THE SECOND HANKEL DETERMINANT FOR STARLIKE AND CONVEX FUNCTIONS OF ORDER ALPHA

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Abstract

In recent years, the study of Hankel determinants for various subclasses of normalised univalent functions $f \in \mathcal{S}$ given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ for $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ has produced many interesting results. The main focus of interest has been estimating the second Hankel determinant of the form $H_{2,2}(f) = a_2a_4 - a_3^2$. A non-sharp bound for $H_{2,2}(f)$ when $f \in \mathcal{K}(\alpha)$, $\alpha \in [0, 1)$ consisting of convex functions of order α was found by Krishna and Ramreddy [7], and later improved by Thomas *et.al* [17]. In this paper, we give the sharp result. Moreover we obtain sharp results for $H_{2,2}(f^{-1})$ for the inverse functions f^{-1} when $f \in \mathcal{K}(\alpha)$, and when $f \in \mathcal{S}^*(\alpha)$, the class of starlike functions of order α. Thus the results in this paper complete the set of problems for the second Hankel determinants of f and f^{-1} for the classes $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$, $\mathcal{S}^*_\beta$ and $\mathcal{K}_\beta$, where $\mathcal{S}^*_\beta$ and $\mathcal{K}_\beta$ are respectively the classes of strongly starlike, and strongly convex functions of order β.

1. Introduction

Denote by $\mathcal{A}$, the class of analytic functions defined in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, with expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

and let $\mathcal{S}$ be the subset of $\mathcal{A}$, consisting of functions which are univalent in $\mathbb{D}$.

For a given $f \in \mathcal{A}$ of the form (1), the qth Hankel determinant $H_{q,n}(f)$ is defined by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \dots & \dots & \dots & \dots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}, \tag{2}$$

where $n, q \in \mathbb{N}$.

In recent years most investigations concerning Hankel determinants for various subclasses of $\mathcal{A}$ have focused on finding estimates for the second Hankel determinant

$$H_{2,2}(f) = a_2a_4 - a_3^2, \tag{3}$$

with some recent results devoted to the third Hankel determinant $H_{3,1}(f)$. Most results have been concerned with subclasses of $\mathcal{S}$.

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The Koebe function $k(z) = z/(1-z)^2$ serves as the extreme function for many coefficient problems in $\mathcal{S}$, but when $f \in \mathcal{S}$, no exact bound for $H_{2,2}(f)$ is known. However it is known [5], that when $f \in \mathcal{S}$, $|H_{2,2}(f)| \le C$, where C is an absolute constant, and that C can be greater than 1, [14]. Thus $k(z)$ is not the extreme function in this case.

In finding bounds for $H_{2,2}(f)$ and $H_{3,1}(f)$ for subclasses of $\mathcal{S}$, most authors have used the method developed by Janteng, Halim and Darus in [6], who found the sharp bound $|H_{2,2}(f)| \le 1$ when $f \in \mathcal{S}^*$, the class of starlike functions, and the sharp bound $|H_{2,2}(f)| \le 1/8$, when $f \in \mathcal{K}$ the class of convex functions.

Although as mentioned above, $|H_{2,2}(f)| \le 1$ when $f \in \mathcal{S}^*$, the exact bound in the case of close-to-convex functions is still unknown, the best result to date being $|H_{2,2}(f)| \le 1.242\ldots$ [14].

We next note the following well-known generalisations of $\mathcal{S}^*$ and $\mathcal{K}$.

For $\alpha \in [0,1)$, we say that f is a starlike (respectively) convex function of order α if, and only if,

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \tag{4}$$

and

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha. \tag{5}$$

We denote these classes by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ respectively, noting that both $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ are natural subclasses of $\mathcal{S}$.

Similarly for $\beta \in (0,1]$, we say that f is a strongly starlike (respectively) strongly convex function of order β if, and only if,

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\beta, \tag{6}$$

and

$$\left|\arg \left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\beta. \tag{7}$$

We denote these classes by $\mathcal{S}^*_\beta$ and $\mathcal{K}_\beta$ respectively, noting again that both $\mathcal{S}^*_\beta$ and $\mathcal{K}_\beta$ are natural subclasses of $\mathcal{S}$.

For $f \in \mathcal{S}^*(\alpha)$, $f \in \mathcal{S}^*_\beta$ and $f \in \mathcal{K}_\beta$, the following sharp bounds for $H_{2,2}(f)$ are known (see [15], [16]).

Theorem 1. ([3, Theorem 2.3]) *Let $\alpha \in [0,1)$. If $f \in \mathcal{S}^*(\alpha)$ and is given by* (1), *then*

$$|H_{2,2}(f)| \le (1-\alpha)^2.$$

Theorem 2. ([2, Theorem 2.3] and [16, Theorem 4.1]) *Let $\beta \in (0,1]$.*

(i) *If $f \in \mathcal{S}^*_\beta$ and is given by* (1), *then*

$$|H_{2,2}(f)| \le \beta^2.$$

(ii) *If $f \in \mathcal{K}_\beta$ and is given by* (1), *then*

$$|H_{2,2}(f)| \leq \begin{cases} \dfrac{\beta^2}{9} & \beta \in (0, 1/3], \\[2ex] \dfrac{\beta(1+\beta)(1+17\beta)}{72(3+\beta)} & \beta \in [1/3, 1]. \end{cases}$$

For $f \in \mathcal{K}(\alpha)$, the best bound for $H_{2,2}(f)$ to date is the following.

Theorem 3. ([17, p. 77]) *Let $\alpha \in [0,1)$. If $f \in \mathcal{K}(\alpha)$ and is given by* (1), *then*

$$|H_{2,2}(f)| \leq \frac{(1-\alpha)^2(36-36\alpha+17\alpha^2)}{144(2-2\alpha+\alpha^2)}. \tag{8}$$

The primary object of this paper is to provide the sharp bound for $H_{2,2}(f)$ when $f \in \mathcal{K}(\alpha)$.

Let $\mathcal{P}$ denote the class of functions p analytic in $\mathbb{D}$, for which $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$, with p given by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \ . \tag{9}$$

The following lemmas for functions in $\mathcal{P}$ are well-known.

Lemma 1. [13, p. 41] *If $p \in \mathcal{P}$, then the sharp inequality $|c_n| \leq 2$ holds for $n \geq 1$.*

Lemma 2. [9, p. 228] (see also [10, p. 254]) *If $p \in \mathcal{P}$ and is given by* (9) *with $c_1 \geq 0$, then*

$$2c_2 = c_1^2 + \zeta(4-c_1^2), \tag{10}$$

and

$$4c_3 = c_1^3 + 2c_1(4-c_1^2)\zeta - c_1(4-c_1^2)\zeta^2 + 2(4-c_1^2)(1-|\zeta|^2)\eta, \tag{11}$$

for some ζ and η such that $|\zeta| \leq 1$ and $|\eta| \leq 1$.

In proving our results, we use the technique developed in [6]. However this method does not always give sharp results, and we will use the following inequalities.

