Research Article

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Coefficient inequalities for a subclass of Bazilevič functions

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Abstract: Let f be analytic in $\mathbb{D} = \{z\colon |z| < 1\}$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, and for $\alpha \geq 0$ and $0 < \lambda \leq 1$, let $\mathcal{B}_1(\alpha, \lambda)$ denote the subclass of Bazilevič functions satisfying $\left|f'(z)\left(\frac{z}{f(z)}\right)^{1-\alpha} - 1\right| < \lambda$ for $0 < \lambda \leq 1$. We give sharp bounds for various coefficient problems when $f \in \mathcal{B}_1(\alpha, \lambda)$, thus extending recent work in the case $\lambda = 1$.

Keywords: univalent functions, Bazilevi, coefficients, inverse, Fekete–Szegö, Hankel determinant

MSC: Primary 30C45, Secondary 30C50

1 Definitions and preliminaries

Denote by $\mathcal{A}$ the class of analytic functions f defined for $z \in \mathbb{D} = \{z\colon |z| < 1\}$, and normalized so that $f(0) = 0$ and $f'(0) = 1$, and by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{D} = \{z\colon |z| < 1\}$. Let f be given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Then, for $\alpha > 0$, it was shown by Bazilevič [1] that if $f \in \mathcal{A}$ and is given by eq. (1), then there exists starlike functions g such that

$$\operatorname{Re} \frac{zf'(z)}{f(z)^{1-\alpha} g(z)^{\alpha}} > 0,$$

it follows that $f \in \mathcal{S}$. We denote this class of Bazilevič functions by $\mathcal{B}(\alpha)$, so that $\mathcal{B}(\alpha) \subset \mathcal{S}$ when $\alpha > 0$.

The case $\alpha = 0$ was subsequently considered by Sheil-Small [2], who showed that $\mathcal{B}(\alpha) \subset \mathcal{S}$ when $\alpha \geq 0$.

Taking $g(z) \equiv z$ gives the class $\mathcal{B}_1(\alpha)$ of Bazilevič functions, which has been the subject of much recent research. We note that $\mathcal{B}_1(0)$ is the class $\mathcal{S}^*$ of starlike functions, and $\mathcal{B}_1(1)$ the well-known class $\mathcal{R}$ of functions whose derivative has positive real part in $\mathbb{D}$.

Thus, $f \in \mathcal{B}_1(\alpha)$, if and only if, for $z \in \mathbb{D}$

$$\operatorname{Re} f'(z)\left(\frac{z}{f(z)}\right)^{1-\alpha} > 0.$$

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Various properties have been obtained for functions in $\mathcal{B}_1(\alpha)$. Among other results, Singh [3] found sharp estimates for the moduli of the first four coefficients and obtained the solution to the Fekete–Szegö problem. Sharp bounds for the second Hankel determinant, the initial coefficients of the function $\log(f(z)/z)$, and the initial coefficients of the inverse function f^{-1} were obtained in [4], and distortion theorems and some length–area results were also obtained in [5–7].

We now define the subclass $\mathcal{B}_1(\alpha, \lambda)$ of $\mathcal{B}_1(\alpha)$, which was introduced in 1996 by Ponnusamy and Singh [8, Theorem 3]. In this article, the authors determine condition on λ so that functions in $\mathcal{B}_1(\alpha, \lambda)$ are starlike in $\mathbb{D}$ (see also [9, Theorem 3] for an extension of this result). Later in [10, Theorems 1 and 2], the authors considered complex values of α and obtained condition on λ such as the functions in $\mathcal{B}_1(\alpha, \lambda)$ are spirallike in $\mathbb{D}$. Two of the present authors in [11] studied the class $\mathcal{B}_1(\alpha, \lambda)$ in the case $\lambda = 1$. Therefore, it is natural to consider the investigation of the problems discussed in this article for the complex values of α in the context of the investigation from [10].

