PATH INDEPENDENCE OF THE ADDITIVE FUNCTIONALS FOR STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY G-LÉVY PROCESSES*

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Abstract. In this study, we are interested in stochastic differential equations driven by G-Lévy processes. We illustrate that a certain class of additive functionals of the equations of interest exhibits the path-independent property, generalizing a few known findings in the literature. The study is ended with many examples.

1. Introduction

Recently, mathematical finance's development forces the consideration of the appearance of a type of process—the G-Brownian motions (See [12, 13, 14]). Then, related theories like stochastic calculus and stochastic differential equations (SDEs for short) driven by G-Brownian motions, are extensively investigated (See [1, 14, 15, 17]). However, in some financial models, volatility uncertainty makes G-Brownian motions inadequate for simulating these models. One crucial and difficult issue is caused by the continuity of their trajectories to the time variable, which is comparably too restrictive from a modeling prospective. To this end, Hu and Peng [4] tackled the problem by introducing G-Lévy processes with discontinuous (right continuous with left limits, also named càdlàg) paths. Later, Paczka [9] defined the Itô-Lévy stochastic integrals, deduced the Itô formula, and introduced SDEs driven by G-Lévy processes, and determined the existence and uniqueness of solutions for the equations under Lipschitz conditions. On the other hand, under some discontinuous conditions, Wang and Yuan [27] showed the solutions' existence of the SDEs driven by G-Lévy processes. Furthermore, under non-Lipschitz conditions, Wang and Gao [26] demonstrated the well-posedness of the SDEs driven by G-Lévy processes and examined their solutions' exponential stability. In this study, following up the line of [26], we introduce additive functionals for the SDEs driven by G-Lévy processes and investigate their path-independent property.

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Fix $T>0$ arbitrarily, we consider the following SDE on $\mathbb{R}^d$:

$$\left\{\begin{array}{l} \mathrm{d}Y_t = b(t,Y_t)\mathrm{d}t + h_{ij}(t,Y_t)\mathrm{d}\langle B^i, B^j\rangle_t + \sigma(t,Y_t)\mathrm{d}B_t + \int_{\mathbb{R}^d\setminus\{0\}} f(t,Y_t,u)L(\mathrm{d}t,\mathrm{d}u), \\ Y_0 = y \in \mathbb{R}^d, \qquad 0<t\leqslant T, \end{array}\right. \tag{1}$$

where B is a G-Brownian motion, $\langle B^i, B^j\rangle_t$ is the mutual variation process of B^i and B^j for $i,j=1,2,\cdots,d$, and $L(\mathrm{d}t,\mathrm{d}u)$ is a G-random measure specified later (See Subsection 2.4). The coefficients $b:[0,T]\times\mathbb{R}^d\mapsto\mathbb{R}^d$, $h_{ij}=h_{ji}:[0,T]\times\mathbb{R}^d\mapsto\mathbb{R}^d$, $\sigma:[0,T]\times\mathbb{R}^d\mapsto\mathbb{R}^{d\times d}$ and $f:[0,T]\times\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})\mapsto\mathbb{R}^d$ are Borel measurable. We take the convention that the repeated indices stand for the summation. Under the assumptions $(\mathbf{H}^1_{b,h,\sigma,f})$-$(\mathbf{H}^2_{b,h,\sigma,f})$ given in Subsection 2.5, by [26, Theorem 3.1], there is a unique solution Y to Eq.(1). Based on this, we introduce additive functionals of Y and give definition of the path independence of these functionals. Under certain conditions, we reveal that these functionals exhibit the path-independent property. Our finding covers those relevant findings obtained in ([18, 21]). We further propose numerous examples to help clarify our finding.

Finally, we would like to indicate our motivations for this study. First, the concept of path independence for additive functionals comes from the mathematical study of economics and finance (c.f. [2, 22]). Due to the necessity of stochastic volatility as the measurement of uncertainty in the modeling of financial markets, SDEs have got a lot of interest from theoretical and practical aspects. Using SDEs, the price dynamics or the wealth growth are modeled. To an equilibrium financial market, there must exist a so-called risk neutral probability measure that is continuous to the given real-world probability measure and it is crucial to evaluate the path-independent property for the associated density process defined using the Radon–Nikodym derivative so that the stochastic optimization problem of the corresponding utility function can be explicitly explained. Otherwise, it would be challenging to deal with the utility functions requiring Itôs stochastic integrals. Our finding proposes a complete characterization of this property with a mathematically sound reason. This is the first motivation. Second, we mention that Ren and Yang [21] demonstrated the additive functionals' path independence for SDEs driven by G-Brownian motions. Since the SDEs driven by G-Lévy processes can fit the exact modeling demand even better, extending the finding of Ren and Yang [21] becomes our second motivation. Finally, by examining some special cases, we surprisingly discovered that one can derive explicit expressions for these additive functionals, which is generally challenging to have (see, e.g., [20]). This point is our third motivation to work out the present study.

The rest of our study is arranged as follows: In Section 2, we propose G-Lévy processes, the Itô-Lévy stochastic integrals, SDEs driven by G-Lévy processes, additive functionals, the path independence, and some related findings. The primary finding and its proof are presented in Section 3. The final section, Section 4, is dedicated to numerous supporting examples.

2. Preliminary

In this section, we present numerous concepts and relevant findings that will be employed in the sequel.

2.1. **Notation.** We first introduce the notations of this study. For convenience, we shall use $|\cdot|$ and $\|\cdot\|$ for norms of vectors and matrices. Furthermore, let $\langle \cdot\,,\cdot\rangle$ represent the scalar product in $\mathbb{R}^d$. Let Q^* represent the matrix Q's transpose.

Let $\mathscr{B}(\mathbb{R}^d)$ be the Borel σ-field on $\mathbb{R}^d$. Let $C_{lip}(\mathbb{R}^d)$ be the set of all Lipschitz continuous functions on $\mathbb{R}^d$ and $C_{b,lip}(\mathbb{R}^d)$ be the collection of all bounded and Lipschitz continuous functions on $\mathbb{R}^d$. Let $C_b^3(\mathbb{R}^d)$ be the space of bounded and three times continuously differentiable functions with bounded derivatives of all orders less than or equal to 3.

2.2. **G-Lévy processes.** In this subsection, we present G-Lévy processes (c.f. [4]).

Let Ω be a given set and $\mathcal{H}$ be a linear space of real functions defined on Ω such that if $X_1,\ldots,X_n\in\mathcal{H}$, then $\phi(X_1,\ldots,X_n)\in\mathcal{H}$ for each $\phi\in C_{lip}(\mathbb{R}^n)$. If $X\in\mathcal{H}$, we call X a random variable.

Definition 2.1. *If a functional $\bar{\mathbb{E}}:\mathcal{H}\mapsto\mathbb{R}$ satisfies: for $X,Y\in\mathcal{H}$,*
(i) $X\geqslant Y\Rightarrow\bar{\mathbb{E}}[X]\geqslant\bar{\mathbb{E}}[Y]$,
(ii) $\bar{\mathbb{E}}[X+Y]\leqslant\bar{\mathbb{E}}[X]+\bar{\mathbb{E}}[Y]$,
(iii) for all $\lambda\geqslant 0,\bar{\mathbb{E}}[\lambda X]=\lambda\bar{\mathbb{E}}[X]$,
(iv) for all $c\in\mathbb{R},\bar{\mathbb{E}}[X+c]=\bar{\mathbb{E}}[X]+c$,
we call $\bar{\mathbb{E}}$ a sublinear expectation on $\mathcal{H}$ and $(\Omega,\mathcal{H},\bar{\mathbb{E}})$ a sublinear expectation space.

Next, we define the distribution of a random vector on $(\Omega,\mathcal{H},\bar{\mathbb{E}})$. For an n-dimensional random vector $X=(X_1,X_2,\cdots,X_n)$ for $X_i\in\mathcal{H}$, $i=1,2,\cdots,n$, set

$$\bar{\mathscr{L}}_X(\phi):=\bar{\mathbb{E}}(\phi(X)),\qquad \phi\in C_{lip}(\mathbb{R}^n),$$

and then we call $\bar{\mathscr{L}}_X$ the distribution of X.

Definition 2.2. *Assume that X_1,X_2 are two n-dimensional random vectors defined on $(\Omega^1,\mathcal{H}^1,\bar{\mathbb{E}}^1)$ and $(\Omega^2,\mathcal{H}^2,\bar{\mathbb{E}}^2)$, respectively. If for all $\phi\in C_{lip}(\mathbb{R}^n)$,*

$$\bar{\mathscr{L}}^1_{X_1}(\phi)=\bar{\mathscr{L}}^2_{X_2}(\phi),$$

where $\bar{\mathscr{L}}^1_{X_1},\bar{\mathscr{L}}^2_{X_2}$ represent the distributions of X_1,X_2, respectively, we say that the distributions of X_1,X_2 are similar.

Definition 2.3. *For two random vectors $Y=(Y_1,Y_2,\cdots,Y_m)$ for $Y_j\in\mathcal{H}$ and $X=(X_1,X_2,\cdots,X_n)$ for $X_i\in\mathcal{H}$, if for all $\phi\in C_{lip}(\mathbb{R}^n\times\mathbb{R}^m)$,*

$$\bar{\mathbb{E}}[\phi(X,Y)]=\bar{\mathbb{E}}[\bar{\mathbb{E}}[\phi(x,Y)]_{x=X}],$$

we say that Y is independent of X.

