv -N_6 N_5 = -\sum_{i=1}^{2N_6} L_i\,, \tag{1.192}$$

finding complete agreement between sugra and CFT.

E.2 Second example

As a second example, let us consider the λ-profile given by

$$\lambda(\eta) = N_c \begin{cases} \eta & 0 \leq \eta \leq k \\ \frac{k(N_5-\eta)}{(N_5-k)} & k < \eta \leq N_5\,. \end{cases} \tag{1.193}$$

The associated quiver and the Hanany-Witten setup are depicted in Figure 1.14. The number of $D4$ and $D6$ branes for the SCFTs of Figure 1.14 is given by

$$\begin{aligned} N_{D4} &= \sum_{r=1}^{k} N_6 r + \sum_{r=1}^{N_5-k-1} \frac{kN_6(N_5-k-r)}{N_5-k} = \frac{N_6 N_5 k}{2}\,, \\ N_{D6} &= \frac{N_5 N_6}{(N_5-k)}\,. \end{aligned} \tag{1.194}$$

Again, we can easily count how many vectormultiplets and hypermultiplets we

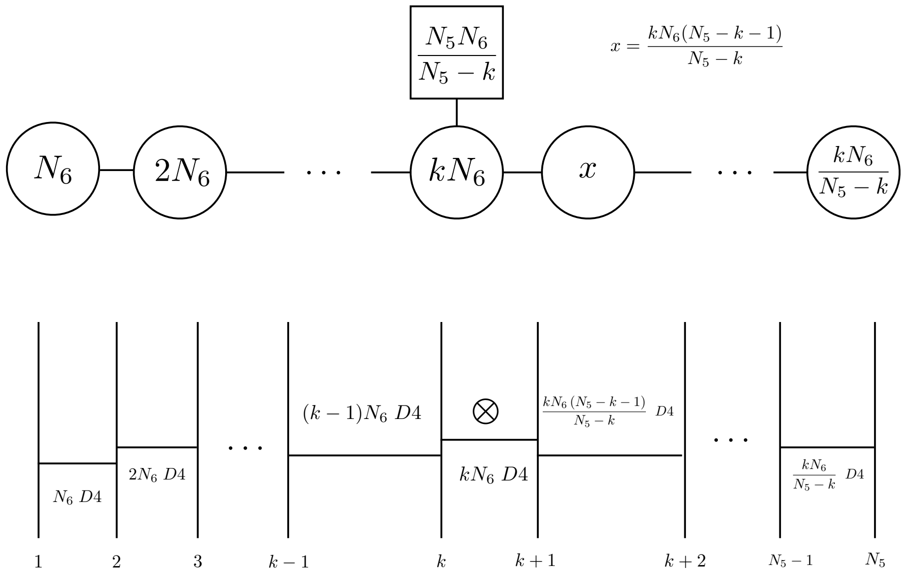

Figure 1.14: The quiver and Hanany-Witten set-up for the profile in equation (1.193).

have. These are given by

$$\begin{aligned}
n_v &= \sum_{r=1}^{k} r^2N_6^2 - 1 + \sum_{r=1}^{N_5-k-1} \frac{k^2N_6^2}{(N_5-k)^2}(N_5-k-r)^2 - 1 \\
&= \frac{1}{6(N_5-k)}\left[2k^2N_5^2N_6^2 + kN_5(N_6^2+6) - 2k^3N_5N_6^2 - 6N_5(N_5-1) - 6k\right] \\
n_h &= \sum_{r=1}^{k} r(r+1)N_6^2 + \left(\frac{kN_6^2N_5}{N_5-k} + \frac{k^2N_6^2}{N_5-k}(N_5-k-1)\right) \\
&\quad + \sum_{r=1}^{N_5-k-2} \frac{k^2N_6^2}{(N_5-k)^2}(N_5-k-r)(N_5-k-r-1) \\
&= \frac{N_6^2k}{3(N_5-k)}\left[5N_5 - k^2(N_5+3) + k(N_5^2+3N_5-3)\right] ,
\end{aligned} \tag{1.195}$$

which lead to the following central charge

$$c = \frac{1}{12(N_5-K)}\left[k^2N_6^2(N_5^2+N_5-1) + 2k(N_6^2N_5+N_5-1) - k^3N_6^2(N_5+1) + 2N_5(N_5-1)\right] . \tag{1.196}$$

Sending both $N_5 \to \infty$ and $N_6 \to \infty$ we find, at leading order,

$$c = \frac{k^2 N_6^2 N_5}{12\pi} \,. \tag{1.197}$$

We can check these values by performing the corresponding holographic calculations from the equations (1.67), (1.70) and (1.88). We find for the $D4$ and $D6$ branes

$$\begin{aligned} N_{D4} &= \frac{2\mu^2}{\pi}\mu^4 \int_0^{\eta_f} \lambda(\eta) d\eta = \frac{N_6 N_5 k}{2} \,, \\ N_{D6} &= \frac{N_6 N_5}{N_5 - k} \end{aligned} \tag{1.198}$$

while for the central charge

$$c = \frac{2\mu^{14}}{\pi^4} \int_0^{\eta_f} \lambda^2(\eta) d\eta = \frac{k^2 N_6^2 N_5}{12\pi} \,. \tag{1.199}$$

The associated linking numbers for the Hanany-Witten set up in Figure 1.14 are given by

$$K_i = -\frac{k N_6}{N_5 - k} \,, \qquad L_j = k \,, \tag{1.200}$$

where $i = 1, \dots, N_5$ and $j = 1, \dots, \frac{N_5 N_6}{N_5 - k}$. We can easily see that the condition (1.72) is satisfied. Using the λ profile in equation (1.193), and the expressions in (1.75) and (1.78), the linking numbers of the $NS5$ and $D6$ branes are given by

$$\sum_{i=1}^{N_5} k_i = \frac{2}{\pi}\mu^6 \lambda'(\eta_f)\eta_f = \frac{2}{\pi}\mu^6 \frac{k N_c N_5}{k - N_5} = -\sum_{i=1}^{N_5 N_6/N_5 - k} L_i \,, \tag{1.201}$$

finding again perfect agreement between sugra and CFT.

E.3 Third example

In our third example we consider the λ-profile given by

$$\lambda(\eta) = N_c \begin{cases} \eta & 0 \le \eta \le k \\ k & k < \eta \le k + q \\ k\frac{(N_5 - \eta)}{N_5 - k - q} & k + q < \eta \le N_5 \end{cases} \tag{1.202}$$

The associated quiver and the Hanany-Witten setup can be seen in Figure 1.15.

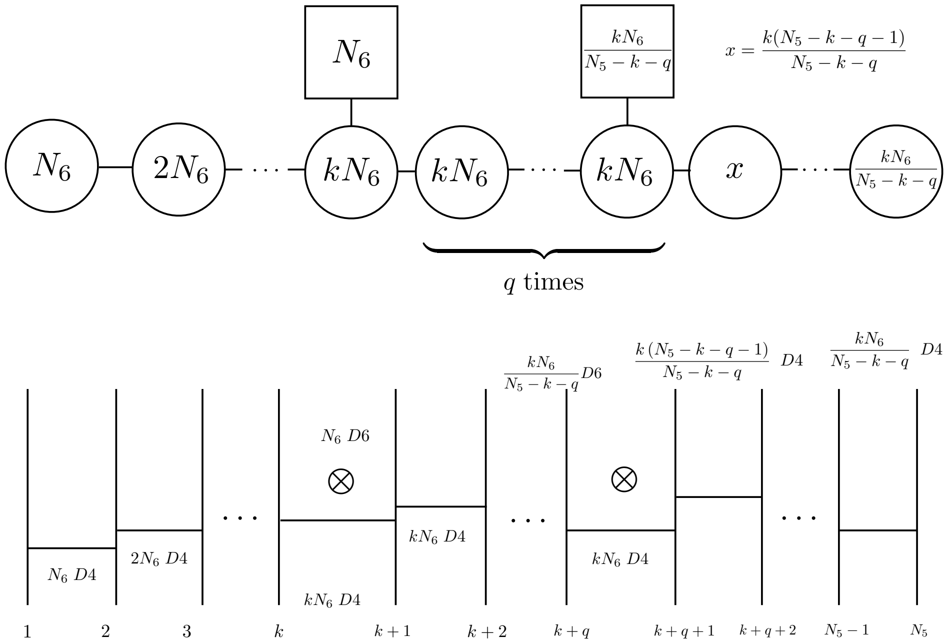

Figure 1.15: The quiver and Hanany-Witten set-up for the profile in equation (1.202).

The number of $D4$ and $D6$ branes is

$$
\begin{aligned}
N_{D4} &= \sum_{r=1}^{k} N_6 r + kqN_6 + \sum_{r=1}^{N_5-k-q-1} \frac{N_6 k}{N_5-k-q}(N_5-k-q-r) \\
&= \frac{kN_6}{2}(N_5+q)\,, \\
N_{D6} &= \frac{(N_5-q)N_6}{N_5-k-q}\,,
\end{aligned}
\tag{1.203}
$$

Again, we count the number of vectors and hypers to find

$$
\begin{aligned}
n_v &= \sum_{r=1}^{k} r^2 N_6^2 - 1 + q(k^2 N_6^2 - 1) + \sum_{r=1}^{N_5-k-q-1} \frac{k^2 N_6^2 (N_5-k-q-r)^2}{(N_5-k-q)^2} \\
&= \frac{1}{6}\left(6 + 2N_5(k^2N_6^2 - 3) + kN_6^2(1 + 4q + \frac{k}{N_5-k-q})\right), \\
n_h &= \sum_{r=1}^{k} r(r+1)N_6^2 + \left(k^2N_6^2 q + kN_6^2 + \frac{k^2N_6^2}{N_5-k-q}\right) \\
&\quad + \sum_{r=1}^{N_5-k-q-1} \frac{k^2N_6^2}{(N_5-k-q)^2}(N_5-k-q-r)(N_5-k-q-r-1) \\
&= \frac{k^2N_6}{3}\left(5 + k(N_5 + 2q + \frac{5}{N_5-k-q})\right),
\end{aligned}
\tag{1.204}
$$

which lead to a central charge

$$
c = \frac{1}{12\pi}\left[2 + 2kN_6^2 + N_5(k^2N_6^2 - 2) + 2k^2N_6^2(q + \frac{1}{N_5-k-q})\right]. \tag{1.205}
$$

At leading order, when $N_5 \to \infty$ and $N_6 \to \infty$, we have

$$
c = \frac{k^2N_6^2N_5}{12\pi} \tag{1.206}
$$

We can check all this by performing holographic computations using again the equations (1.67), (1.70) and (1.88). We find

$$
\begin{aligned}
N_{D4} &= \frac{kN_6}{2}(N_5 + q), \\
N_{D6} &= \frac{N_6(N_5 - q)}{N_5 - k - q},
\end{aligned}
\tag{1.207}
$$

for the number of $D4$ and $D6$ branes, while for the central charge

$$
c = \frac{k^2N_6^2N_5}{12\pi}. \tag{1.208}
$$

The linking numbers for the Hanany-Witten set up in Figure 1.15 are easily seen to be

$$
K_i = -\frac{kN_6}{N_5-k-q}, \quad L_j = k \quad L_n = k + q, \tag{1.209}
$$

where $i = 1, \dots, N_5$, $j = 1, \dots, N_6$ and $n = 1, \dots, kN_6/(N_5 - K - q)$.

Again, we can easily see that the condition (1.72) is satisfied. The holographic linking numbers for the $NS5$ and $D6$ branes are determined using (1.75) and (1.78). With the help of the λ profile in equation (1.202) we find

$$\sum_{i=1}^{N_5} K_i = \frac{2}{\pi}\mu^6\lambda'(\eta_f)\eta_f = \frac{2}{\pi}\mu^6\frac{KN_cN_5}{K+q-N_5} = -\sum_{j=1}^{N_6} L_j - \sum_{n=1}^{N_5N_6/N_5-K-q} L_n\,. \tag{1.210}$$

F Black Holes in Gaiotto Maldacena Backgrounds

In this appendix we will consider the generic Gaiotto-Maldacena class of geometries given in (1.33) with a Schwarzschild black hole profile solution in the AdS sector. In particular, the background metric reads

$$\frac{ds^2_{10}}{\alpha'\mu^2} = 4f_1\left(-r^2g(r)dt^2 + \frac{dr^2}{r^2g(r)} + r^2d\vec{x}^2\right) + \frac{ds^2_{\text{int}}}{\alpha'\mu^2}\,, \tag{1.211}$$

where ds^2_{int} is given by

$$\frac{ds^2_{\text{int}}}{\alpha'\mu^2} = f_2(d\sigma^2 + d\eta^2) + f_3 ds^2_{S^2}(\chi,\xi) + f_4 d\beta^2\,, \tag{1.212}$$

while $g(r)$ is the blackening factor whose precise form is determined by the equations of motion. The functions $f_i(i = 1\ldots 4)$ are still given in equation (1.36), while $\vec{x}$ is a vector in $\mathbb{R}^3$.

The dilaton equation of motion gives a simple equation for the function $g(r)$,

$$r^2g''(r) + 10rg'(r) + 20g(r) - 20 = 0\,. \tag{1.213}$$

The general solution for the equation (1.213) is

$$g(r) = 1 - \frac{c_1}{r^4} + \frac{c_2}{r^5}\,. \tag{1.214}$$

The Einstein equations for the background metric (1.211) force c_2 to be zero, leaving c_1 undetermined. As usual, the potential $V(\sigma,\eta)$ appearing in the various functions f_i still satisfies the same Laplace-like equation (1.32). In order to have a sensible black hole profile for the generic class of geometries we are considering, we will set c_1 to be r_h^4, with r_h being the size of the horizon. The blackening factor $g(r)$ then takes the standard form

$$g(r) = 1 - \frac{r_h^4}{r^4}\,. \tag{1.215}$$

It is now straightforward to compute the temperature of such a black hole. This is given by the general formula

$$T = \frac{1}{2\pi}\sqrt{-\frac{1}{4}g^{tt}g^{rr}(\partial_r g_{tt})^2}\,. \tag{1.216}$$

Evaluating (1.216) on the background (1.211) we get

$$T = \frac{r_h}{\pi}\,. \tag{1.217}$$

Let us now compute the entropy S for this back hole solution. This is given by the standard BH relation

$$S = \frac{A}{4}\,, \tag{1.218}$$

where A is the area of the black hole horizon. This reads

$$A = \int d^8x\sqrt{\tilde{g}_8}\,, \tag{1.219}$$

where $d^8x = d^3\vec{x}d\sigma d\eta d\chi d\xi d\beta$ and $\tilde{g}_8$ is the determinant of the eight-dimensional sub-space in Einstein frame. It is easy to see that S is given by

$$S = 16\pi^2\mathrm{vol}(\mathbb{R}^3)r_h^3\int d\sigma d\eta\sqrt{e^{-4\phi}f_1^3\det g_{\mathrm{int}}}\,, \tag{1.220}$$

where $\det g_{\mathrm{int}} = f_2^2 f_3^2 f_4$. The integral in equation (1.220) reduces to that in equation (1.88): Being both the entropy and the central charge extensive quantities counting the degrees of freedom of the theory, they must be proportional to each other.

G Detailed construction of the deformed backgrounds

In this appendix, we give details about the construction of our new backgrounds in Section 1.3.

G.1 The construction in eleven dimensions

Here, we will derive the gamma-deformed background of Subsection 1.3.2 following the rules discussed in [79]. Let us define the doublet

$$B^a = \begin{pmatrix} A^a \\ -\frac{1}{2}\epsilon^{abc}C_{(1)bc} \end{pmatrix}, \tag{1.221}$$

where A^a and $C_{(1)bc}$ are defined in equation (1.135) and (1.136). For this particular background $C_{(2)}$ and $g_{\mu\nu}dx^\mu dx^\nu$ are invariant under gamma-deformation, while $C_{(3)}$ is identically vanishing and therefore not subjected to any transformation. A non trivial transformation can possibly affect A^a, $C_{(0)}$ and $C_{(1)ab}$ as we discuss below.

According to the rules of [79], the doublet B^a defined above transforms under gamma deformation in the following way

$$B^a \to \Lambda^{-T} B^a\,, \tag{1.222}$$

where $\Lambda \in Sl(2,\mathbb{R})$ is given by

$$\Lambda = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}. \tag{1.223}$$

Here γ is the parameter of the deformation. It is not difficult to see that the only (eight-dimensional) vector transforming is A^a. It transforms in the following way

$$A^a \to A^a = \frac{1}{2}\gamma\epsilon^{abc}C_{(1)bc} \tag{1.224}$$

and, in particular, we have for the transformed A^a

$$A^1 = 0\,, \quad A^2 = -\gamma C_{(1)\xi y} \equiv -\gamma\kappa F_7 \sin\chi d\chi\,, \quad A^3 = \gamma C_{(1)\xi\beta} \equiv \gamma\kappa F_6 \sin\chi d\chi\,. \tag{1.225}$$

Moreover the τ parameter, defined as $\tau \equiv -C_{(0)} + i\widehat{\Delta}^{1/2}$, undergoes a non trivial transformation given by $\tau \to \tau/(1+\gamma\tau)$. This in turn implies[25]

$$\widehat{\Delta} \to \frac{\widehat{\Delta}}{(1+\gamma^2\widehat{\Delta})^2}\,, \quad C_{(0)} \to -\frac{\gamma\widehat{\Delta}}{1+\gamma^2\widehat{\Delta}}\,. \tag{1.226}$$

Inserting these new definitions for the fields into the general equation (1.134) the

[25] In order to get (1.226), we have taken into account that $C_{(0)}$ is equal to zero before the transformation, see (1.136).

background metric and the three-form $A_{(3)}$ take the form

$$\begin{aligned}\frac{ds^2}{\kappa^{2/3}} &= (1+\gamma^2\widehat{\Delta})^{1/3}\left(4F_1 ds^2_{AdS_5} + F_2(d\sigma^2+d\eta^2) + F_3 d\chi^2\right)\\ &\quad + (1+\gamma^2\widehat{\Delta})^{-2/3}\left(F_3\sin^2\chi d\xi^2 + F_4\tilde{\mathcal{D}}\beta^2 + F_5(\tilde{\mathcal{D}}y + \tilde{A}\tilde{\mathcal{D}}\beta)^2\right),\\ A_{(3)} &= \kappa(F_6\tilde{\mathcal{D}}\beta + F_7\tilde{\mathcal{D}}y)\wedge\widehat{\mathrm{vol}}_{S^2} - \frac{\gamma\widehat{\Delta}}{1+\gamma^2\widehat{\Delta}}d\xi\wedge\tilde{\mathcal{D}}\beta\wedge\tilde{\mathcal{D}}y\,,\end{aligned} \tag{1.227}$$

consistent with equation (1.138).

G.2 The TsT transformation of the Gaiotto-Maldacena solution in type IIB

The purpose of this appendix is to provide the details of the construction of the TsT transformed GM solution studied in Subsection 1.3.2, following [16]. The starting point is the type IIB solution in equation (1.142) obtained by performing a T-duality on the GM solution of equation (1.33), (1.34) and (1.35) along the isometric β direction,

$$\begin{aligned}\frac{ds^2}{\alpha'\mu^2} &= 4f_1 ds^2_{AdS_5} + f_2(d\sigma^2+d\eta^2) + f_3(d\chi^2+\sin^2\chi d\xi^2) + f_4^{-1}d\beta^2\,,\\ B_{(2)} &= \mu^2\alpha' f_5\widehat{\mathrm{vol}}_{S^2}\,,\quad e^{2\phi} = \frac{f_8}{\mu^2 f_4}\,,\\ C_{(2)} &= \mu^6\alpha' f_7\widehat{\mathrm{vol}}_{S^2}\,,\quad C_{(0)} = \mu^4 f_6\,.\end{aligned} \tag{1.228}$$

Moreover, following [16], any configuration in IIB supergravity with $U(1)\times U(1)$

isometry can be conveniently written in the form

$$
\begin{aligned}
\frac{ds^2_{IIB}}{\alpha'\mu^2} &= \frac{F}{\sqrt{\Delta}}(D\varphi^1 - CD\varphi^2)^2 + F\sqrt{\Delta}(D\varphi^2)^2 + \left(\frac{e^{2\phi/3}}{F^{1/3}}\right)\mathcal{G}_{\mu\nu}dX^\mu dX^\nu\,,\\
\frac{B_{(2)}}{\alpha'\mu^2} &= B_{12}D\varphi^1\wedge D\varphi^2 + \left[B_{1\mu}(D\varphi^1) + B_{2\mu}(D\varphi^2)\right]\wedge dX^\mu - \frac{1}{2}A^m_\mu B_{m\nu}dx^\mu\wedge dx^\nu\\
&\quad + \frac{1}{2}\tilde{b}_{\mu\nu}dx^\mu\wedge dx^\nu\,,\quad e^{2\phi_B} = e^{2\phi}\\
\frac{C_{(2)}}{\alpha'\mu^6} &= C_{12}D\varphi^1\wedge D\varphi^2 + \left(C_{1\mu}D\varphi^1 + C_{2\mu}D\varphi^2\right)\wedge dX^\mu\\
&\quad - \frac{1}{2}A^m_\mu C_{m\nu}dx^\mu\wedge dx^\nu + \frac{1}{2}\tilde{c}_{\mu\nu}dx^\mu\wedge dx^\nu\,,\quad C_{(0)} = \mu^4 A_0\,,\\
\frac{C_{(4)}}{\alpha'^2\mu^8} &= -\frac{1}{2}(\tilde{d}_{\mu\nu} + B_{12}\tilde{c}_{\mu\nu} - \epsilon^{mn}B_{m\mu}C_{\mu\nu} - B_{12}A^m_\mu C_{m\nu})dx^\mu\wedge dx^\nu\wedge D\varphi^1\wedge D\varphi^2\\
&\quad + \frac{1}{6}(C_{\mu\nu\lambda} + 3(\tilde{b}_{\mu\nu} + A^1_\mu B_{1\nu} - A^2_\mu B_{2\nu})C_{1\lambda})dx^\mu\wedge dx^\nu\wedge dx^\lambda\wedge D\varphi^1\\
&\quad + d_{\mu_1\mu_2\mu_3\mu_4}dx^{\mu_1}\wedge dx^{\mu_2}\wedge dx^{\mu_3}\wedge dx^{\mu_4} + \widehat{d}_{\mu_1\mu_2\mu_3}dx^{\mu_1}\wedge dx^{\mu_2}\wedge dx^{\mu_3}\wedge D\varphi^2,
\end{aligned}
\tag{1.229}
$$

where the indices $m, n = 1, 2$ and all the quantities above defining the fields in the solution are dimensionless quantities. The coordinates $\varphi^{1,2}$ are the two isometric coordinates associated with the two-torus and

$$
D\varphi^1 = d\varphi^1 + \mathcal{A}^{(1)}_\mu dx^\mu,\qquad D\varphi^2 = d\varphi^2 + \mathcal{A}^{(2)}_\mu dx^\mu. \tag{1.230}
$$

For the solution in equation (1.228) we identify $\varphi^1 = \beta$ and $\varphi^2 = \xi$. A direct comparison between (1.228) and (1.229) leads to the following identifications

$$
\begin{aligned}
\mathcal{G}_{\mu\nu}dX^\mu dX^\nu &= e^{-2\phi/3}F^{1/3}\left(4f_1 ds^2_{AdS_5} + f_2(d\sigma^2 + d\eta^2) + f_3 d\chi^2\right),\\
F &= \sqrt{\frac{f_3}{f_4}}\sin\chi,\quad \sqrt{\Delta} = \sqrt{f_3 f_4}\sin\chi,\quad e^{2\phi_B} = e^{2\phi} = \frac{f_8}{\mu^2 f_4},\\
B_{2\chi} &= -f_5\sin\chi\,,\quad C_{2\chi} = -f_7\sin\chi\,,\quad C_0 = A_0 = f_6\,,
\end{aligned}
\tag{1.231}
$$

with the remaining quantities in the solution set to zero. We are now in a position to apply the standard TsT transformation rules [16] to the type IIB background expressed above in equation (1.228). The $Sl(3,\mathbb{R})$ transformation is applied with

$$\Lambda = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We then group the different components of the fields in the solution of equation (1.229) according to their transformation under $Sl(3,\mathbb{R})$. For the scalar sector, the transformed fields are given in terms of the following matrix elements [78],

$$\begin{aligned} &g_{11}^T = \frac{e^{-\phi/3}}{F^{1/3}}\sqrt{1+\gamma^2F^2}\,, \quad g_{12}^T = \frac{\gamma e^{-\phi/3}F^{5/3}}{\sqrt{1+\gamma^2F^2}}\,, \quad g_{22}^T = \frac{e^{-\phi/3}F^{2/3}}{\sqrt{1+\gamma^2F^2}}\,, \\ &g_{31}^T = \frac{e^{2\phi/3}A_0}{F^{1/3}}\,, \quad g_{32}^T = 0\,, \quad g_{33}^T = \frac{e^{2\phi/3}}{F^{1/3}}, \end{aligned} \tag{1.232}$$

In particular, the metric components and the dilaton transform according to

$$\begin{aligned} &F' = \frac{g_{22}^T}{g_{11}^T} = \frac{F}{1+\gamma^2F^2} = \frac{\sqrt{f_3f_4}\sin\chi}{f_4+\gamma^2 f_3\sin^2\chi}\,, \quad \Delta^{(TsT)} = \Delta = f_3f_4\sin^2\chi\,, \\ &e^{2\phi'} = \left(\frac{g_{33}^T}{g_{11}^T}\right)^2 = \frac{e^{2\phi}}{1+\gamma^2F^2} = \frac{f_8}{\mu^2(f_4+\gamma^2 f_3\sin^2\chi)}\,. \end{aligned} \tag{1.233}$$

Moreover, the non-zero components of the NS two-form,

$$B'(2) = B'_{12}(D\varphi^1)'\wedge(D\varphi^2)' + B'_{2\chi}(D\varphi^2)'\wedge d\chi, \tag{1.234}$$

have the following transformation rules

$$B'_{12} = \frac{g_{12}^T}{g_{11}^T} = \frac{\gamma F^2}{1+\gamma^2F^2} = \frac{\gamma f_3\sin^2\chi}{f_4+\gamma^2f_3\sin^2\chi}\,, \quad B'_{2\chi} = B_{2\chi} = -f_5\sin\chi\,, \tag{1.235}$$

whilst

$$(D\varphi^1)' = d\beta + (\mathcal{A}^1_\chi)'d\chi = d\beta - \gamma f_5\sin\chi d\chi, \quad (D\varphi^2)' = d\xi\,. \tag{1.236}$$

The RR potentials, on the other hand, could be formally expressed as

$$\begin{aligned} &A'_0 = \left(\frac{g_{22}^Tg_{11}^T}{g_{33}^T}\right)^{1/2} g_{31}^T = A_0 = f_6\,, \\ &C'_2 = C'_{12}(D\varphi^1)'\wedge(D\varphi^2)' + C'_{2\chi}(D\varphi^2)'\wedge d\chi\,, \end{aligned} \tag{1.237}$$

where the components of the 2-form RR potential transform as

$$\begin{aligned} C'_{12} &= A'_0 B'_{12} - g^T_{32} g^T_{22} g^T_{11} = \frac{\gamma f_3 f_6 \sin^2\chi}{f_4 + \gamma^2 f_3 \sin^2\chi}\,, \\ C'_{2\chi} &= C_{2\chi} = -f_7 \sin\chi\,. \end{aligned} \tag{1.238}$$

The TsT transformed solution is given by equation (1.229) by replacing the original fields by the transformed ones. The final result is the one given in equation (1.144) in the main text.

Chapter 2

— Spin 2 fluctuations in $\frac{1}{4}$ BPS AdS$_3$/CFT$_2$ —

In this chapter we study spin 2 fluctuations around a class of gravity backgrounds with an AdS_3 factor first introduced in [17]. General fluctuations of the supergravity fields are usually classified according to the maximal bosonic subalgebra realised by the background. Because of the holographic duality, such fluctuations are mapped to operators in the dual Quantum Field Theory. The idea is then to classify a (small) part of the spectrum of the operators in the two-dimensional dual Field Theory via holography.

1 Introduction

Superconformal Field Theories (SCFTs) in diverse dimensions, and with different number of supersymmetries, have been object of intense study in the past years and still constitute a rich and fruitful subject. Aside from being interesting in their own right, they play a crucial role in the AdS/CFT duality. In 1997, Maldacena [89] conjectured that d-dimensional SCFTs are dual to AdS_{d+1} backgrounds and since then the AdS/CFT duality has provided a powerful tool to make strongly coupled CFTs more tractable. We have seen an example of holographic duality at work in Chapter 1.

Over the years, very useful has proved to be the correspondence between SCFTs in $d > 2$ with 8 Poincare supercharges and their holographic duals. For instance, the $\mathcal{N} = 4$ three-dimensional field theories studied in [90, 91] have been explored from a holographic perspective in e.g. [92, 93, 94, 72]. In four dimensions, the A-type of quivers of [14] corresponding to $\mathcal{N} = 2$ supersymmetry, already solved in [26], found a holographic realisation in [13]. Further holographic studies were given in e.g. [34, 35, 1] (this was amply discussed in the previous chapter). Also five dimensional SCFTs with

8 supercharges have found a holographic realisation, see for instance [95, 96, 97, 98] while in six dimensions $\mathcal{N} = (0, 1)$ SCFTs were addressed from a QFT and holographic point of view in many papers, see for instance [99, 100, 38].

The case of two-dimensional SCFTs is particularly interesting, as they are intrinsically different from SCFTs in $d > 2$. First of all, their (superconformal) algebra is infinite-dimensional [101]. This makes them much more easy to analyse and, sometimes, they can even be solved exactly [102]. Secondly, they find many applications in string and quantum field theory, e.g. two-dimensional SCFTs make their appearance when quantising the super-Polyakov action, but they also offer a description of several critical phenomena.

In the present chapter we focus on two-dimensional SCFTs with $\mathcal{N} = (0, 4)$ supersymmetry. Their (infinite-dimensional) superconformal algebra was constructed in [103], and studied further in the subsequent papers [104, 105]. By virtue of AdS/CFT, $\mathcal{N} = (0, 4)$ two-dimensional SCFTs are supposed to be dual to type II supergravity backgrounds with an AdS_3 factor. In fact, an infinite family of new solutions in type IIA supergravity with an $AdS_3 \times S^2$ factor, preserving $\mathcal{N} = (0, 4)$ supersymmetry, was recently built in [18, 19] and further explored in [106]. All these solutions relied on the local construction given in [17] which classify local solutions in massive IIA with an $AdS_3 \times S^2$ factor and an $SU(2)$ structure. The authors of [18, 19, 106, 17] identified the backgrounds that are dual to the IR limit of a special class of long quivers. These quivers are, in turn, made of two families of $\mathcal{N} = (4, 4)$ linear quivers coupled by matter fields. We will introduce such quantum field theories and review their main features in Section 2 and Appendix J.

An important part of the study of a class of SCFTs is the spectrum of operators, and understanding how they fit into representations of the superalgebra is a challenging and stimulating problem. In a recent work [107], multiplets for the two-dimensional $\mathcal{N} = 4$ superconformal algebra have been built. These multiplets fall into short and long multiplets. The authors of [107] were mainly interested in applications of two-dimensional $\mathcal{N} = 4$ superconfomal algebra to a numerical bootstrap study. Here, we will rely on their results concerning representations of the $\mathcal{N} = 4$ superconfomal algebra to study holographically the spectrum of operators.

In AdS/CFT, the linearised fluctuations of the supergravity background capture the spectrum of the dual gauge-invariant superconformal operators. Therefore, the main motivation of the present work is to study (some) fluctuations around the backgrounds first presented in [18], in order to holographically reproduce (some of) the spectrum of operators already found in [107]. Constructing linearised fluctuations for the full supergravity background is not an easy task[1]. However, as noticed in [110], for the case of sole spin 2 fluctuations the problem simplifies considerably. It turns out that spin 2 fluctuations, which are given in terms of perturbations of the backgroud metric, solve an equation that depends only on the underlying geometry of the background. This strategy has been applied succesfully in e.g. [111, 112, 113, 114, 115, 116, 117, 118, 119] for the case of four-, five- and six-dimensional SCFTs. We will follow a similar path for the case of a warped AdS_3.

This chapter is organised as follows. In Section 2 we briefly review the background presented in [18] along with the dual CFT interpretation further explored in [19, 106]. In Section 3 we derive an equation for spin 2 fluctuations of the metric background. These are transverse and traceless fluctuations along the AdS_3 part of the geometry and correspond to massive rank-2 tensors. In Section 4 we identify a particular class of solutions. These are the *universal* type of solutions, as they are independent of the background data. As we will see, they are also *minimal* solutions as they correspond to spin 2 fluctuations for which the mass of the graviton is the minimum possible in terms of the angular momentum on the S^2. In Section 5 we discuss the implications for the dual field theory. In particular, we will see that the universal solution corresponding to massless gravitons is dual to the energy momentum tensor operator of the dual field theory. Finally, in Section 6, we will see how to compute the central charge for the $\mathcal{N} = (0, 4)$ long quiver of [19] from the action of the spin 2 fluctuations. We give our conclusions in Section 7. In the Appendices, we give an example of non universal solution (dependent on the background data) and spell out the algebra and superfield construction of $\mathcal{N} = (0, 4)$ two-dimensional superconformal field theories.

2 The gravity backgrounds and dual field theories

In this section we review the global class of solutions first presented in [18] as well as the proposed dual field theories. These new backgrounds are solutions to massive IIA supergravity and have the structure of a warped $AdS_3 \times S^2 \times \mathrm{CY}_2 \times \mathcal{I}_\rho$, with $\mathcal{I}_\rho$ an interval parametrised by a coordinate labelled ρ.

[1] See the seminal papers [108, 109] where the full Kaluza-Klein spectrum was obtained for the non-warped cases of $AdS_5 \times S^5$ and $AdS_3 \times S^3$.

The holographic backgrounds

The N-S sector of the global class of solutions of [18] reads

$$ds^2 = \frac{u}{\sqrt{\hat{h}_4 h_8}}\left(ds^2_{AdS_3} + \frac{\hat{h}_4 h_8}{4\hat{h}_4 h_8 + (u')^2} ds^2_{S^2}\right) + \sqrt{\frac{\hat{h}_4}{h_8}} ds^2_{\text{CY}_2} + \frac{\sqrt{\hat{h}_4 h_8}}{u} d\rho^2\,,$$
$$e^{-\Phi} = \frac{h_8^{3/4}}{2\hat{h}_4^{1/4}\sqrt{u}}\sqrt{4\hat{h}_4 h_8 + (u')^2}\,, \quad H_{(3)} = \frac{1}{2} d\left(-\rho + \frac{uu'}{4\hat{h}_4 h_8 + (u')^2}\right) \wedge \widehat{\text{vol}}_{S^2} + \frac{1}{h_8} d\rho \wedge H_2\,. \tag{2.1}$$

Here Φ is the dilaton, $H_{(3)}$ the N-S three-form and the metric is given in string frame. H_2 is a two form whose explicit form was given in [17]. The functions u, $\hat{h}_4$, h_8 are functions only of the ρ coordinate.[2] A prime denotes a derivative with respect to ρ.

The R-R sector reads

$$F_{(0)} = h_8'\,, \quad F_{(2)} = -H_2 - \frac{1}{2}\left(h_8 - \frac{h_8' u' u}{4h_8\hat{h}_4 + u'^2}\right)\widehat{\text{vol}}_{S^2}\,,$$
$$F_{(4)} = \left(d\left(\frac{u'u}{2\hat{h}_4}\right) + 2h_8 d\rho\right) \wedge \widehat{\text{vol}}_{AdS_3} - \partial_\rho \hat{h}_4 \widehat{\text{vol}}_{\text{CY}_2} - \frac{u'u}{2(4\hat{h}_4 h_8 + (u')^2)} H_2 \wedge \widehat{\text{vol}}_{S^2}\,. \tag{2.2}$$

Higher R-R fluxes are related to $F_{(0)}$, $F_{(2)}$ and $F_{(4)}$ as usual as $F_{(6)} = -\star F_{(4)}$, $F_{(8)} = \star F_{(2)}$, $F_{(10)} = -\star F_{(0)}$, where $\star$ is the ten-dimensional Hodge-dual operator.

It was shown in [17] that supersymmetry is mantained when

$$u'' = 0\,, \qquad H_2 + \star_4 H_2 = 0\,, \tag{2.3}$$

where $\star_4$ is the Hodge-dual on the CY$_2$. In the following we will consider only that class of geometries with $H_2 = 0$. Away from brane sources, the Bianchi identities imply

$$h_8'' = 0\,, \qquad \hat{h}_4'' = 0\,. \tag{2.4}$$

Thus the three functions u, $\hat{h}_4$ and h_8 that appear as warping factors are at most linear in ρ[3]. This will lead to considerable simplifications in the following sections.

[2] A complication of this system is when $\hat{h}_4$ has support on (ρ, CY_2). The more general backgrounds deriving from this assumption are discussed in the original paper [17].

[3] Again, this is true away from brane sources. In the presence of branes, the rhs' of the two equations in (2.4) receive infinite contributions in the form of delta functions. This causes $\hat{h}_4$ and h_8 to be piecewise linear functions.

Following [19], we will be interested in the case of a finite interval $\mathcal{I}_\rho$ where both $\widehat{h}_4$ and h_8 vanish at both ends of the interval. So, to start fixing conventions, let us set $\mathcal{I}_\rho = [0, \rho^*]$ and $\widehat{h}_4(\bar\rho) = h_8(\bar\rho) = 0$, when $\bar\rho$ is equal to 0 and ρ^*. It is convenient [19] to set $\rho^* = 2\pi(P+1)$, with P a large integer. On the other hand, u vanishes only at $\rho = 0$. The general form for $\widehat{h}_4$, h_8 and u is then found to be

$$\widehat{h}_4(\rho) = \Upsilon \begin{cases} \frac{\beta_0}{2\pi}\rho & 0 \le \rho \le 2\pi \\ \beta_0 + \cdots + \beta_{k-1} + \frac{\beta_k}{2\pi}(\rho - 2\pi k) & 2\pi k < \rho \le 2\pi(k+1) \\ \alpha_P - \frac{\alpha_P}{2\pi}(\rho - 2\pi P) & 2\pi P < \rho \le 2\pi(P+1)\,, \end{cases} \tag{2.5}$$

$$h_8(\rho) = \begin{cases} \frac{\nu_0}{2\pi}\rho & 0 \le \rho \le 2\pi \\ \nu_0 + \cdots + \nu_{k-1} + \frac{\nu_k}{2\pi}(\rho - 2\pi k) & 2\pi k < \rho \le 2\pi(k+1) \\ \mu_P - \frac{\mu_P}{2\pi}(\rho - 2\pi P) & 2\pi P < \rho \le 2\pi(P+1)\,, \end{cases} \tag{2.6}$$

where $k = 1, \cdots, P-1$, and

$$u = \frac{b_0}{2\pi}\rho\,. \tag{2.7}$$

Here $\alpha_P = \sum \beta_k$ and $\mu_P = \sum \nu_k$ by continuity of $\widehat{h}_4$ and h_8.

The dual field theories

The background given in (2.1), (2.2) with $\widehat{h}_4$, h_8 and u as in (2.5), (2.6) and (2.7) was found [19] to be dual to the IR limit of the quiver in Figure 2.1. More precisely, the quiver in Figure 2.1 is supposed to flow in the IR to a fixed point whose dynamics is captured by the background above with the warping functions just described[4].

Let us spell out what the building blocks of such a quiver are.[5] An $U(N)$ gauge node of the quiver is denoted by (N): (β_0) stands for an $U(\beta_0)$ gauge node, $(\beta_0 + \beta_1)$ for an $U(\beta_0 + \beta_1)$ gauge node, and so on. There are two rows of gauge groups for the quiver in Figure 2.1. Associated with each gauge node there is a $(4,4)$ vector multiplet. $SU(F)$ flavour groups are denoted by $[F]$. Black lines represent $(4,4)$

[4]Originally, in [19] and [3], $(4,4)$ hypermultiplets between colour and flavour groups (curved black lines in Figure 2.1) were not contemplated. However, open string quantisation suggests that they should be part of the spectrum of massless fields. Further arguments in favour of the possibility of adding such hypermultiplets were given in [120].

[5]Basics of $\mathcal{N} = (0,2)$ and $\mathcal{N} = (0,4)$ superconformal field theories in 2 dimensions are reviewed in Appendix B. For a complete treatment see [121]. Very useful are also [122, 123, 124, 125].

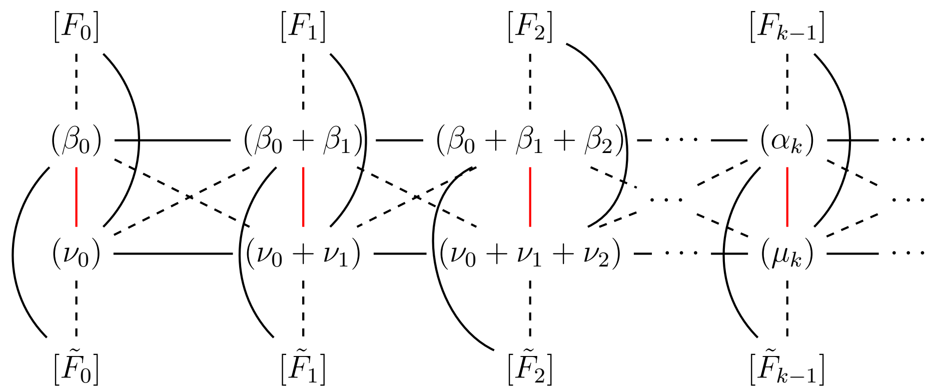

Figure 2.1: The generic quiver whose IR is captured by the background in (2.1) and (2.2). Each gauge node is associated with a $(4,4)$ vector multiplet. Black lines represent $(4,4)$ hypermultiplets. Red lines represent instead $(0,4)$ hypermultiplets and dashed lines $(0,2)$ Fermi multiplets.

hypermultiplets. They transform in the bifundamental representation of the groups they are attached to. Red lines represent $(0,4)$ hypermultiplets, while dashed lines are $(0,2)$ Fermi multiplets. They also carry one fundamental and one anti-fundamental index of the groups they are attached to. The F's and $\tilde{F}$'s are not independent of the other numbers of the quiver: as noticed in [19] the theory is chiral and might suffer from gauge anomalies. The F's and $\tilde{F}$'s can be chosen in such a way gauge anomalies cancel out at each gauge node of the quiver. A straightforward calculation (see [19]) leads to

$$F_{k-1}=\nu_{k-1}-\nu_k\,,\quad \tilde{F}_{k-1}=\beta_{k-1}-\beta_k\,. \tag{2.8}$$

3 Spin 2 fluctuations on $AdS_3\times S^2\times \mathbf{CY}_2\times \mathcal{I}_\rho$

In this section we aim at studying massive spin 2 fluctuations of the AdS_3 metric in (2.1). As we will see, they are composed of a transverse, traceless part along the AdS_3 direction and a scalar mode along the internal manifold. The goal of this section is to find the equation that such a metric fluctuation should solve. Explicit solutions will be given in the following sections.

As mentioned in the introduction, the study of the full KK-spectrum of the warped AdS_3 background in (2.1) and (2.2) is not an easy task. However, in [110] it has been shown that, in the case of a warped AdS_4, the equations for the fluctuation of the metric decouple from all other fluctuations. Moreover, they solve a ten dimensional Laplace equation which depends only on the background metric (such equation will be given later in this section). The analysis done in [110] can be extended to any warped background with an AdS factor, and it is straightforward to apply it to the case we are interested in, namely spin 2 fluctuations of the warped $AdS_3 \times S^2 \times \mathrm{CY}_2 \times \mathcal{I}_\rho$.

Equation for spin 2 fluctuations

To begin with, let us consider the background metric of (2.1) in the Einstein frame. This is achieved, as usual, by multiplying the "string frame" metric of (2.1) by $e^{-\Phi/2}$, being Φ the dilaton. A useful and compact form for it is

$$ds^2 = f_1 e^{-\Phi/2} ds^2_{AdS_3} + \widehat{g}_{ab} dz^a dz^b \,, \tag{2.9}$$

where the warping factor f_1 and the "internal" metric are given by the following expressions

$$f_1 = \frac{u}{\sqrt{h_4 h_8}} \,, \quad \widehat{g}_{ab} dz^a dz^b = e^{-\Phi/2}\left(\frac{u\sqrt{h_4 h_8}}{4h_4 h_8 + (u')^2} ds^2_{S^2} + \sqrt{\frac{h_4}{h_8}} ds^2_{\mathrm{CY}_2} + \frac{\sqrt{h_4 h_8}}{u} d\rho^2\right). \tag{2.10}$$

In the following we will take the CY to be a four-torus T^4 parametrised by 4 angles $\theta_i, (i = 1, \ldots, 4)$. Here, of course, $\theta_i \cong \theta_i + 2\pi$. Let us then consider a symmetric fluctuation h along the AdS_3 part of the ten-dimensional metric

$$ds^2 = f_1 e^{-\Phi/2}(ds^2_{AdS_3} + h_{\mu\nu} dx^\mu dx^\nu) + \widehat{g}_{ab} dz^a dz^b \,. \tag{2.11}$$

h can be decomposed into a transverse traceless fluctuation on AdS_3 and a mode on the internal manifold in the following manner

$$h_{\mu\nu}(x,z) = h^{[tt]}_{\mu\nu}(x)\psi(z). \tag{2.12}$$

Following [110], the transverse traceless fluctuation $h^{[tt]}_{\mu\nu}(x)$ satisfies the following equation of motion on AdS_3

$$\Box^{(2)}_{AdS_3} h^{[tt]}_{\mu\nu}(x) = (M^2 - 2) h^{[tt]}_{\mu\nu}(x) \,, \tag{2.13}$$

where $\Box^{(2)}_{AdS_3}$ is the Laplace operator acting on massive rank-two tensors in AdS_3, see e.g. [126]. The authors of [110] have shown that the linearised Einstein equations reduce to the ten dimensional Laplace equation

$$\frac{1}{\sqrt{-g}}\partial_M \sqrt{-g} g^{MN}\partial_N h_{\mu\nu} = 0\,. \tag{2.14}$$

For the background metric in (2.1), and with $h^{[tt]}_{\mu\nu}$ satisfying the equation (2.13), we get the following equation for the "internal mode" $\psi(z)$

$$\frac{(f_1 e^{-\Phi/2})^{-1/2}}{\hat{g}^{1/2}}\partial_a\left[(f_1 e^{-\Phi/2})^{3/2}\sqrt{\hat{g}}\hat{g}^{ab}\partial_b\right]\psi(z) = -M^2\psi(z)\,. \tag{2.15}$$

Expanding out equation (2.15) we find

$$\left[\left(4+\frac{(u')^2}{\hat{h}_4 h_8}\right)\nabla^2_{S^2} + \frac{u}{\hat{h}_4}(\partial^2_{\theta_1}+\partial^2_{\theta_2^2}+\partial^2_{\theta_3}+\partial^2_{\theta_4}) + \frac{1}{\hat{h}_4 h_8}\frac{d}{d\rho}(u^2\frac{d}{d\rho}) + M^2\right]\psi(z) = 0\,. \tag{2.16}$$

The function ψ can be conveniently decomposed into spherical harmonics on the S^2 and into plane waves on the T^4 in the following fashion

$$\psi = \sum_{l,m,n}\psi_{lmn} Y_{l,m} e^{in\cdot\theta}\,. \tag{2.17}$$

Here n is a shorthand notation for (n_1, n_2, n_3, n_4) and $n\cdot\theta = n_1\theta_1 + n_2\theta_2 + n_3\theta_3 + n_4\theta_4$. The n_i's are of course integers, in order for ψ to be single valued. Substituting (2.17) into (2.16) we get an equation for ψ_{lmn} which reads

$$\frac{1}{\hat{h}_4 h_8}\frac{d}{d\rho}\left(u^2\frac{d\psi_{lmn}}{d\rho}\right) - \left[\left(4+\frac{(u')^2}{\hat{h}_4 h_8}\right)l(l+1) + \frac{u}{\hat{h}_4}n^2 - M^2\right]\psi_{lmn} = 0\,. \tag{2.18}$$

It turns out to be useful to redefine $\psi_{lmn} = u^l\phi_{lmn}$. In this way, equation (2.18) becomes an equation for ϕ_{lmn}, which reads[6]

$$\frac{d}{d\rho}\left(u^{2(l+1)}\frac{d\phi_{lmn}}{d\rho}\right) - n^2\hat{h}_4 h_8 u^{2l}\left(\frac{u}{\hat{h}_4}\right)\phi_{lmn} = -(M^2 - 4l(l+1))\hat{h}_4 h_8 u^{2l}\phi_{lmn}\,, \tag{2.19}$$

or, in a more compact form,

$$S\phi_{lmn} + q(\rho)\phi_{lmn} = -\lambda w(\rho)\phi_{lmn}\,, \tag{2.20}$$

[6] To get the equation (2.19), we need to use $u'' = 0$, which is globally true.

with the differential operator S and the functions q and w given by

$$S = \frac{d}{d\rho}\left(p(\rho)\frac{d}{d\rho}\right)\,, \quad p(\rho) = u^{2(l+1)}\,, \quad q(\rho) = -n^2\widehat{h}_4 h_8 u^{2l}\left(\frac{u}{\widehat{h}_4}\right)\,, \quad w(\rho) = \widehat{h}_4 h_8 u^{2l}\,, \tag{2.21}$$

while the "eingenvalue" λ is

$$\lambda = M^2 - 4l(l+1)\,. \tag{2.22}$$

The equation (2.20), together with the definitions (2.21), defines a *Sturm-Liouville problem*[7]. As we discussed in Section 2, the variable ρ takes values in the finite interval $\mathcal{I}_\rho = [0, 2\pi(P+1)]$ and the function u vanishes only at $\rho = 0$. Therefore we have what in the mathematical literature is known as a singular Sturm-Liouville problem.

