Stability of Numerical Solution to Pantograph Stochastic Functional Differential Equations*

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Abstract

The paper studies the convergence of the numerical solutions for pantograph stochastic functional differential equations which was proposed in [16]. We also show that the approximate solutions have the properties of almost surely polynomial stability and exponential stability.

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1 Introduction

Throughout this article, let $(\Omega, \mathscr{F}, \mathbb{F} := \{\mathscr{F}_t\}_{t\geq 0}, P)$ be a complete probability space, which satisfies the usual conditions. $B(t)$ is a standard d-Brownian motion defined on this probability space. For $x, y \in \mathbb{R}^n$, $|x|$ means the Euclidean norm of x, and use $\langle x, y\rangle$ or $x^T y$ to represent the Euclidean inner product. If A is a matrix, A^T is the transpose of A and $|A|$ represents $\sqrt{\mathrm{Tr}(AA^T)}$. Let $\lfloor a \rfloor$ be the integer parts of a. Moreover, for fixed constant $0 < \underline{q} < 1$, denote by $\mathscr{C} := \mathscr{C}([\underline{q}, 1]; \mathbb{R}^n)$ the set of $\mathbb{R}^n-$valued continuous functions ϕ defined on $[\underline{q}, 1]$ with the norm $\|\phi\| = \sup_{\underline{q}\leq q\leq 1} |\phi(q)|$. For a constant $T > 0$ and a $\mathbb{R}^n-$valued stochastic process ψ, denote $\mathbb{E}\sup_{0\leq t\leq T} |\psi(t)|^p$ by $\|\psi\|_{T,p}$.

In order to solve a problem on the pantograph of an electric locomotive, Ockendon and Tayler [12] proposed pantograph differential equations (PDEs). PDEs then have used in

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many areas such as electric dynamics, the oscillation, dielectric materials and continuum mechanics, which have the following form:

$$\tilde{x}(t) = \tilde{x}_0 + \int_0^t b(\tilde{x}(s), \tilde{x}(qs), s)\mathrm{d}s, t \geq 0.$$

where q is a fixed constant satisfying $0 < q < 1$. Guan et al. [3] studied the oscillatory properties of the solutions. Hou et al. [5] applied PDEs to solve some problems in neural networks. Ahmad and Mukhtar [1] solved a class of cell-growth model by using PDEs. Additionally, many researchers have extended PDEs to pantograph stochastic differential equations (PSDEs), so as to capture the practice problems more precisely. The form of PSDEs is as follows:

$$\bar{x}(t) = \bar{x}_0 + \int_0^t b(\bar{x}(s), \bar{x}(qs), s)\mathrm{d}s + \sigma(\bar{x}(s), \bar{x}(qs), s)\mathrm{d}B(s), t \geq 0$$

where q is a fixed constant satisfying $0 < q < 1$. In real world, many events are random and uncertain. Compared with PDEs, this type of systems including a Gaussian noise with unbounded memory provided more realistic models to simulate phenomena that considered the time-lag or after effect. Afterwards, it aroused great interest in studying different kinds of phenomena in different fields of science, which had a significant impact on the investigation of differential equation with delay incorporating memory or after-effect, especially, in studying of financial variables, group models, corporate claim and predictions about their evolution. For example, Ren et al. [13] used PSDEs to study multi- group models. Eissa and Tian [2] used PSDEs to model the corporate claim value. Milošević [11] studied polynomial stable properties of solution to a class of highly nonlinear PSDEs and the Euler-Maruyama (EM) approximation. Shen et al. [14] investigated the exponential stable properties of highly nonlinear PSDEs. Guo et al. [4] discussed the stable properties of numerical solutions for the PSDEs with variable step size. Song et al. [15] analyzed the pth moment asymptotical ultimate boundedness of PSDEs with time-varying coefficients. Hu et al. [6] established the existence and uniqueness for a class of PSDEs. More related work can be seen in [18, 7, 9, 10, 17].

Recently, authors [16] have developed the fundamental theories for the following pantograph stochastic functional differential equations (PSFDEs):

$$x(t) = x_0 + \int_0^t b(x_s, s)\mathrm{d}s + \sigma(x_s, s)\mathrm{d}B(s), t \geq 0, \tag{1.1}$$

where $x_t = \{x(qt), \underline{q} \leq q \leq 1)\}$ and $0 < \underline{q} < 1$ is a fixed constant. And they have show that this type of equation has some meaningful applications in oscillator systems. The exponential stability and polynomial stability of solutions for PSFDEs are also investigated in [16]. Our main contribution are as follows:

- We study a new type of stochastic functional differential equation. The PSFDEs differs markedly from PSDEs, since the current state of PSFDEs depends on a past segment of its solution while the current state of PSDEs depends only on a past point of its solution (more details can be seen in Section 2 below).

