# Stability of a non-Lipschitz stochastic Riemann-Liouville type fractional differential equation driven by Lévy noise *

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**Abstract**

In this paper, stability of a non-Lipschitz stochastic Riemann-Liouville type fractional differential equation driven by Lévy noise is studied. Three types of stability, namely, stochastic stability, almost sure exponential stability, and moment exponential stability, of the fractional equation are established. Examples are given to illustrate and to support our results.

**Keywords:** Fractional derivative of Riemann-Liouville type; stochastic fractional differential equations with non-Lipschitz coefficients; Lévy noise; stochastic stability; almost sure exponential stability; moment exponential stability.

## 1. Introduction

This paper is concerned with stability behaviours of stochastic Riemann-Liouville type fractional differential equations with non-Lipschitz coefficients, where the equation we are interested involve fractional derivatives of Riemann-Liouville type for the time variable and is driven by Lévy noise.

As is well known, stochastic differential equations play a significant role in modelling evolutions of dynamical systems when taking into account uncertainty features in diverse fields ranging from biology, chemistry, and physics, as well as ecology, economics and finance, and so on (see, for example, Sobczyk [37], Mao [28], Henderson and Plaschko [12] and the references therein). There have been numerous studies addressing the existence and uniqueness of the solutions of stochastic differential equations under more general conditions for coefficients, extending the classical condition of linear growth and Lipschitz condition. One remarkable extension is the non-Lipschitz condition initiated by Fang and Zhang [11], which has been further generalised to various different situations, see e.g. [5], [9], [10], [13], [19], [32], [33], and references therein.

One important issue of studying the well-posedness of stochastic differential equations, after achieving the existence and uniqueness of their solutions, is the stability of the solutions, in particular for the qualitative study and/or for the long time asymptotic behaviour of the solutions, see, e.g., [36]. Roughly, a solution of a stochastic differential equation is stable if it is insensitive for small changes of the initial value

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or the parameters contained in the equation. To date, the stability of stochastic differential equations has been very well developed since the seminal work of Khasminski [16] extending the celebrating concept of stability of (deterministic) dynamic systems introduced by Lyapunov [25]. Various generalisations, such as stochastically stability, stochastically asymptotically stability, moment exponentially stability, almost surely stability, mean square polynomial stability etc. were carried out in a series of works by Mao [26], [27], [29], as well as for different types of stochastic functional equations by, e.g., Li, Dong and Situ [20], Applebaum and Siakalli [6], Siakalli [35], Liu, Foondun and Mao [22], Li and Deng [21], Lu et al [23], Nane and Ni [30], Zhu [39], just mention a few.

On the other hand, fractional derivatives have been used to model physical and engineering processes, which are found to be best described by fractional differential equations, due to the fact that the standard mathematical models of integer-order derivatives, including nonlinear models, do not work adequately in many cases. In the recent years, fractional derivatives and related fractional calculus have played a very important role in modelling arising in diverse fields such as biology, chemistry, economics, mechanics, and also in technology links to signal and image processing, as well as control theory for which stability is pivotal. In contrast to the classical calculus, the fractional derivative of a constant is not zero. Amongst the fractional derivatives, the Riemann-Liouville fractional derivative is the most important extension of classical calculus. Almost all other definitions for the fractional derivative represent a special case of the Riemann-Liouville fractional derivative. Along this line, there has been increasing interest and high demanding for investigating stochastic differential equations with fractional derivatives, from both theoretical and practical aspects, see e.g. [7], [8], [17], [18], [34], and references therein. Recently, a very interesting work [31] by Pedjeu and Ladde studied the following stochastic Riemann-Liouville type fractional differential equations for stochastic modelling of dynamical processes in ecology and epidemiology

$$dx = b(t,x)dt + \sigma_1(t,x)dB(t) + \sigma_2(t,x)(dt)^\alpha, \quad x(t_0) = x_0, \tag{1.1}$$

where $\alpha \in (0,1)$, and $B(t)$ is a standard Brownian motion. By utilising the Picard-Lindelöf successive approximation, the authors showed the existence and uniqueness of the solutions to the equations with Lipschitz coefficients, under the linearly independent time scale. Moreover, Abouagwa, Liu and Li [4] established the existence and uniqueness of solutions for the stochastic fractional differential equations (1.1) with non-Lipschitz coefficients and further investigated the continuous dependence of solutions on the initial value in the sense of mean square stability; Abouagwa and Li [1] discussed the approximation theorem as an averaging principle for the solutions of the stochastic fractional differential equations (1.1) with non-Lipschitz coefficients; Abouagwa and Li [2] obtained the existence and uniqueness of the solutions to the stochastic fractional differential equations driven by Lévy noise by utilising the Carathéodory approximation and stopping time techniques; and further Abouagwa, Cheng and Li [3] utilised an approximation scheme of Carathéodory type to study the existence and uniqueness results for a class of stochastic fractional differential equations with impulses driven by a standard Brownian motion and an independent fractional Brownian motion with Hurst index $\frac{1}{2} < H < 1$ under a non-Lipschitz condition (with the Lipschitz condition as a particular case). Luo, Zhu and Luo [24] studied averaging principle for stochastic fractional differential equations with time-delays. However, to the best of our knowledge, stability analysis of solutions of stochastic fractional differential equations has not been considered in the literature, apart from [4] wherein, as mentioned above, a special consideration of the continuous dependence of solutions on the initial value in the sense of mean square stability was carried out for (1.1). We would like to mention Xiao and Wang [38], the authors studied stability of Caputo-type fractional stochastic differential equations in which stochastic stability and stochastic asymptotical stability were shown by stopping time technique, almost surly exponential stability and p-th moment exponentially stability were obtained by a newly established Itô's formula of Caputo version.

The present paper is thus devoted to the stabilisation problem for the following more general stochastic Riemann-Liouville type fractional differential equation with non-Lipschitz coefficients driven by Lévy noise

$$\begin{cases} dx(t) = u(x(t),t)dt + b(x(t),t)dB(t) + \sigma(x(t),t)(dt)^{\alpha} \\ \qquad\qquad + \int_{|y|<c} h(x(t-),y,t)\tilde{N}(dt,dy), \qquad\qquad 0<\alpha<1, \quad t\in[0,T], \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

where the positive constant $c$ is the maximum allowable jump size, and the mappings $u, \sigma : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$, $b : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n\otimes m}$, $h : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \times [0,T] \to \mathbb{R}^n$ are all progressively measurable with further conditions specified in the sequel. Here we would like to point out that adding Lévy noise to the equation induces non-Gaussian perturbation effect which fits the modelling demanding of many real world problems. The existence and uniqueness of the solutions of (1.2) has been established by Abouagwa and Li in [2]. So our focus here is the stability problem, which has not been considered elsewhere yet. In case that $\sigma \equiv 0$, i.e., the SDE (1.2) is without the fractional derivative term, Applebaum and Siakalli [6] and Siakalli [35] have given a thorough consideration for stabilities of the solutions under Lipschitz coefficients. In the present paper, we would like to study the stability problem for the stochastic Riemann-Liouville type fractional differential equations (1.2) with non-Lipschitz coefficients. Apart from the appearance of the more general (non-Lipschitz) coefficients, the primary challenge here is to conquer the trouble term $(dt)^{\alpha}$ which requires certain novel approach, in particular to integrate with the appearance of Lévy noise $\tilde{N}$. Clearly, the methods used in those aforementioned papers do not apply here in a straightforward manner. Instead, one needs to carefully treat the integrals related to the Riemann-Liouville type fractional derivative and the jump type stochastic integral with respect to Lévy noise simultaneously, towards the stability of the equation (1.2). By tedious calculations and notably utilising stopping time techniques, we are able to establish three types of stability for the solutions of the equation (1.2) : stochastic stability, almost sure exponential stability, and moment exponential stability of (1.2).

