Averaging principle for a type of Caputo fractional stochastic differential equations

Zhongkai Guo[a),], **Junhao Hu**[a)], **Chenggui Yuan**[b)]

[a)]School of Mathematics and Statistics

South-Central University for Nationalities, Wuhan, 430074, China

[b)]Department of Mathematics, Swansea University, Bay campus, SA1 8EN, UK

zkguo@scuec.edu.cn, junhaohu74@163.com, c.yuan@swansea.ac.uk

Abstract

Averaging principle for Caputo fractional stochastic differential equations attracted much attention recently. In this paper, we investigate the averaging principle for a type of Caputo fractional stochastic differential equations. Comparing with the existing literature, we shall use different estimate methods to investigate the averaging principle, which will enrich the development of the theory for Caputo fractional stochastic differential equations.

Keywords: Caputo fractional; Stochastic differential equations; Averaging principle

1 Introduction

The averaging principle is an important method to study complex systems, which has a long history. For ordinary differential equations, the research was originally given by authors such as Bogoliubov [1], Gikhman [2], Volosov [3] and Besjes [4]. And then, Khasminskii developed this method to deal with a class of second order parabolic partial differential equations, see [5]. After that, lots of work concern about the averaging principle for partial differential equations. The averaging principle for stochastic differential equations was firstly considered by Khasminskii in [6]. Since then, many efforts have been devoted to developing this theory for the stochastic system. Here we only highlight [7, 8, 9, 10, 11, 12, 13, 14] and references therein.

On the other hand, because of the memory effect of the fractional derivatives in time, stochastic fractional modeling has come to play an important role in many branches of science and industry where more and more people are concerned about the research on the Caputo stochastic differential equation. Until now, many efforts have been devoted to the existence and uniqueness solution for Caputo stochastic differential equation, see [15, 16, 17] and ect.. However, only a few results have been made by the dynamic approach. For example, in [18], after study the existence and uniqueness of solutions for Caputo fractional differential equations under a weak continuity condition on nonlinear terms, the authors considered the asymptotic behavior of the solution and the existence of attracting set. In [19], under a different framework, when the existence and uniqueness of solutions are established, the asymptotic distance between two distinct solutions is discussed. To the best of our knowledge, the averaging principle for fractional differential equations still has a big challenge, there are only a few papers [12, 20] to investigate the averaging principle for Caputo fractional SDEs. It is worth noting that the averaging principle has been obtained in these two papers by using similar methods but different assumptions.

In this paper, we shall develop a new method and investigate the averaging principle for the following Caputo fractional SDE:

$$\begin{cases} D_t^\alpha u^\epsilon(t) = f(u^\epsilon(t), \frac{t}{\epsilon})dt + g(u^\epsilon(t))dB_t, \quad 0 < t \le T, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

with $\frac{1}{2} < \alpha \le 1$ under the averaging condition

$$\sup_{t\ge 0} \frac{1}{T^{2\alpha-1}} \int_t^{t+T} \|f(u,s) - \overline{f}(u)\|^2 ds \le \kappa(T)(1 + \|u\|^2), \tag{1.2}$$

where $\overline{f}(\cdot)$ is the averaging function of $f(\cdot)$ and $\kappa(\cdot)$ can be seen as a convergence rate function between $f(\cdot)$ and $\overline{f}(\cdot)$ which is a positive bounded function with $\lim_{T\to\infty} \kappa(T) = 0$.

If we let $\alpha = 1$ in equation (1.1) and (1.2), then the equation (1.1) to be a classic SDEs and the condition (1.2) is consistent with the classic averaging condition for SDEs, the averaging principle for such SDEs have been considered by many authors with similar methods, see [21, 22]. It is worth pointing out that the condition $\lim_{T\to\infty} \kappa(T) = 0$ has been imposed in many papers, but it has not been used in the proofs, for details see [7, 23, 24]. Recently, removing the condition $\lim_{T\to\infty} \kappa(T) = 0$, authors[25] established averaging principle for SDEs of neutral type by similar methods as in [7, 23, 24]. However, the assumption of Lipschitz and linear growth condition for $\overline{f}$ still imposed as the other papers, which we will show that this is not necessary, see remark 2.1. In this paper, using a different approach, without assuming that $\overline{f}$ satisfies Lipschitz and linear growth condition, we derive the averaging principle for a type of Caputo fractional SDEs, the treatment and result in our article fully reflects the importance of the condition $\lim_{T\to\infty} \kappa(T) = 0$. For more detail, see section 2 and 3.

