Global well-posedness of 2D stochastic Burgers equations with multiplicative noise

Guoli Zhou∗† Lidan Wang‡ § Jiang-Lun Wu¶

Abstract

In this article, we study 2D stochastic Burgers equations driven by linear multiplicative noise, and with non-periodic boundary conditions. We first apply Galerkin approximation method to show the local existence and uniqueness of strong solutions, we then establish the global well-posedness for strong solutions by utilizing the maximum principle.

MSC: Primary 60H15; Secondary 76S05.
Keywords: Stochastic 2D Burgers equations; global well-posedness; Galerkin approximation; maximum principle

1 Introduction

This article is concerned with 2D Stochastic Burgers equations (SBEs) prescribed on a smooth, bounded, open domain \( D \subset \mathbb{R}^2 \). For arbitrarily fixed \( T > 0 \), let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})\) be a given stochastic basis. Set \( W := \sum_{k=1}^{\infty} \sigma_k B_k(t) \), for \( t \in [0,T] \), \( \sigma_k \in \mathbb{R} \) with \( \sum_{k=1}^{\infty} \sigma_k^2 < \infty \), where \( \{B_k(t)\}_{t \in [0,T]}, k \in \mathbb{N} \) is a sequence of independent, one-dimensional \( \{\mathcal{F}_t\}_{t \in [0,T]} \)-Wiener processes. On the other, let us denote \( \Delta := \partial^2_{x_1} + \partial^2_{x_2} \) the Laplace operator, and \( \nabla := (\partial_{x_1}, \partial_{x_2}) \) the gradient operator. We consider the Cauchy problem for the following 2D SBE driven by linear multiplicative noise, and subject to the Dirichlet boundary condition

\[
\begin{align*}
    du(t) &= \nu \Delta u dt - (u \cdot \nabla) u dt + u(dW(t), \\
    u(t,x) &= 0, \quad t > 0, \; x = (x_1, x_2) \in \partial D, \quad (BC) \\
    u(0,x) &= u_0(x), \quad x = (x_1, x_2) \in D, \quad (IC)
\end{align*}
\]

for the unknowns 2D vector-valued random fields \( u(t,x) = (u_1(t,x), u_2(t,x)) \in \mathbb{R}^2 \) for \((t,x) \in [0,T] \times D\), where the parameter \( \nu > 0 \) stands for the viscosity and \( \circ \) denotes the Stratonovich integral. The mathematical study of the Burgers equation was originated in a series of articles (chronologically) by Forsyth [12], Beteman [3], and Burgers [8]. The case of scalar SBEs (i.e., \( \mathbb{R} \)-valued random fields \( u \)) has been pretty well studied by Bertini, Cancrini and Jona-Lasinio in [4] and Da Prato, Debussche and Temam in [9], just mention a few.

Towards the case of higher dimensional inviscid SBEs, stationary solutions and stationary distributions were constructed by Iturriaga and Khanin in [15]. Based on the stochastic version of Lax formula, Gomes,
Iturriaga, Khanin and Padilla in [13] proved the convergence of stationary distributions for the randomly forced multi-dimensional Burgers equations when viscosity tends to zero. Utilizing some inventive techniques, Brzeźniak, Goldys and Neklyudov [5] established the global well-posedness of multidimensional Burgers equations with additive noise effecting only on one coordinate. Furthermore, the asymptotic behavior of solutions is studied when the viscosity tends to zero. For the potential case, one can see [7, 16, 11] and other references therein.

In the present work, we consider the global well-posedness of 2D SBEs with Dirichlet boundary conditions driven by linear multiplicative noise. Here, we should point out that the absence of the incompressible property and high nonlinearity of 2D SBEs bring difficulties to establish a priori estimates even in $L^2$ space. To overcome the difficulties, we heavily rely on the random version of maximum principle (Lemma 4.1) and an argument of compactness and regularity of the solutions to 2D SBEs (see the proof of Theorem 4.1). In the forthcoming work by the same authors, the global well-posedness of 2D SBEs with periodic boundary conditions is established without the help of Poincaré’s inequality, hence requires more delicate techniques.