Notice also that with the substitution $d\rho/dt = u^{2(l+1)}$, with t a new variable, the equation (2.19) reduces to a Schrödinger-like equation. We will not be studying (2.19) in its Schrödinger form. Our starting point will be equation (2.19) and, as we will see in coming sections, solutions to that equation can be found.

4 Unitarity and a special class of solutions

In this section we will show how a bound for M^2 emerges from equation (2.19). For this bound, we find a particular class of solutions which will be dubbed "universal". Regularity conditions for the mode ψ will also be discussed.

A bound for M^2

To begin with, let us multiply (2.19) by ϕ_{lmn} and then integrate over ρ. The equation we get is

$$\int_{\mathcal{I}_\rho} d\rho\, \phi\frac{d}{d\rho}\left(u^{2(l+1)}\frac{d\phi}{d\rho}\right) - n^2\widehat{h}_4 h_8 u^{2l}\left(\frac{u}{\widehat{h}_4}\right)\phi^2 + (M^2 - 4l(l+1))\widehat{h}_4 h_8 u^{2l}\phi^2 = 0\,, \tag{2.23}$$

where ϕ stands for ϕ_{lmn}. Now, if we integrate by parts the first term we get

$$\int_{\mathcal{I}_\rho} d\rho\left(-\phi'^2 u^{2(l+1)} - n^2 h_8 u^{2l+1}\phi^2 + (M^2 - 4l(l+1))\widehat{h}_4 h_8 u^{2l}\phi^2\right) = -\phi\phi' u^{2(l+1)}\Big|_0^{\rho^*}\,. \tag{2.24}$$

[7]The three functions u, $\widehat{h}_4$ and h_8 are, of course, always positive definite, in order for the background metric in (2.1) to have the correct signature. Therefore $w(\rho)$ is always positive definite. This condition is necessary to have a well defined Sturm-Liouville problem.

Notice that $\phi\phi' u^{2(l+1)}$ vanishes when evaluated at $\rho = 0$ (as u does vanish at $\rho = 0$) as long as ϕ and ϕ' are regular there, while it doesn't when evaluated at $\rho = \rho^*$. In the following, we will focus on the Hilbert space of functions ϕ for which $\phi\phi' u^{2(l+1)}$ vanishes also at $\rho = \rho^*$. Thus, the equation (2.24) reduces to

$$\int_{\mathcal{I}_\rho} d\rho \left(\phi'^2 u^{2(l+1)} + n^2 h_8 u^{2l+1}\phi^2\right) = \left(M^2 - 4l(l+1)\right) \int_{\mathcal{I}_\rho} d\rho\, \widehat{h}_4 h_8 u^{2l}\phi^2 \,, \tag{2.25}$$

Given that u, $\widehat{h}_4$ and h_8 are non-negative, and the integrals finite, we find the following lower bound for M^2

$$M^2 \geq 4l(l+1)\,. \tag{2.26}$$

Universal minimal solution

Let us consider the case where $M^2 = 4l(l+1)$ and $n = 0$. Then, equation (2.19) simply reduces to

$$\frac{d}{d\rho}\Big(u^{2(l+1)}\frac{d\phi_{lm}}{d\rho}\Big) = 0\,, \tag{2.27}$$

which can be integrated to give $\phi'_{lm} = \text{constant}/u^{2(l+1)}$. However, for the class of geometries discussed in Section 2, u vanishes at $\rho = 0$ (it is in fact linear in ρ) and therefore ϕ'_{lm} is not finite at $\rho = 0$. This, in turn, implies that both ϕ_{lm} and $\psi_{lm} = u^l\phi_{lm}$ are not finite at $\rho = 0$ for any l. As we are looking for fluctuations that remain finite everywhere, the only acceptable solution to (2.27) is $\phi_{lm} = \text{constant}$. This in turn implies that

$$\phi_{lm} = \text{constant}\,, \qquad \psi_{lm} = \text{constant} \times u^l\,, \qquad M^2 = 4l(l+1)\,. \tag{2.28}$$

This class of solutions is independent of the form of u, $\widehat{h}_4$ and h_8 and in this sense they are "universal". Moreover, they are the solutions with minimal M for a given l, saturating the bound (2.26), and therefore correspond to "minimal" solutions.

The bound (2.26) for the mass of spin 2 excitations will prove to be very important when discussing quantum field theory implications. In particular, anticipating the discussion in Section 5, the spin 2 fluctuations considered are dual to operators in the field theory with dimension Δ given by the usual AdS/CFT formula, $M^2 = \Delta(\Delta - 2)$. The inequality (2.26) implies for the conformal dimension the following lower bound

$$\Delta \geq \Delta_{\min}\,, \qquad \Delta_{\min} = 2l + 2\,. \tag{2.29}$$

We leave a further discussion of the bound (2.29) for later.

For the case of non universal solutions, i.e. solutions for which $M^2 > 4l(l+1)$, it is necessary to specify what the three functions u, $\widehat{h}_4$ and h_8 are. The general form of these functions has been given in Section 2. An example of non universal solution to (2.19) will be discussed in the Appendix H.

5 Implications for the dual field theory

In this section we identify the operators dual to the spin 2 fluctuations that we have studied in the previous sections. Crucial for this would be the comparison of the spectrum of fluctuations with the spectrum of multiplets built in [107]. The analysis of [107] uses insights developed in [127, 128] for the construction of supermultiplets in $d \geq 3$ and is sketched in Appendix I.

Superconformal multiplets

For each fluctuation of the metric introduced above there is an operator in the dual SCFT. Therefore, we should aim at understanding what kind of operators these metric fluctuations correspond to. As already mentioned, the representations of $\mathcal{N}=4$ superconformal algebra in two dimensions were worked out in [107] and briefly sketched in Appendix I. These representations are labelled by the conformal weight h and $\tilde{h}$ of the $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ conformal algebra and the Dynkin index r of the $SU(2)_R$ R-symmetry. In particular, the scaling dimension Δ of any operator is usually given as the sum of its conformal weights, $\Delta = h+\tilde{h}$. The spin of such operators is determined as the difference between their conformal weights, $s = h-\tilde{h}$. Thus, a state in the superconformal algebra can be represented schematically as

$$[h,\tilde{h}]^{(r)}_{\Delta=h+\tilde{h}}\,. \tag{2.30}$$

The $SU(2)_R$ is realised on the supergravity side as the isometries of S^2. Thus, the quantum number on the S^2, l and m, are related to the R-charges of the corresponding dual operators. In particular, the Dynkin r, which is always an integer, is related to l by $r = 2l$. Therefore, in our construction r will always be a positive, even-integer.

As noted earlier, the mass of a spin 2 bulk field and the scaling dimension of the dual operator are related by the formula $M^2 = \Delta(\Delta-2)$. Thus, the minimal solution

(2.28), for which $M^2 = 4l(l+1)$, corresponds to operators with scaling dimension $\Delta = 2l+2$. Finally, we should stress that, for the type of fluctuations that we are studying, h and $\tilde{h}$ are not really independent. The $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ isometry of AdS_3 (plus gauge invariance) classifies $h_{\mu\nu}$ to have $h - \tilde{h} = \pm 2$.

Having identified all the quantum numbers using the standard holographic map, the (complex) spin 2 fluctuations correspond to operators labelled as

$$[h, h \pm 2]^{(2l)}_{\Delta = 2h \pm 2 = 2l+2} \,. \tag{2.31}$$

For $h = 2$, (2.31) comprises of $[2,0]^{(0)}_{\Delta = h = 2}$. These are the quantum numbers of the holomorphic stress energy tensor. As explained in [107] and in Appendix I, such a state arises as top component descendant in the short multiplet whose conformal primary has $r = 2l = 2$ and $h = r/2 = 1$. Notice also that choosing $h = 0$, (2.31) leads to $[0,2]^{(0)}_{\Delta = \tilde{h} = 2}$, the quantum numbers of the anti-holomorphic stress energy tensor.

For massive solutions with $l > 0$ the spin 2 universal fluctuations (2.28) either correspond to operators which sit as top components in a multiplet whose primary field has dimension $\Delta = 2l+1$, very much as explained in [107] and in Appendix I, or to operators obtained by tensoring chiral primaries in short multiplets with the anti-holomorphic sector of the algebra, just like the anti-holomorphic energy momentum tensor above.

It would be nice to understand how this operators are built from the fields of the SCFT at hand (the SCFTs represented by the quiver in Figure 2.1). More in particular, we expect the operators dual to (2.28) to be given by single traces of products of elementary fields in our SCFT. A step forward for this would be to identify the scalar primary $\mathcal{T}$ in the stress-energy tensor multiplet. This, in turn, can be "dressed" by other fields in order to get an operator whose scaling dimension Δ is equal to $2l+1$ and whose R-charge under the $SU(2)_R$ symmetry is $2l+2$. However, we should also take into account that the SCFTs at hand are inherently strongly coupled and a Lagrangian description for them might not be suitable.

6 Central charge from the spin 2 fluctuations

In this section we will briefly show a possible way to compute the central charge for the theories in (2.1), (2.2). To this end, we should compute the normalisation of the two-point function of the operators dual to the graviton fluctuations studied in Section 4. We have seen in Section 4 that the universal, minimal solution with $l = m = n = 0$ corresponds to a massless graviton and, therefore, the dual operator is the energy momentum tensor. The normalisation of the two-point function for the energy momentum tensor is read off from the effective action for the three-dimensional graviton.

Let us start from the type IIA action written schematically in the Einstein frame as

$$S_{\text{IIA}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}R + \cdots . \tag{2.32}$$

Expanded to second order, and following [117], it leads to an action for $h_{\mu\nu}$ which reads

$$S[h] = \frac{1}{\kappa_{10}^2}\int d^{10}x\sqrt{-g}\, h^{\mu\nu}\frac{1}{\sqrt{-g}}\partial_M\sqrt{-g}g^{MN}\partial_N h_{\mu\nu} + \text{boundary term}. \tag{2.33}$$

Expanding out (2.33) and dropping the boundary term which is not necessary in what follows, we get

$$S[h] = \frac{1}{\kappa_{10}^2}\int d^{10}x(-g_{AdS_3})^{\frac{1}{2}}(\widehat{g})^{\frac{1}{2}}(f_1e^{-\frac{\Phi}{2}})^{\frac{1}{2}}\, h^{\mu\nu}\left\{\Box^{(2)}_{AdS_3} + 2 + \widehat{\Box}\right\} h_{\mu\nu}\,, \tag{2.34}$$

where $\widehat{\Box}$ is the operator on the left-hand side of (2.15). Using the Ansatz[8] $h_{\mu\nu} = (h^{[tt]}_{lmn})_{\mu\nu}Y_{lm}\psi_{lmn}e^{in\cdot\theta}$ we find

$$S[h] = \sum_{lmn} C_{lmn}\int d^3x\sqrt{-g_{AdS_3}}\,(h^{[tt]}_{lmn})^{\mu\nu}\left\{\Box^{(2)}_{AdS_3} + 2 - M^2\right\}(h^{[tt]}_{lmn})_{\mu\nu} \tag{2.35}$$

where the coefficients C_{lmn} are given by[9]

$$C_{lmn} = \frac{16\pi^4}{\kappa_{10}}\int_{\mathcal{I}_\rho} d\rho\sqrt{\widehat{g}}\,(f_1e^{-\frac{\Phi}{2}})^{\frac{1}{2}}|\psi_{lmn}|^2\,. \tag{2.36}$$

The integral in (2.36) is finite for the class of solutions discussed in this paper, namely those fluctuations that are finite everywhere. In particular, if we specialise to the universal, minimal solution (2.28) with $l = m = n = 0$, i.e. $\psi_{lmn} = 1$, (2.36)

[8] Notice that we are using the subscripts l, m and n under h. This is because in some solutions (like the "universal" solution above) M^2 depends on those quantum numbers and so does $h^{[tt]}$ through equation (2.13).

[9] Using the standard normalisation $\int Y_{lm}Y_{l'm'} = \delta_{ll'}\delta_{mm'}$.

evaluated on (2.1) gives[10]

$$C_0 = \frac{1}{4\kappa_{10}^2}\text{vol}_{\text{CY}_2}\int_{\mathcal{I}_\rho} d\rho\, \widehat{h}_4 h_8\,. \tag{2.37}$$

The effective three-dimensional gravitational coupling κ_3 is related to C_0 by $C_0 = 1/\kappa_3^2$. The quadratic action for $h_{\mu\nu}$ computes the two point function of the dual stress tensor, whose coefficient is well known to be proportional to the central charge of the 2d CFT. In fact, (2.37) is equal, modulo an irrelevant numerical factor, to the central charge computed on pag. 12 of [19].

7 Conclusions

In this chapter we have investigated aspects of spin 2 fluctuations around the background $AdS_3 \times S^2 \times \text{CY}_2 \times \mathcal{I}_\rho$ of [17]. An equation for these spin 2 fluctuations has been derived in Section 4, following the general analysis of [110], and we have seen that they fall into two classes, universal and non universal solutions.

The universal solutions, discussed in Section 4, turned out to be particularly interesting, as they are independent of the background data. These fluctuations, and therefore the dual operators, are expected to be present for any of the backgrounds given in (2.1) and (2.2). As we have seen in the main text, they are dual to operators with scaling dimension $\Delta = 2l + 2$, where l is the angular-momentum-charge on the S^2 which realises holographically the $SU(2)_R$ symmetry of the dual field theory.

The non universal solutions are more difficult to analyse as they depend on background data, namely on a specific choice of the functions u, $\widehat{h}_4$ and h_8 given in (2.5), (2.6), (2.7). An example of these is worked out in Appendix H.

Finally, we have seen in Section 6 that the central charge c for the 2d dual quiver field theory can be read off from the normalisation of the action for the spin 2 fluctuations, $h_{\mu\nu}$. The central charge c is essentially determined from the 3-dimensional gravitational coupling constant κ_3, $c \propto \kappa_3^{-2}$. The quadratic actions for $l > 0$ modes compute the two point functions for the corresponding dual operators and, more in particular, the holographic normalisations of these operators. Moreover, computing

[10] Even though, in general AdS backgrounds, gravitons should be treated separately from massive spin 2 fields (see for instance [129]), the normalisation for the three dimensional massless graviton can still be obtained by taking $l = m = n = 0$

higher order interaction terms, one can then compute three point and higher point functions of these operators.

H Example of non universal solution

In this appendix we consider a particular solution to (2.19) which is not universal, namely a solution that does not saturate the bound $M^2 = 4l(l+1)$, still with $n = 0$. In order to solve (2.19), we will have to choose some particular u, $\widehat{h}_4$ and h_8 which, in turn, correspond to a particular background. Let us start off by considering the case of[11]

$$\widehat{h}_4(\rho) = \beta_0 \begin{cases} \rho/2\pi & 0 \le \rho \le \pi P \\ P - \rho/2\pi & \pi P < \rho \le 2\pi P \end{cases}, \qquad h_8 = \widehat{h}_4\,, \qquad u = \frac{\beta_0}{2\pi}\rho\,.$$

A solution to (2.19) must be split into two solutions in the two intervals $\mathcal{I}_I = [0, \pi P]$ and $\mathcal{I}_{II} = (\pi P, 2\pi P]$, as both $\widehat{h}_4$ and h_8 are only piecewise continuous. Moreover, in order to get a smooth solution for the fluctuations, we need to impose continuity of the solution and of its derivative at $\rho = \pi P$.

Equation (2.19) for u, $\widehat{h}_4$, h_8 as before looks like

$$\begin{aligned} &\phi''(\rho) + \frac{2l+2}{\rho}\phi'(\rho) + \lambda\phi(\rho) = 0 && \text{in} \quad \mathcal{I}_I\,, \\ &\phi''(\rho) + \frac{2l+2}{\rho}\phi'(\rho) + \lambda\frac{(P-\rho/2\pi)^2}{\rho^2}\phi(\rho) = 0 && \text{in} \quad \mathcal{I}_{II}\,, \end{aligned} \tag{2.38}$$

where again $\lambda = M^2 - 4l(l+1)$. The general solution of (2.38) in $\mathcal{I}_I$ reads $\phi = c_1\phi_1(\rho) + c_2\phi_2(\rho)$, with

$$\phi_1 = \rho^{-\frac{2l+1}{2}} J_{\frac{2l+1}{2}}\big(\sqrt{\lambda}\rho\big) \quad \text{and} \quad \phi_2 = \rho^{-\frac{2l+1}{2}} Y_{\frac{2l+1}{2}}\big(\sqrt{\lambda}\rho\big)\,. \tag{2.39}$$

J and Y are Bessel functions of the first and second kind, respectively and c_1 and c_2 are integration constants. In order for the solution (and its derivative) to be regular at $\rho = 0$ we must set $c_2 = 0$. On the other hand, the general solution to (2.38) in $\mathcal{I}_{II}$ can be given in terms of the complex function $\phi = \tilde{c}_3\tilde{\phi}_3 + \tilde{c}_4\tilde{\phi}_4$, with

$$\begin{aligned} \tilde{\phi}_3 &= e^{-i\sqrt{\lambda}\frac{\rho}{2\pi}}\rho^{-(l+1/2-i\frac{\gamma}{2})}U\big(\alpha,\gamma,i\sqrt{\lambda}\rho/\pi\big) \\ \tilde{\phi}_4 &= e^{-i\sqrt{\lambda}\frac{\rho}{2\pi}}\rho^{-(l+1/2-i\frac{\gamma}{2})}M\big(\alpha,\gamma,i\sqrt{\lambda}\rho/\pi\big)\,, \end{aligned} \tag{2.40}$$

[11] This is of course a particular example of equations (2.5), (2.6), (2.7).

where U and M are the Kummer's hypergeometric functions, respectively, and α and γ are two complex numbers given by[12]

$$\alpha = \frac{\gamma}{2} - i\sqrt{\lambda}\,, \qquad \gamma = 1 + i\{4M^2P^2 - 4l(l+1)(1+4P^2) - 1\}^{1/2}\,. \tag{2.41}$$

The functions $\tilde{\phi}_3$ and $\tilde{\phi}_4$ are always well defined in $\mathcal{I}_{II}$. Therefore neither $\tilde{c}_3$ nor $\tilde{c}_4$ must be set to zero. Moreover, we can always consider two independent real combinations of $\tilde{\phi}_3$ and $\tilde{\phi}_4$. Let us call them ϕ_3 and ϕ_4. Thus, in $\mathcal{I}_{II}$ the general solution reads $\phi = c_3\phi_3 + c_4\phi_4$, with ϕ_3 and ϕ_4 two real linearly independent functions built from $\tilde{\phi}_3$ and $\tilde{\phi}_4$ above.

We should now match the solution $\phi = c_1\phi_1$ with $\phi = c_3\phi_3 + c_4\phi_4$ at $\rho = \pi P$. This leads to two conditions

$$\phi\Big|_{\pi P^-} = \phi\Big|_{\pi P^+}\,, \quad \phi'\Big|_{\pi P^-} = \phi'\Big|_{\pi P^+} \tag{2.42}$$

As a further condition, we would like to impose that either ϕ or ϕ' vanishes at $\rho = 2\pi P$. This is nothing but the condition discussed around (2.24). Say, for instance, that is ϕ that vanishes at $\rho = 2\pi P$

$$\phi\Big|_{2\pi P} = 0\,. \tag{2.43}$$

We therefore get a system of three equations, (2.42) and (2.43), for three integration constants, c_1, c_3 and c_4. A straightforward calculation shows that such a system has a non trivial solution if and only if the following equation is satisfied

$$\det M = 0 \quad \text{with} \quad M = \begin{pmatrix} \phi_1\Big|_{\pi P} & -\phi_3\Big|_{\pi P} & -\phi_4\Big|_{\pi P} \\ \phi_1'\Big|_{\pi P} & -\phi_3'\Big|_{\pi P} & -\phi_4'\Big|_{\pi P} \\ 0 & \phi_3\Big|_{2\pi P} & \phi_4\Big|_{2\pi P} \end{pmatrix}\,. \tag{2.44}$$

Such an equation could be solved numerically for M^2. Even though we will not attempt at solving it, the expectation is to find a solution of the form

$$M^2 = 4l(l+1) + jf(j,l)\,, \quad j \in \mathbb{Z}_{\geq 0}\,, \tag{2.45}$$

where f is a generic positive function of j and l such that $f(0,l)$ is regular.

[12] The argument of the square root appearing in the definition of γ is positive in the limit of very large P. This is the regime well described by supergravity. Moreover, for generic M and l with $M^2 > 4l(l+1)$, γ is never a negative integer. Thus the Kummer's functions are always well defined.

I $\mathcal{N}=4$ superconformal algebra

In this appendix we review the small $\mathcal{N}=4$ superconformal algebra that was first derived in [103]. We will follow Section 2 of [107].

The algebra we are considering is a graded Lie algebra with an internal $SU(2)_R$ symmetry and reads

$$
\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{2}kn(n^2-1)\delta_{n,-m}\,,\\
&\{G^a_r, G^b_s\} = \{\bar{G}^a_r, \bar{G}^b_s\} = 0\,,\\
&\{G^a_r, \bar{G}^b_s\} = 2\delta^{ab}L_{r+s} - 2(r-s)\sigma_i^{ab}J^i_{r+s} + \frac{1}{2}k(4r^2-1)\delta_{r,-s}\delta^{ab}\,,\\
&[L_n, J^i_m] = -mJ_{n+m}\,,\\
&[L_n, G^a_r] = -\Big(\frac{1}{2}n - r\Big)G^a_{n+r}\,,\\
&[J^i_n, G^a_r] = \frac{1}{2}\sigma^i_{ab}G^b_{n+r}\,,\\
&[J^i_n, J^j_n] = i\epsilon^{ijk}J^k_{n+m} + \frac{1}{2}kn\delta_{n,-m}\,.
\end{aligned}
\tag{2.46}
$$

Here L_n and G^a_r are the generators of superconformal symmetry. G^a_r's carry an $SU(2)_R$ fundamental index, a, and therefore they form an $SU(2)$ complex doublet. J^i_n ($i=1,2,3$) are the $SU(2)_R$ Kac-Moody currents generating the corresponding Kac-Moody loop algebra. σ_i^{ab} are Pauli matrices. Indices n and m run over integer numbers while r belongs to $\mathbb{Z}+1/2$: only for the NS-sector there exists a finite dimensional subalgebra generated by L_0, $L_{\pm 1}$, $G^a_{\pm 1/2}$ and J^i_0 (see below).

In the following we will mainly be interested in the global part of the superconformal algebra. This reads

$$
\begin{aligned}
&[L_{+1}, L_{-1}] = 2L_0 \,, \quad [L_{\pm 1}, L_0] = \pm L_{\pm} \,, \\
&\{G^a_{\pm\frac{1}{2}}, G^b_{\pm\frac{1}{2}}\} = \{\bar{G}^a_{\pm\frac{1}{2}}, \bar{G}^b_{\pm\frac{1}{2}}\} = 0 \,, \\
&\{G^a_{+\frac{1}{2}}, \bar{G}^b_{-\frac{1}{2}}\} = 2\delta^{ab} L_0 - 2\sigma^{ab}_i J^i_0 \,, \quad \{G^a_{\pm\frac{1}{2}}, \bar{G}^b_{\pm\frac{1}{2}}\} = 2\delta^{ab} L_{\pm 1} \\
&\{G^a_{-\frac{1}{2}}, \bar{G}^b_{+\frac{1}{2}}\} = 2\delta^{ab} L_0 + 2\sigma^{ab}_i J^i_0 \,, \\
&[L_0, G^a_{\pm\frac{1}{2}}] = \mp \frac{1}{2} G^a_{\pm\frac{1}{2}} \,, \\
&[L_{\pm 1}, G^a_{\mp\frac{1}{2}}] = \pm G^a_{\pm\frac{1}{2}} \,, \\
&[J^i_0, G^a_{\pm\frac{1}{2}}] = -\frac{1}{2} \sigma^i_{ab} G^b_{\pm\frac{1}{2}} \,, \\
&[J^i_0, J^j_0] = i\epsilon^{ijk} J^k_0 \,.
\end{aligned}
\tag{2.47}
$$

A highest weight state of the superconformal algebra can be specified by the eigenvalues of the mutually commuting operators L_0, $\vec{J}_0^2$ and J^3_0, $|\mathcal{O}_{h,l}\rangle$, satisfying

$$
L_0|\mathcal{O}_{h,j}\rangle = h|\mathcal{O}_{h,j}\rangle \,, \quad \vec{J}_0^2|\mathcal{O}_{h,j}\rangle = j(j+1)|\mathcal{O}_{h,j}\rangle \quad J^3_0|\mathcal{O}_{h,j}\rangle = j|\mathcal{O}_{h,j}\rangle \,, \tag{2.48}
$$

as well as

$$
L_n|\mathcal{O}_{h,j}\rangle = G^a_r|\mathcal{O}_{h,j}\rangle = \bar{G}^a_r|\mathcal{O}_{h,j}\rangle = J^i_n|\mathcal{O}_{h,j}\rangle = 0 \,, \qquad n, r > 0 \,. \tag{2.49}
$$

The correspondence between the heighest weight state $|\mathcal{O}_{h,j}\rangle$ and the corresponding operator of conformal weight h and $SU(2)_R$ spin j is made as usual $|\mathcal{O}_{h,j}\rangle = \mathcal{O}_{h,j}|0\rangle$, where $|0\rangle$ is the conformal vacuum. In the following, it will make no difference the use of $\mathcal{O}_{h,j}$ or $|\mathcal{O}_{h,j}\rangle$.

The operators $G^a_{\frac{1}{2}}$ and $\bar{G}^a_{-\frac{1}{2}}$ can be used to derive the full module $\mathcal{L}_r$ from a superconformal primary state $|\mathcal{O}_{h,j}\rangle$. Being fermionic operators, they can act on a state $|\mathcal{O}_{h,j}\rangle$ until they annihilate. Thus the length of a module is finite and determined by Fermi statistics. In deriving the full modules, we should also make sure that the various representations are unitary. This will lead to shortening conditions and constraints on the allowed values of h and j, as we now shall see.

Singular vectors, short and long multiplets

The superconformal algebra constrains the values the conformal weight h can assume. In particular, in (super)conformal theories unitarity implies a lower bound for the scaling dimension of operators as a function of the other quantum numbers in the algebra. The details of the bound depend of course on the particular theory and corresponding algebra. Let us see how this works in our case[13].

Consider a superconformal primary state of conformal weight h and $SU(2)_R$ spin j and the fact that[14]

$$0 \le |\bar{G}^a_{-\frac{1}{2}}|\mathcal{O}_{h,j}\rangle|^2 + |G^a_{\frac{1}{2}}|\mathcal{O}_{h,j}\rangle|^2 = \langle\mathcal{O}_{h,j}|\{G^a_{\frac{1}{2}}, \bar{G}^a_{-\frac{1}{2}}\}|\mathcal{O}_{h,j}\rangle\,, \qquad \text{no sum over } a\,. \quad (2.50)$$

The superconformal algebra implies

$$0 \le \langle\mathcal{O}_{h,j}|\{G^a_{\frac{1}{2}}, \bar{G}^b_{-\frac{1}{2}}\}|\mathcal{O}_{h,j}\rangle = \langle\mathcal{O}_{h,j}|2L_0 - 2\sigma^{ab}_i J^i_0|\mathcal{O}_{h,j}\rangle = 2(h-j)\delta^{ab}\langle\mathcal{O}_{h,j}|\mathcal{O}_{h,j}\rangle\,. \quad (2.51)$$

To have a unitary theory we should then impose $h \ge j$. Sometimes it is customary to use the Dynkin index (r say) of the representation of the internal group $SU(2)_R$. In our case it is related to the spin j by $r = 2j$. In particular r is always an integer. This is the convention that has been used in [107].

Thus, the algebra implies a lower bound for the conformal weight h in terms of the other quantum number j. When $h = j$, the module gets shortened as there are null-states that need to be modded out. In particular, when the bound is satisfied there are two states that satisfy

$$\bar{G}^1_{-\frac{1}{2}}|\mathcal{O}_{h,j}\rangle = G^2_{-\frac{1}{2}}|\mathcal{O}_{h,j}\rangle = 0\,. \quad (2.52)$$

Therefore, only $\bar{G}^2_{-\frac{1}{2}}$ and $G^1_{-\frac{1}{2}}$ will produce new states. Multiplets of this kind are *short*.

Following [107], we just state the result of considering $h = j$ when $j = 1$ (or equivalently $r = 2$). This is the case of most relevance for our purposes as it will lead us to identify the supermultiplet to which the holomorphic energy momentum tensor belongs. A state with $h = j = 1$ can be labelled as $[h]^{(j)} = [1]^{(1)}$. The structure of the

[13]Unitarity for $\mathcal{N}=2$ and $\mathcal{N}=4$ algebras in two dimensions were also discussed in [130, 131, 132].
[14]Notice that the same conclusion can be reached by sandwiching $\{G^a_{-\frac{1}{2}}, \bar{G}^a_{\frac{1}{2}}\}$.

resulting short multiplet is

$$
\begin{array}{ccccc}
 & & \left[\frac{3}{2}\right]^{\left(\frac{1}{2}\right)} & & \\
 & \nearrow & & \searrow & \\
[1]^{(1)} & & & & [2]^{(0)} \\
 & \searrow & & \nearrow & \\
 & & \left[\frac{3}{2}\right]^{\left(\frac{1}{2}\right)} & &
\end{array}
$$

where $\nearrow$ stands for the action of $G_{-\frac{1}{2}}$, while $\searrow$ stands for the action of $\bar{G}_{-\frac{1}{2}}$.

As noticed in [107], the top component $[2]^{(0)}$ corresponds to the holomorphic energy momentum tensor.

Let us conclude by briefly mentioning the case where $h > j$. In this case there are no null-states and the supermultiplets do not get shortened. All the G's and $\bar{G}$'s contribute to produce new states from the corresponding conformal primary. Such multiplets are therefore *long*. We will not discuss long multiplets any further. A careful analysis can be found in [107].

J $\mathcal{N} = (0,2)$ and $\mathcal{N} = (0,4)$ theories

In this appendix we review some basic facts about $\mathcal{N} = (0,4)$ gauge theories. $\mathcal{N} = (0,4)$ superfields are made from $\mathcal{N} = (0,2)$ superfields, therefore we start by reviewing $\mathcal{N} = (0,2)$ gauge theories. For a complete discussion see [121].

$\mathcal{N} = (0,2)$ **multiplets** Let us list the field components of three types of $\mathcal{N} = (0,2)$ multiplets, namely the vector U, chiral Φ and the Fermi Ψ multiplets

$$\boxed{U : (u_\mu, \zeta_-, D)\,, \qquad \Phi : (\phi, \psi_+)\,, \qquad \Psi : (\psi_-, G)\,.} \tag{2.53}$$

The subscript on the fermions refers to their chiralities under $SO(1,1)$ Lorentz group. D is a real and G a complex auxiliary field.

A *vector* U has the following expansion in superspace[15]

$$U = u_0 - u_1 - 2i\theta^+\bar{\zeta}_- - 2i\bar{\theta}^+\zeta_- + 2\theta^+\bar{\theta}^+ D\,. \tag{2.54}$$

[15] $\mathcal{N} = (0,2)$ superspace is parametrised by two real spacetime coordinates, $x_\pm = x^0 \pm x^1$, and two complex Grassmann variables θ^+ and $\overline{\theta}^+$ subject to a reality constraint.

The corresponding field strength is obtained by means of

$$\Upsilon = [\overline{\mathcal{D}}_+, \mathcal{D}_-] = -\zeta_- - i\theta^+(D - iu_{01}) - i\theta^+\bar{\theta}^+(\mathcal{D}_0 + \mathcal{D}_1)\zeta_- \,, \tag{2.55}$$

where $\overline{\mathcal{D}}_+$ and $\mathcal{D}_-$ are the supercovariant gauge derivatives [121]. It turns out that Υ is a Fermi multiplet – it satisfies $\overline{\mathcal{D}}_+\Upsilon = 0$. We shall give a more precise definition of a Fermi multiplet momentarily.

Mass units for the $\mathcal{N}=(0,2)$ vector multiplet are as follows

$$[u_\mu] = m \quad [\zeta_-] = m^{3/2}\,, \quad [D] = m^2\,, \tag{2.56}$$

while we have $[\theta^+] = m^{-1/2}$ and therefore $[d\theta^+] = m^{1/2}$. The 1d gauge coupling g has unit mass dimension, $[g] = m$. The action for the gauge multiplet reads

$$\begin{aligned} S &= \frac{1}{8g^2}\mathrm{tr}\int d^2x d^2\theta \overline{\Upsilon}\Upsilon \\ &= \frac{1}{g^2}\mathrm{tr}\int d^2x \left(\frac{1}{2}u_{01}^2 + i\overline{\zeta}_-(\mathcal{D}_0 + \mathcal{D}_1)\zeta_- + D^2\right) . \end{aligned} \tag{2.57}$$

A *chiral field* Φ is a superfield that satisfies the following equation

$$\overline{\mathcal{D}}_+\Phi = 0\,, \tag{2.58}$$

and therefore expands out in components as

$$\Phi = \phi + \sqrt{2}\theta^+\psi_+ - i\theta^+\bar{\theta}^+(D_0 + D_1)\phi\,, \tag{2.59}$$

where D_0 and D_1 stand for the time- and space-components of the usual covariant derivative. Mass units for a chiral multiplet are as follows

$$[\phi] = m^0\,, \qquad [\psi_+] = m^{1/2}\,, \tag{2.60}$$

while a kinetic term for it reads

$$\begin{aligned} S &= -\frac{i}{2}\int d^2x d^2\theta \overline{\Phi}(\mathcal{D}_0 - \mathcal{D}_1)\Phi \\ &= \int d^2x \left(-|D_\mu\phi|^2 + i\overline{\psi}_+(D_0 - D_1)\psi_+ - i\sqrt{2}\overline{\phi}\zeta_-\psi_+ + i\sqrt{2}\overline{\psi}_+\overline{\zeta}_-\phi + \overline{\phi}D\phi\right) . \end{aligned} \tag{2.61}$$

A *Fermi superfield* instead satisfies the following equation

$$\overline{\mathcal{D}}_+\Psi = E(\Phi_i)\,, \tag{2.62}$$

where $E(\Phi_i)$ is a holomorphic function of the chiral superfields Φ_i. E should be chosen in such a way it transforms as Ψ under all symmetries. Solving (2.62) leads to the following expansion for Ψ

$$\Psi = \psi_- - \theta^+ G - i\theta^+\bar{\theta}^+(D_0+D_1)\psi_- - \bar{\theta}^+E(\phi_i) - \theta^+\bar{\theta}^+\frac{\partial E}{\partial\phi^i}\psi_{+i}\,, \tag{2.63}$$

where G is an auxiliary complex field. The holomorphic function E can be shown to appear in the Lagrangian as a potential term.

Mass units for a Fermi superfield are as follows

$$[\psi_-] = m^{1/2}\,, \qquad [G] = m\,, \tag{2.64}$$

while a kinetic term for it is given by

$$\begin{aligned} S &= -\frac{1}{2}\int d^2x d^2\theta\overline{\Psi}\Psi \\ &= \int d^2x\left(i\overline{\psi}_-(D_0+D_1)\psi_- + |G^2| - |E(\phi_i)|^2 - \overline{\psi}_-\frac{\partial E}{\partial\phi^i}\psi_{+i} - \overline{\psi}_{+i}\frac{\partial\overline{E}}{\partial\overline{\phi}_i}\psi_-\right)\,. \end{aligned} \tag{2.65}$$

There is also another type of superpontential we can consider for $\mathcal{N}=(0,2)$ theories. For each Fermi multiplet Ψ_a we can introduce a holomorphic function $J^a(\Phi_i)$ such that

$$\begin{aligned} S_J &= \int d^2x d\theta^+\sum_a J^a(\Phi_i)\Psi_a + \text{h.c.} \\ &= \sum_a\int d^2x\, G_aJ^a(\phi_i) + \sum_i\psi_{-a}\frac{\partial J^a}{\partial\phi_i}\psi_{+i} + \text{h.c.} \end{aligned} \tag{2.66}$$

We see that, in analogy to $\mathcal{N}=1$, $d=4$, $\mathcal{W}=J\cdot\Psi$ is integrated over half superspace.

It must be stressed that the superpotentials E and J cannot be introduced independently. It turns out that, in order for supersymmetry to be preserved, they have to satisfy the following constraint

$$E\cdot J = \sum_a E_aJ^a = 0\,. \tag{2.67}$$

Let us now move on to listing $\mathcal{N}=(0,4)$ supermultiplets. They are built from $\mathcal{N}=(0,2)$ supermultiplets.

$\mathcal{N}=(0,4)$ **multiplets** $\mathcal{N}=(0,4)$ supermultiplets are usually given in terms of $\mathcal{N}=(0,2)$ supermultiplets, pretty much as in 4 dimensions $\mathcal{N}=2$ superfields are built from $\mathcal{N}=1$ superfields. Again, let us list them first.

Multiplets	$\mathcal{N}=(0,2)$ building blocks	component fields	$SU(2)_L\times SU(2)_R$
Vector	Vector + Fermi (U,Θ)	(u_μ,ζ_-^a,G^A)	$(1,1),(2,2),(3,1)$
Hyper	Chiral + Chiral $(\Phi,\tilde{\Phi})$	(ϕ^a,ψ_+^b)	$(2,1),(1,2)$
Twisted hyper	Chiral + Chiral $(\Phi',\tilde{\Phi}')$	(ϕ'^a,ψ'^b_+)	$(1,2),(2,1)$
Fermi	Fermi + Fermi $(\Gamma,\tilde{\Gamma})$	(ψ'^a_-,G^b)	$(1,1),(2,2)$

The $\mathcal{N}=(0,4)$ vector multiplet is made of an $\mathcal{N}=(0,2)$ vector multiplet and an adjoint $\mathcal{N}=(0,2)$ Fermi multiplet Θ. The field content is that of a gauge field u_μ and two left-handed fermions ζ_-^a, $a=1,2$, in addition to a triplet of auxiliary fields G^A, $A=1,2,3$. The gauge field is a singlet under the $SU(2)_L\times SU(2)_R$ R-symmetry while the two fermions transform as $(\mathbf{2},\mathbf{2})$. The triplet of auxiliary fields transforms as $(\mathbf{3},\mathbf{1})$ under the R-symmetry. The action for a generic $\mathcal{N}=(0,4)$ vector multiplet is given by the sum of (2.57) with (2.65).

There are two different types of hypermultiplets, the hypermultiplet and the twisted hypermultiplet. Both of them are formed by two $\mathcal{N}=(0,2)$ chiral multiplets, therefore they both contain two complex scalars (ϕ^a) and two right-handed fermions (ψ_+^b). They differ from each other because of the different representations under the R-symmetry group, as we can see from the table above. The kinetic term for the chiral multiplets making up the (twisted) hypermultiplet is again given by (2.61).

If we want to couple the hypermultiplet to the vector multiplet, we should consider the following coupling between the hyper $(\Phi,\tilde{\Phi})$ and the adjoint Fermi field Θ

$$J^\Theta=\Phi\tilde{\Phi}\Rightarrow\mathcal{W}=\tilde{\Phi}\Theta\Phi\,. \tag{2.68}$$

This looks very much like the coupling between the hypermultiplet and the chiral adjoint for four dimensional $\mathcal{N}=2$ theories. On the other hand, coupling a twisted hypermultiplet to the gauge sector requires an E-type of superpotential

$$E_\Theta=\Phi'\tilde{\Phi}'\,, \tag{2.69}$$

with indices in $\Phi'\tilde{\Phi}'$ set to have E_Θ transforming in the adjoint of the gauge group.

Finally, we can have an $\mathcal{N}=(0,4)$ Fermi multiplet, which is made of two $\mathcal{N}=(0,2)$ Fermi multiplets. It contains two left-handed fermions which are singlets of $SU(2)_L \times SU(2)_R$ R-symmetry. The kinetic term for all the fermions is again given by (2.65). No further coupling between Γ, $\tilde{\Gamma}$ and Θ is possible.

As in the quiver of Figure 2.1 there appear also $\mathcal{N}=(4,4)$ vector and chiral multiplets, it is worth mentioning how $\mathcal{N}=(4,4)$ superfields decompose in $\mathcal{N}=(0,4)$ language.

$\mathcal{N}=(4,4)$ **multiplets** There are two types of $\mathcal{N}=(4,4)$ superfields, the vector and the hypermultiplet.

Multiplets	$\mathcal{N}=(0,4)$ building blocks	$\mathcal{N}=(0,2)$ building blocks
Vector	Vector + Twisted Hyper	$(U,\Theta),(\Sigma,\tilde{\Sigma})$
Hyper	Hyper + Fermi	$(\Phi,\tilde{\Phi}),(\Gamma,\tilde{\Gamma})$

The $\mathcal{N}=(4,4)$ vector multipled is comprised of an $\mathcal{N}=(0,4)$ vector multiplet and a $\mathcal{N}=(0,4)$ twisted hypermultiplet. The twisted hypermultiplet is usually denoted as $(\Sigma,\tilde{\Sigma})$. They are coupled to the gauge sector via the E-type potential

$$E_\Theta = [\Sigma,\tilde{\Sigma}]\,. \tag{2.70}$$

$\mathcal{N}=(4,4)$ hypermultiplets are made of an $\mathcal{N}=(0,4)$ hypermultiplet and an $\mathcal{N}=(4,4)$ Fermi multiplet, all in all $(\Phi,\tilde{\Phi}),(\Gamma,\tilde{\Gamma})$. As before, Φ and $\tilde{\Phi}$ are coupled to the gauge sector via

$$\mathcal{W} = \tilde{\Phi}\Theta\Phi\,. \tag{2.71}$$

We conclude this part by saying that there are couplings between $\mathcal{N}=(0,4)$ Fermi multiplets Γ, $\tilde{\Gamma}$, hypermultiplets Φ, $\tilde{\Phi}$ and twisted hypers Σ, $\tilde{\Sigma}$. They involve both superpotential and E-terms

$$\mathcal{W} = \tilde{\Gamma}\tilde{\Sigma}\Phi + \tilde{\Phi}\tilde{\Sigma}\Gamma\,, \tag{2.72}$$

and

$$E_\Gamma = \Sigma\Phi\,,\qquad E_{\tilde{\Gamma}} = -\tilde{\Phi}\Sigma\,. \tag{2.73}$$

It is easy to see that

$$E\cdot J = \tilde{\Phi}[\Sigma,\tilde{\Sigma}]\Phi + \tilde{\Phi}\tilde{\Sigma}\Sigma\Phi - \tilde{\Phi}\Sigma\tilde{\Sigma}\Phi = 0\,. \tag{2.74}$$

Chapter 3

Warped AdS_2 backgrounds and conformal QM

In this chapter, we introduce two new classes of backgrounds with an AdS_2 factor. The first class has the structure of a warped $AdS_2 \times S^2 \times \text{CY}_2$ background while the second has the structure $AdS_2 \times S^3 \times \text{CY}_2$. They are both derived from the $AdS_3 \times S^2 \times \text{CY}_2$ solutions reviewed at the beginning of the previous chapter after some manipulations in supergravity. We give a thorough study of the underlying geometries. A prescription for the dual QMs of the $AdS_2 \times S^2 \times \text{CY}_2$ solutions is given. The dual Quantum Mechanics of the $AdS_2 \times S^3 \times \text{CY}_2$ backgrounds will be presented in [6]. Remarkably, the superalgebra central extension is found in close correspondence with that of the two-dimensional case of $AdS_3 \times S^2 \times \text{CY}_2$.

1 Overview

A major line of research motivated by the Maldacena's conjecture [89] is the study of supersymmetric and conformal field theories in diverse dimensions. However, since the early 2000's a huge effort has also been devoted to the classification of Type II or M-theory backgrounds with AdS_{d+1} factors, see for example [133, 134], which are conjectured to be dual to SCFTs in d dimensions. Since the original formulation of AdS/CFT [89], the Maldacena's conjecture has been explored for theories of any dimensions[1]. We have seen in the previous chapters two examples of SCFTs with dual supergravity solutions in two and four dimensions.