The paper is organized as follows. Some assumptions and basic results are first recalled in Section 2. The solution of convergence in the mean square between (1.1) and the corresponding averaged equation are discussed in Section 3. In Section 4, we will give an example to illustrate our theory. The conclusion is given in Section 5.

Throughout this paper, the letter C is just denoted a positive constant. If the constants are related to certain parameters, we will mark them specifically.

2 Framework and Preliminaries

In this paper, we denote the norm of R^d by $\|\cdot\|$. To obtain the averaging averaging in this paper, we introduce the following hypotheses.

(H1) (Lipschitz condition). For

$$f : R^+ \times R^d \to R^d,\ g : R^d \to R^d,$$

there exists a constant $l_1 > 0$ such that, for every $x, y \in R^d$,

$$\|f(t,x) - f(t,y)\|^2 + \|g(x) - g(y)\|^2 \le l_1\|x-y\|^2.$$

(H2) (Linear growth condition). For

$$f : R^+ \times R^d \to R^d,\ g : R^d \to R^d,$$

there exists a constant $l_2 > 0$ such that, for every $x \in R^d$,

$$\|f(t,x)\|^2 + \|g(x)\|^2 \le l_2(1 + \|x\|^2).$$

(H3) (Averaging condition). For the nonlinear function f, there exists a corresponding averaging function $\overline{f}$ and a convergence rate function κ such that the following averaging condition holds

$$\sup_{t\ge 0} \frac{1}{T^{2\alpha-1}} \int_t^{t+T} \|f(u,s) - \overline{f}(u)\|^2 ds \le \kappa(T)(1 + \|u\|^2), \tag{2.1}$$

where $\kappa(\cdot)$ is positive bounded function with $\lim_{T\to\infty} \kappa(T) = 0$.

Remark 2.1. For every $x, y \in R^d$, and every $T > 0$, Using the relationship between f and $\overline{f}$ in conditions (**H1**) – (**H3**), we can show that $\overline{f}$ also satisfies the Lipschitz condition and the linear growth condition as f.

- Lipschitz condition for $\overline{f}$:

$$\begin{aligned} &\|\overline{f}(x) - \overline{f}(y)\| \\ \le\ & \left\|\frac{1}{T}\int_0^T [f(x,s) - \overline{f}(x)]ds\right\| + \left\|\frac{1}{T}\int_0^T [f(y,s) - \overline{f}(y)]ds\right\| + \left\|\frac{1}{T}\int_0^T [f(x,s) - f(y,s)]ds\right\| \\ \le\ & \frac{\sqrt{\kappa(T)}}{T^{1-\alpha}}(\sqrt{1+\|x\|^2} + \sqrt{1+\|y\|^2}) + \sqrt{l_1}\|x-y\|, \end{aligned} \tag{2.2}$$

  note that $\frac{1}{2} < \alpha \le 1$ and $\lim_{T\to\infty} \kappa(T) = 0$, which show that $\overline{f}$ satisfies the Lipschitz condition as f.

- Linear growth condition for $\overline{f}$:

$$\begin{aligned} \|\overline{f}(x)\| &\le \left\|\frac{1}{T}\int_0^T [f(x,s) - \overline{f}(x)]ds\right\| + \|\frac{1}{T}\int_0^T f(x,s)ds\| \\ &\le \frac{\sqrt{\kappa(T)}}{T^{1-\alpha}}(\sqrt{1+\|x\|^2}) + \sqrt{l_1}\sqrt{1+\|x\|^2}, \end{aligned} \tag{2.3}$$

  note that $\frac{1}{2} < \alpha \le 1$ and $\lim_{T\to\infty} \kappa(T) = 0$, which show that $\overline{f}$ satisfies the growth condition as f.

By the discussion above, we can see that the Lipschitz and linear growth conditions on $\overline{f}$ need not be assumed as some existing literature, which simplifies the assumption on $\overline{f}$.

