The Journal of Analysis
https://doi.org/10.1007/s41478-021-00374-x

ORIGINAL RESEARCH PAPER

On the difference of inverse coefficients of convex functions

Young Jae Sim[1] · Derek K. Thomas[2]

Received: 22 August 2021 / Accepted: 16 November 2021

Abstract
Let f be analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{S}$ be the subclass of normalised univalent functions given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ for $z \in \mathbb{D}$. Let F be the inverse function of f defined in some set $|\omega| \leq r_0(f)$, and be given by $F(\omega) = \omega + \sum_{n=2}^{\infty} A_n \omega^n$. We prove the sharp inequalities $-1/3 \leq |A_4| - |A_3| \leq 1/4$ for the class $\mathcal{K} \subset \mathcal{S}$ of convex functions, thus providing an analogue to the known sharp inequalities $-1/3 \leq |a_4| - |a_3| \leq 1/4$, and giving another example of an invariance property amongst coefficient functionals of convex functions.

Keywords Difference of coefficients · Convex functions

Mathematics Subject Classification 30C45 · 30C50 · 30C55

1 Introduction and definitions

Let $\mathcal{A}$ denote the class of analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f'(0) - 1$. Then for $z \in \mathbb{D}$, $f \in \mathcal{A}$ has the following representation

Communicated by Samy Ponnusamy.

✉ Derek K. Thomas
d.k.thomas@swansea.ac.uk

Young Jae Sim
yjsim@ks.ac.kr

[1] Department of Mathematics, Kyungsung University, Busan 48434, Korea

[2] Department of Mathematics, Swansea University, Bay Campus, Swansea SA1 8EN, UK

Published online: 09 January 2022

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

In 1985, de Branges [1] solved the famous Bieberbach conjecture by showing that if $f \in \mathcal{S}$, then $|a_n| \le n$ for $n \ge 2$, with equality when $f(z) = k(z) := z/(1-z)^2$, or a rotation. It was therefore natural to ask if for $f \in \mathcal{S}$, the inequality $||a_{n+1}| - |a_n|| \le 1$ is true when $n \ge 2$. This was shown not to be the case even when $n = 2$ [2], and that the following sharp bounds hold.

$$-1 \le |a_3| - |a_2| \le \frac{3}{4} + e^{-\lambda_0}(2e^{-\lambda_0} - 1) = 1.029\cdots,$$

where λ_0 is the unique value of λ in $0 < \lambda < 1$, satisfying the equation $4\lambda = e^{\lambda}$.

Hayman [3] showed that if $f \in \mathcal{S}$, then $||a_{n+1}| - |a_n|| \le C$, where C is an absolute constant. The exact value of C is unknown, the best estimate to date being $C = 3.61\cdots$ [4], which because of the sharp estimate above when $n = 2$, cannot be reduced to 1.

Although the expected inequality $||a_{n+1}| - |a_n|| \le 1$ has been verified for $n \ge 2$ by Leung [5] in the case of starlike functions $\mathcal{S}^*$, sharp upper and lower bounds for $|a_{n+1}| - |a_n|$ are only known when $n = 2$ for some subclasses of $\mathcal{S}$, such as the classes $\mathcal{K}$ of convex and close-to-convex functions [6, 7]. An exception to this was provided by Ming and Sugawa [8], who showed that if $f \in \mathcal{K}$, then $-1/3 \le |a_4| - |a_3| \le 1/4$, and that both of these inequalities are sharp. It turns out that finding sharp bounds for $||a_{n+1}| - |a_n||$ in the case of convex functions presents a significantly difficult problem.

If $f \in \mathcal{S}$, then since f is univalent, it possesses an inverse function F, given by $F(\omega) = \omega + \sum_{n=2}^{\infty} A_n \omega^n$ defined in some set $|\omega| \le r_0(f)$.

Little information is known about the the difference of coefficients of the inverse functions for $f \in \mathcal{S}$, and even finding the order of growth of $||A_{n+1}| - |A_n||$ appears to be an open problem. On the other hand sharp upper and lower bounds for $|A_{n+1}| - |A_n|$ have recently been found when $n = 2$ for a number of subclasses of $\mathcal{S}$ [9].

In this paper we will show that if $f \in \mathcal{K}$, then $-1/3 \le |A_4| - |A_3| \le 1/4$, providing another example of an invariance property amongst coefficient functionals for convex functions, noticed in [9–11].

We note first that equating coefficients easily gives

$$A_2 = -a_2, \quad A_3 = 2a_2^2 - a_3 \qquad A_4 = -5a_2^3 + 5a_2a_3 - a_4. \tag{1.2}$$

2 Preliminary Lemmas

Denote by $\mathcal{P}$, the class of analytic functions p with positive real part on $\mathbb{D}$ given by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{2.1}$$

We will use the following lemmas for the coefficients of functions $\mathcal{P}$, given by (2.1).

Lemma 2.1 [2, p. 41] *For $p \in \mathcal{P}$, $|c_n| \leq 2$ for $n \geq 1$. The inequalities are sharp.*

Lemma 2.2 [10] *If $p \in \mathcal{P}$ is of the form (2.1) with $c_1 \geq 0$, then*

$$2c_2 = c_1^2 + (4 - c_1^2)\zeta \tag{2.2}$$

and

$$4c_3 = c_1^3 + (4 - c_1^2)c_1\zeta(2 - \zeta) + 2(4 - c_1^2)(1 - |\zeta|^2)\eta \tag{2.3}$$

for some $\zeta, \eta \in \overline{\mathbb{D}}$.

The next lemma is a special case of more general results due to Choi et al. [12] (see also [13]). Let $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$, and define

$$Y(A, B, C) := \max_{z \in \overline{\mathbb{D}}} \left(|A + Bz + Cz^2| + 1 - |z|^2 \right), \quad A, B, C \in \mathbb{R}.$$

Lemma 2.3 [12] *If $AC \geq 0$, then*

$$Y(A, B, C) = \begin{cases} |A| + |B| + |C|, & |B| \geq 2(1 - |C|), \\ 1 + |A| + \dfrac{B^2}{4(1 - |C|)}, & |B| < 2(1 - |C|). \end{cases}$$

If $AC < 0$, then

$$Y(A, B, C) = \begin{cases} 1 - |A| + \dfrac{B^2}{4(1 - |C|)}, & \left(-4AC(C^{-2} - 1) \leq B^2 \right) \wedge \left(|B| < 2(1 - |C|) \right), \\ 1 + |A| + \dfrac{B^2}{4(1 + |C|)}, & B^2 < \min\left\{ 4(1 + |C|)^2, -4AC(C^{-2} - 1) \right\}, \\ R(A, B, C), & \text{otherwise}, \end{cases}$$

where

$$R(A, B, C) = \begin{cases} |A| + |B| - |C|, & |C|(|B| + 4|A|) \leq |AB|, \\ -|A| + |B| + |C|, & |AB| \leq |C|(|B| - 4|A|), \\ (|C| + |A|)\sqrt{1 - \dfrac{B^2}{4AC}}, & \text{otherwise}. \end{cases} \tag{2.4}$$

We recall that the discriminant Δ_P of the real quadratic polynomial $P(t) = \alpha + 2\beta t + \gamma t^2$ is defined as $\beta^2 - \alpha\gamma$. We will allow a degenerate case such as $\gamma = 0$. The next lemma will also feature in our proof.