Lemma 3. [12, Proposition 6] (see also [4, Theorem 3.1])

For $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$, and for real numbers A, B, C, let

$$Y(A,B,C) = \max\left\{|A + Bz + Cz^2| + 1 - |z|^2 : z \in \overline{\mathbb{D}}\right\}. \tag{12}$$

If $AC \geq 0$, then

$$Y(A,B,C) = \begin{cases} |A| + |B| + |C|, & |B| \geq 2(1-|C|), \\ 1 + |A| + \dfrac{B^2}{4(1-|C|)}, & |B| < 2(1-|C|). \end{cases}$$

If $AC<0$, *then*

$$Y(A,B,C)=\begin{cases} 1-|A|+\dfrac{B^2}{4(1-|C|)}, & -4AC(C^{-2}-1)\le B^2\wedge |B|<2(1-|C|),\\ 1+|A|+\dfrac{B^2}{4(1+|C|)}, & B^2<\min\left\{4(1+|C|)^2,-4AC(C^{-2}-1)\right\},\\ R(A,B,C), & \text{otherwise,}\end{cases} \tag{13}$$

where

$$R(A,B,C)=\begin{cases} |A|+|B|-|C|, & |C|(|B|+4|A|)\le |AB|,\\ -|A|+|B|+|C|, & |AB|\le |C|(|B|-4|A|),\\ (|C|+|A|)\sqrt{1-\dfrac{B^2}{4AC}}, & \text{otherwise.}\end{cases} \tag{14}$$

2. The Second Hankel determinant for convex functions of order alpha

We estimate the second-order Hankel determinant $H_{2,2}(f)$ for $f\in\mathcal{K}(\alpha)$.

Theorem 4. *Let* $\alpha\in[0,1)$ *and* $f\in\mathcal{K}(\alpha)$ *be given by* (1). *Then*

$$|H_{2,2}(f)|\le \frac{(1-\alpha)^2(6+5\alpha)}{48(1+\alpha)}. \tag{15}$$

The inequality is sharp.

Proof. Since $|H_{2,2}(f)|\le 1/8$ when $f\in\mathcal{K}$, [6], it is enough to consider the case $\alpha\in(0,1)$.

Fix $\alpha\in(0,1)$, and let $f\in\mathcal{K}(\alpha)$ be given by (1). Then we can write

$$1+\frac{zf''(z)}{f'(z)}=\alpha+(1-\alpha)p(z),$$

where $p\in\mathcal{P}$, and is given by (9). Equating coefficients we obtain

$$a_2=\frac{1}{2}(1-\alpha)c_1,$$

$$a_3=\frac{1}{6}(1-\alpha)(c_2+(1-\alpha)c_1^2),$$

and

$$a_4=\frac{1}{12}(1-\alpha)(c_3+\frac{3}{2}(1-\alpha)c_1c_2+\frac{1}{2}(1-\alpha)^2c_1^3).$$

Thus

$$H_{2,2}(f)=a_2a_4-a_3^2=-\frac{1}{144}(1-\alpha)^2\Phi, \tag{16}$$

where

$$\Phi=(1-\alpha)^2c_1^4-(1-\alpha)c_1^2c_2+4c_2^2-6c_1c_3.$$

Since both the class $\mathcal{K}(\alpha)$ and the functional $|a_2a_4-a_3^2|$ are rotationally invariant, we may assume that $c_1:=c\in[0,2]$, and so using (10) and (11) in Lemma 2 we have

$$\begin{aligned}\Phi=&\frac{1}{2}\alpha(2\alpha-3)c^4-\frac{1}{2}(3-\alpha)c^2(4-c^2)\zeta+\frac{1}{2}(4-c^2)(8+c^2)\zeta^2\\ &-3c(4-c^2)(1-|\zeta|^2)\eta,\end{aligned} \tag{17}$$

where $\zeta,\ \eta\in\overline{\mathbb{D}}$.

Assume first that $c = 2$. Then

$$|\Phi| = 8\alpha(3 - 2\alpha), \tag{18}$$

and so from (16), we have

$$|H_{2,2}(f)| = \frac{1}{18}\alpha(1-\alpha)^2(3-2\alpha) < \frac{(1-\alpha)^2(6+5\alpha)}{48(1+\alpha)}.$$

Next when $c = 0$, since $\zeta \in \overline{\mathbb{D}}$, we obtain

$$|\Phi| = 16|\zeta|^2 \leq 16, \tag{19}$$

and therefore

$$|H_{2,2}(f)| \leq \frac{1}{9}(1-\alpha)^2 < \frac{(1-\alpha)^2(6+5\alpha)}{48(1+\alpha)}.$$

Now let $c \in (0, 2)$. Applying the triangle inequality in (17), we obtain

$$|\Phi| \leq 3c(4 - c^2)\Gamma(A, B, C),$$

where

$$\Gamma(A, B, C) = |A + B\zeta + C\zeta^2| + 1 - |\zeta|^2, \quad \zeta \in \overline{\mathbb{D}}, \tag{20}$$

with

$$A = \frac{\alpha(2\alpha-3)c^3}{6(4-c^2)}, \quad B = \frac{1}{6}(\alpha-3)c, \quad \text{and} \quad C = \frac{8+c^2}{6c}.$$

We now use Lemma 3, noting that $AC < 0$ holds for $c \in (0, 2)$.

First note that

$$B^2 \geq -4AC(C^{-2} - 1), \quad c \in (0, 2),$$

since

$$12(8 + c^2)[B^2 + 4AC(C^{-2} - 1)] = c^2[8(3 + 6\alpha - 5\alpha^2) + 3(1-\alpha)^2c^2] > 0.$$

Next $|B| > 2(1 - |C|)$, when $c \in (0, 2)$. To see this, define

$$\varphi_1(c) = 16 - 12c + (5 - \alpha)c^2 = 6c(|B| - 2(1 - |C|)),$$

then $\varphi_1'(c) = 0$ when $c = c^* := 6/(5 - \alpha) \in (0, 2)$, and $\varphi_1''(c^*) > 0$. Thus when $c \in (0, 2)$,

$$\varphi_1(c) \geq \varphi_1(c^*) = 16 - \frac{36}{5-\alpha} \geq 7 > 0,$$

which shows that $|B| > 2(1 - |C|)$.

Also

$$|C|(|B| + 4|A|) \geq |AB|, \quad \text{when } c \in (0, 2). \tag{21}$$

To see this, define $\varphi_2 : [0, 4] \to \mathbb{R}$ by

$$\varphi_2(x) = 32(3 - \alpha) - 4(3 - 25\alpha + 16\alpha^2)x - (1 - \alpha)^2(3 + 2\alpha)x^2.$$

Then we have

$$\varphi_2(0) = 32(3 - \alpha) > 0$$

and

$$\varphi_2(4) = 16(3 + 2\alpha + \alpha^2 - 2\alpha^3) > 0.$$

Since φ_2 is concave on $[0,4]$, we get

$$\varphi_2(x) \geq \min\{\varphi_2(0), \varphi_2(4)\} > 0, \quad x \in [0,4],$$

and therefore

$$36(4-c^2)(|C|(|B|+4|A|) - |AB|) = \varphi_2(c^2) \geq 0, \text{ when } c \in (0,2),$$

which gives (21).

