The Wheel of Rational Numbers as an Abstract Data Type

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Abstract. In an arithmetical structure one can make division a total function by defining $1/0$ to be an element of the structure, or by adding a new element such as infinity ∞ or error element $\perp$. A wheel is an algebra in which division is totalised by setting $1/0 = \infty$ but which also contains an error element $\perp$ to help control its use. We construct the wheel of rational numbers as an abstract data type $\mathbb{Q}_w$ and give it an equational specification without auxiliary operators under initial algebra semantics.

Keywords: Rational numbers · Arithmetic structures · Meadows · Wheels · Division by zero · Infinity · Error · Equational specification · Initial algebra semantics

1 Introduction

In arithmetical structures the most important operator that fails to be total is $1/x$ when $x = 0$. That division becomes a total operation is of value for the semantic modelling, specification and verification of computations with computer arithmetics. Among a number of approaches to making division total are arithmetical algebras in which:

(i) $1/0$ behaves as one of the elements in the structure, e.g., 0 or 1;
(ii) $1/0$ behaves as an error element $\perp$, additional to the structure;
(iii) $1/0$ behaves as an infinite element ∞, additional to the structure.

Meadows are an axiomatically defined class of arithmetical algebras first studied in [3,11] in which the internal option (i) was examined, especially $1/0 = 0$. Later, meadows with the external error element option (ii) were introduced in [6], where they were called common meadows. The infinity element option (iii) was discussed in the survey [2]. All three options deliver workable and interesting ways of removing partiality. In the case of (ii) and (iii), basic properties such as

$$\perp + x = \perp, \perp . x = \perp \text{ and } \infty + x = \infty, \infty . x = \infty$$

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begin to shape intuitions about the semantics. However, adding elements to structures commonly cause complications because the new elements must operate sensibly with all the algebraic constants and operations of the structure. For example, in the case of adding infinity, we could easily find

$$0.\infty = \infty,$$

which is unattractive and against some arithmetic intuitions. Thus, a next step in the case of infinity ∞ is to also add an error element $\bot$ so that some unwanted or suspect results can be controlled:

$$0.\infty = \bot \text{ and, indeed, } \infty + \infty = \bot .$$

The idea of adding a single infinity has several precedents, not least the Riemann sphere (from 1857 onwards); the idea of adding an error element is also older than the approach using meadows and is known as a *wheel*. Wheels are an axiomatically defined class of algebras first studied in [13,18]. Along with the key ideas of a wheel, namely, $\frac{1}{0} = \infty$ and that $\frac{1}{\infty} = 0$, come the subtle controlling properties that $0.\infty = \bot$ and $\infty + \infty = \bot$.

The viability or fitness for purpose of any method of totalising division depends upon the axioms for the structures and, subsequently, on their application to a computational problem. Both axiomatisations of meadows and wheels start with familiar axioms for commutative ring-like structures to which axioms for inverse $^{-1}$ are added. Central to both meadows, wheels and other approaches to totalisation is the field of rational numbers and the problem of totalising division; see [11] and for several other options for division by zero [2].

In the case of meadows, the development uses the theory of algebraic specifications of abstract data types, in which a central concept is axiomatisation by a finite set E of equations over a signature Σ, whose initial algebra defines the data type up to isomorphism. Thus, in [11], are finite equational specifications of the rationals with $1/0 = 0$ up to isomorphism, which defines the abstract data type $\mathbb{Q}_0$ of the meadow of rational numbers. Interestingly, the existence of a finite equational specification for the rational numbers was open until [11].

In this paper we examine the basic structure of the rational number arithmetic considered as a wheel. We will extend and adapt some of the axioms of wheels in [13] using techniques from meadows [3] to give an equational specification for the wheel of rational numbers as an abstract data type $\mathbb{Q}_w$: we give a set of equations E_w over the signature Σ_w such that under initial algebra semantics

$$I(\Sigma_w, E_w) \cong \mathbb{Q}_w.$$

The structure of the paper is as follows. In Sect. 2 we list some basic concepts and principles of our general approach, including the methodology of abstract data type theory and the focus of our interest, namely arithmetical structures. In Sect. 3 we introduce a concrete model of a wheel of rationals and in Sect. 4 we give it its initial algebra specification. Section 5 makes some concluding remarks.

We assume that the reader is familiar with the basic algebraic concepts used to model data types: *signature*, *algebra*, *expansion*, *reduct*, *congruence*, *term*, *homomorphism*, *equational theory*, *first order theory*, etc. These basics can be found in several introductions to algebraic methods and abstract data type theory [14–16, 21]. We have chosen to keep our algebraic techniques very simple to focus attention on a new topic for theoretical investigation, namely arithmetic data type theory.