Definition 1.1. Let $f \in \mathcal{S}$ and be given by eq. (1). Then, for $\alpha \geq 0$ and $0 < \lambda \leq 1$, $f \in \mathcal{B}_1(\alpha, \lambda)$, if and only if, for $z \in \mathbb{D}$

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^{1-\alpha} - 1 \right| < \lambda. \tag{2}$$

We note that $f \in \mathcal{B}_1(0, 1)$ reduces to the class of bounded starlike functions considered by Singh [12].

Although the aforementioned definition requires that $\alpha \geq 0$, choosing $\alpha = -1$ gives the class $\mathcal{U}(\lambda)$ of univalent functions defined for $z \in \mathbb{D}$ by

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^{2} - 1 \right| < \lambda.$$

The class $\mathcal{U}(\lambda)$ has been the focus of a great deal of research in recent years (see e.g. [13,14], and for a summary of some known results, see [15]). Although the classes $\mathcal{B}_1(\alpha, \lambda)$ for $\alpha \geq 0$ and $\mathcal{U}(\lambda)$ have similar structural representations, they are fundamentally different in many ways, and we shall see in the following analysis that the methods used in this study cannot be applied to the class $\mathcal{U}(\lambda)$. It is also interesting to note that the only known negative value of α which gives a subset of $\mathcal{S}$ appears to be $\alpha = -1$. See [16,17] and references therein for recent investigation, which also deals with the case $\alpha = -1$ for meromorphic functions.

In this study, we give sharp bounds for the modulus of the coefficients a_n for $f \in \mathcal{B}_1(\alpha, \lambda)$ when $2 \leq n \leq 5$, together with other related results, noting that when $f \in \mathcal{U}(\lambda)$, sharp bounds have been found only for some initial coefficients.

First note that from eq. (2), we can write

$$f'(z)\left(\frac{z}{f(z)}\right)^{1-\alpha} = 1 + \lambda\omega(z), \tag{3}$$

for $z \in \mathbb{D}$, where ω is the Schwarz function.

Next, recall the class $\mathcal{P}$ of functions with positive real part in $\mathbb{D}$, so that $h \in \mathcal{P}$, if, and only if, $\operatorname{Re} h(z) > 0$ for $z \in \mathbb{D}$.

We write

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{4}$$

Thus, as we can write

$$h(z) = \frac{1 + \omega(z)}{1 - \omega(z)},$$

eq. (3) can be written as

$$f'(z)\left(\frac{z}{f(z)}\right)^{1-\alpha} = 1 + \lambda\frac{h(z) - 1}{h(z) + 1}. \tag{5}$$

We shall use the following results concerning the coefficients of $h \in \mathcal{P}$.

Lemma 1.1. [18] *If $h \in \mathcal{P}$, then for some complex valued x with $|x| \leq 1$, and some complex valued ζ with $|\zeta| \leq 1$,*

$$2c_2 = c_1^2 + x(4 - c_1^2),$$
$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)\zeta.$$

Lemma 1.2. [19] *If $h \in \mathcal{P}$, then $|c_n| \leq 2$ for $n \geq 1$, and*

$$\left|c_2 - \frac{\mu}{2}c_1^2\right| \leq \max\{2, 2|\mu - 1|\} = \begin{cases} 2, & 0 \leq \mu \leq 2, \\ 2|\mu - 1|, & \textit{elsewhere}. \end{cases}$$

Lemma 1.3. [19] *If $h \in \mathcal{P}$, then*

$$|c_3 - (1 + \mu)c_1c_2 + \mu c_1{}^3| \leq \max\{2, 2|2\mu - 1|\} = \begin{cases} 2, & 0 \leq \mu \leq 1, \\ 2|2\mu - 1|, & \textit{elsewhere}. \end{cases}$$