Besides examples in $d \geq 2$, it is also of interest the study of backgrounds with an AdS_2 factor dual to one-dimensional quantum mechanical problems. One of the obvious reasons to study AdS_2 backgrounds is that in the proximity of their horizons,

[1] Here it is meant up to six dimensions, as there are no "accidental isomorphisms" for $d > 6$, which makes it difficult to construct spinorial representations of the conformal algebra.

extremal black holes in diverse dimensions, both in flat and AdS spaces, contain an AdS_2 component with an electric field[2].

When introducing AdS_2 backgrounds, it is also worth mentioning that a conformal quantum mechanical theory needs an $Sl(2,\mathbb{R})$ global symmetry only – aside from possible supersymmetry and associated R-symmetry – which can be realised holographically as the group of isometries of the dual AdS_2 factor, $Spin(1,2) = Sl(2,\mathbb{R})$. Remarkably, the analysis of [137, 138, 139] implies that, whilst the isometry of AdS_2 is $Sl(2,\mathbb{R})$, asymptotically the group of symmetry is enlarged to an infinite dimensional Virasoro algebra, and the central extension is proportional to the inverse Newton constant in two dimensions. This is very much similar to the case of AdS_3 holography where the full Virasoro algebra of the dual 2d CFT is realised asymptotically [140].

Similarities between known superconformal algebras in one and two dimensions or the geometric relation between AdS_2 and AdS_3 spacetimes suggest that the study of AdS_3 backgrounds might shed some light on the AdS_2 case. For some recent significant developments involving AdS_2 geometries, see e.g. [141, 142, 143, 144, 145, 146, 147, 148, 149].

Even though many aspects of AdS_2 holography are still being explored, it seems natural to exploit our knowledge about two-dimensional SCFTs dual to warped AdS_3 solutions to tackle somewhat the problem of AdS_2 holography for warped backgrounds. This is essentially the main motivation of the pages that will follow.

In this chapter, we discuss two interesting classes of AdS_2 backgrounds. These – one in the form of a warped $AdS_2 \times S^2 \times \mathrm{CY}_2$ and the other in the form of a warped $AdS_2 \times S^3 \times \mathrm{CY}_2$ – make up the two main sections discussed here. Given that they are both new entries in the classification of AdS_2 spacetimes, we give a thorough study of both following a similar pattern. Some of the material in this chapter will be presented in [5, 6].

2 Warped $AdS_2 \times S^2 \times \mathbf{CY}_2$ backgrounds and dual QM

In this section, we deal with warped $AdS_2 \times S^2 \times \mathrm{CY}_2$ solutions to Type IIB supergravity obtained from the $AdS_3 \times S^2 \times \mathrm{CY}_2$ backgrounds discussed at the beginning

[2]For a review on superconformal quantum mechanics and their connection to black hole physics see [135, 136] and references therein.

of the previous chapter after a T-duality transformation.

For the sake of readability, we write once again the AdS_3 backgrounds of [17, 18, 19] discussed in the previous chapter. We have

$$\begin{aligned} ds^2 &= \frac{u}{\sqrt{\widehat{h}_4 h_8}}\left(ds^2_{AdS_3} + \frac{\widehat{h}_4 h_8}{4\widehat{h}_4 h_8 + (u')^2} ds^2_{S^2}\right) + \sqrt{\frac{\widehat{h}_4}{h_8}} ds^2_{\mathrm{CY}_2} + \frac{\sqrt{\widehat{h}_4 h_8}}{u} d\rho^2 \,, \\ e^{-\Phi} &= \frac{h_8^{3/4}}{2\widehat{h}_4^{1/4}\sqrt{u}}\sqrt{4\widehat{h}_4 h_8 + (u')^2} \,, \quad H_{(3)} = \frac{1}{2} d\left(-\rho + \frac{u u'}{4\widehat{h}_4 h_8 + (u')^2}\right) \wedge \widehat{\mathrm{vol}}_{S^2} + \frac{1}{h_8} d\rho \wedge H_2 \end{aligned} \tag{3.1}$$

for the N-S sector, while the R-R sector reads

$$\begin{aligned} F_{(0)} &= h_8' \,, \quad F_{(2)} = -H_2 - \frac{1}{2}\left(h_8 - \frac{h_8' u' u}{4 h_8 \widehat{h}_4 + (u')^2}\right) \widehat{\mathrm{vol}}_{S^2} \,, \\ F_{(4)} &= \left(d\left(\frac{u' u}{2\widehat{h}_4}\right) + 2 h_8 d\rho\right) \wedge \widehat{\mathrm{vol}}_{AdS_3} - \partial_\rho \widehat{h}_4 \widehat{\mathrm{vol}}_{\mathrm{CY}_2} - \frac{u' u}{2(4\widehat{h}_4 h_8 + (u')^2)} H_2 \wedge \widehat{\mathrm{vol}}_{S^2} \,. \end{aligned} \tag{3.2}$$

We have already argued, following [19], that such backgrounds lead to interesting dual SCFTs when $H_2 = 0$ and the functions h_4, h_8 are given by the general expressions

$$h_4(\rho) = \begin{cases} \frac{\beta_0}{2\pi}\rho & \rho \in [0, 2\pi] \\ \alpha_k + \frac{\beta_k}{2\pi}(\rho - 2\pi k) & \rho \in [2\pi k, 2\pi(k+1)] \\ \alpha_P - \frac{\alpha_P}{2\pi}(\rho - 2\pi P) & \rho \in [2\pi P, 2\pi(P+1)] \,, \end{cases} \tag{3.3}$$

$$h_8(\rho) = \begin{cases} \frac{\nu_0}{2\pi}\rho & \rho \in [0, 2\pi] \\ \mu_k + \frac{\nu_k}{2\pi}(\rho - 2\pi k) & \rho \in [2\pi k, 2\pi(k+1)] \\ \mu_P - \frac{\mu_P}{2\pi}(\rho - 2\pi P) & \rho \in [2\pi P, 2\pi(P+1)] \,. \end{cases} \tag{3.4}$$

where $\widehat{h}_4 = \Upsilon h_4$. Here $k = 1, \ldots, P-1$ with P some, possibly large, integer and the constants α_k, β_k, μ_k, ν_k related to the number of D branes in the background – see [19] for details. In order to ensure continuity of the N-S sector ($g_{\mu\nu}, B_{\mu\nu}$ and Φ) it is just enough to require continuity of $\widehat{h}_4$ and h_8 [19]. This, in turn, implies that

$$\alpha_k = \sum_{j=0}^{k-1} \beta_j \,, \qquad \mu_k = \sum_{j=0}^{k-1} \nu_j \,. \tag{3.5}$$

The associated dual SCFT was briefly sketched in Section 2 of the previous chapter, and is depicted in Figure 2.1. As it will be crucial in the following, we remind the reader that the central charge of the dual 2d SCFT is given by

$$c = 6(n_{hyp} - n_{vec})\,, \tag{3.6}$$

where n_{hyp} is the total number of $\mathcal{N} = (0,4)$ hypermultiplets, while n_{vec} is the total number of $\mathcal{N} = (0,4)$ vector multiplets in the SCFT. It was shown in [19] that the central charge c is captured holographically[3] by

$$c_{holo} = \frac{3\pi}{2G_N}\mathrm{vol}_{\mathrm{CY}_2}\int_0^{2\pi(P+1)} \widehat{h}_4 h_8\, d\rho\,, \tag{3.7}$$

with $G_N = 8\pi^6$ in units where the string length and coupling constant are set to one.

Let us now give the details on the construction of the new $AdS_2 \times S^2 \times \mathrm{CY}_2$ backgrounds in Type IIB supergravity which will take up the first part of this chapter.

2.1 $AdS_2 \times S^2 \times$ CY$_2$ backgrounds in Type IIB

We now introduce a new class of AdS_2 backgrounds in Type IIB supergravity by applying a T-duality transformation on the AdS_3 subspace of the backgrounds reviewed around (3.1) and (3.2).

Using local coordinates, let us write the AdS_3 subspace in (3.1) and (3.2) as an S^1 fibration over AdS_2. We have

$$ds^2_{AdS_3} = \frac{1}{4}\left[(d\psi + \eta)^2 + ds^2_{AdS_2}\right]\,, \tag{3.8}$$

where

$$d\eta = \widehat{\mathrm{vol}}_{AdS_2}\,, \qquad ds^2_{AdS_2} = -dt^2\cosh^2(r) + dr^2\,. \tag{3.9}$$

Let us perform a T-duality transformation to the S^1 fibre direction ψ to obtain a new class of solutions. These have the structure of warped $AdS_2 \times S^2 \times S^1 \times \mathrm{CY}_2$ backgrounds. Upon using the standard Buscher-Rules – see for instance Appendix A

[3]Here, by "holographically" it is meant in the large P limit and large ranks of the gauge and flavour groups. Thus, formula (3.7) captures the central charge of the dual field theory only at leading order.

of [150] – we find that the N-S sector reads

$$
\begin{aligned}
ds^2 &= \frac{u}{\sqrt{\widehat{h}_4 h_8}}\left(\frac{1}{4}ds^2_{AdS_2} + \frac{\widehat{h}_4 h_8}{4\widehat{h}_4 h_8 + (u')^2}ds^2_{S^2}\right) + \sqrt{\frac{\widehat{h}_4}{h_8}}ds^2_{\text{CY}_2} + \frac{\sqrt{\widehat{h}_4 h_8}}{u}(d\rho^2 + 4d\psi^2)\,,\\
e^{-2\Phi} &= \frac{1}{16}\frac{h_8}{\widehat{h}_4}\big(4\widehat{h}_4 h_8 + (u')^2\big)\,, \quad H_{(3)} = \frac{1}{2}d\left(-\rho + \frac{uu'}{4\widehat{h}_4 h_8 + (u')^2}\right)\wedge \widehat{\text{vol}}_{S^2} + \widehat{\text{vol}}_{AdS_2}\wedge d\psi\,.
\end{aligned}
\tag{3.10}
$$

The R-R sector, instead, is given by

$$
\begin{aligned}
F_{(1)} &= h_8' d\psi\,,\\
F_{(3)} &= -\frac{1}{2}\left(h_8 - \frac{h_8' u' u}{4h_8\widehat{h}_4 + (u')^2}\right)\widehat{\text{vol}}_{S^2}\wedge d\psi + \frac{1}{8}\left(d\left(\frac{u'u}{2\widehat{h}_4}\right) + 2h_8 d\rho\right)\wedge \widehat{\text{vol}}_{AdS_2}\,,\\
F_{(5)} &= -\widehat{h}_4'\,\widehat{\text{vol}}_{\text{CY}_2}\wedge d\psi + \frac{\widehat{h}_4' h_8 u^2}{8\widehat{h}_4(4\widehat{h}_4 h_8 + (u')^2)}\widehat{\text{vol}}_{AdS_2}\wedge\widehat{\text{vol}}_{S^2}\wedge d\rho\,,\\
F_{(7)} &= \frac{4\widehat{h}_4^2 h_8 - uu'\widehat{h}_4' + \widehat{h}_4(u')^2}{8\widehat{h}_4 h_8 + 2(u')^2}\widehat{\text{vol}}_{\text{CY}_2}\wedge\widehat{\text{vol}}_{S^2}\wedge d\psi - \left(\frac{\widehat{h}_4}{4} + \partial_\rho\frac{uu'}{16h_8^2}\right)\widehat{\text{vol}}_{AdS_2}\wedge\widehat{\text{vol}}_{\text{CY}_2}\wedge d\rho\,,\\
F_{(9)} &= -\frac{\widehat{h}_4 h_8' u^2}{8\widehat{h}_8(4\widehat{h}_4 h_8 + (u')^2)}\widehat{\text{vol}}_{AdS_2}\wedge\widehat{\text{vol}}_{\text{CY}_2}\wedge\widehat{\text{vol}}_{S^2}\wedge d\rho\,.
\end{aligned}
\tag{3.11}
$$

where $F_{(7)} = -\star F_{(3)} =$ and $F_{(9)} = \star F_{(1)}$, in accord with the fact that higher form fluxes are just given by $F_{(p)} = (-1)^{[p/2]}\star F_{(10-p)}$, with $[p/2]$ the integer part of $p/2$.

The equations of motion of Type IIB supergravity can be checked to be satisfied – see Appendix IV for equations of motion and conventions of Type IIB – whenever we impose the BPS equations and Bianchi identities, $u'' = 0$ and $\widehat{h}_4'' = h_8'' = 0$. A violation of the Bianchi identities is admissible at points where source branes are located. This feature is, of course, inherited from the $AdS_3 \times S^2 \times \text{CY}_2$ parent theory. Likewise, in the following, we will be concerned with profiles for $\widehat{h}_4$, h_8 like those in equations (3.3) and (3.4).

Remark

An interesting relation between the class of AdS_2 solutions just described and the classification of the AdS_2 backgrounds in Type IIB given in [149, 151] will be discussed in [152].

Behaviour of the background near special points

Let us give some details on the asymptotic behaviour of the background (3.10) for the functions $\widehat{h}_4$, h_8 given as in the equations (3.3) and (3.4). We distinguish two cases:

- u is a linear function of ρ, $u = \frac{u_0}{2\pi}\rho$.

  Close to $\rho = 0$, where $\widehat{h}_4$ and h_8 are given by

$$\widehat{h}_4 = \Upsilon \frac{\beta_0}{2\pi}\rho\,, \quad h_8 = \frac{\nu_0}{2\pi}\rho\,. \tag{3.12}$$

  we find a regular background. This reads

$$\begin{aligned} ds^2 &= \frac{u_0}{\sqrt{\Upsilon\beta_0\nu_0}}\left(\frac{1}{4}ds^2_{AdS_2} + \frac{\Upsilon\beta_0\nu_0\rho^2}{4\Upsilon\beta_0\nu_0\rho^2 + u_0^2}ds^2_{S^2}\right) + \sqrt{\frac{\Upsilon\beta_0}{\nu_0}}ds^2_{\text{CY}_2} + \frac{\sqrt{\Upsilon\beta_0\nu_0}}{u_0}(d\rho^2 + 4d\psi^2)\,, \\ e^{-\Phi} &= \frac{1}{16}\frac{\Upsilon\beta_0}{\nu_0}\left(\frac{\Upsilon\beta_0\nu_0}{\pi^2}\rho^2 + \frac{u_0^2}{4\pi^2}\right)\,. \end{aligned} \tag{3.13}$$

  Notice that the metric of the two-sphere together with the "radial" direction ρ defines a ball of finite radius in three dimensions. The two-sphere S^2 shrinks to zero at $\rho = 0$. Thus, we find a regular geometry which "ends" at $\rho = 0$.

  Close to $\rho = 2\pi(P+1)$, the the end of space in the ρ direction, where $\widehat{h}_4$ and h_8 are given by

$$\widehat{h}_4 = \Upsilon\left(\alpha_P - \frac{\alpha_P}{2\pi}(\rho - 2\pi P)\right)\,, \quad h_8 = \left(\mu_P - \frac{\mu_P}{2\pi}(\rho - 2\pi P)\right)\,, \tag{3.14}$$

  we find that the metric and dilaton behave as

$$\begin{aligned} ds^2 &= \frac{2\pi(P+1)u_0}{4\sqrt{\Upsilon\alpha_P\mu_P}}\frac{1}{x}ds^2_{AdS_2} + \sqrt{\frac{\Upsilon\alpha_P}{\mu_P}}ds^2_{\text{CY}_2} \\ &\quad + \sqrt{\Upsilon\alpha_P\mu_P}\frac{2\pi(P+1)}{u_0}x\left(ds^2_{S^2} + \frac{1}{2\pi(P+1)^2u_0}(dx^2 + 4d\psi^2)\right)\,, \\ e^{-2\Phi} &= \frac{1}{16}\frac{\Upsilon\alpha_P}{\mu_P}\frac{u_0^2}{4\pi^2}\,. \end{aligned} \tag{3.15}$$

  where we have set $\rho = 2\pi(P+1) - x$, for small positive x. We find that the N-S sector asymptotes to the superposition of O1 and O5 planes, stretched along

AdS_2 and $AdS_2 \times \text{CY}_2$, respectively[4].

- u is constant everywhere, $u = \frac{u_0}{2\pi}$.

  Close to $\rho = 0$ we find that the N-S sector behaves as

$$
\begin{aligned}
ds^2 &= \frac{u_0}{4\sqrt{\Upsilon \beta_0 \nu_0}} \frac{1}{\rho}(ds^2_{AdS_2} + ds^2_{S^2}) + \sqrt{\frac{\Upsilon \beta_0}{\nu_0}} ds^2_{\text{CY}_2} + \frac{\sqrt{\Upsilon \beta_0 \nu_0}}{u_0} \rho (d\rho^2 + 4 d\psi^2)\,, \\
e^{-2\Phi} &= \frac{1}{4} \frac{\nu_0^2}{4\pi^2} \rho^2\,.
\end{aligned}
\tag{3.16}
$$

  Close to $\rho = 2\pi(P+1)$ we find precisely the same kind of behaviour: Just set $\rho = 2\pi(P+1) - x$, for small positive x, and replace (modulo some unimportant constants) ρ with x in (3.16).

  We find that the N-S sector asymptotes to the superposition of O3 and O7 planes, extended on $AdS_2 \times S^2$ and $AdS_2 \times S^2 \times \text{CY}_2$, respectively.

  In both cases we find that, approaching the boundaries of the interval $\mathcal{I}_\rho$, the S^1 parametrised by ψ becomes of vanishing size. We may T-dualise back along this S^1 to recover the seed backgrounds discussed in (3.1) and (3.2).

From the analysis above, it appears that backgrounds with a linear function u are “less singular” – they are in fact regular at $\rho = 0$. However, in the following, we will often need to rely on an everywhere constant u.

2.2 Fluxes and brane charges

Let us continue the study of the background given in (3.10) and (3.11) by working out its associated Page charges. As already stressed in Chapter 1, it is this kind of charges that is conserved, localised and quantised, even though they are not invariant under (large) gauge transformations. Once again, they will imply the quantisation of (some) constants directly related to the number of branes present in the supergravity background.

Page fluxes

We remind the reader that the k-th Page flux is defined by the polyform $\widehat{F}_{(k)} = (F \wedge e^{-B_{(2)}})_{(k)}$, where $e^{-B_{(2)}}$ is understood through its series expansion. The general

[4] For a detailed account of orientifold planes backreacted on the geometry, see pag. 23ff of [17].

expression for the Page charge of a Dp brane is therefore given by

$$Q_{Dp} = \frac{1}{2\kappa_{10}^2 T_{Dp}} \int_{\mathcal{M}_{8-p}} \widehat{F}_{8-p} = \frac{1}{(2\pi)^{7-p} g_s \alpha'^{\frac{7-p}{2}}} \int_{\mathcal{M}_{8-p}} \widehat{F}_{8-p} \tag{3.17}$$

where, κ_{10}^2 and T_{Dp} are given by[5]

$$T_{Dp} = \frac{1}{(2\pi)^p g_s \alpha'^{\frac{p+1}{2}}}\,, \qquad 2\kappa_{10}^2 = (2\pi)^7 g_s^2 \alpha'^4\,, \tag{3.18}$$

As in the rest of this chapter, we will set $\alpha' = g_s = 1$. Here, $\mathcal{M}_{8-p}$ is any $(8-p)$-dimensional compact manifold transverse to the branes.

Let us give all the Page fluxes in our background

$$\begin{aligned}
\widehat{F}_{(1)} &= h_8'\, d\psi\,,\\
\widehat{F}_{(3)} &= \frac{1}{2}(\rho h_8' - h_8)\,\widehat{\text{vol}}_{S^2}\wedge d\psi + \frac{1}{8}\left(2h_8 + \partial_\rho \frac{uu'}{\widehat{h}_4}\right)\widehat{\text{vol}}_{AdS_2}\wedge d\rho\,,\\
\widehat{F}_{(5)} &= \frac{1}{32}\left(4\rho h_8 + \partial_\rho \frac{\rho u u' - u^2}{\widehat{h}_4}\right)\widehat{\text{vol}}_{AdS_2}\wedge\widehat{\text{vol}}_{S_2}\wedge d\rho - \widehat{h}_4'\widehat{\text{vol}}_{\text{CY}_2}\wedge d\psi\,,\\
\widehat{F}_{(7)} &= \frac{1}{2}(\widehat{h}_4 - \rho\widehat{h}_4')\widehat{\text{vol}}_{S^2}\wedge\widehat{\text{vol}}_{\text{CY}_2}\wedge d\psi - \left(\frac{\widehat{h}_4}{4} + \partial_\rho\frac{uu'}{16h_8}\right)\widehat{\text{vol}}_{AdS_2}\wedge\widehat{\text{vol}}_{\text{CY}_2}\wedge d\rho\,,\\
\widehat{F}_9 &= -\left(\frac{\rho\widehat{h}_4}{8} + \partial_\rho\frac{\rho u u' - u^2}{32 h_8}\right)\widehat{\text{vol}}_{AdS_2}\wedge\widehat{\text{vol}}_{S_2}\wedge\widehat{\text{vol}}_{\text{CY}_2}\wedge d\rho.
\end{aligned} \tag{3.19}$$

In order to fully describe the brane setup corresponding to the backgrounds in (3.10) and (3.11), it is useful to compute the following quantities (that we might refer to as the Bianchi identities for the Page fluxes)

$$\begin{aligned}
d\widehat{F}_{(1)} &= h_8''\, d\rho\wedge d\psi\,,\\
d\widehat{F}_{(3)} &= -\frac{1}{2}h_8''\times(\rho - 2\pi k)\, d\rho\wedge\widehat{\text{vol}}_{S^2}\wedge d\psi\,,\\
d\widehat{F}_{(5)} &= -\widehat{h}_4''\, d\rho\wedge\widehat{\text{vol}}_{\text{CY}_2}\wedge d\psi\,,\\
d\widehat{F}_{(7)} &= -\frac{1}{2}\widehat{h}_4''\times(\rho - 2\pi k)\, d\rho\wedge\widehat{\text{vol}}_{S_2}\wedge\widehat{\text{vol}}_{\text{CY}_2}\wedge d\psi\,,\\
d\widehat{F}_{(9)} &= 0\,,
\end{aligned} \tag{3.20}$$

[5]The tension of $NS5$ branes is given by $T_{NS5} = \frac{1}{(2\pi)^5 g_s^2 \alpha'^3}$. It differs from the tension of a $D5$ brane by a power of the string coupling.

where we have allowed for large gauge transformations of the $B_{(2)}$ field as $B_{(2)} \to B_{(2)} + \pi k \widehat{\text{vol}}_{S^2}$ in each interval $[2\pi k, 2\pi(k+1)]$. The need for a large gauge transformation on the $B_{(2)}$ field whenever we cross an integer multiple of $\rho = 2\pi$ was carefully explained in [17, 19]. It is needed in order to keep the quantity $b_0 = -\frac{1}{(2\pi)^2}\int_{S^2} B_{(2)}$ into the fundamental domain $[0,1)$. In particular, the condition $b_0 \in [0,1)$ partitions the real line spanned by ρ into segments of length 2π. As we shall see momentarily, such large gauge transformations are actually crucial in the construction of the brane web associated with the background (3.10), (3.11).

Let us now make use of the fact that $\widehat{h}_4$ and h_8 are both piecewise linear and continuous functions – see equations (3.3) and (3.4). We find

$$\begin{aligned} h_4'' &= \frac{1}{2\pi}\sum_{k=1}^{P}(\beta_{k-1}-\beta_k)\delta(\rho-2\pi k)\,, \\ h_8'' &= \frac{1}{2\pi}\sum_{k=1}^{P}(\nu_{k-1}-\nu_k)\delta(\rho-2\pi k)\,. \end{aligned} \tag{3.21}$$

This is nothing but the violation of the Bianchi identities quoted above at isolated points. The right hand side of both equations (3.21) tells us that we have $(\beta_{k-1} - \beta_k)$ and $(\nu_{k-1} - \nu_k)$ localised branes at $\rho = 2\pi k$ $(k = 1, \dots, P)$ in the supergravity background, respectively. Let us find out what type of localised branes we are talking about. We will make extensive use of the formula (3.17).

- Let us begin with the case of $D1$ branes. It is fairly easy to see that they behave as colour branes – they are dissolved into fluxes – as $dF_{(7)}$ in equation (3.20) vanishes identically by virtue of the identity $(\rho - 2\pi k)\delta(\rho - 2\pi k) = 0$ for any k. Their charge, in the interval $\mathcal{I}^k_\rho = [2\pi(k-1), 2\pi k]$, is given by[6]

$$\begin{aligned} Q^{(k)}_{D1} &= \frac{1}{(2\pi)^6}\int_{\mathcal{M}_7}\widehat{F}_7 = \frac{\Upsilon \text{vol}_{\text{CY}_2}}{16\pi^4}\frac{\text{vol}_{S^2}}{4\pi}\left(h_4 - h_4'(\rho-2\pi k)\right)\big|_{\rho=2\pi k} \\ &= \alpha_k\,. \end{aligned} \tag{3.22}$$

  where $\mathcal{M}_7 = \text{CY}_2 \times S^2 \times S^1_\psi$. We have chosen Υ such that $\Upsilon \text{vol}_{\text{CY}_2} = (2\pi)^4$. In particular, this implies quantisation of the α's. Positivity of the charges allows us to consider the α's as integer positive quantities.

[6] Notice that, in the definition of $\widehat{F}_{(7)}$, we have $\widehat{h}_4 - \rho\widehat{h}_4' \to \widehat{h}_4 - \widehat{h}_4'(\rho - 2\pi k)$ as the effect of a large gauge transformation of the $B_{(2)}$ field as explained above. Of course, the same holds for the other fluxes to be shown next.

- In our setup there are also $D3$ branes. They behave as physical sources and therefore give rise to a flavour symmetry for the dual CFT. They are stretched along (AdS_2, S^2) and otherwise localised in $(\mathrm{CY}_2, \rho, \psi)$. Their charge is given by

$$\begin{aligned} Q_{D3}^{(k)} &= \frac{1}{16\pi^4}\int_{\mathcal{I}_\rho^k\times\mathcal{M}_5} d\widehat{F}_5 = \frac{\Upsilon \mathrm{vol}_{\mathrm{CY}_2}}{16\pi^4} 2\pi \int_{2\pi(k-1)}^{2\pi k} d\rho h_4'' \\ &= \beta_{k-1} - \beta_k \end{aligned} \tag{3.23}$$

where $\mathcal{M}_5 = \mathrm{CY}_2 \times S^1_\psi$. Quantisation of $D3$ brane charges for any k implies that the β's are quantised as well. Positivity of the brane charges implies in turn $\beta_{k-1} - \beta_k \geq 0$.

- $D5$ branes are also dissolved into fluxes ($dF_{(3)}$, just as $dF_{(7)}$, vanishes identically), and they provide another source of gauge symmetry for the dual CFT. Their charge in the interval $[2\pi(k-1), 2\pi k]$ is given by

$$\begin{aligned} Q_{D5}^{(k)} &= \frac{1}{4\pi^2}\int_{\mathcal{M}_3} \widehat{F}_3 = \frac{\mathrm{vol}_{S^2}}{4\pi}\left(h_8 - h_8'(\rho - 2\pi k)\right)|_{\rho=2\pi k} \\ &= \mu_k\,. \end{aligned} \tag{3.24}$$

Here $\mathcal{M}_3 = S^2 \times S^1_\psi$. Again, $\mu_k \in \mathbb{Z}_{>0}$.

- $D7$ branes are stretched along $(AdS_2, \mathrm{CY}_2, S^2)$ and localised in ρ and ψ. Just like $D3$ branes, they appear in the background as physical sources. Their charge in the interval $[2\pi(k-1), 2\pi k]$ is computed by means of the formula

$$\begin{aligned} Q_{D7}^{(k)} &= \int_{\mathcal{I}_\rho^k\times S^1_\psi} dF_{(1)} = 2\pi\int_{2\pi(k-1)}^{2\pi k} h_8'' d\rho \\ &= \nu_{k-1} - \nu_k\,. \end{aligned} \tag{3.25}$$

which forces us to consider quantised ν's with $\nu_{k-1} - \nu_k \geq 0$.

- Finally, in analogy with [19], we have $NS5$ branes[7]. We find one of them when-

[7] Notice that in our background, differently from [19], we have also fundamental $F1$ strings whose (Maxwell) charge is defined by

$$Q_{F1}^{(M)} = \frac{1}{(2\pi)^5}\int e^{-2\Phi} \star H_{(3)}\,. \tag{3.26}$$

The definition of the associated Page flux is a little more tricky, but we will not need it here.

ever we cross an integer multiple of $\rho = 2\pi$. Their total number is given by

$$Q_{NS5} = \frac{1}{4\pi^2} \int_{\mathcal{I}_\rho \times S^2} H_{(3)} = P + 1 \,. \tag{3.27}$$

We list the brane content of our background in Table 3.1.

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
$D1$	–					–				
$D3$	–						–	–	–	
$D5$	–	–	–	–	–	–				
$D7$	–	–	–	–	–		–	–	–	
$NS5$	–	–	–	–	–					–
$F1$	–									–

Table 3.1: Brane setup, where – mark the spacetime directions spanned by the various branes. x^0 corresponds to the time direction of the ten dimensional spacetime, $x^1, \ldots, x^4$ are the coordinates spanned by the CY_2. x^5 is the field theory space, while x^6, x^7 and x^8 are the coordinates realising an $SO(3)$ rotation symmetry. x^9 realises an $\mathbb{R}$ translation symmetry that, upon compactification, reduces to a $U(1)$.

To sum up, we have $\mu_k = \sum_{j=0}^{k-1} \nu_j$ $D5$ and $\alpha_k = \sum_{j=0}^{k-1} \beta_j$ $D1$ colour branes, stretched between adjacent $NS5$ branes, as well as $\tilde{F}_{k-1} = (\beta_{k-1} - \beta_k)$ $D7$ and $F_{k-1} = (\nu_{k-1} - \nu_k)$ $D3$ flavour branes in the interval $[2\pi(k-1), 2\pi k]$. See Figure 3.1.

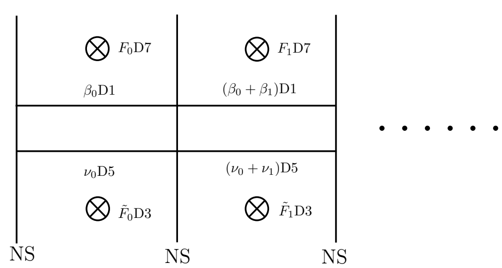

Figure 3.1: The brane web for the background (3.10), (3.11).

In order to give a more refined account of the geometries discussed in this section, we now probe the background (3.10), (3.11) by means of probe $D1$ and $D5$ probe branes, i.e. branes which do not backreact on the geometry, but that allow us to explore it.

2.3 Chern-Simons coupling constants

In this subsection we probe the background given in (3.10) and (3.11) by introducing probe D branes. The action describing the coupling between generic Dp branes and the N-S and R-R closed string fields is the usual DBI + WZ term

$$\begin{aligned} S_{Dp} &= S_{DBI} + S_{WZ}\,, \\ S_{DBI} &= -T_{Dp}\int d^{p+1}\xi \left\{ e^{-\Phi}\left[-\det(g_{ab}+B_{ab}+2\pi\alpha'\mathcal{F}_{ab})\right]^{\frac{1}{2}} \right\} \\ S_{WZ} &= T_{Dp}\int_{p+1}\exp(2\pi\alpha'\mathcal{F}_{(2)}+B_{(2)})\wedge\sum_q C_{(q)}\,, \end{aligned} \tag{3.28}$$

where $\mathcal{F}$ is the gauge field living on the brane. $\mathcal{F} = d\mathcal{A}$ for abelian gauge fields.

- Let us compute the action for a probe $D1$ brane[8]

$$\begin{aligned} S_{D1} &= S_{DBI} + S_{WZ} \\ S_{DBI} &= -T_{D1}\int dt d\rho\, e^{-\Phi}\sqrt{-\det(g_{ind})}\,, \\ S_{WZ} &= T_{D1}\int C_{(2)} + 2\pi C_{(0)}\mathcal{F}_{(2)}\,. \end{aligned} \tag{3.29}$$

The induced metric on the $D1$ brane is given by

$$ds^2_{ind} = -\frac{u}{4\sqrt{\hat{h}_4 h_8}}\cosh^2(r_*)dt^2 + \frac{\sqrt{\hat{h}_4 h_8}}{u}d\rho^2\,, \tag{3.30}$$

while the pullback of the gauge potential $C_{(2)}$ on the worldvolume of the $D1$ brane is given by

$$C_{(2)} = \frac{\sinh(r_*)}{8}\left[\partial_\rho\left(\frac{uu'}{2\hat{h}_4}\right) + 2h_8\right] d\rho\wedge dt\,, \tag{3.31}$$

where r_* just means that we are sitting at a particular value of the radial coordinate r.

[8] Notice that in the DBI action we are not including the $\mathcal{F}$ field nor the pullback of the $B_{(2)}$ as we will not need it in what follows.

We find that the Dirac-Born-Infeld and Wess-Zumino actions read

$$\begin{aligned} S_{DBI} &= -T_{D1} \int_{2\pi k}^{2\pi(k+1)} \sqrt{\frac{h_8}{\widehat{h}_4}(4\widehat{h}_4 h_8 + u'^2)} d\rho \int_{\mathbb{R}} dt \frac{\cosh(r_*)}{8} \,, \\ S_{WZ} &= T_{D1} \int_{2\pi k}^{2\pi(k+1)} d\rho \left[\partial_\rho \left(\frac{uu'}{\widehat{h}_4}\right) + 2h_8\right] \int_{\mathbb{R}} dt \frac{\sinh(r_*)}{8} + 2\pi T_{D1} \int_{\mathcal{I}_\rho^k \times \mathbb{R}} C_{(0)} \mathcal{F}_{(2)} \,. \end{aligned} \tag{3.32}$$

Notice that such a probe $D1$ brane becomes extremal (tension equal charge) when $u' = 0$ and the brane is located at the boundary of AdS_2, i.e. $r_* \to \infty$.

Let us focus on the second term of the Wess-Zumino in (3.32). On the worldvolume of the $D1$ brane, we have that

$$C_{(0)} \mathcal{F}_{(2)} = d(C_{(0)} \mathcal{A}_{(1)}) \,, \tag{3.33}$$

just because $F_{(1)}$, as defined in (3.19), has zero pull-back. Therefore, taking $\mathcal{A}_{(1)}$ as a one-dimensional gauge connection, we find that

$$2\pi T_{D1} \int C_{(0)} \mathcal{F}_{(2)} = \frac{\psi_*}{2\pi}(\nu_k - \nu_{k+1}) \int_{\mathbb{R}} \mathcal{A}_{(1)} \,. \tag{3.34}$$

Such an action describes a Chern-Simons term in one dimension. Invariance under large gauge transformations in homotopically non trivially situations (for instance if we try to formulate the theory on a time-circle S^1) requires the Chern-Simons coefficient $\frac{\psi_*}{2\pi}(\nu_k - \nu_{k+1})$ to be an integer. Given that $(\nu_k - \nu_{k+1})$ is a positive integer number and ψ is 2π-periodic, we demand that

$$\psi_* = \frac{2\pi n}{\nu_k - \nu_{k+1}} \,, \tag{3.35}$$

with $n = 0, \ldots, \nu_k - \nu_{k+1}$. Therefore, when probe D brane are included in the supergravity backgrounds, the $U(1)$ symmetry parametrised by ψ is broken down at best to $\mathbb{Z}_{\nu_k - \nu_{k+1}}$.

- Let us repeat the same steps for the case of a probe colour $D5$ brane stretched along (t, CY_2, ρ) and otherwise located at some fixed values of all the other coordinates.

The action for a probe $D5$ brane is given by

$$
\begin{aligned}
S_{D5} &= S_{DBI} + S_{WZ} \\
S_{DBI} &= -T_{D5} \int e^{-\Phi} \sqrt{-\det(g_{ind})}\,, \\
S_{WZ} &= T_{D5} \int C_{(6)} + 2\pi C_{(4)} \wedge \mathcal{F}_{(2)} + \dots\,,
\end{aligned}
\tag{3.36}
$$

where the terms summarised as "..." in the Wess-Zumino action vanish when reducing down to one dimension and therefore need not be displayed explicitly.

The induced metric on the $D5$ brane is

$$
ds^2_{ind} = -\frac{u}{4\sqrt{\widehat{h}_4 h_8}} \cosh^2(r_*) dt^2 + \frac{\sqrt{\widehat{h}_4 h_8}}{u} d\rho^2 + \sqrt{\frac{\widehat{h}_4}{h_8}} ds^2_{\mathrm{CY}_2}\,,
\tag{3.37}
$$

while the pull-back of the gauge potential $C_{(6)}$ onto the brane is

$$
C_{(6)} = \left(\frac{\widehat{h}_4}{4} + \partial_\rho \frac{u u'}{h_8} \right) \sinh(r_*) d\rho \wedge dt \wedge \widehat{\mathrm{vol}}_{\mathrm{CY}_2}\,.
\tag{3.38}
$$

Thus, for a probe $D5$ brane, we get the Dirac-Born-Infeld and Wess-Zumino actions given by

$$
\begin{aligned}
S_{DBI} &= -T_{D5} \mathrm{vol}_{\mathrm{CY}_2} \int_{2\pi k}^{2\pi(k+1)} d\rho \sqrt{\frac{\widehat{h}_4}{h_8}(4\widehat{h}_4 h_8 + u'^2)} \int_{\mathbb{R}} dt\, \frac{\cosh(r_*)}{8}\,, \\
S_{WZ} &= T_{D5} \mathrm{vol}_{\mathrm{CY}_2} \int_{2\pi k}^{2\pi(k+1)} d\rho \left(\frac{\widehat{h}_4}{4} + \partial_\rho \frac{u u'}{h_8} \right) \int_{\mathbb{R}} dt\, \sinh(r_*) \\
&\quad + 2\pi T_{D5} \int C_{(4)} \wedge \mathcal{F}_{(2)}\,.
\end{aligned}
\tag{3.39}
$$

Again, we assume $u' = 0$ and $r^\star = \infty$. Let us then focus on the last term of the Wess-Zumino action for the $D5$ brane, which is the most relevant for us. On the $D5$ brane we have

$$
\begin{aligned}
2\pi T_{D5} \int C_{(4)} \wedge \mathcal{F}_{(2)} &= 2\pi T_{D5} \int d(C_{(4)} \wedge \mathcal{A}_{(1)}) \\
&= \frac{\Upsilon \mathrm{vol}_{\mathrm{CY}_2}}{(2\pi)^4} \frac{\psi_*}{2\pi} (\beta_k - \beta_{k+1}) \int_{\mathbb{R}} \mathcal{A}_{(1)}\,,
\end{aligned}
\tag{3.40}
$$

where we have used $C_{(4)}|_{D5} = \widehat{h}_4'\, \psi\, \widehat{\mathrm{vol}}_{\mathrm{CY}_2}$. Choosing again Υ such that $\Upsilon \mathrm{vol}_{\mathrm{CY}_2} =$

$(2\pi)^4$, we find a Chern-Simons term of the form

$$S_{CS} = \frac{\psi_*}{2\pi}(\beta_k - \beta_{k+1}) \int_{\mathbb{R}} \mathcal{A}_{(1)} \,. \tag{3.41}$$

Invariance under large gauge transformations of the $\mathcal{A}_{(1)}$ connection implies once again that $U(1)_\psi \to \mathbb{Z}_{\beta_k - \beta_{k+1}}$. If $(\beta_k - \beta_{k+1})$ and $(\nu_k - \nu_{k+1})$ are not proportional to each other or have common divisors, it is reasonable to think that the $U(1)$ symmetry parametrised by ψ is broken completely in supergravity.

Let us now move on to a definition of central charge for our backgrounds.

2.4 Holographic central charge

The definition of "central charge" in conformal quantum mechanics is not free of subtleties. In a one-dimensional theory, there is only one component of the energy-momentum tensor, $T_{\mu\nu}$. Naively, if the theory is conformal, the trace of this quantity is supposed to vanish, which implies $T_{tt} = H = 0$. The system is in its ground state.

A proposal pushed forward in [5] is that the one-dimensional central charge we are going to compute describes the "dynamics of the ground states": It essentially counts the number of degenerate ground states in a quantum-mechanical system. We then may associate this quantity with the "central extension" of the Virasoro algebra that appears asymptotically in the two-dimensional dual gravity, as discussed in [137, 138, 139].

In this subsection we compute the holographic central charge for the class of backgrounds discussed in (3.10) and (3.11) using the rationale of [40, 41]. Being the field theory zero-dimensional, some of the steps in the calculation need some care. The relevant quantity in this case is the volume of the "internal" space (that part of the spacetime "external" to the AdS_2). As we will see in a moment, what we are really computing is the Newton constant in an effective two-dimensional gravity theory,

$$\frac{1}{G_{N,2}} = \frac{\mathcal{V}_{int}}{G_{N,10}} \,. \tag{3.42}$$

The volume of the internal space reads

$$\begin{aligned} \mathcal{V}_{int} &= \int d^8x \sqrt{e^{-4\Phi} \det g_{8,ind}} \\ &= \frac{2\pi\, \mathrm{vol}_{\mathrm{CY}_2} \mathrm{vol}_{S^2}}{8} \int_0^{2\pi(P+1)} \widehat{h}_4 h_8 d\rho \,. \end{aligned} \tag{3.43}$$

If we compare equation (3.43) with (3.7), we see that, under a suitable rescaling, this observable is related to the central charge of the *seed* CFT. In other words, we can think of $\mathcal{V}_{int}$, as defined in (3.43), as the holographic central charge for the class of backgrounds discussed in this section – equations (3.10) and (3.11).

This seems to be compatible also with the results of the paper [153], where it was suggested that the chiral sector remaining when DLCQ is applied to a 2d CFT has the same central extension in the Virasoro algebra.

On purely field theoretical terms, this result tells us that the number of vacua of the $\mathcal{N} = 4$ SCQM corresponds to the expression obtained in [124], namely

$$c_{qm} = 6(n_{hyp} - n_{vec})\,. \tag{3.44}$$

In particular, this is a first hint at defining the dual Quantum Mechanics just as the dimensional reduction from two dimensions of the parent two-dimensional field theory dual to the $AdS_3 \times S^2 \times \mathrm{CY}_2$ solutions mentioned above. The number of $\mathcal{N} = 4$ hyper and vector multiplets in the one dimensional theory are inherited from those in the two dimensional "parent theory". We will have more to say about the dual Quantum Mechanics at the end of this section.

To get the numbers right, we define the central charge of the dual conformal quantum mechanics to be

$$c_{1d} = \frac{3}{2\pi}\frac{1}{G_2} = \frac{3}{2\pi}\frac{\mathcal{V}_{int}}{G_N}\,. \tag{3.45}$$

Explicitly, using that $G_N = 8\pi^6$, c_{1d} is given by

$$c_{1d} = \frac{3}{\pi}\int_0^{2\pi(P+1)} h_4 h_8 d\rho\,, \tag{3.46}$$

in agreement with equation (2.37).

An action functional for the central charge

We now present a "minimisation procedure" in supergravity that leads to the holographic central charge as just discussed around (3.46). The computation is done along the lines of [154]. Later in this section, we will motivate such an extremisation procedure by a more physical point of view.

To give some context, the authors of [154], considered a family of backgrounds in eleven-dimensional supergravity of the form $AdS_2 \times Y_9$, with Y_9 a closed manifold.

Such backgrounds are shown to preserve $\mathcal{N} = 2$ supersymmetry. Aside from the AdS_2 factor, they are rather different from those discussed in this chapter, the main difference being the the amount of supersymmetry ($\mathcal{N} = 2$ as opposed to $\mathcal{N} = 4$) and the absence of a boundary for the internal manifold Y_9. Nevertheless, it is argued in [154] that an "holographic central charge" can be obtained from an extremisation procedure. We now show that the intuition about the central charge developed in [154] coincides with ours.

Following [154], and splitting our background (3.10) and (3.11) as $AdS_2 \times X^8$, with X_8 a manifold with boundary $\partial X_8 = (\{0\}, \{2\pi(P+1)\})$, we define the following forms living on the internal manifold X_8

$$\mathcal{J}_1 = j_1 d\psi\,, \qquad \mathcal{F}_1 = f_1 d\rho\,, \tag{3.47}$$
$$\mathcal{J}_3 = j_3 \widehat{\text{vol}}_{S^2} \wedge d\psi\,, \qquad \mathcal{F}_3 = f_3 \widehat{\text{vol}}_{S^2} \wedge d\rho\,, \tag{3.48}$$
$$\mathcal{J}_5 = j_5 \widehat{\text{vol}}_{\text{CY}_2} \wedge d\psi\,, \qquad \mathcal{F}_5 = f_5 \widehat{\text{vol}}_{\text{CY}_2} \wedge d\rho\,, \tag{3.49}$$
$$\mathcal{J}_7 = j_7 \widehat{\text{vol}}_{\text{CY}_2} \wedge \widehat{\text{vol}}_{S^2} \wedge d\psi\,, \qquad \mathcal{F}_7 = f_7 \widehat{\text{vol}}_{\text{CY}_2} \wedge \widehat{\text{vol}}_{S^2} \wedge d\rho\,, \tag{3.50}$$

obtained from (3.19) just by omitting the AdS_2 volume form, $\widehat{\text{vol}}_{AdS_2}$. For instance, $F_{(1)}$ gives rise to a one-form $\mathcal{J}_1 = h_8' d\psi$ (no AdS_2 factor in the definition of $F_{(1)}$), $F_{(3)}$ gives rise to another one-form, $\mathcal{F}_{(1)}$, after omitting the AdS_2 volume factor as well as a three form $\mathcal{J}_3$ and so on. Just by comparison with (3.19), the functions j_i and f_i are defined in the following manner

$$j_1 = h_8'\,, \qquad f_1 = \frac{h_8}{4} + \partial_\rho \frac{u u'}{16\widehat{h}_4}\,, \tag{3.51}$$
$$j_3 = -\frac{1}{2}(h_8 - h_8'(\rho - 2\pi k))\,, \qquad f_3 = \frac{4\rho h_8}{8} + \partial_\rho \frac{\rho u u' - u^2}{32\widehat{h}_4}\,, \tag{3.52}$$
$$j_5 = -\widehat{h}_4'\,, \qquad f_5 = -\frac{\widehat{h}_4}{4} - \partial_\rho \frac{u u'}{16 h_8}\,, \tag{3.53}$$
$$j_7 = \frac{1}{2}(\widehat{h}_4 - \widehat{h}_4'(\rho - 2\pi k))\,, \qquad f_7 = -\frac{\rho\widehat{h}_4}{8} - \partial_\rho \frac{\rho u u' - u^2}{32 h_8}\,, \tag{3.54}$$

We define also

$$\mathcal{G}_{2i+1} = \mathcal{J}_{2i+1} + i\mathcal{F}_{2i+1}\,, \tag{3.55}$$

with $i = 1, 2, 3$. Notice that the forms $\mathcal{G}_3$ and $\mathcal{G}_7$ satisfy the following Bianchi identities

$$d\mathcal{G}_1 = h_8''\, d\rho \wedge d\psi\,, \quad d\mathcal{G}_5 = -\widehat{h}_4''\, \widehat{\text{vol}}_{\text{CY}_2} \wedge d\rho \wedge d\psi\,. \tag{3.56}$$

As a final step, we now define also the functional[9]

$$\mathcal{C} = \frac{i}{2}\int_{X_8} \mathcal{G}_3\wedge\mathcal{G}_5 - \mathcal{G}_1\wedge\mathcal{G}_7$$
$$= \frac{1}{2}\int_{X_8}\left(\frac{\widehat{h}_4 h_8}{4} + u^2\left(\frac{\widehat{h}_4'^2}{\widehat{h}_4^2}+\frac{h_8'^2}{h_8^2}\right) - 2uu'\left(\frac{\widehat{h}_4'}{\widehat{h}_4}+\frac{h_8'}{h_8}\right)+2u'^2\right)\widehat{\text{vol}}_{\text{CY}_2}\wedge d\rho\wedge\widehat{\text{vol}}_{S^2}\wedge d\psi \tag{3.57}$$

Following [154], we minimise the functional $\mathcal{C}$ imposing the Euler-Lagrange equation for u, which can be shown to be solved if

$$u''=0\,,\qquad h_8''=0\,,\qquad \widehat{h}_4''=0\,. \tag{3.58}$$

As we know already, the first is a BPS equation, the last two are Bianchi identities for the backgrounds given in equations (3.10) and (3.11).

