The Exponential Behavior and Stabilizability of Quasilinear Parabolic Stochastic Partial Differential Equation

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Abstract: In this paper, we study the stability of quasilinear parabolic stochastic partial differential equations with multiplicative noise, which are neither monotone nor locally monotone. The exponential mean square stability and pathwise exponential stability of the solutions are established. Moreover, under certain hypothesis on the stochastic perturbations, pathwise exponential stability can be derived, without utilzing the mean square stability.

Keywords: Quasilinear stochastic partial differential equations; exponential stability; stabilization

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1 Introduction

We are concerned with the following quasilinear stochastic partial differential equation

$$\begin{cases} du + \operatorname{div}(B(u))dt = \operatorname{div}(A(u)\nabla u)dt + \sigma(u)dW(t), & x \in \mathbb{T}^d, t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $W(t), t \geq 0$ is a Q-Wiener process. The coefficients $B : \mathbb{R} \to \mathbb{R}^d$ and $A : \mathbb{R} \to \mathbb{R}^{d \otimes d}$ are appropriate coefficients specified later. This type of deterministic partial differential equations (i.e., when $\sigma = 0$) model the phenomenon of convection-diffusion of ideal fluids and arise in a diverse variety of areas with significant applications, including, for instance, two or three phase flows in porous media or sedimentation-consolidation processes, for more details, we refer to [9] and the references therein. Recently, the stochastic perturbation of this type equations has attracted many attentions on the well posedness problem. The existence and uniqueness of pathwise weak solution of the above stochastic equation was firstly studied in Debussche et al. [7] by utilizing a Yamada-Watanabe type argument and kinetic formulation. Then, based on a new method of applying Itô's formula for the L_1-norm, Hofmanová and Zhang [11] developed a direct approach to establish the existence and uniqueness of the solution. Recently, Dong et al. [6] established the

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large deviation principles and Zhang [17] established a small time large deviation principles for the solution of the equation (1.1).

As is well known, the long-time behaviour of flows is a very interesting and important problem in the theory of fluid dynamics and has been a very well studied topic, see, e.g. [10, 12–14] and the references. It would be very interesting and important to analyse the stochastic effects on a deterministic system. There were some studies for such stochastic equations in fluid mechanics. Caraballo et al. [3] proved that the weak solutions converge exponentially in the mean square and almost surely exponentially to the stationary solutions under some conditions. Moreover, they studied the stabilization of an stationary solution by the appearance of a random disturbance. In [4], the authors generalized the results of [3] to a class of dissipative nonlinear systems that include the 3D LANS-α model. Anh and Da [1] studied the exponential behaviour and stabilizability of a class of abstract nonlinear stochastic evolution equations, which include 2D Navier-Stokes equations. For more literatures, we refer the reader to [2, 5, 8, 15, 16] and the references.

Inspired by the above investigations, in this paper, we analyse the stability and stabilization of solutions of the equation (1.1). We first study the existence, uniqueness and the stability properties of a stationary solution to the corresponding deterministic equation, including both mean square exponential stability and path-wise exponential stability. Then we consider the stabilization of an unstable stationary solution by using a multiplicative Itô noise of sufficient intensity.

The rest of the paper is organized as follows. In Section 2, we present some necessary preliminaries. In Section 3, we show the existence and the uniqueness of the stationary solution. In Section 4, we derive results on the exponential mean square stability and the path-wise exponential stability of the stationary solution. Finally, in Section 5, we analyse the stabilization problem for the stationary solution.

2 Preliminaries

We consider periodic boundary conditions, that is $x \in \mathbb{T}^d$ where $\mathbb{T}^d = [0,1]^d$ denotes the d-dimensional torus. Let C_b^1 be the space of continuously differentiable functions with bounded fist order derivative. For $r \in [1, +\infty]$, $(L^r, \|\cdot\|_{L^r})$ are the Lebesgue spaces. In particular, we write H for $L^2(\mathbb{T}^d)$. Moreover, we also denote the inner product and the normal of H by $(\cdot,\cdot)_H$ and $|\cdot|_H$. Let $H^1 := H^{1,2}(\mathbb{T}^d)$ be the usual Sobolev space of order 1 with the normal $\|u\|_{H^1}^2 = \|u\|_{L^2} + \|\nabla u\|_{L^2}$ and H^{-1} is the topological dual of H^1. We denote the duality between H^1 and H^{-1} by $\langle\cdot,\cdot\rangle$. For any $u, v \in H^1$, defining $((u,v)) := (\nabla u, \nabla v)_H$. Then using Poincáre inequality, we know that there exists a constant $\lambda_1 > 0$, such that $\|u\|^2 := ((u,u)) \geq \lambda_1 |u|_H^2$ and thus $\|u\|_{H^1}^2 \cong \|u\|^2$.