In a recent paper [19], Zhang, Zhou, Guo and Wu studied 3D stochastic Burgers equations with the noise perturbing only one coordinate, with the initial data lying in $L^\infty(D)$. The present article aims to extend the noise to all the coordinates and drops the assumption that the initial data should lie in $L^\infty(D)$. Due to the dimension is 2 and the noise is multiplicative, instead of utilizing the contraction principle argument, we first do a martingale type transform, then adopt the techniques from partial differential equations (PDEs), that is, we first derive energy estimates in a more regular spaces, then establish the local existence of the solutions by applying comparison principle. Also, since the noise perturbs all coordinates, we need to adopt a new version of maximum principle (see Lemma 4.1), and derive a priori estimates to help us obtain the global well-posedness.

The maximum principle stated in Lemma 4.1 is the key tool to establish the global existence of the strong solution to (1.1) with (BC)-(IC). It is well known that the maximum principle should be applied to the classical solutions to differential equations. However, for stochastic partial differential equations and stochastic ordinary differential equations, there is no classical solution. Therefore, we can not consider the global well-posedness for 2D SBEs with nonlinear multiplicative noise. The novelty of the present paper is that we apply the maximum principle to the random Galerkin approximations. Then, utilizing the classic compactness arguments shown in Lemma 2.1 and Lemma 2.2, we obtain a subsequence of the solutions converging almost everywhere with respect to time to the solutions to (1.1) with (BC)-(IC). To refine the almost everywhere result, we make use of the continuity of the local solutions with respect to time, and achieve the result that a priori estimates hold for any time.

Finally, we would like to point out the differences between [5] and the present work. Firstly, the boundary conditions in [5] are periodic, while in our present article the boundary conditions are Dirichlet. Secondly, the noise considered in [5] is additive and the noise exists only in one coordinate, while, in our article we deal with (linear) multiplicative noise which perturbs on all the coordinates. Finally, in [5], the proof of local existence of solutions relies on the semigroup method, while here we use the Galerkin approximation approach.

The rest of this article is organized as follows. Some preliminaries are presented in Section 2, local well-posedness result of the stochastic system is stated and proved in Section 3. In Section 4, we establish the global well-posedness of the 2D stochastic Burgers equation.

2 Preliminaries

We first introduce the notations that will be used throughout this article. For $p \in \mathbb{N}^+$, let $L^p(D; \mathbb{R}^2)$ be the vector-valued $L^p$-space in which the norm is denoted by $\| \cdot \|_p$. When $p = 2$, denote by $\mathbb{H} := L^2(D; \mathbb{R}^2)$ and its associated norm and inner product are $\| \cdot \|_2$ and $\langle \cdot, \cdot \rangle$, respectively. Moreover, when $p = \infty$, $L^\infty(D; \mathbb{R}^2)$ stands for the collection of vector-valued functions which are essentially bounded on $D$ and the corresponding norm is denoted by $\| \cdot \|_\infty$.

For $m \in \mathbb{N}^+$, $(W^{m,p}(D), \| \cdot \|_{m,p})$ is the classical Sobolev space. When $p = 2$, denote by $\mathbb{H}^m(D) = W^{m,2}(D)$, and

$$\|u\|_{m,p}^2 = \sum_{0 \leq \varepsilon \leq m} \int_D |D^\varepsilon u|^2 dx,$$
where $\delta := (\delta_1, \delta_2)$ is a multi-index with nonnegative integers $\delta_1, \delta_2$, and $|\delta| = \delta_1 + \delta_2$. It is well known that $(H^m(D), \| \cdot \|_m)$ is a Hilbert space. Let $C_c^\infty(D)$ be the space of all infinitely differentiable functions on $D$ with compact support. Let $W_0^{1,p}$ be the closure of $C_c^\infty(D)$ in $W^{1,p}(D)$. Set $H_0^1 = W_0^{1,2}$ and let $H^{-1}$ be the dual space of $H_0^1$.

In this article, we simply deal with the case that the viscosity $\nu = 1$. In fact, $\nu$ can be any strictly positive real. Denote by $A := -\Delta$, then $A : H_0^1 \to H^{-1}$ and $D(A) = [H^2 \cap H_0^1]^2$, $A$ is a positive self-adjoint operator with discrete spectrum in $H$, that is, there exists an orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$ in $H$ and a sequence of increasing real values $\{\lambda_k\}_{k \in \mathbb{N}}$ such that $Ae_k = \lambda_k e_k$.