In this paper, we focus on the averaging principle of equation (1.1). The existence and uniqueness of solutions to equation (1.1) have been considered by many authors under conditions (**H1**) and (**H2**), details see the papers[26, 27] and ect..

In the following, we will prepare an inequality to be used in the next section, which can be thought of as a generalization of Gronwall's inequality for singular kernels, see [28].

Lemma 2.1. [28] Suppose $b \geq 0$, $\beta > 0$ and $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$ (some $T \leq +\infty$), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with

$$u(t) \leq a(t) + b\int_0^t (t-s)^{\beta-1}u(s)ds.$$

Then

$$u(t) \leq a(t) + \int_0^t \Big[\sum_{n=1}^{\infty} \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)\Big]ds, \quad 0 \leq t < T,$$

where $\Gamma(\cdot)$ is the Gamma function.

In the following, we cite two lemmas for future use.

Lemma 2.2. [26] Under conditions (**H1**) and (**H2**), for every $x_0 \in L^2(\Omega, H)$, equation (1.1) has a unique solution X_t such that in $X_t \in C([0,T]; L^2(\Omega, H))$ and

$$\sup_{0\leq t\leq T} E\|X_t\|^2 \leq C(l_1, x_0, T, \alpha).$$

Lemma 2.3. [26] Assume that the condition (**H2**) holds. If $u(t)$ is the solution of equation (1.1), then

$$E\|u(t)-u(s)\|^2 \leq C(l_2, T, u_0, \alpha)(t-s)^{2\alpha-1}. \tag{2.4}$$

3 An averaging principle

We now study an averaging principle for the following Caputo fractional stochastic integral equations (SIEs) in R^d:

$$u^\epsilon(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(u^\epsilon(s), \frac{s}{\epsilon})ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(u^\epsilon(s))dB(s), \tag{3.1}$$

where u_0 is a random vector, $B(t)$ is a one dimensional Brownian motion and $\epsilon \in (0, \epsilon_0]$ is a positive small parameter with ϵ_0 a fixed number.

By Remark 2.1 , $\overline{f}$ is also satisfy the Lipschitz and linear growth conditions. Thus, the existence and uniqueness solution for the following Caputo fractional SIEs is still guaranteed.

$$\overline{u}(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{f}(\overline{u}(s))ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(\overline{u}(s))dB(s). \tag{3.2}$$

Now, we turn to the main objectives of this paper, we will prove that the solution of equation (3.1) will converge to the solution of (3.2) in the mean square sense as $\epsilon \to 0$, which presents as follows:

Theorem 3.1. Assume that the Lipschitz condition (**H1**) and the linear growth condition (**H2**) hold. Let $u^\epsilon(t)$ be the unique solution of equation (3.1) and $\overline{u}(t)$ be the unique solution of equation (3.2). Together with the condition (**H3**), then

$$\lim_{\epsilon\to 0} \sup_{0\le t\le T} E\|u^\epsilon(t) - \overline{u}(t)\|^2 = 0. \tag{3.3}$$

In order to prove the Theorem 3.1, we first consider the following lemma.

Lemma 3.1. Assume conditions (**H1**)-(**H3**) hold and $E\|u_0\|^2 < +\infty$, for $\frac{1}{2} < \alpha \le 1$, one has

$$\lim_{\epsilon\to 0} \sup_{0\le t\le T} E\Big\| \int_0^t (t-s)^{\alpha-1} [f(u^\epsilon(s), \frac{s}{\epsilon}) - \overline{f}(u^\epsilon(s))]ds \Big\|^2 = 0. \tag{3.4}$$