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration. Defining $Q \in L(H,H)$ be the operator by $Qe_n = \lambda'_n e_n$, where $\lambda'_n \geq 0,\ n = 1, 2, ...$ satisfies $\sum_{n=1}^{+\infty} \lambda'_n < +\infty$ and $e_n, n = 1, 2, ...$ is a complete orthonormal basis in H. For simplicity, we assume Q is positive-definite. Let $\beta_n(t), n = 1, 2, ...$ be a sequence of real-valued one-dimensional standard Brownian motions independent on $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$. Now we define a Q-Wiener process:

$$W(t) = \sum_{n=1}^{+\infty} \sqrt{\lambda'_n} \beta_n(t) e_n,\ t \geq 0.$$

Let $L_Q(H_0, H)$ be the space of Hilbert-Schmidt operator from H_0 to H, where $H_0 = Q^{1/2}H$ is a Hilbert space with the inner product

$$(u, v)_0 := (Q^{-1/2}u, Q^{-1/2}v),\ \ u, v \in H_0.$$

Defining the norm on the space $L_Q(H_0, H)$ by $|\Phi|^2_{L_Q} = \mathrm{Tr}(\Phi Q\Phi^*)$. For an $L_Q(H_0, H)$ valued predictable process $\Phi(t, \omega),\ 0 \le t \le T$, the stochastic integral $\int_0^T \Phi(t, \omega)dW(t)$ is well-defined if

$$|\Phi|^2_T := \mathbb{E}\int_0^T \mathrm{Tr}(\Phi Q\Phi^*)dt < +\infty.$$

We introduce the following hypotheses.

Hypotheses A. The diffusion matrix A, the flux function B, and the noise in (1.1) satisfy:

(1) $A = (A_{ij})^d_{i,j=1} : \mathbb{R} \to \mathbb{R}^{d\otimes d}$ is of class C^1_b, uniformly positive definite and bounded, i.e. $\delta\mathrm{I} \le A \le C\mathrm{I}$. Moreover, there exists a constant $L_A > 0$, such that

$$|A(u) - A(v)| \le L_A|u - v|.$$

(2) $B = (B_1, ..., B_d) : \mathbb{R} \to \mathbb{R}^d$ is of class C^1_b, and there exists a constant $L_B > 0$, such that

$$|B(u) - B(v)| \le L_B|u - v|,\ \ |B(u)| \le L_B(1 + |u|).$$

(3) $\sigma \in C([0, +\infty) \times H^1; L_Q(H_0, H))$, and there exists a constant $L_\sigma > 0$, such that

$$|\sigma(t, u)|^2_{L_Q} \le L_\sigma(1 + |u|^2_H),\quad |\sigma(t, u) - \sigma(t, v)|^2_{L_Q} \le L_\sigma|u - v|^2_H.$$

Now, we recall the definition of a solution to (1.1) from [11].

Definition 2.1 *An $\mathscr{F}_t$-adapted, H-valued continuous process $(u(t), t \ge 0)$ is called a solution to equation (1.1), if*

(1) $u \in L^2(\Omega, C([0, T], H)) \cap L^2(\Omega, L^2([0, T], H^1))$ for any $T > 0$;

(2) for any $\phi \in C^\infty(\mathbb{T}^d), t > 0$ the following holds almost surely

$$\begin{aligned}&\langle u(t), \phi\rangle - \langle u_0, \phi\rangle - \int_0^t \langle B(u(s)), \nabla\phi\rangle ds\\ &\quad = -\int_0^t \langle A(u(s))\nabla u(s), \nabla\phi\rangle ds + \int_0^t \langle\sigma(u(s))dW(s), \phi\rangle.\end{aligned}$$

According to [6], we have

Theorem 2.1 *Let $u_0 \in L^p(\Omega, \mathscr{F}_0, L^p(\mathbb{T}^d))$ for all $p \in [1, \infty)$. Under the hypotheses A, there exists a unique solution to equation (1.1) that satisfies the following enery inequality*

$$\mathbb{E}\Big(\sup_{0\le t\le T}|u|^2_H\Big) + \int_0^T \mathbb{E}\|u\|^2 dt < +\infty.$$