For any $u \in L^2(D)$, denote by $u_k = \langle u, e_k \rangle$. Given $s \in \mathbb{R}$, the fractional power $(A^s, D(A^s))$ of the operator $(A, D(A))$ is defined by

$$A^s u = \sum_{n=1}^\infty \lambda_n^s u_n e_n, \quad \text{where} \quad u = \sum_{n=1}^\infty u_n e_n; \quad D(A^s) = \{u = \sum_{n=1}^\infty u_n e_n \mid \sum_{n=1}^\infty \lambda_n^{2s} |u_n|^2 < \infty \}.$$ 

We then set $H_s := D(A^{s/2})$ and denote by $\| \cdot \|_s$ the seminorm $|A^{s/2} \cdot|_2$.

Next, let us introduce strong solutions to (1.1) with (BC)-(IC).

**Definition 2.1** (Local strong solutions). Let the stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W)$ be given as before. Suppose $u_0$ is a $\mathbb{H}^1$-valued, $\mathcal{F}_0$-measurable random variable.

(i) A pair $(u, \tau)$ is a local strong pathwise solution to (1.1) with (BC)-(IC) if $\tau$ is a strictly positive stopping time and $u(\cdot, \tau)$ is an $\mathcal{F}_t$-adapted process in $\mathbb{H}^1$ so that

$$u(\cdot, \tau) \in L^2\left(\Omega; C([0, \infty); \mathbb{H}^1)\right), \quad u_{\mathbb{1}_{t \leq \tau}} \in L^2\left(\Omega; L^2_{loc}([0, \infty); \mathbb{H}^2)\right);$$

and for any $t \geq 0$, the following identity holds in $\mathbb{H}$,

$$u(t \wedge \tau) - \int_0^{t \wedge \tau} \Delta u(s) ds + \int_0^{t \wedge \tau} (u \cdot \nabla u)(s) ds = u_0 + \sum_{k=1}^\infty \int_0^{t \wedge \tau} \sigma_k u(s) \circ dB_k(s).$$

(ii) Strong pathwise solutions of (1.1) are said to be unique up to a random positive time $\tau > 0$ if given any pair of solutions $(u^1, \tau), (u^2, \tau)$, which coincide at $t = 0$ on the event $\Omega = \{u^1(0) = u^2(0)\} \subset \Omega$, then

$$\mathbb{P}(\mathbb{1}_\Omega (u^1(t \wedge \tau) - u^2(t \wedge \tau))) = 0; \forall t \geq 0 = 1.$$

**Definition 2.2** (Maximal and global strong solutions).

(i) Let $\xi$ be a positive random variable. We say that $(u, \xi)$ is a maximal pathwise strong solution if $(u, \tau)$ is a local strong pathwise solution for each $\tau < \xi$ and $\sup_{t \in [0, \xi]} \|u\|_1 = \infty$ almost surely on the set $\{\xi < \infty\}$.

(ii) If $(u, \xi)$ is a maximal pathwise strong solution and $\xi = \infty$ a.s., then we we say the solution is global.

Let $\alpha : [0, \infty] \times \Omega \to \mathbb{R}$ be the solution to the following Stratonovich stochastic differential equation:

$$d\alpha(t) = \sum_{k=1}^\infty \sigma_k \alpha(t) \circ dB_k(t), \quad \text{for } t \geq 0, \text{ and } \alpha(0) = 1. \quad (2.2)$$

Applying Itô’s formula, we have for any $t \geq 0$, $\alpha(t) = \exp\{\sum_{k=1}^\infty \sigma_k B_k(t)\}$. By Novikov’s condition and Doob’s maximal inequality, we know that for any $T > 0$,

$$\mathbb{E}\left[\sup_{t \in [0, T]} |\alpha(t)|\right] < \infty.$$ 

We now make change of variables by $v(t) = \alpha(t) u(t)$, then $v(t)$ satisfies the following equation:

$$dv(t) = \Delta v(t) dt + \alpha(t)^{-1}(v(t) \cdot \nabla) v(t) dt, \quad (2.3a)$$
\( v(t, x) = 0, \quad (t, x) \in [0, T] \times \partial D; \quad (2.3b) \)
\( v(0, x) = u_0(x), \quad x = (x_1, x_2) \in D. \quad (2.3c) \)