Let us fix our set up as follows. Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition, i.e., it is increasing and right continuous with $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets of $\mathcal{F}$, $B(t) := (B_1(t), B_2(t), \cdots, B_m(t))^{\mathrm{T}}$ is an $m$-dimensional $\{\mathcal{F}_t\}$-Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and the superscript T stands for the transpose. Given a Lévy measure $\nu$, namely, $\nu$ is a $\sigma$-finite measure on the Borel measurable space $\mathbb{R}^n \setminus \{0\}$ satisfying $\int_{\mathbb{R}^n\setminus\{0\}} (|y|^2 \wedge 1)\nu(dy) < \infty$, we assume that the associated $\{\mathcal{F}_t\}$-Poisson random measure $N$ on $\mathbb{R}^+ \times (\mathbb{R}^n \setminus \{0\})$ is independent of $B(t)$. The associated compensated martingale measure is denoted by $\widetilde{N}(dt,dy) := N(dt,dy) - \nu(dy)dt$. Furthermore, the coefficient $h : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \times [0,T] \to \mathbb{R}^n$ of (1.2) fulfils that the stochastic integral process $\int_0^t \int_{|y|<c} h(x(t-),y,t)\widetilde{N}(dt,dy), t \geq 0$, is an $\mathbb{R}^n$-valued, square integrable $\{\mathcal{F}_t\}$-martingale with the property that $\mathbb{P}(\int_0^t \int_{|y|<c} |h(x(t),y,t)|^2 \nu(dy)ds < \infty) = 1$.

The rest of this paper is organised as follows. In Section 2, we present some preliminaries for fractional derivatives and stochastic fractional differential equations. In Section 3, we discuss the stochastic stability, almost surely stability and moment exponential stability of stochastic Riemann-Liouville type fractional differential equations driven by Lévy noise under non-Lipschitz conditions. We finish our paper by given several examples to illustrate our obtained results.

## 2. Preliminaries

We start with a brief introduction to Riemann-Liouville type fractional derivatives. For more detailed discussions on this concept, the reader is referred to Jumarie [14], [15], Kilbas, Srivastava, Trujillo [17], Samko, Kilbas, Marichev [34]. It is worthwhile to mention the recent monograph [18] by Kolokoltsov wherein another type fractional derivatives named as Caputo type fractional derivatives (together with their related evolution equations) are thoroughly studied.

As usual, we use $L^1[a,b] := L^1([a,b];\mathbb{R}^n)$ to denote the totality of $\mathbb{R}^n$-valued Lebesgue integrable functions on a given finite interval $[a,b] \subset \mathbb{R}$. Throughout the paper, we let $\alpha \in (0,1)$ be arbitrarily fixed.

**Definition 2.1.** (Riemann-Liouville fractional integrals) For any function $f \in L^1([a,b];\mathbb{R}^n)$, the left sided and right sided Riemann-Liouville fractional integrals of order $\alpha$ are defined respectively by the following

$$(I^{\alpha}_{a+}f)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds, \qquad t > a, \tag{2.1}$$

and

$$(I^{\alpha}_{b-}f)(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (t-s)^{\alpha-1}f(s)ds, \qquad t < b, \tag{2.2}$$

for almost all $t \in (a,b)$, where $\Gamma(\alpha) := \int_0^\infty s^{\alpha-1}e^{-s}ds$ stands for the Gamma function.

One can use the Riemann-Liouville fractional integrals to define the Riemann-Liouville fractional derivatives. In fact, we have

**Definition 2.2.** (Riemann-Liouville fractional derivatives) Let $f \in L^1([a,b];\mathbb{R}^n)$. Assume that $f$ is absolutely continuous on $[a,b]$, then its left sided and right sided Riemann-Liouville fractional derivatives are defined respectively as

$$(D^{\alpha}_{a+}f)(t) = \frac{1}{\Gamma(1-\alpha)}[\frac{f(a)}{(t-a)^{\alpha}} + \int_a^t (t-s)^{-\alpha}f'(s)ds], \tag{2.3}$$

and

$$(D^{\alpha}_{b-}f)(t) = \frac{1}{\Gamma(1-\alpha)}[\frac{f(b)}{(b-t)^{\alpha}} - \int_t^b (s-t)^{-\alpha}f'(s)ds]. \tag{2.4}$$

In this paper, we will consider the case that $a = 0$ and $f : [0,T] \to \mathbb{R}^n$ for arbitrarily fixed $T > 0$. So in the sequel, we will use simplified notation $D^{\alpha}f$ for the left-sided Riemann-Liouville fractional derivative $D^{\alpha}_{0+}f$. Recall that we always have $\alpha \in (0,1)$ being arbitrarily fixed. Thus, for $f$ with $D^{\alpha}f$ being well defined (i.e., $f$ is $\alpha$-th differentiable), using integration by parts, one has

$$(D^{\alpha}f)(t) = \frac{1}{\Gamma(1-\alpha)}[\frac{f(0)}{t^{\alpha}} + \int_0^t (t-s)^{-\alpha}f'(s)ds] = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha-1}f(s)ds.$$

Furthermore, viewing $D^\alpha$ as an operator, one can have its inverse operator

$$(D^{-\alpha}f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds. \tag{2.5}$$

Next, interpreting $D^\alpha := \frac{d^\alpha}{(dt)^\alpha}$ and noticing the relation $(d^\alpha f)(t) = \Gamma(1+\alpha)(df)(t)$ (see, e.g., Lemma 3.1 of [15]), one has $\Gamma(1+\alpha)(df)(t) = (D^\alpha f)(t)(dt)^\alpha$. Letting $\sigma(t) = (D^\alpha f)(t)$, then by (2.5)

$$\int_0^t \sigma(s)(ds)^\alpha = \Gamma(1+\alpha)f(t) = \Gamma(1+\alpha)(D^{-\alpha}\sigma)(t) = \frac{\Gamma(1+\alpha)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sigma(s)ds.$$

That is, one can define integrals with respect to $(dt)^\alpha$ in the following manner

$$\int_0^t \sigma(s)(ds)^\alpha = \alpha\int_0^t (t-s)^{\alpha-1}\sigma(s)ds, \quad t \in [0,\infty) \tag{2.6}$$

provided that the usual integral on the right hand side of (2.6) for the integrant $\sigma : [0,T] \to \mathbb{R}^n$ is well defined. For example, one can take $\sigma : [0,T] \to \mathbb{R}^n$ to be a continuous function, then it is clear that the integral on the right hand side of (2.6) is well defined. Thus, we have the Riemann-Liouville fractional integral of $\sigma$ with respect to $(dt)^\alpha$ defined via (2.6).

Following Abouagwa and Li [2], let us present the following assumptions for the equation (1.2) for the establishment of the existence and uniqueness of the solutions.