3 Existence and uniqueness of the stationary solutions

Considering the deterministic version of equation (1.1):

$$\begin{cases} du + \mathrm{div}(B(u))dt = \mathrm{div}(A(u)\nabla u)dt, & x \in \mathbb{T}^d,\ t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{3.2}$$

In this section, we aim to study existence and uniqueness of stationary solution to equation (3.2). We recall a stationary solution to equation (3.2) is an element $u_\infty \in H^1$ such that

$$\mathrm{div}(B(u_\infty)) - \mathrm{div}(A(u_\infty)\nabla u_\infty) = 0, \quad in\ H^{-1}. \tag{3.3}$$

The main result of this section is

Theorem 3.1 *Under hypotheses A. If* $\delta\sqrt{\lambda_1} - L_B > 0$, *then equation* (3.2) *has at least a stationary solution* $u_\infty \in H^1$ *satisfying*

$$\|u_\infty\| \le \frac{\sqrt{\lambda_1}L_B}{\delta\sqrt{\lambda_1} - L_B}. \tag{3.4}$$

Moreover, if further $\delta - \frac{L_B}{\sqrt{\lambda_1}} - \frac{L_A L_B}{\delta\sqrt{\lambda_1} - L_B} > 0$, *then the stationary solution is unique.*

Proof. We will divide the proof into two steps.

Step 1: Existence. Let $v_1, v_2, ..., v_n, ...$ be an orthonormal basis of H^1. For every $m \ge 1$, consider the finite dimensional Hilbert space $V_m = \mathrm{span}\{v_1, ..., v_m\}$ with scalar product $[\cdot,\cdot]$. We define an approximate stationary solution of equation (3.2) in the following form:

$$u_m = \sum_{i=1}^{m} c_{mi} v_i,$$

which satifies

$$\langle \mathrm{div}B(u_m), v_i\rangle - \langle \mathrm{div}(A(u_m)\nabla u_m), v_i\rangle = 0. \tag{3.5}$$

We define a mapping $R_m : V_m \to V_m$ by

$$[R_m u, v] = ((R_m u, v)) := \langle \mathrm{div}(B(u)), v\rangle - \langle \mathrm{div}(A(u)\nabla u), v\rangle, \quad u, v \in V_m.$$

For any $u \in V_m$, we easily obtain that

$$\begin{aligned} [R_m u, u] &= \langle \mathrm{div}(B(u)), u\rangle - \langle \mathrm{div}(A(u)\nabla u), u\rangle \\ &= -\langle B(u), \nabla u\rangle + \langle A(u)\nabla u, \nabla u\rangle \\ &\ge -|B(u)|_H |\nabla u|_H + \delta\|u\|^2 \\ &\ge -L_B(1 + |u|_H)\|u\| + \delta\|u\|^2 \\ &\ge -L_B(1 + \frac{1}{\sqrt{\lambda_1}}\|u\|)\|u\| + \delta\|u\|^2 \\ &= \Big(\delta - \frac{L_B}{\sqrt{\lambda_1}}\Big)\|u\|^2 - L_B\|u\|. \end{aligned}$$

If we choose

$$k > \frac{\sqrt{\lambda_1}L_B}{\delta\sqrt{\lambda_1} - L_B},$$

then $[R_m u, u] \geq 0$ for all $u \in V_m$ such that $\|u\| = k$. Then, by [13, Lemma 1.4], for every $m \geq 1$, there exists $u_m \in V_m, \|u_m\| \leq k$ such that $R_m u_m = 0$. Replacing v_i in (3.5) by u_m, we have

$$\|u_m\| \leq \frac{\sqrt{\lambda_1}L_B}{\delta\sqrt{\lambda_1} - L_B}.$$

Therefore, there exists a subsequence of $\{u_m, m \geq 1\}$ (also denoted by $\{u_m, m \geq 1\}$), such that u_m converges weakly in H^1 and strongly in H to some limit u_∞. Next, we will show that u_∞ is a stationary solution to equation (3.2). Observe that, for any $v \in H^1$, when m tends to infinity,

$$\begin{aligned}
&|\langle \mathrm{div} B(u_m), v\rangle - \langle \mathrm{div} B(u_\infty), v\rangle| \\
&= |\langle B(u_m) - B(u_\infty), \nabla v\rangle| \leq L_B |u_m - u_\infty|_H \|v\| \to 0,
\end{aligned}$$