Imposing (3.58) on (3.57), we are allowed to write $\mathcal{C}$ as

$$\mathcal{C}|_{on-shell} = \frac{\pi\text{vol}_{\text{CY}_2}\text{vol}_{S^2}}{4}\int_0^{2\pi(P+1)}\left(\widehat{h}_4 h_8 + \partial_\rho\mathcal{M}\right)d\rho\,, \tag{3.59}$$

where

$$32\mathcal{M} = 2uu' - u^2\left(\frac{\widehat{h}_4'}{\widehat{h}_4}+\frac{h_8'}{h_8}\right) \tag{3.60}$$

Up to a boundary term (and irrelevant proportionality constants), we recover the expression (3.46) for the holographic central charge. Remarkably, our central charge, as defined in (3.45), was given in terms of N-S fields only, while we recovered the same result with a different procedure involving just R-R fields.

We might wonder what is the physical meaning of such an extremisation procedure. Hopefully the argument given next will help shed some light.

A relation between the central charge and Page fluxes

We now point out a nice link between the holographic central charge given in (3.46) and a particular functional of some suitably defined electric and magnetic fluxes for the ten-dimensional space.

Consider a Dp brane as the electric source for $\widehat{F}_{(p+2)}$ and magnetic source for $\widehat{F}_{(8-p)}$ Page field strengths. We will define "electric" and "magnetic charges" of a Dp brane

[9] Here, $\mathcal{C}$ is viewed as "functional" of the variable u.

q^e_{Dp} and q^m_{Dp} as[10]

$$q^e_{Dp} = \frac{1}{(2\pi)^p}\int \widehat{F}_{(p+2)}\,, \qquad q^m_{Dp} = \frac{1}{(2\pi)^{7-p}}\int \widehat{F}_{(8-p)}\,. \tag{3.61}$$

Some charges will turn out to be infinite as a consequence of the integration over the non-compact AdS spacetime. Thus, in the following a regularisation procedure is understood.

Consider the product of electric and magnetic charges, as given in equation (3.61), for each of the D branes in our background.

Define[11]

$$\mathcal{C} = \sum_{k=0}^{3}(-1)^k q^e_{D(2k+1)} q^m_{D(2k+1)}\,. \tag{3.62}$$

Using the Page fluxes given in (3.19) and (3.132) we find for $D1$ branes[12]

$$\frac{1}{(2\pi)^7}\int \widehat{F}^e_{(3)}\wedge \widehat{F}^m_{(7)} = \frac{1}{(2\pi)^7}\int \frac{1}{2}\left(\widehat{h}_4 - \widehat{h}'_4\rho\right)\left(\frac{h_8}{4} + \partial_\rho\left(\frac{uu'}{16\widehat{h}_4}\right)\right)\widehat{\mathrm{vol}}_{10}\,. \tag{3.63}$$

For $D3$ and $D5$ branes we have instead

$$\frac{1}{(2\pi)^7}\int \widehat{F}^e_{(5)}\wedge \widehat{F}^m_{(5)} = -\frac{1}{(2\pi)^7}\int \widehat{h}'_4\left[\frac{\rho h_8}{8} + \partial_\rho\left(\frac{\rho uu' - u^2}{32\widehat{h}_4}\right)\right]\widehat{\mathrm{vol}}_{10}\,, \tag{3.64}$$

and

$$\frac{1}{(2\pi)^7}\int \widehat{F}^e_{(7)}\wedge \widehat{F}^m_{(3)} = \frac{1}{(2\pi)^7}\int \frac{1}{2}(h_8 - h'_8\rho)\left(\frac{\widehat{h}_4}{4} + \partial_\rho\left(\frac{uu'}{16h_8}\right)\right)\widehat{\mathrm{vol}}_{10}\,, \tag{3.65}$$

respectively. Finally, for the case of $D7$ branes we find

$$\frac{1}{(2\pi)^7}\int \widehat{F}^e_{9}\wedge \widehat{F}^m_{1} = -\frac{1}{(2\pi)^7}\int h'_8\left[\frac{\rho\widehat{h}_4}{8} + \partial_\rho\left(\frac{\rho uu' - u^2}{32h_8}\right)\right]\widehat{\mathrm{vol}}_{10}\,, \tag{3.66}$$

[10] Notice that the definition of the "magnetic charges" q^m_{Dp} coincides with the definition of the Page charges given in (3.17), so they have a direct physical interpretation. On the other hand, the "electric charges" to be associated with a Dp brane represent a sort of electro-magnetic dual.

[11] When computing the product for $D3$ branes, $\int \widehat{F}_5\wedge\widehat{F}_5$ is identically zero, due to the self-duality of the five form field. We then retain in the computation only one of the two components of this product.

[12] Here and in the following, integration is understood over the ten-dimensional spacetime and, as before, it is fully justified when BPS and Bianchi identities hold globally.

Summing up, we find for $\mathcal{C}$ the following final result

$$\mathcal{C} = \mathrm{vol}_{AdS_2} \frac{\mathrm{vol}_{\mathrm{CY}_2}}{16\pi^4} \int_0^{2\pi(P+1)} d\rho \left[\frac{\widehat{h}_4 h_8}{4} + \frac{1}{32} \partial_\rho \left(2uu' - u^2 \frac{(\widehat{h}_4 h_8)'}{\widehat{h}_4 h_8} \right) \right] . \tag{3.67}$$

Up to a boundary term, this is proportional to the central charge given in equation (3.46). We then see that the holographic central charge is proportional to the (regularised) product of suitably defined electric and magnetic charges associated to the Dp branes.

In [137], Hartman and Strominger argued that, in a two-dimensional Maxwell-dilaton quantum gravity on AdS_2 and a constant electric field, the central charge of the dual CFT is a quadratic form of the electric field. It seems tantalising to think that the functional $\mathcal{C}$, defined as a quadratic form of the electro-magnetic Page charges, provides a string theory realisation of the same idea.

2.5 The dual QM

We now aim at defining the CFT dual to the backgrounds found in (3.10) and (3.11). Given that the $AdS_2 \times S^2 \times \mathrm{CY}_2$ backgrounds were obtained by T-dualising the fibre direction of the (warped) $AdS_3 \times S^2 \times \mathrm{CY}_2$ parent theory, and also looking at the brane web in Table (3.1) as compared to the brane web on page 11 of [19], it is natural to conjecture that the one-dimensional superconformal quantum mechanics we are looking for is just given by dimensionally reducing the parent two-dimensional theory down to one time dimension.

We remind the reader that the parent two-dimensional theory is the one depicted in Figure 2.1, in the previous chapter, and is made of $\mathcal{N} = (0,2)$, $\mathcal{N} = (0,4)$, $\mathcal{N} = (4,4)$ supermultiplets. The dual quantum mechanical quiver is then of the same form and is given below.

We give a general discussion on compactification of 2d $\mathcal{N} = (0,4)$ and $\mathcal{N} = (4,4)$ theories in Appendix K.

In two-dimensional language, we have that nodes in the quiver are associated with $(4,4)$ vector multiplets. Black lines denote $(4,4)$ hypermultiplets and they are always found to some bifundamental representation. Likewise, we have red line for $(0,4)$ hypers and dashed lines for $(0,2)$ Fermi's. It is by now clear that ranks of colour and flavour groups are determined by the Page fluxes studied around (3.17). In particular, we have $U(\alpha_k)$ and $U(\mu_k)$ colour groups for each k as well as $SU(F_k)$ and $SU(\tilde{F}_k)$

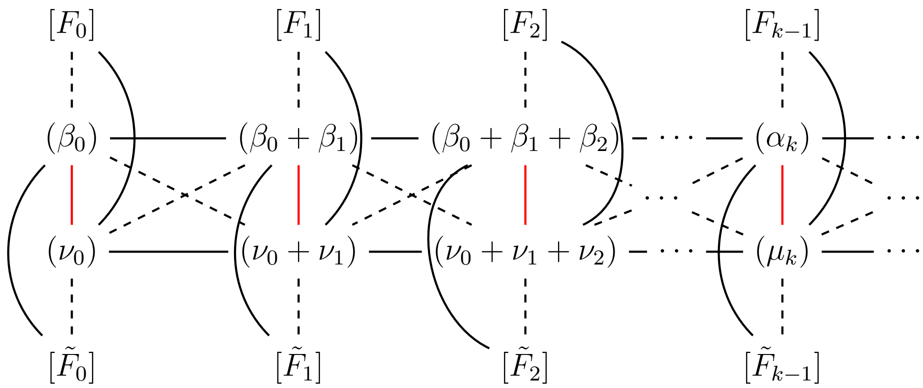

Figure 3.2: Generic quantum mechanical quivers.

flavour groups with $F_{k-1} = \nu_{k-1} - \nu_k$ and $\tilde{F}_{k-1} = \beta_{k-1} - \beta_k$.

One – perhaps obvious – test for our conjecture is given by computing the central charge as given by the formula (3.44). This feature is of course inherited from the parent two-dimensional theory and, as explained in [19], formula (3.44) is reproduced holographically by equation (3.46) in the limit of very long quivers (very large P) and large ranks for the gauge and flavour groups. This has been tested in a substantial number of examples in [19].

Intuitively, large P is needed in order to have weak curvature (almost) everywhere, whereas large ranks for the gauge group just reduce string loop effects.

As a last comment, we say a few words on what should be the correct IR R-symmetry realised by our conformal quantum mechanics. This is sharpened further in Appendix K. A quantum mechanical $\mathcal{N}=4$ theory has in general $SO(4)$ R-symmetry. However, it is consistent to have a conformal quantum mechanics with an $SU(2)$ R-symmetry only. As a matter of fact, our backgrounds realise an $SU(2)$ R-symmetry geometrically on the S^2, as supercharges in supergravity transform as a doublet under it. The correct (global) superalgebra with an $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)_R$ is $\mathfrak{su}(1,1|2)$. We are then led to identify this as the algebra realised by our quantum mechanical quivers.

We now proceed to introducing the second class of AdS_2 backgrounds considered in this chapter. Besides being solutions to Type IIA massive supergravity, we will see that they will bring along some nice features: The backgrounds realise in supergravity the backreaction of the well-known $D0-D4-D4'-D8-F1$ systems preserving eight superconformal symmetries.

3 Warped $AdS_2 \times S^3 \times \mathrm{CY}_2$ backgrounds in Type IIA

In this section we introduce a new family of backgrounds with an AdS_2 factor. These are obtained through a double analytic continuation of the backgrounds already given in (2.1) and (2.2) and reviewed in the previous section around (3.1) and (3.2). We give a thorough study of the underlying geometry in Appendix L. The geometries discussed in this section were given in [4]. More details about the CFT will be given in [6].

3.1 A new family of backgrounds

Let us start off this section by discussing a double analytic continuation of the background just given in (3.1) and (3.2). We can indeed perform an analytic continuation from $AdS_3 \times S^2$ to $AdS_2 \times S^3$ as

$$AdS_3 \to -S^3 \,, \qquad S^2 \to -AdS_2 \,, \,. \tag{3.68}$$

The reason for the minus sign is the following. Starting from AdS_3 in global coordinates[13], after a standard Wick rotation,

$$r \to -i\eta \,, \qquad \phi \to \xi_1 \,, \qquad \tau \to \xi_2 \,, \tag{3.69}$$

one gets, from the AdS_3 metric, minus the metric of a three-sphere

$$-\cosh^2(r)d\tau^2 + dr^2 + \sinh^2(r)d\phi^2 \to -(d\eta^2 + \sin^2(\eta)d\xi_1^2 + \cos^2(\eta)d\xi_2^2) \,. \tag{3.70}$$

Notice that η and ξ_2 are to be taken compact after Wick rotating. Here η runs over the range 0 to $\pi/2$, and ξ_1 and ξ_2 can take any values between 0 and 2π. A similar story pans out for $S^2 \to -AdS_2$. Using local coordinates, say ϕ_1 and ϕ_2, for S^2, the Wick rotation

$$\phi_1 \to i\tilde{r} \,, \qquad \phi_2 \to t \,, \tag{3.71}$$

leads to

$$d\phi_1^2 + \cos^2(\phi_1)d\phi_2^2 \to -(-\cosh^2(\tilde{r})dt^2 + d\tilde{r}^2) \,, \tag{3.72}$$

[13] Here r is the AdS radius in global coordinates, while τ is the global AdS time and ϕ an angular variable with values in the range $[0, 2\pi)$.

where now $\tilde{r}$ and t are to be decompactified after the Wick rotation. As it will be useful in the following, we also notice the following analytic continuation for the volume forms

$$\begin{aligned} \sinh(r)\cosh(r) d\tau \wedge dr \wedge d\phi &\to -\sin(\eta)\cos(\eta) d\eta \wedge d\xi_1 \wedge d\xi_2 \,, \\ \cos(\phi_1) d\phi_1 \wedge d\phi_2 &\to -i\cosh(\tilde{r}) dt \wedge d\tilde{r} \,. \end{aligned} \tag{3.73}$$

We summarize them as $\widehat{\mathrm{vol}}_{AdS_3} \to -\widehat{\mathrm{vol}}_{S^3}$ and $\widehat{\mathrm{vol}}_{S^2} \to -i\,\widehat{\mathrm{vol}}_{AdS_2}$. In order to get a spacetime with the correct signature, we consider also the following analytic continuation for the functions u, $\widehat{h}_4$, h_8

$$u \to -iu \,, \qquad \widehat{h}_4 \to i\widehat{h}_4 \,, \qquad \widehat{h}_8 \to i\widehat{h}_8 \,. \tag{3.74}$$

It should be probably stated that, throughout this chapter, analytic continuation is performed by extending the domain of the real numbers to the domain of the complex numbers through the upper half of the complex plane.

All in all, we are then considering the following analytic continuation

$$\begin{aligned} AdS_3 \to -S^3 \,, \quad S^2 \to -AdS_2 \,, \quad \widehat{\mathrm{vol}}_{\mathrm{AdS_3}} \to -\widehat{\mathrm{vol}}_{\mathrm{S^3}} \,, \quad \widehat{\mathrm{vol}}_{\mathrm{S^2}} \to -i\,\widehat{\mathrm{vol}}_{\mathrm{AdS_2}} \\ u \to -iu \,, \qquad \widehat{h}_4 \to i\widehat{h}_4 \,, \qquad \widehat{h}_8 \to i\widehat{h}_8 \,, \end{aligned} \tag{3.75}$$

together with $\rho \to i\rho$, being ρ the coordinate parametrising $\mathbb{R}$ in (2.1).

Performing (3.75) on (3.1) and (3.2), we get

$$\begin{aligned} ds^2 &= \frac{u}{\sqrt{\widehat{h}_4 h_8}} \left(\frac{\widehat{h}_4 h_8}{4\widehat{h}_4 h_8 - (u')^2} ds^2_{AdS_2} + ds^2_{S^3} \right) + \sqrt{\frac{\widehat{h}_4}{h_8}} ds^2_{\mathrm{CY}_2} + \frac{\sqrt{\widehat{h}_4 h_8}}{u} d\rho^2 \,, \\ e^{-\Phi} &= \frac{h_8^{3/4}}{2\widehat{h}_4^{1/4}\sqrt{u}} \sqrt{4\widehat{h}_4 h_8 - (u')^2} \,, \quad H_{(3)} = -\frac{1}{2} d\left(\rho + \frac{uu'}{4\widehat{h}_4 h_8 - (u')^2} \right) \wedge \widehat{\mathrm{vol}}_{AdS_2} + \frac{1}{h_8} d\rho \wedge H_2 \,. \end{aligned} \tag{3.76}$$

The R-R sector reads

$$\begin{aligned} F_{(0)} &= h_8' \,, \quad F_{(2)} = -H_2 - \frac{1}{2}\left(h_8 + \frac{h_8' u' u}{4h_8\widehat{h}_4 - (u')^2} \right) \widehat{\mathrm{vol}}_{AdS_2} \,, \\ F_{(4)} &= \left(-d\left(\frac{u'u}{2\widehat{h}_4} \right) + 2h_8 d\rho \right) \wedge \widehat{\mathrm{vol}}_{S^3} - \frac{h_8}{u} \widehat{\star}_4 d_4 h_4 \wedge d\rho - \partial_\rho \widehat{h}_4 \widehat{\mathrm{vol}}_{\mathrm{CY}_2} + \frac{u'u}{2(4\widehat{h}_4 h_8 - (u')^2)} H_2 \wedge \widehat{\mathrm{vol}}_{AdS_2} \,. \end{aligned} \tag{3.77}$$

The backgrounds (3.76) and (3.77) solve the massive IIA supergravity equation of motion provided that $u''=0$ globally, and $\widehat{h}_4''=h_8''=0$ away from localised brane sources. The last two conditions come from the Bianchi identities for the RR sector. As discussed in the previous section, violating $\widehat{h}_4''=h_8''=0$ at points means that we can effectively consider piecewise linear and continuous functions $\widehat{h}_4$ and h_8. We will discuss this in much detail later.

Note that it must be that $4\widehat{h}_4 h_8-(u')^2>0$, in order for the metric to be of the correct signature and the dilaton to be real. As in the case of the $AdS_2\times S^2$ backgrounds, in the following, we will be mainly concerned with the case of $H_2=0$ and $\widehat{h}_4=\widehat{h}_4(\rho)$.

3.2 Fluxes and branes

Let us continue the study of the background given in (3.76) and (3.77) by working out its associated Page charges. Once again, quantisation of brane charges will imply the quantisation of some constants appearing in the definition of $\widehat{h}_4$ and h_8.

Page fluxes

The definition of Page fluxes and associated charges was given around equation (3.17). Thus, we go straight to listing all the Page fluxes in our background

$$\begin{aligned}
\widehat{F}_{(0)}&=h_8'\,,\\
\widehat{F}_{(2)}&=-\frac{1}{2}(h_8-\rho h_8')\widehat{\text{vol}}_{AdS_2}\,,\\
\widehat{F}_{(4)}&=-\widehat{h}_4'\widehat{\text{vol}}_{\text{CY}_2}-\left(2h_8-\partial_\rho\frac{uu'}{2\widehat{h}_4}\right)\widehat{\text{vol}}_{S^3}\wedge d\rho\,,\\
\widehat{F}_{(6)}&=\frac{1}{2}(\widehat{h}_4-\rho\widehat{h}_4')\widehat{\text{vol}}_{AdS_2}\wedge\widehat{\text{vol}}_{\text{CY}_2}+\left(-\rho h_8+\partial_\rho\frac{\rho uu'-u^2}{4\widehat{h}_4}\right)\widehat{\text{vol}}_{AdS_2}\wedge\widehat{\text{vol}}_{S^3}\wedge d\rho\,,\\
\widehat{F}_{(8)}&=\left(2\widehat{h}_4-\partial_\rho\frac{uu'}{2h_8}\right)\widehat{\text{vol}}_{\text{CY}_2}\wedge\widehat{\text{vol}}_{S^3}\wedge d\rho\,,\\
\widehat{F}_{(10)}&=\left(\rho\widehat{h}_4-\partial_\rho\frac{\rho uu'-u^2}{4h_8}\right)\widehat{\text{vol}}_{AdS_2}\wedge\widehat{\text{vol}}_{S^3}\wedge\widehat{\text{vol}}_{\text{CY}_2}\wedge d\rho\,.
\end{aligned}\tag{3.78}$$

Given these fluxes, we can now aim at studying what kind of branes are present in our supergravity background and how their charge is computed.

- Let us begin with the case of $D0$ branes. These are electric sources for a two form, $\widehat{F}_{(2)}$ say, which, as a matter of fact, has electric nature ($\widehat{F}_{(2)}$ has a "leg" along the time direction t, see (3.78)). On the other hand, they can also be thought of as magnetic sources for $\widehat{F}_8$ as given in (3.78). From the fact that $d\widehat{F}_8$ vanishes identically, the $D0$ branes are dissolved into fluxes and are to be thought of as "colour" branes. Their charge is given by

$$Q_{D0} = \frac{1}{(2\pi)^7}\text{vol}_{\text{CY}_2}\text{vol}_{S^3}\int_0^{2\pi(P+1)} d\rho\left(2\widehat{h}_4 - \partial_\rho \frac{uu'}{2h_8}\right) \tag{3.79}$$

- Consider now the case of $D2$ branes. They are supposed to source magnetically $\widehat{F}_{(6)}$. However, $\widehat{F}_{(6)}$ has a purely electric nature. We conclude that there are no $D2$ branes.

- Let us move on to the case of $D4$ branes.

  From the definition of $\widehat{F}_{(4)}$ given in (3.78) we identify two four-manifolds supporting fluxes of $\widehat{F}_{(4)}$. These are $\mathcal{M}_4' = \text{CY}_2$ and $\mathcal{M}_4 = S^3 \times \mathcal{I}_\rho$. We find

$$\begin{aligned} Q_{D4'} &= \frac{\text{vol}_{\text{CY}_2}}{(2\pi)^3}\int_{\mathcal{I}_\rho} d\rho\,\widehat{h}_4''\,, \\ Q_{D4} &= \frac{\text{vol}_{S^3}}{(2\pi)^3}\int_0^{2\pi(P+1)} d\rho\left(2h_8 - \partial_\rho\frac{uu'}{2\widehat{h}_4}\right). \end{aligned} \tag{3.80}$$

  Note that

$$d\widehat{F}_{(4)} = \widehat{h}_4''\,d\rho \wedge \widehat{\text{vol}}_{\text{CY}_2}\,. \tag{3.81}$$

  Thus, the $D4$ branes counted by $Q_{D4'}$ provide localised sources if we allow the "Bianchi identity" $\widehat{h}_4'' = 0$ to be violated at points. They are localised somewhere in the ρ direction. We refer to them as "flavour" $D4$ branes, and denote them as $D4'$. Being localised in ρ and transverse to the CY, they are naturally seen to warp $AdS_2 \times S^3$.

  The branes whose charge is given by Q_{D4} are instead dissolved into fluxes and therefore do not provide additional physical sources. They must be thought of as "colour" $D4$ branes.

- The fact that $\widehat{F}_8$ has no electric components tells us that there are no $D6$ branes.

- Let us consider $D8$ branes. They are electric sources for $\widehat{F}_{(10)}$, which is necessarily of electric nature. Note also that $d\widehat{F}_{(0)} \neq 0$ if $h_8'' \neq 0$. According to (3.82), we have a natural definition of $D8$ branes as objects localised somewhere in the ρ direction. This, in turn, leads to the fact that $D8$ branes are not dissolved into fluxes, and effectively behave as flavour branes. They are stretched along $AdS_2 \times S^3 \times \text{CY}_2$ and their charges is given by

$$Q_{D8} = 2\pi \int_{\mathcal{I}_\rho} d\widehat{F}_{(0)} \,, \qquad d\widehat{F}_{(0)} = h_8'' d\rho \,, \tag{3.82}$$

- The electric nature of $H_{(3)}$ tells us that there are fundamental strings but not $NS5$ branes. Fundamental strings are stretched in the (t, r) directions and the fact that $d(e^{-2\phi} \star H_{(3)})$ is non-vanishing tells us that these $F1$ are physical objects in the background.

The brane web characterising our background is given in Table 3.2.

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
$D0$	–									
$D4$	–	–	–	–	–					
$D4'$	–						–	–	–	–
$D8$	–	–	–	–	–		–	–	–	–
$F1$	–					–				

Table 3.2: Brane setup, where – mark the spacetime directions spanned by the various branes. x^0 corresponds to the time direction of the ten dimensional spacetime, $x^1, \ldots, x^4$ are the coordinates spanned by the CY_2, x^5 is the field theory space. The coordinates x^6, …, x^9 are the coordinates realising an $SO(4)$ rotational symmetry.

We now argue that the quantisation of brane charges comes naturally when $u' = 0$. Let us see how this plays out.

Charge quantisation

In order to discuss charge quantisation in more detail, it turns out to be useful to consider again the profiles (3.3) and (3.4) for $\widehat{h}_4$ and h_8. We report them here for the sake of readibility[14].

$$h_4(\rho) = \begin{cases} \frac{\beta_0}{2\pi}\rho & \rho \in [0, 2\pi] \\ \alpha_k + \frac{\beta_k}{2\pi}(\rho - 2\pi k) & \rho \in [2\pi k, 2\pi(k+1)] \\ \alpha_P - \frac{\alpha_P}{2\pi}(\rho - 2\pi P) & \rho \in [2\pi P, 2\pi(P+1)]\,, \end{cases} \tag{3.83}$$

$$h_8(\rho) = \begin{cases} \frac{\nu_0}{2\pi}\rho & \rho \in [0, 2\pi] \\ \mu_k + \frac{\nu_k}{2\pi}(\rho - 2\pi k) & \rho \in [2\pi k, 2\pi(k+1)] \\ \mu_P - \frac{\mu_P}{2\pi}(\rho - 2\pi P) & \rho \in [2\pi P, 2\pi(P+1)]\,, \end{cases} \tag{3.84}$$

We consider an everywhere constat u, $u = u_0$. Here $k = 1, \ldots, P-1$ with P some, possibly large, integer. When $u' = 0$, the second equation in (3.80) and equation (3.79) reduce to

$$Q_{D4} = \left(2\frac{\mathrm{vol}_{S^3}}{8\pi^3}\right)\int_0^{2\pi(P+1)} d\rho\, h_8\,, \quad Q_{D0} = \left(2\frac{\mathrm{vol}_{S^3}}{8\pi^3}\right)\left(\frac{\Upsilon \mathrm{vol}_{\mathrm{CY}_2}}{16\pi^4}\right)\int_0^{2\pi(P+1)} d\rho\, h_4\,. \tag{3.85}$$

Using $\mathrm{vol}_{S^3} = 2\pi^2$ and choosing Υ such that $\Upsilon\mathrm{vol}_{\mathrm{CY}_2} = 16\pi^4$, we get for the total number of colour $D4$ and $D0$ branes

$$Q_{D4} = \frac{1}{2\pi}\int_0^{2\pi(P+1)} d\rho\, h_8\,, \quad Q_{D0} = \frac{1}{2\pi}\int_0^{2\pi(P+1)} d\rho\, h_4\,. \tag{3.86}$$

We can find similar formulas for flavour branes. Computing $d\widehat{F}_{(4)}$, with $\widehat{F}_{(4)}$ defined as in (3.78), we find

$$d\widehat{F}_{(4)} = \widehat{h}_4''\, d\rho \wedge \widehat{\mathrm{vol}}_{\mathrm{CY}_2}\,. \tag{3.87}$$

For the class of solutions defined in (3.83) and (3.84) we get

$$d\widehat{F}_{(4)} = \Upsilon \sum_{k=1}^{P}\left(\frac{\beta_{k-1} - \beta_k}{2\pi}\right)\delta(\rho - 2\pi k) d\rho \wedge \mathrm{vol}_{\mathrm{CY}_2}\,. \tag{3.88}$$

Using again $\Upsilon\mathrm{vol}_{\mathrm{CY}_2} = 16\pi^4$, (3.88) tells us that there are semi-localised flavour $D4$ branes at $\rho = 2\pi k$. Their number is counted by

$$Q_{D4}^{(k)} = \beta_{k-1} - \beta_k\,. \tag{3.89}$$

[14] Why this is at all possible, also for the case of $AdS_2 \times S^3 \times \mathrm{CY}_2$ backgrounds, is explained carefully in Appendix L.

Thus, if there is a change in the slope of $\widehat{h}_4$ at $\rho = 2\pi k$, so that $\beta_{k-1} - \beta_k \neq 0$, we will have $D4$ flavour branes. In particular, we see also that it must be $\beta_{k-1} > \beta_k$, in order to have positive charge.

A similar analysis goes through for the $D8$ flavour branes. We find at $\rho = 2\pi k$

$$Q_{D8}^{(k)} = \nu_{k-1} - \nu_k \,. \tag{3.90}$$

The total number of flavour $D4$ and $D8$ can be found either by summing (3.89) and (3.90) over all $k = 1, \dots, P$ or by using (3.80) and (3.82). In any case the result is

$$\begin{aligned} Q_{D4'} &= 2\pi\Big(h_4'(0) - h_4'(2\pi(P+1))\Big)\,, \\ Q_{D8} &= 2\pi\Big(h_8'(0) - h_8'(2\pi(P+1))\Big)\,. \end{aligned} \tag{3.91}$$

3.3 Holographic central charge

In this section we discuss the holographic central charge associated with the background (3.76) and (3.77). This section is very similar in spirit to the corresponding section for the $AdS_2 \times S^2$ backgrounds. We just give a few more details on why it is difficult at first to define a holographic central charge in one dimension. The reader might feel free to skip to the main results. Also this time it is possible to "derive" the central charge from an action functional. This is reviewed in Appendix L.

Let us recast the metric in (3.76) in the following fashion

$$ds^2 = a(r,\theta)\big(-dt^2 + b(r)dr^2\big) + g_{ij}(r,\theta)d\theta^i d\theta^j\,, \tag{3.92}$$

with θ denoting collectively the coordinates parametrising the internal manifold and

$$a(r,\theta) = \frac{u\sqrt{\widehat{h}_4 h_8}}{4\widehat{h}_4 h_8 - (u')^2} r^2\,, \quad b(r) = \frac{1}{r^4}\,. \tag{3.93}$$

Following [41], we can define the quantities

$$\mathcal{V}_{\text{int}} = \int d\theta \sqrt{e^{-4\Phi}\det(g_{ij})a(r,\theta)^d}\,, \qquad \widehat{H} = \mathcal{V}_{\text{int}}^2 \tag{3.94}$$

The holographic central charge, when the dual CFT is $(d+1)$-dimensional, is then defined as

$$c_{\text{holo}} = \frac{d^d}{G_N} b(r)^{d/2} \frac{\widehat{H}^{\frac{2d+1}{2}}}{(\widehat{H}')^d}\,. \tag{3.95}$$

We note that, in our case,

$$\det(g_{ij}) = \frac{u^2 \widehat{h}_4}{h_8^3}\,, \qquad e^{-4\Phi} = \frac{h_8^3}{16\widehat{h}_4 u^2}\left(4\widehat{h}_4 h_8 - (u')^2\right)^2\,, \tag{3.96}$$

while $d = 0$. Thus we find for $\mathcal{V}_{\text{int}}$

$$\mathcal{V}_{\text{int}} = \frac{\text{vol}_{\text{CY}_2}\text{vol}_{S^3}}{4}\int_0^{2\pi(P+1)} d\rho \big(4\widehat{h}_4 h_8 - (u')^2\big)\,. \tag{3.97}$$

Note that $\mathcal{V}_{\text{int}}$, and hence $\widehat{H}$, is r-independent. Because of $\widehat{H}' = 0$ and $d = 0$, the formula (3.95) is not immediately meaningful. The only quantity that seems to make sense is therefore $\mathcal{V}_{\text{int}}$, just as discussed previously. In particular, for the class of solutions for which $u =$ constant everywhere, we find

$$\mathcal{V}_{\text{int}} = \text{vol}_{\text{CY}_2}\text{vol}_{S^3}\int_0^{2\pi(P+1)} d\rho\, \widehat{h}_4 h_8\,. \tag{3.98}$$

This coincides, modulo unimportant constant factors, to the 2d central charge derived in (2.37) and in the previous section. We are then led to interpret (3.98) as the quantity to be matched with the central charge for the field theories dual to the family of backgrounds in (3.76) and (3.77).

Closing remarks

We finish off this section by giving a couple of important remarks about the $AdS_2 \times S^3 \times \text{CY}_2$ and its dual field theories.

The quantum-mechanical dual theory will be presented in [6]. The construction of the dual UV QM can be easily carried out by employing open string quantisation for the system $D0 - D4 - D4' - D8 - F1$ discussed previously. The latter is readily done by following the rules of e.g. [155, 156]. Similar systems, discussing webs of $D0$-$D4$-$F1$ were discussed, for instance, in [157, 158, 159].

It is remarkable to have a fully backreacted geometry involving $D0$ and $D4$ branes. In particular, in the case where the Bianchi identities are taken to hold globally and the geometries are then smooth everywhere – see Appendix L for details on this point – we have a fully backreacted solution of an instantonic configuration with fundamental strings [158].

The $AdS_2 \times S^3 \times \text{CY}_2$ backgrounds discussed in this section were obtained from an analytical continuation of the $AdS_3 \times S^2 \times \text{CY}_2$ solutions of [17]. There, the supercharges were given to belong to the $(\mathbf{1},\mathbf{2};\mathbf{2})$ of $Sl(2,\mathbb{R})\times Sl(2,\mathbb{R})\times SU(2)_R$, realised geometrically on the AdS_3 and S^2 factors. Thus, it seems natural to postulate that the supersymmetry generators are singlets of one of the $SU(2)$ of $SO(4)=SU(2)\times SU(2)$ after the analytical continuation. We then make the assertion that the dual CFT just provides another realisation of the $\mathfrak{su}(1,1|2)$ superalgebra in one dimension, and that the second $SU(2)$ gives a flavour non R-symmetry.

4 Conclusions

In this chapter we have introduced two new families of AdS_2 backgrounds. The first family has the structure of a warped $AdS_2\times S^2\times \mathrm{CY}_2$ background while the second has the structure $AdS_2\times S^3\times \mathrm{CY}_2$ and they are both obtained after some manipulation of the $AdS_3\times S^2\times \mathrm{CY}_2$ system proposed in [17]. They are argued to realise an $\mathfrak{su}(1,1|2)$ superconformal algebra for a one-dimensional quantum-mechanical problem. A prescription for the dual CFT of the $AdS_2\times S^2\times \mathrm{CY}_2$ backgrounds is readily given as dimensional reduction of the "parent" two-dimensional CFT dual to the $AdS_3\times S^2\times \mathrm{CY}_2$ background of [19]. The definition of the dual QM for the $AdS_2\times S^3\times \mathrm{CY}_2$ system is more subtle and will be given in [6]. In both cases, we gave a prescription for computing the central charge. Remarkably, it follows from a minimisation principle which involves a functional given by the product of suitably defined "electric" and "magnetic" charges.

Many questions remain unanswered. It would be nice to test our conjectures in a number of different ways. It is believed that an index might capture the holographic central charge as defined in this chapter. Papers which compute superconformal indices for quantum mechanical problems comprise of e.g. [160, 161, 157]. Also, it would be interesting to consider the limit of everywhere smooth geometries and explore compactification down to AdS_2 spacetime. Fluctuations – spin 2 in the case of warped spacetime – of the backgrounds could be studied and dual operators might be constructed explicitly. Finally, it would be nice to understand what black holes have a near-horizon geometry in the class described in this section, and what the freedom of choosing our backgrounds implies for black-hole physics.

K Dimensional reduction to Quantum Mechanical systems

Here, we discuss in some detail how to dimensionally reduce $\mathcal{N} = (0,2)$ and $\mathcal{N} = (0,4)$ systems down to one dimension. This appendix is mostly relevant for the $AdS_2 \times S^2 \times$ CY_2 solutions to Type IIB supergravity discussed in the main text.

K.1 Reduction of $\mathcal{N} = (0,2)$ supersymmetry multiplets to one dimension

In $\mathcal{N} = 2$ Quantum Mechanics we have two real supercharges with an $SO(2)$ R-symmetry. Equivalently, they can be rearranged as two complex supercharges Q and $\overline{Q}$ with a reality constraint, and $U(1)$ R-symmetry. They satisfy the algebra

$$Q^2 = \overline{Q}^2 = 0\,, \quad \{Q, \overline{Q}\} = H\,, \tag{3.99}$$

with H the hamiltonian. Moreover, if we denote by J the R-symmetry generator we have

$$[J, Q] = -Q\,, \quad [J, \overline{Q}] = \overline{Q}\,, \quad [J, H] = 0\,. \tag{3.100}$$

Let us see what $\mathcal{N} = 2$ supermultiplets in quantum mechanics are relevant to us. Much of the construction is obtained from the dimensional reduction of 2d $\mathcal{N} = (0,2)$ supersymmetric systems.

As discussed already in Appendix J of the previous chapter, the 2d $\mathcal{N} = (0,2)$ vector multiplet consists of a two-dimensional gauge field v_μ, a left-handed (complex) fermionic field ζ_- and a real auxiliary field D. They are all valued in the adjoint representation of the corresponding gauge group G. In the following we will just set $\zeta_- \equiv \zeta$, as there is no chirality in 1d. After reduction, we have $v_\mu = (v_t, \sigma)$, where v_t is the one dimensional gauge field and σ a real scalar. The supersymmetric kinetic term for an $\mathcal{N} = 2$ vector multiplet in quantum mechanics is

$$L_{\text{vector}} = \frac{1}{2g^2}\mathrm{tr}\left[(D_t\sigma)^2 + i\bar{\zeta}D_t^{(+)}\zeta + D^2\right]\,. \tag{3.101}$$

where $D_t^{(\pm)} = D_t \pm i\sigma$ and D_t is the usual covariant derivative $D_t = \partial_t + iv_t$ for fields in generic representation of the gauge group. Notice that this is nothing but the dimensional reduction to one dimension of the lagrangian (2.57) given in the previous chapter.

A 2d $\mathcal{N}=(0,2)$ chiral multiplet consists of a complex scalar boson and a right-handed (complex) fermionic field ψ_+ in some unitary representation of the gauge group. As before, we will only be concerned with the fundamental and adjoint representations. Again, in going down to 1d we will drop the sub-index. The supersymmetric kinetic term for an $\mathcal{N}=2$ chiral multiplet in quantum mechanics reads

$$L_{\text{chiral}} = D_t\bar{\phi}D_t\phi + i\bar{\psi}D_t^{(-)}\psi + \bar{\phi}(D-\sigma^2)\phi - i\sqrt{2}\bar{\phi}\lambda\psi - i\sqrt{2}\bar{\psi}\bar{\lambda}\phi\,. \tag{3.102}$$

Again, this is just the dimensional reduction of the action (2.61).

A 2d $\mathcal{N}=(0,2)$ fermi multiplet consists of a left-handed (complex) fermion ψ_- and an auxiliary field G. In the following, we will make the identification $\psi_- \equiv \eta$ and $\psi_{+,i} \equiv \psi_i$ if $(\phi_i, \psi_{+,i})$ is a chiral multiplet. The lagrangian for a generic Fermi multiplet reads

$$L_{\text{fermi}} = i\bar{\eta}D_t^{(+)}\eta + |G^2| - |E(\phi_i)|^2 - \bar{\eta}\frac{\partial E}{\partial \phi_i}\psi_i - \bar{\psi}_i\frac{\partial E}{\partial \bar{\phi}_i}\eta\,, \tag{3.103}$$

to be compared with (2.65) in the previous chapter.

In addition to the E-term potentials it is possible, for each Fermi multiplet Ψ_a, to introduce a holomorphic function $J^a(\Phi_i)$ which gives rise to a interactions of the form

$$L_J = G^a J_a(\phi_i) + \sum_i \eta_a \frac{\partial J^a}{\partial \phi^i}\psi_i + \text{h.c.}\,. \tag{3.104}$$

As remarked already in Appendix J, the superpotentials E and J cannot be introduced independently. In order for supersymmetry to be preserved, they must satisfy $\sum_a E_a J^a = 0$.

K.2 $\mathcal{N}=4$ supersymmetric systems

The $\mathcal{N}=4$ supermultiplets that are relevant to our construction are given just by dimensional reduction of $\mathcal{N}=(0,4)$ and $\mathcal{N}=(4,4)$ supermultiplets. Two-dimensional $\mathcal{N}=(0,4)$ supermultiplets are given in terms of $\mathcal{N}=(0,2)$ multiplets as follows[15].

[15] See also Appendix J for further details.

Multiplets	$\mathcal{N}=(0,2)$ building blocks	component fields	$SU(2)_L \times SU(2)_R$
Vector:	Vector + Fermi (U,Θ)	(u_μ, ζ_-^a, G^A)	$(\mathbf{1},\mathbf{1}),(\mathbf{2},\mathbf{2}),(\mathbf{3},\mathbf{1})$
Hyper:	Chiral + Chiral $(\Phi,\tilde{\Phi})$	(ϕ^a, ψ_+^b)	$(\mathbf{2},\mathbf{1}),(\mathbf{1},\mathbf{2})$
Twisted hyper:	Chiral + Chiral $(\Phi',\tilde{\Phi}')$	(ϕ'^a, ψ'^b_+)	$(\mathbf{1},\mathbf{2}),(\mathbf{2},\mathbf{1})$
Fermi	Fermi: + Fermi $(\Gamma,\tilde{\Gamma})$	$(\psi'^a_-, \tilde{G}^b)$	$(\mathbf{1},\mathbf{1}),(\mathbf{2},\mathbf{2})$

An $\mathcal{N}=(0,4)$ supersymmetric system enjoys an $SO(4)$ R-symmetry. Thus, supersymmetries can be given in terms of a vector $Q^i \in \mathbf{4}$ of $SO(4)$. Of course, we can exploit the isomorphism $SO(4) = SU(2)_L \times SU(2)_R$ for which the $\mathbf{4}$ decomposes as $\mathbf{4} = (\mathbf{2},\mathbf{2})$. The supercharges are given by a matrix Q^{ij} with a reality constraint

$$Q^{ij} = \epsilon^{ik}\epsilon^{jl} Q^\dagger_{jl}\,, \tag{3.105}$$

with $i,j,k,l = 1,2$. Of course, as it is clear from above, all fields in each supermultiplet are organised under representation of $SU(2)_L \times SU(2)_R$.

Also, $\mathcal{N}=(4,4)$ supermultiplets decompose in terms of $\mathcal{N}=(0,4)$ supermultiplets in the following fashion

Multiplets	$\mathcal{N}=(0,4)$ building blocks	$\mathcal{N}=(0,2)$ building blocks
Vector	Vector + Twisted Hyper	$(U,\Theta),(\Sigma,\tilde{\Sigma})$
Hyper	Hyper + Fermi	$(\Phi,\tilde{\Phi}),(\Gamma,\tilde{\Gamma})$

The dimensional reduction of the 2d theory depicted in Figure 2.1 is then readily done according to the rules above. In particular, a two-dimensional gauge field always reduces to one-component gauge field plus a scalar in one dimension. Scalars and fermions remain untouched. In the case of the fermions, this is due to the fact that in both one and two dimensions the minimal spinor representation is one-dimensional.

Before ending this section, let us give one remark about the R-symmetry of the IR theory.

K.3 R-symmetry

The R-symmetry group of supersymmetric $\mathcal{N} = 4$ quantum mechanics is $SO(4) = SU(2)_L \times SU(2)_R$. As we flow to the IR and hit a fixed point, given that it exists, we should find that our quantum mechanics realises some classified superconformal algebra. When $\mathcal{N} = 4$, we have essentially two possibilities. See Table 3.3.

supersymmetry	superalgebra	R-symmetry
$\mathcal{N} = 1$	$\mathfrak{osp}(1\|2)$	1
$\mathcal{N} = 2$	$\mathfrak{su}(1,1\|1)$	$U(1)$
$\mathcal{N} = 3$	$\mathfrak{osp}(3\|2)$	$SU(2)$
$\mathcal{N} = 4$	$\mathfrak{su}(1,1\|2)$	$SU(2)$
	$\mathfrak{d}(2,1;\alpha)$	$SU(2) \times SU(2)$
$\mathcal{N} = 5$	$\mathfrak{osp}(5\|2)$	$SO(5)$
$\mathcal{N} = 6$	$\mathfrak{su}(1,1\|3)$	$SU(3) \times U(1)$
	$\mathfrak{osp}(6\|2)$	$SO(6)$
$\mathcal{N} = 7$	$\mathfrak{osp}(7\|2)$	$SO(7)$
	$\mathfrak{g}(3)$	$G(2)$
$\mathcal{N} = 8$	$\mathfrak{osp}(8\|2)$	$SO(8)$
	$\mathfrak{su}(1,1\|4)$	$SU(4) \times U(1)$
	$\mathfrak{osp}(4^*\|4)$	$SU(2) \times SO(5)$
	$\mathfrak{f}(4)$	$SO(7)$

Table 3.3: Simple superalgebras that contain an $\mathfrak{sl}(2,\mathbb{R})$.

The $\mathfrak{d}(2,1;\alpha)$ global algebra is often referred to as *large* superconformal algebra, and α is a parameter which parametrises the relative strength of the two Kac-Moody levels, k_- and k_+ of the $SU(2)$ R-symmetries. In the $AdS_2 \times S^2 \times \mathrm{CY}_2$ we have only one $SU(2)$ (realised geometrically on the S^2). Given that in the parent $AdS_3 \times S^2$ supersymmetries were in the $(\mathbf{1}, \mathbf{2}; \mathbf{2})$ of $Sl(2,\mathbb{R}) \times Sl(2,\mathbb{R}) \times SU(2)_R$, we are naturally led to the conclusion that the superalgebra realised by our IIB backgrounds and the dual field theories is the $\mathfrak{su}(1,1|2)$ superalgebra[16].

Also, superalgebras in one and two dimensions are closely related – each chiral sector of a 2d SCFT provides a superalgebra and its realisation for a 1d superconformal QM – and this makes it possible to identify central charges in 1d and 2d [153].

[16] The $\mathfrak{su}(1,1|2)$ is also realised by taking the limit $\alpha \to \infty$ in the $\mathfrak{d}(2,1;\alpha)$ algebra.

L Further aspects of the $AdS_2 \times S^3 \times$ CY$_2$ solutions

In this appendix, we give further details on the $AdS_2 \times S^3 \times \mathrm{CY}_2$ backgrounds. In particular, we give a careful study of the geometry near special points. Also, we compute Chern-Simons coupling constants from probe branes. Finally, we give an expression for the holographic central charge in terms of R-R fluxes.

L.1 Behaviour of the background near special points

Let us begin our study of the behaviour of the background (3.76), (3.77) close to particular points. As we are concerned with spacetimes with finite extention in the ρ direction (ρ belongs to a compact interval, call it $\mathcal{I}_\rho$), either or both $\widehat{h}_4$ and h_8 vanish at both ends of $\mathcal{I}_\rho$. We then have three possibilities for each endpoint. We discuss them in turn[17].