Next, we introduce Lévy processes on $(\Omega,\mathcal{H},\bar{\mathbb{E}})$.

Definition 2.4. *Let $X=(X_t)_{t\geqslant 0}$ be a d-dimensional càdlàg process on $(\Omega,\mathcal{H},\bar{\mathbb{E}})$. If X satisfies*
(i) $X_0=0$;
(ii) for $t,s\geqslant 0$, the increment $X_{s+t}-X_t$ is independent of $(X_{t_1},X_{t_2},\cdots,X_{t_n})$, for any n and $0\leqslant t_1<t_2\cdots<t_n\leqslant t$;
(iii) the distribution of $X_{s+t}-X_t$ does not depend on t, namely, the distribution solely depends on the difference $s=(s+t)-t$;
we then call X a Lévy process.

Definition 2.5. *Assume that X is a d-dimensional Lévy process. If there exists a decomposition $X_t = X_t^c + X_t^d$ for $t \geqslant 0$, where (X_t^c, X_t^d) represent a 2d-dimensional Lévy process satisfying*

$$\lim_{t\downarrow 0} \frac{\bar{\mathbb{E}}|X_t^c|^3}{t} = 0, \quad \bar{\mathbb{E}}|X_t^d| \leqslant Ct, \quad t \geqslant 0, \quad C \geqslant 0,$$

we call X a G-Lévy process.

In the following, we characterize G-Lévy processes by partial differential equations.

Theorem 2.6. *Assume that X is a d-dimensional G-Lévy process. Then for $g \in C_b^3(\mathbb{R}^d)$ with $g(0) = 0$, set*

$$G_X[g(\cdot)] := \lim_{t\downarrow 0} \frac{\bar{\mathbb{E}}[g(X_t)]}{t},$$

and then, G_X has the following Lévy–Khintchine representation

$$G_X[g(\cdot)] = \sup_{(\nu,\zeta,Q)\in\mathcal{U}} \left\{ \int_{\mathbb{R}^d\setminus\{0\}} g(u)\nu(\mathrm{d}u) + \langle \partial_x g(0), \zeta\rangle + \frac{1}{2} tr[\partial_x^2 g(0)QQ^*] \right\}, \tag{2}$$

where $\mathcal{U}$ is a subset of $\mathcal{M}(\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$, $\mathcal{M}(\mathbb{R}^d \setminus \{0\})$ is the collection of all measures on $(\mathbb{R}^d \setminus \{0\}, \mathscr{B}(\mathbb{R}^d \setminus \{0\}))$, $\mathbb{R}^{d\times d}$ is the set of all $d \times d$ matrices and $\mathcal{U}$ satisfies

$$\sup_{(\nu,\zeta,Q)\in\mathcal{U}} \left\{ \int_{\mathbb{R}^d\setminus\{0\}} |u|\nu(\mathrm{d}u) + |\zeta| + \frac{1}{2} tr[QQ^*] \right\} < \infty. \tag{3}$$

Theorem 2.7. *Suppose that X is a d-dimensional G-Lévy process. Then, for $\varphi \in C_{b,lip}(\mathbb{R}^d)$, $v(t,x) := \bar{\mathbb{E}}[\varphi(x + X_t)]$ is the unique viscosity solution of the following partial integro-differential equation:,*

$$\begin{aligned} 0 &= \partial_t v(t,x) - G_X[v(t,x+\cdot) - v(t,x)] \\ &= \partial_t v(t,x) - \sup_{(\nu,\zeta,Q)\in\mathcal{U}} \Big\{ \int_{\mathbb{R}^d\setminus\{0\}} [v(t,x+u) - v(t,x)]\nu(\mathrm{d}u) + \langle \partial_x v(t,x), \zeta\rangle \\ &\qquad + \frac{1}{2} tr[\partial_x^2 v(t,x) QQ^*] \Big\} \end{aligned}$$

with the first condition $v(0,x) = \varphi(x)$.

A natural, converse question is, if we have a set $\mathcal{U}$ satisfying (3), is there a d-dimensional G-Lévy process having the Lévy–Khintchine representation (2) with similar set $\mathcal{U}$? The answer is affirmative. We take $\Omega := D_0(\mathbb{R}^+, \mathbb{R}^d)$, where $D_0(\mathbb{R}^+, \mathbb{R}^d)$ is the space of all càdlàg functions $\mathbb{R}_+ \ni t \mapsto \omega_t \in \mathbb{R}^d$ with $\omega_0 = 0$, equipped with the Skorokhod topology.

Theorem 2.8. *Suppose that $\mathcal{U}$ satisfies (3). Then, there exists a sublinear expectation $\bar{\mathbb{E}}$ on Ω such that the canonical process X is a d-dimensional G-Lévy process having the Lévy-Khintchine representation (2) with a similar set $\mathcal{U}$.*

2.3. **A capacity.** In this subsection, we present a notion of capacity and related definitions (c.f. [9]).

First of all, fix a set $\mathcal{U}$ satisfying (3) and $T > 0$ and take $\Omega_T := D_0([0,T], \mathbb{R}^d)$ and the sublinear expectation $\bar{\mathbb{E}}$ in Theorem 2.8. Therefore, we know that $(\Omega_T, \mathcal{H}, \bar{\mathbb{E}})$ is a sublinear expectation space. Here, we study this space. Let X represent the canonical process on the space, i.e. $X_t(\omega)=\omega_t, t\in[0,T]$. Therefore, X represent a d-dimensional G-Lévy process. Set

$$Lip(\Omega_T):=\{\phi(X_{t_1},X_{t_2},\cdots,X_{t_n}):n\in\mathbb{N},t_1,t_2,\cdots,t_n\in[0,T],\phi\in C_{b,lip}(\mathbb{R}^{d\times n})\}.$$

Let $L^p_G(\Omega_T)$ be the completion of $Lip(\Omega_T)$ under the norm $\|\cdot\|_p:=(\bar{\mathbb{E}}|\cdot|^p)^{1/p}, p\geqslant 1$.

Let

$$\mathcal{V}:=\{\nu\in\mathcal{M}(\mathbb{R}^d\setminus\{0\}):\exists(\zeta,Q)\in\mathbb{R}^d\times\mathbb{R}^{d\times d}\text{ such that }(\nu,\zeta,Q)\in\mathcal{U}\}$$

and let $\mathcal{G}$ be the set of all the Borel measurable functions $g:\mathbb{R}^d\mapsto\mathbb{R}^d$ with $g(0)=0$.

Assumption:

$(\mathbf{H}^1_{\mathcal{V}})$ There exists a measure $\mu\in\mathcal{M}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d\setminus\{0\}}|z|\mu(\mathrm{d}z)<\infty,\quad \mu(\{0\})=0,$$

and for all $\nu\in\mathcal{V}$ there exists a function $g_\nu\in\mathcal{G}$ satisfying

$$\nu(A)=\mu(g_\nu^{-1}(A)),\quad\forall A\in\mathscr{B}(\mathbb{R}^d\setminus\{0\}).$$

$(\mathbf{H}^2_{\mathcal{V}})$

$$\sup_{\nu\in\mathcal{V}}\nu(\mathbb{R}^d\setminus\{0\})<\infty.$$

Let $(\tilde{\Omega},\mathscr{F},\mathbb{P})$ be a probability space supporting a d-dimensional Brownian motion W and a Poisson random measure $N(\mathrm{d}t,\mathrm{d}z)$ with the intensity measure $\mu(\mathrm{d}z)\mathrm{d}t$. Let

$$\mathscr{F}_t:=\sigma\left\{W_s,N((0,s],A):0\leqslant s\leqslant t,A\in\mathscr{B}(\mathbb{R}^d\setminus\{0\})\right\}\vee\mathcal{N},\quad\mathcal{N}:=\{U\in\mathscr{F},\mathbb{P}(U)=0\}.$$

We present the following set.