and

$$\begin{aligned}
&|\langle \mathrm{div}(A(u_m)\nabla u_m - A(u_\infty)\nabla u_\infty), v\rangle| \\
&= |\langle A(u_m)\nabla u_m - A(u_\infty)\nabla u_\infty, \nabla v\rangle| \\
&\leq |\langle A(u_m)(\nabla u_m - \nabla u_\infty), \nabla v\rangle| + |\langle (A(u_m) - A(u_\infty))\nabla u_\infty, \nabla v\rangle| \\
&\leq C|\langle \nabla u_m - \nabla u_\infty, \nabla v\rangle| + L_A |u_m - u_\infty|_H \|u_\infty\| \|v\| \to 0.
\end{aligned}$$

In other words,

$$\langle \mathrm{div} B(u_\infty), v_i\rangle - \langle \mathrm{div}(A(u_\infty)\nabla u_\infty), v_i\rangle = 0, \ i = 1, 2, \cdots. \tag{3.6}$$

In particular, (3.6) is also true for any $v \in V_m$. A continuity argument shows that u_∞ is a solution of (3.3).

Step 2: Unique. Let u_1 and u_2 be two solutions of equation (3.3). Then for any $v \in H^1$,

$$\mathrm{div}(B(u_i)) - \mathrm{div}(A(u_i)\nabla u_i) = 0, \ i = 1, 2.$$

Let $v = u_1 - u_2$, then

$$\begin{aligned}
0 &= \langle \mathrm{div}(B(u_1) - B(u_2)), u_1 - u_2\rangle - \langle \mathrm{div}(A(u_1)\nabla u_1 - A(u_2)\nabla u_2), u_1 - u_2\rangle \\
&= -\langle \nabla(u_1 - u_2), B(u_1) - B(u_2)\rangle + \langle \nabla(u_1 - u_2), A(u_1)\nabla u_1 - A(u_2)\nabla u_2\rangle \\
&\geq -L_B\|u_1 - u_2\||u_1 - u_2|_H + \langle \nabla(u_1 - u_2), A(u_1)(\nabla u_1 - \nabla u_2)\rangle \\
&\quad + \langle \nabla(u_1 - u_2), (A(u_1) - A(\nabla u_2))\nabla u_2\rangle \\
&\geq -\frac{L_B}{\sqrt{\lambda_1}}\|u_1 - u_2\|^2 + \delta\|u_1 - u_2\|^2 - L_A\|u_2\||u_1 - u_2|_H\|u_1 - u_2\| \\
&\geq -\frac{L_B}{\sqrt{\lambda_1}}\|u_1 - u_2\|^2 + \delta\|u_1 - u_2\|^2 - \frac{L_A}{\sqrt{\lambda_1}}\|u_2\|\|u_1 - u_2\|^2 \\
&\geq \Big(-\frac{L_B}{\sqrt{\lambda_1}} + \delta - \frac{L_A L_B}{\delta\sqrt{\lambda_1} - L_B}\Big)\|u_1 - u_2\|^2.
\end{aligned}$$

Since $\delta - \frac{L_B}{\sqrt{\lambda_1}} - \frac{L_A L_B}{\delta\sqrt{\lambda_1} - L_B} > 0$, we get $u_1 = u_2$. The proof is completed.

4 The exponential stability of the solutions

In this section, we discuss the exponential stability in the mean square and almost sure exponential stability of weak solutions to the stochastic equation (1.1).

Theorem 4.1 *Assuming that the conditions of Theorem 3.1 hold and* $\sigma(t, u_\infty) = 0,\ \forall t \geq 0$. *If*

$$\delta - \frac{L_B}{\sqrt{\lambda_1}} - \frac{L_A L_B}{\delta\sqrt{\lambda_1} - L_B} - \frac{L_\sigma}{2\lambda_1} > 0. \tag{4.7}$$

Then

$$E|u(t) - u_\infty|_H^2 \leq E|u(0) - u_\infty|_H^2 e^{-\lambda t},\ \forall t \geq 0.$$

That is, the stationary solution u_∞ to (1.1) is exponentially stable in mean square.

Remark 1. The condition $\sigma(t, u_\infty) = 0,\ \forall t \geq 0$ implies that the stationary solution u_∞ of (3.2) is also a solution of the stochastic perturbed equation (1.1).