**Assumption 2.3.** *There exists a function $R_1 : [0,T] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that*

A1. $R_1(\cdot, z) : [0,T] \to \mathbb{R}^+$ *is locally integrable for all* $z \in \mathbb{R}^+$ *and* $R_1(t,\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous, non-decreasing, concave for any* $t \in [0.T]$ *with the property that* $R_1(t,0) = 0$.

B1. *the following inequality holds*

$$|u(x_1,t) - u(x_2,t)|^2 + ||b(x_1,t) - b(x_2,t)||^2 + |\sigma(x_1,t) - \sigma(x_2,t)|^2 \\ + \int_{|y|<c} |h(x_1,y,t) - h(x_2,y,t)|^2\nu(dy) \le R_1(t, |x_1 - x_2|^2),\ t \in [0,T],\ x \in \mathbb{R}^n.$$

C1. *if a non-negative continuous function* $z : [0,T] \to \mathbb{R}^+$ *fulfils the following inequality*

$$z(t) \le K\int_0^t R_1(s, z(s))ds, \quad \forall t \in [0,T]$$

*for some positive constant* $K$, *then* $z(t) = 0$ *for all* $t \in [0,T]$, *namely,* $z \equiv 0$.

**Assumption 2.4.** *There exists a function $R_2 : [0,T] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that*

A2. $R_2(\cdot, z) : [0,T] \to \mathbb{R}^+$ *is locally integrable for each* $z \in \mathbb{R}^+$ *and* $R_2(t,\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous, non-decreasing, concave for every* $t \in [0,T]$.

B2. *the following inequality holds*

$$|u(x,t)|^2 + \|b(x,t)\|^2 + |\sigma(x,t)|^2 + \int_{|y|<c} |h(x,y,t)|^2\nu(dy) \le R_2(t, |x|^2),\ t \in [0,T],\ x \in \mathbb{R}^n.$$

C2. *for any constant* $\varrho$, *there is a global solution* $z(t), t \in [0,T]$, *of the following deterministic differential equation*

$$\frac{dz(t)}{dt} = \varrho R_2(t, z(t)), \quad t \in (0,T]$$

*for any given initial value* $z(0) = z_0 > 0$.

According to Abouagwa and Li [2], under Assumptions 2.3 –2.4, there exists a unique solution of the stochastic fractional differential equation (1.2). In order to consider the stability problem, we further assume that the coefficients of (1.2) fulfil the following

$$u(0,t) = b(0,t) = \sigma(0,t) = h(0,y,t) = 0$$

for $y \in \mathbb{R}^n \setminus \{0\}$ with $|y| < c$ and for all $t \geq 0$. Clearly, the above condition implies that $x(t) \equiv 0$ (with initial value $x(0) = 0$) is the trivial solution of (1.2).

Next, let us reformulate several different types of stability of our stochastic fractional differential equations by following Mao [28] (see also Applebaum and Siakalli [6]). Let $x(t; x_0)$ denote the solution of the equation (1.2) with the initial value $x(0) = x_0 \in \mathbb{R}^n$.

**Definition 2.5.** The trivial solution of the equation (1.2) is said to be

(I) stochastically stable or stable in probability, if for any $\varepsilon \in (0, 1)$ and $r > 0$, there exists $\delta = \delta(\varepsilon, r) > 0$ such that

$$\mathbb{P}\{|x(t; x_0)| < r, \ \forall t > 0\} \geq 1 - \varepsilon$$

whenever $|x_0| < \delta$. Otherwise, it is said to be stochastically unstable;

(II) almost surely exponentially stable, if

$$\limsup_{t \to \infty} \frac{1}{t} \log |x(t; x_0)| < 0, \qquad a.s.$$

for all $x_0 \in \mathbb{R}^n$;

(III) $p$th moment exponentially stable, if there are positive constants $\lambda$ and $C$ such that

$$\mathbb{E}|x(t; x_0)|^p \leq C|x_0|^p \exp(-\lambda t),$$

for all $x_0 \in \mathbb{R}^n$. When $p = 2$, this is called the exponentially stable in mean square.

Similarly to Lemma 3.1 of Pedjeu and Ladde [31], one can derive an Itô formula for the stochastic fractional differential equation (1.2). Let $\frac{1}{2} < \alpha < 1$ and let $x(t)$ fulfil

$$dx = u(x,t)dt + b(x,t)dB(t) + \sigma(x,t)(dt)^\alpha + \int_{|y|<c} h(x,y,t)\widetilde{N}(dt,dy).$$

For $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R})$, that is, $V_t(x,t) = \frac{\partial V(x,t)}{\partial t}, V_x(x,t) = \left(\frac{\partial V(x,t)}{\partial x_1}, \cdots, \frac{\partial V(x,t)}{\partial x_n}\right)$ and $V_{xx}(x,t) = \left(\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j}\right)_{n\times n}$ are continuous for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$, one has the following

$$\begin{aligned} dV(x,t) =& J_1V(x,t)dt + J_2V(x,t)dB(t) + J_3V(x,t)(dt)^\alpha \\ &+ \int_{|y|<c} (V(x + h(x,y,t),t) - V(x,t))\widetilde{N}(dt,dy), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} J_1V(x,t) =& V_t(x,t) + V_x(x,t)u(x,t) + \frac{1}{2}Tr[b^T(x,t)V_{xx}(x,t)b(x,t)] \\ &+ \int_{|y|<c} (V(x+h(x,y,t),t) - V(x,t) - V_x(x,t)h(x,y,t)\nu(dy), \end{aligned} \tag{2.8}$$

$$J_2V(x,t) = V_x(x,t)b(x,t), \tag{2.9}$$

$$J_3V(x,t) = V_x(x,t)\sigma(x,t). \tag{2.10}$$

Towards the stability, we need the following growth conditions for the coefficients of (1.2)

**Assumption 2.6.** *For every $\theta > 0$, there exists constant $K_\theta > 0$, such that*

$$|u(x,t)| + \|b(x,t)\| + 2\int_{|y|<c} \frac{|h(x,y,t)|(|x| + |h(x,y,t)|)}{|x+h(x,y,t)|}\nu(dy) \leq K_\theta|x|, \tag{2.11}$$

*and*

$$|\sigma(x,t)| \leq K_\theta|x|^2 \qquad for \quad 0 < |x| \leq \theta, \quad t \in \mathbb{R}^+. \tag{2.12}$$

With this condition in hand, we have the following lemma

**Lemma 2.7.** *Under the assumption 2.6 , the solution $x(t) = x(t;x_0)$ of* (1.2) *enjoys the following property*

$$\mathbb{P}\{\omega \in \Omega : x(t) \neq 0 \; \textit{for all} \; t \geq 0\} = 1, \;\; \textit{provided that the initial} \quad x_0 \neq 0. \tag{2.13}$$

*Proof.* The proof essentially follows the procedure of the proof of Lemma 3.2 in Applebaum and Siakalli [6], but here we need to deal with the fractional term $(dt)^\alpha$ by examining the role of the condition (2.12).