Definition 2.9. *$\mathcal{A}^{\mathcal{U}}_{0,T}$ is a set of all the processes $\theta_t=(\theta^d_t,\theta^{1,c}_t,\theta^{2,c}_t)$ for $t\in[0,T]$ satisfying*

(i) $(\theta^{1,c}_t,\theta^{2,c}_t)$ is an $\mathscr{F}_t$-adapted process and θ^d is an $\mathscr{F}_t$-predictable random field on $[0,T]\times\mathbb{R}^d$,

(ii) For $\mathbb{P}$-a.s. ω and a.e. $t\in[0,T]$,

$$(\theta^d(t,\cdot)(\omega),\theta^{1,c}_t(\omega),\theta^{2,c}_t(\omega))\in\left\{(g_\nu,\zeta,Q)\in\mathcal{G}\times\mathbb{R}^d\times\mathbb{R}^{d\times d}:(\nu,\zeta,Q)\in\mathcal{U}\right\},$$

(iii)

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T\Big(|\theta^{1,c}_t|+\|\theta^{2,c}_t\|^2+\int_{\mathbb{R}^d\setminus\{0\}}|\theta^d(t,z)|\mu(\mathrm{d}z)\Big)\mathrm{d}t\right]<\infty.$$

For $\theta\in\mathcal{A}^{\mathcal{U}}_{0,T}$, set

$$B^{0,\theta}_t:=\int_0^t\theta^{1,c}_s\mathrm{d}s+\int_0^t\theta^{2,c}_s\mathrm{d}W_s+\int_0^t\int_{\mathbb{R}^d\setminus\{0\}}\theta^d(s,z)N(\mathrm{d}s,\mathrm{d}z),\quad t\in[0,T],$$

and by [9, Corollary 14], it holds that for $\xi\in L^1_G(\Omega_T)$

$$\bar{\mathbb{E}}[\xi]=\sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,T}}\mathbb{E}^{\mathbb{P}^\theta}[\xi],\quad\mathbb{P}^\theta=\mathbb{P}\circ(B^{0,\theta}_\cdot)^{-1}.$$

Then define

$$\bar{C}(D):=\sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,T}}\mathbb{P}^\theta(D),\quad D\in\mathscr{B}(\Omega_T),$$

and $\bar{C}$ is a capacity. For $D \in \mathscr{B}(\Omega_T)$, if $\bar{C}(D) = 0$, we call D a polar set. Therefore, if a property holds outside a polar set, the property holds quasi-surely ($q.s.$ for short). In the following, we characterize the property for paths of G-Lévy processes.

Theorem 2.10. [9, Proposition 16] *For each finite interval $[s,t], 0 < s < t \leqslant T$, the canonical process X has finite number of jumps q.s..*

2.4. **The Itô integrals with respect to G-Lévy processes.** In this subsection, we present the Itô integrals to G-Lévy processes under the framework outlined in the above subsection.

Although the Itô integrals for G-Brownian motions have been presented using Peng in the profound studies [14, 15] (See more recent paper [17]), we must introduce two related spaces for defining the Itô integrals for G-Lévy processes. Take $0 = t_0 < t_1 < \cdots < t_N = T$. Let $p \geqslant 1$ be fixed. Set

$$\mathcal{M}_G^{p,0}(0,T) := \Big\{ \eta_t(\omega) = \sum_{j=1}^{N} \xi_{j-1}(\omega) 1_{[t_{j-1},t_j)}(t); \xi_{j-1}(\omega) \in L_G^p(\Omega_{t_{j-1}}) \Big\}.$$

Let $\mathcal{M}_G^p(0,T)$ and $\mathcal{H}_G^p(0,T)$ denote the completion of $\mathcal{M}_G^{p,0}(0,T)$ under the norm

$$\|\eta\|_{\mathcal{M}_G^p(0,T)} = \left(\int_0^T \bar{\mathbb{E}} |\eta_t|^p \mathrm{d}t \right)^{\frac{1}{p}} \text{ and } \|\eta\|_{\mathcal{H}_G^p(0,T)} = \left(\bar{\mathbb{E}} \left(\int_0^T |\eta_t|^2 \mathrm{d}t \right)^{\frac{p}{2}} \right)^{\frac{1}{p}},$$

, respectively. Let $\mathcal{M}_G^p([0,T], \mathbb{R}^d)$, and $\mathcal{H}_G^p([0,T], \mathbb{R}^d)$ be the collection of all the processes

$$\eta_t = (\eta_t^1, \eta_t^2, \cdots, \eta_t^d), \quad t \in [0,T], \quad \eta^i \in \mathcal{M}_G^p(0,T) \text{ and } \mathcal{H}_G^p(0,T),$$

, respectively.

Furthermore, we introduce the Itô integrals with stopping times. For $0 \leqslant t \leqslant T$, set

$$\Omega_t := D_0([0,t], \mathbb{R}^d), \quad \mathscr{B}_t := \mathscr{B}(\Omega_t),$$

and then $(\mathscr{B}_t)_{t\in[0,T]}$ is a filtration.

Definition 2.11. *A stopping time τ for the filtration $(\mathscr{B}_t)_{t\in[0,T]}$ is a mapping from Ω_T to $[0,T]$ satisfying*

$$\{\tau \leqslant t\} \in \mathscr{B}_t, \quad 0 \leqslant t \leqslant T.$$

Theorem 2.12. [6, Lemma 4.2] *For each stopping time τ and $\eta \in \mathcal{M}_G^p(0,T)$, it holds that $I_{[0,\tau]}\eta \in \mathcal{M}_G^p(0,T)$.*

Remark 2.13. *In fact, Theorem 2.12 is different from Lemma 4.2 in* [6] *since here $\Omega = D_0(\mathbb{R}_+, \mathbb{R}^d)$ and there $\Omega = C_0(\mathbb{R}_+, \mathbb{R}^d)$, where $C_0(\mathbb{R}^+, \mathbb{R}^d)$ is the space of all continuous functions $\mathbb{R}_+ \ni t \mapsto \omega_t \in \mathbb{R}^d$ with $\omega_0 = 0$, equipped with the uniformly convergence topology. However, we carefully confirm the proof of Lemma 4.2 in* [6] *and discover that the sample paths' continuity is not employed there, namely, the finding of Lemma 4.2 in* [6] *holds for $\Omega = D_0(\mathbb{R}_+, \mathbb{R}^d)$. Thus, we cite this finding without alteration.*

Next, we present the Itô integrals for random measures. First of all, define a G-random measure: for any $0 \leqslant t \leqslant T$ and $A \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$,

$$\kappa_t := X_t - X_{t-}, \quad L((0,t], A) := \sum_{0<s\leqslant t} I_A(\kappa_s), \quad q.s.,$$

and then, using Theorem 2.10, we know that it is well-defined. Therefore, we define the Itô integral for the random measure $L(\mathrm{d}t, \mathrm{d}u)$. Let $\mathcal{H}_G^S([0,T]\times(\mathbb{R}^d\setminus\{0\}))$ be the collection of all the processes defined on $[0,T]\times(\mathbb{R}^d\setminus\{0\})\times\Omega$ with the form

$$f(s,u)(\omega)=\sum_{k=1}^{n-1}\sum_{l=1}^{m}\phi_{k,l}(\omega_{t_1},\omega_{t_2}-\omega_{t_1},\cdots,\omega_{t_k}-\omega_{t_{k-1}})I_{[t_k,t_{k+1})}(s)\psi_l(u), n,m\in\mathbb{N},$$

where $0\leqslant t_1<\cdots<t_n\leqslant T$ is a partition of $[0,T]$, $\phi_{k,l}\in C_{b,lip}(\mathbb{R}^{d\times k})$ and $\{\psi_l\}_{l=1}^m\subset C_{b,lip}(\mathbb{R}^d)$ are functions with disjoint supports and $\psi_l(0)=0$.

Definition 2.14. *For any $f\in\mathcal{H}_G^S([0,T]\times(\mathbb{R}^d\setminus\{0\}))$, set*

$$\int_0^t\int_{\mathbb{R}^d\setminus\{0\}}f(s,u)L(\mathrm{d}s,\mathrm{d}u):=\sum_{0<s\leqslant t}f(s,\kappa_s),\quad q.s..$$

By [9, Theorem 28], we have that $\int_0^t\int_{\mathbb{R}^d\setminus\{0\}}f(s,u)L(\mathrm{d}s,\mathrm{d}u)\in L_G^2(\Omega_T)$. Let $\mathcal{H}_G^2([0,T]\times(\mathbb{R}^d\setminus\{0\}))$ be the completion of $\mathcal{H}_G^S([0,T]\times(\mathbb{R}^d\setminus\{0\}))$ with respect to the norm $\|\cdot\|_{\mathcal{H}_G^2([0,T]\times(\mathbb{R}^d\setminus\{0\}))}$, where

$$\|f\|_{\mathcal{H}_G^2([0,T]\times(\mathbb{R}^d\setminus\{0\}))}:=\left(\bar{\mathbb{E}}\left[\int_0^T\sup_{\nu\in\mathcal{V}}\int_{\mathbb{R}^d\setminus\{0\}}|f(s,u)|^2\nu(\mathrm{d}u)\mathrm{d}s\right]\right)^{1/2},\quad f\in\mathcal{H}_G^S([0,T]\times(\mathbb{R}^d\setminus\{0\})).$$

Therefore, using [9, Corollary 29], it holds that for $f\in\mathcal{H}_G^2([0,T]\times(\mathbb{R}^d\setminus\{0\}))$,

$$\int_0^t\int_{\mathbb{R}^d\setminus\{0\}}f(s,u)L(\mathrm{d}s,\mathrm{d}u)=\sum_{0<s\leqslant t}f(s,\kappa_s),\quad q.s.. \tag{4}$$

Let $\mathcal{H}_G^2([0,T]\times(\mathbb{R}^d\setminus\{0\}),\mathbb{R}^d)$ be the space of all the processes

$$f(t,u)=\Big(f^1(t,u),f^2(t,u),\cdots,f^d(t,u)\Big),\quad f^i\in\mathcal{H}_G^2([0,T]\times(\mathbb{R}^d\setminus\{0\})).$$

2.5. **Stochastic differential equations driven by G-Lévy processes.** In this subsection, we present SDEs driven by G-Lévy processes and related additive functionals.