We first consider the Galerkin approximation of (2.3). For \( n \in \mathbb{N} \), let \( P_n \) be the projection from \( H \) onto the subspace expanded by \( \{e_1, e_2, \ldots, e_n\} \), that is, for \( u \in H \),
\[
P_n(u) = P_n \left( \sum_{k=1}^{\infty} \langle u, e_k \rangle e_k \right) = \sum_{k=1}^{n} \langle u, e_k \rangle e_k.
\]

Then we obtain the Galerkin approximation of (2.3) as follows:
\[
dv_n(t, x) = \Delta v_n(t, x) dt - \alpha(t)^{-1} P_n [ (v_n \cdot \nabla) v_n(t, x) ] dt, \quad (t, x) \in [0, T] \times D, \quad (2.4a)
\]
\[
v_n(t, x) = 0, \quad (t, x) \in [0, T] \times \partial D; \quad (2.4b)
\]
\[
v_n(0, x) = u_n(0, x), \quad x = (x_1, x_2) \in D. \quad (2.4c)
\]

Since (2.4) is a locally Lipschitz system of stochastic ODEs, there exists the unique local solution \( v_n \) to (2.4) with \( \tau_{n, \omega} \) as the maximal existence time of \( v_n \). Obviously, \( v_n \in C([0, \tau_{n, \omega}] \times D) \). To end this section, we present Aubin-Lions lemma and Lions-Magenes lemma as follows. One can refer to [18] for proof details.

**Lemma 2.1 (Aubin-Lions, [1]).** Let \( B_0, B, B_1 \) be Banach spaces such that \( B_0, B_1 \) are reflexive and \( B_0 \subset B \subset B_1 \). Define for \( 0 < T < \infty \),
\[
X := \{ h \in L^2([0, T]; B_0), \frac{dh}{dt} \in L^2([0, T]; B_1) \}.
\]

Then \( X \) is a Banach space equipped with the norm \( \| h \|_{L^2([0, T]; B_0)} + \| \frac{dh}{dt} \|_{L^2([0, T]; B_1)} \). Moreover, \( X \subset L^2([0, T]; B) \).

**Lemma 2.2 (Lions-Magenes, [17]).** Let \( V, H, V' \) be three Hilbert spaces such that \( V \subset H = H' \subset V' \), where \( H' \) and \( V' \) are the dual spaces of \( H \) and \( V \) respectively. Suppose \( u \in L^2(0, T; V) \) and \( u' \in L^2(0, T; V') \). Then \( u \) is almost everywhere equal to a function continuous from \([0, T]\) into \( H \).

## 3 Local well-posedness of stochastic Burgers equations

In this section, we use the Galerkin approximation method to show the local existence of the solutions to (2.3).

**Proposition 3.1.** Assume that the initial data \( u_0 \in \mathbb{H}^1 \) is \( \mathcal{F}_0 \)-measurable, there exists a random variable \( \tau > 0 \) such that the unique strong solution \( v \) to the equation (2.3) on the interval \([0, \tau]\) satisfies
\[
\sup_{t \in [0, \tau]} \left\| v(t) \right\|^2_1 + \int_0^\tau \left\| v(t) \right\|^2_2 dt < \infty, \quad \mathbb{P}\text{-a.s.} \quad \omega \in \Omega. \tag{3.1}
\]

Moreover, \( v \) is Lipschitz continuous with respect to the initial data in \( \mathbb{H}^1 \).