- We are the first to give the definition of the approximate solution for PSFDEs. The approximate solution converges strongly to the analytical solution in finite time interval.
- The numerical solutions preserve the exponential stability and polynomial stability of the analytical solution under the certain conditions.

We end this part by presenting our organization in this paper. In Section 2, we show the convergence of the numerical solutions. In Section 3, the exponential stability and polynomial stability of numerical solutions are discussed and several examples are presented in Subsection 3.1 and Subsection 3.2.

2 The EM Method and Strong Convergence

Consider the E.q.(1.1). For the sake of simplicity, we assume $b(0,t) = \sigma(0,t) = 0$. In [16], we have shown that the analytic solution to (1.1) is exponential stable and polynomial stable under some conditions. In this paper, we will prove that the Euler-Maruyama(EM) method can inherit exponential stable properties and polynomial stable properties of the analytical solution.

Choose a step size $\Delta \in (0,1)$ and define the discrete EM approximate solution $y(m) = y(m\Delta) \approx x(m\Delta)$ by setting $y(0) = x_0, y_0 = x_0$ and forming

$$y(m+1) = y(m) + b(\bar{y}_m, m\Delta)\Delta + \sigma(\bar{y}_m, m\Delta)\Delta B(m),\ m = 0, 1, 2, \dots \tag{2.1}$$

where $y(m) = y(m\Delta), \Delta B(m) = B((m+1)\Delta) - B(m\Delta)$, and $\bar{y}_m$ is a $\mathscr{C}([\underline{q}, 1]; \mathbb{R}^n)-$valued random variable given as follows:

$$y_m(u) = \begin{cases} y((\lfloor m\underline{q} \rfloor + j)\Delta) + \frac{u-(\lfloor m\underline{q}\rfloor + j)\Delta}{\Delta}[y((\lfloor m\underline{q}\rfloor + j+1)\Delta) - y((\lfloor m\underline{q}\rfloor + j)\Delta)], \\ \qquad \text{for } (\lfloor m\underline{q}\rfloor + j)\Delta \le u \le (\lfloor m\underline{q}\rfloor + j+1)\Delta, j = 1, \cdots, m - \lfloor m\underline{q}\rfloor - 1; \\ y(\lfloor m\underline{q}\rfloor\Delta) + \frac{u-\lfloor m\underline{q}\rfloor\Delta}{\Delta}[y((\lfloor m\underline{q}\rfloor + 1)\Delta) - y(\lfloor m\underline{q}\rfloor\Delta)], \\ \qquad \text{for } m\underline{q}\Delta \le u \le (\lfloor m\underline{q}\rfloor + 1)\Delta, \end{cases} \tag{2.2}$$

$$\bar{y}_m(q) = y_m(qm\Delta). \tag{2.3}$$

Thus,

$$|y_m(u)| \le |y((\lfloor m\underline{q}\rfloor + j)\Delta)| \vee |y((\lfloor m\underline{q}\rfloor + j+1)\Delta)|, (\lfloor m\underline{q}\rfloor + j)\Delta \le u \le (\lfloor m\underline{q}\rfloor + j+1)\Delta, \tag{2.4}$$

where $\lfloor a \rfloor$ is the integer parts of a. From (2.4), we have

$$\|y_m\| \le \sup_{q\in[\underline{q},1]} |y(\lfloor mq \rfloor\Delta)|. \tag{2.5}$$

In order to analyze the continuous-time approximation , for $t \in [0,T]$, we define

$$z(t) = \sum_{m=0}^{\infty} y(m) 1_{[m\Delta,(m+1)\Delta)}(t),\ \bar{t} = \sum_{m=0}^{\infty} m\Delta 1_{[m\Delta,(m+1)\Delta)}(t),$$

$$\bar{z}(q,t) = \sum_{m=0}^{\infty} y(\lfloor qm \rfloor) 1_{[m\Delta,(m+1)\Delta)}(t),$$

where $1_{[a,b)}(\cdot)$ represents the indicator function on interval $[a,b)$. We also define $\bar{z}_t$ a segment process on $\mathscr{C}([\underline{q},1];\mathbb{R}^n)$ as the following

$$\bar{z}_t(q) = \sum_{m=0}^{\infty} y_m(qm\Delta) 1_{[m\Delta,(m+1)\Delta)}(t),\ q \in [\underline{q},1].$$

The continuous-time approximation $\{Y(t), t \geq 0\}$ is defined as $Y(0) = x_0$ and

$$Y(t) = x_0 + \int_0^t b(\bar{z}_s, \bar{s})\mathrm{d}s + \int_0^t \sigma(\bar{z}_s, \bar{s})\mathrm{d}B(s). \tag{2.6}$$

For the future use, we make the following hypothesis.