2 Preliminaries on ADTs

2.1 The Approach of ADTs

The theory of abstract data types is based upon the following principle:

Principle. *In programming, data is characterised by the operations and tests we can use on the data. All the data can be constructed and accessed by applying the operations to given constants. An interface to the data type is a syntactic declaration of these constant, operator and test names. What is known to the programmer about the implementation of the data type is only a set of properties of the constants, operators and tests. The interface and set of properties is called a specification of an abstract data type.*

These ideas about programming are faithfully modelled by the algebraic and logical theories of general algebras and relational structures. In particular, the interface is modelled by a signature Σ and the properties modelled by an axiomatic theory T.

Principle. *An abstract data type is an isomorphism class of algebras of common signature. Each algebra is a possible representation or construction or implementation of the data type. The algebras for which all the data can be constructed by applying their operations to their constants are the minimal algebras.*

Thus, to specify an abstract data type is to specify an isomorphism class of algebras. In the case of the field of rational numbers, the standard notation $\mathbb{Q}$ will stand for the isomorphism type and Q will stand for a particular representative or construction or implementation of the rational isomorphism type. Thus, Q is *a* concrete data type and $\mathbb{Q}$ is *the* abstract data type.

In general, conditions are added to this idea of an abstract data type, of which finite, computable, semicomputable abstract data types are common; notice such properties must be isomorphism invariants of algebras. Computable algebras have important roles in modelling data [19], and especially in classical field theory [20].

Let us expand on the ideas introduced above.

The interface is a signature Σ and, typically, the properties is a set T of first order axioms about the constants, operations and tests in Σ. A Σ-algebra

A is Σ-minimal if all the elements of A can be constructed by applying the operations to the constants of A. The pair (Σ, T) is an axiomatic specification and $Alg(\Sigma, T)$ is the class of all Σ-structures that satisfy the axioms of T. Of particular importance is the case when the axiomatisations consist of a set E of equations only. Such an equational specification (Σ, E) has an initial algebra $I(\Sigma, E)$ in the class $Alg(\Sigma, E)$ that is unique up to isomorphism. The initial algebra has an important representative structure. Let $T(\Sigma)$ be the set of all closed terms over the signature Σ. Define a congruence on $T(\Sigma)$ for any $t_1, t_2 \in T(\Sigma)$,

$$t_1 \equiv t_2 \Longleftrightarrow E \vdash t_1 = t_2$$

Then we have

$$I(\Sigma, E) \cong T(\Sigma)/\equiv .$$

Thus the

Specification Problem. *Given a Σ-algebra A representing an implementation of an abstract data type, can we find an equational specification (Σ, E) such that $I(\Sigma, E) \cong T(\Sigma)/\equiv \cong A$.*

The general specification problem for computable, semicomputable and cosemicomputable abstract data types has been studied in depth [1,9,10]. In general, auxiliary data, operators and even sorts may be needed. For example, small equational specifications exist for *all* computable data types, provided some auxiliary operations may appear in the specification; indeed, general theory [9] shows that the rational number data types studied here are computable and, therefore, can all be specified with 6 auxiliary functions and 4 equations only! However, these general theoretical results use advanced methods from computability theory and do not yield recognisable and useable axiomatisations. Here the specifications are close to the algebra of rational numbers and do not use auxiliary operators.

2.2 Arithmetic Structures

The signature Σ_m for meadows contains the ring operations and an inverse $^{-1}$ operation. It is richer than signatures commonly used for working with fields and skew structures as often inverse $^{-1}$ is not an explicit operation having an axiomatisation. Binary division $-/-$ seems to be rarely used but is useful to have as a derived function, $\frac{x}{y} = x.y^{-1}$, that can be eliminated. In some cases of interest to us, it is convenient to add other binary operations such as subtraction as a binary operation. To give an hint of what we have in mind for the term 'arithmetic structures':

Definition 1. *Σ is an* arithmetic signature *if it extends the meadow signature, i.e., $\Sigma_m \subset \Sigma$.*

Consider the following signature for wheels which simply adds two characteristic constants, ∞ for infinity and $\perp$ for error, to the signature Σ_m for meadows:

signature	Σ_w
sorts	num
constants	$0 :\to num$
	$1 :\to num$
	$\infty :\to num$
	$\perp :\to num$
operations	$+ : num \times num \to num$
	$. : num \times num \to num$
	$- : num \to num$
	$^{-1} : num \to num$
	$/ : num \times num \to num$
end	

Clearly, a wheel signature is an arithmetic signature.

3 The Wheel of Rationals

In working with arithmetic structures as abstract data types, the distinction between concrete constructions of algebras and their isomorphism type becomes important and subtle. There are lots of ways of constructing the rationals from the integers, and the integers from the naturals.

3.1 Basic Constructions with Rationals

We start with a specific ring Q of rational numbers with unit made from some specific copy of the integers Z. This is an algebra that is not minimal.

We begin the construction of the rationals as follows: let

$$SFP = \{(n, m) : n, m \in Z, m > 0, and\ gcd(n, m) = 1\}.$$

SFP stands for *simplified fracpairs.*[1] Note we seek to avoid equivalence classes in this construction.