Assume in the contrary that (2.13) is not true, then there exist $x_0 \neq 0$ and a stopping time $\tau_1$ with $\mathbb{P}(\tau_1 < \infty) > 0$, where

$$\tau_1 = \inf\{t \geq 0 : |x(t)| = 0\}.$$

Set

$$B = \{\omega \in \Omega : \tau_1(\omega) \leq T \quad and \quad |x(t)(\omega)| \leq \theta - 1 \quad for \quad all \quad 0 < t < \tau_1(\omega)\}.$$

Since the paths of $x(t)$ are almost surely right continuous with left limits, there exist $T > 0$ and $\theta > 1$ such that $\mathbb{P}(B) > 0$. Next, we can define another stopping time $\zeta$ as

$$\zeta = \inf\{t \geq 0 : |x(t)| \leq \varepsilon \quad or \quad |x(t)| \geq \theta\}, for \quad each \quad 0 < \varepsilon < |x_0|.$$

Applying the Itô formula to $e^{-\upsilon(T\wedge\zeta)}|x(T\wedge\zeta)|^{-1}$ with $\upsilon = 2K_\theta + K_\theta^2$, we have

$$\begin{aligned}
&e^{-\upsilon(T\wedge\zeta)}|x(T\wedge\zeta)|^{-1}\\
&= |x_0|^{-1} + \int_0^{T\wedge\zeta} e^{-\upsilon s}\frac{-\upsilon|x(s-)|^2}{|x(s-)|^3}ds + \int_0^{T\wedge\zeta} e^{-\upsilon s}\frac{-x(s-)u(x(s-),s)}{|x(s-)|^3}ds\\
&\quad + \int_0^{T\wedge\zeta} e^{-\upsilon s}\frac{\|b(x(s-),s)\|^2}{|x(s-)|^3}ds + \int_0^{T\wedge\zeta} e^{-\upsilon s}\frac{-x(s-)\sigma(x(s-),s)}{|x(s-)|^3}(dt)^\alpha\\
&\quad + \int_0^{T\wedge\zeta}\int_{|y|<c} e^{-\upsilon s}[\frac{1}{|x(s-)+h(x(s-),y,s)|} - \frac{1}{|x(s-)|} + \frac{x(s-)h(x(s-),y,s)}{|x(s-)|^3}\nu(dy)ds\\
&\quad + \int_0^{T\wedge\zeta} e^{-\upsilon s}\frac{-x(s-)b(x(s-),s)}{|x(s-)|^3}dB(s)\\
&\quad + \int_0^{T\wedge\zeta}\int_{|y|<c} e^{-\upsilon s}[\frac{1}{|x(s-)+h(x(s-),y,s)|} - \frac{1}{|x(s-)|}]\widetilde{N}(ds,dy)\\
&\le |x_0|^{-1} + \int_0^{T\wedge\zeta}\alpha\frac{|\sigma(x(s-),s)|}{|x(s-)|^2}(t-s)^{\alpha-1}ds + \int_0^{T\wedge\zeta} e^{-\upsilon s}\frac{-x(s-)b(x(s-),s)}{|x(s-)|^3}dB(s)\\
&\quad + \int_0^{T\wedge\zeta} e^{-\upsilon s}\{\frac{-\upsilon}{|x(s-)|} + \frac{|u(x)|}{|x(s-)|^2} + \frac{\|b(x(s-),s)\|^2}{|x(s-)|^3}\\
&\quad + \int_{|y|<c}[\frac{1}{|x(s-)+h(x(s-),y,s)|} - \frac{1}{|x(s-)|} + \frac{x(s-)h(x(s-),y,s)}{|x(s-)|^3}\nu(dy)\}ds\\
&\quad + \int_0^{T\wedge\zeta}\int_{|y|<c} e^{-\upsilon s}[\frac{1}{|x(s-)+h(x(s-),y,s)|} - \frac{1}{|x(s-)|}]\widetilde{N}(ds,dy)\\
&\le |x_0|^{-1} + K_\theta(T-0)^\alpha + \int_0^{T\wedge\zeta} e^{-\upsilon s}(\frac{-\upsilon}{|x(s-)|} + \frac{2K_\theta + K_\theta^2}{|x(s-)|})ds\\
&\quad + \int_0^{T\wedge\zeta} e^{-\upsilon s}\frac{-x(s-)b(x(s-),s)}{|x(s-)|^3}dB(s)\\
&\quad + \int_0^{T\wedge\zeta}\int_{|y|<c} e^{-\upsilon s}[\frac{1}{|x(s-)+h(x(s-),y,s)|} - \frac{1}{|x(s-)|}]\widetilde{N}(ds,dy).
\end{aligned}$$

Utilising the following inequality (see, e.g., Lemma 3.1 in Applebaum and Siakalli [6] or Lemma 3.4.2 in Siakalli [35])

$$\frac{1}{|x+y|} - \frac{1}{|x|} + \frac{xy}{|x+y|} \le \frac{2|y|}{|x|^2}\frac{(|y|+|x|)}{|x+y|}$$

for $x, y, x+y \neq 0$, we get

$$\begin{aligned}
&\int_{|y|<c}\frac{1}{|x(s-)+h(x(s-),y,s)|} - \frac{1}{|x(s-)|} + \frac{x(s-)h(x(s-),y,s)}{|x(s-)|^3}\nu(dy)\\
&\le \int_{|y|<c}\frac{2|h(x(s-),y,s)|}{|x(s-)|^2}\frac{(|h(x(s-),y,s)| + |x(s-)|)}{|h(x(s-),y,s)+x(s-)|}\nu(dy)\\
&= \frac{1}{|x(s-)|^2}\int_{|y|<c}2|h(x(s-),y,s)|\frac{(|h(x(s-),y,s)| + |x(s-)|)}{|h(x(s-),y,s)+x(s-)|}\nu(dy)\\
&\le \frac{K_\theta|x(s-)|}{|x(s-)|^2} = \frac{K_\theta}{|x(s-)|}.
\end{aligned}$$

Hence, we have

$$\mathbb{E}[e^{-\upsilon(T\wedge\zeta)}|x(T\wedge\zeta)|^{-1}] \le |x_0|^{-1} + K_\theta T^\alpha.$$

If $\omega \in B$, then $\zeta \le T$ and $|x(\zeta(\omega))| < \varepsilon$, then

$$\begin{aligned}\mathbb{E}[e^{-\upsilon(T-0)}\varepsilon^{-1}1_B] &\le \mathbb{E}[e^{-\upsilon(\zeta-0)}|x(\zeta(\omega))|^{-1}1_B] \\ &\le \mathbb{E}[e^{-\upsilon(\zeta-0)}|x(\zeta(\omega))|^{-1}] \le |x_0|^{-1} + K_\theta T^\alpha.\end{aligned}$$

Hence,

$$\mathbb{P}(B) = \varepsilon e^{\upsilon(T-0)}(|x_0|^{-1} + K_\theta T^\alpha).$$

Let $\varepsilon \to 0$, we get $\mathbb{P}(B) = 0$, which contradicts to the assumption, thus the desired result must be true. □

We end up this section by stating the following exponential martingale inequality (cf. for instance [6], [35]) which will be used in the sequel.