(H) For any $\varphi_1, \varphi_2 \in \mathscr{C}([\underline{q},1];\mathbb{R}^n)$, we have

$$|b(\varphi_1,t) - b(\varphi_2,t)|^2 \vee |\sigma(\varphi_1,t) - \sigma(\varphi_2,t)|^2 \leq K\|\varphi_1 - \varphi_2\|^2,$$

in which K a positive constant.

Lemma 2.1. *Assume* (H). *Then, we have*

$$\|Y(t)\|_{T,p} \vee \|x(t)\|_{T,p} \leq K_{1,p},$$

in which $K_1 > 0$ is a constant only depending on $K, x_0, T, p \geq 2$ but being independent of Δ.

By similar method to that of Lemma 3.2 in Mao [8], this lemma can be easily proved. We omit it here. Now, we state the second lemma in this section.

Lemma 2.2. *Assume (H). Then, we derived*

$$\|Y - z\|_{T,2} \leq K_2\Delta,\ \mathbb{E}\sup_{0\leq t\leq T}\|Y_t - \bar{z}_t\|^2 \leq K_2\Delta,$$

in which $K_2 > 0$ is a constant only depending on $K, K_{1,2}, K_{1,4}, x_0, T$ but being independent of Δ, i.e., $K_2 = 9\left[2KK_{1,2} + d^{\frac{3}{2}}K^2K_{1,4}\frac{32}{3\sqrt{3}}\right]$.

Proof. $\forall t\in[0,T]$, let $m=\lfloor\frac{t}{\Delta}\rfloor, l=\lfloor\frac{T}{\Delta}\rfloor$, obviously, $m\le l$. Then, $t\in[m\Delta,(m+1)\Delta)$, thus, we have

$$Y(t)-z(t)=b(\bar z_{m\Delta},m\Delta)(t-m\Delta)+\sigma(\bar z_{m\Delta},m\Delta)(B(t)-B(m\Delta)).$$

According to assumption (H), we obtain

$$|Y(t)-z(t)|^2\le 2K\|\bar z_{m\Delta}\|^2\Delta^2+2|\sigma(\bar z_{m\Delta},m\Delta)(B(t)-B(m\Delta))|^2.$$

Thus,

$$\begin{aligned}
\mathbb{E}\sup_{m\Delta\le t\le(m+1)\Delta}|Y(t)-z(t)|^2 &\le 2K\mathbb{E}[\sup_{m\Delta\le t\le(m+1)\Delta}\|\bar z_t\|^2]\Delta^2\\
&\quad+2\mathbb{E}[\sup_{m\Delta\le t\le(m+1)\Delta}|\sigma(\bar z_{m\Delta},m\Delta)(B(t)-B(m\Delta))|^2]\\
&\le 2K\mathbb{E}[\sup_{m\Delta\le t\le(m+1)\Delta}\|\bar z_t\|^2]\Delta^2\\
&\quad+2(\mathbb{E}\sup_{m\Delta\le t\le(m+1)\Delta}|\sigma(\bar z_{m\Delta},m\Delta)|^4)^{\frac12}(\mathbb{E}\sup_{m\Delta\le t\le(m+1)\Delta}|B(t)-B(m\Delta)|^4)^{\frac12}\\
&\le 2KK_{1,2}\Delta^2+2KK_{1,4}^{\frac12}(\mathbb{E}\sup_{m\Delta\le t\le(m+1)\Delta}|B(t)-B(m\Delta)|^4)^{\frac12}\\
&\le 2KK_{1,2}\Delta^2+2d^{\frac12}KK_{1,4}^{\frac12}\left(\mathbb{E}\sup_{t\in[m\Delta,(m+1)\Delta\wedge T]}|B_j(t)-B_j(m\Delta)|^4\right)^{\frac12}
\end{aligned}\tag{2.7}$$