Lemma 1.4. [19] *Let $h \in \mathcal{P}$. If $0 \leq B \leq 1$ and $B(2B - 1) \leq D \leq B$, then*

$$|c_3 - 2Bc_1c_2 + Dc_1^3| \leq 2.$$

Lemma 1.5. [20] *If $h \in \mathcal{P}$, and α_1, α_2, β_1 and β_2 satisfy $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$ and*

$$8\alpha_1(1 - \alpha_1)((\alpha_2\beta_2 - 2\beta_1)^2 + (\alpha_2(\alpha_1 + \alpha_2) - \beta_2)^2) + \alpha_2(1 - \alpha_2)(\beta_2 - 2\alpha_1\alpha_2)^2 \leq 4\alpha_2^2(1 - \alpha_2)^2\alpha_1(1 - \alpha_1), \tag{6}$$

then

$$|\beta_1c_1^4 + \alpha_1c_2^2 + 2\alpha_2c_1c_3 - (3/2)\beta_2c_1^2c_2 - c_4| \leq 2.$$

2 Initial coefficients

We first give sharp bounds for some initial coefficients for $f \in \mathcal{B}_1(\alpha, \lambda)$, extending those given in [11].

Theorem 2.1. *Let $f \in \mathcal{B}_1(\alpha, \lambda)$ for $\alpha \ge 0$ and $0 < \lambda \le 1$ and be given by eq.* (1).
Then, for $2 \le n \le 5$,

$$|a_n| \le \frac{\lambda}{(\alpha + n - 1)}.$$

The inequalities are sharp.

Proof. Equating coefficients in eq. (5) gives

$$\begin{aligned}
a_2 &= \frac{c_1\lambda}{2(1+\alpha)},\\
a_3 &= \frac{\lambda}{2(2+\alpha)}\left(c_2 - \frac{(2+4\alpha+2\alpha^2-2\lambda+\alpha\lambda+\alpha^2\lambda)}{4(1+\alpha)^2}c_1^2\right),\\
a_4 &= \frac{\lambda}{2(3+\alpha)}\left(c_3 - \frac{(4+6\alpha+2\alpha^2-3\lambda+2\alpha\lambda+\alpha^2\lambda)}{2(1+\alpha)(2+\alpha)}c_1c_2 + (12+42\alpha+54\alpha^2+30\alpha^3+6\alpha^4-18\lambda\right.\\
&\quad -24\alpha\lambda+12\alpha^2\lambda+24\alpha^3\lambda+6\alpha^4\lambda+6\lambda^2-13\alpha\lambda^2+6\alpha^4\lambda+6\lambda^2-13\alpha\lambda^2-2\alpha^2\lambda^2+7\alpha^3\lambda^2\\
&\quad \left.+2\alpha^4\lambda^2)\frac{c_1^3}{24(1+\alpha)^3(2+\alpha)}\right),\\
a_5 &= \frac{-\lambda}{2(4+\alpha)}\left(Mc_1^4 + Nc_2^2 + 2Qc_1c_3 - \frac{3}{2}Rc_1^2c_2 - c_4\right),
\end{aligned} \tag{7}$$

where

$$N = \frac{8+8\alpha+2\alpha^2-4\lambda+3\alpha\lambda+\alpha^2\lambda}{4(2+\alpha)^2},$$

$$Q = \frac{6+8\alpha+2\alpha^2-4\lambda+3\alpha\lambda+\alpha^2\lambda}{4(1+\alpha)(3+\alpha)},$$

$$\begin{aligned}
M = {}&(288+1536\alpha+3432\alpha^2+4152\alpha^3+2928\alpha^4+1200\alpha^5+264\alpha^6+24\alpha^7-528\lambda-1764\alpha\lambda-1704\alpha^2\lambda\\
&+444\alpha^3\lambda+1872\alpha^4\lambda+1284\alpha^5\lambda+360\alpha^6\lambda+36\alpha^7\lambda+288\lambda^2+24\alpha\lambda^2-1068\alpha^2\lambda^2-744\alpha^3\lambda^2+552\alpha^4\lambda^2\\
&+696\alpha^5\lambda^2+228\alpha^6\lambda^2+24\alpha^7\lambda^2-48\lambda^3+212\alpha\lambda^3-116\alpha^2\lambda^3-295\alpha^3\lambda^3+37\alpha^4\lambda^3+149\alpha^5\lambda^3+55\alpha^6\lambda^3\\
&+6\alpha^7\lambda^3)/(192(1+\alpha)^4(2+\alpha)^2(3+\alpha)),
\end{aligned}$$