**Lemma 2.8.** *(Exponential martingale inequality) Let $T, \lambda, c, \beta$ be positive constants and let $B_c = \{y \in \mathbb{R}^n : 0 < |y| < c\}$. Assume that $b : [0,T] \times \Omega \to \mathbb{R}^n$ fulfils $\mathbb{P}[\int_0^T |b(t)|^2 dt < \infty] = 1$ and $h : [0,T] \times B_c \times \Omega \to \mathbb{R}^n$ satisfies $\mathbb{P}[\int_0^T \int_{|y|<c} |h(s,y)|^2 \nu(dy)ds < \infty] = 1$, then*

$$\begin{aligned}\mathbb{P}[\sup_{0\le t\le T}\{\int_0^t b(s)dB(s) - \frac{\lambda}{2}\int_0^t |b(s)|^2 ds + \int_0^t \int_{|y|<c} h(s,y)\widetilde{N}(ds,dy) \\ - \frac{1}{\lambda}\int_0^t \int_{|y|<c} [\exp(\lambda h(s,y)) - 1 - \lambda h(s,y)]\nu(dy)ds\} > \beta] \le \exp(-\lambda\beta).\end{aligned} \tag{2.14}$$

## 3. Main results

This section is devoted to the stability problem of the stochastic fractional differential equation (1.2). As mentioned in the introduction, [6] and [35] obtained very interesting results on stability for (1.2) without the fractional term $(dt)^\alpha$. Here we will follow [6] to exam the stability for our fractional equation (1.2). Our focus is to treat the fractional term $(dt)^\alpha$ and the non-Lipschitz coefficients. Recall that $J_1, J_2, J_3$ were defined respectively by (2.8), (2.9), and (2.10) in Section 2. Our first main result reads as follows

**Theorem 3.1.** *The trivial solution of* (1.2) *is stable in probability, provided that there is a positive definite function $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^+)$ such that*

$$J_1 V(x,t) \le 0 \quad \textit{and} \quad J_3 V(x,t) \le 0, \quad \forall (x,t) \in (\mathbb{R}^n, \mathbb{R}^+). \tag{3.1}$$

*Proof.* By the definition of a positive definite function, we know that $V(0,t) = 0$, and we can have a continuous, increasing function $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\mu(0) = 0$ and $\mu(x) > 0$ for $x > 0$, such that

$$\mu(|x|) \le V(x,t) \qquad \textit{for all} \quad (x,t) \in (\mathbb{R}^n, \mathbb{R}^+). \tag{3.2}$$

Let $\varepsilon \in (0,1)$ and $r > 0$, $|x| < h$, and assume $r < h$, since $V(x,t)$ is a continuous function, there exists a $\delta = \delta(\varepsilon, r) > 0$ such that

$$\sup_{|x|<\delta} V(x,0) \le \varepsilon\mu(r), \tag{3.3}$$

then fix $x_0 \in \mathbb{R}^n$ with $|x_0| < \delta$, we define the stopping time

$$\tau_2 = \inf\{t \geq 0 : |x(t)| \geq r\}.$$

Using the Itô formula, for $t \geq 0$, we have

$$\begin{aligned} V(x(t\wedge\tau_2), t\wedge\tau_2) =& V(x_0,0) + \int_0^{t\wedge\tau_2} J_1V(x(s-),s)ds \\ &+ \int_0^{t\wedge\tau_2} J_2V(x(s-),s)dB(s) + \int_0^{t\wedge\tau_2} J_3V(x(s-),s)(ds)^\alpha \\ &+ \int_0^{t\wedge\tau_2}\int_{|y|<c} [V(x(s-)+h(x(s-),y,s),s) - V(x(s-),s)]\widetilde{N}(ds,dy). \end{aligned}$$

By (2.6), we have

$$\begin{aligned} V(x(t\wedge\tau_2), t\wedge\tau_2) =& V(x_0,0) + \int_0^{t\wedge\tau_2} J_1V(x(s-),s)ds \\ &+ \int_0^{t\wedge\tau_2} J_2V(x(s-),s)dB(s) + \int_0^{t\wedge\tau_2} \alpha J_3V(x(s-),s)(t-s)^{\alpha-1}ds \\ &+ \int_0^{t\wedge\tau_2}\int_{|y|<c} [V(x(s-)+h(x(s-),y,s),s) - V(x(s-),s)]\widetilde{N}(ds,dy). \end{aligned}$$

Taking the expectation on both sides, we have

$$\mathbb{E}V(x(t\wedge\tau_2), t\wedge\tau_2) = V(x_0,0) + \mathbb{E}\int_0^{t\wedge\tau_2} (J_1V(x(s-),s) + \alpha J_3V(x(s-),s)(t-s)^{\alpha-1})ds.$$

By the condition $J_1V(x,t) \leq 0 \quad and \quad J_3V(x,t) \leq 0$, it follows that

$$\mathbb{E}V(x(t\wedge\tau_2), t\wedge\tau_2) \leq V(x_0,0). \tag{3.4}$$

Since $x|(t\wedge\tau_2)| = |x(\tau_2)| = r$ for $t \geq \tau_2$, and $\mu(|x|) \leq V(x,t)$, we can obtain

$$V(x_0,0) \geq \mathbb{E}V(x(t\wedge\tau_2), t\wedge\tau_2) \geq \mathbb{E}[1_{\{t\geq\tau_2\}}V(x(\tau_2),\tau_2)] \geq \mu(r)\mathbb{P}\{t \geq \tau_2\}.$$

When $t \to \infty$ we get $\mathbb{P}\{\tau_2 < \infty\} \leq \varepsilon$, so

$$\mathbb{P}\{|x(t;0,x_0)| < r \quad for\ all\ t > 0\} \geq 1 - \varepsilon.$$

This completes the proof. □

The next main result gives the almost surely exponential stability of the solution.

**Theorem 3.2.** *Let the assumption 2.6 hold and let $p > 0$, $\alpha_1 > 0$, $\alpha_2 \in \mathbb{R}$, $\alpha_3 \geq 0$, $\alpha_4 \geq 0$, $\alpha_5 > 0$. Assume that there is $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^+)$ such that, for $x_0 \neq 0$, the following*

*(1)* $\alpha_1|x(t)|^p \leq V(x,t)$,
*(2)* $J_1V(x,t) \leq \alpha_2 V(x,t)$,
*(3)* $|J_2V(x,t)|^2 \geq \alpha_3(V(x,t))^2$,
*(4)* $J_3V(x,t) \leq \alpha_4 V(x,t)$,
*(5)*$\int_{|y|<c}[log(\frac{V(x+h(x,y,t))}{V(x,t)}) - \frac{V(x+h(x,y,t))-V(x,t)}{V(x,t)}]\nu(dy) \leq -\alpha_5$

*hold for the solution $x(t)$ of* (1.2) *and* $\forall(x,t)\in\mathbb{R}^n\times\mathbb{R}^+$*, then*

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)|\le\frac{1}{p}(\alpha_2-\frac{1}{2}\alpha_3-\alpha_5). \tag{3.5}$$

*If in addition that* $\alpha_2<\frac{1}{2}\alpha_3+\alpha_5$*, then the trivial solution of* (1.2) *is almost surely exponentially stable.*