**Proof.** For any \( t \in (0, \tau_{n, \omega}) \), taking inner product of (2.4) with \(-\Delta v_n \) in \( L^2([0, t] \times D) \), then by Hölder’s inequality, Young’s inequality, interpolation inequality and Sobolev imbedding theorem,
\[
\left\| v_n(t) \right\|^2_1 + 2 \int_0^t \left\| v_n(s) \right\|^2_2 ds \leq \left\| u_0 \right\|^2_1 + \int_0^t \alpha^{-1}(s) \int_D |(v_n \cdot \nabla) v_n(s, x)| |\Delta v_n(s, x)| dx ds
\]
\[
\leq \left\| u_0 \right\|^2_1 + \varepsilon \int_0^t \left\| v_n(s) \right\|^2_2 ds + C_\varepsilon \int_0^t \alpha^{-2}(s) \left\| \nabla v_n(s) \right\|^2_0 \left\| v_n(s) \right\|^2_1 ds
\]
\[
\leq \left\| u_0 \right\|^2_1 + \varepsilon \int_0^t \left\| v_n(s) \right\|^2_2 ds + C \int_0^t \alpha^{-4}(s) \left\| v_n(s) \right\|^2_0 ds
\]
\[
\leq \left\| u_0 \right\|^2_1 + \varepsilon \int_0^t \left\| v_n(s) \right\|^2_2 ds + C \int_0^t \alpha^{-4}(s) \left\| v_n(s) \right\|^2_0 ds.
\]
For $t \in [0, 1]$, we have
\[
\|v_n(t)\|^2_t + \int_0^t \|v_n(s)\|^2 ds \leq \|u_0\|^2 + C \int_0^t \left[ \sup_{t \in [0,1]} \alpha^{-4/3}(s) \|v_n(s)\|^3 \right] ds \\
=: \|u_0\|^2 + \int_0^t \left[ K \|v_n(s)\|^3 \right] ds.
\] (3.2)

Applying comparison theorem (see Theorem III-5-1 in [14]), with $t_\ast := \frac{1}{2K\|u_0\|^3}$, one can get
\[
\|v_n(t)\|^2 \leq \|u_0\|^2 \left[ 1 - 2tK\|u_0\|^3 \right]^{1/2} = \|u_0\|^2 \left( 1 - H_t^{-1} \right)^{1/2}.
\] (3.3)

Hence, for any $n$, the estimate in (3.3) rules out the blowup of $v_n$ in $H^1$ before the time $t_\ast$. We can choose $\tau(\omega) := \frac{t(\omega)}{2} > 0$ such that $\tau(\omega)$ does not depend on $n \in \mathbb{N}$. It follows from (3.2) and (3.3) that $v_n$ are uniformly bounded in $L^\infty([0, \tau]; H^1) \cap L^2([0, \tau]; H^2)$. Now back to (2.4), by Hölder inequality and Sobolev imbedding theorem,
\[
|\partial_s v_n| \leq \alpha^{-1} |v_n \cdot \nabla v_n| + |\Delta v_n| \leq \alpha^{-1} |v_n|_{\infty} |\nabla v_n| + \|v_n\|_2 \leq c \alpha^{-1} \|v_n\|_{1/2}^3 \|v_n\|_{1/2}^2 + \|v_n\|_2,
\] (3.4)
where $c$ is a constant that is independent of $n, s$. Thus, $\partial_s v_n$ is uniformly bounded in $L^2([0, \tau]; H^1)$. By Lemma 2.1 and Lemma 2.2, there exists a subsequence of $v_n$, which, for convenience, is still denoted by $v_n$, such that $v_n$ converges to $v$ in $L^2([0, \tau]; H^1)$ and $v \in C([0, \tau]; H^1)$. Following a standard argument, it can be verified that $v$ is the strong solution to (2.3) and the estimate (3.1) follows analogously from (3.2) and (3.3). It remains to show the uniqueness. Let $v_1, v_2$ be two strong solutions to (2.3) with $v_1(0) = v_2(0) = u_0$, and let $\bar{v} = v_1 - v_2$, then we have
\[
\left( \frac{1}{2} \partial_s \|\bar{v}\|^2 + \|\bar{v}\|^2 \right) \leq \alpha^{-1} (\bar{v} \cdot \nabla v_1, \Delta \bar{v}) + \alpha^{-1} (v_2 \cdot \nabla \bar{v}, \Delta \bar{v}) \\
\leq \alpha^{-1} (\bar{v} \bar{v}, \|v_2\|^2 + \|\bar{v}\|^2) + \alpha^{-1} (\bar{v} \bar{v}, \|v_2\|_{1/2}^2 \|v_2\|_1 + \|\bar{v}\|_{1/2}^2 \|v_1\|_1) \\
\leq c \|\bar{v}\|^3 + c \alpha^{-1} \|v_2\|^2 \|v_1\|_1 + c \alpha^{-1} \|v_2\|^2 \|v_1\|_1.
\]
By Gronwall’s inequality, with $\bar{v}(0) = 0$, $\|\bar{v}\|_1 = 0$ for $s \in [0, \tau]$. The Lipschitz continuity of the local strong solution with respect to the initial data in $H^1$ also follows from the above estimate. \qed