$$\begin{aligned}
R = {}&(72+240\alpha+306\alpha^2+186\alpha^3+54\alpha^4+6\alpha^5-88\lambda-118\alpha\lambda+40\alpha^2\lambda+112\alpha^3\lambda+48\alpha^4\lambda\\
&+6\alpha^5\lambda+24\lambda^2-46\alpha\lambda^2-21\alpha^2\lambda^2+26\alpha^3\lambda^2+15\alpha^4\lambda^2+2\alpha^5\lambda^2)/(12(1+\alpha)^2(2+\alpha)^2(3+\alpha)).
\end{aligned}$$

The inequality for $|a_2|$ is trivial.

For a_3, we apply Lemma 1.2 with

$$\mu = \frac{(2 + 4\alpha + 2\alpha^2 - 2\lambda + \alpha\lambda + \alpha^2\lambda)}{2(1 + \alpha)^2}.$$

Since $0 \le \mu \le 2$ for $\alpha \ge 0$ and $0 < \lambda \le 1$, the inequality for $|a_3|$ follows.

For a_4, we use Lemma 1.4 with

$$B = \frac{(4 + 6\alpha + 2\alpha^2 - 3\lambda + 2\alpha\lambda + \alpha^2\lambda)}{4(1 + \alpha)(2 + \alpha)},$$

and

$$\begin{aligned} D = {} & (12 + 42\alpha + 54\alpha^2 + 30\alpha^3 + 6\alpha^4 - 18\lambda - 24\alpha\lambda + 12\alpha^2\lambda + 24\alpha^3\lambda + 6\alpha^4\lambda + 6\lambda^2 \\ & - 13\alpha\lambda^2 + 6\alpha^4\lambda + 6\lambda^2 - 13\alpha\lambda^2 - 2\alpha^2\lambda^2 + 7\alpha^3\lambda^2 + 2\alpha^4\lambda^2)/(24(1 + \alpha)^3(2 + \alpha)). \end{aligned}$$

Since $0 \le B \le 1$, and $B(2B - 1) \le D \le B$, when $\alpha \ge 0$ and $0 < \lambda \le 1$, the inequality for $|a_4|$ follows.

For a_5, we apply Lemma 1.5 with α_1, α_2, β_1 and β_2 the respective coefficients of a_5 in eq. (7). Since $0 < \alpha_1 < 1$ and $0 < \alpha_2 < 1$, for $\alpha \ge 0$ and $0 < \lambda \le 1$, then by expanding both sides and subtracting, it is easily seen that the conditions (6) of Lemma 1.5 are satisfied (the detailed proof of this step can be found in [21]), and so the inequality for $|a_5|$ follows. The inequality of $|a_i|$ is sharp on choosing $c_i = 2$ when $2 \le i \le 5$, and $c_j = 0$ when $i \ne j$. □

3 Inverse coefficients

Since $\mathcal{B}_1(\alpha, \lambda) \subset \mathcal{S}$, inverse functions f^{-1} exist, and so we can write

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n, \tag{8}$$

valid in some disk $|w| \le r_0(f)$. It is an easy exercise to show that

$$\begin{aligned} A_2 &= -a_2, \\ A_3 &= 2a_2^2 - a_3, \\ A_4 &= -5a_2^3 + 5a_2a_3 - a_4. \end{aligned} \tag{9}$$

We first prove the following.