Lemma 2.4 *Let $P(t)$ and $Q(t)$ be (possibly degenerate) real quadratic polynomials. Suppose that $P(t) > 0$ and $Q(t) > 0$ on an interval $I \subset \mathbb{R}$ and that $\Delta_P > 0$. If there exists a positive constant T such that*

(i) $\Delta_Q \geq T^{3/2}\Delta_P$, and
(ii) $TP(t) \geq Q(t)$ for $t \in I$,

then the function $G(t) = \sqrt{P(t)} - \sqrt{Q(t)}$ is convex on I.

3 Preliminary results

In order to prove the main result in this paper we will need the following two theorems.

Theorem 3.1 *Let $f \in \mathcal{K}$ and be given by (1.1). Then*

$$|A_4| \leq \begin{cases} \dfrac{1}{24}(4 + 21s^2 + s^3), & \text{if } 0 \leq s \leq \dfrac{2}{7}, \\ s(1 - 2s^2), & \text{if } \dfrac{2}{7} \leq s \leq \sqrt{\dfrac{5}{59}}, \\ \dfrac{(1 + 5s^2)\sqrt{25 - s^2}}{12\sqrt{6}}, & \text{if } \sqrt{\dfrac{5}{59}} \leq s \leq \sqrt{\dfrac{5}{11}}, \\ \dfrac{1}{3}s(2 + s^2), & \text{if } \sqrt{\dfrac{5}{11}} \leq s \leq 1, \end{cases} \tag{3.1}$$

where $s = |a_2| \in [0, 1]$. All the inequality are sharp.

Proof We note at this point that if $f \in \mathcal{K}$, then $|a_2| \leq 1$, and we can write $1 + zf''(z)/f'(z) = p(z)$, for some $p \in \mathcal{P}$ with the form (2.1). So equating coefficients gives

$$a_2 = \frac{1}{2}c_1, \quad a_3 = \frac{1}{6}(c_1^2 + c_2) \quad \text{and} \quad a_4 = \frac{1}{24}(c_1^3 + 3c_1c_2 + 2c_3). \tag{3.2}$$

Hence from (1.2) we obtain

$$A_4 = \frac{1}{24}(-6c_1^3 + 7c_1c_2 - 2c_3). \tag{3.3}$$

Since $\mathcal{P}$ and $|A_4|$ are rotationally invariant, using Lemma 2.1, we may assume that $c_1 = c$ with $0 \leq c \leq 2$. Now using Lemma 2.2, we obtain

$$48A_4 = -6c^3 + 5c(4 - c^2)\zeta + c(4 - c^2)\zeta^2 - 2(4 - c^2)(1 - |\zeta|^2)\eta, \tag{3.4}$$

where $\eta \in \overline{\mathbb{D}}$. If $a_2 = c/2 = 1$, then $A_4 = -1$, and so the inequality (3.1) is true when $s = |a_2| = 1$. Also if $a_2 = c/2 = 0$, then $|A_4| = (1 - |\zeta|^2)|\eta|/6 \le 1/6$, and so the inequality (3.1) is true when $s = |a_2| = 0$.

We now assume that $0 < a_2 < 1$, or equivalently, $0 < c < 2$. Then, since $|\eta| \le 1$, (3.4) implies that

$$48|A_4| \le 2(4 - c^2)[|A + B\zeta + C\zeta^2| + (1 - |\zeta|^2)], \tag{3.5}$$

where

$$A = \frac{-3c^3}{4 - c^2}, \quad B = \frac{5}{2}c, \quad C = \frac{1}{2}c.$$

We note that $AC < 0$.

(a) We first consider the case $0 < a_2 \le 2/7$. Since $a_2 = c/2$, we have $0 < c \le 4/7$. and the condition $-4AC(C^{-2} - 1) \le B^2$ and $|B| < 2(1 - |C|)$ in Lemma 2.3 is satisfied. Therefore from Lemma 2.3 we obtain

$$48|A_4| \le 2(4 - c^2)\left[1 - |A| + \frac{B^2}{4(1 - |C|)}\right] = \frac{1}{4}(32 + 42c^2 + c^3). \tag{3.6}$$

We note that for $c \ge 4/7$ (i.e. the case $a_2 \ge 2/7$), the conditions $|B| \ge 2(1 - |C|)$ and $-4AC(C^{-2} - 1) \ge B^2$ are valid. Hence from Lemma 2.3 we have

$$\max_{\zeta \in \overline{\mathbb{D}}} \left\{ |A + B\zeta + C\zeta^2| + (1 - |\zeta|^2) \right\} = R(A, B, C),$$

where R is given by (2.4).

(b) For the case $2/7 \le a_2 \le \sqrt{5/59}$, the conditions $4/7 \le c \le 2\sqrt{5/59}$ and $|AB| \le |C|(|B| - 4|A|)$ are valid, and so Lemma 2.3 gives

$$48|A_4| \le 2(4 - c^2)(-|A| + |B| + |C|) = 12c(2 - c^2). \tag{3.7}$$

(c) For the case $\sqrt{5/11} \le a_2 < 1$ we have $2\sqrt{5/11} \le c < 2$, and the condition $|C|(|B| + 4|A|) \le |AB|$ is satified. Therefore Lemma 2.3 gives

$$48|A_4| \le 2(4 - c^2)(|A| + |B| - |C|) = 2c(8 + c^2). \tag{3.8}$$

(d) For the case $\sqrt{5/59} \le a_2 \le \sqrt{5/11}$, we have $2\sqrt{5/59} \le c \le 2\sqrt{5/11}$, and the conditions $|C|(|B| + 4|A|) \ge |AB|$ and $|AB| \ge |C|(|B| - 4|A|)$ are satified. Therefore Lemma 2.3 gives

$$48|A_4| \le 2(4 - c^2)(|A| + |C|)\sqrt{1 - \frac{B^2}{4AC}} = \frac{1}{2}c(4 + 5c^2)\sqrt{\frac{50}{3c^2} - \frac{1}{6}}. \tag{3.9}$$

Finally replacing c in (3.6), (3.7), (3.8) and (3.9) by $2s$, we obtain the inequalities in (3.1).