The additive identity is uniquely defined by $(0, 1)$ – note that elements such as $(0, 2)$ are not in SFP. The multiplicative unit is $(1, 1)$. Note that $(1, 0)$ is not in SFP We define the operations in stages starting with addition:

$$(n, m) + (p, q) = (a, b)$$

[1] For information on fracpairs see [7].

where

$$a = \frac{np + mq}{gcd(np + mq, mq)}$$

and

$$b = \frac{mq}{gcd(np + mq, mq)}$$

Secondly, we define multiplication:

$$(n, m).(p, q) = (a, b)$$

where

$$a = \frac{np}{gcd(np, mq)}$$

and

$$b = \frac{mq}{gcd(np, mq)}$$

Thirdly, we define additive inverse:

$$-(n, m) = (-n, m).$$

Let

$$Q = (SFP \mid (0, 1), (1, 1), +, -, .)$$

3.2 A Wheel of Rationals

To build a wheel Q_w of rational numbers from Q we need to add elements that behave like infinity and error. The element $(1, 0)$ will represent infinity ∞ and the element $(0, 0)$ will represent error $\perp$.

The elements $(1, 0)$ and $(0, 0)$ are not in the set SFP and so we define

$$SFP_w = SFP \cup \{(1, 0), (0, 0)\}$$

and extend the operations of Q as follows:

For error, if $(n, m) \in SFP$ then

$$(n, m) + (0, 0) = (0, 0)$$
$$(0, 0) + (n, m) = (0, 0)$$
$$(n, m).(0, 0) = (0, 0)$$
$$(0, 0).(n, m) = (0, 0)$$
$$-(0, 0) = (0, 0)$$
$$(0, 0) + (0, 0) = (0, 0)$$
$$(0, 0).(0, 0) = (0, 0)$$

For infinity, if $(n,m) \in SFP$ and $n \neq 0$ then

$$\begin{gathered}(n,m)+(1,0)=(1,0)\\(1,0)+(n,m)=(1,0)\\(n,m).(1,0)=(1,0)\\(1,0).(n,m)=(1,0)\\(1,0)+(1,0)=(0,0)\\(1,0).(1,0)=(1,0)\\(1,0).(0,1)=(0,0)\\(0,1).(1,0)=(0,0)\\-(1,0)=(1,0)\end{gathered}$$

Error and infinity combine as follows:

$$\begin{gathered}(1,0)+(0,0)=(0,0)\\(0,0)+(1,0)=(0,0)\\(1,0).(0,0)=(0,0)\\(0,0).(1,0)=(0,0)\end{gathered}$$

Thus the structure Q is extended to the algebra

$$Q[(1,1),(0,0)] = (SFP_w \mid (0,1),(1,1),+,-,.)$$

The new elements are not named constants.

At this point we expand the algebra $Q[(1,1),(0,0)]$ with inverse operation $^{-1}$ defined by:

$$\begin{gathered}(n,m)^{-1}=(m,n) \text{ for } n>0\\(n,m)^{-1}=(-m,-n) \text{ for } n<0\\(0,0)^{-1}=(0,0)\\(1,0)^{-1}=(0,1)\\(0,1)^{-1}=(1,0)\end{gathered}$$

This extended structure

$$Q_w = (SFP_w \mid (0,1),(1,1),(1,0),(0,0),+,-,.,^{-1})$$

is a wheel of rational numbers.

Lemma 1. *The algebra Q_w is a Σ_w-minimal algebra.*

Definition 2. *The* abstract data type $\mathbb{Q}_w$ of the wheel of rational numbers *is the isomorphism class of the Σ_w-minimal algebra Q_w.*

4 Initial Algebra Specification of the Wheel of Rationals

We now give an equational specification (Σ_w, E_w) and prove that it defines our wheel of rationals.

Table 1. E_w: an initial algebra specification of the abstract data type of wheels

$$x + y = y + x \tag{1}$$
$$(x + y) + z = x + (y + z) \tag{2}$$
$$x + 0 = x \tag{3}$$
$$x.y = y.x \tag{4}$$
$$x.(y.z) = (x.y).z \tag{5}$$
$$x.1 = x \tag{6}$$
$$(x.y)^{-1} = x^{-1}.y^{-1} \tag{7}$$
$$(x^{-1})^{-1} = x \tag{8}$$
$$x.x^{-1} = 1 + 0.x^{-1} \tag{9}$$
$$\frac{x}{y} = x.y^{-1} \tag{10}$$
$$(x + y).z + 0.z = x.z + y.z \tag{11}$$
$$0.0 = 0 \tag{12}$$
$$0^{-1} = \infty \tag{13}$$
$$\infty + 1 = \infty \tag{14}$$
$$(-x).\infty = x.\infty \tag{15}$$
$$-\infty = \infty \tag{16}$$
$$0.\infty = \perp \tag{17}$$
$$- \perp = \perp \tag{18}$$
$$x + \perp = \perp \tag{19}$$
$$x + (-x) = 0.x \tag{20}$$
$$x^2 = x.x \tag{21}$$
$$1 + 0.(x + y + z + u) = \frac{x^2+y^2+z^2+u^2+1}{x^2+y^2+z^2+u^2+1} \tag{22}$$
$$\infty + 0.(x + y + z + u) = \frac{x^2+y^2+z^2+u^2+1}{0} + 0.(x + y + z + u) \tag{23}$$