According to Doob's martingale inequality, we get

$$\mathbb{E}\sup_{m\Delta\le t\le(m+1)\Delta}|B_j(t)-B_j(k\Delta)|^4\le\frac{256}{27}\Delta^2.\tag{2.8}$$

By (2.7) and (2.8), we arrive at

$$\begin{aligned}
\mathbb{E}\sup_{m\Delta\le t\le(m+1)\Delta}|Y(t)-z(t)|^2&\le 2KK_{1,2}\Delta^2+2d^{\frac32}KK_{1,4}^{\frac12}\frac{16}{3\sqrt3}\Delta\\
&\le\left[2KK_{1,2}+d^{\frac32}K^2K_{1,4}\frac{32}{3\sqrt3}\right]\Delta.
\end{aligned}\tag{2.9}$$

Next, we prove the second result.

$$\begin{aligned}
&\mathbb{E}\sup_{m\Delta\le t\le(m+1)\Delta}\sup_{\underline{q}\le q\le1}|Y(qt)-\bar z_t(q)|^2\\
&\le 3\left[2KK_{1,2}+d^{\frac32}K^2K_{1,4}\frac{32}{3\sqrt3}\right]\Delta+3\mathbb{E}\sup_{t\in[m\Delta,(m+1)\Delta\wedge T]}\sup_{\underline{q}\le q\le1}|z(qt)-\bar z(t,q)|^2\\
&\quad+3\mathbb{E}\sup_{t\in[m\Delta,(m+1)\Delta\wedge T]}\sup_{\underline{q}\le q\le1}|\bar z(t,q)-\bar z_t(q)|^2=:I_1+I_2+I_3.
\end{aligned}\tag{2.10}$$

It is obvious that $qt \in [m_t^q\Delta, (m_t^q+1)\Delta]$, where $m_t^q = \lfloor \frac{qt}{\Delta} \rfloor$. Thus, for $m = \lfloor \frac{t}{\Delta} \rfloor$, we get

$$|z(qt) - \bar{z}(q,t)| = |y(m_t^q\Delta) - y(\lfloor qm \rfloor \Delta)|. \tag{2.11}$$

Since $\frac{qt}{\Delta} \in [qm, q(m+1))$, it can be seen that $\lfloor qm \rfloor \le m_t^q \le \lfloor q(m+1) \rfloor \le \lfloor qm \rfloor + 1$. Then, $m_t^q - \lfloor qm \rfloor \le 1$. By (2.11), one has

$$I_2 = 3\mathbb{E} \sup_{t\in[m\Delta,(m+1)\Delta\wedge T]} \sup_{\underline{q}\le q\le 1} |z(qt) - \bar{z}(t,q)|^2 \le 3\mathbb{E} \sup_{0\le k\Delta\le T} |y((k+1)\Delta\wedge T) - y(k\Delta)|^2. \tag{2.12}$$

Next, we calculate I_3. From (2.2), we have

$$I_3 = 3\mathbb{E} \sup_{t\in[m\Delta,(m+1)\Delta\wedge T]} \sup_{\underline{q}\le q\le 1} |\bar{z}(t,q) - \bar{z}_t(q)|^2 \le 3\mathbb{E} \sup_{0\le k\Delta\le T} |y((k+1)\Delta\wedge T) - y(k\Delta)|^2. \tag{2.13}$$

By using similar procedure as used method in the proof of first result, one can see that

$$\mathbb{E} \sup_{0\le k\Delta\le T} |y((k+1)\Delta\wedge T) - y(k\Delta)|^2 \le \left[2KK_{1,2} + d^{\frac{3}{2}}K^2K_{1,4}\frac{32}{3\sqrt{3}}\right]\Delta. \tag{2.14}$$

(2.10), (2.11), (2.13), (2.12) and (2.14) lead to the second result. □

The following result reveals that the numerical solutions converge to the true solution.