Next in (14), a computation gives

$$|AB| - |C|(|B| - 4|A|) = \frac{k_2c^4 + k_1c^2 - k_0}{36(4-c^2)},$$

where

$$k_0 = 32(3-\alpha), \quad k_1 = 12 + 92\alpha - 64\alpha^2,$$

and

$$k_2 = 3 + 20\alpha - 17\alpha^2 + 2\alpha^3.$$

Let $\varphi_3(x) = k_2x^2 + k_1x - k_0$, and

$$\xi = \frac{-k_1 + \sqrt{\Delta}}{2k_2} \tag{22}$$

be the unique positive root of φ_3, where

$$\Delta = 48(27 + 198\alpha - 45\alpha^2 - 184\alpha^3 + 80\alpha^4) > 0, \quad \text{for } \alpha \in (0,1).$$

Then $\xi < 4$. Therefore for $c^{**} := \sqrt{\xi} \in (0,2)$, it follows that $|AB| = |C|(|B| - 4|A|)$.

Moreover $|AB| \leq |C|(|B| - 4|A|)$, when $c \in (0, c^{**}]$, and $|AB| \geq |C|(|B| - 4|A|)$, when $c \in [c^{**}, 2)$.

We now consider the following cases.

A. First suppose that $c \in (0, c^{**}]$. Then $|AB| \leq |C|(|B| - 4|A|)$, and by Lemma 3 we obtain

$$|\Phi| \leq 3c(4-c^2)[-|A| + |B| + |C|] = \psi_1(c^2),$$

where

$$\psi_1(x) = 16 + 2(2-\alpha)x - (1+\alpha)(2-\alpha)x^2.$$

We note that $\psi_1'(x) = 0$ only when $x = \tau := 1/(1+\alpha)$, and it is easily seen that $\tau \in (0, \xi)$, and since the function ψ_1 has negative leading coefficient, it follows that

$$\psi_1(x) \leq \psi_1(\tau) = \frac{3(6+5\alpha)}{1+\alpha}, \quad x \in (0, \xi],$$

and so

$$|\Phi| \leq \frac{3(6+5\alpha)}{1+\alpha}, \tag{23}$$

when $c \in (0, c^{**}]$.

B. Next suppose that $c \in [c^{**}, 2)$. Then by Lemma 3 we have

$$\begin{aligned}|\Phi| &\le 3c(4-c^2)(|A|+|C|)\sqrt{1-\frac{B^2}{4AC}}\\ &= \frac{1}{4\sqrt{\alpha(3-2\alpha)}}g_1(c^2)\sqrt{g_2(c^2)} =: \frac{1}{4\sqrt{\alpha(3-2\alpha)}}F(c^2),\end{aligned}$$

where g_1 and g_2 are defined by

$$g_1(x) = 32 - 4x - (1-\alpha)(1-2\alpha)x^2, \tag{24}$$

and

$$g_2(x) = \frac{12(3+6\alpha-5\alpha^2) - 9(1-\alpha)^2 x}{8+x}, \tag{25}$$

respectively, and where

$$2F'(x)(g_2(x))^{1/2} = 2g_1'(x)g_2(x) + g_1(x)g_2'(x).$$

Moreover

$$\frac{1}{12}(8+x)^2[2g_1'(x)g_2(x) + g_1(x)g_2'(x)] = m_0 + m_1 x + m_2 x^2 + m_3 x^3 =: g(x),$$

where

$$\begin{aligned}m_0 &= -96(1+\alpha)(5-3\alpha),\\ m_1 &= 4(-9-18\alpha+159\alpha^2-216\alpha^3+80\alpha^4),\\ m_2 &= 3(1-\alpha)(9-42\alpha+67\alpha^2-30\alpha^3),\end{aligned}$$

and

$$m_3 = 3(1-\alpha)^3(1-2\alpha).$$

Therefore the inequality $F(c^2) \le F(\xi)$ will hold for all $c \in [c^{**}, 2)$ provided $g(x) \le 0$ for $x \in [0, 4]$.

We now consider the following sub-cases.

B(1) When $\alpha = 1/2$, we have

$$g(x) = -504 - x + \frac{3}{2}x^2 \le g(4) = -484 < 0, \quad \text{when } x \in [0,4].$$

B(2) When $\alpha \in (0, 1/2)$, we have $m_3 > 0$, and using $x^3 \le 4x^2$ gives

$$g(x) \le m_0 + m_1 x + (m_2 + 4m_3)x^2 =: h_1(x), \text{ when } x \in [0,4].$$

Note that

$$h_1(0) = m_0 = -96(1+\alpha)(5-3\alpha) < 0,$$

and

$$h_1(4) = m_0 + 4m_1 + 16m_2 + 64m_3 = -16\alpha(3-2\alpha)(81-150\alpha+97\alpha^2) < 0.$$

Therefore since

$$m_2 + 4m_3 = 3(1-\alpha)(13-58\alpha+87\alpha^2-38\alpha^3) > 0,$$

h_1 is convex on $[0,4]$, and so

$$g(x) \le h_1(x) \le \max\{h_1(0); h_1(4)\} < 0, \text{ when } x \in [0,4].$$

B(3) Now suppose that $\alpha \in (1/2, 1)$, then $m_3 < 0$. Thus

$$g(x) \le m_0 + m_1 x + m_2 x^2 =: h_2(x), \quad \text{when } x \in [0,4].$$

Using a similar argument to case B(2), we have $h_2(x) < 0$ for $x \in [0,4]$, and so $g(x) < 0$, when $x \in [0,4]$.

Thus when $c \in [c^{**}, 2)$, we have shown that

$$|\Phi| \le \frac{1}{4\sqrt{\alpha(3-2\alpha)}} g_1(\xi)\sqrt{g_2(\xi)} = \psi_1(\xi) \le \frac{3(6+5\alpha)}{1+\alpha}, \tag{26}$$

and (15) is proved.

To see that (15) is sharp, define $\tilde{f} \in \mathcal{A}$ so that

$$1 + \frac{z\tilde{f}''(z)}{\tilde{f}'(z)} = \alpha + (1-\alpha)\tilde{p}(z),$$

where

$$\tilde{p}(z) = \frac{1-z^2}{1-\tilde{\tau} z + z^2}$$

with $\tilde{\tau} = \sqrt{\tau} = 1/\sqrt{1+\alpha}$. Since $\tilde{\tau} \in (0,1)$, $\tilde{p} \in \mathcal{P}$, and so $\tilde{f} \in \mathcal{K}(\alpha)$.