Asymptotic region $\rho \to 0$

Let us begin by focussing on the region close to $\rho = 0$, where the space begins in the ρ direction. In the following we will assume that u is a constant function everywhere[18], $u = u_0$.

- $\widehat{h}_4(0) = 0\,, h_8(0) \neq 0\,.$ Let us set, close to $\rho = 0$,

$$\widehat{h}_4 = \Upsilon \frac{\beta_0}{2\pi}\rho\,, \qquad h_8 = \mu_0 + \frac{\nu_0}{2\pi}\rho\,. \tag{3.106}$$

  The background metric and the dilaton given in (3.76) have the asymptotic form as $\rho \to 0$

$$\begin{aligned} ds^2 &= \sqrt{\frac{\pi u_0^2}{8\Upsilon\beta_0\mu_0}}\rho^{-1/2}\left(ds^2_{AdS_2} + 4ds^2_{S^3}\right) + \sqrt{\frac{\Upsilon\beta_0}{2\pi\mu_0}}\rho^{1/2}(ds^2_{\mathrm{CY}_2} + \mu_0 d\rho^2)\,, \\ e^{-4\Phi} &= \mu_0^5\frac{\beta_0}{2\pi}\rho\,. \end{aligned} \tag{3.107}$$

  This is nothing but the background around an O4 plane located at $\rho = 0$ and stretched along $AdS_2 \times S^3$.

[17] Here, and just for the analysis given here for the background geometry, we allow either $\widehat{h}_4$ or h_8 to be non-vanishing at $\rho = 0$ and $\rho = 2\pi(P+1)$, but not simultaneously. This kind of behaviour was also considered in detail in [120].

[18] This was motivated by the requirement of flux quantisation.

- $\widehat{h}_4(0) \neq 0\,, h_8(0) = 0\,.$ Let us consider, still close to $\rho = 0$,

$$\widehat{h}_4 = \Upsilon\left(\alpha_0 + \frac{\beta_0}{2\pi}\rho\right)\,, \qquad h_8 = \frac{\nu_0}{2\pi}\rho\,. \tag{3.108}$$

The metric and the dilaton, close to $\rho = 0$, read

$$\begin{aligned} ds^2 &= \sqrt{\frac{\pi u_0^2}{8\Upsilon\alpha_0\nu_0}}\frac{1}{\rho^{1/2}}\left(ds^2_{AdS_2} + 4ds^2_{S^3}\right) + \sqrt{\frac{\Upsilon\alpha_0\nu_0}{2\pi u_0^2}}\frac{1}{\rho^{1/2}}ds^2_{\text{CY}_2} + \sqrt{\frac{\Upsilon\alpha_0\nu_0}{2\pi u_0^2}}\rho^{1/2}d\rho^2\,, \\ e^{-4\Phi} &= \left(\frac{\nu_0}{2\pi}\right)^5\frac{\Upsilon\alpha_0}{u_0^2}\rho^5\,. \end{aligned} \tag{3.109}$$

We recognise the background of an O8 plane at $\rho = 0$ and stretched along $AdS_2 \times S^3 \times \text{CY}_2$.

- $\widehat{h}_4(0) = 0\,, h_8(0) = 0\,.$ Now, $\widehat{h}_4$ and h_8 are given, as ρ approaches zero, by

$$\widehat{h}_4 = \Upsilon\frac{\beta_0}{2\pi}\rho\,, \qquad h_8 = \frac{\nu_0}{2\pi}\rho\,. \tag{3.110}$$

For very small values of ρ, the metric and the dilaton are given by the following expressions

$$\begin{aligned} ds^2 &= \frac{\pi u_0}{\sqrt{\Upsilon\beta_0\nu_0}\,\rho}\frac{1}{\rho}\left(ds^2_{AdS_2} + 4ds^2_{S^3}\right) + \sqrt{\frac{\Upsilon\beta_0}{\nu_0}}ds^2_{\text{CY}_2} + \frac{\sqrt{\Upsilon\beta_0\nu_0}}{2\pi}\rho\,d\rho^2\,, \\ e^{-4\Phi} &= \left(\frac{\nu_0}{2\pi}\right)^5\frac{\Upsilon\beta_0}{2\pi u_0^2}\rho^6\,. \end{aligned} \tag{3.111}$$

We recognise here a superposition of O4 and O8 planes.

Asymptotic region $\rho \to 2\pi(P+1)$

In order for the space to close off at some value $\rho = \bar{\rho}$, we need either or both $\widehat{h}_4$ and h_8 to vanish at $\bar{\rho}$. Again we have three cases. From charge quantisation (to be discussed later) it is also useful to set once again $\bar{\rho} = 2\pi(P+1)$, with P an integer. Again, we assume $u = u_0$.

- $\widehat{h}_4(\bar{\rho}) = 0\,, h_8(\bar{\rho}) \neq 0\,.$ Now $\widehat{h}_4$ and h_8 are given by

$$\widehat{h}_4 = \Upsilon\left(\alpha_P - \frac{\alpha_P}{2\pi}(\rho - 2\pi P)\right)\,, \qquad h_8 = \mu_P + \frac{\nu_P}{2\pi}(\rho - 2\pi P)\,. \tag{3.112}$$

Let us define $x = 2\pi(P+1) - \rho$ and expand for small positive x

$$\begin{aligned} ds^2 &= \sqrt{\frac{\pi u_0^2}{8\Upsilon\alpha_P(\mu_P+\nu_P)}}x^{-1/2}\left(ds^2_{AdS_2} + 4ds^2_{S^3}\right) + \sqrt{\frac{\Upsilon\alpha_P}{2\pi(\mu_P+\nu_P)}}x^{1/2}\left(ds^2_{\mathrm{CY}_2} + \frac{\mu_P+\nu_P}{a_0}dx^2\right)\,, \\ e^{-4\Phi} &= \frac{(\mu_P+\nu_P)^5}{u_0^2}\frac{\Upsilon\alpha_P}{2\pi}x\,. \end{aligned} \tag{3.113}$$

Again, we identify the metric of an O4 plane stretched along $AdS_2 \times S^3$.

- $\widehat{h}_4(\bar\rho) \neq 0\,, h_8(\bar\rho) = 0\,.$ In this case we have

$$\widehat{h}_4 = \Upsilon\left(\alpha_P + \frac{\beta_P}{2\pi}(\rho - 2\pi P)\right)\,, \qquad h_8 = \mu_P - \frac{\mu_P}{2\pi}(\rho - 2\pi P)\,. \tag{3.114}$$

Expanding again for small and positive $x = 2\pi(P+1) - \rho$ we find

$$\begin{aligned} ds^2 &= \sqrt{\frac{\pi u_0^2}{8\Upsilon\mu_P(\alpha_P+\beta_P)}}x^{-1/2}\left(ds^2_{AdS_2} + 4ds^2_{S^3} + 4\frac{\Upsilon(\alpha_P+\beta_P)}{a_0}ds^2_{\mathrm{CY}_2}\right) \\ &\quad + \sqrt{\frac{\Upsilon\mu_P(\alpha_P+\beta_P)}{2\pi a_0^2}}x^{1/2}dx^2\,, \\ e^{-4\Phi} &= \left(\frac{\mu_P}{2\pi}\right)^5\frac{\Upsilon(\alpha_P+\beta_P)}{u_0^2}x^5\,. \end{aligned} \tag{3.115}$$

Again, we recognise the metric of an O8 plane located at $\rho = 2\pi(P+1)$ and stretched along $AdS_2 \times S^3 \times \mathrm{CY}_2$.

- $\widehat{h}_4(\bar\rho) = 0\,, h_8(\bar\rho) = 0\,.$ In this case we have

$$\widehat{h}_4 = \Upsilon\left(\alpha_P - \frac{\alpha_P}{2\pi}(\rho - 2\pi P)\right)\,, \qquad h_8 = \mu_P - \frac{\mu_P}{2\pi}(\rho - 2\pi P)\,. \tag{3.116}$$

Expanding again for small x as before we find

$$\begin{aligned} ds^2 &\simeq \sqrt{\frac{\pi^2 a_0^2}{4\Upsilon\alpha_P\mu_P}}x^{-1}\left(ds^2_{AdS_2} + 4ds^2_{S^3}\right) + \sqrt{\frac{\Upsilon\alpha_P}{\mu_P}}ds^2_{\mathrm{CY}_2} + \sqrt{\frac{\Upsilon\alpha_P\mu_P}{4\pi^2 a_0^2}}x dx^2\,, \\ e^{-4\Phi} &\simeq \left(\frac{\mu_P}{2\pi}\right)^5\frac{\Upsilon\alpha_P}{2\pi a_0^2}x^6\,. \end{aligned} \tag{3.117}$$

This time we recognise the background generated by the superposition of an O4 and an O8 plane at $\rho = 2\pi(P+1)$.

L.2 Chern-Simons coupling constants

We now probe the background in (3.76) and (3.77) by introducing probe D branes. The action describing the coupling of a generic Dp brane to the N-S and R-R closed string fields is the DBI + WZ term given already in (3.28).

- Let us begin by considering probe $D0$ branes. The field theory living on a $D0$ brane is $(0+1)-$dimensional. We can define a metric for such a $(0+1)-$dimensional field theory from the pullback of (3.76),

$$ds^2_{ind} = -\frac{u\sqrt{\widehat{h}_4 h_8}}{4\widehat{h}_4 h_8 - (u')^2} dt^2 \,. \tag{3.118}$$

Note that the pullback of $B_{(2)}$ and the field strength $\mathcal{F}_{ab}$ on the $D0$ brane are automatically zero, due to their (anti)symmetric properties. Given that

$$e^{-\Phi}\sqrt{\det g_{ind}} = \frac{h_8(\rho_*)}{2}\cosh(r_*) \,, \tag{3.119}$$

where the "$*$" simply refers to the fact that we are keeping r and ρ fixed at some values, for a probe $D0$ brane we find the following action

$$S_{D0} = -T_{D0}\int_{\mathbb{R}} dt e^{-\Phi}\sqrt{\det g_{ind}} + T_{D0}\int C_{(1)} \,, \tag{3.120}$$

which leads to

$$S_{D0} = -T_{D0}\frac{h_8(\rho_*)}{2}\cosh(r_\star)\int_{\mathbb{R}} dt + T_{D0}\frac{\mu_k}{2}\sinh(r_\star)\int_{\mathbb{R}} dt \,, \tag{3.121}$$

If we choose $\rho_* = 2\pi k$, we find

$$S_{D0} = T_{D0}\frac{\mu_k}{2}(\sinh(r_*) - \cosh(r_*))\int_{\mathbb{R}} dt \,. \tag{3.122}$$

We find that the brane is "calibrated" only when $r_* = \infty$, for which, however, we have a vanishing action.

Consider instead the coupling of a $D0$ brane to a one form, call it $\mathcal{A}_{(1)}$. An action describing such a coupling is

$$S = 2\pi T_{D0} \int F_{(0)} \mathcal{A}_{(1)} \,. \tag{3.123}$$

If, as in (3.77), $F_{(0)} = h_8'$, and using (3.84), we get

$$S = 2\pi T_{D0} \frac{\nu_k}{2\pi} \int_{\mathbb{R}} \mathcal{A}_{(t)} dt \,. \tag{3.124}$$

Given that $T_{D0} = 1$, we find that the action (3.124) describes a Chern-Simons term with $k_{CS} = \nu_k$. If we were to make sense of the action (3.124) also on a topologically non-trivial one-dimensional manifold, say on a circle S^1, we would find that it describes a gauge invariant action if and only if ν_k is an integer. Moreover, if parity is to be a symmetry of our system we can take, without loss of generality, ν_k to be positive as well.

- Let us move on to the case of a probe $D4$ branes stretched along (t, CY_2). The induced metric on the worldvolume of such a $D4$ brane, from the pullback of (3.76), reads

$$ds^2_{ind} = -\frac{u \widehat{h}_4 h_8}{4 \widehat{h}_4 h_8 - (u')^2} dt^2 + \sqrt{\frac{\widehat{h}_4}{h_8}} ds^2_{\mathrm{CY}_2} \,. \tag{3.125}$$

The pullback of $B_{(2)}$ vanishes, whereas we will ignore for now the field strength $\mathcal{F}_{ab}$ along the brane[19]. Given that

$$e^{-\Phi} \sqrt{\det g_{ind}} = \frac{\widehat{h}_4(\rho_*)}{2} \cosh(r_*) \,, \tag{3.126}$$

for a probe colour $D4$ brane the DBI action reads

$$S_{DBI} = -T_{D4} \mathrm{vol}_{\mathrm{CY}_2} \frac{\widehat{h}_4(\rho_*)}{2} \cosh(r_*) \int_{\mathbb{R}} dt \,. \tag{3.127}$$

Using that $T_{D4} = 1/(2\pi)^4$ and $\widehat{h}_4 = \Upsilon h_4$ we find

$$S_{DBI} = -\frac{\Upsilon \mathrm{vol}_{\mathrm{CY}_2}}{(2\pi)^4} \frac{h_4(\rho_*)}{2} \cosh(r_*) \int_{\mathbb{R}} dt \,. \tag{3.128}$$

[19] An α' expansion of the DBI action would produce a Maxwell kinetic term for $\mathcal{F}$. Eventually we will be interested in the dimensional reduction to 1 dimension where such a kinetic term would be absent.

Let us consider now the WZ part of the D4 brane action

$$S_{WZ} = T_{D4} \int C_{(5)} + 2\pi C_{(3)} \wedge \mathcal{F}_{(2)} + 4\pi^2 C_{(1)} \wedge \mathcal{F}_{(2)} \wedge \mathcal{F}_{(2)} \,. \tag{3.129}$$

From (3.19), $C_{(5)}|_{D4} = -\frac{1}{2}(\widehat{h}_4 - \widehat{h}_4'(\rho - 2\pi k)) \sinh r dt \wedge \widehat{\text{vol}}_{\text{CY}_2}$, where $dC_{(5)} = \widehat{F}_{(6)}$. The second term in the action, $2\pi C_{(3)} \wedge F_{(2)}$, understood as $2\pi \widehat{F}_4 \wedge \mathcal{A}_{(1)}$ after an integration by parts, will contribute as a Chern-Simons term, as we will see in a moment. Moreover, on the brane, we have $\widehat{F}_4 = \widehat{h}_4' \widehat{\text{vol}}_{\text{CY}_2}$.

Therefore, if we place the $D4$ brane at $\rho = 2\pi k$, its action reads

$$\begin{aligned} S &= S_{DBI} + T_{D4} \int C_{(5)} + 2\pi T_{D4} \int \widehat{F}_{(4)} \wedge \mathcal{A}_{(1)} \,, \\ &= T_{D4} \Upsilon \text{vol}_{\text{CY}_2} \frac{\alpha_k}{2} (\sinh(r_*) - \cosh(r_*)) \int_{\mathbb{R}} dt + 2\pi T_{D4} \int \widehat{F}_{(4)} \wedge \mathcal{A}_{(1)} \,. \end{aligned} \tag{3.130}$$

The Chern-Simons action reads explicitly

$$2\pi T_{D4} \int_{\text{CY}_2 \times \mathbb{R}} \widehat{F}_{(4)} \wedge \mathcal{A}_{(1)} = 2\pi T_{D4} \widehat{h}_4' \text{vol}_{\text{CY}_2} \int_{\mathbb{R}} \mathcal{A}_{(1)} \,. \tag{3.131}$$

Using that $\widehat{h}_4 = \Upsilon h_4$ and choosing Υ, as usual, such that $\Upsilon T_{D4} \text{vol}_{\text{CY}_2} = 1$ we find a CS action with level $k_{CS} = \beta_k$. Again, invariance under large gauge transformations implies that β_k has to be an integer. Parity allows us to take it to be positive.

L.3 Central charge from "electric-magnetic" charges

We mimic what we have done in Subsection 2.4 in order to obtain the holographic central charge of the $AdS_2 \times S^3 \times \text{CY}_2$ backgrounds from fluxes or, equivalently, from an extremisation procedure.

Again, consider a Dp brane as the electric source for $\widehat{F}_{(p+2)}$ and magnetic source for $\widehat{F}_{(8-p)}$ Page field strengths and define

$$q^e_{Dp} = \frac{1}{(2\pi)^p} \int \widehat{F}_{(p+2)} \,, \quad q^m_{Dp} = \frac{1}{(2\pi)^{7-p}} \int \widehat{F}_{(8-p)} \,. \tag{3.132}$$

When integrating over the non-compact AdS spacetime a regularisation procedure is understood.

Let us split each and every Page flux in (3.78) into their electric magnetic components as

$$\widehat{F}_{(k)} = \widehat{F}^{e}_{(k)} + \widehat{F}^{m}_{(k)}\,, \tag{3.133}$$

where the electric component of a Page flux has a leg along the time direction. Let us define the quantity $\mathcal{Q}^{(p)}$, for a Dp brane, as

$$\mathcal{Q}^{(p)} = \frac{1}{(2\pi)^7}\int \widehat{F}^{e}_{(p+2)} \wedge \widehat{F}^{m}_{(8-p)}\,. \tag{3.134}$$

We find that, for $D0$ branes, $(2\pi)^7\mathcal{Q}^{(0)}$ is given by

$$\int \widehat{F}^{e}_{(2)} \wedge \widehat{F}^{m}_{(8)} = -\int (h_8 - \rho h_8')\left(h_4 - \frac{1}{4}\partial_\rho\left(\frac{uu'}{h_8}\right)\right)\widehat{\mathrm{vol}}_{10}\,, \tag{3.135}$$

while, for $D8$ branes, $(2\pi)^7\mathcal{Q}^{(8)}$ reads

$$\int \widehat{F}^{e}_{(10)} \wedge \widehat{F}^{m}_{(0)} = \int h_8'\left(\rho\widehat{h}_4 - \frac{1}{4}\partial_\rho\frac{\rho uu' - u^2}{h_8}\right)\widehat{\mathrm{vol}}_{10} \tag{3.136}$$

In our setup, we have both colour and flavour branes, so two terms contribute to $(2\pi)^7\mathcal{Q}^{(4)}$. These are given by

$$\int \widehat{F}^{e}_{(6)} \wedge \widehat{F}^{m}_{(4)} = -\int (\widehat{h}_4 - \rho\widehat{h}_4')\left(h_8 - \frac{1}{4}\partial_\rho\left(\frac{uu'}{\widehat{h}_4}\right)\right) + \int \widehat{h}_4'\left(\rho h_8 - \frac{1}{4}\partial_\rho\left(\frac{\rho uu' - u^2}{\widehat{h}_4}\right)\right)\,, \tag{3.137}$$

Computing $\mathcal{C} = \sum \mathcal{Q}^{(p)}$ we get

$$\frac{\mathrm{vol}_{\mathrm{CY}_2}}{(2\pi)^4}\frac{\mathrm{vol}_{S^3}}{(2\pi)^3}\mathrm{vol}_{AdS_2}\int_0^{2\pi(P+1)} d\rho\left[-4\widehat{h}_4 h_8 + u'^2 + \partial_\rho\left(2\rho\widehat{h}_4 h_8 - \frac{uu'}{2} + \frac{u^2 - 2\rho uu'}{4}\frac{(\widehat{h}_4 h_8)'}{\widehat{h}_4 h_8}\right)\right]\,, \tag{3.138}$$

which, up to a boundary term, is proportional to (3.97).

It is straightforward to show that the very same quantity, $\mathcal{C}$, can be obtained by a minimisation procedure in supergravity [6].

Part II

QFT/QFT dualities in Chern-Simons theories

Chapter 1

— Phases of $U(N_c)$ QCD$_3$ from type 0 strings —

We now abandon the realm of SCFTs to land on three dimensional Chern-Simons theories. We also abandon holography to discuss a special class of dualities in Quantum Field Theory. More in particular, in this chapter, we deal with a Seiberg-like duality for three-dimensional non-supersymmetric Chern-Simons theories.

The outline of the chapter is as follows. In Section 1, we give a brief overview of Chern-Simons theories in three dimensions. First, we introduce the Chern-Simons functional and discuss its gauge invariance. We then move on to discussing the fermion path integral. We will see that integrating-out massive fermions in three dimensions produces a shift in the Chern-Simons level. We conclude Section 1 with a brief account of IR dualities in Chern-Simons theories.

In Section 2 we add supersymmetry to the game and discuss $\mathcal{N}=2$ Chern-Simons theories. A nice duality between two $\mathcal{N}=2$ Chern-Simons theories, first discussed by Giveon and Kutasov in [20], has a natural "stringy" origin and it will be reviewed at the end of Section 2. The Giveon-Kutasov-Chern-Simons duality will offer a nice guiding principle in order to embed non-supersymmetric Chern-Simons dualities into string theory. We will treat the case of pure Chern-Simons theories with unitary gauge groups coupled to fundamental matter and this is, in essence, the main goal of this chapter. The reader that has already some knowledge of generic supersymmetric Chern-Simons theories might feel free to skip directly to Section 3.

Starting with Section 3, we present material from [2]. We give a broad introduction and summary on how $U(N_c)$ QCD$_3$ emerges at low energies from the theory living on a particular brane configuration in Type 0B string theory, which we call OQCD$_3$. We also give a general overview on how the different phases of QCD$_3$ are captured in such a string theory setting. In Section 4 we review the essential properties of type 0B string theory and its brane configurations. In Section 5 we discuss in detail the brane configuration that leads to OQCD$_3$ and propose a Seiberg duality. In Section 6 we show how the phase diagram of the electric theory manifest itself in the magnetic, and in Section 7 we focus on QED_3. Section 8 is devoted to conclusions.

Seiberg duality and its "origin" in string theory is reviewed in Appendix M.

1 A bird's eye view on Chern-Simons theories

The aim of this section is to review a few basic facts about Chern-Simons theories in three dimensions. The literature about Chern-Simons theories is huge. Here we follow the 2019 TASI lectures by Moore[1] and the first part of the paper by Witten [162]. A nice older review is [163], while a modern review comprehensive also of all the modern development in Chern-Simons dualities is [164].

Chern-Simons theories are quantum gauge theories involving a particular and subtle action principle. Remarkably, they lead to quantum field theories where many questions can be answered explicitly.

Consider a gauge theory for a Lie group G on a manifold M. A connection on a G-bundle is denoted by A, and transforms under gauge transformations as

$$A \to g^{-1}(d+A)g \,, \tag{1.1}$$

while the curvature $F = dA + A^2$ transforms as $F \to g^{-1}Fg$.

Let us define $\mathcal{P}$, an invariant polynomial of order two on the Lie algebra $\mathfrak{g}$,

$$\mathcal{P} = \mathrm{tr}F^2 \,. \tag{1.2}$$

It is easy to see that $\mathcal{P}$ is gauge invariant and globally defined. Moreover, it can be written as a total derivative in the following manner

$$\mathrm{tr}F^2 = d\,\mathrm{tr}\left(AdA + \frac{2}{3}A^3\right) \,. \tag{1.3}$$

The term in parenthesis is known as Chern-Simons form in three dimensions[2] and is the starting point for Chern-Simons theories. Consider a three-dimensional manifold

[1] Available at: Introduction To Chern-Simons Theories.

[2] Chern-Simons forms exist in any odd dimension. See e.g. [165] pag. 444.

M_3 with an orientation[3] $\mathfrak{o}(M_3)$. We can define a quantum field theory with action

$$S_{CS} = \frac{k}{4\pi} \int_{M_3} \mathrm{tr} \left(AdA + \frac{2}{3} A^3 \right) \tag{1.4}$$

and path integral[4]

$$\int_{\mathcal{A}/\mathcal{G}} e^{iS_{CS}} \,. \tag{1.5}$$

Here the path integration is made over all inequivalent gauge configurations $\mathcal{A}/\mathcal{G}$, where $\mathcal{G}$ is the space of gauge transformations – not to be confused with G which is finite dimensional instead.

Note that if we choose an abelian gauge group G, say $U(1)$ or $\mathbb{R}$, the Chern-Simons action reduces to

$$S_{CS} = \frac{k}{4\pi} \int_{M_3} AdA \,, \tag{1.6}$$

as the term $A \wedge A \wedge A$ vanishes identically.

Along with (1.4), we can of course also consider the Yang-Mills action in three dimensions

$$S_{YM} = \frac{1}{2e^2} \int \mathrm{tr} F \wedge \star F \,. \tag{1.7}$$

In three dimensions, e^2 has dimension of mass and, therefore, the Yang-Mills action behaves as an irrelevant term for the Chern-Simons theory. The variation of the Chern-Simons action gives

$$\delta S_{CS} = \frac{k}{2\pi} \int \mathrm{tr} \delta A F \,, \tag{1.8}$$

and the equations of motion for the full Chern-Simons-Yang-Mills theory read

$$\frac{1}{e^2} D_A \star F + \frac{k}{2\pi} F = 0 \,, \tag{1.9}$$

where $D_A = d + A$ is the covariant derivative. Applying $D_A \star$ from the left and using

[3] The request that M_3 has an orientation is often stated as taking a manifold with trivial first Stiefel-Whitney class, $w_1(TM_3) = 0$, with TM_3 the tangent bundle over M_3.

[4] Strictly speaking, the path integral in (1.5) is not quite correct. Gauge fixing is needed and even though the action (1.4) is topological – the metric of the three-manifold M_3 never appears in the definition – the Fadeev-Popov determinant is not. This leads to the frame anomaly for the path integral [166]. We will not address this issue here. See for instance the lecture notes Introduction To Chern-Simons Theories.

again the equation of motion and the Bianchi identity $D_A F = 0$, we get

$$\Box_A F = \left(\frac{ke^2}{2\pi}\right)^2 F\,, \tag{1.10}$$

where $\Box_A = \{D_A, \star D_A \star\}$ is the covariant Laplacian. The quantity $\frac{ke^2}{2\pi}$ behaves like a mass for the gauge field A, $m_{CS} = \frac{ke^2}{2\pi}$. Solutions to the linearised (1.9) can be found explicitly in the form of plane waves for A,

$$A_i \sim \epsilon_i e^{ip\cdot x} + c.c.\,. \tag{1.11}$$

and they define the degrees of freedom of our theory. Solutions for A are of two types: either plane waves with energy $\omega = \sqrt{\vec{p}^2 + m^2}$, or modes with zero energy and, by the equations of motion, flat connection $F = 0$. The zero modes are not local degrees of freedom, thus a Chern-Simons theory appears to be gapped.

We have already stressed that the gauge coupling e^2 has dimensions of mass and therefore the Yang-Mills term of the Lagrangian is irrelevant in the IR. Let us motivate this a bit more. Under a conformal rescaling of the metric $g_{\mu\nu} \to \Lambda^2(x) g_{\mu\nu}$, the Yang-Mills Lagrangian scales as $\mathcal{L}_{YM} \to \Lambda(x)^{-1}\mathcal{L}_{YM}$, while the Chern-Simons term is left invariant (the Chern-Simons term is topological, the metric $g_{\mu\nu}$ never appears in the definition). The long distance limit is achieved by keeping the metric $g_{\mu\nu}$ fixed while sending $\Lambda \to \infty$. In this limit, the Yang-Mills action formally vanishes, while the Chern-Simons term remains. Note that the scaling $g_{\mu\nu} \to \Lambda^2(x) g_{\mu\nu}$ is equivalent to $e^2 \to \Lambda(x) e^2$. Thus, taking long distances is equivalent to considering strong coupling.

The Chern-Simons mass m_{CS} has been defined as $m_{CS} = ke^2/2\pi$. Thus, in the strong coupling limit, $e^2 \to \infty$, also $m \to \infty$ and the propagator in Chern-Simons theories goes to zero at low energies, forcing all Feynman diagrams and local correlators to vanish. Thus, it seems that Chern-Simons theories have no gauge invariant observables and the theory is trivial. This is no quite true, as we can form observables from Wilson line operators along a closed loop γ

$$W(R,\gamma) = \mathrm{tr}_R \mathrm{P}\exp\left(i \oint_\gamma A\right)\,. \tag{1.12}$$

We can then compute correlation functions of these operators to get topological invariants[5][6].

1.1 Chern-Simons action and quantisation of the level

We address now an issue that has been ignored so far: the gauge invariance of the Chern-Simons term. In the action (1.4), the gauge connection A appears explicitly and we may wonder if such an action is gauge invariant. It turns out that, as we shall see, that the action can be rendered gauge invariant, but a few subtleties must be taken care of.

On a closed three-dimensional Euclidean manifold, the action (1.4) makes sense only if the gauge G bundle is trivial (A is globally defined). However, even in that case, a gauge transformation

$$A \to A^g = g^{-1}(A + d)g\,, \tag{1.13}$$

with $g(x) \in G$ a local gauge transformation, the Chern-Simons action (1.4) transforms as

$$S[A] \to S[A] + 2\pi k N[g]\,, \tag{1.14}$$

where $N[g]$ is given by

$$N[g] = \frac{1}{24\pi^2}\int \mathrm{tr}(g^{-1}dg)^3\,, \tag{1.15}$$

and is the winding number of the gauge transformation. When the gauge group G is compact and simple (e.g. $SU(n)$ with $n \geq 2$), $N[g] \in \mathbb{Z}$ (it corresponds to the homotopy classification $\pi_3(G) = \mathbb{Z}$). In quantum mechanics we do not need the action S to be single-valued, but rather e^{iS}, which is the right factor that appears in the path integral. Thus, choosing the Chern-Simons coefficient $k \in \mathbb{Z}$, we find that e^{iS} is single-valued even under large gauge transformations ($k \neq 0$).

In topologically non-trivial situations, like for instance the case of $G = U(1)$, it is difficult to interpret the Chern-Simons functional as the gauge connection A may have string singularities (Dirac string) and is not globally defined. A procedure that avoids this problem is as follows.

Given a three-dimensional manifold M_3 and a gauge G-bundle E, we can always

[5] There is a subtlety concerning correlation functions of Wilson loops which relies on the choice of a frame. We will not addredd this issue here. See Introduction To Chern-Simons Theories.

[6] The most celebrated is probably the Jones polynomial found by Witten [166] when the gauge group $G = SU(2)$ and the trace is taken over the defining representation of G.

construct a four-manifold Y_4 and a gauge bundle on it $\tilde{E}$ such that $\partial Y_4 = M_3$ and the bundle $\tilde{E}$ restricts to E on M_3. Then, we can define the Chern-Simons action as

$$S_{CS} \equiv \frac{k}{4\pi} \int_{Y_4} \mathrm{tr} F \wedge F \,, \tag{1.16}$$

which is manifestly gauge-invariant, as only F appears in the definition. The problem we run into by defining the Chern-Simons action in this way, is that the definition (1.16) seems to depend on the extension $(M_3, E) \to (Y_4, \tilde{E})$. A different choice would seem to produce a different result. But this is not quite true. Consider a different extension, $(Y_4', \tilde{E}')$. We can glue together the first extension and the orientation-reversal of the second extension (this is needed in order for the glueing to work). We obtain a closed four-dimensional manifold $Y_4 + Y_4'$ and the difference between the two Chern-Simons actions is

$$S_{CS}[A] - S_{CS}'[A] = 2\pi k \cdot \frac{1}{2} \int_{Y_4 + Y_4'} \mathrm{tr} \left(\frac{F}{2\pi} \wedge \frac{F}{2\pi} \right) . \tag{1.17}$$

On a closed spin four-manifold $\frac{1}{2} \int_{Y_4 + Y_4'} \mathrm{tr} \left(\frac{F}{2\pi} \wedge \frac{F}{2\pi} \right)$ is always an integer and, thus, different extensions produce the same result for the Chern-Simons functional modulo an integer multiple of 2π.

Implicit in the last formula is the choice of a spin-structure for M_3 that extends to Y_4 (and Y_4'). It is always possible for an oriented three-dimensional manifold M_3 to choose a spin structure and the Chern-Simons functional (1.16) will depend on such spin structure. If M_3 is not necessarily spin, $\int F \wedge F/(2\pi)^2$ is not an integer in general, and a finer quantisation of k is required. For instance, if we consider an $SO(3)$ bundle over $\mathbb{C}P^2$ it is possible to show that k must be divisible by four (in units in which an arbitrary integer is allowed for $SU(2)$).

1.2 Fermion path integral

One question we might naturally pose is what happens if we choose the Chern-Simons level k to be equal to zero. In that case we do not have a Chern-Simons term and our theory simply reduces to Yang-Mills theory in three dimensions, which is trivial in the IR. The aim of this section is to show that this is not quite correct if there are fermions coupled to the gauge field A. As a matter of fact, integrating out massive fermions produces a shift of the Chern-Simons level and such a shift can be computed exactly. Thus, in a theory with massive interacting fermions, even if no Chern-Simons term is present in the original Lagrangian, it will be generated in the

IR. Loosely speaking, Chern-Simons terms are unavoidable in three dimensions. Here we follow closely [162].

Consider a massless Dirac fermion in 2+1 dimensions[7]

$$S = \int_Y d^3x \bar{\psi} i \not{D} \psi \,. \tag{1.18}$$

Here $\not{D} = \sum_{\mu=0}^{3} \gamma^\mu D_\mu$ is the Dirac operator and ψ couples to the electromagnetic potential A. Upon a choice of a frame bundle on Y, in local Lorentz coordinates, it is possible to pick three real gamma matrices satisfying the algebra $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$, with $\eta_{ab} = \mathrm{diag}(-1, +1, +1)$. Then, the "curved" gamma matrices γ_μ are related to the "flat" γ_a as usual by $\gamma_\mu = e^a_\mu \gamma_a$.

To discuss fermion path integrals and anomalies, it is convenient to consider the theory on a Euclidean manifold. Thus, let us Wick rotate[8] the action (1.18). The operator $\mathcal{D} = i\not{D}$ is hermitian in Euclidean signature and the partition function for ψ reads

$$Z_\psi = \det \mathcal{D} = \prod_i \lambda_i \,, \tag{1.19}$$

where λ_i are eigenvalues for $\mathcal{D}$ and are all real. The fact that the partition function is real has a nice interpretation. The theory of a massless Dirac fermion is invariant under time reversal symmetry. Reversing the orientation of the (Euclidean) spacetime gives the complex conjugate partition function. Therefore, a T-invariant theory has real Z.

Even if we are able to claim that Z_ψ is a real partion function, there is no way to establish its sign. This follows from the fact that the number of eigenvalues λ_i is infinite, and there are potentially infinite positive as well infinite negative eigenvalues. In fact, it turns out that, if we try to formally define[9] Z_ψ to be positive, there is a clash with gauge invariance. The reason for this can be traced back to a beautiful theorem in mathematics, the Atiyah-Patodi-Singer (APS) theorem. Let us see briefly why this is to be the case.

[7] To physically motivate the action (1.18), we might consider the case of a topological insulator on a $(3+1)$-dimensional manifold X with spatial boundary Y. Y is $(2+1)$-dimensional and ψ is a boundary mode living on Y.

[8] For what matters, just send $t \to -it$, $\gamma^0 \to -i\gamma^0$ and $A_t \to iA_t$. We find in this way the Dirac algebra $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$. The action (1.18) is formally unchanged.

[9] Witten points out [162] another problem related to picking a sign for the path integral. A well-defined theory should determine the sign of the path integral for any Y. This issue is resolved by regularising the path integral as we now show.

Let us pick up a metric and a gauge field, (g_0, A_0), and define Z_ψ to be positive at $g = g_0$ and $A = A_0$. Define g_0^ϕ and A_0^ϕ the diffeo/gauge transformed metric and gauge field (ϕ can be understood as the set of parameters that defines the transformation). Because of diffeo/gauge invariance, the partition function Z_ψ should be left invariant by the transformation. Let us interpolate between (g_0, A_0) and $(g_0^\phi, A_0^\phi$ as

$$g_s = (1-s)g_0 + s g_0^\phi \,, \qquad A_s = (1-s)A_0 + s A_0^\phi \,, \tag{1.20}$$

with $0 \leq s \leq 1$. Notice that g_s is an acceptable metric for our spacetime Y, as from its very definition it is positive definite if g_0 (and then g_0^ϕ) is. Of course, g_s and A_s are gauge transformed versions of (g_0, A_0) only for $s = 1$. In other words, (1.20) is not a gauge transformation, in general.

Let us now evolve (g_s, A_s) from $s = 0$ to $s = 1$. Gauge invariance implies that the partition function Z_ψ should have the same sign at $s = 1$ and $s = 0$. Moreover, the spectrum at $s = 0$ and $s = 1$ should be the same. However, between $s = 0$ and $s = 1$ there might be a "spectral flow" for which, even though the spectrum is unchanged, one negative (or positive) eigenvalue crosses zero and the partition function changes sign. Of course, this is only possible because the eigenvalues are infinitely many. The APS theorem serves to precisely count the number of eigenvalues that cross zero and, when this number is odd, the partition function changes sign. In this situation we would have

$$Z[g_0, A_0] = -Z[g_0^\phi, A_0^\phi] \,, \tag{1.21}$$

and gauge invariance is lost. For the situation at hand (topological insulator) a negative number of eigenvalues crosses zero and gauge invariance is spoiled.

As we would like to preserve gauge invariance, we need to find a way out. The solution to this problem boils down to giving up a real partition function. As a matter of fact, quantising the theory not in a T-invariant way is perfectly reasonable: even the addition of a mass term for the fermions would spoil T-invariance. Moreover, as it stands, the partition function Z_ψ needs some regularisation, as it is given by an infinite product. This can be achieved by introducing Pauli-Villars fields χ. We can think of them as fermions – satisfying the massive Dirac equation $(i\not{D} + iM)\chi = 0$ – with bosonic statistics. Z_ψ then reads

$$Z_\psi = \prod_i \frac{\lambda_i}{\lambda_i + iM} \,. \tag{1.22}$$

Notice that parity sends $\lambda_i \to -\lambda_i$ and, therefore, $Z_\psi \to (Z_\psi)^*$, consistently with what we discussed before.

For large positive[10] M, the partition function Z_ψ can be written as

$$Z_\psi = |Z_\psi| \exp\left(-i\frac{\pi}{2}\sum_i \mathrm{sign}(\lambda_i)\right) , \tag{1.23}$$

or

$$Z_\psi = |Z_\psi| \exp\left(-i\frac{\pi}{2}\eta\right) , \tag{1.24}$$

where η is the regularised APS invariant

$$\eta = \lim_{\epsilon\to 0^+}\sum_i \mathrm{sign}(\lambda_i) e^{-\epsilon\lambda_i^2} . \tag{1.25}$$

Notice that, since the spectrum is gauge invariant, also η is gauge invariant. Therefore, if a spectral flow is to take place as before, we would find this time a gauge invariant partition function for ψ. Also, the partition function is supposed to change sign whenever an eigenvalues changes sign. This is indeed the case as if we send one of the eigenvalues $\lambda_i \to -\lambda_i$, η jumps by a factor 2.

Had we chosen large negative M we would find

$$Z_\psi = |Z_\psi| \exp\left(+i\frac{\pi}{2}\eta\right) . \tag{1.26}$$

This is also perfectly allowed and consistent with gauge and Poincaré – in fact conformal – invariance, and the plus or minus sign is a convention here. Of course T-invariance is by now lost as Z_ψ is no longer real: the Pauli-Villars regulator was not T-invariant in the first place.

The APS theorem says also that η can be rewritten as

$$\exp(-i\pi\eta) = \exp\left(-iS_{CS}[A] - 2i\mathrm{CS}_{\mathrm{grav}}[g]\right) , \tag{1.27}$$

where $\mathrm{CS}_{\mathrm{grav}}$ is defined as $\mathrm{CS}_{\mathrm{grav}} = 2\pi\frac{\widehat{A}}{2}$, being $\widehat{A}$ the Dirac genus defined as

$$\widehat{A} = -\frac{1}{48}\int_X \frac{\mathrm{tr} R\wedge R}{(2\pi)^2} , \tag{1.28}$$

with X, as before, a four manifold such that $\partial X = Y$.

[10] We discuss what changes if we take large negative M in a moment.

We might be tempted to say that the regularisation of the fermion path integral has produced a Chern-Simons term at level $-1/2$. However, this is not correct as fractional Chern-Simons terms are not gauge invariant whereas η is perfectly well defined and invariant under gauge transformations. Yet, we will use the notation $U(1)_{-\frac{1}{2}}$ to indicate the theory of a massless fermion coupled to a $U(1)$ gauge field. The "$-1/2$" refers to the regularisation scheme adopted to regularise the fermion path integral. This is particularly enlightening when we give a mass m to the fermion ψ. As is well known, integrating out ψ at low energies produces a shift in the Chern-Simons level by $\mathrm{sign}(m)/2$ [167, 168]. Therefore, integrating out ψ at low energies we find

$$Z_\psi = \begin{cases} 1 & \text{if } m \gg 0 \\ e^{-i\pi\eta} & \text{if } m \ll 0 \end{cases} \tag{1.29}$$

The last equation is always well defined in terms of Chern-Simons terms and implies that the effective action at low energies reads

$$\delta S = -\frac{1}{4\pi}\int d^3x AdA - 2\mathrm{CS}_{\mathrm{grav}}\,, \tag{1.30}$$

when $m \ll 0$.

So far we have assumed the gauge group G to be $U(1)$. In general, we can have also other gauge groups and fermions transforming under arbitrary representations. How does the shift of Chern-Simons level work in such more general situations? An explicit computation reveals that the fermion determinant for a fermion that belongs to a representation R of the gauge group G is given by

$$\det\mathcal{D} = |\det\mathcal{D}|\exp(-i\frac{\pi}{2}\eta)\,, \qquad R \text{ complex representation}\,, \tag{1.31}$$

if the representation R is complex, or

$$\det\mathcal{D} = |\det\mathcal{D}|\exp(-i\frac{\pi}{4}\eta)\,, \qquad R \text{ real representation}\,, \tag{1.32}$$

if the representation R is real. The APS theorem says that

$$\pi\eta = 2x_R S_{CS} + 2\dim R\,\mathrm{CS}_{\mathrm{grav}}\,, \tag{1.33}$$

where x_R is the Dinkin index of the representation R, normalised such that $\mathrm{tr}_R(t^a_R t^b_R) =$

$-2x_R\delta^{ab}$.

Thus, integrating out a single fermion of mass m shifts the Chern-Simons level by $\mathrm{sign}(m)x_R$ for complex representations or $\mathrm{sign}(m)x_R/2$ for real representations. This will be important later for our study of OQCD_3, where we will have to integrate out fermions in the two-index antisymmetric representation of the gauge group.

Remarks

- The Chern-Simons theories we will consider in the following all arise from fluctuations of threebranes in flat spacetime. Therefore, the three manifold M_3 for us will be simply $\mathbb{R}^{1,2}$ (or $\mathbb{R}^3$ upon continuation to Euclidean signature) and no gravitational Chern-Simons term appears in the Lagrangian.

- When we consider systems of fermions $\psi \oplus \tilde{\psi}$, with $\tilde{\psi}$ transforming under any symmetries as the complex conjugate of ψ, and with an action which is the complex conjugate of that of ψ, the partition function is $Z_\psi \overline{Z_\psi} = |Z_\psi|^2$ and there is no anomaly. This will be relevant later when considering fermions $\lambda \oplus \tilde{\lambda}$ in the $\begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ representation of $U(N_c)$.

1.3 Physical implications: IR dualities

As we already outlined at the beginning of this chapter, over the last few years Chern-Simons theories have received one more push forward as new kind of dualities between Chern-Simons-like theories in three dimensions have been uncovered, the most famous example being probably level rank duality, see [169] which can be proved exactly. Coupling matter with non-abelian gauge fields, Aharony proposed [21] the following dualities

$$\begin{aligned} SU(N_c)_K \oplus N_f \text{ scalars} &\longleftrightarrow U(K)_{-N_c+N_f/2} \oplus N_f \text{ fermions} \\ U(N_c)_K \oplus N_f \text{ scalars} &\longleftrightarrow SU(K)_{-N_c+N_f/2} \oplus N_f \text{ fermions} \\ U(N_c)_{K,K+N_c} \oplus N_f \text{ scalars} &\longleftrightarrow U(K)_{-N_c+N_f/2,-N_c+N_f/2-K} \oplus N_f \text{ fermions} \end{aligned} \tag{1.34}$$

which are believed to hold whenever

$$N_f \leq N_c\,. \tag{1.35}$$

In all these dualities, the scalars have quartic interactions and, as we flow to the IR, we allow to tune the fermion and boson masses to reach a critical point, if it exists. Such dualities cannot be established rigorously, except for some special cases.

All these correspondences are similar in spirit to Seiberg duality [170], which is conjectured to hold in the deep IR after reaching an interacting fixed point. However, this time the dualities are formulated between non-supersymmetric theories and this makes them somewhat special.

The dualities above have been generalised to gauge groups different from SU or U (like SO or Sp) and with matter in representations different from the fundamental. For a comprehensive review on the existing cases, see [164].

One of the main goals of this chapter is to embed the unitary duality between Chern-Simons theories (third line in (1.34)) into string theory. Before diving into that, let us give some details about supersymmetric Chern-Simons theory and how they are realised on branes in string theory. This will help us formulate our cojectures.

2 Additional structure: Supersymmetry

An additional structure that can be added to Chern-Simons theories is supersymmetry. In the following, we will be concerned mainly with $\mathcal{N} = 2$ supersymmetry. Remember that a minimal spinor in three dimensions is an object with half the degrees of freedom of a minimal spinor in four dimensions. Thus, $\mathcal{N} = 2$ in three dimensions simply means that we are dealing with a theory with four supercharges, and it is very much related to $\mathcal{N} = 1$ supersymmetry in four dimensions. In fact, $\mathcal{N} = 2$ supersymmetric theories in three dimensions can very often be understood as dimensional reduction from four dimensions.

A good reference – and perhaps the first paper dealing with supersymmetric Chern-Simons theories systematically – is that of Schwarz [171]. We will draw results from there.

$\mathcal{N} = 2$ gauge multiplet

Let us begin by considering the gauge multiplet for $\mathcal{N} = 2$ supersymmetric theories. As outlined before, it is readily obtained by dimensionally reducing the $\mathcal{N} = 1$ gauge multiplet in four dimensions. The latter contains a gauge field A_μ, a real Majorana fermion χ and a real scalar D. Upon reduction to three dimensions, the gauge field A_μ

gives rise to a 3d gauge field, that we still call A_μ, and a real scalar σ from the fourth component of the 4d gauge field, A_3. The real Majorana fermion χ can be recast as a two component (complex) Dirac fermion in three dimensions, and we still have the real scalar D. Notice that off-shell we have four bosonic and four fermionic degrees of freedom. This was expected as we are describing a theory with four supercharges. If we go on-shell we find two bosonic and two fermionic degrees of freedom.

The Chern-Simons lagrangian made out of a vector multiplet reads

$$\mathcal{L} = \mathrm{tr}\left(AdA + \frac{2}{3}A^3 - \bar{\chi}\chi + 2D\sigma \right) . \tag{1.36}$$

Notice that both the gauge field A_μ and the scalar σ have dimension 1, while the fermion χ has dimension 3/2 and the scalar D has dimension 2. So each term in the lagrangian has dimension 3, as it should be. All the fields, being part of the same supermultiplet, belong to the same representation of the gauge group G, the adjoint.