Theorem 2.3. *Assume* (H). *It holds that*

$$\|x - Y\|_{T,2} \le K_4\Delta,$$

where x *is the solution of E.q.* (1.1), Y *is defined in* (2.6) *and*

$$K_4 := (4T^2KK_2 + 16KTK_2)e^{(4TK+16K)T}.$$

Proof. By (H), Lemmas 2.1 and 2.2, we compute

$$\begin{aligned}
&\|x - Y\|_{t,2} \\
&\le 2T\mathbb{E}\int_0^t |b(x_r,r) - b(\bar{z}_r,r)|^2\mathrm{d}r + 2\mathbb{E}\sup_{0\le r\le t}\left|\int_0^r \sigma(x_r,r) - \sigma(\bar{z}_r,r)\mathrm{d}B(r)\right|^2 \\
&\le 2T\mathbb{E}\int_0^t |b(x_r,r) - b(\bar{z}_r,r)|^2\mathrm{d}r + 8\mathbb{E}\int_0^t |\sigma(x_r,r) - \sigma(\bar{z}_r,r)|^2\mathrm{d}r \\
&\le 4TK\mathbb{E}\int_0^t \|x_r - Y_r\|^2\mathrm{d}r + 4TK\mathbb{E}\int_0^t \|Y_r - \bar{z}_r\|^2\mathrm{d}r \\
&+ 16K\mathbb{E}\int_0^t \|x_r - Y_r\|^2\mathrm{d}r + 16K\mathbb{E}\int_0^t \|Y_r - \bar{z}_r\|^2\mathrm{d}r
\end{aligned}$$

$$
\begin{aligned}
&\leq 4TK\mathbb{E}\int_0^t \|x-Y\|_{r,2}\mathrm{d}r + 4TK\mathbb{E}\int_0^t \|Y_r-\bar{z}_r\|^2\mathrm{d}r\\
&+16K\int_0^t \|x-Y\|_{r,2}\mathrm{d}r + 16K\mathbb{E}\int_0^t \|Y_r-\bar{z}_r\|^2\mathrm{d}r\\
&\leq (4T^2KK_2+16KTK_2)\Delta + (4TK+16K)\int_0^t \|x-Y\|_{r,2}\mathrm{d}r.
\end{aligned}
$$

Gronwall's inequality leads to required result. □

3 Stability of Numerical Solutions

In the section, we shall investigate the exponential stable properties and polynomial stable properties for the numerical solutions.

3.1 Exponential Stable Properties of Numerical Solution

We need the following assumptions.

(H1) For any $\varphi, \phi \in \mathscr{C}([\underline{q}, 1]; \mathbb{R}^n)$, there exists a probability measures ν on $[\underline{q}, 1]$ with positive constants λ_1, λ_2 such that

$$
\begin{aligned}
&2\langle \varphi(1)-\phi(1), b(\varphi,t)-b(\phi,t)\rangle + |\sigma(\varphi,t)-\sigma(\phi,t)|^2\\
&\leq -\lambda_1|\varphi(1)-\phi(1)|^2 + \lambda_2\int_{\underline{q}}^1 e^{-\beta t}|\varphi(q)-\phi(q)|^2\mathrm{d}\nu(q).
\end{aligned}
\tag{3.1}
$$

(H2) For any $\varphi, \phi \in \mathscr{C}([\underline{\theta}, 1]; \mathbb{R}^n)$, there exists a probability measures ν on $[\underline{\theta}, 1]$ with positive constants λ_3 and λ_4 such that

$$
\begin{aligned}
&|b(\varphi,t)-b(\phi,t)|^2 \vee |\sigma(\varphi,t)-\sigma(\phi,t)|^2\\
&\leq \lambda_3|\varphi(1)-\phi(1)|^2 + \lambda_4\int_{\underline{\theta}}^1 e^{-\beta t}|\varphi(q)-\phi(q)|^2\mathrm{d}\nu(q),
\end{aligned}
$$

where $\beta > 0$ is a constant satisfying $0 < \frac{1-\beta}{\underline{q}} < 1$.

We can see that (H2) implies (H).

Theorem 3.1. *Assume* (H1) *and* (H2)*. If the following conditions are established:*

i) there exist some positive constants $\bar{C}, \alpha_0 < 1$ satisfying $1 < \bar{C} \leq e^{\alpha_0}$ and a sufficiently small constant $\lambda_0 > 0$ such that

$$
H(\bar{C},\lambda_0) = -\lambda_1 + \alpha_0 + 2\lambda_2\Big(\Big\lfloor\frac{1}{\underline{q}}\Big\rfloor+1\Big)\bar{C}^{\frac{\lambda_0}{\underline{q}}} + \lambda_3\lambda_0 + 2\lambda_4\Big(\Big\lfloor\frac{1}{\underline{q}}\Big\rfloor+1\Big)\bar{C}^{\frac{\lambda_0}{\underline{q}}}\lambda_0 \leq 0.
$$

ii) $\Delta \in (0, \lambda_0)$ *is small enough satisfying the following inequality:*

$$\left[\lambda_2\Big(\Big\lfloor\frac{1}{\underline{q}}\Big\rfloor+1\Big)\bar{C}^{\frac{\Delta}{\underline{q}}}+\lambda_4\Big(\Big\lfloor\frac{1}{\underline{q}}\Big\rfloor+1\Big)\bar{C}^{\frac{\Delta}{\underline{q}}}\Delta\right]\bar{C}^{\Delta}\Delta \le \frac{1}{2}.$$