First, we introduce some notations. Let $\mathbb{S}^d$ be the space of all $d\times d$ symmetric matrices. For $A\in\mathbb{S}^d$, set

$$G(A):=\frac{1}{2}\sup_{Q\in\mathcal{Q}}\mathrm{tr}[QQ^*A],$$

where $\mathcal{Q}$ is a nonempty, bounded, closed, and convex subset of $\mathbb{R}^{d\times d}$. Then $G:\mathbb{S}^d\mapsto\mathbb{R}$ is a monotonic, sublinear, and positive homogeneous functional (c.f. [15]). We choose $\mathcal{U}\subset\mathcal{M}(\mathbb{R}^d\setminus\{0\})\times\{0\}\times\mathcal{Q}$ satisfying (3) and still work under the framework of Subsection 2.3. Therefore, the canonical process $X_\cdot$ can be denoted as $X_\cdot=B_\cdot+X_\cdot^d$, where $B_\cdot$ is a G-Brownian motion associated with $\mathcal{Q}$ and $X_\cdot^d$ is a pure jump G-Lévy process associated with $\mathcal{M}(\mathbb{R}^d\setminus\{0\})$.

Next, consider Eq.(1), i.e.

$$\left\{\begin{array}{l}\mathrm{d}Y_t=b(t,Y_t)\mathrm{d}t+h_{ij}(t,Y_t)\mathrm{d}\langle B^i,B^j\rangle_t+\sigma(t,Y_t)\mathrm{d}B_t+\int_{\mathbb{R}^d\setminus\{0\}}f(t,Y_t,u)L(\mathrm{d}t,\mathrm{d}u),0<t\leqslant T,\\ Y_0=y\in\mathbb{R}^d.\end{array}\right.$$

We assume the following:

$(\mathbf{H}^1_{b,h,\sigma,f})$ There exists a constant $C_1>0$ such that for any $t\in[0,T]$ and $x,y\in\mathbb{R}^d$,

$$
\begin{aligned}
&|b(t,x)-b(t,y)|^2+|h_{ij}(t,x)-h_{ij}(t,y)|^2+\|\sigma(t,x)-\sigma(t,y)\|^2\\
+&\sup_{\nu\in\mathcal{V}}\int_{\mathbb{R}^d\setminus\{0\}}|f(t,x,u)-f(t,y,u)|^2\nu(\mathrm{d}u)\leqslant C_1\rho(|x-y|^2),
\end{aligned}
$$

where $\rho:(0,+\infty)\mapsto(0,+\infty)$ is a continuous, increasing and concave function so that

$$
\rho(0+)=0,\quad \int_0^1\frac{dr}{\rho(r)}=+\infty.
$$

$(\mathbf{H}^2_{b,h,\sigma,f})$ There exists a constant $C_2>0$ such that for any $t\in[0,T]$

$$
|b(t,0)|^2+|h_{ij}(t,0)|^2+\|\sigma(t,0)\|^2+\sup_{\nu\in\mathcal{V}}\int_{\mathbb{R}^d\setminus\{0\}}|f(t,0,u)|^2\nu(\mathrm{d}u)\leqslant C_2.
$$

By [26, Theorem 3.1], we know that under $(\mathbf{H}^1_{b,h,\sigma,f})$-$(\mathbf{H}^2_{b,h,\sigma,f})$, Eq.(1) has a unique solution $Y_\cdot$ with

$$
\bar{\mathbb{E}}\left[\sup_{t\in[0,T]}|Y_t|^2\right]<\infty. \tag{5}
$$

Then we introduce the following additive functional

$$
\begin{aligned}
F_{s,t} \;:=\;& \alpha\int_s^t G(g_1)(r,Y_r)\mathrm{d}r+\beta\int_s^t g_1^{ij}(r,Y_r)\mathrm{d}\langle B^i,B^j\rangle_r+\int_s^t\langle g_2(r,Y_r),\mathrm{d}B_r\rangle\\
&+\int_s^t\int_{\mathbb{R}^d\setminus\{0\}}g_3(r,Y_r,u)L(\mathrm{d}r,\mathrm{d}u)+\gamma\int_s^t\sup_{\nu\in\mathcal{V}}\int_{\mathbb{R}^d\setminus\{0\}}g_3(r,Y_r,u)\nu(\mathrm{d}u)\mathrm{d}r,\\
&0\leqslant s<t\leqslant T,
\end{aligned} \tag{6}
$$

where $\alpha,\beta,\gamma\in\mathbb{R}$ are three constants and

$$
\begin{aligned}
&g_1:[0,T]\times\mathbb{R}^d\mapsto\mathbb{R}^{d\times d},\quad g_1^{ij}=g_1^{ji},\\
&g_2:[0,T]\times\mathbb{R}^d\mapsto\mathbb{R}^d,\\
&g_3:[0,T]\times\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})\mapsto\mathbb{R},
\end{aligned}
$$

are Borel measurable, and $g_1(t,x),g_2(t,x),g_3(t,x,u)$ are continuous in x so that $F_{s,t}$ is well-defined.

Definition 2.15. *The additive functional $F_{s,t}$ is called path independent, if there exists a scalar function*

$$
V:[0,T]\times\mathbb{R}^d\mapsto\mathbb{R},
$$

such that for any $s\in[0,T]$ and $Y_s\in L^2_G(\Omega_s)$, the solution $(Y_t)_{t\in[s,T]}$ of Eq.(1) satisfies

$$
F_{s,t}=V(t,Y_t)-V(s,Y_s). \tag{7}
$$

3. The main result

In this section, we state and demonstrate the primary finding under the framework of Subsection 2.5. Then, we compare our findings with some known findings.

3.1. **The main result and it's proof.** In this subsection, we state and demonstrate the primary finding. Let us begin with a key lemma.

Lemma 3.1. *Assume that $\mathcal{Q}$ is bounded away from 0 to Z_t is a 1-dimensional G-Itô-Lévy process, i.e.*

$$Z_t = \int_0^t \Gamma_s \mathrm{d}s + \int_0^t \Phi_{ij}(s)\mathrm{d}\langle B^i, B^j\rangle_s + \int_0^t \langle \Psi_s, \mathrm{d}B_s\rangle + \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} K(s,u)L(\mathrm{d}s, \mathrm{d}u), \quad (8)$$

where $\Gamma \in \mathcal{M}_G^1(0,T), \Phi_{ij} \in \mathcal{M}_G^1(0,T), \Phi_{ij} = \Phi_{ji}, i,j = 1,2,\cdots,d, \Psi \in \mathcal{H}_G^1([0,T],\mathbb{R}^d), K \in \mathcal{H}_G^2([0,T]\times(\mathbb{R}^d\setminus\{0\}))$. Then $Z_t = 0$ for all $t \in [0,T]$ q.s. if and only if $\Gamma_t = 0, \Phi_{ij}(t) = 0, \Psi_t = 0$ a.e.×q.s. on $[0,T]\times\Omega_T$ and $K(t,u) = 0$ a.e.×a.e.×q.s. on $[0,T]\times(\mathbb{R}^d\setminus\{0\})\times\Omega_T$.

Proof. Sufficiency is direct if one inserts $\Gamma_t = 0, \Phi_{ij}(t) = 0, \Psi_t = 0, K(t,u) = 0$ into (8). We prove necessity. If $Z_t = 0$ for any $t \in [0,T]$, we get that

$$0 = \int_0^t \Gamma_s \mathrm{d}s + \int_0^t \Phi_{ij}(s)\mathrm{d}\langle B^i, B^j\rangle_s + \int_0^t \Psi_s^* \mathrm{d}B_s + \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} K(s,u)L(\mathrm{d}s, \mathrm{d}u). \quad (9)$$

By taking the quadratic process with $\int_0^t \Psi_s^* \mathrm{d}B_s$ on two sides of (9), it holds that

$$\begin{aligned} 0 &= \langle \int_0^\cdot \Psi_s^*\mathrm{d}B_s, \int_0^\cdot \Psi_s^*\mathrm{d}B_s\rangle_t = \int_0^t \Psi_s^i\Psi_s^j \mathrm{d}\langle B^i, B^j\rangle_s = \int_0^t \mathrm{tr}(\Psi_s\Psi_s^*\mathrm{d}\langle B\rangle_s) \\ &= \int_0^t \mathrm{tr}(\mathrm{d}\langle B\rangle_s \Psi_s\Psi_s^*) = \int_0^t \langle \mathrm{d}\langle B\rangle_s\Psi_s, \Psi_s\rangle, \end{aligned}$$

where

$$\langle B\rangle := \begin{pmatrix} \langle B^1, B^1\rangle & \langle B^1, B^2\rangle \cdots \langle B^1, B^d\rangle \\ \vdots & \vdots \qquad\qquad \vdots \\ \langle B^d, B^1\rangle & \langle B^d, B^2\rangle \cdots \langle B^d, B^d\rangle \end{pmatrix}.$$

Note that $\mathcal{Q}$ is bounded away from 0. Therefore, there exists a constant $\iota > 0$ such that $\langle B\rangle_s \geqslant \iota s I_d$ and then

$$0 = \int_0^t \langle \mathrm{d}\langle B\rangle_s\Psi_s, \Psi_s\rangle \geqslant \iota \int_0^t \langle \Psi_s, \Psi_s\rangle \mathrm{d}s.$$