We now show that the inequalities in (3.1) are sharp by constructing extreme functions for each case.

Given $p \in \mathcal{P}$, let $f_p \in \mathcal{K}$ be defined by

$$1 + \frac{zf_p''(z)}{f_p'(z)} = p(z). \tag{3.10}$$

For the case $2/7 \le s \le \sqrt{5/59}$, we consider a function $p_1 \in \mathcal{P}$ defined by $p_1(z) = (1 + 2sz + z^2)/(1 - z^2)$. Then it is easy to see that the coefficients of p_1 are given by $c_1 = 2s$, $c_2 = 2$ and $c_3 = 2s$. From (3.3), we obtain $A_4 = s(1 - 2s^2)$.,and so the inequality (3.1) in this case is sharp for $f_{p_1} \in \mathcal{K}$.

In a similar way we can see that the inequality (3.1) in the case $\sqrt{5/11} \le s \le 1$ is sharp for $f_{p_2} \in \mathcal{K}$, where

$$p_2(z) = \frac{1 - z^2}{1 - 2sz + z^2} = 1 + 2sz + (4s^2 - 2)z^2 + (8s^3 - 6s)z^3 + \cdots.,$$

and the inequality (3.1) in the case $0 \le s \le 2/7$ is sharp for $f_{p_3} \in \mathcal{K}$, where

$$\begin{aligned} p_3(z) &= \frac{2 - 3sz + 3sz^2 - 2z^3}{2 - 7sz - 7sz^2 + 2z^3} \\ &= 1 + 2sz + s(5 + 7s)z^2 + \frac{1}{2}(-4 + 49s^2 + 49s^3)z^3 + \cdots. \end{aligned}$$

Finally we consider the case $\sqrt{5/59} \le s \le \sqrt{5/11}$. Let $p_4 \in \mathcal{P}$ be defined by

$$p_4(z) = \frac{1 + s(\xi + 1)z + \xi z^2}{1 + s(\xi - 1)z - \xi z^2},$$

where $\xi = e^{i\theta}$ with

$$\theta = \arccos\left(\frac{5(1 - 7s^2)}{24s^2}\right), \tag{3.11}$$

then it is easy to see that the coefficients of p_4 are given by

$$c_1 = 2s, \quad c_2 = 2(s^2(1 - \xi) + \xi) \quad \text{and} \quad c_3 = 2s[s^2(1 - \xi)^2 + (2 - \xi)\xi],$$

which by (3.3) implies that

$$A_4 = \frac{1}{6}s[\xi(5 + \xi) - s^2(6 + 5\xi + \xi^2)]. \tag{3.12}$$

Since $\xi = e^{i\theta}$, simple computations using (3.11) show that

$$
\begin{aligned}
&|\xi(5+\xi) - s^2(6+5\xi+\xi^2)|^2 \\
&= |(1-7s^2)\cos\theta + 5(1-s^2) + i(1+5s^2)\sin\theta|^2 \\
&= -24s^2(1-s^2)\cos^2\theta + 10(1-s^2)(1-7s^2)\cos\theta + 25(1-s^2)^2 + (1+5s^2)^2 \\
&= \frac{(25-s^2)(1+5s^2)^2}{24s^2}.
\end{aligned}
$$

Hence from (3.12) we obtain

$$
|A_4| = \frac{1}{6}s|\xi(5+\xi) - s^2(6+5\xi+\xi^2)| = \frac{(1+5s^2)\sqrt{25-s^2}}{12\sqrt{6}},
$$

and so (3.1) in the case $\sqrt{5/59} \le s \le \sqrt{5/11}$ is sharp for $f_{p_4} \in \mathcal{K}$, which completes the proof of Theorem 3.1. □

We next denote by $\mathcal{K}_+$ the class of functions $f \in \mathcal{A}$ with non-negative second coefficient, and prove the following.

Theorem 3.2 *Let $f \in \mathcal{K}_+$ and be given by* (1.1)*. Then*

$$
|A_3 + A_4| \le \begin{cases} \frac{1}{3}(1-s)(-1+2s+6s^2), & \frac{4}{7} \le s \le 1, \\ \frac{1}{24}(8-16s+25s^2+s^3), & 0 \le s \le \frac{4}{7}, \end{cases} \tag{3.13}
$$

where $s = a_2 \in [0,1]$. The inequalities are sharp.

Proof Since $f \in \mathcal{K}_+$ we can write $1 + zf''(z)/f'(z) = p(z)$, for some $p \in \mathcal{P}$ given by (2.1). From (3.2) we obtain (3.3) and

$$
A_3 = 2a_2^2 - a_3 = \frac{1}{6}(2c_1^2 - c_2).
$$

We also note that $c_1 \in [0,2]$ since $a_2 \ge 0$, and so

$$
A_3 + A_4 = \frac{1}{24}(8c_1^2 - 6c_1^3 + 7c_1c_2 - 4c_2 - 2c_3).
$$

Thus using Lemma 2.2 we have

$$
48(A_3 + A_4) = 6c^2(2-c) + (5c-4)(4-c^2)\zeta + c(4-c^2)\zeta^2 - 2(4-c^2)(1-|\zeta|^2)\eta, \tag{3.14}
$$

where $\zeta,\ \eta \in \overline{\mathbb{D}}$.

If $c = 2s = 2$, then $A_3 + A_4 = 0$, hence (3.13) is true when $s = 1$.

Next assume that $0 \le s < 1$ (i.e. $0 \le c < 2$). Then since $|\eta| \le 1$, (3.14) implies that

$$
48|A_3 + A_4| \le 2(4-c^2)[|A + B\zeta + C\zeta^2| - (1-|\zeta|^2)],
$$

where

$$A = \frac{3c^2}{2+c}, \quad B = \frac{5c-4}{2}, \quad C = \frac{1}{2}c.$$

We note that $AC \geq 0$.