Proof of Theorem 4.1. Let λ be sufficiently small such that

$$\frac{L_B}{\sqrt{\lambda_1}} - \delta + \frac{\lambda + L_\sigma}{2\lambda_1} + \frac{L_A L_B}{\delta\sqrt{\lambda_1} - L_B} < 0.$$

Observe that

$$\begin{aligned} u(t) - u_\infty = u_0 - u_\infty &- \int_0^t (\operatorname{div}(B(u(s))) - \operatorname{div}(B(u_\infty))ds \\ &+ \int_0^t (\operatorname{div}(A(u(s))\nabla u(s)) - \operatorname{div}(A(u_\infty)\nabla u_\infty)ds \\ &+ \int_0^t (\sigma(u(s)) - \sigma(u_\infty))dW(s). \end{aligned}$$

Then by Itô formula, we get

$$\begin{aligned} e^{\lambda t}E|u(t) - u_\infty|_H^2 = E|u_0 - u_\infty|_H^2 &+ \lambda \int_0^t e^{\lambda s}E|u(s) - u_\infty|_H^2 ds \\ &- 2\int_0^t e^{\lambda s}E\langle \operatorname{div}(B(u(s))) - \operatorname{div}(B(u_\infty), u(s) - u_\infty\rangle ds \\ &+ 2\int_0^t e^{\lambda s}E\langle \operatorname{div}(A(u(s))\nabla u(s)) - \operatorname{div}(A(u_\infty)\nabla u_\infty), u(s) - u_\infty\rangle ds \\ &+ \int_0^t e^{\lambda s}E|\sigma(s, u(s))|_{L_Q}^2 ds. \end{aligned}$$

Let $w(t) = u(t) - u_\infty$, it is easy to see that

$$-\langle \operatorname{div}(B(u) - B(u_\infty)), w\rangle = \langle B(u) - B(u_\infty), \nabla w\rangle \leq L_B|u - u_\infty|_H\|w\| \leq \frac{L_B}{\sqrt{\lambda_1}}\|w\|^2 \tag{4.8}$$

and

$$
\begin{aligned}
&\langle \operatorname{div}(A(u)\nabla u - A(u_\infty)\nabla u_\infty), w\rangle \\
&= -\langle A(u)\nabla u - A(u_\infty)\nabla u_\infty, \nabla w\rangle \\
&= -\langle A(u)(\nabla u - \nabla u_\infty), \nabla w\rangle - \langle (A(u) - A(u_\infty))\nabla u_\infty, \nabla w\rangle \\
&\le -\delta\|w\|^2 + L_A|w|_H\|u_\infty\|\|w\| \\
&\le \Big(-\delta + \frac{L_A L_B}{\delta\sqrt{\lambda_1} - L_B}\Big)\|w\|^2.
\end{aligned}
\tag{4.9}
$$

Consequently, we have

$$
\begin{aligned}
e^{\lambda t}E|u(t) - u_\infty|_H^2 \le\ & E|u_0 - u_\infty|_H^2 + \lambda\int_0^t e^{as}E|u(s) - u_\infty|_H^2 ds \\
&+ 2\Big(\frac{L_B}{\sqrt{\lambda_1}} - \delta + \frac{L_A L_B}{\delta\sqrt{\lambda_1} - L_B}\Big)\int_0^t e^{\lambda s}E\|u(s) - u_\infty\|^2 ds \\
&+ \int_0^t e^{\lambda s}E|\sigma(s, u(s))|_{L_Q}^2 ds \\
\le\ & E|u_0 - u_\infty|_H^2 + \int_0^t e^{\lambda s}E|\sigma(s, u(s))|_{L_Q}^2 ds \\
&+ \Big(2\frac{L_B}{\sqrt{\lambda_1}} - 2\delta + \frac{\lambda}{\lambda_1} + \frac{2L_A L_B}{\delta\sqrt{\lambda_1} - L_B}\Big)\int_0^t e^{\lambda s}E\|u(s) - u_\infty\|^2 ds \\
\le\ & E|u_0 - u_\infty|_H^2 \\
&+ \Big(2\frac{L_B}{\sqrt{\lambda_1}} - 2\delta + \frac{\lambda + L_\sigma}{\lambda_1} + \frac{2L_A L_B}{\delta\sqrt{\lambda_1} - L_B}\Big)\int_0^t e^{\lambda s}E\|u(s) - u_\infty\|^2 ds \\
\le\ & E|u_0 - u_\infty|_H^2.
\end{aligned}
$$

The proof is completed.