From this, we know that $\Psi_t = 0$ a.e.×q.s., and (9) becomes

$$0 = \int_0^t \Gamma_s \mathrm{d}s + \int_0^t \Phi_{ij}(s)\mathrm{d}\langle B^i, B^j\rangle_s + \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} K(s,u)L(\mathrm{d}s, \mathrm{d}u).$$

Next, set

$$\tau_0 = 0, \quad \tau_n := \inf\{t > \tau_{n-1} : \kappa_t \neq 0\}, \quad n = 1,2,\cdots,$$

and $\{\tau_n\}$ is a stopping time sequence with respect to $(\mathscr{B}_t)_{t\geqslant 0}$ and $\tau_n \uparrow \infty$ as $n \to \infty$ q.s. (c.f. Theorem 2.10). Thus, by (4) it holds that for $t \in [0, \tau_1 \wedge T)$,

$$0 = \int_0^{\tau_1\wedge T} \Gamma_s \mathrm{d}s + \int_0^{\tau_1\wedge T} \Phi_{ij}(s)\mathrm{d}\langle B^i, B^j\rangle_s,$$

i.e.,

$$-\int_0^{\tau_1\wedge T} \Gamma_s \mathrm{d}s = \int_0^{\tau_1\wedge T} \Phi_{ij}(s)\mathrm{d}\langle B^i, B^j\rangle_s.$$

By the deduction similar to that in [21, Corollary 1] or [24, Corollary 3.5], one can have that

$$\bar{\mathbb{E}}\int_0^{\tau_1\wedge T}(\mathrm{tr}[\Phi_s\Phi_s])^{1/2}\,\mathrm{d}s=\bar{\mathbb{E}}\int_0^{\tau_1\wedge T}|\Gamma_s|\mathrm{d}s=0.$$

Based on this, we know that $\Phi_t=0,\Gamma_t=0$ for $t\in[0,\tau_1\wedge T)$. If $\tau_1\geqslant T$, the proof is over; if $\tau_1<T$, we continue. For $t=\tau_1$, (9) goes to

$$\Phi_t=0,\quad \Gamma_t=0,\quad K(t,\kappa_t)=0.$$

For $t\in[\tau_1,\tau_2\wedge T)$, by means the same as to the above for $t\in[0,\tau_1\wedge T)$, we get that $\Phi_t=0,\Gamma_t=0$ for $t\in[\tau_1,\tau_2\wedge T)$. If $\tau_2\geqslant T$, the proof is over; if $\tau_2<T$, we continue till $T\leqslant\tau_n$. Thus, we obtain that $\Gamma_t=0,\Phi_{ij}(t)=0,\Psi_t=0$ a.e.×q.s. on $[0,T]\times\Omega_T$ and $K(t,u)=0$ a.e.×a.e.×q.s. on $[0,T]\times(\mathbb{R}^d\setminus\{0\})\times\Omega_T$. The proof is complete. □

The primary finding in this section is the following theorem.

Theorem 3.2. *Assume that $\mathcal{Q}$ is bounded away from 0 to b,h,σ,f satisfy ($\mathbf{H}^1_{b,h,\sigma,f}$)-($\mathbf{H}^2_{b,h,\sigma,f}$). Assume further that the Borel measurable function g_3 in the additive functional $F_{s,t}$ defined by (6) is smooth in (t,x) uniformly for u. Then $F_{s,t}$ is path independent in the sense of (7) with the associated scalar function $V\in C_b^{1,2}([0,T]\times\mathbb{R}^d)$, if and only if (V,g_1,g_2,g_3) satisfies the following:*

$$\begin{cases}\partial_tV(t,x)+\frac{\partial}{\partial x_k}V(t,x)b^k(t,x)=\alpha G(g_1)(t,x)+\gamma\sup\limits_{\nu\in\mathcal{V}}\int_{\mathbb{R}^d\setminus\{0\}}g_3(t,x,u)\nu(\mathrm{d}u),\\ \frac{\partial}{\partial x_k}V(t,x)h^k_{ij}(t,x)+\frac{1}{2}\frac{\partial^2}{\partial x_k\partial x_l}V(t,x)\sigma^{ki}(t,x)\sigma^{lj}(t,x)=\beta g_1^{ij}(t,x),\\ (\sigma^*\partial_xV)(t,x)=g_2(t,x),\\ V\Big(t,x+f(t,x,u)\Big)-V(t,x)=g_3(t,x,u),\\ t\in[0,T],x\in\mathbb{R}^d,u\in\mathbb{R}^d\setminus\{0\}.\end{cases}\tag{10}$$

Proof. First, we prove the necessity. On one hand, since $F_{s,t}$ is path independent in the sense of (7), by Definition 2.15 it holds that

$$\begin{aligned}V(t,Y_t)-V(s,Y_s)\ =\ &\alpha\int_s^tG(g_1)(r,Y_r)\mathrm{d}r+\beta\int_s^tg_1^{ij}(r,Y_r)\mathrm{d}\langle B^i,B^j\rangle_r+\int_s^t\langle g_2(r,Y_r),\mathrm{d}B_r\rangle\\&+\int_s^t\int_{\mathbb{R}^d\setminus\{0\}}g_3(r,Y_r,u)L(\mathrm{d}r,\mathrm{d}u)+\gamma\int_s^t\sup_{\nu\in\mathcal{V}}\int_{\mathbb{R}^d\setminus\{0\}}g_3(r,Y_r,u)\nu(\mathrm{d}u)\mathrm{d}r.\end{aligned}\tag{11}$$

On the other hand, by applying the Itô formula for G-Itô-Lévy processes ([9, Theorem 32]) to $V(t,Y_t)$, one can generate that

$$\begin{aligned}V(t,Y_t)-V(s,Y_s)\ =\ &\int_s^t\partial_rV(r,Y_r)\mathrm{d}r+\int_s^t\frac{\partial}{\partial x_k}V(r,Y_r)b^k(r,Y_r)\mathrm{d}r\\&+\int_s^t\frac{\partial}{\partial x_k}V(r,Y_r)h^k_{ij}(r,Y_r)\mathrm{d}\langle B^i,B^j\rangle_r+\int_s^t\langle(\sigma^*\partial_xV)(r,Y_r),\mathrm{d}B_r\rangle\\&+\int_s^t\int_{\mathbb{R}^d\setminus\{0\}}\Big(V(r,Y_r+f(r,Y_r,u))-V(r,Y_r)\Big)L(\mathrm{d}r,\mathrm{d}u)\\&+\frac{1}{2}\int_s^t\frac{\partial^2}{\partial x_k\partial x_l}V(r,Y_r)\sigma^{ki}(r,Y_r)\sigma^{lj}(r,Y_r)\mathrm{d}\langle B^i,B^j\rangle_r.\end{aligned}\tag{12}$$

By (5) and $(\mathbf{H}^1_{b,h,\sigma,f})$-$(\mathbf{H}^2_{b,h,\sigma,f})$, one can verify that

$$
\begin{aligned}
&\partial_r V(r,Y_r)+\frac{\partial}{\partial x_k}V(r,Y_r)b^k(r,Y_r)\in\mathcal{M}^1_G(0,T),\\
&\frac{\partial}{\partial x_k}V(r,Y_r)h^k_{ij}(r,Y_r)+\frac{1}{2}\frac{\partial^2}{\partial x_k\partial x_l}V(r,Y_r)\sigma^{ki}(r,Y_r)\sigma^{lj}(r,Y_r)\in\mathcal{M}^1_G(0,T),\\
&(\sigma^*\partial_x V)(r,Y_r)\in\mathcal{H}^1_G([0,T],\mathbb{R}^d),\\
&V(r,Y_r+f(r,Y_r,u))-V(r,Y_r)\in\mathcal{H}^2_G([0,T]\times(\mathbb{R}^d\setminus\{0\})).
\end{aligned}
$$

Thus, by (11), (12), and Lemma 3.1 we know that

$$
\begin{cases}
\partial_r V(r,Y_r)+\frac{\partial}{\partial x_k}V(r,Y_r)b^k(r,Y_r)=\alpha G(g_1)(r,Y_r)+\gamma\sup\limits_{\nu\in\mathcal{V}}\int_{\mathbb{R}^d\setminus\{0\}}g_3(r,Y_r,u)\nu(\mathrm{d}u),\\
\frac{\partial}{\partial x_k}V(r,Y_r)h^k_{ij}(r,Y_r)+\frac{1}{2}\frac{\partial^2}{\partial x_k\partial x_l}V(r,Y_r)\sigma^{ki}(r,Y_r)\sigma^{lj}(r,Y_r)=\beta g_1^{ij}(r,Y_r),\\
(\sigma^*\partial_x V)(r,Y_r)=g_2(r,Y_r),\\
V\Big(r,Y_r+f(r,Y_r,u)\Big)-V(r,Y_r)=g_3(r,Y_r,u),\quad a.e.\times q.s..
\end{cases}
$$

Now, we insert $r=s, Y_r=x\in\mathbb{R}^d$ into the above equalities and get that

$$
\begin{cases}
\partial_s V(s,x)+\frac{\partial}{\partial x_k}V(s,x)b^k(s,x)=\alpha G(g_1)(s,x)+\gamma\sup\limits_{\nu\in\mathcal{V}}\int_{\mathbb{R}^d\setminus\{0\}}g_3(s,x,u)\nu(\mathrm{d}u),\\
\frac{\partial}{\partial x_k}V(s,x)h^k_{ij}(s,x)+\frac{1}{2}\frac{\partial^2}{\partial x_k\partial x_l}V(s,x)\sigma^{ki}(s,x)\sigma^{lj}(s,x)=\beta g_1^{ij}(s,x),\\
(\sigma^*\partial_x V)(s,x)=g_2(s,x),\\
V\Big(s,x+f(s,x,u)\Big)-V(s,x)=g_3(s,x,u),\\
s\in[0,T], x\in\mathbb{R}^d, u\in\mathbb{R}^d\setminus\{0\}.
\end{cases}
$$

Since s,x are arbitrary, we have (10).