4.1 Axioms

The complete set of equations in E_w is in Table 1. Notice that Eqs. 1–8 are familiar; the effects of the new elements on the standard operations start to appear in Eqs. 9, 11 and 20. To get the feel of these axioms, we prove carefully some basic identities that will be needed as lemmas later on.

First, recall the properties of numerals. Let $\underline{n} = 1 + 1 + \ldots + 1$ (n-times). Then the following is easy to check:

Lemma 2. $(\Sigma_w, E_w) \vdash \underline{n} + \underline{m} = \underline{n+m}$ *and* $(\Sigma_w, E_w) \vdash \underline{n}.\underline{m} = \underline{n.m}$.

Lemma 3. $(\Sigma_w, E_w) \vdash 0.\underline{n} = 0$.

Proof. We do this by induction on n. As basis, note that if $n = 0$ or $n = 1$ then the lemma is true by the axioms. Suppose the lemma is true for $n = k$ and consider $n = k + 1$. Using the axioms we deduce that

$$
\begin{aligned}
0.\underline{k+1} &= \underline{k+1}.0 && \text{by commutativity}\\
&= (\underline{\mathrm{k}} + 1).0 && \text{by definition of numerals}\\
&= (\underline{\mathrm{k}} + 1).0 + 0.0 && \text{by axioms 3 and 13}\\
&= 0.\underline{\mathrm{k}} + 0.1 && \text{by axiom 11}\\
&= 0 + 0 && \text{by induction hypothesis}\\
&= 0.
\end{aligned}
$$

□

Lemma 4. $(\Sigma_w, E_w) \vdash \underline{m}.\underline{m}^{-1} = 1$.

Proof. Applying Lagrange's Theorem to $m - 1$, let $m = p^2 + q^2 + r^2 + s^2 + 1$. Then:

$$
\begin{aligned}
\frac{\underline{\mathrm{m}}}{\underline{\mathrm{m}}} &= \frac{\underline{\mathrm{p}}^2 + \underline{\mathrm{q}}^2 + \underline{\mathrm{r}}^2 + \underline{\mathrm{s}}^2 + 1}{\underline{\mathrm{p}}^2 + \underline{\mathrm{q}}^2 + \underline{\mathrm{r}}^2 + \underline{\mathrm{s}}^2 + 1}\\
&= 1 + 0.(\underline{\mathrm{p}} + \underline{\mathrm{q}} + \underline{\mathrm{r}} + \underline{\mathrm{s}}) && \text{by axiom 22}\\
&= 1 + 0.(\underline{p + q + r + s}) && \text{by Lemma 2}\\
&= 1 + 0 && \text{by Lemma 3}\\
&= 1
\end{aligned}
$$

□

Lemma 5. $(\Sigma_w, E_w) \vdash 0.\underline{m}^{-1} = 0$.

Proof.

$$
\begin{aligned}
0.\underline{m}^{-1} &= (0.\underline{m}).\underline{m}^{-1} && \text{by Lemma 3}\\
&= 0.(\underline{m}.\underline{m}^{-1}) && \text{by axiom 5}\\
&= 0.1 && \text{by Lemma 3}\\
&= 0 && \text{by axiom 6}
\end{aligned}
$$

□

Lemma 6. $(\Sigma_w, E_w) \vdash 0.(\underline{n}.\underline{m}^{-1}) = 0$.

Proof.

$$\begin{aligned} 0.(\underline{n}.\underline{m}^{-1}) &= (0.\underline{n}).\underline{m}^{-1}) && \text{by axiom 5} \\ &= 0.\underline{m}^{-1} && \text{by Lemma 3} \\ &= 0 && \text{by Lemma 5} \end{aligned}$$

□

Turning to ∞ we have these basic properties:

Lemma 7. $(\Sigma_w, E_w) \vdash \infty - 1 = \infty$.

Proof.

$$\begin{aligned} \infty - 1 &= (\infty + 1) - 1 && \text{by axiom 14} \\ &= \infty + (1 - 1) && \text{by axiom 2} \\ &= \infty + 0 && \text{by axiom 20} \\ &= \infty && \text{by axiom 3} \end{aligned}$$

□

Lemma 8. $(\Sigma_w, E_w) \vdash \infty^{-1} = 0$.