$\mathcal{N} = 2$ matter multiplet

As it will be useful in applications to OQCD_3, it is appropriate to discuss also how matter can be coupled to the $\mathcal{N} = 2$ gauge multiplet. In particular, we would like to add to the game the $\mathcal{N} = 2$ matter multiplet. Again, this is readily obtained by dimensional reduction of an $\mathcal{N} = 1$ chiral multiplet of four dimensional theories which, in turn, contains a complex scalar ϕ^a of dimension 1/2 a complex Dirac two-component fermion ψ^a and an auxiliary scalar F^a of dimension 3/2. The index a labels the representation R of the gauge group G the fields ϕ, ψ and F belong to, and runs from 1 to $\dim(R)$, the dimension of the representation. In more mathematical terms, the fields of the matter multiplets can be understood as sections of the associated vector bundles in the appropriate representation. So for instance, if $S(M)$ is the spin bundle over the three-dimensional manifold M_3, ψ is a section of the product bundle $S(M) \otimes E$, with E the bundle associated to the principal G-bundle by the representation R. For the scalars we replace the spin bundle by complex or real line bundles. Of course, the same was true for the gauge multiplet, where the relevant representation was the adjoint.

The lagrangian for the matter multiplet reads

$$\mathcal{L} = \partial_\mu \phi_a \partial^\mu \phi^a + i\bar{\psi}_a \partial\!\!\!/ \psi^a + F_a F^a + \mathcal{W}_F + \mathcal{W}_F^* , \tag{1.37}$$

where $\mathcal{W}_F$ is the superpotential that gives rise to interaction terms like $\phi^2\psi^2$ or $\phi^3 F$. When the representation R is complex, it is customary to set $(\phi^a)^* = \phi_a$ (same for the other fields), and therefore all the terms in the lagrangian above are gauge singlets.

When the matter multiplet is coupled to the gauge multiplet ordinary derivatives are replaced by covariant derivatives and Yukawa terms coupling fermions and bosons of gauge and matter multiplets need to be added in order for the lagrangian to be $\mathcal{N} = 2$ supersymmetric [171]

$$\begin{aligned}\mathcal{L} = {} & D_\mu\phi_a D^\mu\phi^a + i\bar{\psi}_a \not{D}\psi^a + F_a F^a + \mathcal{W}_F + \mathcal{W}_F^* \\ & - \phi_a\sigma^2\phi^a + \phi_a D\phi^a - \bar{\psi}_a\sigma\psi^a + i\phi_a\bar{\chi}\phi^a - i\bar{\psi}_a\chi\phi^a \,.\end{aligned} \tag{1.38}$$

2.1 String theory embedding and duality

The $\mathcal{N} = 2$ Chern-Simons theory discussed above can be embedded into string theory. We will now spend a few words on what is the correct brane setup that gives rise to $\mathcal{N} = 2$ Chern-Simons theory and how to "derive" a dual theory for it, known also as Giveon-Kutasov duality [20].

Consider the brane web given in Table 1.1 in Type IIB String Theory. Such a brane

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
$NS5$	–	–	–	–	–	–				
$NS5'$	–	–	–	–					–	–
$D3$	–	–	–				–			
$D5$	–	–	–					–	–	–

Table 1.1: Brane web electric theory, before recombination of fivebranes.

web preserves $\mathcal{N} = 2$ supersymmetry in three dimensions (four supersymmetries).

When an $NS5'$ brane intersects k $D5$ branes in the $(3,7)$ plane, they locally combine into a $(1,k)$ fivebrane at an angle θ with respect to the $NS5$ brane, see Figure 1.1. θ is related to k via the relation $\tan\theta = g_s k$, with g_s the string coupling. Supersymmetry is mantained for any length of $(1,k)$ fivebrane segment. When the length of the segment goes to infinity we are left with a $(1,k)$ bound state of fivebranes at an angle θ in the $(3,7)$ plane.

We consider a system with $N_f + k$ $D5$ branes, see the left of Figure 1.2. The low energy theory is that of $U(N_c)$ gauge theory with $N_f + k$ flavours of chiral fields Q^i, $\tilde{Q}_i$ in the fundamental representation of the gauge group. We then move k $D5$ branes

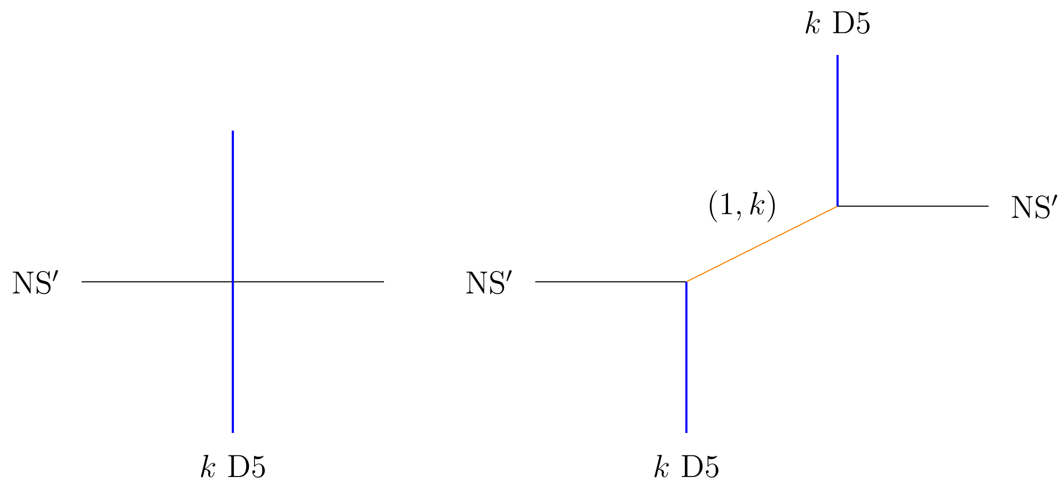

Figure 1.1: Recombination for fivebranes.

towards the $NS5'$ brane in order to have a $(1, k)$ bound state. The final brane setup is given by the Figure 1.2, to the right. The operation for which we can replace an $NS5'$ and k $D5$ branes with a $(1, k)$ fivebrane corresponds to giving mass to k chiral fields and sending the mass to infinity, i.e. integrating out k flavours of chiral fields. As we have seen in Section 1.2 integrating out a fermion charged under the underlying gauge symmetry produces a shift of the Chern-Simons level. It is not difficult to see, using formula (1.33) with $x_R = \frac{1}{2}$ and $\mathrm{CS}_{\mathrm{grav}} = 0$ for each fermions that we integrate out, that the shift is given precisely by k. Thus, what we get after integrating out k flavours of chiral fields is a $U(N_c)$ Chern-Simons theory with N_f flavours Q^i, $\tilde{Q}_i$ $(i = 1, \ldots, N_f)$ and Chern-Simons level k.

Figure 1.2: Electric brane configuration.

The global symmetry of the gauge Chern-Simons theory is $SU(N_f) \times SU(N_f) \times U(1)_a \times U(1)_R$. The first three factors are realised on the brane web starting with the configuration of right Figure 1.2, moving all N_f $D5$ branes to the $(1,k)$-brane, and performing separate $U(N_f)$ rotations on the $D5$ branes with $x_7 > 0$ and $x_7 < 0$. The $U(1)_R$ is a subgroup of the ten-dimensional Lorentz group preserved by the brane system.

Duality: the magnetic theory

Following [27], we swap the $NS5$ and $(1,k)$ branes to get the brane setup in Figure 1.3.

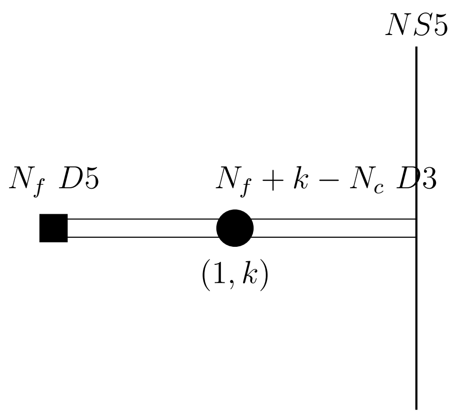

Figure 1.3: Magnetic brane configuration.

As a consequence of the Hanany-Witten transition [28], we have $N_f + k - N_c$ threebranes between the $NS5$ and $(1,k)$ fivebranes. The low energy theory is that of a Chern-Simons theory with gauge group $U(N_f + k - N_c)$ and N_f flavours q_i and $\tilde{q}_i$. The N_f $D3$ branes, which give rise to the flavour symmetry, are free to slide the in the (x^8, x^9) direction. Therefore, we have in the magnetic theory also $N_f \times N_f$ gauge-singlet massless scalars. Because of supersymmetry, what we actually have is $N_f \times N_f$ chiral singlets M^i_j.

Giveon and Kutasov [27] proposed such a theory as the magnetic dual of the electric theory described above upon identifying

$$M^i_j = Q^i \tilde{Q}_j \,. \tag{1.39}$$

The duality just proposed can be checked by studying the structure of the moduli spaces of both theories and/or by adding deformations, like masses for the quarks, on both sides. For further details see [27].

We are now ready to start studying how non-supersymmetric Chern-Simons dualities can be embedded in string theory. We begin now by giving a broad overview on how QCD$_3$ emerges from particular brane configuration in Type 0B string theory and how the different phases of QCD$_3$ are captured by string theory. Details are spelled out in later sections.

3 Introduction and summary of OQCD$_3$

String theory has long been a source of insight for investigations in strong coupling dynamics of quantum field theory. In particular, dualities in field theories often follow from properties of the corresponding brane configuration in string theory, as we have seen in the previous section. Having independent evidence from field theory and string theory is a step in verifying dualities. Most of the effort so far has been largely focused on supersymmetric theories in various dimensions, owing to the fact that non-perturbative phenomena in both string theory and field theory are better understood in that setting.

One may naturally ponder the ubiquity of dualities in generic QFTs, and their relationship to string theory. Indeed, recent years have seen progress made on the field theory front for non-supersymmetric gauge theories in three dimensions. There has been significant progress in the understanding of the phase diagram of QCD$_3$ with a Chern-Simons term.

Consider a $U(N_c)$ theory with N_f massless Dirac fermions and a level K Chern-Simons term. As we have outlined in Subsection 1.3, it was argued in [21](see also [172, 173, 169, 174, 175]) that for $N_f/2 \leq K$ the theory admits a dual description in terms of a gauge theory coupled to scalars as follows[11]

$$U(N_c)_{K,K\pm N_c} \oplus N_f \text{ fermions} \longleftrightarrow U\left(K+\frac{N_f}{2}\right)_{-N_c,-N_c\mp(K+N_f/2)} \oplus N_f \text{ scalars}\,. \quad (1.40)$$

However, one may wonder whether something changes for $N_f/2 > K$. In the case of $SU(N_c)$ gauge symmetry, it was conjectured in [22] that when $N^\star > N_f/2 > K$ the theory admits a flavour symmetry breaking phase where

$$U(N_f) \to U(N_f/2-K) \times U(N_f/2+K)\,. \quad (1.41)$$

[11] We have switched the role of N_f and K relative to the previous sections.

A similar picture was developed in [22] also for $SO(N)$ and $Sp(N)$ gauge theories. For $N_f \geq N^\star$ the theory is expected to flow to a CFT[12].

Following [177], which concerned the symplectic gauge group, it is proposed [2] that the infrared phase diagram of $U(N_c)$ QCD$_3$ can be understood in terms of a non-SUSY Seiberg duality. Our proposal involves a modification of the UV theory, i.e. we start with a UV theory, which we refer to as the *electric* theory, whose Lagrangian is more complicated than QCD$_3$. This theory flows in the IR to QCD$_3$. The electric theory also admits a Seiberg dual description, which we refer to as the *magnetic* theory. The various IR phases of the electric theory (and so of QCD$_3$) can then be identified with the phases of the magnetic dual. In particular both the bosonized phase and the symmetry breaking phase, which will be our main focus, can be understood in terms of the condensation of a scalar field, namely the dual "squark", in the magnetic theory.

Our proposal of Seiberg duality is motivated by string theory[13]. In order to realise $U(N_c)$ QCD$_3$ we embed the gauge theory in a Hanany-Witten brane configuration of type 0B string theory. The brane configuration consists of N_c $D3$ branes suspended between an $NS5$ branes and a $(1, k)$ fivebrane. In addition, there exits N_f flavour branes and an O$'$3 orientifold plane. It is similar to the corresponding supersymmetric brane configuration of Giveon and Kutasov in type IIB [20], reviewed in the previous section.

By swapping the fivebranes we obtain the brane configuration that realises the magnetic Seiberg dual. The relation between field theory and string theory phenomena teaches us about non-supersymmetric brane dynamics. The aforementioned squark condensation translate into a reconnection of colour and flavour branes.

Our Seiberg duality proposal is supported by planar equivalence [182, 183]: when N_c, N_f, k are taken to infinity the electric theory becomes equivalent to a supersymmetric theory and the magnetic theory becomes equivalent to a supersymmetric theory. The electric and magnetic theories form an $\mathcal{N} = 2$ supersymmetric Giveon-Kutasov dual pair. Therefore, there exists a limit in which our non-supersymmetric dual pair becomes a known supersymmetric dual pair.

In the following we will always denote the bare CS level by k. In addition, we

[12] In the 't Hooft limit, when $N_c \to \infty$ and K, N_f are kept fixed, the theory exhibits rich vacua [176]. The discussion of this limit is beyond the scope of this paper.

[13] Other approaches to obtain 3d duality with relation to string theory are given in [178, 179], while the possibility of relating these dualities to supersymmetric dualities were explored in [180, 181].

define the frequently occurring combination

$$\kappa \equiv k - N_c + 2 \, , \qquad K \equiv \kappa - \frac{N_f}{2} \tag{1.42}$$

4 Overview of type 0B

In this section we review aspects of $D3$ branes and O$'$3 planes in type 0 string theory. For the relevant background we refer the reader to [184].

Type 0B string theory can be obtained by a $\mathbb{Z}_2$ orbifold of type IIB, with the $\mathbb{Z}_2$ action generated by $(-1)^{F_s}$, the mod 2 spacetime fermion number operator. The untwisted sector is therefore identical to the bosonic sector of the parent type IIB theory. The twisted sector is composed of a tachyon in the NS-NS sector as well as a new full set of R-R fields. The tachyon will eventually be projected out by the orientifold action. The doubled set of R-R fields lead in effect to a doubling of the D-brane spectrum. In particular there are now two types of threebranes which we denote by $D3$ and D3$'$ respectively.

The worldvolume theory on a stack of n $D3$ and m $D3'$ branes was worked out in [185, 186]. It is a $U(n) \times U(m)$ gauge theory with 3 complex scalars in the adjoint representation, and a pair of bifundamental Weyl fermions.

In order to project out the closed string tachyon we make use of the $\Omega(-1)^{f_R}$ projection [187, 188]. Here, Ω is worldsheet parity and $(-1)^{f_R}$ is the operator that counts the number of right moving worldsheet fermions mod 2. Combining this with reflection in 6 spatial directions I_6 we get an O$'3^{\pm}$ orientifold, the (3+1) dimensional fixed hyperplane with respect to the $\Omega(-1)^{f_R} I_6$ action. The existence of two types of orientifold planes follows from the fact that the NS-NS two form can have a non-trivial Wilson line $\exp\left(i \int B\right)$ and the signs are chosen to reflect the R-R charge of the orientifold plane. Note that unlike the O3-planes of type IIB we do not have the additional possibilities associated with the R-R discrete torsion. Under the action of Ω, $D3$ turns into $D3'$, thus requiring an equal number of each type of brane. In fact Ω projects out half of the doubled set of R-R fields in the closed string sector.

We are interested in stacks of N $D3$ branes (together with their image N $D3'$s) on top of O$'3^{\pm}$, with the worldvolume directions of $D3$ and $D3'$ parallel to that of the O$'3^{\pm}$-plane (see Table 4.2). The worldvolume theory of such a configuration was worked out in [186]. In both cases one has a $U(N)$ gauge field and 6 adjoint scalars parameterising the directions transverse to the worldvolume. There are also a pair of Weyl fermions which transform in the 2-index symmetric or antisymmetric representation of $U(N)$ in the configuration with $\mathrm{O}'3^+$ and $\mathrm{O}'3^-$ respectively. We will denote these theories by $\mathcal{Y}^+$ ($\Box\!\Box$), $\mathcal{Y}^-$ ($\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$) respectively, highlighting the orientifold type on which they live as well as the representation of the worldvolume fermions (the two features relevant for our purposes). We summarise this in Table 1.2. The Lagrangian for these theories can be obtained by subjecting the component fields of $\mathcal{N}=4$ SYM, collectively denoted by φ, to the constraints

$$J\varphi J^T = (-1)^F \varphi \,, \tag{1.43}$$

where $(-1)^F$ is the mod 2 fermion number operator and J is the symplectic form

$$J = \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix} . \tag{1.44}$$

The choice of gauge group for the $\mathcal{N}=4$ theory descends to the choice of fermion representation (Figure 1.4); starting from the *parent* theory with $SO(2N)$ gauge group one lands on $\mathcal{Y}^-$ ($\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$), and the supersymmetric $Sp(N)$ theory leads to $\mathcal{Y}^+$ ($\Box\!\Box$)[189].

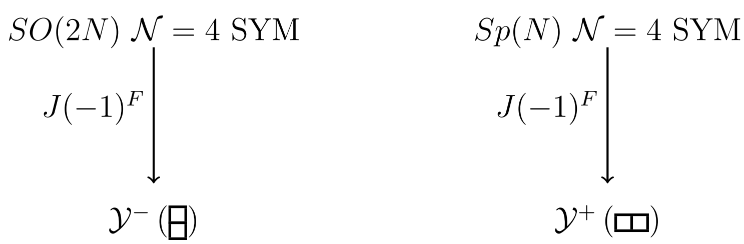

Figure 1.4: The "orientifold" daughters of $\mathcal{N}=4$ SYM.

$\mathcal{Y}^-$ ($\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$)	$U(N)$	$SO(6)$
B_μ^-	adj	$\cdot$
X_-	adj	$\mathbf{6}_v$
ξ_-	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix} \oplus \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}$	$\mathbf{4}_s \oplus \mathbf{4}_c$

$\mathcal{Y}^+$ ($\Box\!\Box$)	$U(N)$	$SO(6)$
B_μ^+	adj	$\cdot$
X_+	adj	$\mathbf{6}_v$
ξ_+	$\Box\!\Box \oplus \overline{\Box\!\Box}$	$\mathbf{4}_s \oplus \mathbf{4}_c$

Table 1.2: The field content of the world volume theory of N $D3$ branes on top of an $\mathrm{O}'3^\pm$ plane.

The Möbius amplitude for a single $D3$ and its image $D3'$ separated by a distance

$2|X_\pm|$ across the $\mathrm{O}'3^\pm$ is [186]

$$\mathcal{A}_\mathcal{M} = \pm \frac{V_4}{(8\pi^2\alpha')^2} \int_0^\infty \frac{dt}{2t^3} \frac{f_2^8(iq)}{f_1^8(iq)} \exp\left(\frac{-2tX_\pm^2}{\pi\alpha'}\right) , \tag{1.45}$$

where $q = e^{-\pi t}$ and the $f_i(q)$ are defined as in [190]. We would like to extract the charge of the orientifold plane as well as the brane-orientifold potential. We note that the integrand in (1.45) is, up to a sign, identical to the case analysed in [191]. We will state the relevant results in the following. For large separation $X_\pm$, the leading order term as $t \to 0$ is given by

$$\mathcal{A}_\mathcal{M} \sim \pm\pi V_4 G_6(X_\pm^2) , \tag{1.46}$$

where $G_6(X_\pm^2) = (4\pi^3)^{-1}|X_\pm|^{-4}\Gamma(2)$ is the 6d scalar propagator. We see that the long range potential between the branes and $\mathrm{O}'3^-$ ($\mathrm{O}'3^+$) is attractive (repulsive). For small $X_\pm$, (1.46) is no longer a valid approximation, instead one can expand the exponential in (1.45) around $X_\pm = 0$

$$\mathcal{A}_\mathcal{M} = \pm\left[\Lambda - MX_\pm^2 + \mathcal{O}\left(X_\pm^4\right)\right] , \tag{1.47}$$

where the coefficients Λ, M are both positive, with the explicit form given in [191]. From this, it follows that there is a short range attractive (repulsive) force between the branes and $\mathrm{O}'3^-$ ($\mathrm{O}'3^+$) plane. The nature of the interaction at short and long distances from the orientifold is similar. Therefore, the theory with fermions in the antisymmetric (symmetric) representation is perturbatively stable (unstable). Note that instabilities of non-perturbative nature may still arise, but are less straightforward to detect in string theory. Instead, we may rely on the field theory analysis and try to revert some lessons back to the brane setup (as in Section 6.2).

Notice that the (in)stability of the brane configuration translates in the worldvolume field theory to statements about the vev of the scalars $X_\pm$. This is obvious from the second term in (1.47), where the sign of the mass term for the scalars is positive (negative) for the theory with anti-symmetric (symmetric) fermions. In the Field Theory, this is encoded in the 1-loop Coleman-Weinberg potential, which gets unequal contributions from the bosons and fermions in each theory.

As observed in [192], the threebranes in type 0 carry the following charge and tension

$$Q_{D3} = \sqrt{\pi}, \qquad T_{D3} = \frac{\sqrt{\pi}}{\sqrt{2}\kappa_{10}} . \tag{1.48}$$

It is then a matter of comparing (1.46) with $4V_4G_6(X_\pm^2)T_{\mathrm{O}'3^\pm}T_{D3}\kappa_{10}^2$ to see that the orientifold charge and tension are

$$Q_{\mathrm{O}'3^\pm} = \pm\frac{Q_{D3}}{2}, \qquad T_{\mathrm{O}'3^\pm} = \pm\frac{T_{D3}}{2}\ . \tag{1.49}$$

This is clearly different from the situation in type II theories where an O$p^\pm$ plane carries $\pm 2^{p-5}$ units of Dp brane charge. The charges (1.49) of the O$'3^\pm$ relative to the $D3$ will be crucial in constructing seiberg dual pairs in the next section.

4.1 A pseudo-moduli space

The discussion in the previous section shows that the $\mathcal{Y}^+$ ($\square\square$) theory is unstable, namely the $D3$s are repelled away from the orientifold. But the analysis tells us nothing about where the stable vacuum of the theory lies. In a non-SUSY setup, the scalar vevs, or correspondingly the coordinates of the branes are not to be viewed as moduli but are rather dictated by the dynamics of the theory. Generically one expects a scalar potential $V(X_+)$ to be induced via loop corrections. It is however useful to have a completely kinematical discussion of the possible *pseudo-moduli* of the brane system before imposing the dynamical constraints. We will examine the situation both in string theory and field theory.

Using the $U(N)$ matrices, the most generic vev for the scalars X_+ takes the diagonal form

$$\langle X_+\rangle = \mathrm{diag}\,(a_1, a_2, \cdots, a_N)\,; \qquad a_i \in \mathbb{R}\ . \tag{1.50}$$

From a field theoretic point of view, depending on the specific values of the eigenvalues a_i we encounter 3 possibilities:

(i) The a_i are all distinct. In this case the gauge group is broken to its $U(1)^N$ maximal torus and the worldvolume fermions all become massive. There are also adjoint (charge 0) scalars for each $U(1)$ factor in $U(1)^N$

(ii) When n of the N eigenvalues become exactly degenerate there is an enhanced $U(n)$ symmetry. The breaking pattern in this case takes the form

$$U(N) \to U(n) \times U(1)^{N-n}\ . \tag{1.51}$$

All worldvolume fermions are massive but there are scalars in the adjoint of the unbroken gauge group. A special case of this type is when all the eigenvalues coincide and the entire gauge symmetry is unbroken.

(iii) There is a more exotic possibility. Consider the situation where n eigenvalues take the opposite sign of an exactly degenerate set of m eigenvalues, i.e.

$$\langle X_+ \rangle = \text{diag}\Big(\overbrace{v, \cdots, v}^{n}, \overbrace{-v, \cdots, -v}^{m}, a_1, \cdots, a_{N-(n+m)}\Big) . \tag{1.52}$$

The unbroken gauge symmetry is now $U(n) \times U(m) \times U(1)^{N-(n+m)}$. As in the cases (i), (ii) above there are scalars transforming in the adjoint of the unbroken gauge symmetry. Unlike those cases, there are now also massless fermions thanks to the cancellation between the positive and negative eigenvalues of equal magnitude. These fermions transform in the bi-fundamental of the non-abelian $U(n) \times U(m)$ factor of the unbroken gauge group.

From the string theory perspective, case (i) corresponds to a configuration where all branes are at distinct points away from the orientifold, that is, none of the $D3$s coincide. Case (ii) corresponds to n $D3$ branes coinciding in the bulk (away from the orientifold). Case (iii) is more interesting. Suppose that $v > 0$, then in the brane picture v denotes the coordinates of n $D3$ branes in the transverse space. On the other hand giving negative vevs to m of the scalars corresponds to separating m $D3$s from the orientifold in the negative direction. But only the quotient space, i.e. the positive direction is physical. When we send m $D3$s to a negative point in the transverse space, their image $D3$'s are given positive coordinates and appear in the physical space. So we see that case (iii) corresponds to n $D3$s and m $D3$'s coinciding at coordinate v in the bulk. The worldvolume theory of this configuration beautifully matches what one would expect from field theory discussed in (iii).

4.2 Hanany-Witten setup

We are interested in Hanany-Witten setups to study 3d theories, which requires the introduction of $NS5$ branes. Our construction is the non-SUSY analogue of the 3d $\mathcal{N} = 2$ setup in type IIB (see e.g. [27]). In particular, we have $NS5$ branes which are non-parallel in two of their spatial coordinates as in Table 4.2, we distinguish them by referring to one as an $NS5'$. The orientifold charge is switched from $\text{O}'3^+$ to $\text{O}'3^-$ and vice versa on either side of an $NS5$ or $NS5'$ which intersects the orientifold. We

	3	4	5	6	7	8	9
$NS5$	3	4	5				
$NS5'$	3					8	9
$D3$				$\|6\|$			
O$'$3				6			
$D5$					7	8	9
$(1,k)$	$\begin{bmatrix}3\\7\end{bmatrix}_\theta$					8	9

Table 1.3: The various extended objects and their orientation in $\mathbb{R}^{1,9}$. All objects extend along the shared $x^{0,1,2}$ directions as well as those indicated above.

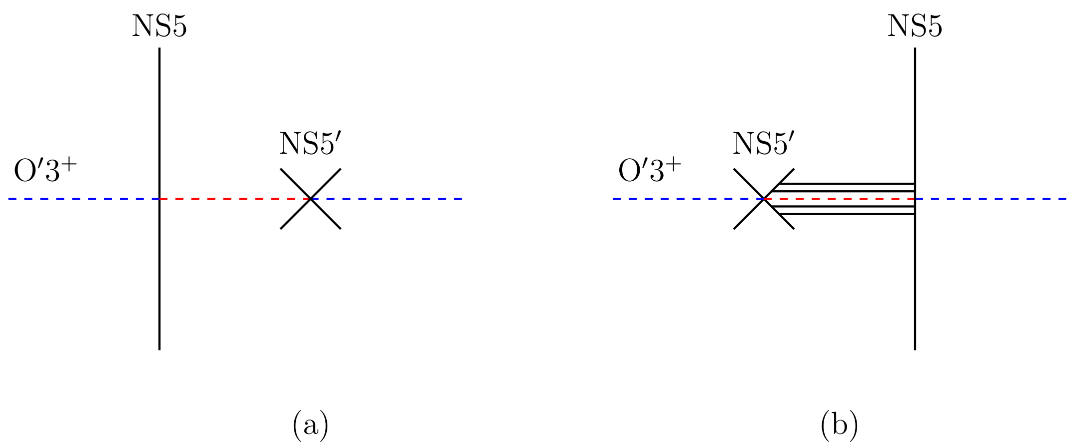

Figure 1.5: The Hanany-Witten effect. In passing from the configuration (a) to (b) a pair of $D3$s are created between the non-parallel $NS5$s.

will only consider configurations where the orientifold is asymptotically O$'3^+$ and label only the asymptotic charge of the orientifold plane in our diagrams (see Figure 1.5).

Seiberg duality has a standard string theory derivation [193] which follows from a rearrangement of non-parallel $NS5$ branes in the Hanany-Witten setup. In constructions without an orientifold, it is possible to achieve this rearrangement without the need for the $NS5$ branes to intersect. This is done by using the freedom to separate them in a direction mutually transverse to the $NS5$ and $NS5'$. In the presence of an orientifold, the $NS5$s are bound to the orientifold plane and this is no longer possible. The $NS5$ branes will inevitably intersect as we try to move them past one another [194].

The result of moving non-parallel fivebranes through one another in the presence of an orientifold is well understood. This is the so called Hanany-Witten transition [15].

In type IIB constructions with an orientifold this amounts to the creation/annihilation of a $D3$ between the $NS5$ and $NS5'$ depending on the orientifold type, a fact that follows from imposing the conservation of linking number. In the absence of $D5$ branes the linking number of an $NS5$ is proportional to the difference of the net $D3$ brane charges ending on it from the left and right respectively. Following the discussion around (1.49) it is easy to see that for the type 0 configuration of Figure 1.5 the linking number of the $NS5$ and $NS5'$ are conserved provided a *pair* of $D3$s are created in between them as we go from (a) to (b). This is twice the corresponding situation in type IIB as one would expect from the fact that the charge of $\mathrm{O}'3^{\pm}$ relative to the type 0 $D3$ is a factor of two greater than the type IIB analogue.

In the next section we discuss the Hanany-Witten setup that leads to the non-SUSY gauge theories of interest with and without flavours.

5 3d dualities from non-supersymmetric brane configurations

In this section we consider Hanany-Witten setups that lead to three-dimensional CS theories. See Figure 1.6 and 1.7. The construction is analogous to [27]. The difference here, besides being in type 0B, is the presence of the $\mathrm{O}'3$ orientifold discussed previously.

In Section 5.1 we consider the setup of Figure 1.6. The low-energy theory of such a configuration is that of non-SUSY analogue of $\mathcal{N}=2$ CS theories without flavours of (s)quarks. Such a setup turns out to be meaningful for the discussion of 3d dualities without matter. These dualities are also known in the literature as level-rank dualities.

In Section 5.2 we consider the addition of N_f flavour D5-branes, see Figure 1.7. The low-energy theory emerging from such a brane configuration includes quarks and squarks in the fundamental representation of the gauge group.

5.1 Level-rank duality

We begin by discussing how level-rank duality is realised in our setup. The discussion follows that of [195], and we provide a more refined account. In particular, we will be more careful about the CS level of the $U(1)$ factor of the gauge group.

The starting point is the brane configuration (a) of Figure 1.6 with N_c $D3$ branes stretched between an $NS5$ brane and a $(1,k)$ 5-brane. We will refer to this as the

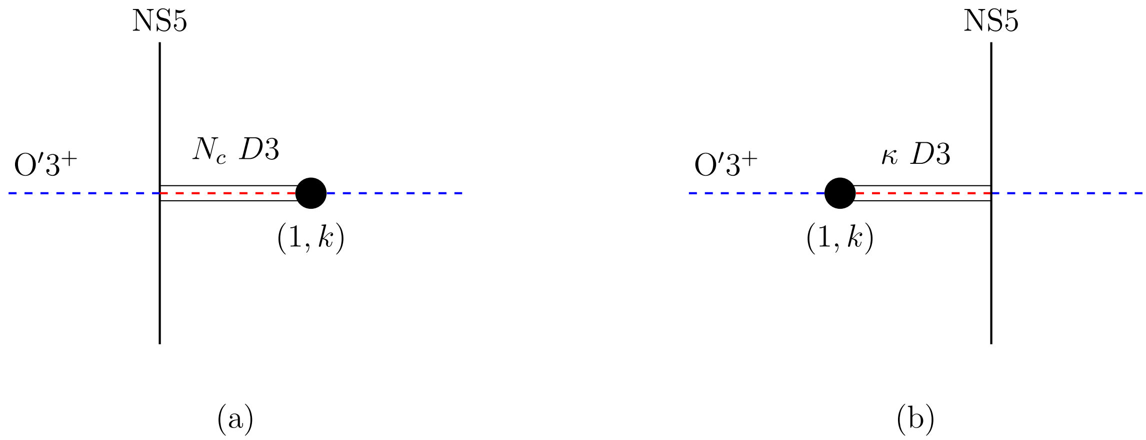

Figure 1.6: The brane setup for the (a) electric and (b) magnetic theory which give rise to level-rank duality.

electric theory. The worldvolume theory is the dimensional reduction of the $\mathcal{Y}^-$ ($\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$) subject to suitable boundary conditions. There is a $U(N_c)$ gauge field A_μ with a YM term and level k CS interactions, as well as a real scalar σ in the adjoint of $U(N_c)$ and two antisymmetric (complex) Dirac fermions in the $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ and the $\overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}$ of $U(N_c)$, respectively. The Lagrangian takes the following form[14][15]

$$\begin{aligned}\mathcal{L}^{(E)}_{N_f=0} =&\frac{1}{g_e^2}\mathrm{tr}\left[-\frac{1}{2}(F_{\mu\nu})^2+|D_\mu\sigma|^2+i\bar{\lambda}\not{D}\lambda+i\bar{\tilde{\lambda}}\not{D}\tilde{\lambda}-i\bar{\lambda}\sigma\lambda-i\bar{\tilde{\lambda}}\sigma\tilde{\lambda}+D^2\right]\\&+\frac{k}{4\pi}\mathrm{tr}\left[\epsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho+\frac{2i}{3}A_\mu A_\nu A_\rho\right)+2D\sigma-\bar{\lambda}\lambda-\bar{\tilde{\lambda}}\tilde{\lambda}\right]\,.\end{aligned}\tag{1.53}$$

Here $F_{\mu\nu}$ is the gauge field strength and $D_\mu\equiv\partial_\mu-i[A_\mu,\cdot]$ is the covariant derivative. The covariant derivative is understood to act on the various fields in the representations of $U(N_c)$ they belong to. D is the auxiliary field of the vector multiplet borrowed from the supersymmetric parent theory. It belongs to the adjoint representation of the gauge group just like the gauge field and scalar gaugino. This Lagrangian can be thought of as the 3d $\mathcal{N}=2$ $SO(2N_c)_k$ supersymmetric YM-CS theory where the fields have been subjected to the $J(-1)^F$ projection.

[14]Such a Lagrangian is understood as descending from its parent $\mathcal{N}=2$ counterpart. In the large N limit we expect to recover a supersymmetric CS theory. The following rule is expected to hold: $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}\oplus\overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}\to$ adj.

[15]With respect to the previous sections, we have slightly changed conventions, choosing hermitian generators for the (sub)algebra $\mathfrak{su}(N_c)$. That explains the factor $i=\sqrt{-1}$ that appears in the Chern-Simons term.

	$U(N_c)_k$
A_μ	adj
σ	adj
λ	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
$\tilde{\lambda}$	$\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$

	$U(\kappa)_{-k}$
a_μ	adj
s	adj
l	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
$\tilde{l}$	$\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$

Table 1.4: The field content of the worldvolume theories of the brane constructions in Figure 1.6.

It is straightforward to obtain the Seiberg dual of this theory following e.g. [194, 27] with a slight modification that takes into account the effect discussed in Figure 1.5. After reshuffling the $NS5$ and $(1,k)$ fivebrane we arrive at the configuration (b) in Figure 1.6, where the number of colour $D3$s is now $\kappa \equiv k - N_c + 2$. We refer to this as the *magnetic* theory. The worldvolume theory is now that of a gauge field a_μ with YM term and level $-k$ CS interactions as well as a real adjoint scalar s and antisymmetric Dirac fermions l and $\tilde{l}$. The Lagrangian is

$$\begin{aligned}\mathcal{L}^{(M)}_{N_f=0} =& \frac{1}{g_m^2}\mathrm{tr}\left[-\frac{1}{2}(f_{\mu\nu})^2 + |D_\mu s|^2 + i\bar{l}\not{D}l + i\bar{\tilde{l}}\not{D}\tilde{l} - i\bar{l}sl - i\bar{\tilde{l}}s\tilde{l} + D^2\right] \\ &+ \frac{k}{4\pi}\mathrm{tr}\left[\epsilon^{\mu\nu\rho}\left(a_\mu\partial_\nu a_\rho + \frac{2i}{3}a_\mu a_\nu a_\rho\right) + 2Ds - \bar{l}l - \bar{\tilde{l}}\tilde{l}\right] .\end{aligned} \tag{1.54}$$

We are interested in the IR dynamics of these theories. In the absence of supersymmetry, the scalars on the two sides are expected to acquire a 1-loop mass of the order of the cutoff [195]

$$m_\sigma^2 \sim g_e^2\Lambda, \quad m_s^2 \sim g_m^2\Lambda . \tag{1.55}$$

As in the discussion following (1.47) this translates to an attractive force between the branes and the orientifolds, signalling perturbative stability of the configuration. At energies well below the cutoff scales, the scalars are decoupled and do not play a role. Note that the scalars also have tree level CS masses, but we expect them to be subleading due to the stringy nature of the masses in (1.55). After integrating out the scalars we are left with gauge fields and antisymmetric fermions, both of which have tree-level CS masses $M_{CS} = \pm g^2 k$ where the sign of the mass follows from the sign of the bare CS levels in (1.53) and (1.54). Due to the lack of supersymmetry, also the gauginos (the antisymmetric fermions) get a mass at one-loop and can be integrated out. Integrating out the antisymmetric fermions shift the levels of the $U(1)$

and $SU(N_c)$ (resp. $SU(\kappa)$) factors of the gauge group by disproportionate amounts. As a result the IR of the electric theory is a $U(N_c)_{K_1,K_2}$ CS TQFT where

$$K_1 = k - N_c + 2 \equiv \kappa, \qquad K_2 = k - 2N_c + 2 \equiv \kappa - N_c \, . \tag{1.56}$$

Here, the Dynkin index of the antisymmetric representation is $(N_c - 2)/2$, whereas its dimension is $N_c(N_c - 1)/2$. In order to get the result (1.56), we have also taken into account that the gauginos have charge 2 under the $U(1)$ inside the $U(N_c)$.

The IR of the magnetic theory is described by a $U(\kappa)_{L_1,L_2}$ CS TQFT with

$$L_1 = -k + \kappa - 2 = -N_c \, , \qquad L_2 = -k + 2\kappa - 2 = -N_c + \kappa \, . \tag{1.57}$$

Putting everything together we end up with the TQFTs $U(N_c)_{\kappa,\kappa-N_c}$ and $U(\kappa)_{-N_c,-N_c+\kappa}$, In fact, these theories are dual to each other. Therefore, in the IR, we recover the following level-rank duality

$$U(N_c)_{\kappa,\kappa-N_c} \longleftrightarrow U(\kappa)_{-N_c,-N_c+\kappa} \, . \tag{1.58}$$

5.2 Including flavours

We can include flavours in the discussion by adding $D5$ branes to the setup, the worldvolume directions spanned by the flavour $D5$ branes are as in Table 4.2. The IR phases of the electric theory turn out to be richer than the cases studied above and are nicely encoded in terms of the dual magnetic theory. We begin by analysing each theory separately semi-classically before mapping out the phase diagram.

Electric theory

The flavoured electric theory is realised on the brane configuration (a) of Figure 1.7. The worldvolume theory on the $D3$ branes now includes N_f complex scalars Φ and N_f Dirac fermions Ψ. The relevant flavour symmetry emerging from the branes is an $SU(N_f)$ group. The representations of the scalars and fermions with respect to the gauge and flavour groups are listed in Table 1.5. These are essentially determined by their coupling to the antisymmetric gauginos, see later (1.60).

The tree level Lagrangian is given by

$$\mathcal{L}^{(E)} = \mathcal{L}^{(E)}_{N_f=0} + \mathcal{L}_{\text{matter}} \, , \tag{1.59}$$

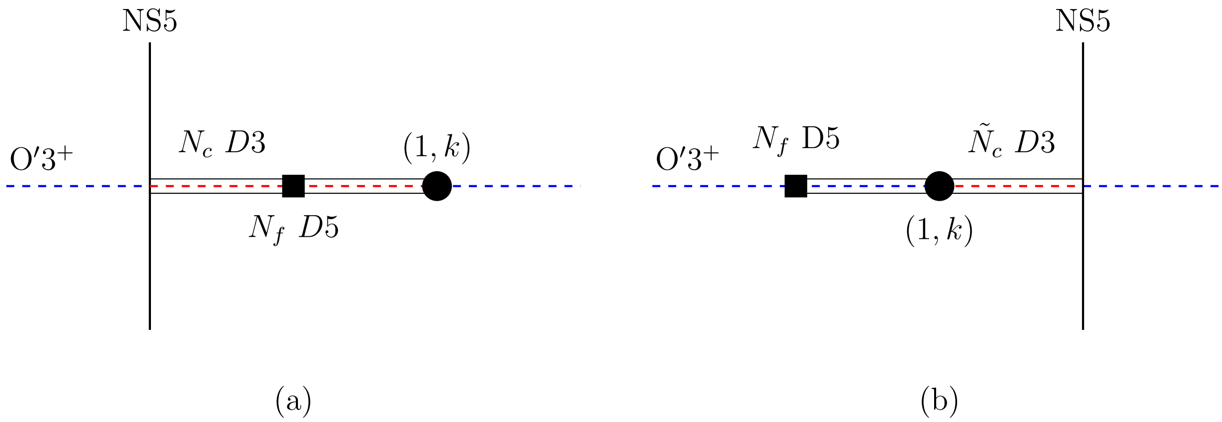

Figure 1.7: The brane setup for the (a) electric and (b) magnetic theory of our proposal. Here $\tilde{N}_c = N_f + k + 2 - N_c$.

Electric Theory

	$U(N_c)_k$	$SU(N_f)$
A_μ	adj	·
σ	adj	·
λ	$\begin{array}{c}\square\\\square\end{array}$	·
$\tilde{\lambda}$	$\overline{\begin{array}{c}\square\\\square\end{array}}$	·
Φ	$\overline{\square}$	$\square$
Ψ	$\square$	$\square$

Magnetic Theory

	$U(\tilde{N}_c)_{-k}$	$SU(N_f)$
a_μ	adj	·
s	adj	·
l	$\begin{array}{c}\square\\\square\end{array}$	·
$\tilde{l}$	$\overline{\begin{array}{c}\square\\\square\end{array}}$	·
ϕ	$\square$	$\overline{\square}$
ψ	$\overline{\square}$	$\overline{\square}$
M	·	adj
χ	·	$\square\square$
$\tilde{\chi}$	·	$\overline{\square\square}$

Table 1.5: The field content of the electric and magnetic theory.

where $\mathcal{L}^{(E)}_{N_f=0}$ is, as before, given by (1.53). The additional flavour terms are described by

$$\begin{aligned}\mathcal{L}_{\text{matter}} =& |D_\mu \Phi^a_i|^2 + i\bar{\Psi}^{ai}(\not{D}\Psi)_{ai} - \bar{\Phi}^i_a(\sigma^2)^a_b\Phi^b_i + \bar{\Phi}^i_a(D^2)^a_b\Phi^b_i \\ & - \Psi_{ai}\sigma^a_b\bar{\Psi}^{bi} - (i\lambda_{[ab]}\Phi^a_i\bar{\Psi}^{bi} + i\tilde{\lambda}^{[ab]}\bar{\Phi}^i_a\Psi_{bi} + \text{h.c.})\ .\end{aligned} \tag{1.60}$$

Here $a, b = 1, \cdots, N_c$ are colour indices and $i, j = 1, \cdots, N_f$ are flavour indices. The interactions with the gauginos fix the representations of the (s)quark fields to be as in Table 1.5.

The fate of the scalar σ of the gauge multiplet of the electric theory is similar to the flavourless case. The one-loop corrections to the scalar propagator get positive contributions from its coupling to itself and to the gauge field and negative contributions from its coupling to the gaugino λ. Since there are more bosonic than fermionic degrees of freedom, the vacuum $\langle\sigma\rangle = 0$ is stable; σ does not play a role in the IR dynamics of the theory and can be integrated out.

A similar story pans out for the squark Φ. Indeed, the squark couples to the gauge field A_μ, the scalar σ and the gaugino λ. Since there are more bosonic than fermionic degrees of freedom, one expects the squark to acquire a positive mass $M_\Phi^2 > 0$ and decouple from the IR physics.

For a non-zero level $k \neq 0$, the gauge field and the gaugino acquire a Chern-Simons mass $M_{CS} = g^2 k$. We therefore expect the IR physics to be dominated by the topological CS theory coupled to N_f fundamental quarks, i.e. QCD_3 with N_f quark flavours.[16] The IR levels of the electric theory are shifted by the gaugino as in (1.56), as well as the fundamental quarks. In summary, using the dictionary (1.42) we have

$$\text{electric IR:} \qquad U(N_c)_{K,K-N_c} \oplus N_f \text{ fermions} , \tag{1.61}$$

which is nothing but the left hand side of (1.40).

On the other hand, when $k = 0$, the IR theory is that of YM theory coupled to the gaugino and the fundamental quarks. It is less straightforward to say anything concrete about the IR dynamics of this theory.

Magnetic theory

The flavoured magnetic theory lives on the configuration (b) of Figure 1.7. It is obtained from the flavoured electric theory by the standard Giveon-Kutasov move [27, 194] modified so as to account for the brane creation described in Figure 1.5. One can easily verify that the resulting number of colour branes between the $NS5$ and the $(1, k)$ fivebrane is

$$\tilde{N}_c = N_f + k - N_c + 2 \equiv N_f + \kappa . \tag{1.62}$$

The magnetic field content is given in Table 1.5. This can be obtained in a similar fashion to the electric theory, i.e. by subjecting the theory on the $D3$ branes in Table 1.2 to the appropriate boundary conditions. We have a gauge multiplet identical to the

[16] Integrating out the gauge sector is somewhat more natural in the semiclassical regime $k \gg 1$. We expect this to remain true also at finite k, unless something drastic happens.

magnetic theory of the $N_f = 0$ case. The matter multiplet consists of a complex scalar ϕ and a Dirac fermion ψ. Their representations with respect to the gauge and flavour groups are given in Table 1.5. There are in addition new degrees of freedom, which have no analogue on the electric side, corresponding to the motion of the flavour $D3$ branes along the $x^{8,9}$ directions. These give rise to two gauge singlets; the meson M which is an $SU(N_f)$ adjoint and its fermionic partners, the "mesinos" χ transforming as $\square\!\square$ of $SU(N_f)$ and $\tilde{\chi}$ transforming as $\overline{\square\!\square}$ of $SU(N_f)$.