Next, we show adequacy. By the Itô formula for G-Itô-Lévy processes to $V(t,Y_t)$, we have (12). Then one can use (10) to (12) to get (11). That is, $F_{s,t}$ is path independent in the sense of (7). The proof is complete. □

3.2. **Comparison with some known results.** In this subsection, we compare our findings with some known findings.

First, if we take $f(t,x,u)=0$ in Eq.(1) and $g_3(t,x,u)=0$ in (6), Theorem 3.2 becomes [21, Theorem 2]. Thus, our finding is more general.

Second, we take $\mathcal{M}(\mathbb{R}^d\setminus\{0\})=\{\nu\}, \mathcal{Q}=\{I_d\}$ in Subsection 2.5. Therefore, B is a classical Brownian motion with $\langle B^i,B^j\rangle_t=I_{i=j}t$ and $L(\mathrm{d}t,\mathrm{d}u)$ is a classical Poisson random measure. Then Eq.(1) goes into

$$
\mathrm{d}Y_t=b(t,Y_t)\mathrm{d}t+\sum_{i=1}^d h_{ii}(t,Y_t)\mathrm{d}t+\sigma(t,Y_t)\mathrm{d}B_t+\int_{\mathbb{R}^d\setminus\{0\}}f(t,Y_t,u)L(\mathrm{d}t,\mathrm{d}u). \tag{13}
$$

Note that $\nu(\mathbb{R}^d\setminus\{0\})<\infty$. Thus, by (5), $(\mathbf{H}^1_{b,h,\sigma,f})$-$(\mathbf{H}^2_{b,h,\sigma,f})$, and [10, Theorem 13], we generate that

$$
\int_0^t\int_{\mathbb{R}^d\setminus\{0\}}f(s,Y_s,u)\tilde{L}(\mathrm{d}s,\mathrm{d}u):=\int_0^t\int_{\mathbb{R}^d\setminus\{0\}}f(s,Y_s,u)L(\mathrm{d}s,\mathrm{d}u)-\int_0^t\int_{\mathbb{R}^d\setminus\{0\}}f(s,Y_s,u)\nu(\mathrm{d}u)\mathrm{d}s
$$

is a $\mathscr{B}_t$-martingale, where $\mathscr{B}_t := \sigma\{\omega_s, 0 \leqslant s \leqslant t\}, 0 \leqslant t \leqslant T$. Therefore, we can rewrite Eq.(13) to get that

$$\begin{aligned}
\mathrm{d}Y_t &= b(t, Y_t)\mathrm{d}t + \sum_{i=1}^{d} h_{ii}(t, Y_t)\mathrm{d}t + \int_{\mathbb{R}^d\setminus\{0\}} f(t, Y_t, u)\nu(\mathrm{d}u)\mathrm{d}t + \sigma(t, Y_t)\mathrm{d}B_t \\
&\quad + \int_{\mathbb{R}^d\setminus\{0\}} f(t, Y_t, u)\tilde{L}(\mathrm{d}t, \mathrm{d}u).
\end{aligned}$$

This is a classical stochastic differential equation with jumps. Therefore, using [16, Theorem 1.2], it holds that under $(\mathbf{H}^1_{b,h,\sigma,f})$-$(\mathbf{H}^2_{b,h,\sigma,f})$, the above equation exhibit a unique solution.

In the following, note that $G(g_1) = \frac{1}{2}\mathrm{tr}(g_1) = \frac{1}{2}\sum\limits_{i=1}^{d} g_1^{ii}$. Therefore, $F_{s,t}$ can be denoted as:

$$\begin{aligned}
F_{s,t} &= \int_s^t \left(\frac{\alpha}{2} + \beta\right)\sum_{i=1}^{d} g_1^{ii}(r, Y_r)\mathrm{d}r + \int_s^t \langle g_2(r, Y_r), \mathrm{d}B_r\rangle + \int_s^t \int_{\mathbb{R}^d\setminus\{0\}} g_3(r, Y_r, u)\tilde{L}(\mathrm{d}r, \mathrm{d}u) \\
&\quad + \int_s^t \int_{\mathbb{R}^d\setminus\{0\}} (1+\gamma)g_3(r, Y_r, u)\nu(\mathrm{d}u)\mathrm{d}r.
\end{aligned}$$

This is just right (3) in [20] without the distribution of Y_r for $r \in [s, t]$. Therefore, in this case Definition 2.15 and Theorem 3.2 are Definition 2.1 and Theorem 3.2 in [20] without the distribution of Y_r for $r \in [s, t]$, respectively. Thus, our finding overlaps [20, Theorem 3.2] in some sense.

4. Some examples

In this final section, we use our findings for some interesting examples. In the first three examples, we explicitly propose the additive functionals $F_{s,t}$. Then the density process's path independence for the Girsanov transformation under the G-setting is explained in the fourth example. In the fifth example, our findings are employed to finance.

Example 4.1. *If $b(t, x) = 0, h_{ij}(t, x) = 0, \sigma(t, x) = I_d, f(t, x, u) = u$, Eq.(1) becomes*

$$\mathrm{d}Y_t = \mathrm{d}B_t + \int_{\mathbb{R}^d\setminus\{0\}} uL(\mathrm{d}t, \mathrm{d}u) = \mathrm{d}X_t.$$

Therefore, by Theorem 3.2, it holds that $F_{s,t}$ is path independent in the sense of (7) if and only if (V, g_1, g_2, g_3) satisfies the following:

$$\begin{cases}
\partial_t V(t, x) = \alpha G(g_1)(t, x) + \gamma \sup\limits_{\nu\in\mathcal{V}} \int_{\mathbb{R}^d\setminus\{0\}} g_3(t, x, u)\nu(\mathrm{d}u), \\
\frac{1}{2}\partial_x^2 V(t, x) = \beta g_1(t, x), \\
\partial_x V(t, x) = g_2(t, x), \\
V(t, x+u) - V(t, x) = g_3(t, x, u).
\end{cases}$$

We take $\alpha = 1, \beta = \frac{1}{2}, \gamma = 1$, and furthermore generate that (V, g_1, g_2, g_3) satisfies the following:

$$\begin{cases} \partial_t V(t,x) = G(\partial_x^2 V)(t,x) + \sup\limits_{\nu\in\mathcal{V}} \int_{\mathbb{R}^d\setminus\{0\}} (V(t,x+u) - V(t,x))\nu(\mathrm{d}u), \\ \partial_x^2 V(t,x) = g_1(t,x), \\ \partial_x V(t,x) = g_2(t,x), \\ V(t,x+u) - V(t,x) = g_3(t,x,u). \end{cases} \tag{14}$$

That is, V satisfies the following partial integro-differential equation:

$$\partial_t V(t,x) - G(\partial_x^2 V)(t,x) - \sup_{\nu\in\mathcal{V}} \int_{\mathbb{R}^d\setminus\{0\}} \Big(V(t,x+u) - V(t,x)\Big)\nu(\mathrm{d}u) = 0. \tag{15}$$

Note that, by Theorem 2.7, it holds that for $\phi \in C_{b,lip}(\mathbb{R}^d)$, $V(t,x) = \bar{\mathbb{E}}[\phi(x + X_t)]$ is the unique viscosity solution of Eq.(15) with the initial condition $V(0,x) = \phi(x)$. If we further assume that $V(t,x) = \bar{\mathbb{E}}[\phi(x+X_t)]$ is a classical solution of Eq.(15) with the initial condition $V(0,x) = \phi(x)$, i.e., if $V \in C_b^{1,2}([0,T]\times\mathbb{R}^d)$, then from (14), it follows that

$$\begin{aligned} &g_1(t,x) = \partial_x^2 V(t,x) = \partial_x^2 \bar{\mathbb{E}}[\phi(x+X_t)], \\ &g_2(t,x) = \partial_x V(t,x) = \partial_x \bar{\mathbb{E}}[\phi(x+X_t)], \\ &g_3(t,x,u) = V(t,x+u) - V(t,x) = \bar{\mathbb{E}}[\phi(x+u+X_t)] - \bar{\mathbb{E}}[\phi(x+X_t)]. \end{aligned}$$

That is, we can find g_1, g_2, g_3.