Proof. Clearly, $\infty^{-1} = (0^{-1})^{-1} = 0$ by axioms 13 and 8. □

Lemma 9. $(\Sigma_w, E_w) \vdash \infty + \underline{m} = \infty$.

Proof. By induction on m. If $\underline{m} = 0$ and $\underline{m} = 1$ then the lemma follows from the axioms 3 and 14, respectively. Suppose $\underline{m} = \underline{k+1}$. Then $\infty + (\underline{k+1}) = (\infty + \underline{k}) + 1$ and, by the induction hypothesis on k, we have $\infty + 1 = \infty$. □

Lemma 10. $(\Sigma_w, E_w) \vdash \infty.\underline{m} = \infty$ *and* $(\Sigma_w, E_w) \vdash \infty.\underline{m}^{-1} = \infty$.

Proof. Consider the first statement. Applying Lagrange's Theorem, let $m = p^2 + q^2 + r^2 + s^2 + 1$. Then

$$\begin{aligned} \infty.\underline{m} &= \frac{\underline{m}}{0} && \text{by axiom 13} \\ &= \frac{\underline{\underline{p^2 + q^2 + r^2 + s^2 + 1}}}{0} && \text{by substitution} \\ &= \frac{\underline{p}^2 + \underline{q}^2 + \underline{r}^2 + \underline{s}^2 + 1}{0} + 0.(\underline{p} + \underline{q} + \underline{r} + \underline{s} + 1) && \text{by Lemma 2 and axiom 3} \\ &= \infty + 0.(\underline{p} + \underline{q} + \underline{r} + \underline{s} + 1) && \text{by axiom 23} \\ &= \infty. \end{aligned}$$

Next, consider the second statement.

$$\begin{aligned}
\infty.\underline{m}^{-1} &= (0^{-1}.\underline{m}^{-1}) && \text{by axiom 13}\\
&= (0.0)^{-1} && \text{by axiom 7}\\
&= 0^{-1} && \text{by axiom 12}\\
&= \infty && \text{by axiom 13}
\end{aligned}$$

□

Lemma 11. $(\Sigma_w, E_w) \vdash \infty.(\underline{n}.\underline{m}^{-1}) = \infty.$

Proof.

$$\begin{aligned}
\infty.(\underline{n}.\underline{m}^{-1}) &= (\infty.\underline{n}).\underline{m}^{-1} && \text{by axiom 5}\\
&= \infty.\underline{m}^{-1} && \text{by Lemma 10}\\
&= \infty. && \text{by Lemma 10}
\end{aligned}$$

□

Lemma 12. $(\Sigma_w, E_w) \vdash \infty.\infty = \infty.$

Proof.

$$\begin{aligned}
\infty.\infty &= (0^{-1}.0^{-1}) && \text{by axiom 13}\\
&= (0.0)^{-1} && \text{by axiom 7}\\
&= 0^{-1} && \text{by Lemma 3}\\
&= \infty && \text{by axiom 13}
\end{aligned}$$

□

4.2 Remarks on the Equations

The equations of E_w, displayed in Table 1, build on a number of sources and required adaptations. The basic axioms are those of commutative rings. Some of the axioms are of common meadows which introduce $\perp$: e.g., axioms 9 and 20. Axioms 22 and 23 are adaptations of the data generating axioms for the rationals in [11]. Axiom 11 was used by Setzer and Carlström [13,18].

Our set of axioms is intended to be informative and practical. It is not intended to be minimal. For instance, axiom 22 implies axiom 13 (by setting the variables = 0). As with all established axiom systems, properties that are lemmas can often also serve as axioms, whence some axioms can become lemmas.

Axioms from [13] that are true but which we do not use are:

$$\frac{x}{y} + z + 0.y = \frac{x + yz}{y} \tag{24}$$

$$(x + 0.y).z = x.z + 0.y \tag{25}$$

$$(x + 0.y)^{-1} = x^{-1} + 0.y \tag{26}$$

In the Introduction we suggested some identities about ∞ and $\perp$ were either to be expected or are desirable or undesirable. In fact, all identities involving ∞ and $\perp$ offer opportunities for mathematical investigation and possibly new technical insights and semantic perspectives.

4.3 Equational Specification Theorem

Theorem 1. *The initial algebra $I(\Sigma_w, E_w)$ of the equations in E_w is isomorphic to the wheel Q_w of rational numbers.*

Proof. We will take the standard term representation $T(\Sigma_w, E_w)$ of initial algebra $I(\Sigma_w, E_w)$ and show that $T(\Sigma_w, E_w) \cong Q_w$. Recall from 2.1 that $T(\Sigma_w, E_w) = T(\Sigma_w)/\equiv_{E_w}$ and that for $t_1, t_2 \in T(\Sigma_w)$,

$$t_1 \equiv_{E_w} t_2 \Longleftrightarrow E_w \vdash t_1 = t_2.$$

To work with the congruence we define a transversal Tr of unique representatives of the equivalence classes of $\equiv_{E_w}$. Let

$$\text{Tr} = \{\underline{n}.\underline{m}^{-1}, \perp, \infty \mid (n, m) \in SFP\}.$$

Lemma 13. $(\Sigma_w, E_w) \models Q_w$

Proof. To prove soundness we inspect each axiom and show its validity in Q_w. This involves 23 equations, often with many case distinctions each. We give some examples to illustrate the pattern of reasoning.