Then, the approximate solution $y(m)$ *has the properties as follows:*

$$\limsup_{m\to\infty}\frac{1}{m}\ln|y(m)|^2 \le -\alpha,$$
$$\limsup_{m\to\infty}\frac{1}{m}\ln\mathbb{E}[|y(m)|^2] \le -\alpha,$$

where α *is a constant satisfying* $e^{\alpha} = \bar{C}$.

Proof. By virtue of (2.1), we have

$$\begin{aligned}|y(m+1)|^2 &\le |y(m)|^2 + |b(\bar{y}_m, m\Delta)|^2\Delta^2 + 2y^T(k)b(\bar{y}_m, m\Delta)\Delta\\ &+ |\sigma(\bar{y}_m, m\Delta)|^2\Delta + |\sigma(\bar{y}_m, m\Delta)|^2((\Delta B(m))^2 - \Delta)\\ &+ 2y^T(m)\sigma(\bar{y}_m, m\Delta)\Delta B(m) + 2b(\bar{y}_m, m\Delta)\sigma(\bar{y}_m, m\Delta)\Delta B(m)\Delta\\ &\le |y(m)|^2 + |b(\bar{y}_m, m\Delta)|^2\Delta^2 + 2y^T(m)b(\bar{y}_m, m\Delta)\Delta\\ &+ |\sigma(\bar{y}_m, m\Delta)|^2\Delta + M(m),\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}M(m) &= |\sigma(\bar{y}_m, m\Delta)|^2((\Delta B(m))^2 - \Delta) + 2y^T(m)\sigma(\bar{y}_m, m\Delta)\Delta B(m)\\ &+ 2b(\bar{y}_m, m\Delta)\sigma(\bar{y}_m, m\Delta)\Delta B(m)\Delta.\end{aligned}$$

By (H1) and (H2), one can see that

$$\begin{aligned}|y(m+1)|^2 - |y(m)|^2 &\le \Big(-\lambda_1|y(m)|^2 + \lambda_2\int_{\underline{q}}^1 e^{-\beta m\Delta}|y_m(mq\Delta)|^2\mathrm{d}\nu(q)\Big)\Delta\\ &+ \Big(\lambda_3|y(m)|^2 + \lambda_4\int_{\underline{q}}^1 e^{-\beta m\Delta}|y_m(mq\Delta)|^2\mathrm{d}\nu(q)\Big)\Delta^2 + M(m).\end{aligned} \tag{3.3}$$

Multiplying $C^{(j+1)\Delta}$ on both sides of the inequality (3.3) yields that

$$\begin{aligned}&C^{(j+1)\Delta}|y(j+1)|^2 - C^{j\Delta}|y(j)|^2\\ &= C^{(j+1)\Delta}(1-\frac{1}{C^{\Delta}})|y(j)|^2\\ &+ \Big(-\lambda_1 C^{(j+1)\Delta}|y(j)|^2 + \lambda_2\int_{\underline{q}}^1 C^{(j+1)\Delta}e^{-\beta j\Delta}|y_j(jq\Delta)|^2\mathrm{d}\nu(q)\Big)\Delta\\ &+ \Big(\lambda_3 C^{(j+1)\Delta}|y(j)|^2 + \lambda_4\int_{\underline{q}}^1 C^{(j+1)\Delta}e^{-\beta j\Delta}|y_j(jq\Delta)|^2\mathrm{d}\nu(q)\Big)\Delta^2 + M(j),\end{aligned} \tag{3.4}$$

where C is a constant satisfying $1 < C \leq e^{\alpha_0}$. Since $1 - C^{-\Delta} < \alpha_0 \Delta$, summing both sides of (3.4) from $j = 0$ to $j = m - 1$, we obtain