The tree level Lagrangian for this theory is

$$\mathcal{L}^{(M)} = \mathcal{L}^{(M)}_{N_f=0} + \mathcal{L}_{\text{matter}} \,, \tag{1.63}$$

where $\mathcal{L}^{(M)}_{N_f=0}$ is as in (1.54). The matter Lagrangian is

$$\begin{aligned} \mathcal{L}_{\text{matter}} =& |D_\mu \phi^i_a|^2 + i\bar{\psi}(\not{D}\psi)^{ai} - \bar{\phi}^a_i (s^2)^b_a \phi^i_b + \bar{\phi}^a_i D^b_a \phi_{bi} - \psi^{ai}(s)^b_a \bar{\psi}_{bi} \\ & - \left(i\tilde{l}^{[ab]}\phi^i_a \bar{\psi}_{bi} + i l_{[ab]}\bar{\phi}^a_i \psi^{bi} + \text{h.c.}\right) + |\partial_\mu M^i_j|^2 + i\bar{\chi}^{\{ij\}}\not{\partial}\chi_{\{ij\}} \\ & - y^2 \bar{\phi}^a_i \phi^i_a \bar{\phi}^b_j \phi^j_b - y^2 \phi^i_a \bar{M}^j_i M^k_j \bar{\phi}^a_k - y\Big(\chi_{\{ij\}}\phi^i_a \psi^{aj} + \tilde{\chi}^{\{ij\}}\bar{\phi}^a_i \bar{\psi}_{aj} + \text{h.c.}\Big) \\ & - y\left(\psi^{ai} M^j_i \bar{\psi}_{aj} + \text{h.c.}\right) \,. \end{aligned} \tag{1.64}$$

Note that in addition to the magnetic gauge coupling g_m, we now have another coupling constant y which controls interactions between the (s)quarks and the meson multiplet.

The scalar s of the magnetic gauge multiplet gets a positive mass and decouples, just as it did in the flavourless case. This signals the stability of the colour branes near the orientifold.

The squark ϕ couples to the gauge multiplet as well as the meson multiplet. There are more bosonic than fermionic degrees of freedom in the gauge multiplet, and more fermionic than bosonic degrees of freedom in the meson multiplet. Therefore, the squark aquires a 1-loop mass of the form

$$M^2_\phi \sim (-y^2 + g^2_m)\Lambda \,. \tag{1.65}$$

The two effects compete and the squark may become massive or tachyonic. Since at large k the gauge field becomes heavy and decouples we operate under the assumption that in this limit the squark is tachyonic.

The matter Lagrangian (1.64) for the magnetic theory includes a coupling between the meson field and the scalar quarks

$$y^2 \phi_a^i \bar{M}_i^j M_j^k \bar{\phi}_k^a \, . \tag{1.66}$$

If the meson acquires a vev of the form $\langle \bar{M}_i^j M_j^k \rangle = u^2 \delta_i^k$ the squark ϕ becomes massive. If the squark acquires a vev $\langle \phi_a^i \rangle = v \delta_a^i$, and flavour symmetry is unbroken, the mesons become massive. Therefore, the most likely scenario is that in all phases [177]

$$M_\phi^2 M_M^2 < 0 \, . \tag{1.67}$$

In the following we will always work with this assumption in mind. This will be crucial in obtaining the phase diagram of QCD_3.

6 Phase diagram

As we saw in (1.61), the IR theory on the electric brane configuration is precisely QCD_3. In this section we argue that the conjectured phase diagram of QCD_3 can be understood in terms of the dual magnetic description. Many of the features are similar to the symplectic case analysed in [177]. For this reason we will be somewhat brief and focus only on the details which are new to the unitary theory.

6.1 Region I: Bosonization

We start with the region of the parameter space where $\kappa \equiv k + 2 - N_c \geq N_f$. This corresponds to region I in the phase diagram of Figure 1.8. In this region the rank of the magnetic gauge group $\tilde{N}_c = N_f + \kappa$ is automatically positive. Following the discussion around (1.65), the N_f squarks are assumed to be tachyonic throughout this region. This is reasonable as one can go to arbitrarily large values of k while keeping N_f fixed. In this regime the gauge sector becomes heavy and decouples from the dynamics. The main contribution to the mass of the squark (ϕ) comes from the meson multiplet, which is indeed negative. Thus, our main assumption is that this remains true as we move to finite k.

Let us then assume that the magnetic squarks condense. In the brane configuration, this corresponds to Higgsing N_f colour $D3$ branes via reconnection to N_f flavour $D3$ branes. This is the Higgs mechanism in the string theory language. The world-volume of the N_f Higgsed $D3$ branes no longer supports a gauge multiplet as they end on D5s

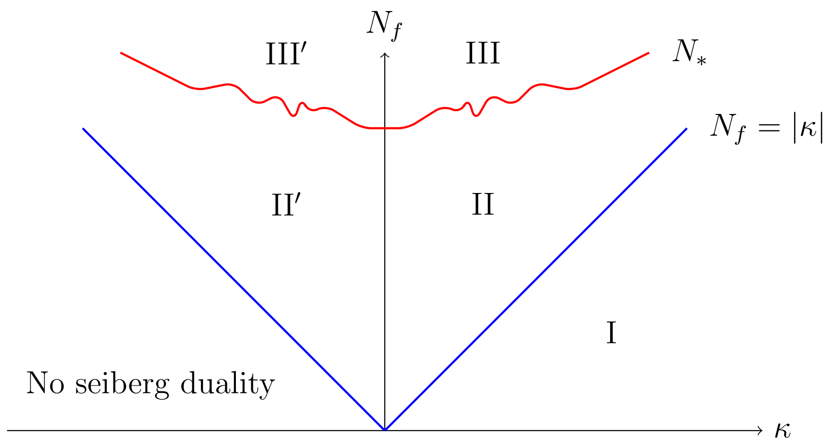

Figure 1.8: Phase diagram of QCD_3.

from one side and end on the $NS5$ brane from the other. However, we still have κ colour $D3$ branes which support a $U(\kappa)_{-k}$ gauge theory with massive gauge field and massive gauginos. The CS mass is still proportional to k, and we can integrate out the gauge field and gauginos at energies below g^2k. The reconnection preserves the original $SU(N_f)$ global symmetry. We will shortly argue, from the field theory side, that there are N_f scalars in the fundamental after the Higgsing. In the brane set-up these can only come from open strings stretched between the colour branes and N_f Higgsed $D3$ branes.

Let us try to understand the phenomenon described in the last paragraph in terms of the field theory description of the magnetic theory. Indeed, the Higgsing corresponds to giving a colour-flavour locking vev to the magnetic squark without breaking the global $U(N_f)$. The gauge symmetry breaking pattern is given by

$$U(\kappa + N_f) \to U(\kappa)\ , \tag{1.68}$$

leaving the gauginos in the $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ and $\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ of the Higgsed gauge group as well as N_f fundamental squarks. The N_f magnetic quarks become massive due to Yukawa terms. In addition, the meson and the mesino all become massive due to interactions like (1.66) and can be integrated out.

The IR levels get shifted after integrating out the gaugino according to (1.57) so that, using the dictionary (1.42), the IR of the magnetic theory in this region of the parameter space is described by

$$\text{magnetic IR:} \quad U\left(K+\frac{N_f}{2}\right)_{-N_c,-N_c+K+\frac{N_f}{2}} \oplus N_f \text{ scalars} . \qquad (1.69)$$

Such a bosonic dual is described in the IR by a Lagrangian that contains, in addition to a CS term with appropriate levels and coupling between the scalars and gauge field, also self-interactions for the squarks. These correspond to mass terms of the form $\bar{\phi}_i^a \phi_a^i$ as well as quartic interaction of the form (single-trace) $(\bar{\phi}_i^a \phi_a^j)(\bar{\phi}_j^b \phi_b^k)$ and (double-trace) $(\bar{\phi}_i^a \phi_a^i)^2$. These terms can be generated, if not already present, by the RG flow consistently with global symmetries.

As a final step, tuning the mass terms both in the electric IR theory in (1.61) and in the magnetic IR theory in (1.69), we recover a well-established duality. This is nothing but the duality (1.40).

6.2 Symmetry breaking

When $N^\star > N_f > \kappa$, which corresponds to region II and II′ in the phase diagram of Figure 1.8, we expect rather different dynamics for the system and we anticipate breaking of the flavour symmetry. As we shall see, the physics in these regions is still captured by a tachyonic squark, colour-flavour locking and brane reconnection, but the implications and the resulting physics will be different with respect to region I.

Region II′

Let us begin with region II' in the phase diagram of Figure 1.8. In this region $\kappa < 0$. Therefore, on the magnetic side, there are less colour $D3$ branes than flavour $D3$ branes: $\tilde{N}_c = N_f + \kappa < N_f$. We will assume that the squarks condense also in this case. Nonetheless, squark condensation leads in this case to a fully Higgsed gauge group. Once again this is realised in string theory by reconnecting $N_f + \kappa$ colour and flavour $D3$ branes (we stress that $\kappa < 0$ here). After the Higgsing, we are left with $|\kappa|$ flavour $D3$ branes stretched between the $D5$ brane and the $(1, k)$ fivebrane, as well as the $N_f + \kappa$ connected $D3$ branes. The latter no longer support a gauge multiplet and therefore gauge symmetry is fully broken.

The global symmetry now consists of a $U(N_f + \kappa)$ factor corresponding to the symmetry on the $N_f+\kappa$ reconnected branes as well as a $U(\kappa)$ factor from the remaining flavour $D3$ branes. Using the dictionary (1.42) we have that in this region the global symmetry breaking pattern is

$$SU(N_f) \to S\left[U\left(\frac{N_f}{2}+K\right)\times U\left(\frac{N_f}{2}-K\right)\right] . \tag{1.70}$$

This symmetry breaking pattern is the one anticipated in [22]. As a consequence, the IR physics of this phase is described in terms of the Grassmannian

$$\mathcal{M}\left(K+\frac{N_f}{2}, N_f\right) = \frac{SU(N_f)}{S\left[U\left(\frac{N_f}{2}+K\right)\times U\left(\frac{N_f}{2}-K\right)\right]} \tag{1.71}$$

corresponding to the symmetry breaking pattern given in (1.70). Such a Grassmannian will be essentially parametrised by[17]

$$N_f^2-1-\left[(N_f+\kappa)^2+\kappa^2-1\right] = 2|\kappa|(N_f-|\kappa|) = 2\left(\frac{N_f}{2}+K\right)\left(\frac{N_f}{2}-K\right) \tag{1.72}$$

massless Nambu-Goldstone bosons. We identify the Nambu-Goldstone bosons as the massless modes of open strings stretched between the two stacks of flavour branes.

Region II

When $0<\kappa<N_f<N^\star$ (or $0<K+\frac{N_f}{2}<N_f<N^\star$), after reconnection the theory in the IR is

$$U\left(K+\frac{N_f}{2}\right)_{-N_c,-N_c+K+\frac{N_f}{2}} \oplus N_f\ \phi\ . \tag{1.73}$$

Naively, we seem to have a puzzle: instead of obtaining a theory of massless Nambu-Goldstone bosons we obtain bosonization. The NG theory we are seeking is nothing but the effective description of (1.73) for large negative masses of the squarks ϕ. According to the field theory analysis of Komargodski and Seiberg [22] upon condensation of the squarks we land on the symmetry breaking phase.

Indeed, after reconnection, the scalars in the bosonic dual (1.73) correspond to scalar modes of the open strings in the brane configuration. Therefore our proposal is that these scalars are tachyonic and are to be stabilised via open string tachyon condensation. We do not know whether a nice geometric picture emerges after this condensation. Regardless, in the field theory limit one eventually lands on the Grass-

[17] In order to be consistent with the UV symmetries one must also include CS terms in the effective description. The required modification is discussed in detail in [22].

mannian $\mathcal{M}(N_f, \kappa)$. This picture is consistent with the mass deformations of the brane setup, already discussed in [177].

7 Comments about QED$_3$

The discussion of the phase diagram in the preceding sections holds for a general number of colours N_c. However, "accidents" happen when $N_c = 1, 2$ that modify parts of the discussion. In the case of $N_c = 2$ the electric gaugino is a singlet of the $SU(2)$ factor of the gauge group, but it carries charge 2 under the abelian factor. Because of this, some intermediate steps taken to arrive at the general phase diagram in Figure 1.8 are slightly modified, the end result is however unaffected and the phase diagram of Figure 1.8 is the correct picture for $N_c \geq 2$.

On the other hand, we start to see deviations from the general picture of Figure 1.8 for $N_c = 1$ i.e. QED$_3$. In particular, as we shall see momentarily, when $k = 0$ there is no symmetry breaking phase. This in turn suggests that no symmetry breaking can occur for non-zero k since the window for which a Grassmannian phase exists in the IR is maximised for $k = 0$ [22].

7.1 QED$_3$ with vanishing CS-term

When the electric gauge group is $U(1)$, there is no electric gaugino. Therefore, the IR of the electric theory is $U(1)_0$ theory coupled to N_f fermions. The magnetic dual has a gauge group $U(N_f + 1)$ with vanishing CS level at tree-level. Previously, squark condensation led to masses being generated for the quarks, meson and the mesino, due to the presence of Yukawa interactions. However, in this case after reconnection we have a $U(1)$ gauge theory with no CS term and N_f massless Dirac fermions. The reason that in this specific case the fermions do not acquire a mass is that there is no gluino when the gauge group is $U(1)$ and no Yukawa term. In the absence of supersymmetry and without fine-tuning the squarks acquire a mass. So, we end up with a magnetic theory that admits the same matter content as the electric theory.

The brane setup is such that the flavour branes coincide and hence flavour symmetry remains unbroken. Thus, our magnetic theory predicts no spontaneous breaking of $U(N_f)$. This is consistent with existing conjectures about the IR behaviour of QED$_3$ [196].

8 Conclusions

In this chapter, we discussed QCD_3 based on a unitary group and its embedding in string theory. The UV field theory on the brane configuration consists of fields that acquire a mass and decouple as the theory flows to the IR. The advantage of having such a UV theory is that it admits a Seiberg duality. The magnetic Seiberg dual leads to new insights about QCD_3. In particular the bosonized theory admits a simple realisation as a magnetic dual of the electric fermionic theory. While in the electric side scalar quarks acquire a mass and decouple, in the magnetic side the fermionic quarks acquire a mass due to Yukawa coupling and decouple.

The Seiberg dual also enables us to gain a better understanding of the symmetry breaking phase. Triggered by condensation of the dual squark the magnetic gauge theory is completely Higgsed and flavour symmetry gets broken.

In addition, we learned about the abelian theory, with or without a Chern-Simons term. The level k (with $k \geq 0$) $U(1)$ theory with N_f flavours admits a magnetic dual that upon Higgsing flows to another $U(1)$ theory with $k' = -k$ and N_f flavours. Flavour symmetry is not broken, as expected from field theory analysis. For $k = 0$ the theory looks self-dual. While for $N_f = 2$ the self duality is well understood [22], for $N_f \neq 2$ the naive self-duality deserves further investigation.

We haven't discussed the regime of $N_f > N^\star$. This regime is hard to analyse both in field theory and in string theory. As in the symplectic case [177] we anticipate that it is described by meson condensation.

M Basics of Seiberg duality

In this appendix we review some basic facts about Seiberg duality [170] and its stringy origin [193]. Before getting started, let us just say that there is a vast literature on $\mathcal{N} = 1$ supersymmetric gauge theories and Seiberg duality. We point out [197, 198, 199, 200, 201, 202]. See also the last chapter of the second volume of [33]. Here, we will only present a few basic facts without claiming to give a complete treatment. The reader who is interested in these topics might wish to consult the references.

Concretely, we consider four-dimensional $\mathcal{N} = 1$ gauge theory with gauge group[18] G coupled to $\mathcal{N} = 1$ matter (SQCD). Recall that in $\mathcal{N} = 1$ supersymmetric theories

[18]In the following, we will be concerned only with $G = SU(N_c)$, even though other choices for compact Lie groups are possible.

we have a principal G-bundle[19] $P \to \mathbb{R}^{1,3|4}$. SQCD is then identified by the gauge group G, a (generically complex) representation R of G and a classical superpotential $\mathcal{W}$, which is G-invariant. Quarks Q are N_f copies (flavours) of chiral sections of the associated bundle $P \times_G R \to \mathbb{R}^{1,3|4}$.

In this appendix we consider (anti-)fundamental representations only. Thus, quarks Q are just N_f copies of fundamental fields, Q^i_a, with a a fundamental index of G and $i = 1, \ldots, N_f$, while antiquarks belong to the conjugate representation, $\tilde{Q}^a_i$.

The global symmetry of the theory is $U(N_f)_L \times U(N_f)_R \times U(1)_X$, where the two $U(N_f)$'s are different rotations of the Q's and $\tilde{Q}$'s, while $U(1)_X$ is the $U(1)$ that acts on the fermionic coordinates θ and $\bar{\theta}$. Writing $U(N_f)_L \times U(N_f)_R$ as $SU(N_f)_L \times SU(N_f)_R \times U(1)_A \times U(1)_B$, with $U(1)_A$ and $U(1)_B$ the anti-diagonally and diagonally embedded $U(1)$ subgroups of $U(N_f)_L \times U(N_f)_R$, we find that $U(1)_A$ is quantum mechanically anomalous – it is an axial symmetry – while $U(1)_B$ is quantum-mechanically exact – it is usually called "baryon number". Also $U(1)_X$ is anomalous. It is possible to combine $U(1)_A$ and $U(1)_X$ into a non-anomalous $U(1)_R$. We will refer to $U(1)_R$ as the R-symmetry of SQCD. It is an exact symmetry and the quarks Q and $\tilde{Q}$ have charge $1 - N_c/N_f$ under it. See Table 1.6.

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
Q	$\overline{N_f}$	$\mathbf{1}$	-1	$\frac{N_f - N_c}{N_f}$
$\tilde{Q}$	$\mathbf{1}$	N_f	1	$\frac{N_f - N_c}{N_f}$

Table 1.6: Anomaly free global symmetry for $SU(N_c)$ SQCD.

The fermions ψ_Q, $\psi_{\tilde{Q}}$ and λ transform as in Table 1.7. It is easy to see that the path integral measure $D\psi_Q D\psi_{\tilde{Q}} D\lambda$ is invariant under the full global symmetry.

[19]Working in superspace, we have four "bosonic" Minkowski coordinates x^μ ($\mu = 0, 1, 2, 3, 4$) – with Minkowski metric $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ – and "fermionic" coordinates θ_α, $\bar{\theta}_{\dot{\alpha}}$, $\alpha, \dot{\alpha} = 1, 2$. The "4" in $\mathbb{R}^{1,3|4}$ simply stands for the number of fermionic coordinates. A superfield Q has a natural expansion in superspace given by $Q = \phi + \sqrt{2}\theta\psi + \theta^2 F$.

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
ψ_Q	$\overline{N_f}$	$\mathbf{1}$	-1	$-\frac{N_c}{N_f}$
$\psi_{\tilde{Q}}$	$\mathbf{1}$	N_f	1	$-\frac{N_c}{N_f}$
λ	$\mathbf{1}$	$\mathbf{1}$	0	1

Table 1.7: Transformation laws for the fermions under the quantum global symmetry of $SU(N_c)$ SQCD.

M.1 The conformal window, non-trivial IR fixed points

SQCD with gauge group $G = SU(N_c)$ and N_f fundamental flavours has a β function which at two loops reads

$$\beta(g) = -\frac{g^3}{16\pi^2}(3N_c - N_f) + \frac{g^5}{128\pi^4}\left(2N_cN_f - 3N_c^2 - \frac{N_f}{N_c}\right) + \mathcal{O}(g^7)\,. \tag{1.74}$$

We see that the two-loops contribution is positive at $N_f = 3N_c$. Consider the limit where both $N_c, N_f \to \infty$, together with $3N_c - N_f = \epsilon$ fixed for some small $\epsilon > 0$. The β function has a zero at

$$g_*^2 N_c = \frac{8\pi^2}{3}\frac{\epsilon}{N_c}\,. \tag{1.75}$$

The reason for considering $g_*^2 N_c$ instead of g_* is that, as 't Hooft proved a long time ago, when N_c is large enough the actual coupling is $\lambda = g^2 N_c$, rather than g^2, and that perturbation theory breaks down when λ is not small, even though $g \ll 1$.

Therefore, what we have found is a perturbative IR fixed point, and the use of formula (1.74) is indeed justified. Such IR fixed point is usually referred to as Banks-Zacks fixed point.

Seiberg made concrete the idea that such a fixed point persists even for other values of N_f, and in particular that it is possible to reach an IR strongly coupled fixed point in the "conformal window" $\frac{3N_c}{2} < N_f < 3N_c$. We have already argued that when $N_f \geq 3N_c$ the theory becomes IR free. We will see in a moment that there cannot be any fixed points for $N_f \leq \frac{3}{2}N_c$.

Let us recall a few facts about the superconformal algebra in four dimensions. The Lie algebra of the conformal group in $d \geq 3$ is generated by the translations $P_\mu = \partial_\mu$, Lorentz generators $M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$, dilatation $D = -x^\mu\partial_\mu$ and special conformal transformations $K_\mu = 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu$. The commutation relations for the generators are

$$[D, P_\mu] = P_\mu\,, \quad [D, K_\mu] = -K_\mu\,, \quad [K_\mu, P_\nu] = 2(\eta_{\mu\nu} D - M_{\mu\nu})\,, \tag{1.76}$$

besides the usual Lorentz algebra for P_μ and $M_{\mu\nu}$.

We will work in a basis where P_μ and K_μ are the adjoint of each other[20], $P_\mu^\dagger = K_\mu$. We can define a vacuum (or heighest weight state) $|\mathcal{O}\rangle$ which is an eigenstate of D (whose eigenavalue we also denote D) and $K_\mu|\mathcal{O}\rangle = 0$. If $\mathcal{O}$ is a scalar we have that $M_{\mu\nu}|\mathcal{O}\rangle = 0$. The norm of $P_\mu|\mathcal{O}\rangle$ is easily computed by using the algebra (1.76) to give

$$\langle\mathcal{O}|K_\mu P_\mu|\mathcal{O}\rangle = 2dD\,. \tag{1.77}$$

Unitarity implies $D \geq 0$. The equality is saturated if and only if $P_\mu|\mathcal{O}\rangle = 0$, i.e. if $\mathcal{O}$ is a constant operator (the identity or a multiple of it). If we compute instead the norm of the state $P_\mu P^\mu|\mathcal{O}\rangle$ using the algebra (1.76) we get

$$\langle\mathcal{O}|K^\nu K_\nu P^\mu P_\mu|\mathcal{O}\rangle = 8dD(D - \frac{d-2}{2})\,. \tag{1.78}$$

For $d = 4$ unitary requires $D \geq 1$. The equality is saturated if and only if $P^2|\mathcal{O}\rangle = 0$, i.e. $\mathcal{O}$ is a free operator satisfying the Klein-Gordon equation.

As we already discussed in Chapter 1, when we add supersymmetry in four dimensions the algebra has extra (odd) generators Q_α, $\bar{Q}_{\dot\alpha}$ and S^α, $\bar{S}^{\dot\alpha}$. The Q's have dimension $\frac{1}{2}$ while the S's have dimension $-\frac{1}{2}$ and the algebra gets extended as to include the following commutation relations

$$\begin{aligned} &[Q_\alpha, K^\mu] = \sigma^\mu_{\alpha\dot\alpha}\bar{S}^{\dot\alpha}\,, \qquad [\bar{Q}_{\dot\alpha}, K^\mu] = \bar\sigma^\mu_{\dot\alpha\alpha} S^\alpha\,, \\ &\{Q_\alpha, S^\beta\} = (\sigma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu} + 2\delta_\alpha^\beta \left(D - \frac{3}{2}R\right)\,. \end{aligned} \tag{1.79}$$

The operators Q and S are conjugate to each other in radial quantisation, $Q^\dagger = S$, in a similar manner as P and K are conjugate to each other. An operator $\mathcal{O}$ is said to be chiral if it satisfies $\bar{Q}_{\dot\alpha}|\mathcal{O}\rangle = 0$.

Consider now a scalar operator which is also an heighest weight state, $K_\mu|\mathcal{O}\rangle =$

[20] This can be motivated by considering radial quantisation, after analytical continuation to Euclidean signature. In a Euclidean space there is no preferred direction for time. Thus, we can identify $\tau = \log r$, with τ the Euclidean time and r radial distance from the origin. Since time is imaginary, complex conjugation just sends $\tau \to -\tau$, which correspond to an inversion $r \to \frac{1}{r}$. Thus, loosely speaking, $P_\mu^\dagger$ = Inversion $\cdot\, P_\mu \cdot$ Inversion, which is known to generate a special conformal transformation.

$S^\alpha|\mathcal{O}\rangle = \bar{S}^{\dot{\alpha}}|\mathcal{O}\rangle = 0$. Then the norm of $Q_\alpha|\mathcal{O}\rangle$ is computed using the superconformal algebra to be

$$\langle\mathcal{O}|S^\alpha Q_\alpha|\mathcal{O}\rangle = 4\left(D - \frac{3}{2}R\right) . \tag{1.80}$$

Unitary implies that $D \geq \frac{3}{2}R$ and the equality is saturated for scalar chiral operators. As we shall see momentarily, this has important consequences. Consider for instance the meson operator $M^i_j = Q^i_a \tilde{Q}^a_j$. It has classical dimension 2 (because Q and $\tilde{Q}$ have classical dimension 1). Using that $R(Q) = R(\tilde{Q}) = 1 - N_c/N_f$, the quantum dimension of M at the IR fixed point is given by

$$D(M) = \frac{3}{2}R(M) = 3\left(1 - \frac{N_c}{N_f}\right) . \tag{1.81}$$

The difference between the classical dimension of M and its quantum dimension is called "anomalous dimension" γ, $D(M) = 2 + \gamma$. Perturbatively, $\gamma(g)$ is given by

$$\gamma(g) = -\frac{g^2}{8\pi^2}\frac{N_c^2 - 1}{N_c} + \mathcal{O}(g^4) . \tag{1.82}$$

At the IR fixed point (1.75), we have that $D(M) = 2 - \frac{\epsilon}{3N_c}$. This is consistent with

$$D(M) = 3\left(1 - \frac{N_c}{N_f}\right) = 2 - \frac{\epsilon}{3N_c} , \tag{1.83}$$

when $3N_c - N_f = \epsilon$ as before.

In fact, we can do even better than just perturbative considerations. It turns out that for $\mathcal{N} = 1$ SQCD there exists an exact formula for the gauge coupling β function. This is the NSVZ formula [203, 204, 205], and in the case of N_f fundamental quarks it is given by

$$\beta(g) = -\frac{g^3}{16\pi^2}\frac{3N_c - N_f(1-\gamma)}{1 - \frac{N_c g^2}{8\pi^2}} . \tag{1.84}$$

At second order in the gauge coupling, (1.84) is consistent with (1.74). A zero of (1.84) implies that γ is given by

$$\gamma = 1 - \frac{3N_c}{N_f} , \tag{1.85}$$

and the quantum dimension of M is given by

$$D(M) = 2 + \gamma = 3 - \frac{3N_c}{N_f} , \tag{1.86}$$

consistent with (1.81), that was derived from the superconformal algebra as a condition on chiral operators. Notice that $\gamma = 1 - 3N_c/N_f$ in the range $\frac{3N_c}{2} < N_f < 3N_c$ is in general an arbitrary number, not necessarily small. We are not in the perturbative regime anymore.

We are now ready to motivate the lower bound for the conformal window $\frac{3N_c}{2} < N_f < 3N_c$. Unitarity implies $D(M) \geq 1$ and, because of (1.86), $N_f \geq \frac{3}{2}N_c$. Thus, $D(M) > 1$ in the range $\frac{3N_c}{2} < N_f < 3N_c$, and the IR fixed point of SQCD is non-trivial. When $N_f = \frac{3}{2}N_c$, $D(M) = 1$ and M (as maybe the whole low energy theory) is free. When $N_f < \frac{3}{2}N_c$, $D(M) < 1$ and there cannot be any fixed point.

M.2 The magnetic theory

Seiberg proposed that associated with the electric theory there should be a dual magnetic theory in the regime $N_c + 2 \leq N_f \leq \frac{3}{2}N_c$ which describes the same physics at long distances. In particular, the magnetic theory is an $\mathcal{N} = 1$ SQCD theory with gauge group $SU(N_f - N_c)$ and N_f flavours of quarks q_i, $\tilde{q}^i$ and a magnetic meson M^i_j which is a singlet under the gauge symmetry and couples to the quarks through the superpotential

$$\mathcal{W} = M^i_j q_i \tilde{q}^j \,. \tag{1.87}$$

The meson M^i_j is identified with the operator $Q^i\tilde{Q}_j$ of the electric theory. while the magnetic $q_i\tilde{q}^j$ vanishes by the equation of motion of M^i_j. Also the baryons of the electric and magnetic theory are mapped to each other

$$b^{j_1 \dots j_{N_c}} = \epsilon^{i_1 \dots i_{N_f - N_c} j_1 \dots j_{N_c}} B_{i_1 \dots i_{N_f - N_c}} \,, \tag{1.88}$$

where $B_{i_1 \dots i_{N_f - N_c}}$ are the baryons of the electric theory

$$B_{i_1 \dots i_{N_f - N_c}} = \epsilon_{i_1 \dots i_{N_f - N_c} j_1 \dots j_{N_c}} \epsilon^{a_1 \dots a_{N_c}} Q^{j_1}_{a_1} \dots Q^{j_{N_c}}_{a_{N_c}} \,. \tag{1.89}$$

Antibaryons are constructed from $\tilde{Q}$ and matched with the antibaryons of the magnetic theory in a similar way.

The exact quantum global symmetry is still $SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R$, where the quarks and meson transform according to Table 1.8.

The fact that the electric and magnetic theory do not have the same gauge symmetry is not a problem, as gauge symmetries are not really physical symmetries of a

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
q	N_f	$\mathbf{1}$	$-\frac{N_c}{N_f-N_c}$	$\frac{N_c}{N_f}$
$\tilde{q}$	$\mathbf{1}$	$\overline{N_f}$	$\frac{N_c}{N_f-N_c}$	$\frac{N_c}{N_f}$
M	$\overline{N_f}$	N_f	0	$2\frac{N_f-N_c}{N_f}$

Table 1.8: Quantum global symmetry of $SU(N_f - N_c)$ MSQCD.

theory but just redundancies in its description.

The fermions ψ_q, $\psi_{\tilde{q}}$, $\psi_{\tilde{\lambda}}$ and ψ_M transform according to the following table.

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
ψ_q	N_f	$\mathbf{1}$	$-\frac{N_c}{N_f-N_c}$	$\frac{N_c}{N_f} - 1$
$\psi_{\tilde{q}}$	$\mathbf{1}$	$\overline{N_f}$	$\frac{N_c}{N_f-N_c}$	$\frac{N_c}{N_f} - 1$
ψ_M	$\overline{N_f}$	N_f	0	$1 - 2\frac{N_c}{N_f}$
$\psi_{\tilde{\lambda}}$	$\mathbf{1}$	$\mathbf{1}$	0	1

Table 1.9: Transformation laws for the fermions under the quantum global symmetry of $SU(N_f - N_c)$ MSQCD.

It is easy to see that the path integral measure $D\psi_q D\psi_{\tilde{q}} D\psi_{\tilde{\lambda}}$ of the fermions charged under the gauge symmetry is invariant under the full quantum global symmetry.

Seiberg duality allows us to study $SU(N_c)$ SQCD in the regime $N_c+2 \leq N_f \leq \frac{3}{2}N_c$ by switching to the magnetic description. The magnetic $SU(N_f - N_c)$ is IR free when $N_f < \frac{3}{2}N_c$. Thus, the strongly coupled $SU(N_c)$ SQCD is in fact free in the dual magnetic variables.

Several checks for Seiberg duality have been given [170]. Among these we have

- 't Hooft anomaly matching for the global symmetries
- the duality is preserved upon giving masses to the quarks
- the duality is an involution: applying the duality twice we get back to the original $SU(N_c)$ SQCD
- the two theories have the same moduli spaces of vacua

- the superconformal index of the two theories match [206].

M.3 String theory embedding of Seiberg duality

Elitzur, Giveon and Kutasov found in [193] a string theory embedding for Seiberg duality. It is very much similar in spirit to the Giveon-Kutasov duality discussed in [20], thus we will brief and just sketch the dual brane configurations.

We start in Type IIA string theory with the brane web on the left of Figure 1.9.

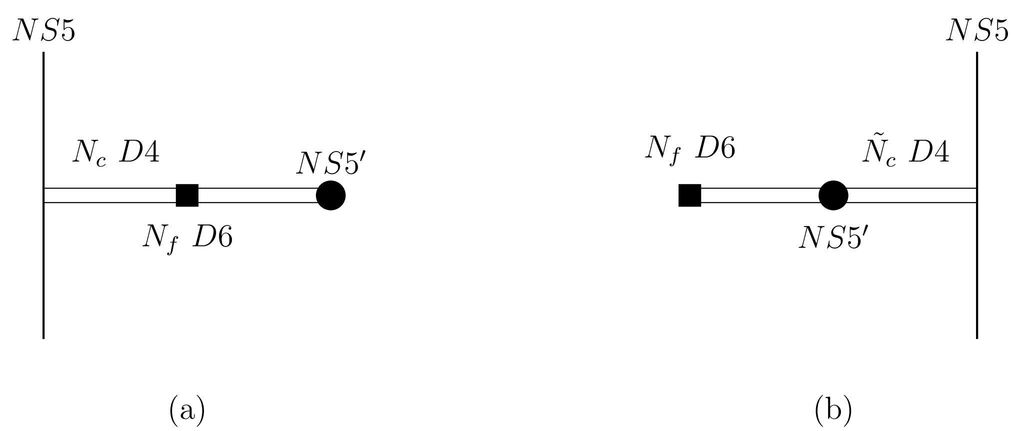

Figure 1.9: The brane setup for the (a) electric and (b) magnetic Seiberg dual theories. Here $\tilde{N}_c = N_f - N_c$.

Both brane setups in Figure 1.9 are easily shown to preserve 4 supercharges, or $\mathcal{N} = 1$ supersymmetry in four dimensions. The low energy theory for the brane system on the left is that of an $U(N_c)$ SQCD with N_f flavours of quarks arising from strings stretched between the N_c colour branes and the N_f flavour $D6$ branes. Exchanging the $NS5$ and the $D6$ branes we get the configuration on the right in Figure 1.9, where the low energy theory is that of $U(N_f - N_c)$ SQCD with N_f flavours of quarks and a gauge singlet M^i_j which correspond to the motion of the $D4$ flavour branes along (x^8, x^9).

Field theory results, like the study of the moduli spaces of vacua of both theories or deformations on both sides of the duality, can be performed on the brane systems and shown to reproduce the field theory analysis [193].

Part III

Conclusions and future directions

Epilogue

We have finally come to the end of our long journey through dualities in Quantum Field Theories and string theories.

We started off in Part I, Chapter 1 by discussing the general holographic duals to a broad class of four-dimensional superconformal theories. We have seen that, in order to find gravity solutions in M-theory that capture the physics of four-dimensional field theories – not only gauge theories – we should solve a complicated non-linear Toda equation and impose boundary conditions on its solutions. We have also seen that the problem simplifies considerably if we assume an extra $U(1)$ isometry of the eleven-dimensional background. This allows us to reduce the problem to Type IIA string theory. The equation to be solved in Type IIA to find gravity backgrounds is linear, and often referred to as Laplace equation. The class of dual field theories is then seen to emerge from fluctuations on systems of intersecting $D4$, $D6$ and $NS5$ branes. We gave a number of formulas in supergravity that give the number of branes as well as the Linking Numbers and the central charges of the dual field theories. These were also tested in many examples of increasing level of complexity. As a final application of holography for four-dimensional field theories, we introduced marginal deformation that are supposed to break half of the supersymmetry. It would be nice in the future to find general solutions of the eleven-dimensional Toda equation that allow us to tackle the problem in M-theory, where more general classes of field theories can be studied. It would be also of interest to introduce in the gravity backgrounds relevant deformations and study the corresponding RG flow. The IR physics is expected to be very rich – depending on the deformation – and this would provide a more dynamical way of breakig supersymmetry.

In Part I, Chapter 2, we studied spin 2 fluctuations around a broad class of backgrounds with an AdS_3 factors preserving 8 superconformal symmetries. We have seen that spin 2 fluctuations are rather special, as they decouple from other fluctuations allowing us to study a protected sector of excitations without solving complicated nonlinear partial differential equations in supergravity. We identified a class of solutions which are not sensitive to the details of the geometry. This is remarkable because, via the holographic duality, we were able to identify a class of operators in the dual field theory for any geometry in the class of warped AdS_3 backgrounds. Among these operators, we found the holomorphic and anti-holomorphic energy momentum tensor, which indeed are supposed to exist in any CFT. From the quadratic action of massless spin 2 fluctuations we were able also to reproduce the holographic central charge. We know indeed that the quadratic action of the massless graviton computes the two-point function of the energy momentum tensor. It would be nice in the future to construct the dual operators in the gauge theory explicitly. The generality of the formalism suggests that all we have done in this chapter can be reproduced in many other cases of holographic dualities.

In Part I, Chapter 3, we introduced two new classes of warped backgrounds in Type IIA and IIB supergravity with an AdS_2 factor: AdS_2 holography stands out as a paradigmatic example of duality as it is intimately connected with black hole physics in flat or anti de-Sitter spacetimes. We gave a careful study of the underlying geometries. These appear to be regular everywhere except at points where physical sources or orientifold planes are located. Physical branes provide flavour symmetry for the dual field theory and, therefore, they are needed in order to describe the most general dual Quantum Mechanics. Interestingly, we were able to give a prescription for the central charge in terms of the NS-NS supergravity fields. We proposed that such a central charge counts the number of degenerate ground states of the system. Also, we proposed a minimisation principle, when a suitable functional is given, that leads to the same result for the central charge entirely formulated in terms of fields in the R-R sector. Such a functional is shown to emerge from the product of some electric and magnetic charges associated with all branes in the supergravity background. In the case of the gravity solutions in Type IIB, we were able also to identify the dual Quantum Mechanics, which we discussed in some detail. It is suspected that the central charge defined here could be related to some superconformal indices computable by means of localisation techniques. Also, given that we have explicit AdS_2 solutions in string theory, it would be interesting to connect them with black hole physics.

In Part II, Chapter 1 we moved on to dualities between three-dimensional Quantum Field Theories. In particular, we focussed on the intriguing example of Chern-Simons theories. After a general introduction to Chern-Simons theories in three-dimensions, we discussed in detail the case of duality between Chern Simons theories with unitary gauge symmetry. The main goal was to motivate such a duality from string theory and see what we could learn from it. We saw that a suitable string theory embedding exists in Type 0B string theory. In particular, a brane web made of intersections of $D3$, $D5$ and $NS5$ branes, along with an orientifold threeplane, is shown to lead in the IR to QCD in three dimensions. The dual theory, obtained after rearranging branes in the original setup, provides us with the bosonic dual theory that captures the infrared physics of QCD_3. The phase diagram of QCD_3 is then sectioned in four different parts. We showed how to reproduce bosonisation in the classical picture as well as the flavour symmetry breaking phases in the quantum regime. Our results are consistent with the existing literature.

It should be clear that *duality* is a powerful tool to map out problems in physics. For one thing, they allow us – in certain cases – to study problems that we would not know how to tackle, even though it is not clear – at least to the author – why dualities exist at all. For another thing, dualities pinpoint our ignorance in describing certain physical systems. Perhaps, pointing out our deficiencies in the description of the real world is the best teaching dualities could give us.

Part IV

Appendices

Type II supergravities

In this appendix we review some basic facts about type II supergravities. In particular, we spell out what the bosonic actions are, and write down explicitly the equations of motion for the various fields. All supergravity backgrounds discussed in the main text are solutions to the equations of motions quoted in the following. Standard references are the books of Polchinski [155, 156].

Perturbative superstring theory requires the spacetime to be ten-dimensional, in order to be consistent at the quantum level, and it turns out that $\mathbb{R}^{1,9}$ defines an acceptable vacuum. Also, the spectrum of closed string theory around $\mathbb{R}^{1,9}$ comprises of the graviton plus other fields with spin less than 2, besides an infinite tower of massive modes with masses

$$m_n^2 \sim \frac{n}{\alpha'} \,. \tag{1}$$

There are two known consistent superstring theories with 32 supercharges. They are called type IIA and IIB superstring theories. Type IIA string theory has $\mathcal{N} = (1,1)$ supersymmetry and therefore is non-chiral, whereas Type IIB has $\mathcal{N} = (2,0)$ and is therefore chiral[1].

The *supergravity* approximation to string theory is generally true at low energies and weak string coupling. More in detail, we are allowed to approximate string theory with classical supergravity whenever

$$R \ll \frac{1}{\alpha'} \,, \qquad E \ll \frac{1}{\sqrt{\alpha'}} \,, \qquad g_s \ll 1 \,, \tag{2}$$

where R is the scalar curvature of the ten-dimensional spacetime and g_s the string coupling. Such regime of coupling is usually referred to as the $\alpha' \to 0$ weak coupling

[1]Remember that, in Minkowski signature, we can impose both Majorana and Weyl conditions only in dimension 2 mod 8. Therefore, a minimal spinor in 10 dimensions has 16 real independent components.

limit of string theory. Intuitively sending $\alpha' = l_s^2 \to 0$ simply means taking the "point-particle limit" of string theory. Also, in this limit all massive modes in (1) decouple from the dynamics and we are left with only massless modes.

Let us now discuss the field content of the two supergravities.

A Type IIB supergravity

The massless content of type IIB supergravity is organised in the graviton multiplet

$$\mathcal{G} = (g_{\mu\nu}, \phi, B_{\mu\nu}, C, C_{\mu\nu}, C_{\mu\nu\rho\sigma}, \Psi^a_\mu, \psi^a)\,, \tag{3}$$

where

- $g_{\mu\nu}$ is the graviton
- ϕ is the dilaton
- $B_{(2)}$ is the N-S two-form
- $C_{(0)} = C$ is a R-R zero-form, that is an axion with $C_{(0)} \cong C_{(0)} + 2\pi$
- $C_{(2)}$ is a R-R two-form
- $C_{(4)}$ is a R-R four-form
- Ψ^a_μ, ψ^a, with $a = 1, 2$, are two gravitinos and two chiral fermions[2].

Notice that the spin bundle over $\mathcal{M}_{10}$ has real structure and decomposes as $S^+(\mathcal{M}_{10}) \oplus S^-(\mathcal{M}_{10})$. Thus, Ψ^a can be thought of as a section of $S^+(\mathcal{M}_{10}) \otimes T^*(\mathcal{M}_{10})$, with $T^*(\mathcal{M}_{10})$ the cotangent bundle, and ψ^a a section of $S^+(\mathcal{M}_{10})$.

The bosonic action of type IIB supergravity in string frame is given by[3]

$$\begin{aligned} S_{IIB} =& \frac{1}{2\kappa_{10}^2}\int_{\mathcal{M}_{10}} d^{10}x\sqrt{-g}\left[e^{-2\phi}\left(R + 4|\partial\phi|^2 - \frac{1}{12}|H|^2\right) - \frac{1}{2}\left(F_{(1)}^2 + \frac{F_{(3)}^2}{3!} + \frac{1}{2}\frac{F_{(5)}^2}{5!}\right)\right] \\ & - \frac{1}{4\kappa_{10}^2}\int C_{(4)} \wedge H_{(3)} \wedge dC_{(2)}\,, \end{aligned} \tag{4}$$

[2] Fermionic indices are not shown explicitly.

[3] As in the main text, we have $2\kappa_{10}^2 = (2\pi)^7 g_s^2 \alpha'^4$.

where the field strengths are given by

$$H_{(3)} = dB_{(2)}\,, \quad F_{(1)} = dC_{(0)}\,, \quad F_{(3)} = dC_{(2)} - C_{(0)}H_{(3)}\,, \quad F_{(5)} = dC_{(4)} - H_{(3)} \wedge C_{(2)}\,. \tag{5}$$

Moreover, $F_{(5)}$ must be supplemented with a self-duality condition (not derived from the action)

$$F_{(5)} = \star F_{(5)}\,. \tag{6}$$

The Bianchi identities for the field strengths are

$$dH_{(3)} = 0\,, \qquad dF_{(1)} = 0\,, \qquad dF_{(3)} = H_{(3)} \wedge F_{(1)}\,, \qquad dF_{(5)} = H_{(3)} \wedge F_{(3)}\,. \tag{7}$$

The Einstein's equations coming from the action above read

$$\begin{aligned} &R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi - \frac{1}{4} H^2_{\mu\nu} \\ &\quad = e^{2\phi}\left[\frac{1}{2}(F^2_{(1)})_{\mu\nu} + \frac{1}{4}(F^2_{(3)})_{\mu\nu} + \frac{1}{96}(F^2_{(5)})_{\mu\nu} - \frac{1}{4} g_{\mu\nu}\left(F^2_{(1)} + \frac{1}{6}F^2_{(3)}\right)\right]\,, \end{aligned} \tag{8}$$

where $H^2_{\mu\nu} = H_{\mu\rho\sigma}H_\nu^{\rho\sigma}$ and so on. Notice also that $F^2_{(5)} = 0$ because of the self-duality condition $F_{(5)} = \star F_{(5)}$.

The dilaton equation reads

$$R + 4\nabla^2\phi - 4(\partial\phi)^2 - \frac{1}{12}H^2_{(3)} = 0\,. \tag{9}$$

Finally, we give the equations of motion for the field strengths

$$\begin{aligned} &d\left(e^{-2\phi} \star H_{(3)}\right) - F_{(1)} \wedge \star F_{(3)} - F_{(3)} \wedge \star F_{(5)} = 0\,, \\ &d \star F_{(1)} + H_{(3)} \wedge \star F_{(3)} = 0\,, \\ &d \star F_{(3)} + H_{(3)} \wedge \star F_{(5)} = 0\,, \\ &d \star F_{(5)} - H_{(3)} \wedge \star F_{(3)} = 0\,. \end{aligned} \tag{10}$$

B Massive type IIA supergravity

The N-S sector of massive type IIA supergravity is the same as that of type IIB supergravity, while the R-R sector is made of odd-form potentials. They altogether are given by

$$\mathcal{G} = (g_{\mu\nu}, \phi, B_{\mu\nu}, C_\mu, C_{\mu\nu\rho}, \Psi^a_\mu, \psi^a)\,, \tag{11}$$

where

- $C_{(1)}$ is a R-R one-form
- $C_{(3)}$ is a R-R three-form
- Ψ^a_μ, ψ^a, with $a = \pm$, are two gravitinos and two chiral fermions.

$\Psi^\pm$ can be thought of as a section of $S^\pm(\mathcal{M}_{10}) \otimes T^*(\mathcal{M}_{10})$, with $T^*(\mathcal{M}_{10})$ the cotangent bundle, and $\psi^\pm$ is a section of $S^\pm(\mathcal{M}_{10})$.