First, consider if any one of the variables is the error element $(0, 0)$. Note that all but one of the equations have the property that the variables that appear on the left side of the equality sign also appear on the right, and vice versa. The definition of the operations in the error case of Sect. 3.2 shows that $(0, 0)$ propagates. Thus, all these equations are valid if one of the variables is $(0, 0)$. The equation left is $x + \perp = \perp$ which is valid by definition.

Thus we need only consider the equations when their variables have values that are rationals or infinity. Note that infinity does not always propagate, which can lead to many case distinctions in the equations with several variables.

1. Consider associativity: $x + (y + z) = (x + y) + z$. If all variables are rationals then the equation is easily seen to be valid. There are three cases involving infinity $(1, 0)$.

If exactly one variable is $(1,0)$ then both sides of the equation evaluate to $(1,0)$.
If exactly two variables are $(1,0)$ then both sides evaluate to $(0,0)$.
If all three variables are $(1,0)$ then both sides evaluate to $(0,0)$.

2. Consider $(x+0.y).z = x.z+0.y$. Suppose y is the infinite element $(1,0)$. Then $0.y = (0,0)$ and since error propagates both sides evaluate to $(0,0)$. Suppose $y \neq (1,0)$. Then $0.y = (0,1)$ and the equation reduces to the value of $x.z$ on both sides.
3. Consider the four squares equation. Suppose one of x, y, z, u is the infinite element $(1,0)$. Then, on the LHS, the sums of squares numerator and denominator evaluate to the infinite $(1,0)$ and their quotient is the error element $(0,0)$. On the RHS, the sum of the variables x, y, z, u is $(1,0)$ and its product with 0 is the error element $(0,0)$, and so the equation holds.

   Next suppose more than one of x, y, z, u is the infinite element $(1,0)$. Then both the sum of squares of the variables on the LHS, and the sum of the variables on the RHS, are both the error element $(0,0)$ and since error propagates the equation holds. □

Now $T(\Sigma_w, E_w)$ is the initial algebra of the class of models of (Σ_w, E_w). By Lemma 13, Q_w is such a model and so by initiality, there exists a unique surjective homomorphism $\phi : T(\Sigma_w, E_w) \longrightarrow Q_w$. We have to show that ϕ is an isomorphism. We do this by proving by induction that every term $t \in T(\Sigma_w)$ reduces to an element t_0 of the transversal Tr, i.e.,

$$(\Sigma_w, E_w) \vdash t = t_0.$$

We deal with the constants in the base case and the operator symbols in the induction step. There are several case distinctions and the argument uses the 25 equations in various subtle ways.

Basis Case: The Constants. Clearly, the transversal contains the constants $\bot$, ∞. We will show that the constant 0 is $\underline{0}.\underline{1}^{-1}$ using the axioms of $x.1 = x$ and associativity:

$$0 = 0.1 = 0.(1.1^{-1}) = 0.1^{-1}.$$

Last we show the constant 1 is $\underline{1}.\underline{1}^{-1}$ using the four squares axiom:

$$\frac{1}{1} = \frac{0^2+0^2+0^2+0^2+1}{0^2+0^2+0^2+0^2+1} = 1 + 0.(0+0+0+0) = 1$$

.

Induction Step. There are four operators $-, ^{-1}, =, .$ to consider.

Additive Inverse. Consider the leading operator symbol $-$ and term $t = -s$. By induction, the subterm s reduces to one of three cases: $s = \infty$, $s = \bot$ and $s = \underline{n}.\underline{m}^{-1}$ in the transversal Tr. From the axioms, the first two cases of $-s$ are immediate as $-\infty = \infty$ and $-\bot = \bot$, which are in the transversal. The last case is quite involved, however.

Suppose $t = -(\underline{n}.\underline{m}^{-1})$. We will show that $t = (-\underline{n}.\underline{m}^{-1})$, which is in the transversal. We begin with some lemmas of use here and later on.