Remark 2. Let $\sigma \equiv 0$, then according to Theorem 4.1, if $\delta - \frac{L_B}{\sqrt{\lambda_1}} - \frac{L_A L_B}{\delta\sqrt{\lambda_1} - L_B} > 0$ holds, we can get that the stationary solution u_∞ to (3.2) is exponentially stable in mean square, i.e.

$$|u(t) - u_\infty|_H^2 \le e^{-\lambda t}|u(0) - u_\infty|_H^2,\ \ for\ some\ \lambda > 0.$$

Theorem 4.2 *Under the conditions of Theorem 4.1, the stationary solution u_∞ to (1.1) is almost surely exponentially stable. That is, there exists a constant $\gamma > 0$, such that*

$$\limsup_{t\to\infty}\frac{1}{t}\log|u(t) - u_\infty|_H^2 \le -\gamma, \quad almost\ surely.$$

Proof. Let N be a natural number. For any $t \ge N$, using the Itô formula, we get

$$
\begin{aligned}
|u(t) - u_\infty|_H^2 = |u_N - u_\infty|_H^2 &- 2\int_N^t \langle \operatorname{div}(B(u(s))), u(s) - u_\infty\rangle ds \\
&+ 2\int_N^t \langle \operatorname{div}(A(u(s))\nabla u(s)), u(s) - u_\infty\rangle ds \\
&+ \int_N^t |\sigma(s, u(s))|_{L_Q}^2 ds + 2\int_N^t (u(s) - u_\infty, \sigma(s, u(s))dW(s)).
\end{aligned}
\tag{4.10}
$$

For the stochastic term, it follows from Burkholder-Davis-Gundy inequalities that

$$\begin{aligned}
&2E\left[\sup_{N\le t\le N+1}\int_N^t (u(s)-u_\infty,\sigma(s,u(s))dW(s)\right]\\
&\le c_1E\left[\int_N^{N+1}|u(N)-u_\infty|_H^2|\sigma(s,u(s))|_{L_Q}^2ds\right]^{\frac{1}{2}}\\
&\le c_1E\left[\sup_{N\le s\le N+1}|u(s)-u_\infty|_H^2\int_N^{N+1}|\sigma(s,u(s))|_{L_Q}^2ds\right]^{\frac{1}{2}}\\
&\le \frac{1}{2}E\Big[\sup_{N\le s\le N+1}|u(s)-u_\infty|_H^2\Big]+\frac{c_1^2}{2}E\int_N^{N+1}|\sigma(s,u(s))|_{L_Q}^2ds,
\end{aligned} \tag{4.11}$$

where $c_1>0$. Combining (4.11) with (4.10), one gets

$$\begin{aligned}
E\left[\sup_{N\le t\le N+1}|u(t)-u_\infty|_H^2\right]\le& E|u_N-u_\infty|_H^2\\
&+2\Big(\frac{L_B}{\sqrt{\lambda_1}}-\delta+\frac{L_AL_B}{\delta\sqrt{\lambda_1}-L_B}\Big)\int_N^{N+1}E\|u(s)-u_\infty\|^2ds\\
&+\frac{1}{2}E\Big[\sup_{N\le s\le N+1}|u(s)-u_\infty|_H^2\Big]+\frac{c_1^2+2}{2}E\int_N^{N+1}|\sigma(s,u(s))|_{L_Q}^2ds.
\end{aligned}$$

Since $\delta-\frac{L_B}{\sqrt{\lambda_1}}-\frac{L_AL_B}{\delta\sqrt{\lambda_1}-L_B}>0$, it follows form Theorem 4.1 that

$$\begin{aligned}
E\left[\sup_{N\le t\le N+1}|u(t)-u_\infty|_H^2\right]&\le 2E|u_N-u_\infty|_H^2+c_1^2E\int_N^{N+1}|\sigma(s,u(s))|_{L_Q}^2ds\\
&\le 2E|u_N-u_\infty|_H^2+c_1^2L_\sigma E\int_N^{N+1}E|u(s)-u_\infty|_H^2ds\\
&\le 2E|u_0-u_\infty|_H^2e^{-\lambda N}+c_1^2L_\sigma E|u(0)-u_\infty|_H^2E\int_N^{N+1}e^{-\lambda s}ds\\
&\le 2E|u_0-u_\infty|_H^2e^{-\lambda N}+\frac{c_1^2L_\sigma}{\lambda}E|u(0)-u_\infty|_H^2(1-e^{-\lambda})e^{-\lambda N}\\
&=:Me^{-\lambda N}.
\end{aligned}$$