Furthermore

$$a_2 = \frac{1-\alpha}{2\sqrt{1+\alpha}}, \quad a_3 = -\frac{\alpha(1-\alpha)}{2(1+\alpha)},$$

and

$$a_4 = -\frac{(1-\alpha)(6+11\alpha-7\alpha^2)}{24(1+\alpha)^{3/2}},$$

which implies that

$$H_{2,2}(\tilde{f}) = -\frac{(1-\alpha)^2(6+5\alpha)}{48(1+\alpha)}.$$

Thus (15) is sharp for the extreme function $\tilde{f}$. □

3. Hankel determinants for the inverse functions

Since the classes $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$, $\mathcal{S}^*_\beta$ and $\mathcal{K}_\beta$ are all subsets of $\mathcal{S}$, inverse functions f^{-1} exist in some neighbourhood of the origin. A classical result of Löwner [11] shows that if $f \in \mathcal{S}$, and f^{-1} is given by

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n, \tag{27}$$

then for $n \ge 2$, $|d_n| \le K_n$, where K_n are the coefficients of the inverse of the Koebe function, and that this inequality is sharp.

Sharp bounds for the two initial coefficients of the inverse functions when $f \in \mathcal{S}^*(\alpha)$ were found by Krzyż, Libera and Złotkiewicz in [8], and similar results for the initial coefficients of the inverse coefficients when $f \in \mathcal{S}^*_\beta$ were obtained by Ali in [1], and for $f \in \mathcal{K}_\beta$ by Thomas and Verma in [16].

In this section we find sharp bounds for $H_{2,2}(f^{-1})$, when $f \in \mathcal{S}^*(\alpha)$, and when $f \in \mathcal{K}(\alpha)$, thus completing the set of problems for the second Hankel determinants of f and f^{-1} for the classes $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$, $\mathcal{S}^*_\beta$ and $\mathcal{K}_\beta$.

We first note that if f^{-1} is the inverse function of f, and f and f^{-1} are given by (1) and (27) respectively, then comparing coefficients in $f(f^{-1}(w)) = w$ gives

$$d_2 = -a_2, \; d_3 = -a_3 + 2a_2^2, \; d_4 = -a_4 + 5a_2a_3 - 5a_2^3, \tag{28}$$

so that

$$H_{2,2}(f^{-1}) = d_2d_4 - d_3^2 = a_2a_4 - a_3^2 - a_2^2a_3 + a_2^4. \tag{29}$$

Before stating and proving our results, we note some properties of the following functions, which we will use in our proofs.

Let $\Phi_i : (1/10, 1/2) \to \mathbb{R}$ $(i = 1, 2, 3, 4)$ be defined by

$$\Phi_1(x) = (1-\alpha)^2, \quad \Phi_2(x) = \frac{1}{15}(19 - 42x + 24x^2), \tag{30}$$

$$\Phi_3(x) = \frac{1}{3}(1-x)^2(1-2x)(9-10x), \tag{31}$$

and

$$\Phi_4(x) = \frac{4(1-2x)^{1/2}(9-10x)^{1/2}\phi_1(x)}{(17-30x)^2}\left(\frac{\phi_2(x)}{\phi_3(x)}\right)^{1/2}, \tag{32}$$

where

$$\phi_1(x) = (9-10x)(11 - 33x + 24x^2) + (5x-3)\sqrt{\phi(x)},$$

$$\phi_2(x) = -253 + 1092x - 1548x^2 + 720x^3 + 2\sqrt{\phi(x)},$$

and

$$\phi_3(x) = 2\left(-86 + 339x - 432x^2 + 180x^3 - \sqrt{\phi(x)}\right),$$

with

$$\phi(x) = 511 - 2706x + 5292x^2 - 4536x^3 + 1440x^4.$$

Let $\alpha_0 = 0.232\cdots$ be a zero of the polynomial q defined by $q(x) = 100x^4 - 340x^3 + 401x^2 - 188x + 26$. Then the following equalities, which can be verified by direct computation hold.

$$\max\{\Phi_i(x) : i = 1,2,3,4\} = \begin{cases} \Phi_2(x), & \text{when } x \in (1/10, \alpha_0], \\ \Phi_3(x), & \text{when } x \in [\alpha_0, 1/2). \end{cases} \tag{33}$$

Theorem 5. *Let $\alpha \in [0,1)$, $\alpha_0 = 0.232\ldots$ be the root of the polynomial $q(x) = 100x^4 - 340x^3 + 401x^2 - 188x + 26 = 0$ in $[0,1)$, and f^{-1} be the inverse function of $f \in \mathcal{S}^*(\alpha)$. Then*

$$|H_{2,2}(f^{-1})| \le \begin{cases} \frac{1}{3}(1-\alpha)^2(1-2\alpha)(9-10\alpha), & \text{if } \alpha \in [0, \alpha_0], \\ \frac{1}{15}(19 - 42\alpha + 24\alpha^2), & \text{if } \alpha \in [\alpha_0, \frac{2}{3}], \\ (1-\alpha)^2, & \text{if } \alpha \in [\frac{2}{3}, 1). \end{cases} \tag{34}$$

The first inequality is sharp for the inverse function of $f \in \mathcal{S}^*(\alpha)$, where f is given by

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}}, \quad z \in \mathbb{D}, \tag{35}$$

the second inequality is sharp for the inverse function of $f \in \mathcal{S}^*(\alpha)$, where f, is given by

$$f(z) = \frac{z}{((1-z)^{1+\tau/2}(1+z)^{1-\tau/2})^{1-\alpha}}, \quad z \in \mathbb{D}, \tag{36}$$

and where

$$\tau = \sqrt{\frac{2(2-3\alpha)}{5(1-\alpha)^2}}, \tag{37}$$

and the third inequality is sharp for the inverse function of f, where $f \in \mathcal{S}^*(\alpha)$ is given by

$$f(z) = \frac{z}{(1-z^2)^{1-\alpha}}, \quad z \in \mathbb{D}. \tag{38}$$

Proof. Let $\alpha \in [0,1)$, and $f \in \mathcal{S}^*(\alpha)$ be given by (1). Then there exists $p \in \mathcal{P}$ given by (9) such that

$$\frac{zf'(z)}{f(z)} = \alpha + (1-\alpha)p(z). \tag{39}$$

Equating coefficients in (39) we obtain

$$a_2 = (1-\alpha)c_1, \quad a_3 = \frac{1}{2}(1-\alpha)\left(c_2 + (1-\alpha)c_1^2\right), \tag{40}$$

and

$$a_4 = \frac{1}{3}(1-\alpha)\left(c_3 + \frac{3}{2}(1-\alpha)c_1c_2 + \frac{1}{2}(1-\alpha)^2c_1^3\right). \tag{41}$$

Now let f^{-1} be the inverse of f with expansion (27). Then from (29), (40) and (41) we obtain

$$d_2d_4 - d_3^2 = \frac{1}{12}(1-\alpha)^2(4c_1c_3 - 3c_2^2 - 6(1-\alpha)c_1^2c_2 + 5(1-\alpha)^2c_1^4). \tag{42}$$

From Lemma 2, (42) becomes

$$\begin{aligned} d_2d_4 - d_3^2 = \frac{1}{12}(1-\alpha)^2\Big(&\frac{1}{4}(1-2\alpha)(9-10\alpha)c_1^4 - \frac{1}{2}(5-6\alpha)c_1^2(4-c_1^2)\zeta \\ &- \frac{1}{4}(12+c_1^2)(4-c_1^2)\zeta^2 + 2c_1(4-c_1^2)(1-|\zeta|^2)\eta\Big), \end{aligned} \tag{43}$$

with $\zeta,\, \eta \in \overline{\mathbb{D}}$.