Theorem 3.1. *If $f \in \mathcal{B}_1(\alpha, \lambda)$, with inverse coefficients given by eq.* (8), *then for* $0 < \lambda \le 1$,

$$\begin{aligned} |A_2| &\le \frac{\lambda}{1 + \alpha} \quad \textit{when} \ \ \alpha \ge 0, \\ |A_3| &\le \frac{\lambda}{2 + \alpha} \quad \textit{when} \ \ 0 \le \alpha \le \frac{1}{2}(1 + \sqrt{17}), \ \ \textit{and} \ \ 0 < \lambda \le \frac{2(1 + \alpha)^2}{(2 + \alpha)(3 + \alpha)}, \\ &\le \frac{\lambda}{2 + \alpha} \quad \textit{when} \ \ \alpha > \frac{1}{2}(1 + \sqrt{17}), \ \ \textit{and} \ \ 0 < \lambda \le 1, \\ &\le \frac{(3 + \alpha)\lambda^2}{2(1 + \alpha)^2} \quad \textit{when} \ \ 0 \le \alpha \le \frac{1}{2}(1 + \sqrt{17}), \ \ \textit{and} \ \ \frac{2(1 + \alpha)^2}{(2 + \alpha)(3 + \alpha)} < \lambda \le 1. \end{aligned}$$

All the inequalities are sharp.

Proof. Substituting eq. (7) into eq. (9) gives

$$\begin{aligned} 2(1+\alpha)A_2 &= -\lambda c_1, \\ 2(2+\alpha)A_3 &= -\lambda\left(c_2 - \frac{2(1+\alpha)^2 + (2+\alpha)(3+\alpha)\lambda}{4(1+\alpha)^2}c_1^2\right). \end{aligned} \tag{10}$$

The inequality for $|A_2|$ is trivial, since $|c_1| \le 2$.

For A_3, we use Lemma 1.2 with

$$\mu = \frac{2(1+\alpha)^2 + (2+\alpha)(3+\alpha)\lambda}{2(1+\alpha)^2},$$

and the inequalities for $|A_3|$ easily follow. The inequality for $|A_2|$ is sharp when $c_1 = 2$. The first and second inequalities for $|A_3|$ are sharp on choosing $c_1 = 0$ and $c_2 = 2$. The third inequality for $|A_3|$ is sharp when $c_1 = c_2 = 2$. □

When $\lambda = 1$, obtaining sharp bounds for $|A_4|$ follows relatively easily from an application of Lemmas 1.3 and 1.4 [11]. However, finding sharp bounds when $0 < \lambda \le 1$ appears to be a much more difficult problem, as the next theorem demonstrates.

The inequalities for $|A_4|$ for $f \in \mathcal{B}_1(\alpha, \lambda)$ are complicated, and in the interest of brevity, we omit many of the detailed calculations. Also, to simplify the analysis and presentation of the results, we define $\Gamma_i(\alpha)$ for $i = 1, 2, 3$ as follows:

$$\begin{aligned} \Gamma_1(\alpha) &= \sqrt{\frac{3(1+\alpha)^3}{(2+\alpha)(3+\alpha)(4+\alpha)}}, \\ \Gamma_2(\alpha) &= \frac{2(1+\alpha)(2+\alpha)}{(3+\alpha)(4+\alpha)}, \\ \Gamma_3(\alpha) &= \sqrt{\frac{6(1+\alpha)^3(2+\alpha)^2}{(3+\alpha)(4+\alpha)(20+33\alpha+12\alpha^2+\alpha^3)}}. \end{aligned}$$

We also denote the positive real root of the equation $21 + 17\alpha - 2\alpha^3 = 0$ by $\alpha_1^* = 3.40366\ldots$, $\alpha_2^* = \frac{1}{2}(1+\sqrt{33}) = 3.37228\ldots$, and the positive real root of the equation $4 + 9\alpha - \alpha^3 = 0$ by $\alpha_3^* = 3.20147\ldots$.

Theorem 3.2. *If $f \in \mathcal{B}_1(\alpha, \lambda)$, with inverse coefficients given by eq.* (8), *then*

$$|A_4| \le \frac{\lambda}{3+\alpha}, \tag{11}$$

when either (i) $\alpha > \alpha_1^*$ and $0 < \lambda \le 1$; (ii) $\alpha_2^* \le \alpha \le \alpha_1^*$ and $0 < \lambda \le \Gamma_1(\alpha)$; (iii) $\alpha_3^* < \alpha < \alpha_2^*$ and $0 < \lambda \le \Gamma_1(\alpha)$; or (iv) $0 \le \alpha \le \alpha_3^*$, and $0 < \lambda \le \Gamma_3(\alpha)$.