4 Global well-posedness of stochastic Burgers equations

To establish the global well-posedness, we utilize the maximal principle stated as follows:

**Lemma 4.1.** If $v_n$ is a solution to (2.4) on the time interval $[0, t]$, then $\sup_{s \in [0, t]} |v_n(s)|_\infty \leq |v_n(0)|_\infty$.

**Proof.** For any $\beta > 0$, set $f(s, x) := e^{-\beta s} v_n(s, x)$ for any $s \in [0, t]$ and $x \in D$. Taking inner product of (2.4) with $v_n$ on both sides gives that
\[
\partial_s |v_n(s)|^2 + 2\alpha(s) |v_n(s) \cdot \nabla v_n(s)|^2 - 2(\Delta v_n \cdot v_n)(s) = 0.
\]
With $|v_n(s)|^2 = |f(s)|^2 e^{2\beta s}$ and $2\Delta f(s) \cdot f(s) = |f(s)|^2 - 2|\nabla f|^2$, we get that
\[
\partial_s |f(s)|^2 + 2\beta |f(s)|^2 + 2\alpha(s) |f(s) \cdot \nabla |f(s)|^2 - \Delta |f(s)|^2 + 2|\nabla f|^2 = 0.
\] (4.1)

Note that if $|f(s, x)|$ achieves the local maximum for $(s, x) \in (0, t) \times D$, then the left-hand side of (4.1) is strictly positive unless $|f(t, x)| \equiv 0$. Therefore, $|f(s)|_{\infty} \leq |f(0)|_{\infty}$, and this yields that
\[
|v_n(s)|_{\infty} \leq e^{\beta s} |v_n(0)|_{\infty}, \quad \text{for any } s \in (0, t].
\]
The result follows by letting $\beta$ go to 0. \qed
Theorem 4.1. For any $F_0$-adapted initial data $u_0 \in H^1$ and any $T > 0$, there exists a unique global strong solution $v$ to (2.3) in the sense of Definition 2.2. Furthermore, $v \in C([0,T];H^1) \cap L^2([0,T];H^2)$ and $v$ is Lipschitz continuous with respect to initial data in $H^1$.

Proof. Taking inner product of (2.4) with $-\Delta v_n$ in $H$ and applying Young’s inequality gives that
\begin{equation}
\partial_t \|v_n\|^2 \leq 2\alpha(t)^{-1} \int_D (v_n \cdot \nabla) v_n \Delta v_n \, dx - 2\|v_n\|_{L^2}^2 \leq \alpha^{-2}(t)\|v_n\|^2_{H^2} \|v_n\|^2_1 - \|v_n\|^2.
\end{equation}
By Gronwall’s inequality and Lemma 4.1, we have
\begin{equation}
\|v_n(t)\|_1 + \int_0^t \|v_n(s)\|^2_2 \, ds \leq \|u_0\|^2_1 \exp \left\{ c \|u_0\|^2_1 \int_0^t \alpha(s)^{-2} \, ds \right\}.
\end{equation}
Now let $\tau$ be the maximum existence time of the local strong solution $v$ to (2.3). Hence, for any $\hat{\tau} < \tau$, $\{v_n\}$ is uniformly bounded in $L^2([0, \hat{\tau}];H^2)$. By Lemma 2.1 and Lemma 2.2, there exists a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$, converging to $v$ in $L^2([0, \hat{\tau}];H^1)$.

Acknowledgements We are very grateful to Prof. Zdzislaw Brzezniak for his stimulation and encouragement. We also express our heartfelt thanks to the anonymous referees who devoted much time to reading the article and provided many insightful suggestions.

References