*Proof.* Define $Z(t)=\log(V(x,t))$ and apply Itô formula to $Z(t)$, we have for each $t\ge 0$,

$$\begin{aligned}
&\log|V(x(t),t)|\\
&=\log|V(x_0,0)|+\int_0^t\frac{V_x(x(s-),s)}{V(x(s-),s)}u(x(s-),s)ds+\int_0^t\frac{V_x(x(s-),s)}{V(x(s-),s)}b(x(s-),s)dB(s)\\
&\quad+\int_0^t[\frac{1}{2}\frac{V_{xx}(x(s-),s)b(x(s-),s)^2}{V(x(s-),s)}-\frac{1}{2}\frac{(V_x(x(s-),s)b(x(s-),s))^2}{V(x(s-),s)^2}]ds\\
&\quad+\int_0^t\frac{V_x(x(s-),s)\sigma(x(s-),s)}{V(x(s-),s)}(ds)^\alpha\\
&\quad+\int_0^t\int_{|y|<c}[\log(V(x(s-)+h(x(s-),y,s),s))-\log(V(s-),s)]\widetilde{N}(ds,dy)\\
&\quad+\int_0^t\int_{|y|<c}[\log(V(x(s-)+h(x(s-),y,s),s))\\
&\qquad\qquad-\log(V(s-),s)-\frac{V_x(x(s-),s)h(x(s-),y,s)}{V(x(s-),s)}]\nu(dy)ds\\
&=\log|V(x_0,0)|+\int_0^t\frac{V_x(x(s-),s)}{V(x(s-),s)}b(x(s-),s)dB(s)+\int_0^t\frac{V_x(x(s-),s)\sigma(x(s-),s)}{V(x(s-),s)}(ds)^\alpha\\
&\quad+\int_0^t\{\frac{V_x(x(s-),s)}{V(x(s-),s)}u(x(s-),s)+\frac{1}{2}\frac{V_{xx}(x(s-),s)b(x(s-),s)^2}{V(x(s-),s)}\\
&\quad+\int_{|y|<c}[\frac{V(x(s-)+h(x(s-),y,s),s)-V(x(s-),s)}{V(x(s-),s)}-\frac{V_x(x(-),s)h(x(s-),y,s)}{V(x(s-),s)}]\nu(dy)\}ds\\
&\quad+\int_0^t\int_{|y|<c}[\log(\frac{V(x(s-)+h(x(s-),y,s),s)}{V(x(s-),s)})]\widetilde{N}(ds,dy)\\
&\quad+\int_0^t\int_{|y|<c}[\log(\frac{V(x(s-)+h(x(s-),y,s),s)}{V(x(s-),s)})\\
&\qquad\qquad-\frac{V(x(s-)+h(x(s-),y,s),s)-V(x(s-),s)}{V(x(s-),s)}]\nu(dy)ds
\end{aligned}$$

$$\begin{aligned}
&=\log|V(x_0,0)|+\int_0^t\frac{J_1V((x(s-),s)}{V(x(s-),s)}ds\\
&\quad+\int_0^t\frac{J_2V(x(s-),s)}{V(x(s-),s)}dB(s)+\int_0^t\frac{J_3V(x(s-),s)}{V(x(s-),s)}(ds)^\alpha\\
&\quad-\frac{1}{2}\int_0^t\frac{(J_2V(x(s-),s))^2}{(V(x(s-),s))^2}ds+\int_0^t\int_{|y|<c}[\log(\frac{V(x(s-)+h(x(s-),y,s),s)}{V(x(s-),s)})]\widetilde{N}(ds,dy)\\
&\quad+\int_0^t\int_{|y|<c}[\log(\frac{V(x(s-)+h(x(s-),y,s),s)}{V(x(s-),s)})\\
&\quad-\frac{V(x(s-)+h(x(s-),y,s),s)-V(x(s-),s)}{V(x(s-),s)}]\nu(dy)ds.
\end{aligned}$$

Define

$$M(t) = \int_0^t \frac{V_x(x(s-),s)}{V(x(s-),s)} b(x(s-),s)dB(s) + \int_0^t \int_{|y|<c} [log(\frac{V(x(s-)+h(x(s-),y,s),s)}{V(x(s-),s)})]\widetilde{N}(ds,dy), \tag{3.6}$$

where the first integral is a continuous martingale and the second is a local martingale. Hence we can use exponential martingale inequality for $T = n$, $c = \varepsilon$, $\beta = \varepsilon n$, here $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$, for every integer $n \geq 0$, we have

$$\begin{aligned} \mathbb{P}[\sup_{0\leq t\leq n}\{M(t) &- \frac{\varepsilon}{2}\int_0^t \frac{(V_x(x(s-),s)b(x(s-),s))^2}{V(x(s-),s)^2}ds \\ &- \frac{1}{\varepsilon}\int_0^t \int_{|y|<c} [(\frac{V(x(s-)+h(x(s-),y,s),s)}{V(x(s-),s)})^\varepsilon - 1 \\ &- \varepsilon \log(\frac{V(x(s-)+h(x(s-),y,s),s)}{V(x(s-),s)})]\nu(dy)ds\} > \varepsilon n] \leq e^{-\varepsilon^2 n}. \end{aligned} \tag{3.7}$$

By the Borel-Cantelli lemma, and due to the $\sum_{n=1}^\infty e^{-\varepsilon^2 n} < \infty$, we have

$$\begin{aligned} \mathbb{P}[\limsup_{n\to\infty}[\sup_{0\leq t\leq n}\{M(t) &- \frac{\varepsilon}{2}\int_0^t \frac{(V_x(x(s-),s)b(x(s-),s))^2}{V(x(s-),s)^2}ds \\ &- \frac{1}{\varepsilon}\int_0^t \int_{|y|<c} [(\frac{V(x(s-)+h(x(s-),y,s),s)}{V(x(s-),s)})^\varepsilon - 1 \\ &- \varepsilon \log(\frac{V(x(s-)+h(x(s-),y,s),s)}{V(x(s-),s)})]\nu(dy)ds\}] < \varepsilon n] = 1. \end{aligned} \tag{3.8}$$

Hence for almost all $\omega \in \Omega$ there exists an integer $N$ such that for all $n \geq N$, $0 \leq t \leq n$,

$$\begin{aligned} M(t) \leq &\frac{\varepsilon}{2}\int_0^t \frac{(V_x(x(s-),s)b(x(s-),s))^2}{V(x(s-),s)^2}ds + \varepsilon n \\ &+ \frac{1}{\varepsilon}\int_0^t \int_{|y|<c} [(\frac{V(x(s-)+h(x(s-),y,s),s)}{V(x(s-),s)})^\varepsilon - 1 \\ &- \varepsilon \log(\frac{V(x(s-)+h(x(s-),y,s),s)}{V(x(s-),s)})]\nu(dy)ds. \end{aligned} \tag{3.9}$$

Then using condition (2),(3),(4), we can get

$$\begin{aligned} \log|V(x(t),t)| \leq \log|V(x_0,0)| &+ \alpha_2(t-0) - \frac{1}{2}(1-\varepsilon)\alpha_3(t-0) + \varepsilon n - \int_0^t \alpha_4(ds)^\alpha \\ &+ \int_0^t \int_{|y|<c} [\log(\frac{V(x(s-)+h(x(s-),y,s),s)}{V(x(s-),s)}) \\ &\quad - \frac{V(x(s-)+h(x(s-),y,s),s) - V(x(s-),s)}{V(x(s-),s)}]\nu(dy)ds \\ &+ \frac{1}{\varepsilon}\int_0^t \int_{|y|<c} [(\frac{V(x(s-)+h(x(s-),y,s),s)}{V(x(s-),s)})^\varepsilon \\ &\quad - 1 - \varepsilon \log(\frac{V(x(s-)+h(x(s-),y,s),s)}{V(x(s-),s)})]\nu(dy)ds, \end{aligned} \tag{3.10}$$

for $n \geq N$, $0 \leq t \leq n$. Let $\varepsilon \to 0$, we get

$$\begin{aligned}\log|V(x(t),t)| \leq \log|V(x_0,0)| + \alpha_2(t-0) - \frac{1}{2}\alpha_3(t-0) + \varepsilon n - \int_0^t \alpha_4(ds)^\alpha \\ + \int_0^t \int_{|y|<c} [\log(\frac{V(x(s-)+h(x(s-),y,s),s)}{V(x(s-),s)}) \\ - \frac{V(x(s-)+h(x(s-),y,s),s) - V(x(s-),s)}{V(x(s-),s)}]\nu(dy)ds.\end{aligned} \tag{3.11}$$