Proof. Let

$$b_i = i\sqrt{\epsilon},\ 0 \le i \le N-1,\ 0 < T - b_{N-1} \le \sqrt{\epsilon},\ b_N = T,$$

be a partition of $[0,T]$. It is easy to see that $T \le N\sqrt{\epsilon} < T + \sqrt{\epsilon}$.
Define

$$X_i = \int_{b_i}^{b_{i+1}} (t-s)^{\alpha-1} [f(u^\epsilon(s), \frac{s}{\epsilon}) - \overline{f}(u^\epsilon(s))]ds = \int_{b_i}^{b_{i+1}} (t-s)^{\alpha-1} I(u^\epsilon(s), \frac{s}{\epsilon})ds,$$

it follows that,

$$\begin{aligned} &\Big\| \int_0^t (t-s)^{\alpha-1} [f(u^\epsilon(s), \frac{s}{\epsilon}) - \overline{f}(u^\epsilon(s))]ds \Big\|^2 \\ \le\ & N\Big\| \int_{[\frac{t}{\sqrt{\epsilon}}]\sqrt{\epsilon}}^t (t-s)^{\alpha-1} [f(u^\epsilon(s), \frac{s}{\epsilon}) - \overline{f}(u^\epsilon(s))]ds \Big\|^2 + N\sum_{i=0}^{N-2} \|X_i\|^2. \end{aligned}$$

By the condition (**H2**) and Remark 2.1,

$$E\Big\| \int_{[\frac{t}{\sqrt{\epsilon}}]\sqrt{\epsilon}}^t (t-s)^{\alpha-1} [f(u^\epsilon(s), \frac{s}{\epsilon}) - \overline{f}(u^\epsilon(s))]ds \Big\|^2$$

$$\leq C\int_{[\frac{t}{\sqrt{\epsilon}}]\sqrt{\epsilon}}^{t}(t-s)^{2\alpha-2}ds\int_{[\frac{t}{\sqrt{\epsilon}}]\sqrt{\epsilon}}^{t}(1+E\|u^{\epsilon}(s)\|^2)ds$$

$$\leq C(\alpha)|t-[\frac{t}{\sqrt{\epsilon}}]\sqrt{\epsilon}|^{2\alpha}(1+\sup_{0\leq t\leq T}E\|u^{\epsilon}(t)\|^2)$$

$$\leq C(\alpha)\epsilon^{\alpha}(1+\sup_{0\leq t\leq T}E\|u^{\epsilon}(t)\|^2). \tag{3.5}$$

Then, since $E\|u_0\|^2<+\infty$, by Lemmas 2.2 and (3.5), it follows that

$$E\Big\|\int_0^t(t-s)^{\alpha-1}[f(u^{\epsilon}(s),\frac{s}{\epsilon})-\overline{f}(u^{\epsilon}(s))]ds\Big\|^2$$

$$\leq C(\alpha)\epsilon^{\alpha}N+NE\sum_{i=0}^{N-2}\|X_i\|^2$$

$$\leq C(\alpha)\epsilon^{\alpha-\frac{1}{2}}(T+\sqrt{\epsilon})+NE\sum_{i=0}^{N-2}\|X_i\|^2. \tag{3.6}$$

Thanks to condition (**H1**), (**H3**) and Remark 2.1, we can verify that

$$\begin{aligned}\|X_i\|^2 &= \Big\|\int_{b_i}^{b_{i+1}}(t-s)^{\alpha-1}[f(u^{\epsilon}(s),\frac{s}{\epsilon})-\overline{f}(u^{\epsilon}(s))]ds\Big\|^2\\
&\leq 3\Big\|\int_{b_i}^{b_{i+1}}(t-s)^{\alpha-1}[f(u^{\epsilon}(b_i),\frac{s}{\epsilon})-\overline{f}(u^{\epsilon}(b_i)]ds\Big\|^2\\
&\quad+3\Big\|\int_{b_i}^{b_{i+1}}(t-s)^{\alpha-1}[f(u^{\epsilon}(s),\frac{s}{\epsilon})-f(u^{\epsilon}(b_i),\frac{s}{\epsilon})]ds\Big\|^2\\
&\quad+3\Big\|\int_{b_i}^{b_{i+1}}(t-s)^{\alpha-1}[\overline{f}(u^{\epsilon}(s))-\overline{f}(u^{\epsilon}(b_i)]ds\Big\|^2\\
&\leq 3\Big|\int_{b_i}^{b_{i+1}}(t-s)^{2\alpha-2}ds\Big|\int_{b_i}^{b_{i+1}}\|f(u^{\epsilon}(b_i),\frac{s}{\epsilon})-\overline{f}(u^{\epsilon}(b_i)\|^2ds\\
&\quad+3\Big|\int_{b_i}^{b_{i+1}}(t-s)^{2\alpha-2}ds\Big|\int_{b_i}^{b_{i+1}}\|f(u^{\epsilon}(s),\frac{s}{\epsilon})-f(u^{\epsilon}(b_i),\frac{s}{\epsilon})\|^2ds\\
&\quad+3\Big|\int_{b_i}^{b_{i+1}}(t-s)^{2\alpha-2}ds\Big|\int_{b_i}^{b_{i+1}}\|\overline{f}(u^{\epsilon}(s))-\overline{f}(u^{\epsilon}(b_i)\|^2ds\\
&\leq C(\alpha)\Big|(t-b_i)^{2\alpha-1}-(t-b_{i+1})^{2\alpha-1}\Big|\,\epsilon\int_{\frac{b_i}{\epsilon}}^{\frac{b_{i+1}}{\epsilon}}\|f(u^{\epsilon}(b_i),s)-\overline{f}(u^{\epsilon}(b_i)\|^2ds\\
&\quad+C(\alpha,l_1)\Big|(t-b_i)^{2\alpha-1}-(t-b_{i+1})^{2\alpha-1}\Big|\int_{b_i}^{b_{i+1}}\|u^{\epsilon}(s)-u^{\epsilon}(b_i)\|^2ds. \qquad (3.7)\end{aligned}$$