Since both the class $\mathcal{S}^*(\alpha)$ and the functional $|H_{2,2}(f^{-1})|$ are rotationally invariant, we can assume that $c_1 = c \in [0,2]$. Thus the triangle inequality gives

$$|H_{2,2}(f^{-1})| \le \frac{1}{12}(1-\alpha)^2\left[\frac{1}{4}\left|(1-2\alpha)(9-10\alpha)\right|c^4 + (4-c^2)h(c,|\zeta|)\right], \tag{44}$$

where $h : [0,2]\times[0,1] \to \mathbb{R}$ is defined by

$$h(x,y) = 2x + \frac{1}{2}|5-6\alpha|x^2y + \frac{1}{4}(2-x)(6-x)y^2.$$

Moreover it can be easily seen that $h(\cdot, y)$ is increasing on $[0,1]$, and so

$$|H_{2,2}(f^{-1})| \le \frac{1}{12}(1-\alpha)^2K(c), \tag{45}$$

where

$$K(c) = k_1c^4 + k_2c^2 + 12,$$

and where

$$k_1 = \frac{1}{4}\left(|(1-2\alpha)(9-10\alpha)| - 2|5-6\alpha| - 1\right), \quad \text{and} \quad k_2 = 2\left(|5-6\alpha| - 1\right). \tag{46}$$

We note that $k_1 < 0$, and distinguish various cases.

I. When $\alpha \in [2/3, 1)$, clearly $k_2 < 0$, and so $K(c) \leq K(0) = 12$, when $c \in [0,2]$. Thus from (45), we deduce that $|H_{2,2}(f^{-1})| \leq (1-\alpha)^2$.

II. When $\alpha \in [0, 1/10]$, the function K becomes

$$K(c) = (5\alpha^2 - 4\alpha - \frac{1}{2})c^4 + 4(2-3\alpha)c^2 + 12,$$

and since $5\alpha^2 - 4\alpha - 1/2 < 0$, it follows that

$$K'(c) \geq 8c(1-\alpha)(1-10\alpha) \geq 0, \quad \text{for } c \in [0,2].$$

Hence $K(c) \leq K(2)$ for $c \in [0,2]$, and so from (45) we deduce that

$$|H_{2,2}(f^{-1})| \leq \frac{1}{12}(1-\alpha)^2 K(2) = \frac{1}{3}(1-\alpha)^2(1-2\alpha)(9-10\alpha).$$

III. When $\alpha \in [1/2, 2/3)$, the function K becomes

$$K(c) = -5(1-\alpha)^2c^4 + 4(2-3\alpha)c^2 + 12. \tag{47}$$

It is easy to see that K has a unique local maximum at $c = \tau$, where τ is given by (37). Thus

$$K(c) \leq K(\tau) = 12 + \frac{4(2-3\alpha)^2}{5(1-\alpha)^2}, \quad \text{for } c \in [0,2],$$

and so from (45), we obtain

$$|H_{2,2}(f^{-1})| \leq \frac{1}{15}(19 - 42\alpha + 24\alpha^2).$$

IV. Next suppose that $\alpha \in (1/10, 1/2)$, where we will use Lemma 3.

First note that when $c = 0$, since $\zeta \in \overline{\mathbb{D}}$, we have

$$|H_{2,2}(f^{-1})| = (1-\alpha)^2|\zeta|^2 \leq (1-\alpha)^2 = \Phi_1(\alpha), \tag{48}$$

and that when $c = 2$,

$$|H_{2,2}(f^{-1})| = \frac{1}{3}(1-\alpha)^2(1-2\alpha)(9-10\alpha) = \Phi_3(\alpha),$$

where, Φ_1 and Φ_3 are defined in (30) and (31).

Now let $c \in (0,2)$. Applying the triangle inequality in (43), we obtain

$$|H_{2,2}(f^{-1})| \leq \frac{1}{6}(1-\alpha)^2c(4-c^2)\Gamma(A,B,C), \tag{49}$$

where Γ is defined by (20), with

$$A=\frac{(1-2\alpha)(9-10\alpha)c^3}{8(4-c^2)},\quad B=-\frac{(5-6\alpha)c}{4}\quad\text{and}\quad C=-\frac{12+c^2}{8c}.$$

Note that $AC<0$ when $\alpha\in(1/10,1/2)$, and $c\in(0,2)$, and so we again need to check the various cases in Lemma 3.

The inequalities $|B|\geq 2(1-|C|)$, and $B^2\geq -4A(1-C^2)/C$ are equivalent to

$$|5-6\alpha|c^2\geq -(2-c)(6-c),\quad\text{and}\quad (5-6\alpha)^2\geq -\frac{(1-2\alpha)(9-10\alpha)(36-c^2)}{12+c^2},$$

respectively, which are valid when $\alpha\in(1/10,1/2)$, and $c\in(0,2)$, which shows that the first two cases in (13) are false.

We next note that the first case in (14) is not true. To see this we observe that the inequality $|AB|\geq|C|(|B|+4|A|)$ can be written as $\varphi_1(c^2)\leq 0$, where

$$\varphi_1(x)=(30\alpha^3-57\alpha^2+36\alpha-8)x^2+4(30\alpha^2-39\alpha+11)x+12(5-6\alpha).$$

since $\varphi_1''(x)=2(30\alpha^3-57\alpha^2+36\alpha-8)<0$ for $x\in[0,4]$, we have

$$\varphi_1(x)\geq\min\{\varphi_1(0),\varphi_1(4)\},\quad\text{when } x\in[0,4]. \tag{50}$$

Moreover since

$$\varphi_1(0)=12(5-6\alpha)>0,\quad\text{and}\quad \varphi_1(4)=12(1-2\alpha)(1+2\alpha)(9-10\alpha)>0,$$

(50) implies that $\varphi_1(x)>0$ holds when $x\in[0,4]$, and so $|AB|\geq|C|(|B|+4|A|)$ is false.