Example 4.2. *If $b(t,x) = 0, h_{ij}(t,x) = 0, \sigma(t,x) = 0, f(t,x,u) = u$, Eq.(1) goes into*

$$\mathrm{d}Y_t = \int_{\mathbb{R}^d\setminus\{0\}} uL(\mathrm{d}t, \mathrm{d}u) = \mathrm{d}X_t^d.$$

That is, $Y_.$ is a pure jump process. From Theorem 3.2, it follows that $F_{s,t}$ is path independent in the sense of (7) if and only if (V, g_1, g_2, g_3) satisfies the following:

$$\begin{cases} \partial_t V(t,x) = \alpha G(g_1)(t,x) + \gamma \sup\limits_{\nu\in\mathcal{V}} \int_{\mathbb{R}^d\setminus\{0\}} g_3(t,x,u)\nu(\mathrm{d}u), \\ 0 = \beta g_1(t,x), \\ 0 = g_2(t,x), \\ V(t,x+u) - V(t,x) = g_3(t,x,u). \end{cases}$$

Letting $\alpha = 1, \beta = 1, \gamma = 1$, one can have that

$$\begin{cases} \partial_t V(t,x) = \sup\limits_{\nu\in\mathcal{V}} \int_{\mathbb{R}^d\setminus\{0\}} (V(t,x+u) - V(t,x))\nu(\mathrm{d}u), \\ 0 = g_1(t,x), \\ 0 = g_2(t,x), \\ V(t,x+u) - V(t,x) = g_3(t,x,u). \end{cases}$$

By the deduction similar to that in Example 4.1, we generate that

$$g_3(t,x,u) = V(t,x+u) - V(t,x) = \bar{\mathbb{E}}[\phi(x+u+X_t^d)] - \bar{\mathbb{E}}[\phi(x+X_t^d)].$$

Thus, we give out g_1, g_2, g_3.

Example 4.3. *If $d = 1$, Eq.(1) becomes*

$$\mathrm{d}Y_t = b(t,Y_t)\mathrm{d}t + h(t,Y_t)\mathrm{d}\langle B,B\rangle_t + \sigma(t,Y_t)\mathrm{d}B_t + \int_{\mathbb{R}^d\setminus\{0\}} f(t,Y_t,u)L(\mathrm{d}t,\mathrm{d}u).$$

So, it follows from Theorem 3.2 that $F_{s,t}$ is path independent in the sense of (7) if and only if (V, g_1, g_2, g_3) satisfies the following:

$$\begin{cases} \partial_t V(t,x) + \frac{\partial}{\partial x}V(t,x)b(t,x) = \alpha G(g_1)(t,x) + \gamma \sup\limits_{\nu\in\mathcal{V}} \int_{\mathbb{R}^d\setminus\{0\}} g_3(t,x,u)\nu(\mathrm{d}u), \\ \frac{\partial}{\partial x}V(t,x)h(t,x) + \frac{1}{2}\frac{\partial^2}{(\partial x)^2}V(t,x)\sigma^2(t,x) = \beta g_1(t,x), \\ (\sigma\partial_x V)(t,x) = g_2(t,x), \\ V\Big(t, x+f(t,x,u)\Big) - V(t,x) = g_3(t,x,u). \end{cases}$$

Inserting $\alpha = 1, \beta = 0, \gamma = 0$ in the above equations, we get that

$$\begin{cases} \partial_t V(t,x) + \partial_x V(t,x)b(t,x) = G(g_1)(t,x), \\ \partial_x V(t,x)h(t,x) + \frac{1}{2}\partial_x^2 V(t,x)\sigma^2(t,x) = 0, \\ (\sigma\partial_x V)(t,x) = g_2(t,x), \\ V\Big(t, x+f(t,x,u)\Big) - V(t,x) = g_3(t,x,u). \end{cases}$$

Thus, if $\sigma(t,x) \neq 0$, the unique solution of the above second equation is

$$V(t,x) = V(t,0) + \partial_x V(t,0)\int_0^x e^{-2\int_0^z \frac{h(t,v)}{\sigma^2(t,v)}\mathrm{d}v}\mathrm{d}z.$$

Besides, since $\mathcal{Q}$ is bounded away from 0, G is invertible. Thus,

$$g_1(t,x) = G^{-1}\Big(\partial_t V(t,x) + b(t,x)\partial_x V(t,0)e^{-2\int_0^x \frac{h(t,v)}{\sigma^2(t,v)}\mathrm{d}v}\Big),$$

$$g_2(t,x) = \sigma(t,x)\partial_x V(t,0)e^{-2\int_0^x \frac{h(t,v)}{\sigma^2(t,v)}\mathrm{d}v},$$

$$g_3(t,x,u) = \partial_x V(t,0)\int_x^{x+f(t,x,u)} e^{-2\int_0^z \frac{h(t,v)}{\sigma^2(t,v)}\mathrm{d}v}\mathrm{d}z.$$

That is, we also give out g_1, g_2, g_3 in the case.

Example 4.4. *If $f(t,x,u) = 0$, Eq.(1) goes into*

$$\mathrm{d}Y_t = b(t,Y_t)\mathrm{d}t + h_{ij}(t,Y_t)\mathrm{d}\langle B^i, B^j\rangle_t + \sigma(t,Y_t)\mathrm{d}B_t.$$

That is, $Y.$ is a quasi-continuous process. Therefore, it follows from Theorem 3.2 that $F_{s,t}$ is path independent in the sense of (7) if and only if (V, g_1, g_2, g_3) satisfies the following:

$$\begin{cases} \partial_t V(t,x) + \frac{\partial}{\partial x_k}V(t,x)b^k(t,x) = \alpha G(g_1)(t,x), \\ \frac{\partial}{\partial x_k}V(t,x)h_{ij}^k(t,x) + \frac{1}{2}\frac{\partial^2}{\partial x_k\partial x_l}V(t,x)\sigma^{ki}(t,x)\sigma^{lj}(t,x) = \beta g_1^{ij}(t,x), \\ (\sigma^*\partial_x V)(t,x) = g_2(t,x), \\ 0 = g_3(t,x,u). \end{cases}$$

We take $\alpha = 0, \beta = \frac{1}{2}$ and obtain that

$$\begin{cases} \partial_t V(t,x) + \frac{\partial}{\partial x_k}V(t,x)b^k(t,x) = 0, \\ \frac{\partial}{\partial x_k}V(t,x)h_{ij}^k(t,x) + \frac{1}{2}\frac{\partial^2}{\partial x_k\partial x_l}V(t,x)\sigma^{ki}(t,x)\sigma^{lj}(t,x) = \frac{1}{2}g_1^{ij}(t,x), \\ (\sigma^*\partial_x V)(t,x) = g_2(t,x). \end{cases} \tag{16}$$

Next, assume that $\sigma(t,x)$ is invertible and there exists a $\varepsilon > 0$ such that

$$\bar{\mathbb{E}}\left[\exp\left(\frac{1}{2}(1+\varepsilon)\int_0^T \Big((\sigma^{-1}h_{ij})(t,Y_t)\Big)^*(\sigma^{-1}h_{ij})(t,Y_t)\mathrm{d}\langle B^i, B^j\rangle_t\right)\right] < \infty.$$

Set

$$g_1^{ij}(t,x) := \Big((\sigma^{-1}h_{ij})(t,x)\Big)^*(\sigma^{-1}h_{ij})(t,x), \quad g_2(t,x) := \sum_{i,j=1}^d(\sigma^{-1}h_{ij})(t,x),$$

and then

$$-F_{0,t} = -\frac{1}{2}\int_0^t \Big((\sigma^{-1}h_{ij})(r,Y_r)\Big)^*(\sigma^{-1}h_{ij})(r,Y_r)\mathrm{d}\langle B^i,B^j\rangle_r - \sum_{i,j=1}^d\int_0^t\langle(\sigma^{-1}h_{ij})(r,Y_r),\mathrm{d}B_r\rangle$$

is the density process of the Girsanov transformation under the G-setting (See [8, Theorem 5.3]*). Concretely speaking, set*

$$\begin{aligned}D_t &:= \exp\Big\{-\sum_{i,j=1}^d\int_0^t\langle(\sigma^{-1}h_{ij})(r,Y_r),\mathrm{d}B_r\rangle \\ &\qquad -\frac{1}{2}\int_0^t\Big((\sigma^{-1}h_{ij})(r,Y_r)\Big)^*(\sigma^{-1}h_{ij})(r,Y_r)\mathrm{d}\langle B^i,B^j\rangle_r\Big\}, \\ \check{B}_t &:= B_t + \int_0^t(\sigma^{-1}h_{ij})(r,Y_r)\mathrm{d}\langle B^i,B^j\rangle_r,\end{aligned}$$

and define

$$\begin{aligned}&\check{C}_{b,lip}(\Omega_T) := \left\{\varphi(\check{B}_{t_1},\cdots,\check{B}_{t_n}) : n\in\mathbb{N}, t_1,\cdots,t_n\in[0,T], \varphi\in C_{b,lip}(\mathbb{R}^{d\times n})\right\}, \\ &\check{\mathbb{E}}[Z] := \bar{\mathbb{E}}[ZD_T], \quad \forall Z\in\check{C}_{b,lip}(\Omega_T).\end{aligned}$$

Let $\check{\mathcal{H}}$ be the completion of $\check{C}_{b,lip}(\Omega_T)$ with respect to the norm $\check{\mathbb{E}}[|\cdot|]$. Then we extend $\check{\mathbb{E}}$ to a sublinear expectation on $\check{\mathcal{H}}$ still denoted as $\check{\mathbb{E}}$. Thus, $\check{B}_\cdot$ is a G-Brownian motion on the sublinear expectation space $(\Omega_T,\check{\mathcal{H}},\check{\mathbb{E}})$.