Let $\epsilon_N=e^{-\frac{1}{4}\lambda N}$, then Chebyshev inequality implies that

$$P\left(\sup_{N\le t\le N+1}|u(t)-u_\infty|^2>\epsilon_N\right)\le\frac{1}{\epsilon_N^2}E\left[\sup_{N\le t\le N+1}|u(t)-u_\infty|_H^2\right]\le Me^{-\frac{\lambda N}{2}}.$$

Then Borel-Cantelli lemma implies that

$$\limsup_{t\to\infty}\frac{1}{t}\log|u(t)-u_\infty|_H^2\le-\gamma,\quad \textit{almost surely}$$

holds for $\gamma=\frac{\lambda}{8}$. We thus complete the proof.

5 Stabilization of the solutions

In the previous section, we studied the exponentially stable in mean square of equation (1.1). However, for a giving stochastic ordinary differential equation, it may be path-wise exponentially stable but not exponential stable in the mean square. Consequently, in this section, we want to obtain path-wise exponential stability results by avoiding the method of using the mean square stability. To this end, we also need some additional hypotheses on the stochastic perturbation.

Hypotheses B. Assuming that

$$\widetilde{Q}\psi(t,u) := \mathrm{Tr}[(\psi_u(u)\otimes\psi_u(u))(\sigma(t,u)Q\sigma(t,u)^*)] \geq \varrho(t)^2|u(t)-u_\infty|_H^4,$$

where $\psi(u) = |u-u_\infty|_H^2$ and $\varrho(t)$ is a nonnegative continuous function such that there exists $\varrho_0 > 2L_\sigma$ satisfying

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t \varrho(s)ds \geq \varrho_0.$$

Theorem 5.1 *Under Hypotheses A and B. Let u_∞ be the the stationary solution of (3.2), if u_∞ is small such that*

$$\delta - \frac{L_B}{\sqrt{\lambda_1}} - \frac{L_A\|u_\infty\|}{\sqrt{\lambda_1}} - \frac{L_\sigma}{2\lambda_1} + \frac{\varrho_0}{4\lambda_1} > 0. \tag{5.12}$$

Then u_∞ is almost surely exponentially stable.

Proof. By using the Itô formula to the function $\log|u(t)-u_\infty|_H^2$, we get

$$\begin{aligned}
\log|u(t)-u_\infty|_H^2 = {}& \log|u(0)-u_\infty|_H^2 - 2\int_0^t \frac{\langle \mathrm{div}(B(u(s))) - \mathrm{div}(B(u_\infty)), u(s)-u_\infty\rangle}{|u(t)-u_\infty|_H^2}ds \\
& + 2\int_0^t \frac{\langle \mathrm{div}(A(u(s))\nabla u(s)) - \mathrm{div}(A(u_\infty)\nabla u_\infty), u(s)-u_\infty\rangle}{|u(t)-u_\infty|_H^2}ds \\
& + \int_0^t \frac{|\sigma(s,u(s))|_{L_Q}^2}{|u(t)-u_\infty|_H^2}ds - \frac{1}{2}\int_0^t \frac{\widetilde{Q}\psi(s,u(s))}{|u(s)-u_\infty|_H^4}ds \\
& + 2\int_0^t \frac{1}{|u(t)-u_\infty|_H^2}(u(s)-u_\infty, \sigma(s,u(s))dW(s)) \\
\leq {}& \log|u(0)-u_\infty|_H^2 + 2\Big(\frac{L_B}{\sqrt{\lambda_1}} - \delta + \frac{L_A}{\sqrt{\lambda_1}}\|u_\infty\| + \frac{L_\sigma}{\lambda_1}\Big)\lambda_1 t \\
& - \frac{1}{2}\int_0^t \frac{\widetilde{Q}\psi(s,u(s))}{|u(s)-u_\infty|_H^4}ds + 2\int_0^t \frac{(u(s)-u_\infty, \sigma(s,u(s))dW(s))}{|u(t)-u_\infty|_H^2}.
\end{aligned}$$