$$
\begin{aligned}
& C^{m\Delta}|y(m)|^2 \\
&= x_0 + \sum_{0 \leq j \leq m-1} C^{(j+1)\Delta}(1 - \frac{1}{C^{\Delta}})|y(j)|^2 + \Big(- \lambda_1 \sum_{0 \leq j \leq m-1} C^{(j+1)\Delta}|y(j)|^2 \\
&+ \lambda_2 \sum_{0 \leq j \leq m-1} \int_{\underline{q}}^{1} C^{(j+1)\Delta} e^{-\beta j \Delta}|y_j(jq\Delta)|^2 \mathrm{d}\nu(q) \Big) \Delta \\
&+ \Big(\lambda_3 \sum_{0 \leq j \leq m-1} C^{(j+1)\Delta}|y(j)|^2 + \lambda_4 \int_{\underline{q}}^{1} \sum_{0 \leq j \leq m-1} C^{(j+1)\Delta} e^{-\beta j \Delta}|y_j(jq\Delta)|^2 \mathrm{d}\nu(q) \Big) \Delta^2 \\
&+ \sum_{0 \leq j \leq m-1} M(j),
\end{aligned}
\tag{3.5}
$$

where $\sum_{0 \leq j \leq m-1} M(j)$ is a martingale. Firstly, we compute

$$
\sum_{0 \leq j \leq m-1} \int_{\underline{q}}^{1} C^{(j+1)\Delta} e^{-\beta j \Delta}|y_j(jq\Delta)|^2 \mathrm{d}\nu(q) = \int_{\underline{q}}^{1} \sum_{0 \leq j \leq m-1} C^{(j+1)\Delta} e^{-\beta j \Delta}|y_j(jq\Delta)|^2 \mathrm{d}\nu(q). \tag{3.6}
$$

It is not difficult to see that

$$
\begin{aligned}
& \sum_{0 \leq j \leq m-1} C^{(j+1)\Delta} e^{-\beta j \Delta}|y(\lfloor jq \rfloor)|^2 \\
&\qquad = |y(0)| \sum_{0 \leq j \leq m-1} 1_{\lfloor qj \rfloor = 0} C^{(j+1)\Delta} e^{-\beta j \Delta} + |y(1)| \sum_{0 \leq j \leq m-1} 1_{\lfloor qj \rfloor = 1} C^{(j+1)\Delta} e^{-\beta j \Delta} \\
&\qquad + \cdots + |y(\lfloor q(m-1) \rfloor)| \sum_{0 \leq j \leq m-1} 1_{\lfloor qj \rfloor = \lfloor q(m-1) \rfloor} C^{(j+1)\Delta} e^{-\beta j \Delta},
\end{aligned}
\tag{3.7}
$$

and

$$
\begin{aligned}
& \sum_{0 \leq j \leq m-1} C^{(j+1)\Delta} e^{-\beta j \Delta}|y(\lfloor jq \rfloor + 1)|^2 \\
&\qquad = |y(1)| \sum_{0 \leq j \leq m-1} 1_{\lfloor qj \rfloor = 0} C^{(j+1)\Delta} e^{-\beta j \Delta} + |y(2)| \sum_{0 \leq j \leq m-1} 1_{\lfloor qj \rfloor = 1} C^{(j+1)\Delta} e^{-\beta j \Delta} \\
&\qquad + \cdots + |y(\lfloor q(m-1) \rfloor + 1)| \sum_{0 \leq j \leq m-1} 1_{\lfloor qj \rfloor = \lfloor q(m-1) \rfloor} C^{(j+1)\Delta} e^{-\beta j \Delta}.
\end{aligned}
\tag{3.8}
$$

Additionally, noting that $\lfloor qj \rfloor = i \Leftrightarrow \frac{i}{q} \leq j < \frac{i+1}{q}$, for any $i = 0, 1, \cdots, \lfloor q(m-1) \rfloor$, which implies

$$
\begin{cases}
\frac{i}{q} \leq j \leq \frac{i+1}{q} - 1, \frac{1}{q} \in \mathbb{N}, \\
\lfloor \frac{i}{q} \rfloor + 1 \leq j \leq \lfloor \frac{i+1}{q} \rfloor \leq \lfloor \frac{i}{q} \rfloor + \lfloor \frac{1}{q} \rfloor + 1, \frac{1}{q} \notin \mathbb{N}.
\end{cases}
$$