Noting that for $0<\beta<1$ and $0<a\leq b\leq T$,

$$\Big|b^{\beta}-a^{\beta}\Big|\leq(b-a)^{\beta}.$$

Then, if $\frac{1}{2} < \alpha \leq 1$, we can state that

$$\Big|(t-b_i)^{2\alpha-1} - (t-b_{i+1})^{2\alpha-1}\Big| \leq (b_{i+1}-b_i)^{2\alpha-1}.$$

This, together with (3.7), yields that

$$\begin{aligned}
\|X_i\|^2 &\leq C(\alpha,T)(\sqrt{\epsilon})^{2\alpha-1}\epsilon\int_{\frac{b_i}{\epsilon}}^{\frac{b_{i+1}}{\epsilon}} \|f(u^\epsilon(b_i),s) - \overline{f}(u^\epsilon(b_i))\|^2 ds \\
&\quad + C(\alpha,l_1,T)\sqrt{\epsilon}\int_{b_i}^{b_{i+1}} \|u^\epsilon(s) - u^\epsilon(b_i)\|^2 ds \\
&\leq C(\alpha,T)\epsilon\kappa(\frac{1}{\sqrt{\epsilon}})(1+\|u^\epsilon(b_i)\|^2) + C(\alpha,l_1,T)\sqrt{\epsilon}\int_{b_i}^{b_{i+1}} \|u^\epsilon(s) - u^\epsilon(b_i)\|^2 ds.
\end{aligned}$$

On account of Lemma 2.3, we conclude that

$$\begin{aligned}
N\sum_{i=0}^{N-2} E\|X_i\|^2 &\leq C(\alpha,T)\epsilon N\sum_{i=0}^{N-2} E[\kappa(\frac{1}{\sqrt{\epsilon}})(1+\|u^\epsilon(b_i)\|^2)] \\
&\quad + C(\alpha,l_1,T)\sqrt{\epsilon}N\sum_{i=0}^{N-2} E\int_{b_i}^{b_{i+1}} \|u^\epsilon(s) - u^\epsilon(b_i)\|^2 ds \\
&\leq C(\alpha,l_1,T)\epsilon N^2[\kappa(\frac{1}{\sqrt{\epsilon}}) + \epsilon^{\alpha-\frac{1}{2}}] \\
&\leq C(\alpha,l_1,T)(T+\sqrt{\epsilon})^2[\kappa(\frac{1}{\sqrt{\epsilon}}) + \epsilon^{\alpha-\frac{1}{2}}], \qquad (3.8)
\end{aligned}$$

we here used Lemma 2.3 in the second inequality.