Next note that

$$|C|(|B|-4|A|)-|AB|=\frac{l_2c^4-2l_1c^2+l_0}{8(4-c^2)}, \tag{51}$$

where

$$l_2=(30\alpha-17)(1-\alpha)^2,\quad l_1=2(30\alpha^2-45\alpha+16),\quad\text{and}\quad l_0=12(5-6\alpha).$$

Let $L(x)=l_2x^2-2l_1x+l_0$, and $\xi=(l_1-\sqrt{\Delta})/l_2$ be a zero of L, where for $\alpha\in(1/10,1/2)$,

$$\Delta=l_1^2-l_0l_2=4(511-2706\alpha+5292\alpha^2-4536\alpha^3+1440\alpha^4)>0.$$

Since $\xi\in(0,4)$, $L(x)\geq 0$, when $x\in[0,\xi]$, and $L(x)\leq 0$, when $x\in[\xi,4]$. Hence from (51), we deduce that

$$\begin{cases}|C|(|B|-4|A|)\geq|AB|, & \text{when } c\in[0,\hat{c}],\\ |C|(|B|-4|A|)\leq|AB|, & \text{when } c\in[\hat{c},4],\end{cases}$$

where $\hat{c}=\sqrt{\xi}\in(0,2)$.

We now consider the following sub-cases.

IV(a) Suppose first that $c\in(0,\hat{c}]$. Then by (49) and Lemma 3, we obtain

$$|H_{2,2}(f^{-1})|\leq\frac{1}{6}(1-\alpha)^2c(4-c^2)[-|A|+|B|+|C|]=\varphi_2(c^2),$$

where

$$\varphi_2(x)=\frac{1}{12}(1-\alpha)^2[-5(1-\alpha)^2x^2+4(2-3\alpha)x+12].$$

It is easily seen that

$$\varphi_2(x) \le \varphi_2(\tau) = \frac{1}{15}(19 - 42\alpha + 24\alpha^2),$$

where $\tau \in (0, \xi)$ is given by (37). Thus

$$|H_{2,2}(f^{-1})| \le \frac{1}{15}(19 - 42\alpha + 24\alpha^2) = \Phi_2(\alpha) \tag{52}$$

when $c \in (0, \hat{c}]$, where Φ_2 is given by (30).

IV(b) Next, let $c \in [\hat{c}, 2)$. Then by (49) and Lemma 3, we have

$$\begin{aligned} |H_{2,2}(f^{-1})| &\le \frac{1}{6}(1-\alpha)^2 c(4-c^2)(|A|+|C|)\sqrt{1 - \frac{B^2}{4AC}} \\ &= \frac{1}{3}(1-\alpha)^2 g_1(c^2)\sqrt{g_2(c^2)}, \end{aligned} \tag{53}$$

where

$$g_1(x) = (1-\alpha)(2-5\alpha)x^2 - 2x + 12,$$

and

$$g_2(x) = \frac{24\alpha^2 - 36\alpha + 13 - (1-\alpha)^2 x}{(1-2\alpha)(9-10\alpha)(12+x)}.$$

Now define $G(x) = g_1(x)\sqrt{g_2(x)}$. Then

$$2(1-2\alpha)(9-10\alpha)(12+x)^2\sqrt{g_2(x)}G'(x) = M(x) := m_3x^3 + m_2x^2 + m_1x + m_0, \tag{54}$$

where

$$m_3 = -4(1-\alpha)^3(2-5\alpha),$$
$$m_2 = -38 + 163\alpha - 161\alpha^2 - 24\alpha^3 + 60\alpha^4,$$
$$m_1 = 2(647 - 3948\alpha + 8772\alpha^2 - 8352\alpha^3 + 2880\alpha^4),$$

and

$$m_0 = -12(77 - 204\alpha + 132\alpha^2).$$

We note that

$$M'(0) = m_1 > 0$$

and

$$M'(4) = 2(303 - 2240\alpha + 6112\alpha^2 - 6816\alpha^3 + 2640\alpha^4) > 0.$$

When $\alpha \in (1/10, 2/5]$, since $m_3 \le 0$, the function M' is concave on $[0,4]$. So

$$M'(x) \ge \min\{M'(0), M'(4)\} > 0, \quad \text{when } \ x \in [0,4]. \tag{55}$$

When $\alpha \in (2/5, 1/2)$, $m_3 > 0$ and

$$M''(x) = 6m_3x + 2m_2 \ge 2m_2 > 0, \quad \text{when } x \in [0,4].$$

Thus M' is increasing on $[0,4]$, and $M'(x) \ge M'(0) > 0$ holds for $x \in [0,4]$. Therefore when $\alpha \in (1/10, 1/2)$, the function M is increasing on $[0,4]$, so that

$$M(x) \le M(4) = 4(1-2\alpha)(9-10\alpha)(87 - 388\alpha + 284\alpha^2), \quad \text{when } x \in [0,4].$$

Now let $\alpha_1 := (97 - 4\sqrt{202})/142 = 0.282\cdots$ be the zero of the polynomial $87 - 388x + 284x^2$ in $[0,1]$.

IV(b1) When $\alpha \in [\alpha_1, 1/2)$, $M(x) \le M(4) \le 0$ for $x \in [0,4]$, and so by (54), $G'(x) \le 0$ for $x \in [0,4]$. This, together with (53), gives

$$|H_{2,2}(f^{-1})| \le \frac{1}{3}(1-\alpha)^2 G(c^2) \le \frac{1}{3}(1-\alpha)^2 G(\xi) = \Phi_4(\alpha), \quad \text{for } c \in [\hat{c}, 2), \tag{56}$$

where Φ_4 is defined by (32). Thus from (48), (52), and (56) we obtain

$$|H_{2,2}(f^{-1})| \le \max\{\Phi_1(\alpha), \Phi_2(\alpha), \Phi_4(\alpha)\},$$

and so from (33), we have

$$|H_{2,2}(f^{-1})| \le \Phi_2(\alpha), \quad \text{for } c \in [0,2].$$

IV(b2) Next assume that $\alpha \in (1/10, \alpha_1]$. Since $M(0) = m_0 < 0$, $M(4) > 0$ and M is increasing on $[0,4]$, there is a unique $\mu \in (0,4)$ such that $M(\mu) = 0$. Hence from (54), $G(\mu)$ is the unique local minimum, and G is convex on $[0,4]$. This implies that

$$G(x) \le \max\{G(\xi), G(4)\}, \quad x \in [\xi, 4],$$

which gives

$$|H_{2,2}(f^{-1})| \le \max\{\Phi_4(\alpha), \Phi_3(\alpha)\},$$

where Φ_3 and Φ_4 are defined by (31) and (32). Thus from (48), (52), (56) and (33), we obtain

$$\begin{aligned} |H_{2,2}(f^{-1})| &\le \max\{\Phi_i(\alpha) : i = 1,2,3,4\} \\ &= \begin{cases} \Phi_1(\alpha), & \text{when } \alpha \in (1/10, \alpha_0], \\ \Phi_2(\alpha), & \text{when } \alpha \in [\alpha_0, \alpha_1], \end{cases} \end{aligned}$$

which gives (34).

We finally find the extreme functions.