Also,

$$|A_4| \le \frac{(2+\alpha)(4+\alpha)\lambda^3}{3(1+\alpha)^3}, \tag{12}$$

when either (v) $\alpha_2^* \le \alpha \le \alpha_1^*$ and $\Gamma_1(\alpha) \le \lambda \le 1$; (vi) $\alpha_3^* < \alpha < \alpha_2^*$ and $\Gamma_1(\alpha) < \lambda \le \Gamma_2(\alpha)$; or (vii) $\alpha_3^* < \alpha < \alpha_2^*$ and $\Gamma_2(\alpha) < \lambda \le 1$.

All the inequalities are sharp.

We note that using the aforementioned lemmas, Theorem 3.2 proves that sharp inequalities for $|A_4|$ are established for $\alpha \geq 0$ and $0 < \lambda \leq 1$, apart from the intervals $0 \leq \alpha \leq \alpha_3^\star$ and $\Gamma_3(\alpha) < \lambda \leq 1$, where the methods used fail.

Proof. Again on substituting eq. (7) in eq. (9) we have

$$A_4 = -\frac{\lambda}{2(3+\alpha)}\Bigg(c_3 - \frac{(4+6\alpha+2\alpha^2+12\lambda+7\alpha\lambda+\alpha^2\lambda)}{2(1+\alpha)(2+\alpha)}c_1c_2 + \frac{c_1^3}{12(1+\alpha)^3(2+\alpha)}(6+21\alpha+27\alpha^2+15\alpha^3 \\ + 3\alpha^4+36\lambda+93\alpha\lambda+81\alpha^2\lambda+27\alpha^3\lambda+3\alpha^4\lambda+48\lambda^2+76\alpha\lambda^2+44\alpha^2\lambda^2+11\alpha^3\lambda^2+\alpha^4\lambda^2)\Bigg). \tag{13}$$

To find the maximum of the modulus of eq. (13), we first use Lemmas 1.3 and 1.4.

Let

$$B = \frac{(4+6\alpha+2\alpha^2+12\lambda+7\alpha\lambda+\alpha^2\lambda)}{4(1+\alpha)(2+\alpha)},$$

and

$$D = \frac{1}{12(1+\alpha)^3(2+\alpha)}(6+21\alpha+27\alpha^2+15\alpha^3+3\alpha^4+36\lambda+93\alpha\lambda+81\alpha^2\lambda+27\alpha^3\lambda+3\alpha^4\lambda+48\lambda^2+76\alpha\lambda^2 \\ +44\alpha^2\lambda^2+11\alpha^3\lambda^2+\alpha^4\lambda^2)$$

To see that eq. (11) holds in cases (i)–(iv), we use Lemma 1.4, noting that a long computation shows that both $0 \leq B \leq 1$ and $B(2B-1) \leq D \leq B$ are valid in all cases. This proves inequality (11).

For inequality (12), we write eq. (13) as

$$\frac{-\lambda}{2(3+\alpha)}(c_3 - 2Bc_1c_2 + Bc_1^3) = \frac{-\lambda}{2(3+\alpha)}(c_3 - 2Bc_1c_2 + Bc_1^3 + (D-B)c_1^3). \tag{14}$$

In this case, we can therefore apply Lemma 1.4, provided that both $0 \leq B \leq 1$ and $D - B \geq 0$ are valid, and again a long computation shows that these inequalities are valid in cases (v) and (vi).