(a) First we consider the case $8/7 \leq c < 2.$, then it is easy to see that $|B| \geq 2(1 - |C|)$ is valid, and so by Lemma 2.3 we obtain

$$\begin{aligned} 48|A_3 + A_4| &\leq 2(4-c^2)(|A| + |B| + |C|) \\ &= 4(2-c)(-2+2c+3c^2). \end{aligned} \tag{3.15}$$

(b) Next consider the case $0 \leq c < 8/7$, then the condition $|B| < 2(1 - |C|)$ is satisfied, and so by Lemma 2.3, we obtain

$$\begin{aligned} 48|A_3 + A_4| &\leq 2(4-c^2)(1 + |A| + \frac{B^2}{4(1-|C|)}) \\ &= \frac{1}{4}(64 - 64c + 50c^2 + c^3). \end{aligned} \tag{3.16}$$

Replacing c in (3.15) and (3.16) by $2s$, we therefore obtain (3.13) when $s \in [0, 1)$.

Now we shall show the inequality (3.13) is sharp. For given $p \in \mathcal{P}$, we recall the function f_p defined by (3.10). For the case $4/7 \leq s \leq 1$, we can easily check that the equality in (3.13) holds for $f_{p_1} \in \mathcal{K}$, where $p_1(z) = (1 + 2sz + z^2)/(1 - z^2)$. For the case $0 \leq s \leq 4/7$, we can check the inequality (3.13) is sharp for $f_{p_5} \in \mathcal{K}$, where $p_5 \in \mathcal{P}$ is defined by

$$\begin{aligned} p_5(z) &= \frac{(1-z)(2+(4-3s)z+2z^2)}{(1+z)(2-7sz+2z^2)} \\ &= 1 + 2sz + (-2+3s+7s^2)z^2 + \frac{1}{2}s(-24+21s+49s^2)z^3 + \cdots. \end{aligned}$$

Now the proof of Theorem 3.2 is finished. □

Proposition 3.3 *For a fixed constant $c \in [1, 2]$, define $F_c : [0, 1] \times [-1, 1] \to \mathbb{R}$ by*

$$\begin{aligned} F_c(r,t) = {} & 16(1 + 6r^2 + r^4) - 4c^2(1 + 31r^2 + 2r^4 + 24rt + 10r^3t) \\ & + c^4[36 + r^4 + 60rt + 10r^3t + r^2(13 + 24t^2)]. \end{aligned} \tag{3.17}$$

Then $F_c(r,t) \geq 0$ for all $(r,t) \in [0,1] \times [-1,1]$.

Proof We first show that F_c does not have any critical points in $(0,1) \times (-1,1)$. Since

$$\frac{\partial F_c}{\partial t}(r,t) = 2c^2r[-4(12+5r^2) + c^2(30 + 5r^2 + 24rt)],$$

$(\partial F_c)/(\partial t)(r,t) = 0$ when $t = t_0$, where

$$t_0 = \frac{48 + 20r^2 - 5c^2(6 + r^2)}{24c^2 r}.$$

Furthermore a simple computation gives

$$\frac{\partial F_c}{\partial r}(r,t)\bigg|_{t=t_0} = -\frac{1}{6} r\varphi_c(r),$$

where $\varphi_c(r) = (4 - c^2)^2 r^2 - 6(32 - 108c^2 + c^4)$. It is now easy to see that $\varphi_c(r) > 0$ for $r \in (0,1)$. Thus the system of equations $(\partial F_c)/(\partial t) = 0 = (\partial F_c)/(\partial r) = 0$ has no roots in $(0,1) \times (-1,1)$.

Next we show that $F_c \geq 0$ on the boundary of $[0,1] \times [-1,1]$.

(i) We first note that $F_c(0,t) = 4(4 - c^2 + 9c^4) > 0$.

(ii) Let $g_1(t) = F_c(1,t)$, $t \in [-1,1]$. If $c \leq 2\sqrt{17/59}$, then g_1 is decreasing on $[-1,1]$, since

$$g_1'(t) = -136c^2 + 70c^4 + 48c^4 t \leq c^2(-136 + 118c^2) \leq 0, \quad t \in [-1,1].$$

Thus $g_1(t) \geq g_1(1) = 16(8 - 17c^2 + 9c^4) \geq 0$, $t \in [-1,1]$. If $c > 2\sqrt{17/59}$, we have $g_1'(t_0) = 0$, where $t_0 = (68 - 35c^2)/(24c^2) \in (-1,1)$. Since $g_1''(t_0) > 0$ we have

$$g_1(t) \geq g_1(t_0) = \frac{1}{24}(-1552 + 1496c^2 - 25c^4) \geq 0, \quad t \in [-1,1].$$

(iii) For a fixed $r \in [0,1]$, define $G_1 : [1,4] \to \mathbb{R}$ by $G_1(x) = b_2 x^2 + b_1 x + b_0$, where

$$b_2 = (6 - 5r + r^2)^2, \quad b_1 = -4(1 - 24r + 31r^2 - 10r^3 + 2r^4) \quad \text{and}$$
$$b_0 = 16(1 + 6r^2 + r^4).$$

Then since $b_2 \geq 0$,

$$G_1'(x) = 2b_2 x + b_1 \geq 2b_2 + b_1 = 68 - 24r - 50r^2 + 20r^3 - 6r^4 > 0, \quad x \in [1,4],$$

and so $G_1(x) \geq G_1(1) = 48 + 36r + 9r^2 + 30r^3 + 9r^4 > 0$. Thus $G_1(c^2) > 0$ holds for all $c \in [1,4]$, and $0 \leq r \leq 1$, and we obtain the inequality $F_c(r,-1) > 0$ for $r \in [0,1]$ since $F_c(r,-1) = G_1(c^2)$.

(iv) Now let G_2 be defined by $G_2(x) = b_2 x^2 + b_1 x + b_0$, where

$$b_2 = (6 + 5r + r^2)^2, \quad b_1 = -4(1 + 24r + 31r^2 + 10r^3 + 2r^4) \quad \text{and}$$
$$b_0 = 16(1 + 6r^2 + r^4).$$

By the same methods used in (iii) we have

$$G_2(x) \geq G_2(1) = 3(1-r)(16+4r+7r^2-3r^3) \geq 0, \quad x \in [1,4],$$

and so the identity $F_c(r,1) = G_2(c^2)$ shows that $F_c(r,1) \geq 0$, $r \in [0,1]$, which completes the proof of Proposition 3.3.

□

Since all the proofs of the inequalities in the following proposition are similar to those used in the proof of Proposition 3.3 above, we omit the details.

Proposition 3.4 *Define* $G_1 : [0,2] \times [0,1] \to \mathbb{R}$ *by*

$$G_1(c,r) = -6c^2 - 5r(4-c^2) + (4-c^2)r^2.$$

Then $G_1(c,r) \leq 0$ for all $(c,r) \in [0,2] \times [0,1]$.