On the interval $\alpha \in [0, \alpha_0]$, consider the function $f \in \mathcal{S}^*(\alpha)$ defined by (35). Then $a_2 = 2(1-\alpha)$, $a_3 = (1-\alpha)(3-2\alpha)$ and $a_4 = 2(1-\alpha)(2-\alpha)(3-2\alpha)/3$. Hence by (29), we obtain $H_{2,2}(f^{-1}) = (1-\alpha)^2(1-2\alpha)(9-10\alpha)/3$.

On the interval $\alpha \in [\alpha_0, 2/3]$, consider the function $f \in \mathcal{S}^*(\alpha)$ defined by (36). Then $a_2 = (1-\alpha)\tau$, $a_3 = (1-\alpha)(2 + (1-\alpha)\tau^2)/2$ and

$$a_4 = \frac{1}{6}(1-\alpha)\tau\left(8 - 6\alpha + (1-\alpha)^2\tau^2\right).$$

Again using (29) we obtain $H_{2,2}(f^{-1}) = -(19 - 42\alpha + 24\alpha^2)/15$, which gives equality in this case.

Finally on the interval $\alpha \in [2/3, 1)$, consider the function f defined by (38). Then clearly $H_{2,2}(f^{-1}) = -(1-\alpha)^2$. This completes the proof of Theorem 5. □

Theorem 6. *Let $\alpha \in [0,1)$, and f^{-1} be the inverse of $f \in \mathcal{K}(\alpha)$. Then*

$$|H_{2,2}(f^{-1})| \leq \begin{cases} \dfrac{1}{96}(12-28\alpha+19\alpha^2), & \text{if } \alpha \in [0,2/5], \\ \dfrac{1}{9}(1-\alpha)^2, & \text{if } \alpha \in [2/5,4/5], \\ \dfrac{\alpha(1-\alpha)^2(19\alpha-8)}{48(1+\alpha)(2\alpha-1)}, & \text{if } \alpha \in [4/5,1). \end{cases} \tag{57}$$

All the inequalities are sharp, with equality when $\alpha \in [0,2/5]$ for the inverse function of $f \in \mathcal{K}(\alpha)$ given by

$$f'(z) = \frac{1}{((1-z)^{1+\tau/2}(1+z)^{1-\tau/2})^{1-\alpha}}, \quad z \in \mathbb{D}, \tag{58}$$

when $\alpha \in [2/5,4/5]$ for the inverse function of $f \in \mathcal{K}(\alpha)$ given by

$$f'(z) = \frac{1}{(1-z^2)^{1-\alpha}}, \quad z \in \mathbb{D}, \tag{59}$$

and when $\alpha \in [4/5,1)$ for the inverse function of $f \in \mathcal{K}(\alpha)$ given by

$$f'(z) = \frac{1}{(1-\mu z+z^2)^{1-\alpha}}, \quad z \in \mathbb{D}, \tag{60}$$

where

$$\tau = \sqrt{\frac{2-5\alpha}{2(1-\alpha)^2}} \quad \textit{and} \quad \mu = \sqrt{\frac{5\alpha-4}{(1+\alpha)(2\alpha-1)}}. \tag{61}$$

Proof. Let $\alpha \in [0,1)$, and $f \in \mathcal{K}(\alpha)$ be given by (1), and let f^{-1} be the inverse of f given by (27).

Since $zf' \in \mathcal{S}^*(\alpha)$, (29), (40) and (41) give

$$H_{2,2}(f^{-1}) = \frac{1}{144}(1-\alpha)^2(6c_1c_3 - 4c_2^2 - 5(1-\alpha)c_1^2c_2 + 2(1-\alpha)^2c_1^4). \tag{62}$$

Then from Lemma 2, we obtain

$$\begin{aligned} H_{2,2}(f^{-1}) = \frac{1}{288}(1-\alpha)^2\Big(&\alpha(4\alpha-3)c_1^4 + (5\alpha-3)c_1^2(4-c_1^2)\zeta \\ &- (8+c_1^2)(4-c_1^2)\zeta^2 + 6c_1(4-c_1^2)(1-|\zeta|^2)\eta\Big), \end{aligned} \tag{63}$$

with ζ, $\eta \in \overline{\mathbb{D}}$.

Again we can assume that $c_1 := c \in [0,2]$, and so applying the triangle inequality, we obtain

$$|H_{2,2}(f^{-1})| \leq \frac{1}{288}(1-\alpha)^2[\alpha|4\alpha-3|c^4 + (4-c^2)h(c,|\zeta|)], \tag{64}$$

where $h : [0,2] \times [0,1] \to \mathbb{R}$ is defined by

$$h(x,y) = 6x + |5\alpha-3|x^2y + (2-x)(4-x)y^2.$$

Since the function $h(\cdot,y)$ increases in $[0,1]$, we obtain

$$|H_{2,2}(f^{-1})| \leq \frac{1}{72}(1-\alpha)^2K(c), \tag{65}$$

where

$$K(c) = k_1c^4 + k_2c^2 + 8,$$

and where

$$k_1 = \frac{1}{4}\left(\alpha|4\alpha-3| - |5\alpha-3| - 1\right) \quad \text{and} \quad k_2 = |5\alpha-3| - 1.$$

We again consider various cases.

I. When $\alpha \in [0, 2/5]$, $K(c) = -(1-\alpha)^2c^4 + (2-5\alpha)c^2 + 8$, and it can be easily seen that $K(c) \le K(\tau)$, when $c \in [0,2]$, where τ is given by (61). Hence by (65), we obtain

$$|H_{2,2}(f^{-1})| \le \frac{1}{96}(12 - 28\alpha + 19\alpha^2).$$

II. When $\alpha \in [2/5, 4/5]$, $K(c) \le K(0) = 8$, when $c \in [0,2]$, and so by (65)

$$|H_{2,2}(f^{-1})| \le \frac{1}{9}(1-\alpha)^2.$$

III. When $\alpha \in (4/5, 1)$, let

$$\Psi(\alpha) := \frac{\alpha(1-\alpha)^2(19\alpha-8)}{48(1+\alpha)(2\alpha-1)}.$$

We now show that $|H_{2,2}(f^{-1})| \le \Psi(\alpha)$.