Substituting (3.8) into (3.6), we obtain

$$\begin{aligned}
&\sup_{0\leq t\leq T} E\Big\|\int_0^t (t-s)^{\alpha-1}[f(u^\epsilon(s),\frac{s}{\epsilon}) - \overline{f}(u^\epsilon(s))]ds\Big\|^2 \\
&\leq C(\alpha)\epsilon^{\alpha-\frac{1}{2}}(T+\sqrt{\epsilon}) + C(\alpha,l_1,T)(T+\sqrt{\epsilon})^2[\kappa(\frac{1}{\sqrt{\epsilon}}) + \epsilon^{\alpha-\frac{1}{2}}] \\
&\leq C(\alpha,l_1,T)[\kappa(\frac{1}{\sqrt{\epsilon}}) + \epsilon^{\alpha-\frac{1}{2}}]. \qquad (3.9)
\end{aligned}$$

The conclusion follows from (3.9) by letting ϵ tend to zero. □

Now, we shall prove the main result of this paper.

Proof of Theorem 3.1 : Using the following elementary inequality

$$\|a+b+c\|^2 \leq 3\|a\|^2 + 3\|b\|^2 + 3\|c\|^2,$$

we have

$$E\|u^\epsilon(t) - \overline{u}(t)\|^2 \leq \frac{3}{\Gamma(\alpha)^2}E\Big\|\int_0^t (t-s)^{\alpha-1}[f(u^\epsilon(s),\frac{s}{\epsilon}) - \overline{f}(u^\epsilon(s))]ds\Big\|^2$$

$$
\begin{aligned}
&+\frac{3}{\Gamma(\alpha)^2}E\Big\|\int_0^t (t-s)^{\alpha-1}[\overline{f}(u^\epsilon(s))-\overline{f}(\overline{u}(s))]ds\Big\|^2\\
&+\frac{3}{\Gamma(\alpha)^2}E\Big\|\int_0^t (t-s)^{\alpha-1}[g(u^\epsilon(s))-g(\overline{u}(s))]dB(s)\Big\|^2.
\end{aligned}
$$

Applying the Hölder inequality, the Itô isometry inequality and the condition (**H1**), one can see that

$$
\begin{aligned}
& E\|u^\epsilon(t)-\overline{u}(t)\|^2\\
\leq\ & \frac{3}{\Gamma(\alpha)^2}E\Big\|\int_0^t (t-s)^{\alpha-1}[f(u^\epsilon(s),\frac{s}{\epsilon})-\overline{f}(u^\epsilon(s))]ds\Big\|^2\\
&+\frac{3Tl_1}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2\alpha-2}E\|u^\epsilon(s)-\overline{u}(s)\|^2ds+\frac{12l_1}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2\alpha-2}E\|u^\epsilon(s)-\overline{u}(s)\|^2ds\\
=\ & \frac{3}{\Gamma(\alpha)^2}E\Big\|\int_0^t (t-s)^{\alpha-1}[f(u^\epsilon(s),\frac{s}{\epsilon})-\overline{f}(u^\epsilon(s))]ds\Big\|^2\\
&+\frac{3Tl_1+12l_1}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2\alpha-2}E\|u^\epsilon(s)-\overline{u}(s)\|^2ds\\
\leq\ & C(\alpha,l_1,T)[\kappa(\frac{1}{\sqrt{\epsilon}})+\epsilon^{\alpha-\frac{1}{2}}]+\frac{3Tl_1+12l_1}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2\alpha-2}E\|u^\epsilon(s)-\overline{u}(s)\|^2ds.
\end{aligned}
$$

Thus, by Lemma 2.1 we obtain

$$
\begin{aligned}
& E\|u^\epsilon(t)-\overline{u}(t)\|^2\\
\leq\ & C(\alpha,l_1,T)[\kappa(\frac{1}{\sqrt{\epsilon}})+\epsilon^{\alpha-\frac{1}{2}}]\Big(1+\int_0^t\sum_{n=1}^{\infty}\frac{(\frac{3Tl_1+12l_1}{\Gamma(\alpha)^2}\Gamma(2\alpha-1))^n}{\Gamma(2n\alpha-n)}(t-s)^{n(2\alpha-1)-1}ds\Big)\\
\leq\ & C(\alpha,l_1,T)[\kappa(\frac{1}{\sqrt{\epsilon}})+\epsilon^{\alpha-\frac{1}{2}}]\Big(1+\sum_{n=1}^{\infty}\frac{(\frac{3Tl_1+12l_1}{\Gamma(\alpha)^2}\Gamma(2\alpha-1)T^{2\alpha-1})^n}{\Gamma(2n\alpha-n+1)}\Big)\\
=\ & C(\alpha,l_1,T)[\kappa(\frac{1}{\sqrt{\epsilon}})+\epsilon^{\alpha-\frac{1}{2}}]\Big(1+E_{2\alpha-1,1}(\frac{3Tl_1+12l_1}{\Gamma(\alpha)^2}\Gamma(2\alpha-1)T^{2\alpha-1})\Big).
\end{aligned}
$$