We are able to now state and prove our main result.

4 Main result

Theorem 4.1 *Let* $f \in \mathcal{K}$ *and be given by* (1.1). *Then*

$$-\frac{1}{3} \leq |A_4| - |A_3| \leq \frac{1}{4}. \tag{4.1}$$

Both inequalities are sharp.

Proof Since $f \in \mathcal{K}$ we again write $1 + zf''(z)/f'(z) = p(z)$ for some $p \in \mathcal{P}$ with the form (2.1), then equating coefficients gives

$$a_2 = \frac{1}{2}c_1, \quad a_3 = \frac{1}{6}(c_1^2 + c_2) \quad \text{and} \quad a_4 = \frac{1}{24}(c_1^3 + 3c_1c_2 + 2c_3).$$

Thus from (1.2) we have

$$|A_4| - |A_3| = \frac{1}{24}|-6c_1^3 + 7c_1c_2 - 2c_3| - \frac{1}{6}|2c_1^2 - c_2|.$$

Since $\mathcal{P}$ and $|A_4| - |A_3|$ are rotationally invariant, using Lemma 2.1, we may assume that $c_1 = c$ with $0 \leq c \leq 2$. Moreover by Theorem 3.1, $|A_4| - |A_3| \leq |A_4| \leq 1/4$ when $0 \leq c \leq 2\sqrt{5/59}$, and so it is enough to consider c satisfying $2\sqrt{5/59} \leq c \leq 2$.

We now use Lemma 2.2, and the fact that $|\eta| \leq 1$ to obtain

$$\begin{aligned} &48(|A_4| - |A_3|) \\ &\leq |-6c^3 + 5c(4-c^2)\zeta + c(4-c^2)\zeta^2| - |12c^2 - 4(4-c^2)\zeta| + 2(4-c^2)(1-|\zeta|^2) \\ &=: \Psi_U(c,\zeta), \end{aligned} \tag{4.2}$$

where $c \in [2\sqrt{5/59}, 2]$ and $\zeta \in \overline{\mathbb{D}}$. We now prove that

$$\Psi_U(c,\zeta) \leq 12$$

for $c \in [2\sqrt{5/59}, 2]$ and $\zeta \in \overline{\mathbb{D}}$. Note that since $\Psi_U(c,0) = 8 - c^2(14 - 6c) \leq 8$, we may assume that $\zeta \in \overline{\mathbb{D}}$ with $\zeta \neq 0$.

I(a) We first assume that $2\sqrt{5/59} \leq c \leq 1$, then

$$\begin{aligned}&\Psi_U(c,\zeta)\\ &\leq |-6c^2 + 5(4-c^2)\zeta + (4-c^2)\zeta^2| - |12c^2 - 4(4-c^2)\zeta| + 2(4-c^2)(1-|\zeta|^2).\end{aligned} \tag{4.3}$$

Putting $\zeta = re^{i\theta}$, $r \in (0,1]$, $\theta \in \mathbb{R}$, and $t = \cos\theta \in [-1,1]$, we have

$$\begin{aligned}&|-6c^2 + 5(4-c^2)\zeta + (4-c^2)\zeta^2| - |12c^2 - 4(4-c^2)\zeta|\\ &= \sqrt{b_0 + b_1 t + b_2 t^2} - \sqrt{d_0 + d_1 t}\\ &=: \Lambda_1(c,r,t),\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}b_0 &= b_0(c,r) := 36c^4 + 25(4-c^2)^2 r^2 + (4-c^2)^2 r^4 + 12c^2(4-c^2)r^2,\\ b_1 &= b_1(c,r) := -60c^2(4-c^2)r + 10(4-c^2)^2 r^3,\\ b_2 &= b_2(c,r) := -24c^2(4-c^2)r^2,\\ d_0 &= d_0(c,r) := 16[9c^4 + r^2(4-c^2)^2],\\ d_1 &= d_1(c,r) := -96rc^2(4-c^2).\end{aligned} \tag{4.5}$$

(i) Now assume that Λ_1 is increasing with respect to $t \in [-1,1]$. Then by (4.3) and (4.4), we have

$$\Psi_U(c,\zeta) \leq \Lambda_1(c,r,1) + 2(4-c^2)(1-r^2) =: H_1(c,r),$$

where

$$\begin{aligned}H_1(c,r) = &|-6c^2 + 5(4-c^2)r + (4-c^2)r^2|\\ &- |12c^2 - 4(4-c^2)r| + 2(4-c^2)(1-r^2).\end{aligned}$$

Let

$$k_1 = \sqrt{\frac{4r}{r+3}} \quad \text{and} \quad k_2 = \sqrt{\frac{4r(5+r)}{6+5r+r^2}}. \tag{4.6}$$

We note that $0 < k_1 \leq k_2 \leq \sqrt{2}$ for $r \in (0,1]$ and consider the following cases.

(a) If $c \leq k_1$, then

$$H_1(c,r) = 8 + 4r - 4r^2 + c^2(4 - r + r^2)$$
$$\leq \frac{12(2 + 3r - r^2)}{3 + r} \leq 12, \quad r \in (0,1].$$

(b) If $k_1 \leq c \leq k_2$, then

$$H_1(c,r) = 8 + 36r - 4r^2 + c^2(-20 - 9r + r^2)$$
$$\leq \frac{12(2 + 3r - r^2)}{3 + r} \leq 12, \quad r \in (0,1].$$

(c) If $c \geq k_2$, then

$$H_1(c,r) = c^2(-8 + r + 3r^2) + 4(2 - r - 3r^2)$$
$$\leq \frac{48(1 - 3r - 2r^2)}{6 + 5r + r^2} \leq 8 < 12, \quad r \in (0,1].$$

Thus by (a), (b) and (c), $H_1(c,r) \leq 12$ follows, and $\Psi_U(c,\zeta) \leq 12$ holds.

(ii) Next assume that Λ_1 is decreasing with respect to $t \in [-1,1]$. Then by Proposition 3.4 and the inequality $12c^2 + 4(4 - c^2)r \geq 0$, we obtain

$$\begin{aligned}\Psi_U(c,\zeta) &\leq \Lambda_1(c,r,-1) + 2(4 - c^2)(1 - r^2)\\ &= |G_1(c,r)| - |12c^2 + 4(4 - c^2)r| + 2(4 - c^2)(1 - r^2)\\ &= 4(2 + r - 3r^2) + c^2(-8 - r + 3r^2)\\ &\leq 4(2 + r - 3r^2) < 12,\end{aligned}$$

since $-8 - r + 3r^2 < 0$, and G_1 is the function defined by (3.5). Thus we obtain $\Psi_U(c,\zeta) < 12$.