In addition to this, there is also a zero-form field strength $F_{(0)} = \star F_{(10)}$. Its equation of motion is

$$d \star F_{(10)} = 0\,, \tag{12}$$

and, being a scalar, it is just a constant $\star F_{(10)} =$ constant. Thus, there are no propagating degrees of freedom. $F_{(0)}$ appears in the action – see below – only quadratically and with no derivatives. It can in principle be integrated out, at the cost of introducing a nonlinear dependence on $B_{(2)}$. When $F_{(0)} \neq 0$ we have what is known as "massive type IIA supergravity". If, instead, $F_{(0)} = 0$ we have massless type IIA supergravity. We remind the reader that ony massless type IIA supergravity can be uplifted to eleven-dimensional supergravity.

The bosonic action of type IIA supergravity in string frame is given by

$$\begin{aligned} S_{IIA} =& \frac{1}{2\kappa^2} \int_{\mathcal{M}_{10}} d^{10}x \sqrt{-g} \left[e^{-2\phi} \left(R + 4|\partial\phi|^2 - \frac{1}{12}|H|^2 \right) - \frac{1}{2} \left(F_{(0)}^2 + \frac{F_{(2)}^2}{2!} + \frac{F_{(4)}^2}{4!} \right) \right] \\ & - \frac{1}{4\kappa^2} \int dC_{(3)} \wedge dC_{(3)} \wedge B_{(2)} + \frac{F_{(0)}}{3} dC_{(3)} \wedge B_{(2)}^3 + \frac{F_{(0)}^2}{20} B_{(2)}^5\,, \end{aligned} \tag{13}$$

where the field strengths are given by

$$H_{(3)} = dB_{(2)}\,, \quad F_{(2)} = dC_{(1)} + F_{(0)} B_{(2)}\,, \quad F_{(4)} = dC_{(3)} - H_{(3)} \wedge C_{(1)} + \frac{F_{(0)}}{2} B_{(2)} \wedge B_{(2)}\,. \tag{14}$$

The coefficients have been set as to realise the gauge transformations

$$B_{(2)} \to B_{(2)} + d\Lambda\,, \quad C_{(1)} \to C_{(1)} - F_{(0)}\Lambda\,, \quad C_{(3)} \to C_{(3)} - F_{(0)}\Lambda \wedge B_{(2)}\,, \tag{15}$$

with Λ a one-form. The Bianchi identities for the field strengths are

$$dH_{(3)} = 0\,, \qquad dF_{(2)} = F_{(0)} H_{(3)}\,, \qquad dF_{(4)} = H_{(3)} \wedge F_{(2)}\,. \tag{16}$$

The Einstein's equations coming from the action above read

$$\begin{aligned} &R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi - \frac{1}{4} H^2_{\mu\nu} \\ &\quad = e^{2\phi}\left[\frac{1}{2}(F^2_{(2)})_{\mu\nu} + \frac{1}{12}(F^2_{(4)})_{\mu\nu} - \frac{1}{4} g_{\mu\nu}\left(\frac{1}{2}F^2_{(2)} + \frac{1}{24}F^2_{(4)} + F^2_{(0)}\right)\right]\,, \end{aligned} \tag{17}$$

where, again, $H^2_{\mu\nu} = H_{\mu\rho\sigma} H_\nu^{\rho\sigma}$ and so on.

The dilaton equation is the same as in the type IIB case

$$R + 4\nabla^2\phi - 4(\partial\phi)^2 - \frac{1}{12} H^2_{(3)} = 0\,. \tag{18}$$

Finally, the equations of motion for the field strengths read

$$\begin{aligned} &d\left(e^{-2\phi} \star H_{(3)}\right) - F_{(2)} \wedge \star F_{(4)} - \frac{1}{2} F_{(4)} \wedge F_{(4)} - F_{(0)} \wedge \star F_{(2)} = 0\,, \\ &d \star F_{(2)} + H_{(3)} \wedge \star F_{(4)} = 0\,, \\ &d \star F_{(4)} + H_{(3)} \wedge F_{(4)} = 0\,. \end{aligned} \tag{19}$$

All Bianchi identities above (for both Type IIA and IIB) are given when there are no sources. It turns out to be consistent with supersymmetry to violate Bianchi identities by means of localised brane sources, as discussed in the main text.

Higher rank forms, like $F_{(6)}$, $F_{(8)}$ and $F_{(10)}$ for Type IIA or $F_{(7)}$ and $F_{(9)}$ for Type IIB supergravity, do not appear explicitly in (4) or (13). They are related to lower rank forms via

$$F_{(p)} = (-1)^{[p/2]} \star F_{(10-p)}\,, \tag{20}$$

in Minkowski signature. The constraint (20) is solved by giving the supergravity actions in terms of low rank R-R forms.

Bibliography

[1] C. Nunez, D. Roychowdhury, S. Speziali, and S. Zacarias, "Holographic Aspects of Four Dimensional $\mathcal{N} = 2$ SCFTs and their Marginal Deformations," *Nucl. Phys.* **B943** (2019) 114617, arXiv:1901.02888 [hep-th].

[2] M. Akhond, A. Armoni, and S. Speziali, "Phases of $\mathbf{U(N_c)}$ QCD_3 from Type 0 Strings and Seiberg Duality," *JHEP* **09** (2019) 111, arXiv:1908.04324 [hep-th].

[3] S. Speziali, "Spin 2 fluctuations in 1/4 BPS AdS_3/CFT_2," *JHEP* **03** (2020) 079, arXiv:1910.14390 [hep-th].

[4] Y. Lozano, C. Nunez, A. Ramirez, and S. Speziali, "M-strings and AdS_3 solutions to M-theory with small $\mathcal{N} = (0, 4)$ supersymmetry," arXiv:2005.06561 [hep-th].

[5] Y. Lozano, C. Nunez, A. Ramirez, and S. Speziali, "New AdS_2 backgrounds and $\mathcal{N} = 4$ Conformal Quantum Mechanics," arXiv:2011.00005 [hep-th].

[6] Y. Lozano, C. Nunez, A. Ramirez, and S. Speziali, "AdS_2 duals to ADHM quivers with Wilson lines," arXiv:2011.13932 [hep-th].

[7] J. Polchinski, "Dualities of Fields and Strings," *Stud. Hist. Phil. Sci. B* **59** (2017) 6–20, arXiv:1412.5704 [hep-th].

[8] F. Quevedo, "Duality and global symmetries," *Nucl. Phys. B Proc. Suppl.* **61** (1998) 23–41, arXiv:hep-th/9706210.

[9] M. Ammon and J. Erdmenger, *Gauge/gravity duality: Foundations and applications*. Cambridge University Press, Cambridge, 4, 2015.

[10] H. Nastase, *Introduction to the ADS/CFT Correspondence.* Cambridge University Press, 9, 2015.

[11] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," *Phys. Rept.* **323** (2000) 183–386, arXiv:hep-th/9905111.

[12] H. Lin, O. Lunin, and J. M. Maldacena, "Bubbling AdS space and 1/2 BPS geometries," *JHEP* **10** (2004) 025, arXiv:hep-th/0409174.

[13] D. Gaiotto and J. Maldacena, "The Gravity duals of N=2 superconformal field theories," *JHEP* **10** (2012) 189, arXiv:0904.4466 [hep-th].

[14] D. Gaiotto, "N=2 dualities," *JHEP* **08** (2012) 034, arXiv:0904.2715 [hep-th].

[15] A. Hanany and E. Witten, "Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics," *Nucl. Phys.* **B492** (1997) 152–190, arXiv:hep-th/9611230 [hep-th].

[16] O. Lunin and J. M. Maldacena, "Deforming field theories with U(1) x U(1) global symmetry and their gravity duals," *JHEP* **05** (2005) 033, arXiv:hep-th/0502086.

[17] Y. Lozano, N. T. Macpherson, C. Nunez, and A. Ramirez, "AdS_3 solutions in Massive IIA with small $\mathcal{N} = (4, 0)$ supersymmetry," arXiv:1908.09851 [hep-th].

[18] Y. Lozano, N. T. Macpherson, C. Nunez, and A. Ramirez, "1/4 BPS $\mathrm{AdS}_3/\mathrm{CFT}_2$," arXiv:1909.09636 [hep-th].

[19] Y. Lozano, N. T. Macpherson, C. Nunez, and A. Ramirez, "Two dimensional $\mathcal{N} = (0, 4)$ quivers dual to AdS_3 solutions in massive IIA," arXiv:1909.10510 [hep-th].

[20] A. Giveon and D. Kutasov, "Seiberg Duality in Chern-Simons Theory," *Nucl. Phys.* **B812** (2009) 1–11, arXiv:0808.0360 [hep-th].

[21] O. Aharony, "Baryons, monopoles and dualities in Chern-Simons-matter theories," *JHEP* **02** (2016) 093, arXiv:1512.00161 [hep-th].

[22] Z. Komargodski and N. Seiberg, "A symmetry breaking scenario for QCD_3," *JHEP* **01** (2018) 109, arXiv:1706.08755 [hep-th].

[23] J. Wess and J. Bagger, *Supersymmetry and supergravity*. Princeton University Press, Princeton, NJ, USA, 1992.

[24] F. Dolan and H. Osborn, "On short and semi-short representations for four-dimensional superconformal symmetry," *Annals Phys.* **307** (2003) 41–89, arXiv:hep-th/0209056.

[25] L. Bianchi and M. Lemos, "Superconformal surfaces in four dimensions," *JHEP* **06** (2020) 056, arXiv:1911.05082 [hep-th].

[26] E. Witten, "Solutions of four-dimensional field theories via M theory," *Nucl. Phys.* **B500** (1997) 3–42, arXiv:hep-th/9703166 [hep-th]. [,452(1997)].

[27] A. Giveon and D. Kutasov, "Brane dynamics and gauge theory," *Rev. Mod. Phys.* **71** (1999) 983–1084, arXiv:hep-th/9802067 [hep-th].

[28] A. Hanany and E. Witten, "Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics," *Nucl. Phys. B* **492** (1997) 152–190, arXiv:hep-th/9611230.

[29] N. Seiberg and E. Witten, "Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD," *Nucl. Phys. B* **431** (1994) 484–550, arXiv:hep-th/9408099.

[30] E. Witten, "Monopoles and four manifolds," *Math. Res. Lett.* **1** (1994) 769–796, arXiv:hep-th/9411102.

[31] L. F. Alday, D. Gaiotto, and Y. Tachikawa, "Liouville Correlation Functions from Four-dimensional Gauge Theories," *Lett. Math. Phys.* **91** (2010) 167–197, arXiv:0906.3219 [hep-th].

[32] J. D. Moore, "Lecture notes on seiberg-witten invariants (revised second edition),".

[33] P. Deligne, P. Etingof, D. Freed, L. Jeffrey, D. Kazhdan, J. Morgan, D. Morrison, and E. Witten, eds., *Quantum fields and strings: A course for mathematicians. Vol. 1, 2*. 10, 1999.

[34] R. A. Reid-Edwards and j. Stefanski, B., "On Type IIA geometries dual to N = 2 SCFTs," *Nucl. Phys.* **B849** (2011) 549–572, arXiv:1011.0216 [hep-th].

[35] O. Aharony, L. Berdichevsky, and M. Berkooz, "4d N=2 superconformal linear quivers with type IIA duals," *JHEP* **08** (2012) 131, arXiv:1206.5916 [hep-th].

[36] G. Itsios, H. Nastase, C. Nunez, K. Sfetsos, and S. Zacarias, "Penrose limits of Abelian and non-Abelian T-duals of $AdS_5 \times S^5$ and their field theory duals," *JHEP* **01** (2018) 071, arXiv:1711.09911 [hep-th].

[37] C. Nunez, D. Roychowdhury, and D. C. Thompson, "Integrability and non-integrability in $\mathcal{N} = 2$ SCFTs and their holographic backgrounds," *JHEP* **07** (2018) 044, arXiv:1804.08621 [hep-th].

[38] S. Cremonesi and A. Tomasiello, "6d holographic anomaly match as a continuum limit," *JHEP* **05** (2016) 031, arXiv:1512.02225 [hep-th].

[39] D. Marolf, "Chern-Simons terms and the three notions of charge," in *International Conference on Quantization, Gauge Theory, and Strings: Conference Dedicated to the Memory of Professor Efim Fradkin*, pp. 312–320. 6, 2000. arXiv:hep-th/0006117.

[40] I. R. Klebanov, D. Kutasov, and A. Murugan, "Entanglement as a probe of confinement," *Nucl. Phys. B* **796** (2008) 274–293, arXiv:0709.2140 [hep-th].

[41] N. T. Macpherson, C. Nunez, L. A. Pando Zayas, V. G. J. Rodgers, and C. A. Whiting, "Type IIB supergravity solutions with AdS_5 from Abelian and non-Abelian T dualities," *JHEP* **02** (2015) 040, arXiv:1410.2650 [hep-th].

[42] Y. Bea, J. D. Edelstein, G. Itsios, K. S. Kooner, C. Nunez, D. Schofield, and J. A. Sierra-Garcia, "Compactifications of the Klebanov-Witten CFT and new AdS_3 backgrounds," *JHEP* **05** (2015) 062, arXiv:1503.07527 [hep-th].

[43] C. Nunez, J. M. Penin, D. Roychowdhury, and J. Van Gorsel, "The non-Integrability of Strings in Massive Type IIA and their Holographic duals," *JHEP* **06** (2018) 078, arXiv:1802.04269 [hep-th].

[44] A. D. Shapere and Y. Tachikawa, "Central charges of N=2 superconformal field theories in four dimensions," *JHEP* **09** (2008) 109, arXiv:0804.1957 [hep-th].

[45] Y. Lozano and C. Nunez, "Field theory aspects of non-Abelian T-duality and $\mathcal{N} = 2$ linear quivers," *JHEP* **05** (2016) 107, arXiv:1603.04440 [hep-th].

[46] K. Sfetsos and D. C. Thompson, "On non-abelian T-dual geometries with Ramond fluxes," *Nucl. Phys. B* **846** (2011) 21–42, arXiv:1012.1320 [hep-th].

[47] D. M. Hofman and J. Maldacena, "Conformal collider physics: Energy and charge correlations," *JHEP* **05** (2008) 012, arXiv:0803.1467 [hep-th].

[48] R. Borsato and L. Wulff, "On non-abelian T-duality and deformations of supercoset string sigma-models," *JHEP* **10** (2017) 024, arXiv:1706.10169 [hep-th].

[49] K. Sfetsos and D. C. Thompson, "Spacetimes for λ-deformations," *JHEP* **12** (2014) 164, arXiv:1410.1886 [hep-th].

[50] S. Demulder, K. Sfetsos, and D. C. Thompson, "Integrable λ-deformations: Squashing Coset CFTs and $AdS_5 \times S^5$," *JHEP* **07** (2015) 019, arXiv:1504.02781 [hep-th].

[51] Y. Lozano, E. O Colgain, D. Rodriguez-Gomez, and K. Sfetsos, "Supersymmetric AdS_6 via T Duality," *Phys. Rev. Lett.* **110** no. 23, (2013) 231601, arXiv:1212.1043 [hep-th].

[52] G. Itsios, C. Nunez, K. Sfetsos, and D. C. Thompson, "On Non-Abelian T-Duality and new N=1 backgrounds," *Phys. Lett. B* **721** (2013) 342–346, arXiv:1212.4840 [hep-th].

[53] A. Barranco, J. Gaillard, N. T. Macpherson, C. Nunez, and D. C. Thompson, "G-structures and Flavouring non-Abelian T-duality," *JHEP* **08** (2013) 018, arXiv:1305.7229 [hep-th].

[54] N. T. Macpherson, "Non-Abelian T-duality, G_2-structure rotation and holographic duals of $N = 1$ Chern-Simons theories," *JHEP* **11** (2013) 137, arXiv:1310.1609 [hep-th].

[55] Y. Lozano, E. O Colgain, and D. Rodriguez-Gomez, "Hints of 5d Fixed Point Theories from Non-Abelian T-duality," *JHEP* **05** (2014) 009, arXiv:1311.4842 [hep-th].

[56] J. Gaillard, N. T. Macpherson, C. Nunez, and D. C. Thompson, "Dualising the Baryonic Branch: Dynamic SU(2) and confining backgrounds in IIA," *Nucl. Phys. B* **884** (2014) 696–740, arXiv:1312.4945 [hep-th].

[57] E. Caceres, N. T. Macpherson, and C. Nunez, "New Type IIB Backgrounds and Aspects of Their Field Theory Duals," *JHEP* **08** (2014) 107, arXiv:1402.3294 [hep-th].

[58] Y. Lozano and N. T. Macpherson, "A new $\mathrm{AdS}_4/\mathrm{CFT}_3$ dual with extended SUSY and a spectral flow," *JHEP* **11** (2014) 115, arXiv:1408.0912 [hep-th].

[59] K. Sfetsos and D. C. Thompson, "New $\mathcal{N} = 1$ supersymmetric AdS_5 backgrounds in Type IIA supergravity," *JHEP* **11** (2014) 006, arXiv:1408.6545 [hep-th].

[60] K. Kooner and S. Zacarias, "Non-Abelian T-Dualizing the Resolved Conifold with Regular and Fractional D3-Branes," *JHEP* **08** (2015) 143, arXiv:1411.7433 [hep-th].

[61] T. R. Araujo and H. Nastase, "$\mathcal{N} = 1$ SUSY backgrounds with an AdS factor from non-Abelian T duality," *Phys. Rev. D* **91** no. 12, (2015) 126015, arXiv:1503.00553 [hep-th].

[62] Y. Lozano, N. T. Macpherson, J. Montero, and E. O. Colgain, "New $AdS_3 \times S^2$ T-duals with $\mathcal{N} = (0, 4)$ supersymmetry," *JHEP* **08** (2015) 121, arXiv:1507.02659 [hep-th].

[63] Y. Lozano, N. T. Macpherson, and J. Montero, "A $\mathcal{N} = 2$ supersymmetric AdS_4 solution in M-theory with purely magnetic flux," *JHEP* **10** (2015) 004, arXiv:1507.02660 [hep-th].

[64] N. T. Macpherson, C. Nunez, D. C. Thompson, and S. Zacarias, "Holographic Flows in non-Abelian T-dual Geometries," *JHEP* **11** (2015) 212, arXiv:1509.04286 [hep-th].

[65] L. A. Pando Zayas, V. G. Rodgers, and C. A. Whiting, "Supergravity solutions with AdS_4 from non-Abelian T-dualities," *JHEP* **02** (2016) 061, arXiv:1511.05991 [hep-th].

[66] H. Dimov, S. Mladenov, R. C. Rashkov, and T. Vetsov, "Non-abelian T-duality of Pilch-Warner background," *Fortsch. Phys.* **64** (2016) 657–673, arXiv:1511.00269 [hep-th].

[67] Y. Lozano, N. T. Macpherson, and J. Montero, "AdS_6 T-duals and type IIB $AdS_6 \times S^2$ geometries with 7-branes," *JHEP* **01** (2019) 116, arXiv:1810.08093 [hep-th].

[68] R. Terrisse, D. Tsimpis, and C. A. Whiting, "D-branes and non-Abelian T-duality," *Nucl. Phys. B* **947** (2019) 114733, arXiv:1811.05800 [hep-th].

[69] G. Itsios, C. Nunez, and D. Zoakos, "Mesons from (non) Abelian T-dual backgrounds," *JHEP* **01** (2017) 011, arXiv:1611.03490 [hep-th].

[70] Y. Lozano, C. Nunez, and S. Zacarias, "BMN Vacua, Superstars and Non-Abelian T-duality," *JHEP* **09** (2017) 008, arXiv:1703.00417 [hep-th].

[71] G. Itsios, Y. Lozano, J. Montero, and C. Nunez, "The AdS_5 non-Abelian T-dual of Klebanov-Witten as a $\mathcal{N} = 1$ linear quiver from M5-branes," *JHEP* **09** (2017) 038, arXiv:1705.09661 [hep-th].

[72] Y. Lozano, N. T. Macpherson, J. Montero, and C. Nunez, "Three-dimensional $\mathcal{N} = 4$ linear quivers and non-Abelian T-duals," *JHEP* **11** (2016) 133, arXiv:1609.09061 [hep-th].

[73] J. van Gorsel and S. Zacarias, "A Type IIB Matrix Model via non-Abelian T-dualities," *JHEP* **12** (2017) 101, arXiv:1711.03419 [hep-th].

[74] N. Itzhaki, J. M. Maldacena, J. Sonnenschein, and S. Yankielowicz, "Supergravity and the large N limit of theories with sixteen supercharges," *Phys. Rev. D* **58** (1998) 046004, arXiv:hep-th/9802042.

[75] H. Lin and J. M. Maldacena, "Fivebranes from gauge theory," *Phys. Rev. D* **74** (2006) 084014, arXiv:hep-th/0509235.

[76] O. Aharony, M. Berkooz, and S.-J. Rey, "Rigid holography and six-dimensional $\mathcal{N} = (2, 0)$ theories on $\mathrm{AdS}_5 \times S^1$," *JHEP* **03** (2015) 121, arXiv:1501.02904 [hep-th].

[77] J. M. Maldacena and C. Nunez, "Supergravity description of field theories on curved manifolds and a no go theorem," *Int. J. Mod. Phys. A* **16** (2001) 822–855, arXiv:hep-th/0007018.

[78] U. Gursoy and C. Nunez, "Dipole deformations of N=1 SYM and supergravity backgrounds with U(1) x U(1) global symmetry," *Nucl. Phys. B* **725** (2005) 45–92, arXiv:hep-th/0505100.

[79] J. P. Gauntlett, S. Lee, T. Mateos, and D. Waldram, "Marginal deformations of field theories with AdS(4) duals," *JHEP* **08** (2005) 030, arXiv:hep-th/0505207.

[80] I. Bah, C. Beem, N. Bobev, and B. Wecht, "Four-Dimensional SCFTs from M5-Branes," *JHEP* **06** (2012) 005, arXiv:1203.0303 [hep-th].

[81] I. Bah and N. Bobev, "Linear quivers and $\mathcal{N} = 1$ SCFTs from M5-branes," *JHEP* **08** (2014) 121, arXiv:1307.7104 [hep-th].

[82] K. A. Intriligator and B. Wecht, "The Exact superconformal R symmetry maximizes a," *Nucl. Phys. B* **667** (2003) 183–200, arXiv:hep-th/0304128.

[83] I. Bah, "AdS5 solutions from M5-branes on Riemann surface and D6-branes sources," *JHEP* **09** (2015) 163, arXiv:1501.06072 [hep-th].

[84] I. Bah, "Quarter-BPS AdS_5 solutions in M-theory with a T^2 bundle over a Riemann surface," *JHEP* **08** (2013) 137, arXiv:1304.4954 [hep-th].

[85] V. Bashmakov, M. Bertolini, and H. Raj, "On non-supersymmetric conformal manifolds: field theory and holography," *JHEP* **11** (2017) 167, arXiv:1709.01749 [hep-th].

[86] A. Hanany, M. J. Strassler, and A. M. Uranga, "Finite theories and marginal operators on the brane," *JHEP* **06** (1998) 011, arXiv:hep-th/9803086.

[87] A. Hanany and A. M. Uranga, "Brane boxes and branes on singularities," *JHEP* **05** (1998) 013, arXiv:hep-th/9805139.

[88] A. Karch, D. Lust, and A. Miemiec, "N=1 supersymmetric gauge theories and supersymmetric three cycles," *Nucl. Phys. B* **553** (1999) 483–510, arXiv:hep-th/9810254.

[89] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Int. J. Theor. Phys.* **38** (1999) 1113–1133, arXiv:hep-th/9711200 [hep-th]. [Adv. Theor. Math. Phys.2,231(1998)].

[90] D. Gaiotto and E. Witten, "Supersymmetric Boundary Conditions in N=4 Super Yang-Mills Theory," *J. Statist. Phys.* **135** (2009) 789–855, arXiv:0804.2902 [hep-th].

[91] D. Gaiotto and E. Witten, "S-Duality of Boundary Conditions In N=4 Super Yang-Mills Theory," *Adv. Theor. Math. Phys.* **13** no. 3, (2009) 721–896, arXiv:0807.3720 [hep-th].

[92] E. D'Hoker, J. Estes, M. Gutperle, and D. Krym, "Exact Half-BPS Flux Solutions in M-theory. I: Local Solutions," *JHEP* **08** (2008) 028, arXiv:0806.0605 [hep-th].

[93] E. D'Hoker, J. Estes, and M. Gutperle, "Exact half-BPS Type IIB interface solutions. I. Local solution and supersymmetric Janus," *JHEP* **06** (2007) 021, arXiv:0705.0022 [hep-th].

[94] B. Assel, C. Bachas, J. Estes, and J. Gomis, "Holographic Duals of D=3 N=4 Superconformal Field Theories," *JHEP* **08** (2011) 087, arXiv:1106.4253 [hep-th].

[95] E. D'Hoker, M. Gutperle, A. Karch, and C. F. Uhlemann, "Warped $AdS_6 \times S^2$ in Type IIB supergravity I: Local solutions," *JHEP* **08** (2016) 046, arXiv:1606.01254 [hep-th].

[96] E. D'Hoker, M. Gutperle, and C. F. Uhlemann, "Holographic duals for five-dimensional superconformal quantum field theories," *Phys. Rev. Lett.* **118** no. 10, (2017) 101601, arXiv:1611.09411 [hep-th].

[97] E. D'Hoker, M. Gutperle, and C. F. Uhlemann, "Warped $AdS_6 \times S^2$ in Type IIB supergravity II: Global solutions and five-brane webs," *JHEP* **05** (2017) 131, arXiv:1703.08186 [hep-th].

[98] O. Bergman, D. Rodriguez-Gomez, and C. F. Uhlemann, "Testing AdS_6/CFT_5 in Type IIB with stringy operators," *JHEP* **08** (2018) 127, arXiv:1806.07898 [hep-th].

[99] F. Apruzzi, M. Fazzi, A. Passias, A. Rota, and A. Tomasiello, "Six-Dimensional Superconformal Theories and their Compactifications from Type IIA Supergravity," *Phys. Rev. Lett.* **115** no. 6, (2015) 061601, arXiv:1502.06616 [hep-th].

[100] D. Gaiotto and A. Tomasiello, "Holography for (1,0) theories in six dimensions," *JHEP* **12** (2014) 003, arXiv:1404.0711 [hep-th].

[101] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, "Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory," *Nucl. Phys.* **B241** (1984) 333–380. [,605(1984)].

[102] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997. http://www-spires.fnal.gov/spires/find/books/www?cl=QC174.52.C66D5::1997.

[103] M. Ademollo *et al.*, "Supersymmetric Strings and Color Confinement," *Phys. Lett.* **62B** (1976) 105–110.

[104] M. Ademollo *et al.*, "Dual String with U(1) Color Symmetry," *Nucl. Phys.* **B111** (1976) 77–110.

[105] M. Ademollo *et al.*, "Dual String Models with Nonabelian Color and Flavor Symmetries," *Nucl. Phys.* **B114** (1976) 297–316.

[106] Y. Lozano, N. T. Macpherson, C. Nunez, and A. Ramirez, "AdS_3 solutions in massive IIA, defect CFTs and T-duality," arXiv:1909.11669 [hep-th].

[107] F. Kos and J. Oh, "2d small N=4 Long-multiplet superconformal block," *JHEP* **02** (2019) 001, arXiv:1810.10029 [hep-th].

[108] H. J. Kim, L. J. Romans, and P. van Nieuwenhuizen, "Mass spectrum of chiral ten-dimensional n=2 supergravity on S^5," *Phys. Rev. D* **32** (Jul, 1985) 389–399. https://link.aps.org/doi/10.1103/PhysRevD.32.389.

[109] S. Deger, A. Kaya, E. Sezgin, and P. Sundell, "Spectrum of D = 6, N=4b supergravity on AdS in three-dimensions x S**3," *Nucl. Phys.* **B536** (1998) 110–140, arXiv:hep-th/9804166 [hep-th].

[110] C. Bachas and J. Estes, "Spin-2 spectrum of defect theories," *JHEP* **06** (2011) 005, arXiv:1103.2800 [hep-th].

[111] I. R. Klebanov, S. S. Pufu, and F. D. Rocha, "The Squashed, Stretched, and Warped Gets Perturbed," *JHEP* **06** (2009) 019, arXiv:0904.1009 [hep-th].

[112] J. Schmude and O. Vasilakis, "Superconformal Symmetry in the Kaluza-Klein Spectrum of Warped AdS(3)," *JHEP* **10** (2016) 096, arXiv:1605.00636 [hep-th].

[113] Y. Pang, J. Rong, and O. Varela, "Spectrum universality properties of holographic Chern-Simons theories," *JHEP* **01** (2018) 061, arXiv:1711.07781 [hep-th].

[114] J.-M. Richard, R. Terrisse, and D. Tsimpis, "On the spin-2 Kaluza-Klein spectrum of $\mathrm{AdS}_4 \times S^2\,(\mathcal{B}_4)$," *JHEP* **12** (2014) 144, arXiv:1410.4669 [hep-th].

[115] K. Chen, M. Gutperle, and C. F. Uhlemann, "Spin 2 operators in holographic 4d $\mathcal{N} = 2$ SCFTs," *JHEP* **06** (2019) 139, arXiv:1903.07109 [hep-th].

[116] G. Itsios, J. M. Penin, and S. Zacarias, "Spin-2 excitations in Gaiotto-Maldacena solutions," arXiv:1903.11613 [hep-th].

[117] M. Gutperle, C. F. Uhlemann, and O. Varela, "Massive spin 2 excitations in $AdS_6 \times S^2$ warped spacetimes," *JHEP* **07** (2018) 091, arXiv:1805.11914 [hep-th].

[118] A. Passias and P. Richmond, "Perturbing $AdS_6 \times_w S^4$: linearised equations and spin-2 spectrum," *JHEP* **07** (2018) 058, arXiv:1804.09728 [hep-th].

[119] A. Passias and A. Tomasiello, "Spin-2 spectrum of six-dimensional field theories," *JHEP* **12** (2016) 050, arXiv:1604.04286 [hep-th].

[120] K. Filippas, "Holography for 2d $\mathcal{N} = (0, 4)$ quantum field theory," arXiv:2008.00314 [hep-th].

[121] E. Witten, "Phases of N=2 theories in two-dimensions," *Nucl. Phys.* **B403** (1993) 159–222, arXiv:hep-th/9301042 [hep-th]. [AMS/IP Stud. Adv. Math.1,143(1996)].

[122] E. Witten, "Sigma models and the ADHM construction of instantons," *J. Geom. Phys.* **15** (1995) 215–226, arXiv:hep-th/9410052 [hep-th].

[123] D. Tong, "The holographic dual of $AdS_3 \times S^3 \times S^3 \times S^1$," *JHEP* **04** (2014) 193, arXiv:1402.5135 [hep-th].

[124] P. Putrov, J. Song, and W. Yan, "(0,4) dualities," *JHEP* **03** (2016) 185, arXiv:1505.07110 [hep-th].

[125] S. Franco, D. Ghim, S. Lee, R.-K. Seong, and D. Yokoyama, "2d (0,2) Quiver Gauge Theories and D-Branes," *JHEP* **09** (2015) 072, arXiv:1506.03818 [hep-th].

[126] A. Polishchuk, "Massive symmetric tensor field on AdS," *JHEP* **07** (1999) 007, arXiv:hep-th/9905048 [hep-th].

[127] C. Cordova, T. T. Dumitrescu, and K. Intriligator, "Multiplets of Superconformal Symmetry in Diverse Dimensions," *JHEP* **03** (2019) 163, arXiv:1612.00809 [hep-th].

[128] C. Cordova, T. T. Dumitrescu, and K. Intriligator, "Deformations of Superconformal Theories," *JHEP* **11** (2016) 135, arXiv:1602.01217 [hep-th].

[129] G. E. Arutyunov and S. A. Frolov, "Quadratic action for Type IIB supergravity on AdS(5) x S**5," *JHEP* **08** (1999) 024, arXiv:hep-th/9811106 [hep-th].

[130] T. Eguchi and A. Taormina, "Unitary Representations of $N = 4$ Superconformal Algebra," *Phys. Lett.* **B196** (1987) 75.

[131] T. Eguchi and A. Taormina, "Character Formulas for the $N = 4$ Superconformal Algebra," *Phys. Lett.* **B200** (1988) 315.

[132] T. Eguchi and A. Taormina, "On the Unitary Representations of $N = 2$ and $N = 4$ Superconformal Algebras," *Phys. Lett.* **B210** (1988) 125–132.

[133] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, "Supersymmetric AdS(5) solutions of M theory," *Class. Quant. Grav.* **21** (2004) 4335–4366, arXiv:hep-th/0402153.

[134] J. Gutowski and G. Papadopoulos, "Supersymmetry of AdS and flat backgrounds in M-theory," *JHEP* **02** (2015) 145, arXiv:1407.5652 [hep-th].

[135] R. Britto-Pacumio, J. Michelson, A. Strominger, and A. Volovich, "Lectures on Superconformal Quantum Mechanics and Multi-Black Hole Moduli Spaces," *NATO Sci. Ser. C* **556** (2000) 255–284, arXiv:hep-th/9911066.

[136] J. Michelson and A. Strominger, "The Geometry of (super)conformal quantum mechanics," *Commun. Math. Phys.* **213** (2000) 1–17, arXiv:hep-th/9907191.

[137] T. Hartman and A. Strominger, "Central Charge for AdS(2) Quantum Gravity," *JHEP* **04** (2009) 026, arXiv:0803.3621 [hep-th].

[138] M. Cadoni and S. Mignemi, "Asymptotic symmetries of AdS(2) and conformal group in d = 1," *Nucl. Phys. B* **557** (1999) 165–180, arXiv:hep-th/9902040.

[139] M. Alishahiha and F. Ardalan, "Central Charge for 2D Gravity on AdS(2) and AdS(2)/CFT(1) Correspondence," *JHEP* **08** (2008) 079, arXiv:0805.1861 [hep-th].

[140] J. D. Brown and M. Henneaux, "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity," *Commun. Math. Phys.* **104** (1986) 207–226.

[141] M. van Beest, S. Cizel, S. Schafer-Nameki, and J. Sparks, "$\mathcal{I}/c$-Extremization in M/F-Duality," *SciPost Phys.* **9** (2020) 029, arXiv:2004.04020 [hep-th].

[142] D. Lüst and D. Tsimpis, "AdS_2 type-IIA solutions and scale separation," *JHEP* **07** (2020) 060, arXiv:2004.07582 [hep-th].

[143] R. K. Gupta and A. Sen, "Ads(3)/CFT(2) to Ads(2)/CFT(1)," *JHEP* **04** (2009) 034, arXiv:0806.0053 [hep-th].

[144] N. Kim, "Comments on AdS_2 solutions from M2-branes on complex curves and the backreacted Kähler geometry," *Eur. Phys. J. C* **74** no. 2, (2014) 2778, arXiv:1311.7372 [hep-th].

[145] A. Castro and W. Song, "Comments on AdS_2 Gravity," arXiv:1411.1948 [hep-th].

[146] D. Anninos, D. M. Hofman, and J. Kruthoff, "Charged Quantum Fields in AdS_2," *SciPost Phys.* **7** no. 4, (2019) 054, arXiv:1906.00924 [hep-th].

[147] D. Anninos and D. M. Hofman, "Infrared Realization of dS_2 in AdS_2," *Class. Quant. Grav.* **35** no. 8, (2018) 085003, arXiv:1703.04622 [hep-th].

[148] D. Corbino, "Warped AdS_2 and $SU(1,1|4)$ symmetry in Type IIB," arXiv:2004.12613 [hep-th].

[149] M. Chiodaroli, E. D'Hoker, and M. Gutperle, "Open Worldsheets for Holographic Interfaces," *JHEP* **03** (2010) 060, arXiv:0912.4679 [hep-th].

[150] S. Kachru, M. B. Schulz, P. K. Tripathy, and S. P. Trivedi, "New supersymmetric string compactifications," *JHEP* **03** (2003) 061, arXiv:hep-th/0211182.

[151] M. Chiodaroli, M. Gutperle, and D. Krym, "Half-BPS Solutions locally asymptotic to AdS(3) x S**3 and interface conformal field theories," *JHEP* **02** (2010) 066, arXiv:0910.0466 [hep-th].

[152] Y. Lozano, C. Nunez, and A. Ramirez *to appear* .

[153] V. Balasubramanian, J. de Boer, M. Sheikh-Jabbari, and J. Simon, "What is a chiral 2d CFT? And what does it have to do with extremal black holes?," *JHEP* **02** (2010) 017, arXiv:0906.3272 [hep-th].

[154] C. Couzens, J. P. Gauntlett, D. Martelli, and J. Sparks, "A geometric dual of c-extremization," *JHEP* **01** (2019) 212, arXiv:1810.11026 [hep-th].

[155] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12, 2007.

[156] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12, 2007.

[157] C. Hwang, J. Kim, S. Kim, and J. Park, "General instanton counting and 5d SCFT," *JHEP* **07** (2015) 063, arXiv:1406.6793 [hep-th]. [Addendum: JHEP 04, 094 (2016)].

[158] D. Tong and K. Wong, "Instantons, Wilson lines, and D-branes," *Phys. Rev. D* **91** no. 2, (2015) 026007, arXiv:1410.8523 [hep-th].

[159] B. Assel and A. Sciarappa, "On monopole bubbling contributions to 't Hooft loops," *JHEP* **05** (2019) 180, arXiv:1903.00376 [hep-th].

[160] K. Hori, H. Kim, and P. Yi, "Witten Index and Wall Crossing," *JHEP* **01** (2015) 124, arXiv:1407.2567 [hep-th].

[161] C. Cordova and S.-H. Shao, "An Index Formula for Supersymmetric Quantum Mechanics," arXiv:1406.7853 [hep-th].

[162] E. Witten, "Fermion Path Integrals And Topological Phases," *Rev. Mod. Phys.* **88** no. 3, (2016) 035001, arXiv:1508.04715 [cond-mat.mes-hall].

[163] G. V. Dunne, "Aspects of Chern-Simons theory," in *Les Houches Summer School in Theoretical Physics, Session 69: Topological Aspects of Low-dimensional Systems*. 7, 1998. arXiv:hep-th/9902115.

[164] C. Turner, "Dualities in 2+1 Dimensions," *PoS* **Modave2018** (2019) 001, arXiv:1905.12656 [hep-th].

[165] M. Nakahara, *Geometry, topology and physics*. 2003.

[166] E. Witten, "Quantum Field Theory and the Jones Polynomial," *Commun. Math. Phys.* **121** (1989) 351–399.

[167] A. Redlich, "Parity Violation and Gauge Noninvariance of the Effective Gauge Field Action in Three-Dimensions," *Phys. Rev. D* **29** (1984) 2366–2374.

[168] A. Niemi and G. Semenoff, "Axial Anomaly Induced Fermion Fractionization and Effective Gauge Theory Actions in Odd Dimensional Space-Times," *Phys. Rev. Lett.* **51** (1983) 2077.

[169] P.-S. Hsin and N. Seiberg, "Level rank Duality and Chern-Simons Matter Theories," *JHEP* **09** (2016) 095, arXiv:1607.07457 [hep-th].

[170] N. Seiberg, "Electric - magnetic duality in supersymmetric nonAbelian gauge theories," *Nucl. Phys. B* **435** (1995) 129–146, arXiv:hep-th/9411149.

[171] J. H. Schwarz, "Superconformal Chern-Simons theories," *JHEP* **11** (2004) 078, arXiv:hep-th/0411077.

[172] O. Aharony, G. Gur-Ari, and R. Yacoby, "d=3 Bosonic Vector Models Coupled to Chern-Simons Gauge Theories," *JHEP* **03** (2012) 037, arXiv:1110.4382 [hep-th].

[173] S. Giombi, S. Minwalla, S. Prakash, S. P. Trivedi, S. R. Wadia, and X. Yin, "Chern-Simons Theory with Vector Fermion Matter," *Eur. Phys. J.* **C72** (2012) 2112, arXiv:1110.4386 [hep-th].

[174] A. Karch and D. Tong, "Particle-Vortex Duality from 3d Bosonization," *Phys. Rev.* **X6** no. 3, (2016) 031043, arXiv:1606.01893 [hep-th].

[175] J. Murugan and H. Nastase, "Particle-vortex duality in topological insulators and superconductors," *JHEP* **05** (2017) 159, arXiv:1606.01912 [hep-th].

[176] A. Armoni, T. T. Dumitrescu, G. Festuccia, and Z. Komargodski, "Metastable Vacua in Large-N QCD_3," arXiv:1905.01797 [hep-th].

[177] A. Armoni and V. Niarchos, "Phases of QCD_3 from Non-SUSY Seiberg Duality and Brane Dynamics," *Phys. Rev.* **D97** no. 10, (2018) 106001, arXiv:1711.04832 [hep-th].

[178] K. Jensen and A. Karch, "Embedding three-dimensional bosonization dualities into string theory," *JHEP* **12** (2017) 031, arXiv:1709.07872 [hep-th].

[179] R. Argurio, M. Bertolini, F. Bigazzi, A. L. Cotrone, and P. Niro, "QCD domain walls, Chern-Simons theories and holography," *JHEP* **09** (2018) 090, arXiv:1806.08292 [hep-th].

[180] S. Kachru, M. Mulligan, G. Torroba, and H. Wang, "Nonsupersymmetric dualities from mirror symmetry," *Phys. Rev. Lett.* **118** no. 1, (2017) 011602, arXiv:1609.02149 [hep-th].

[181] G. Gur-Ari and R. Yacoby, "Three Dimensional Bosonization From Supersymmetry," *JHEP* **11** (2015) 013, arXiv:1507.04378 [hep-th].

[182] A. Armoni, M. Shifman, and G. Veneziano, "Exact results in non-supersymmetric large N orientifold field theories," *Nucl. Phys.* **B667** (2003) 170–182, `arXiv:hep-th/0302163 [hep-th]`.

[183] A. Armoni, M. Shifman, and G. Veneziano, "From superYang-Mills theory to QCD: Planar equivalence and its implications," in *From fields to strings: Circumnavigating theoretical physics. Ian Kogan memorial collection (3 volume set)*, M. Shifman, A. Vainshtein, and J. Wheater, eds., pp. 353–444. 2004. `arXiv:hep-th/0403071 [hep-th]`. `http://weblib.cern.ch/abstract?CERN-PH-TH-2004-022`.

[184] C. Angelantonj and A. Sagnotti, "Open strings," *Phys. Rept.* **371** (2002) 1–150, `arXiv:hep-th/0204089 [hep-th]`. [Erratum: Phys. Rept.376,no.6,407(2003)].

[185] R. Blumenhagen, A. Font, and D. Lust, "Nonsupersymmetric gauge theories from D-branes in type 0 string theory," *Nucl. Phys.* **B560** (1999) 66–92, `arXiv:hep-th/9906101 [hep-th]`.

[186] R. Blumenhagen, A. Font, and D. Lust, "Tachyon free orientifolds of type 0B strings in various dimensions," *Nucl. Phys.* **B558** (1999) 159–177, `arXiv:hep-th/9904069 [hep-th]`.

[187] A. Sagnotti, "Surprises in open string perturbation theory," *Nucl. Phys. Proc. Suppl.* **56B** (1997) 332–343, `arXiv:hep-th/9702093 [hep-th]`.

[188] A. Sagnotti, "Some properties of open string theories," in *Supersymmetry and unification of fundamental interactions. Proceedings, International Workshop, SUSY 95, Palaiseau, France, May 15-19, 1995*, pp. 473–484. 1995. `arXiv:hep-th/9509080 [hep-th]`.

[189] M. Unsal and L. G. Yaffe, "(In)validity of large N orientifold equivalence," *Phys. Rev.* **D74** (2006) 105019, `arXiv:hep-th/0608180 [hep-th]`.

[190] J. Polchinski, S. Chaudhuri, and C. V. Johnson, "Notes on D-branes," `arXiv:hep-th/9602052 [hep-th]`.

[191] A. M. Uranga, "Comments on nonsupersymmetric orientifolds at strong coupling," *JHEP* **02** (2000) 041, `arXiv:hep-th/9912145 [hep-th]`.

[192] I. R. Klebanov and A. A. Tseytlin, "D-branes and dual gauge theories in type 0 strings," *Nucl. Phys.* **B546** (1999) 155–181, arXiv:hep-th/9811035 [hep-th].

[193] S. Elitzur, A. Giveon, and D. Kutasov, "Branes and N=1 duality in string theory," *Phys. Lett.* **B400** (1997) 269–274, arXiv:hep-th/9702014 [hep-th].

[194] N. J. Evans, C. V. Johnson, and A. D. Shapere, "Orientifolds, branes, and duality of 4-D gauge theories," *Nucl. Phys.* **B505** (1997) 251–271, arXiv:hep-th/9703210 [hep-th].

[195] A. Armoni and E. Ireson, "Level-rank duality in Chern Simons theory from a non-supersymmetric brane configuration," *Phys. Lett.* **B739** (2014) 387–390, arXiv:1408.4633 [hep-th].

[196] C. Cordova, P.-S. Hsin, and N. Seiberg, "Time-Reversal Symmetry, Anomalies, and Dualities in (2+1)d," *SciPost Phys.* **5** no. 1, (2018) 006, arXiv:1712.08639 [cond-mat.str-el].

[197] K. A. Intriligator and N. Seiberg, "Lectures on supersymmetric gauge theories and electric-magnetic duality," *Subnucl. Ser.* **34** (1997) 237–299, arXiv:hep-th/9509066.

[198] M. E. Peskin, "Duality in supersymmetric Yang-Mills theory," in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 96): Fields, Strings, and Duality*, pp. 729–809. 2, 1997. arXiv:hep-th/9702094.

[199] M. J. Strassler, "An Unorthodox introduction to supersymmetric gauge theory," in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2001): Strings, Branes and EXTRA Dimensions*, pp. 561–638. 9, 2003. arXiv:hep-th/0309149.

[200] M. J. Strassler, "The Duality cascade," in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2003): Recent Trends in String Theory*, pp. 419–510. 5, 2005. arXiv:hep-th/0505153.

[201] J. Terning, *Modern supersymmetry: Dynamics and duality*. 4, 2006.

[202] M. A. Shifman, "Nonperturbative gauge dynamics in supersymmetric theories: A primer," in *NATO Advanced Study Institute on Confinement, Duality and Nonperturbative Aspects of QCD*, pp. 477–544. 6, 1997.

[203] V. Novikov, M. A. Shifman, A. Vainshtein, and V. I. Zakharov, "Exact Gell-Mann-Low Function of Supersymmetric Yang-Mills Theories from Instanton Calculus," *Nucl. Phys. B* **229** (1983) 381–393.

[204] V. Novikov, M. A. Shifman, A. Vainshtein, and V. I. Zakharov, "The beta function in supersymmetric gauge theories. Instantons versus traditional approach," *Sov. J. Nucl. Phys.* **43** (1986) 294.

[205] N. Arkani-Hamed and H. Murayama, "Holomorphy, rescaling anomalies and exact beta functions in supersymmetric gauge theories," *JHEP* **06** (2000) 030, arXiv:hep-th/9707133.

[206] F. Dolan and H. Osborn, "Applications of the Superconformal Index for Protected Operators and q-Hypergeometric Identities to N=1 Dual Theories," *Nucl. Phys. B* **818** (2009) 137–178, arXiv:0801.4947 [hep-th].