By (2.6) and Condition (5), we have

$$\begin{aligned}\log|V(x(t),t)| &\leq \log|V(x_0,0)| + \alpha_2(t-0) - \frac{1}{2}\alpha_3(t-0) - \int_0^t \alpha\alpha_4(t-s)^{\alpha-1}ds \\ &\quad + \int_0^t \int_{|y|<c} [\log(\frac{V(x(s-)+h(x(s-),y,s),s)}{V(x(s-),s)}) \\ &\quad - \frac{V(x(s-)+h(x(s-),y,s),s) - V(x(s-),s)}{V(x(s-),s)}]\nu(dy)ds \\ &\leq \log|V(x_0,0)| + \alpha_2(t-0) - \frac{1}{2}\alpha_3(t-0) - \alpha_4(t-0)^\alpha - \alpha_5(t-0) \\ &= \log|V(x_0,0)| + (\alpha_2 - \frac{1}{2}\alpha_3 - \alpha_5)(t-0) - \alpha_4(t-0)^\alpha.\end{aligned} \tag{3.12}$$

For almost all $\omega \in \Omega$, $0+n-1 \leq t \leq 0+n$, $n \geq N$, we have

$$\frac{1}{t}\log|V(x(t),t)| = \frac{1}{t}\log|V(x_0,0)| + \frac{1}{t}(\alpha_2 - \frac{1}{2}\alpha_3 - \alpha_5)(t-0) - \frac{1}{t}\alpha_4(t-0)^\alpha. \tag{3.13}$$

By condition (1), we have

$$\log|x(t)| \leq \frac{1}{p}\log|\frac{V(x(t),t)}{\alpha_1}|.$$

So,

$$\begin{aligned}&\frac{1}{t}\log|x(t)| \\ &\leq \frac{1}{p}\frac{\log|V(x_0,0)| - \log(\alpha_1)}{t} + \frac{1}{p}\frac{(\alpha_2 - \frac{1}{2}\alpha_3 - \alpha_5)(t-0)}{t} - \frac{1}{p}\frac{\alpha_4(t-0)^\alpha}{t}.\end{aligned} \tag{3.14}$$

Then

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \leq \frac{1}{p}(\alpha_2 - \frac{1}{2}\alpha_3 - \alpha_5). \tag{3.15}$$

This completes the proof. □

**Example 3.3.** Consider the following stochastic fractional differential equation on $\mathbb{R}$

$$dx(t) = -\frac{1}{2}x(t-)ds + x(t-)dB(s) + \int_{|y|<1} x(t-)^2 y\widetilde{N}(ds,dy) + \frac{-1}{3}x(t-)(dt)^{\frac{2}{3}}, \tag{3.16}$$

with $x_0 = 1$, where the Lévy measure $\nu$ is given by $\nu(dy) = \frac{dy}{1+y^2}$ for all $y \in \mathbb{R}$, $\alpha = \frac{2}{3}$ and $c = 1$.

We choose the Lyapunov function $V(x,t)=x^2$, then

$$J_1V(x,t) = -x^2 + x^2 + \int_{|y|<1}[(x+xy^2)^2 - x^2 - 2x^2y^2]\nu(dy)$$
$$= \int_{|y|<1} x^2y^4\nu(dy).$$

By the numerical computation,

$$\int_{|y|<1} x^2y^4\nu(dy) = \int_{|y|<1}\frac{x^2y^4}{1+y^2}dy \approx 0.238x^2,$$

$$J_2V(x,t) = V_x(x,t)b(x,t) = 2x^2,$$

$$J_3V(x,t) = V_x(x,t)\sigma(x,t) = \frac{-2}{3}x^2,$$

$$\int_{|y|<1}[\log(\frac{x+xy^2}{x})^2 - \frac{(x+xy^2)^2 - x^2}{x^2}]\nu(dy)$$
$$= \int_{|y|<1}[2\log(1+y^2) - \frac{(x+xy^2)^2 - x^2}{x^2}]\frac{1}{1+y^2}dy \approx -3.189.$$

So $\alpha_1 = 1$, $\alpha_2 = 0.25$, $\alpha_3 = 2$, $\alpha_4 = 6$, $\alpha_5 = 3.20$, then $\alpha_2 - \frac{1}{2}\alpha_3 - \alpha_5 = -3.95$. Hence, the trivial solution of stochastic fractional differential equation (3.16) is almost surely exponentially stable.

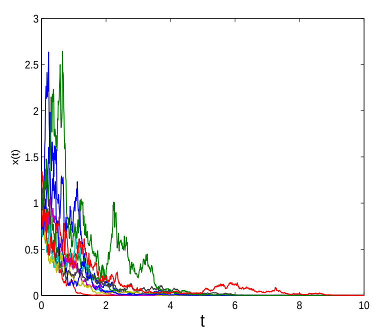


Figure 1: almost surely exponentially stable

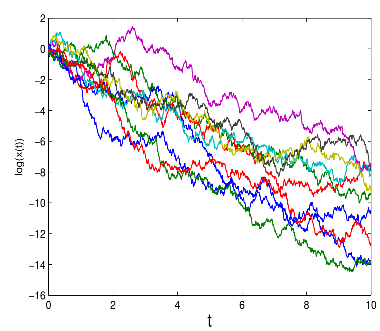


Figure 2: Stable rate

From Figure 1, one can easily see that the numerical solution is almost surely exponentially stable. In order to show the stable rate is exponent, we take the $x(t)$ is $\log x(t)$, which can be seen from Figure 2.

Next, let us discuss the moment exponential stability of the stochastic fractional differential equation (1.2).

**Theorem 3.4.** *Let $p$, $c_1$, $c_2$, $\lambda$ be positive constants. Assume that there is $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^+)$ such that the following*

$$(i) c_1|x(t)|^p \le V(x(t),t) \le c_2|x(t)|^p,$$
$$(ii) J_1V(x(t),t) \le -\lambda V(x(t),t),$$
$$(iii) J_3V(x(t),t) \le 0$$