We first note that by (64),

$$|H_{2,2}(f^{-1})| = \frac{1}{9}(1-\alpha)^2 < \Psi(\alpha), \quad \text{when } c = 0, \tag{66}$$

and

$$|H_{2,2}(f^{-1})| = \frac{1}{18}\alpha(1-\alpha)^2(4\alpha-3) < \Psi(\alpha), \quad \text{when } c = 2. \tag{67}$$

Now let $c \in (0,2)$. Then by (63),

$$|H_{2,2}(f^{-1})| \le \frac{1}{48}(1-\alpha)^2c(4-c^2)\Gamma(A,B,C), \tag{68}$$

where Γ is defined in (20), with

$$A = \frac{\alpha(4\alpha-3)c^3}{6(4-c^2)}, \quad B = \frac{(5\alpha-3)c}{6} \quad \text{and} \quad C = -\frac{c^2+8}{6c}.$$

Note that $AC < 0$. Simple computations show that $|B| > 2(1-|C|)$, and $B^2 > -4A(1-C^2)/C$ hold for $\alpha \in (4/5, 1)$, and $c \in (0,2)$. Thus the two first cases in (13) are not possible. Also, the first case in (14) is not possible. Indeed, the inequality $|AB| \ge |C|(|B| + 4|A|)$ is equivalent to $\varphi_1(c^2) \le 0$, where

$$\varphi_1(x) = (3-20\alpha)(1-\alpha)^2x^2 + 4(32\alpha^2 - 29\alpha + 3)x + 32(5\alpha-3),$$

and since φ_1 is concave on the interval $[0,4]$,

$$\varphi_1(0) = 32(5\alpha-3) > 0, \quad \text{and} \quad \varphi_1(4) = 80\alpha(3-\alpha)(4\alpha-3) > 0,$$

it follows that

$$\varphi_1(x) \geq \min\{\varphi_1(0), \varphi_1(4)\} > 0, \quad x \in [0,4].$$

Next note that

$$36(4-c^2)[|C|(|B|-4|A|)-|AB|] = l_2c^4 + 2l_1c^2 + l_0, \tag{69}$$

where

$$l_2 = 3 - 2\alpha + 11\alpha^2 - 20\alpha^3, \quad l_1 = 2(3+19\alpha-32\alpha^2), \quad \text{and} \quad l_0 = 32(5\alpha-3).$$

Let $L(x) = l_2x^2 + 2l_1x + l_0$, and $\xi = -(l_1 + \sqrt{\Delta})/l_2$ be the greatest zero of L, where

$$\Delta = l_1^2 - l_0l_2 = 12(27 - 18\alpha + 171\alpha^2 - 712\alpha^3 + 608\alpha^4) > 0.$$

Note that $l_2 < 0$ and $\xi \in (0,4)$. Therefore $L(x) \geq 0$, when $x \in [0,\xi]$, and $L(x) \leq 0$, when $x \in [\xi, 4]$. Thus from (69), we obtain

$$\begin{cases} |C|(|B|-4|A|) \geq |AB|, & \text{when } c \in (0,\hat{c}], \\ |C|(|B|-4|A|) \leq |AB|, & \text{when } c \in [\hat{c},4), \end{cases}$$

where $\hat{c} = \sqrt{\xi} \in (0,2)$.

III(a) Let $c \in (0,\hat{c}]$. Then by (68) and Lemma 3, we obtain

$$|H_{2,2}(f^{-1})| \leq \frac{1}{48}(1-\alpha)^2c(4-c^2)[-|A|+|B|+|C|] = \varphi_2(c^2), \tag{70}$$

where

$$\varphi_2(x) = \frac{1}{144}(1-\alpha)^2[-(1+\alpha)(2\alpha-1)x^2 + 2(5\alpha-4)x + 16]. \tag{71}$$

Note that $\varphi_2'(x) = 0$, when $x = \mu^2$, where μ is given by (61). Since $\mu^2 \in (0,\xi)$, and $\varphi_2''(x) = (1-\alpha)^2(1+\alpha)(1-2\alpha)/72 < 0$, $\varphi_2(x) \leq \varphi_2(\mu^2)$, when $x \in [0,\xi]$, and so from (70) we have

$$|H_{2,2}(f^{-1})| \leq \varphi_2(\mu^2) = \Psi(\alpha). \tag{72}$$

III(b) Next let $c \in [\hat{c}, 2)$. Then by (68) and Lemma 3, we obtain

$$\begin{aligned} |H_{2,2}(f^{-1})| &\leq \frac{1}{48}(1-\alpha)^2c(4-c^2)(|A|+|C|)\sqrt{1-\frac{B^2}{4AC}} \\ &= \frac{1}{192}(1-\alpha)^2g_1(c^2)\sqrt{g_2(c^2)}, \end{aligned} \tag{73}$$

where

$$g_1(x) = -(1-\alpha)(1+4\alpha)x^2 - 4x + 32,$$

and

$$g_2(x) = \frac{76\alpha^2 - 72\alpha + 12 - 3(1-\alpha)^2x}{3\alpha(4\alpha-3)(8+x)}.$$

It is easily seen that g_1, and g_2 are decreasing on $[\xi, 4]$, and so from (73) we obtain

$$|H_{2,2}(f^{-1})| \leq \frac{1}{192}(1-\alpha)^2g_1(\xi)\sqrt{g_2(\xi)} = \varphi_2(\xi) \leq \Psi(\alpha), \tag{74}$$

where φ_2 is defined by (71).

Thus from (66), (67), (72) and (74), we have

$$|H_{2,2}(f^{-1})| \leq \Psi(\alpha),$$

which is inequality (57).

We now show that the inequalities are sharp.

When $\alpha \in [0, 2/5]$, consider $f \in \mathcal{K}(\alpha)$ satisfying (58). Then $a_2 = (1-\alpha)\tau/2$, $a_3 = (1-\alpha)(2+(1-\alpha)\tau^2)/6$, and

$$a_4 = \frac{1}{24}(1-\alpha)\tau\left(8 - 6\alpha + (1-\alpha)^2\tau^2\right).$$

Hence from (29) we obtain $H_{2,2}(f^{-1}) = -(12 - 28\alpha + 19\alpha^2)/96$.

When $\alpha \in [2/5, 4/5]$, consider the function $f \in \mathcal{K}(\alpha)$ defined by (59). Then $a_2 = a_4 = 0$, and $a_3 = (1-\alpha)/3$, and so from (29) we have $H_{2,2}(f^{-1}) = -(1-\alpha)^2/9$.

Finally when $\alpha \in [4/5, 1)$, consider the function f satisfying (60). Then

$$a_2 = \frac{1}{2}(1-\alpha)\mu, \quad a_3 = \frac{1}{3}\left(-1 + \alpha + \frac{1}{2}(2-\alpha)(1-\alpha)\mu^2\right),$$

and

$$a_4 = -\frac{1}{24}\mu(2-\alpha)(1-\alpha)(6-(3-\alpha)\mu^2).$$

Hence from (29) we have

$$H_{2,2}(f^{-1}) = -\frac{\alpha(1-\alpha)^2(19\alpha - 8)}{48(1+\alpha)(2\alpha - 1)}.$$

This completes the proof of Theorem 6. □

When $\alpha = 0$ we deduce the following corollaries.

Corollary 1. *Let f^{-1} be the inverse of $f \in \mathcal{S}^*$. Then*

$$|H_{2,2}(f^{-1})| \leq 3.$$

Equality holds for rotations of $f(z) = z/(1-z)^2$.

Corollary 2. *Let f^{-1} be the inverse of $f \in \mathcal{K}$. Then*

$$|H_{2,2}(f^{-1})| \leq \frac{1}{8}.$$

Equality holds for rotations of $f(z) = 2z/(1 - z + \sqrt{1-z^2})$.

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