In addition, by (16), we know that, $F_{0,t}$ is path independent in the sense of (7) if and only if the following holds:

$$\begin{cases}\partial_t V(t,x) + \frac{\partial}{\partial x_k}V(t,x)b^k(t,x) = 0, \\ \frac{\partial}{\partial x_k}V(t,x)h_{ij}^k(t,x) + \frac{1}{2}\frac{\partial^2}{\partial x_k\partial x_l}V(t,x)\sigma^{ki}(t,x)\sigma^{lj}(t,x) = \frac{1}{2}\Big((\sigma^{-1}h_{ij})(t,x)\Big)^*(\sigma^{-1}h_{ij})(t,x), \\ (\sigma^*\partial_x V)(t,x) = \sum_{i,j=1}^d(\sigma^{-1}h_{ij})(t,x).\end{cases}$$

These are different from the classical case (See [25, Theorem 2.1]*). This difference comes from random quadratic variation processes of G-Brownian motions. That's just why it makes sense to investigate G-Brownian motions.*

In the following example, we consider a model in finance.

Example 4.5. *For $d=1$, the stock price S_t is governed by*

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \kappa(t,S_t)\sigma S_t\mathrm{d}\langle B\rangle_t + \sigma S_t\mathrm{d}B_t,$$

where the risk free interest rate r and the volatility σ are constants, $\kappa:[0,T]\times\mathbb{R}\mapsto\mathbb{R}$ is a $C^{1,2}$-function. Then, we deduce some conditions under, which the Girsanov transformation's density process under the G-setting is path independent.

Assume that $\sigma\neq 0$ and there exists a $\varepsilon>0$ such that

$$\bar{\mathbb{E}}\left[\exp\left(\frac{1}{2}(1+\varepsilon)\int_0^T\kappa^2(t,S_t)\mathrm{d}\langle B\rangle_t\right)\right] < \infty.$$

Based on Example 4.4, we know that the density process of the Girsanov transformation under the G-setting is as follows:

$$-F_{0,t} = -\frac{1}{2}\int_0^t \kappa^2(r, S_r)\mathrm{d}\langle B\rangle_r - \int_0^t \kappa(r, S_r)\mathrm{d}B_r.$$

Furthermore, $F_{0,t}$ is path independent in the sense of (7), i.e.

$$V(t, S_t) - V(0, S_0) = \frac{1}{2}\int_0^t \kappa^2(r, S_r)\mathrm{d}\langle B\rangle_r + \int_0^t \kappa(r, S_r)\mathrm{d}B_r,$$

if and only if the following holds:

$$\begin{cases} \partial_t V(t,x) + \partial_x V(t,x) rx = 0, \\ \partial_x V(t,x)\kappa(t,x)\sigma x + \frac{1}{2}\partial_x^2 V(t,x)\sigma^2 x^2 = \frac{1}{2}\kappa^2(t,x), \\ \sigma x \partial_x V(t,x) = \kappa(t,x). \end{cases}$$

Put $r = 0$ and $\kappa(t,x) = \kappa(x)$ and then by the above equations, $\kappa(x)$ satisfies the following ordinary differential equation:

$$\kappa'(x)\sigma x - \sigma\kappa(x) + \kappa^2(x) = 0.$$

We solve it and obtain that

$$\kappa(x) = \frac{\sigma x}{x - c_0},$$

where $c_0 \neq 0$ is a constant. Therefore, $V(x) = \ln(x - c_0) + c_1$, where c_1 is a constant. That is, if we take this $V(x)$, the density process of the Girsanov transformation under the G-setting is path independent.

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References

[1] F. Gao: Pathwise properties and homeomorphic flows for stochastic differential equations driven by G-Brownian motion, *Stochastic Processes and their Applications*, 119(2009)3356-3382.

[2] S. Hodges and A. Carverhill: Quasi mean reversion in an efficient stock market: The characterisation of economic equilibria which support Black-Scholes option pricing, *Econom J.*, 103(1993)395-405.

[3] M. Hu, S. Ji, S. Peng and Y. Song: Comparison theorem, Feynman-Kac formula and Girsanov transformation for BSDEs driven by G-Brownian motion, *Stochastic Processes and their Applications*, 124(2014)1170-1195.

[4] M. Hu and S. Peng: G-Lévy processes under sublinear expectations, *Probability, Uncertainty and Quantitative Risk*, 6(2021)1-22.

[5] M. Hu, F. Wang and G. Zheng: Quasi-continuous random variables and processes under G-expectation framework, *Stochastic Proc. Appl.*, 126(8)(2016)2367-2387.

[6] X. Li and S. Peng: Stopping times and related Itô's calculus with G-Brownian motion, *Stochastic Proc. Appl.*, 121(2011) 1492-1508.

[7] G. Liu: Girsanov theorem for G-Brownian motion: the degenerate case, *Journal of Theoretical Probability*, 34(2021)125-140.

[8] E. Osuka: Girsanovs formula for G-Brownian motion, *Stoch. Process. Appl.*, 123(2013)1301-1318.

[9] K. Paczka: It calculus and jump diffusions for G-Lévy processes, arXiv:1211.2973v3.

[10] K. Paczka: G-martingale representation in the G-Lévy setting, arXiv:1404.2121v1.

[11] K. Paczka: On the properites of Poisson random measures associated with a G-Lévy process, arXiv:1411.4660v1.

[12] S. Peng: Filtration consistent nonlinear expectations and evaluations of contingent claims, *Acta Mathematicae Applicatae Sinica*, English Series 20(2)(2004)1-24.

[13] S. Peng: Nonlinear expectations and nonlinear Markov chains, *Chin. Ann. Math.*, 26B(2)(2005)159-184.

[14] S. Peng: Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation, *Stochastic Processes and their Applications*, 118(2008)2223-2253.

[15] S. Peng: *Nonlinear Expectations and Stochastic Calculus under Uncertainty with Robust CLT and G-Brownian Motion*. Probability Theory and Stochastic Modelling 95, Springer, 2019.

[16] H. Qiao: Euler-Maruyama approximation for SDEs with jumps and non-Lipschitz coefficients, *Osaka Journal of Mathematics*, 51(2014)47-66.

[17] H. Qiao: The cocycle property of stochastic differential equations driven by G-Brownian motion, *Chinese Annals of Mathematics, Series B*, 36(2015)147-160.

[18] H. Qiao and J.-L. Wu: Characterising the path-independence of the Girsanov transformation for non-Lipschitz SDEs with jumps, *Statistics and Probability Letters*, 119(2016)326-333.

[19] H. Qiao and J.-L. Wu: On the path-independence of the Girsanov transformation for stochastic evolution equations with jumps in Hilbert spaces, *Discrete and Continuous Dynamical Systems-B*, 24(2019)1449-1467.

[20] H. Qiao and J.-L. Wu: Path independence of the additive functionals for McKean-Vlasov stochastic differential equations with jumps, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 24(2021)2150006 (20 pages).

[21] P. Ren and F. Yang: Path independence of additive functionals for stochastic differential equations under G-framework, *Front. Math. China*, 14 (2019)135-148.

[22] E. M. Stein and J. C. Stein: Stock price distributions with stochastic volatility: An analytic approach, *Rev. Financ. Stud.*, 4(1991)727-752.

[23] Y. Song: Properties of hitting times for G-martingales and their applications, *Stoch. Proc. Appl.*, 121(8) (2011)1770-1784.

[24] Y. Song: Uniqueness of the representation for G-martingales with finite variation, *Electron J. Probab.*, 17(2012)1-15.

[25] A. Truman, F.-Y. Wang, J.-L. Wu, W. Yang: A link of stochastic differential equations to nonlinear parabolic equations, *SCIENCE CHINA Mathematics*, 55 (2012)1971-1976.

[26] B. Wang and H. Gao: Exponential stability of solutions to stochastic differential equations driven by G-Lévy process, *Applied Mathematics & Optimization*, 83(2021)1191-1218.

[27] B. Wang and M. Yuan: Existence of solution for stochastic differential equations driven by G-Lévy process with discontinuous coefficients, *Advances in Difference Equations*, 2017(2017)Article number:188.

[28] J.-L. Wu and W. Yang: On stochastic differential equations and a generalised Burgers equation, pp 425-435 in *Stochastic Analysis and Its Applications to Finance: Essays in Honor of Prof. Jia-An Yan (eds. T. S. Zhang, X. Y. Zhou)*, Interdisciplinary Mathematical Sciences, Vol. 13, World Scientific, Singapore, 2012.

[29] J. Xu, H. Shang and B. Zhang: A Girsanov type theorem under G-framework, *Stoch. Anal. Appl.*, 29 (2011)386-406.