A simple calculation shows that

$$D - B = \frac{1}{12}\left(\frac{(2+\alpha)(3+\alpha)(4+\alpha)\lambda^2}{(1+\alpha)^3} - 3\right),$$

and so we obtain from Lemma 1.4 and the inequality $|c_1| \leq 2$ that

$$|A_4| \leq \frac{\lambda}{2(3+\alpha)}\left(2 + \frac{2}{3}\left(\frac{(2+\alpha)(3+\alpha)(4+\alpha)\lambda^2}{(1+\alpha)^3} - 3\right)\right) = \frac{(2+\alpha)(4+\alpha)\lambda^3}{3(1+\alpha)^3}.$$

We are therefore left to prove eq. (12) in case (vii), where we use Lemma 1.3, with

$$\mu = \frac{(3+\alpha)(4+\alpha)\lambda}{2(1+\alpha)(2+\alpha)}.$$

Then, $\mu > 1$ when $0 \leq \alpha < \alpha_2^\star$, and $\Gamma_2(\alpha) < \lambda \leq 1$, and $D - \mu \geq 0$ when $\alpha_3^\star \leq \alpha < \alpha_2^\star$ and $\Gamma_2(\alpha) < \lambda \leq 1$, and so both inequalities are satisfied when $\alpha_3^\star \leq \alpha < \alpha_2^\star$ and $\Gamma_2(\alpha) < \lambda \leq 1$.

Thus, writing

$$A_4 = -\frac{\lambda}{2(3+\alpha)}(c_3 - (\mu+1)c_1c_2 + \mu c_1^3 + (D-\mu)c_1^3),$$

Lemma 1.3, and the inequality $|c_1| \le 2$, gives

$$|A_4| \le \frac{\lambda}{2(3+\alpha)}(2|2\mu - 1| + 8(D-\mu)) = \frac{(2+\alpha)(4+\alpha)\lambda^3}{3(1+\alpha)^3},$$

provided $\alpha_3^\star \le \alpha < \alpha_2^\star$ and $\Gamma_2(\alpha) < \lambda \le 1$, which gives inequality (12) in case (vii). □

4 The logarithmic coefficients

The logarithmic coefficients γ_n of f are defined in $\mathbb{D}$ by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z_n. \tag{15}$$

Differentiating eq. (15) and equating coefficients give

$$\begin{aligned} \gamma_1 &= \frac{1}{2}a_2, \\ \gamma_2 &= \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \\ \gamma_3 &= \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right), \\ \gamma_4 &= \frac{1}{8}(-a_2^4 + 4a_2^2a_3 - 2a_3^2 - 4a_2a_4 + 4a_5). \end{aligned} \tag{16}$$

Using the same techniques as in the proof of Theorem 2.1, it is possible to prove the following (proofs can be found in [21]).

Theorem 4.1. *Let $f \in \mathcal{B}_1(\alpha, \lambda)$ for $\alpha \ge 0$ and $0 < \lambda \le 1$ with logarithmic coefficients given by eq.* (16)*. Then, for* $1 \le n \le 4$,

$$|\gamma_n| \le \frac{\lambda}{2(n+\alpha)}. \tag{17}$$

All the inequalities are sharp.

5 The second Hankel determinant

The qth Hankel determinant of f is defined for $q \ge 1$ and $n \ge 1$ as follows and has been extensively studied (see e.g. [22–25])

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1}\cdots & a_{n+q+1} \\ a_{n+1} & \cdots & \vdots \\ \vdots & & \\ a_{n+q-1} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

We prove the following, noting that the result is valid for $\alpha \geq 0$.

Theorem 5.1. *If $f \in \mathcal{B}_1(\alpha, \lambda)$, then for $\alpha \geq 0$, and $0 < \lambda \geq 1$,*

$$H_2(2) = |a_2a_4 - a_3^2| \leq \frac{\lambda^2}{(2+\alpha)^2}.$$

The inequality is sharp.

Proof. We use the idea first developed in [23].