(iii) Now we assume that Λ_1 is neither increasing nor decreasing with respect to $t \in [-1,1]$, and let $\lambda_1(t) = \Lambda_1(\cdot,\cdot,t)$. Then there exists $t_1 \in (-1,1)$ such that $\lambda_1'(t_1) = 0$, which implies that

$$\Lambda_1(\cdot,\cdot,t_1) = \lambda_1(t_1) = \left(\frac{b_1 + 2b_2t_1 - d_1}{d_1}\right)\sqrt{d_0 + d_1t_1}.$$

Since $b_2 < 0$,

$$\begin{aligned}b_1 + 2b_2t_1 - d_1 &> b_1 + 2b_2 - d_1\\ &= 2r(4 - c^2)[20r^2 + c^2(18 - 24r - 5r^2)] > 0.\end{aligned} \tag{4.7}$$

and so using $d_1 < 0$ and (4.7), it follows that $\lambda_1(t_1) \leq 0$. Hence it follows from (4.3), (4.4), (i) and (ii) that

$$\Psi_U(c,\zeta) \le \max_{t\in\Omega_1} \Lambda_1(c,r,t) + 2(4-c^2)(1-r^2) \le 12,$$

where

$$\Omega_1 = \{-1,1\} \cup \{t_1 \in (-1,1) : \lambda_1'(t_1) = 0\}.$$

I(b) Now assume that $1 \le c \le 2$, then by (4.2) we have

$$\Psi_U(c,\zeta) \le (4-c^2)\left[\tilde{\Psi}(c,\zeta) + 2(1-|\zeta|^2)\right] \le 3\left[\tilde{\Psi}(c,\zeta) + 2(1-|\zeta|^2)\right],$$

where $\tilde{\Psi}(c,\zeta)$ is defined by

$$\tilde{\Psi}(c,\zeta) = \left|\frac{-6c^3}{4-c^2} + 5c\zeta + c\zeta^2\right| - \left|\frac{12c^2}{4-c^2} - 4\zeta\right|.$$

We now show that

$$\tilde{\Psi}(c,\zeta) + 2(1-|\zeta|^2) \le 4. \tag{4.8}$$

A computation putting $\zeta = re^{i\theta}$, $r \in (0,1]$ and $\theta \in \mathbb{R}$, gives

$$\tilde{\Psi}(c,\zeta) = \frac{c}{4-c^2}\sqrt{b_0 + b_1 t + b_2 t^2} - \frac{1}{4-c^2}\sqrt{d_0 + d_1 t},$$

where $t = \cos\theta \in [-1,1]$ and b_i, $i \in \{0,1,2\}$, and $d_j, j \in \{0,1\}$, are given by (4.5). So the inequality (4.8) is equivalent to

$$\begin{aligned} &c^2(b_0 + b_1 t + b_2 t^2) - (d_0 + d_1 t) - 4(1+r^2)^2(4-c^2)^2 \\ &\quad\le 4(1+r^2)(4-c^2)\sqrt{d_0 + d_1 t}. \end{aligned} \tag{4.9}$$

Next write

$$\begin{aligned} &c^2(b_0 + b_1 t + b_2 t^2) - (d_0 + d_1 t) \\ &\quad - 4(1+r^2)^2(4-c^2)^2 = -(4-c^2)F_c(r,t), \end{aligned}$$

where F_c is defined by (3.17). Since $c \in [1,2]$, Proposition 3.3, shows that $F_c(r,t) \ge 0$, which gives

$$c^2(b_0 + b_1 t + b_2 t^2) - (d_0 + d_1 t) - 4(1+r^2)^2(4-c^2)^2 \le 0.$$

Therefore since $4(1+r^2)(4-c^2)\sqrt{d_0+d_1 t} \ge 0$, (4.9) is true. Thus (4.8) follows, and $\Psi_U(c,\zeta) \le 12$ holds.

Thus the proof of the upper bound in (4.1) is complete.

II. We next prove the lower bound in (4.1). Since $\mathcal{P}$ and $|A_3| - |A_4|$ are rotationally invariant, using Lemma 2.1 we may assume that $c_1 = c$, with $0 \le c \le 2$. Moreover by Theorem 3.2, $|A_3| - |A_4| \le |A_3 + A_4| \le 1/3$ hold when $0 \le c \le 4/3$ or

$\sqrt{2} \le c \le 2$. Hence it is enough to consider c satisfying $4/3 \le c \le \sqrt{2}$.

Using Lemma 2.2 and the fact that $|\eta| \le 1$, we obtain

$$\begin{aligned} &48(|A_3| - |A_4|) \\ &\le |12c^2 - 4(4-c^2)\zeta| - |-6c^3 + 5c(4-c^2)\zeta + c(4-c^2)\zeta^2| + 2(4-c^2)(1-|\zeta|^2) \\ &=: \Psi_L(c,\zeta), \end{aligned}$$

where $\zeta \in \overline{\mathbb{D}}$, and so we must show that

$$\Psi_L(c,\zeta) \le 16 \tag{4.10}$$

holds for all $c \in [4/3, \sqrt{2}]$ and $\zeta \in \overline{\mathbb{D}}$.

Note that we can assume that $\zeta \neq 0$, since

$$\Psi_L(c,0) = 8 + 2c^2(5-3c) < 16.$$

Also since $c \ge 4/3$, we have

$$\begin{aligned} &|12c^2 - 4(4-c^2)\zeta| - |-6c^3 + 5c(4-c^2)\zeta + c(4-c^2)\zeta^2| \\ &< |12c^2 - 4(4-c^2)\zeta| - \frac{4}{3}|-6c^2 + 5(4-c^2)\zeta + (4-c^2)\zeta^2| \\ &= 4(4-c^2)[|A^* - \zeta| - \frac{1}{3}|-2A + 5\zeta + \zeta^2|], \end{aligned} \tag{4.11}$$

where $A^* = 3c^2/(4-c^2)$. Moreover putting $\zeta = re^{i\theta}$, $r \in (0,1]$, $\theta \in \mathbb{R}$, and $t = \cos\theta \in [-1,1]$, we have

$$|A^* - \zeta| - \frac{1}{3}|-2A^* + 5\zeta + \zeta^2| = \sqrt{d_0 + d_1 t} - \sqrt{(b_0 + b_1 t + b_2 t^2)/9} =: \Lambda_2(c,r,t), \tag{4.12}$$

where

$$\begin{aligned} d_0 = A^2 + r^2, \quad d_1 = -2Ar, \\ b_0 = 25r^2 + (2A^* + r^2)^2, \quad b_1 = 10r(-2A^* + r^2), \quad b_2 = -8A^* r^2. \end{aligned}$$