*hold for the solution $x(t)$ of* (1.2) *and* $\forall(x,t)\in\mathbb{R}^n\times\mathbb{R}^+$, *then*

$$\mathbb{E}[|x(t)|^p]\le\frac{c_2}{c_1}|x_0|^pe^{-\lambda t}. \tag{3.17}$$

*That is, the trivial solution of* (1.2) *is pth moment exponentially stable.*

*Proof.* Fix any $x_0\neq 0$, for every $n\geq|x_0|$, define the stopping time

$$\tau_n=inf\{t\geq 0:|x(t)|\geq n\}.$$

Obviously, $\tau_n\uparrow\infty$ almost surely. Apply Itô formula to $e^{\lambda(t\wedge\tau_n)}V(x(t),t)$, for $t\geq 0$, we have

$$\begin{aligned}
&e^{\lambda(t\wedge\tau_n)}V(x(t),t)\\
&=V(x_0,0)+\int_0^{t\wedge\tau_n}\lambda e^{\lambda s}V(x(s-),s)ds\\
&\quad+\int_0^{t\wedge\tau_n}e^{\lambda s}[V_x(x(s-),s)u(x(s-),s)+\frac{1}{2}V_{xx}(x(s-),s)(b(x(s-),s))^2]ds\\
&\quad+\int_0^{t\wedge\tau_n}\int_{|y|<c}e^{\lambda s}[V(x(s-)+h(x(s-),y,s),s)\\
&\qquad\qquad-V(x(s-),s)-V_x(x(s-),s)h(x(s-),y,s)]\nu(dy)ds\\
&\quad+\int_0^{t\wedge\tau_n}e^{\lambda s}V_x(x(s-),s)\sigma(x(s-),s)(ds)^\alpha+\int_0^{t\wedge\tau_n}e^{\lambda s}V_x(x(s-),s)b(x(s-),s)dB(s)\\
&\quad+\int_0^{t\wedge\tau_n}\int_{|y|<c}e^{\lambda s}[V(x(s-)+h(x(s-),y,s),s)-V(x(s-),s)]\widetilde{N}(ds,dy).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{E}[e^{\lambda t\wedge\tau_n}V(x(t),t)]=V(x_0,0)&+\int_0^{t\wedge\tau_n}\lambda e^{\lambda s}[\lambda V(x(s-),s)+J_1V(x(s-),s)]ds\\
&+\int_0^{t\wedge\tau_n}\lambda e^{\lambda s}\alpha J_3V(x(s-),s)(t-s)^{\alpha-1}ds.
\end{aligned}$$

By Conditions (ii) and (iii), we have

$$\mathbb{E}[e^{\lambda t\wedge\tau_n}V(x(t),t)]\le V(x_0,0).$$

It comes from condition (i), we have

$$c_1\mathbb{E}[|x(t\wedge\tau_n)|^pe^{\lambda(t\wedge\tau_n)}]\le\mathbb{E}[V(x(t\wedge\tau_n),t\wedge\tau_n)e^{\lambda(t\wedge\tau_n)}]\le V(x_0,0)\le c_2|x_0|^p.$$

Hence,

$$\mathbb{E}[|x(t\wedge\tau_n)|^pe^{\lambda(t\wedge\tau_n)}]\le\frac{c_2}{c_1}|x_0|^p,$$

which implies

$$\mathbb{E}[1_{[0,\tau_n]}|x(t)|^pe^{\lambda t}]\le\frac{c_2}{c_1}|x_0|^p.$$

For each $t\geq 0$, $1_{[0,\tau_n]}|x(t)|^pe^{\lambda t}$ forms a monotonic increasing sequence of random variables, by the monotone convergence theorem, we have

$$\mathbb{E}[|x(t)|^p e^{\lambda t}] = \lim_{n\to\infty} \mathbb{E}[|x(t\wedge\tau_n)|^p e^{\lambda(t\wedge\tau_n)}] \le \frac{c_2}{c_1}|x_0|^p. \tag{3.18}$$

This completes the proof. □

**Example 3.5.** Consider the following stochastic fractional differential equation on $\mathbb{R}$

$$dx(t) = -x(t-)ds + \frac{1}{2}x(t-)dB(s) + \int_{|y|<1} x(t-)^2 y\widetilde{N}(ds,dy) - 3x(t-)(dt)^{\frac{2}{3}} \tag{3.19}$$

with $x_0 = 1$, where the Lévy measure $\nu$ is given by $\nu(dy) = \frac{dy}{1+y^2}$ for all $y \in \mathbb{R}\setminus\{0\}$, $\alpha = \frac{2}{3}$ and $c = 1$.

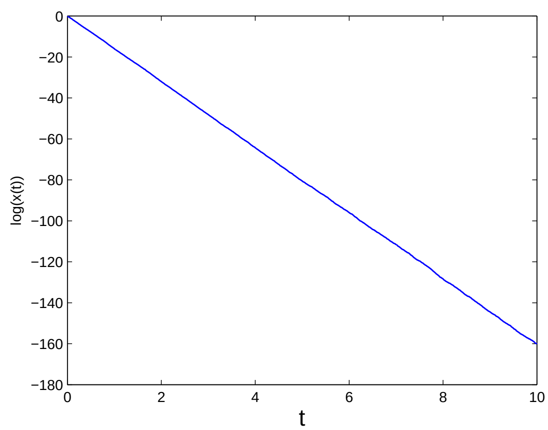


Figure 3: Example 3.5 moment exponentially stable.

The Figure 3 illustrates the moment exponential stability of the numerical solution produced by the truncated EM method, 100 sample paths are generated.

We choose the Lyapunov function $V(x,t) = x^2$, then

$$\begin{aligned} J_1V(x,t) &= -2x^2 + \frac{1}{4}x^2 + \int_{|y|<1} [(x+xy^2)^2 - x^2 - 2x^2y^2]\nu(dy) \\ &= -1.75x^2 + \int_{|y|<1} x^2y^4\nu(dy). \end{aligned}$$

By the numerical computation,

$$\int_{|y|<1} x^2y^4\nu(dy) = \int_{|y|<1} \frac{x^2y^4}{1+y^2}dy \approx 0.238x^2.$$

So,

$$J_1V(x,t) \le -1.5V(x,t),$$

and

$$J_3V(x,t) = -6x^2 \le 0.$$

Thus the trivial solution of stochastic fractional differential equation (3.19) is moment exponentially stable.

*Remark* 3.6. As we have shown in this section, one can use the Itô formula to derive explicit expression for the composition of the Lyapunov type function $V$ and the solutions of non-Lipschitz stochastic Riemann-Liouville type fractional differential equations with jumps. This, together with the introducing of stopping times for the solutions as jump type stochastic processes, then leads to clear conditions to ensure the three types of stability of (1.2). Comparing to Applebaum and Siakalli [6] and Siakalli [35] where stability for (1.2) without Riemann-Liouville fractional differentiation has been achieved under Lipschitz condition assumption, we find more subtle conditions to ensure the stability for the equation eqrefsec1-eq1.2 with non-Lipschitz coefficients. The three examples presented here further explicate the sufficiency of our conditions.

## 4. Conclusion

In this paper, we discuss three types of stability of a stochastic Riemann-Liouville type fractional differential equations with non-Lipschitz coefficients driven by Lévy noise in which we have established stochastic stability, almost sure exponential stability, and moment exponential stability, respectively. As mentioned in the introduction, the equation we are concerned is with non-Lipschitz coefficients and involves the Riemann-Liouville type fractional derivative and the Lévy noise simultaneously, which makes difficult to use the existing methods directly. What we have done here is to utilise stopping time technique and careful derivations of the integrals related to the Riemann-Liouville type fractional derivative and the jump type stochastic integral with respect to Lévy noise. We hope our method presented in this paper could be potentially applicable to stochastic fractional equations involving other type fractional derivatives. Our future work based on the current paper is to investigate numerical solutions to the stochastic Riemann-Liouville type fractional differential equation studied here, and to derive relevant numerical averaging principles, as well as to study drift parameter estimates of the concerned equation involving drift coefficient with unknown parameter for the modelling purpose and derive.

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