Finally, we get

$$
\sup_{0\leq t\leq T}E\|u^\epsilon(t)-\overline{u}(t)\|^2\leq C(\alpha,l_1,T)[\kappa(\frac{1}{\sqrt{\epsilon}})+\epsilon^{\alpha-\frac{1}{2}}]. \tag{3.10}
$$

This completes the proof.

Remark 3.1. By equation (3.10), we see that, the convergence rate relate to the convergence rate function $\kappa(\cdot)$ which is different from others paper results by the similar methods as [21, 22].

Remark 3.2. Using the Chebyshev inequality, we can also derive the converge result in the sense of convergence in probability.

4 Example

In this section, we will present one examples to illustrate our theory. Consider the following Caputo fractional SDEs

$$(4.1)\qquad \begin{cases} D_t^{\frac{3}{4}} u^\epsilon(t) = 4cos^2(\frac{t}{\epsilon})u^\epsilon(t)dt + u^\epsilon(t)dB_t, \quad 0 < t \le T, \\ u(0) = u_0. \end{cases}$$

Here $\alpha = \frac{3}{4}$, $E\|u_0\|^2 < +\infty$. and $f(u^\epsilon(t), \frac{t}{\epsilon}) = 4cos^2(\frac{t}{\epsilon})u^\epsilon(t)$, $g(u^\epsilon(t)) = u^\epsilon(t)$.
Let

$$\overline{f}(u) = \frac{1}{\pi}\int_0^\pi 4cos^2(t)udt = 2u,$$

considered the following averaged equation,

$$(4.2)\qquad D_t^{\frac{3}{4}} u(t) = 2u(t)dt + u(t)dB_t.$$

It is easy to check out that condition **H3** is satisfied and $k(T) = \frac{1}{T^{\frac{3}{4}}}$. By Theorem 3.1, we known that, the solution of (4.1) can be approximated by the solution of equation (4.2) in the sense of mean square and also probability. Moreover, the optimal convergence rate is $\frac{1}{4}$.

5 Conclusion

In this paper, an averaging principle for a type of Caputo fractional SDEs has been established. Under a different averaging condition, we derive an averaging principle for a type of Caputo fractional SDEs by a different estimate method. we prove that the solution of averaged Caputo fractional SDEs converge to that of the standard one in the sense of mean square and also in probability. Our results enriched the averaging principle for Caputo fractional stochastic differential equations.

Acknowledgements
The authors would like to thanks the editor and the referees for their suggestions and comments.

References

[1] N.N. Bogoliubov, Y.A. Mitropolsky, Asymptotic Methods in the Theory of Non-Linear Oscillations, *Gordon and Breach Science Publishers, New York*, 1961.

[2] I.I. Gikhman, On a theorem of N.N. Bogoliubov, *Ukrainian Math. J.* **4** (1952) 215-219.

[3] V.M. Volosov, Averaging in systems of ordinary differential equations, *Russian Math. Surveys* **17** (1962) 1-126.

[4] J.G. Besjes, On the asymptotic methods for non-linear differential equations, *J. Méc.* **8** (1969) 357-373.

[5] R.Z. Khasminskii, Principle of averaging of parabolic and elliptic differentials for Markov process with small diffusion, *Theory Probab. Appl.* **8** (1963) 1-21.

[6] R.Z. Khasminskii, On an averging principle for Itô stochastic differential equations, *Kibernetica*, **4** (1968) 260-279.

[7] Y. Xu, J. Duan and W. Xu, An averaging principle for stochastic dynamical sysytems with Lévy noise, *Physica D* **240** (2011) 1395-1401.