Equating coefficients in eq. (7) gives

$$H_2(2) = \frac{c_1^4\lambda^2}{16(1+\alpha)(2+\alpha)^2(3+\alpha)} - \frac{c_1^2c_2\lambda^2}{4(1+\alpha)(2+\alpha)^2(3+\alpha)} - \frac{c_2^2\lambda^2}{4(2+\alpha)^2} + \frac{c_1c_3\lambda^2}{4(1+\alpha)(3+\alpha)} - \frac{(1-\alpha)c_1^4\lambda^4}{192(1+\alpha).^3} \tag{18}$$

Next applying Lemma 1.1, noting that $H_2(2)$ is rotationally invariant, and again writing $c_1 := c$, so that $0 \leq c \leq 2$, it follows that

$$H_2(2) = -\frac{c^2(4-c_\star^2)\zeta^2\lambda^2}{16(1+\alpha)(3+\alpha)} - \frac{(4-c^2)^2\zeta^2\lambda^2}{16(2+\alpha)^2} - \frac{(1-\alpha)c^4\lambda^4}{192(1+\alpha)^3} + \frac{c(4-c^2)\lambda^2(1-|\zeta|^2)\eta}{8(1+\alpha)(3+\alpha)}. \tag{19}$$

Taking the modulus in eq. (19), and noting that $|\eta| \leq 1$, gives

$$H_2(2) \leq \frac{c^2(4-c^2)\,|\zeta|^2\lambda^2}{16(1+\alpha)(3+\alpha)} + \frac{(4-c^2)^2|\zeta|^2\lambda^2}{16(2+\alpha)^2} + \frac{|1-\alpha|c^4\lambda^4}{192(1+\alpha)^3} + \frac{c(4-c^2)\lambda^2(1-|\zeta|^2)}{8(1+\alpha)(3+\alpha)} := \psi(\alpha, \lambda, |\zeta|, c). \tag{20}$$

Since the derivative of $\psi(\alpha, \lambda, |\zeta|, c)$ with respect to $|\zeta|$ is positive, we deduce from eq. (20) that

$$H_2(2) \leq \frac{c^2(4-c^2)\lambda^2}{16(1+\alpha)(3+\alpha)} + \frac{(4-c^2)^2\lambda^2}{16(2+\alpha)^2} + \frac{|1-\alpha|c^4\lambda^4}{192(1+\alpha)^3} := \psi_1(\alpha, \lambda, c). \tag{21}$$

Thus, we must find the maximum value of $\psi_1(\alpha, \lambda, c)$, when $0 \leq c \leq 2$.

Elementary calculus shows that $\psi'_1(\alpha, \lambda, c) = 0$ has three roots in $0 \leq c \leq 2$, but the only valid root is at $c = 0$.

Since $\psi_1(\alpha, \lambda, 0) = \frac{\lambda^2}{(2+\alpha)^2}$ and $\psi_1(\alpha, \lambda, 2) = \frac{\lambda^4|1-\alpha|}{12(1+\alpha)^3}$, the proof of the theorem is complete on noting that $\psi_1(\alpha, \lambda, 0) \geq \psi_1(\alpha, \lambda, 2)$ when $\alpha \geq 0$ and $0 < \lambda \leq 1$.

The inequality is sharp on choosing $c_1 = 0$ and $c_2 = c_3 = 2$ in eq. (18). □

6 A Fekete–Szegö theorem

We finally give a sharp Fekete–Szegö inequality for $B_1(\alpha,\lambda)$ omitting the proof, which is a straightforward application of Lemma 1.2.

Theorem 6.1. *Let $f \in \mathcal{B}_1(\alpha, \lambda)$. Then, for $\alpha \geq 0$, $0 < \lambda \leq 1$, and $\nu \in \mathbb{R}$,*

$$|a_3 - \nu a_2^2| \leq \begin{cases} -\dfrac{\lambda^2(\alpha - 1 + 2\nu)}{2(1+\alpha)^2}, & \text{if } \nu \leq -\dfrac{5\alpha + 3\alpha^2}{4 + 2\alpha}, \\ \dfrac{\lambda}{2+\alpha}, & \text{if } -\dfrac{5\alpha + 3\alpha^2}{4 + 2\alpha} \leq \nu \leq \dfrac{4 + 3\alpha + \alpha^2}{4 + 2\alpha}, \\ \dfrac{\lambda^2(\alpha - 1 + 2\nu)}{2(1+\alpha)^2}, & \text{if } \nu \geq \dfrac{4 + 3\alpha + \alpha^2}{4 + 2\alpha}. \end{cases}$$

All the inequalities are sharp.

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