In order to apply Lemma 2.4, we put

$$\Delta_1 = d_1^2, \quad \Delta_2 = \frac{1}{81}(b_1^2 - 4b_0 b_2) \quad \text{and} \quad T = \frac{5}{8}c^2.$$

We now show that

$$\Delta_2 \ge T^{3/2}\Delta_1, \tag{4.13}$$

and

$$T(d_0 + d_1 t) \geq \frac{1}{9}(b_0 + b_1 t + b_2 t^2), \quad t \in [-1, 1]. \tag{4.14}$$

First let

$$G(c, r) = \frac{\Delta_2}{\Delta_1} = \frac{(100 - c^2)[-4r^2 + c^2(-6 + r^2)]^2}{729c^4(4 - c^2)}. \tag{4.15}$$

Then it is easy to see that

$$G(c, r) \geq G(c, 0) = \frac{4(100 - c^2)}{81(4 - c^2)} \geq G\left(\frac{4}{3}, 0\right) = \frac{884}{405} > 2. \tag{4.16}$$

Since $c \leq \sqrt{2}$, we obtain

$$T^{3/2} = \left(\frac{5}{8}c^2\right)^{3/2} \leq \left(\frac{5}{4}\right)^{3/2} = \frac{5\sqrt{5}}{8} < 2. \tag{4.17}$$

Thus by (4.15), (4.16) and (4.17), we obtain (4.13).

Next we will show that (4.14) holds. To do this, we let

$$-\frac{1}{9}b_2 t^2 + \left(-\frac{1}{9}b_1 + Td_1\right)t + Td_0 - \frac{1}{9}b_0 = \frac{1}{72(4 - c^2)^2}G_2(t),$$

where

$$\begin{aligned} G_2(t) = {} & 45c^6(9 + r^6 rt) - 128r^2(25 + r^2 + 10rt) \\ & - 8c^4[36 + r^4 + 195rt + 10r^3 t + r^2(58 + 24t^2)] \\ & + 16c^2 r[4r^3 + 120t + 40r^2 t + r(121 + 48t^2)]. \end{aligned}$$

We consider the following 5 cases.

(a) Write

$$G_2(-1) = B_1 + B_2 r + B_3 r^2 + B_4 r^3 + B_5 r^4,$$

where

$$\begin{aligned} & B_1 = -288c^4 + 405c^6, \quad B_2 = -1920c^2 + 1560c^4 - 270c^6, \\ & B_3 = -3200 + 2704c^2 - 656c^4 + 45c^6, \quad B_4 = 1280 - 640c^2 + 80c^4, \\ & B_5 = -128 + 64c^2 - 8c^4. \end{aligned}$$

Then it is easy to check that

$$B_2 > 0, \quad B_4 > 0, \quad B_3 < 0, \quad B_5 < 0.$$

Therefore we obtain

$$\begin{aligned}B_1 + B_2 r + B_3 r^2 + B_4 r^3 + B_5 r^4 &\geq B_1 + B_3 + B_5 \\ &= -3328 + 2768c^2 - 952c^4 + 450c^6 \\ &> 0.\end{aligned}$$

Hence $G_2(-1) > 0$.

(b) We next show that $G_2(1) > 0$.
First note that

$$G_2(1) = B_6 + B_7 s + B_8 s^2 + B_9 s^3 =: H_1(s),$$

where

$$\begin{aligned}B_6 &= -128r^2(5 + r^2), \\ B_7 &= 16r(120 + 169r + 40r^2 + 4r^3), \\ B_8 &= -8(36 + 195r + 82r^2 + 10r^3 + r^4), \\ B_9 &= 45(3 + r)^2,\end{aligned}$$

and $s = c^2 \in [16/9, 2]$, and since $B_8 < 0$ and $B_9 > 0$, we have

$$\begin{aligned}H_1'(s) &= B_7 + 2B_8 s + 3B_9 s^2 \\ &\geq B_7 + 4B_8 + \frac{256}{27} B_9 \\ &= \frac{16}{3}(504 - 330r + 95r^2 + 60r^3 + 6r^4) \\ &> 0.\end{aligned}$$

Thus H_1 is increasing with respect to $s \in [16/9, 2]$, and so

$$\begin{aligned}H_1(s) &\geq H_1\left(\frac{16}{9}\right) \\ &= \frac{128}{81}(864 - 135r^2 - 250r^3 - 25r^4) > 0.\end{aligned}$$

Thus $G_2(1) > 0$ as claimed.

(c) We note that G_2 has its (unique) critical value at $t = t_0$, where

$$t_0 = \frac{5(-48c^2 + 27c^4 + 32r^2 - 8c^2r^2)}{192c^2 r}.$$

We now show that $G_2(t_0) > 0$, in which case $G_2(t) > 0$ for all $t \in [-1, 1]$, and (4.14) follows. Write

$$G_2(t_0) = \frac{1}{192c^2} G_3(c^2, r^2), \tag{4.18}$$

where

$$\begin{aligned} G_3(x,y) &= B_{10} + B_{11}y + B_{12}y^2, \\ B_{10} &= B_{10}(x) = 9x^2(-25600 + 29056x - 6660x^2 + 2025x^3), \\ B_{11} &= B_{11}(x) = -48x(6400 - 944x - 344x^2 + 45x^3), \\ B_{12} &= B_{12}(x) = -64(100 - x)(4 - x)^2. \end{aligned}$$