[8] P. Gao, Averaging principle for stochastic Korteweg-de Vries equation, *J. Differential Equations*, **267** (2019) 6872-6909.

[9] S. Cerrai, A Khasminkii type averaging principle for stochastic reaction-diffusion equations. *Ann. Appl. Probab.*, **19** (2009) 899-948.

[10] C. E. Bréhier, Strong and weak orders in averaging for SPDEs, *Stoch. Process. Appl.*, **122** (2012) 2553-2593.

[11] W. Xu, W. Xu, K. Lu, An averaging principle for stochastic differential equations of fractional order $0 < \alpha < 1$, *Fractional Calculus and Applied Analysis*, **23(3)** (2020) 908-919.

[12] W. Xu, J. Duan, W. Xu, An averaging principle for fractional stochastic differential equations with Lévy noise, *Chaos: An Interdisciplinary Journal of Nonlinear Science*, **30(8)** (2020) 083126.

[13] Z. Guo, G. Lv, J. Wei, Averaging principle for stochastic differential equations under a weak condition, *Chaos* **30** (2020) 123139.

[14] W. Mao, B. Chen, S. You, On the averaging principle for SDEs driven by G-Brownian motion with non-Lipschitz coefficients, *Advances in Difference Equation*, **71** (2021) 1-16.

[15] R. Sakthivel, P. Revathi, and Y. Ren, Existence of solutions for nonlinear fractional stochastic differential equations, Nonlinear Anal. Theory Methods Appl. **81** (2013) 70-86.

[16] J. Xu, Z. Zhang, and T. Caraballo, Mild solutions to time fractional stochastic 2D-Stokes equations with bounded and unbounded delay, *J. Dyn. Differ. Equ.* (2019) 1-21.

[17] Y. Zhou and F. Jiao, Existence of mild solutions for fractional neutral evolution equations, *Comput. Math. Appl.* **59(3)** (2010) 1063-1077.

[18] Y. Wang, J. Xu, and P. E. Kloeden, Asymptotic behavior of stochastic lattice systems with a Caputo fractional time derivative, *Nonlinear Anal. Theory Methods Appl.* **135** (2016) 205-222.

[19] D. Son, P. Huong, P. E. Kloeden, and H. Tuan, Asymptotic separation between solutions of Caputo fractional stochastic differential equations, *Stoch. Anal. Appl.* **36** (2018) 654-664 .

[20] W. Xu, W. Xu, S. Zhang, The averaging principle for stochastic differential equations with Caputo fractional derivative, *Appl. Math. Lett.*, **93** (2019) 79-84.

[21] W. Mao, L. Hu, S. You and X. Mao, The averaging method for multivalued SDEs with jumps and non-Lipschitz coefficients, *Discrete & Continuous Dynamical Systems-B*, **24(9)** (2019) 4937.

[22] W. Mao, S. You, X. Wu and X. Mao, On the averaging principle for stochastic delay differential equations with jumps, *Advances in Difference Equations*, **2015(1)** (2015) 1-19.

[23] B. Pei, Y. Xu, J. Wu, Stochastic averaging for stochastic differential equations driven by fractional Brownian motion and standard Brownian motion, *Appl. Math. Lett.* **100** (2020) 106006.

[24] D. Luo, Q. Zhu, Z. Luo, An averaging principle for stochastic fractional differential equations with time-delays, *Appl. Math. Lett.* **105** (2020) 106290.

[25] X. He, S. Han, J. Tao, Averaging principle for SDEs of neutral type driven by G-Brownian motion, *Stochastics and Dynamics*, **19(01)** (2019) 1950004.

[26] W. Wang, S. Chen, Z. Guo, X. Yan, A note on the continuity for Caputo fractional stochastic differential equations, *Chaos: An Interdisciplinary Journal of Nonlinear Science*, **30(7)** (2020) 073106.

[27] T. Doan, P. Kloeden, P. Huong, H. Tuan, Asymptotic separation between solutions of caputo fractional stochastic differential equations. Stochastic Analysis and Applications **36** (2018) 654-664.

[28] H. Ye, J. Gao, Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, *Journal of Mathematical Analysis and Applications*, **328(2)** (2007) 1075-1081.