To complete the case we need these identities:

Lemma 14. *$(\underline{n}.\underline{m}^{-1}) + (-(\underline{n}.\underline{m}^{-1})) = 0$ and $(\underline{n}.\underline{m}^{-1}) + (\underline{-n}).\underline{m}^{-1} = 0$.*

Proof. First, we show that $(\underline{n}.\underline{m}^{-1}) + (-(\underline{n}.\underline{m}^{-1})) = 0$.

$$\begin{aligned}
(\underline{n}.\underline{m}^{-1}) + (-(\underline{n}.\underline{m}^{-1})) &= 0.(\underline{n}.\underline{m}^{-1}) && \text{by axiom 20}\\
&= (0.\underline{n}).\underline{m}^{-1} && \text{by axiom 5}\\
&= 0.\underline{m}^{-1} && \text{by Lemma 3}\\
&= (0.\underline{m}).\underline{m}^{-1} && \text{by Lemma 3}\\
&= 0.(\underline{m}.\underline{m}^{-1}). && \text{by axiom 5}\\
&= 0.1 && \text{by Lemma 4}\\
&= 0 && \text{by axiom 6}
\end{aligned}$$

Next we show the second identity, $(\underline{n}.\underline{m}^{-1}) + (\underline{-n}).\underline{m}^{-1} = 0$.

$$\begin{aligned}
(\underline{n}.\underline{m}^{-1}) + (-\underline{n}).\underline{m}^{-1} &= (\underline{n} + (-\underline{n})).\underline{m}^{-1} + 0.\underline{m}^{-1} && \text{by axiom 11}\\
&= (\underline{n} + (-\underline{n})).\underline{m}^{-1} && \text{by Lemma 5}\\
&= (0.\underline{n}).\underline{m}^{-1} && \text{by axiom 20}\\
&= 0.(\underline{n}.\underline{m}^{-1}) && \text{by axiom 5}\\
&= 0. && \text{by Lemma 6}
\end{aligned}$$

□

Given the Lemma 14, by subtracting the above equations, it follows that

$$-(\underline{n}.\underline{m}^{-1}) - (\underline{-n}).\underline{m}^{-1} = 0$$

and, thus,

$$-(\underline{n}.\underline{m}^{-1}) = (\underline{-n}).\underline{m}^{-1}$$

which is in the transversal. This completes the basis. □

Multiplicative Inverse. Consider the leading operator symbol $^{-1}$ and term $t = s^{-1}$. By induction, the subterm s reduces to one of four cases: $s = \infty$, $s = \perp$, $s = \underline{0}.\underline{m}^{-1}$ and $s = \underline{n}.\underline{m}^{-1}$ with $n \neq 0$ in the transversal Tr.

If $s = \perp$ then

$$\begin{aligned}
s^{-1} &= (0.\infty)^{-1} && \text{by axiom 17}\\
&= 0^{-1}.\infty^{-1} && \text{by axiom 7}\\
&= \infty.0 && \text{by axiom 17 and Lemma 8}\\
&= 0.\infty && \text{by axiom 4}\\
&= \perp && \text{by axiom 17}
\end{aligned}$$

which is in the transversal.

If $s = \infty$ then $s^{-1} = 0$ by Lemma 8.
If $s = \underline{0}.\underline{m}^{-1}$ then

$$\begin{aligned} s^{-1} &= (\underline{0}.\underline{m}^{-1})^{-1} & \\ &= 0^{-1} & \text{by Lemma 5} \\ &= \infty & \text{by axiom 13} \end{aligned}$$

which is in the transversal.
If $s = \underline{n}.\underline{m}^{-1}$ for $n \neq 0$ then

$$\begin{aligned} s^{-1} &= (\underline{n}.\underline{m}^{-1})^{-1} & \\ &= \underline{n}^{-1}.(\underline{m}^{-1})^{-1} & \text{by axiom 7} \\ &= \underline{n}^{-1}.\underline{m} & \text{by axiom 8} \\ &= \underline{m}.\underline{n}^{-1} & \text{by axiom 4} \end{aligned}$$

which is in the transversal.

Addition. Consider the leading operator symbol $+$ and term $t = r + s$. For notational ease, we write a for $\underline{n}$ and b for $\underline{m}$. By induction, the subterms r, s reduce to one of six cases in the table below:

r	s
$\perp$	$\perp$
$\perp$	∞
$\perp$	$a.b^{-1}$
∞	∞
∞	$a.b^{-1}$
$a.b^{-1}$	$c.d^{-1}$

The first three cases where $r =\perp$ follow from the axiom $x+ \perp=\perp$ and commutativity. The next case uses axioms 20 followed by axiom 17:

$$\infty + \infty = \infty + (-\infty) = 0.\infty =\perp .$$

Now

$$\begin{aligned} \infty + a.b^{-1} &= \infty.b^{-1} + a.b^{-1} & \text{by Lemma 10} \\ &= (\infty + a).b^{-1} + 0.b^{-1} & \text{by axiom 11} \\ &= \infty.b^{-1} + 0 & \text{by Lemma 9} \\ &= \infty.b^{-1} & \text{by axiom 3} \\ &= \infty & \text{by Lemma 10} \end{aligned}$$

which is in the transversal.