Let

$$M(t) = 2\int_0^t \frac{(u(s)-u_\infty, \sigma(s,u(s))dW(s))}{|u(t)-u_\infty|_H^2},$$

then the exponential martingale inequality implies that for any positive $T>0$, $0<\varepsilon<1$ and integer $k\geq 1$, we have

$$P\left\{\omega : \sup_{[0,T]}\Big[M(t) - \frac{\varepsilon}{2}\int_0^t \frac{\widetilde{Q}\psi(s,u(s))}{|u(s)-u_\infty|_H^4}ds\Big] > \frac{2\log k}{\varepsilon}\right\} \leq \frac{1}{k^2}.$$

According to Borel-Cantelli lemma, there exists an integer $k_0(\omega) > 0$ for almost all $\omega \in \Omega$ such that

$$2\int_0^t \frac{(u(s)-u_\infty, \sigma(s,u(s))dW(s))}{|u(t)-u_\infty|_H^2} \leq \frac{2\log k}{\varepsilon} + \frac{\varepsilon}{2}\int_0^t \frac{\widetilde{Q}\psi(s,u(s))}{|u(s)-u_\infty|_H^4} ds$$

for all $0 < t \leq k,\ k \geq k_0(\omega)$. Consequently,

$$\begin{aligned}\log|u(t)-u_\infty|_H^2 \leq \log|u(0)-u_\infty|_H^2 + 2\Big(\frac{L_B}{\sqrt{\lambda_1}} - \delta + \frac{L_A}{\sqrt{\lambda_1}}\|u_\infty\| + \frac{L_\sigma}{2\lambda_1}\Big)\lambda_1 t \\ + \frac{2\log k}{\varepsilon} - \frac{1-\varepsilon}{2}\int_0^t \frac{\widetilde{Q}\psi(s,u(s))}{|u(s)-u_\infty|_H^4} ds, \quad k \leq t \leq k+1.\end{aligned}$$

By using Hypotheses B, we get

$$\begin{aligned}\frac{1}{t}\log|u(t)-u_\infty|_H^2 \leq \frac{1}{t}\Big[\frac{2\log k}{\varepsilon} + \log|u(0)-u_\infty|_H^2\Big] - \frac{1-\varepsilon}{2t}\int_0^t \varrho(s)ds \\ + 2\Big(\frac{L_B}{\sqrt{\lambda_1}} - \delta + \frac{L_A}{\sqrt{\lambda_1}}\|u_\infty\| + \frac{L_\sigma}{2\lambda_1}\Big)\lambda_1, \quad k \leq t \leq k+1.\end{aligned}$$

Therefore,

$$\limsup_{t\to+\infty} \frac{1}{t}\log|u(t)-u_\infty|_H^2 \leq 2\Big(\frac{L_B}{\sqrt{\lambda_1}} - \delta + \frac{L_A}{\sqrt{\lambda_1}}\|u_\infty\| + \frac{L_\sigma}{2\lambda_1}\Big)\lambda_1 - \frac{1-\varepsilon}{2}\varrho_0.$$

By taking $\gamma = 2\Big(\delta\lambda_1 - L_B\sqrt{\lambda_1} - L_A\sqrt{\lambda_1}\|u_\infty\| - \frac{L_\sigma}{2} + \frac{\varrho_0}{4}\Big)$ and letting $\varepsilon \to 0$, one gets

$$\limsup_{t\to+\infty} \frac{1}{t}\log|u(t)-u_\infty|_H^2 \leq -\gamma.$$

This completes the proof of the theorem.

Remark 3. According to (3.4), a sufficient condition ensuring (5.12) is

$$\delta - \frac{L_B}{\sqrt{\lambda_1}} - \frac{L_A L_B}{\delta\sqrt{\lambda_1} - L_B} - \frac{L_\sigma}{2\lambda_1} + \frac{\varrho_0}{4\lambda_1} > 0.$$

Let's consider a special case. When $\sigma(t,u) = \sigma(u-u_\infty)$ and W is a one-dimensional Brownian motion. Then a basic calculus implies that $L_\sigma = \sigma^2,\ \varrho_0 = 4\sigma^2$, the above condition becomes

$$\delta - \frac{L_B}{\sqrt{\lambda_1}} - \frac{L_A L_B}{\delta\sqrt{\lambda_1} - L_B} > -\frac{\sigma^2}{2\lambda_1}.$$

This condition is weaker than the condition in Remak 2. So we do not know whether the stationary solution u_∞ to (3.2) is exponentially stable or not. Therefore, a multiplicative Itô noise of sufficient intensity will improve the stability condition in the deterministic sense.

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