It is easy to see that $B_{11} < 0$ and $B_{12} < 0$, so

$$\begin{aligned} G_3(x,y) &\geq G_3(x,1) = B_{10} + B_{11} + B_{12} \\ &= -102400 - 254976x - 192000x^2 + 278080x^3 - 62100x^4 + 18225x^5 > 0. \end{aligned}$$

holds for $x \in [16/9, 2]$ and $y \in [0, 1]$. Thus we obtain $G_2(t_0) > 0$, by (4.18). Hence by Lemma 2.4, the function Λ_2 is convex on $[-1, 1]$, and the convexity of Λ_2 (on $[-1, 1]$), (4.11) and (4.12) implies that

$$\Psi_L(c,\zeta) \leq \max\{\Psi_L(c,r); \Psi_L(c,-r)\}, \tag{4.19}$$

where $c \in [4/3, \sqrt{2}]$ and $r \in (0, 1]$. Thus we will obtain (4.10) if we show that $\Psi_L(c,r) \leq 4$ and $\Psi_L(c,-r) \leq 4$ for $c \in [4/3, \sqrt{2}]$ and $r \in (0, 1]$, where

$$\begin{aligned} \Psi_L(c,-r) &= \left|3c^2 + r(4-c^2)\right| \\ &\quad - \frac{1}{3}\left|-6c^2 - 5r(4-c^2) + r^2(4-c^2)\right| + \frac{1}{2}(4-c^2)(1-r^2) \end{aligned}$$

and

$$\begin{aligned} \Psi_L(c,r) &= \left|3c^2 - r(4-c^2)\right| \\ &\quad - \frac{1}{3}\left|-6c^2 + 5r(4-c^2) + r^2(4-c^2)\right| + \frac{1}{2}(4-c^2)(1-r^2). \end{aligned}$$

(d) Clearly $3c^2 + r(4-c^2) \geq 0$ holds. Also

$$\begin{aligned} &-6c^2 - 5r(4-c^2) + r^2(4-c^2) = -(6 - 5r + r^2)c^2 - 20r + 4r^2 \\ &\leq -20r + 4r^2 < 0. \end{aligned}$$

Hence

$$\begin{aligned} \Psi_L(c,-r) &= 3c^2 + r(4-c^2) \\ &\quad + \frac{1}{3}[-6c^2 - 5r(4-c^2) + r^2(4-c^2)] + \frac{1}{2}(4-c^2)(1-r^2) \\ &= \frac{1}{6}[4(3 - 4r - r^2) + c^2(3 + 4r + r^2)] \\ &\leq \frac{1}{3}(9 - 4r - r^2) < 4, \end{aligned}$$

which gives $\Psi_L(c,-r) \leq 4$.

(e) Next clearly $3c^2 - r(4 - c^2) > 4(c^2 - 1) > 0$. If $-6c^2 + 5r(4 - c^2) + r^2(4 - c^2) \leq 0$, then, since $c \leq \sqrt{2}$, we have

$$\begin{aligned}\Psi_L(c, r) &= \frac{1}{6}[4(3 + 4r - r^2) + c^2(3 - 4r + r^2)] \\ &\leq \frac{1}{3}(9 + 4r - r^2) \leq 4.\end{aligned}$$

If $-6c^2 + 5r(4 - c^2) + r^2(4 - c^2) \geq 0$, then

$$c^2 \leq \frac{2(5r + r^2)}{6 + 5r + r^2}.$$

Using this inequality, we see that

$$\begin{aligned}\Psi_L(c, r) &= \frac{1}{6}[4(3 - 16r - 5r^2) + c^2(27 + 16r + 5r^2)] \\ &\leq \frac{12(1 + 3r)}{6 + 5r + r^2} \leq 4.\end{aligned}$$

Hence we obtain the desired inequality $\Psi_L(c, r) \leq 4$,and by (4.19), the inequality (4.10) is established. Thus the proof of the lower bound in (4.1) is complete.

We end the proof of Theorem 4.1 by giving extreme functions for the inequalities in (4.1).

First consider $f_1 \in \mathcal{K}$ defined by $1 + zf_1''(z)/f_1'(z) = (1 + z^2)/(1 - z^2)$. Comparing coefficients, we obtain $a_2 = 0$, $a_3 = 1/3$, $a_4 = 0$, and so from (1.2) we have $A_4 = 0$, $A_3 = -1/3$, and $|A_4| - |A_3| = -1/3$, so that the lower bound in (4.1) is sharp for f_1.

Next consider $f_2 \in \mathcal{K}$ defined by $1 + zf_2''(z)/f_2'(z) = (1 + z + z^2)/(1 - z^2)$. Then the coefficients of f_2 are given by $a_2 = 1/2$, $a_3 = 1/2$, $a_4 = 3/8$, and so from (1.2) we have $A_4 = 1/4$, $A_3 = 0$ and $|A_4| - |A_3| = 1/4$. Thus the upper bound in (4.1) is sharp for f_2. □

References

1. De Branges, L. 1985. A proof of the Bieberbach conjecture. *Acta Mathematics* 154 (1–2): 137–152.
2. Duren, P.L. 1983. *Univalent functions*. Berlin: Springer.
3. Hayman, W.K. 1963. On successive coefficients of univalent functions. *Journal of London Mathematical Society* 38: 228–243.
4. Grinspan, A.Z. 1976. *The sharpening of the difference of the moduli of adjacent coefficients of schlicht functions*, In Some problems in modern function theory Proceedings on Conference Modern Problems of Geometric Theory of Functions, Inst. Math., Acad. Sci. USSR, Novosibirsk, (Russian), Akad. Nauk SSSR Sibirsk. Otdel. Inst. Mat., Novosibirsk, 41–45.
5. Leung, Y. 1978. Successive coefficients of starlike functions. *Bulletin of the London Mathematical Society* 10: 193–196.
6. Sim, Y.J., and D.K. Thomas. 2020. On the difference of coefficients of starlike and convex functions. *Mathematics* 8 (1521): 11. https://doi.org/10.3390/math8091521.
7. Koepf, W. 1987. On the Fekete–Szego problem for close to convex functions. *Proceedings of the American Mathematical Society* 101: 89–95.
8. Ming, L., and T. Sugawa. 2016. A note on successive coefficients of convex functions. *Computational Methods and Function Theory*. https://doi.org/10.1007/s40315-016-0177-8.
9. Sim, Y.J., and D.K. Thomas. 2020. On the difference of inverse coefficients of univalent functions. *Symmetry*. https://doi.org/10.3390/sym12122040.
10. Libera, R.J., and E.J. Złotkiewicz. 1983. Coefficient bounds for the inverse of a function with derivatives in $\mathcal{P}$. *Proceedings of the American Mathematical Society* 87: 251–257.
11. Thomas, D.K., and Serika Verma. 2017. Invariance of the coefficients of strongly convex functions. *Bulletin of the Australian Mathematical Society* 95: 236–445.
12. Choi, J.H., Y.C. Kim, and T. Sugawa. 2007. A general approach to the Fekete–Szegö problem. *Journal of Mathematical Society in Japan* 59: 707–727.
13. Ohno, R., and T. Sugawa. *Coefficient estimates of analytic endomorphisms of the unit disk fixing a point with applications to concave functions*, arXiv: 1512.03148v1, pp. 13.