Consider the last case:

$$\begin{aligned}
a.b^{-1}+c.d^{-1} &= a.b^{-1}.1+c.d^{-1}.1 && \text{by axiom 6}\\
&= a.b^{-1}.d.d^{-1}+c.d^{-1}.b.b^{-1} && \text{by Lemma 4}\\
&= a.d.b^{-1}.d^{-1}+c.b.d^{-1}.b^{-1} && \text{by axiom 4}\\
&= (a.d+c.b).b^{-1}.d^{-1}+0.b^{-1}.d^{-1} && \text{by axiom 11}\\
&= \frac{a.d+c.b}{bd}+0 && \text{by axiom 7}\\
&= \frac{a.d+c.b}{bd}. && \text{by axiom 3}
\end{aligned}$$

To finish this deduction: let $p = gcd(a.d+c.b, bd)$. Choose p', p'' such that $p.p' = a.d+c.b$ and $p.p'' = b.d$. Then $\frac{a.d+c.b}{bd} = \frac{p.p'}{p.p''} = \frac{p'}{p''}$, which is in the transversal.

Multiplication. Consider the leading operator symbol . and term $t = r.s$. By induction, the subterms r, s reduce to one of six cases as in the table above; each case will need an argument.

$$\begin{aligned}
\bot\,.\,\bot &= (\infty.0).(\infty.0) && \text{by axiom 17}\\
&= \infty.\infty.0 && \text{by axioms 5, 4, 12}\\
&= \infty.0 && \text{by lemma 12}\\
&= \bot && \text{by axiom 17}
\end{aligned}$$

which is in the transversal. The next case is straightforward:

$$\begin{aligned}
\bot\,.\infty &= (\infty.0).\infty && \text{by axiom 17}\\
&= (\infty.\infty).0 && \text{by axioms 5, 4}\\
&= \infty.0 && \text{by lemma 12}\\
&= \bot && \text{by axiom 17}
\end{aligned}$$

which is in the transversal.

$$\begin{aligned}
\bot\,.a.b^{-1} &= (\infty.0).a.b^{-1} && \text{by axiom 17}\\
&= \infty.(0.a.b^{-1}) && \text{by axioms 5}\\
&= \infty.0 && \text{by lemma 6}\\
&= \bot && \text{by axiom 17}
\end{aligned}$$

which is in the transversal.

Consider $\infty.a.b^{-1}$.

$$\begin{aligned}
\infty.(a.b^{-1}) &= (\infty.a).b^{-1} && \text{by axiom 5}\\
&= (\infty.b^{-1}) && \text{by lemma 10}\\
&= \infty && \text{by lemma 10}
\end{aligned}$$

which is in the transversal.

Finally, the last case is this deduction: let $p = gcd(ac, bd)$

$$\begin{aligned}
a.b^{-1}.c.d^{-1} &= (a.c).(b.d)^{-1} && \text{by axioms 4 and 7}\\
&= (p.p').(p.p'')^{-1} && \text{by substitution}\\
&= (p.p').(p^{-1}.p''^{-1}) && \text{by axiom 7}\\
&= (p.p^{-1}).(p'p''^{-1}) && \text{by axiom 4}\\
&= (1.p'p''^{-1}) && \text{by Lemma 4}\\
&= p'p''^{-1} && \text{by axiom 6}
\end{aligned}$$

which is in the transversal.

This concludes the proof of the theorem. □

5 Concluding Remarks

Using a general conceptual framework for analysing numerical data types with total operations, we have given a mathematical model of the wheel of rational numbers. The concept of a wheel was introduced by Anton Setzer in unpublished notes [18]. It was motivated by Jens Blanck's lectures on exact real number computations, based on domains and rational number intervals, and Per Martin Löff's suggestion to allow 0 in denominators of elements of quotient fields. Later wheels were studied in greater generality and published by Jesper Carlström [13]. Carlström generalised the constructions to semirings, developed equations and identities, and considered the class of wheels.

The concept of a meadow emerged when we were making an algebraic specification of the rational numbers in [11]; we studied the axiomatic class of meadows in [3]. A substantial series of papers has built an algebraic theory of meadows with different properties, e.g., [4–6,8,12].

The ideas in this paper suggest problems and topics for further study. In the case of the meadow programme on totalisation, algebraic specifications for other methods of totalisation in arithmetic structures could be tackled. An obvious candidate is *transrational arithmetic* due to James Anderson, as described in [2,17]. Transrational arithmetic provides signed infinities, i.e., $+\infty$ and $-\infty$, in addition to an error element (which is called nullity and denoted by Φ instead of $\perp$). Thus, while wheels are aimed at reasoning for exact real arithmetics based on intervals, the transrationals are aimed at floating point arithmetics. Other semantic interpretations of infinities and errors are conceivable that could lead to interesting arithmetic data types.

As with most algebraic specification problems, there is also the search for new specifications with good term rewriting properties for arithmetic structures. In the specific case of wheels, can a basis theorem for the class of wheels be provided? In the case of common meadows, which feature $\perp$ but not ∞, a basis theorem has been obtained in [6]. The case of wheels seems to be much harder, however.

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