Then, it yields that the number of those $j \in \{0, 1, 2, \cdots, m-1\}$ such that $\lfloor qj \rfloor = i$, for some $i \in \{0, 1, 2, \cdots, \lfloor q(m-1) \rfloor\}$, is at most $\lfloor \frac{1}{q} \rfloor + 1$. Moreover, the greatest j for which $\lfloor qj \rfloor = i$ is less than $\frac{i+1}{q}$ and greater that $\frac{i}{q}$. By (3.7), we derive that

$$\begin{aligned}
&\sum_{0 \le j \le m-1} C^{(j+1)\Delta} e^{-\beta j \Delta} |y(\lfloor qj \rfloor)|^2 \\
&\le \left(\left\lfloor \frac{1}{q} \right\rfloor + 1 \right) \sum_{0 \le j \le \lfloor q(m-1) \rfloor} C^{(\frac{j+1}{q}+1)\Delta} e^{-\beta \frac{j}{q} \Delta} |y(j)|^2 \\
&\le \left(\left\lfloor \frac{1}{q} \right\rfloor + 1 \right) \sum_{0 \le j \le \lfloor q(m-1) \rfloor} C^{(\frac{1-\beta}{q} j\Delta + \frac{\Delta}{q} + \Delta)} |y(j)|^2 \\
&\le \left(\left\lfloor \frac{1}{q} \right\rfloor + 1 \right) C^{\frac{\Delta}{q}} \sum_{0 \le j \le \lfloor q(m-1) \rfloor} C^{(j+1)\Delta} |y(j)|^2 \\
&\le \left(\left\lfloor \frac{1}{q} \right\rfloor + 1 \right) C^{\frac{\Delta}{q}} \sum_{0 \le j \le m-1} C^{(j+1)\Delta} |y(j)|^2, \qquad (3.9)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{0 \le j \le m-1} C^{(j+1)\Delta} e^{-\beta j \Delta} |y(\lfloor qj \rfloor + 1)|^2 \\
&\le \left(\left\lfloor \frac{1}{q} \right\rfloor + 1 \right) \sum_{0 \le j \le \lfloor q(m-1) \rfloor} C^{(\frac{j+1}{q}+1)\Delta} e^{-\beta \frac{j}{q} \Delta} |y(j+1)|^2 \\
&\le \left(\left\lfloor \frac{1}{q} \right\rfloor + 1 \right) \sum_{0 \le j \le \lfloor q(m-1) \rfloor} C^{(\frac{1-\beta}{q} j\Delta + \frac{\Delta}{q} + \Delta)} |y(j+1)|^2 \\
&\le \left(\left\lfloor \frac{1}{q} \right\rfloor + 1 \right) C^{\frac{\Delta}{q}} \sum_{0 \le j \le \lfloor q(m-1) \rfloor} C^{(j+1)\Delta} |y(j+1)|^2 \\
&\le \left(\left\lfloor \frac{1}{q} \right\rfloor + 1 \right) C^{\frac{\Delta}{q}} \sum_{1 \le j \le m} C^{j\Delta} |y(j)|^2, \qquad (3.10)
\end{aligned}$$

Combining with (3.7) and (3.8), one has

$$\begin{aligned}
&C^{m\Delta} |y(m)|^2 \\
&\le x_0 + \sum_{0 \le j \le m-1} C^{(j+1)\Delta} \Big(1 - \frac{1}{C^{\Delta}} \Big) |y(j)|^2 + \Big[-\lambda_1 \sum_{0 \le j \le m-1} C^{(j+1)\Delta} |y(j)|^2 \\
&+ \lambda_2 \left(\left\lfloor \frac{1}{q} \right\rfloor + 1 \right) C^{\frac{\Delta}{q}} \sum_{0 \le j \le m-1} C^{(j+1)} |y(j)|^2 \Big] \Delta \\
&+ \Big[\lambda_3 \sum_{0 \le j \le m-1} C^{(j+1)\Delta} |y(j)|^2 + \lambda_4 \left(\left\lfloor \frac{1}{q} \right\rfloor + 1 \right) C^{\frac{\Delta}{q}} \sum_{0 \le j \le m-1} C^{(j+1)} |y(j)|^2 \Big] \Delta^2
\end{aligned}$$

$$
\begin{aligned}
&+\lambda_2\bigg(\left\lfloor\frac{1}{q}\right\rfloor+1\bigg)C^{\frac{\Delta}{q}}\sum_{1\le j\le m}C^{j\Delta}|y(j)|^2\Delta+\lambda_4\bigg(\left\lfloor\frac{1}{q}\right\rfloor+1\bigg)C^{\frac{\Delta}{q}}\sum_{1\le j\le m}C^{j\Delta}|y(j)|^2\Delta^2\\
&+\sum_{0\le j\le m-1}M(j)\\
&\le x_0+\bigg[-\lambda_1+\alpha_0+\lambda_2\bigg(\left\lfloor\frac{1}{q}\right\rfloor+1\bigg)C^{\frac{\Delta}{q}}\\
&\qquad\qquad+\lambda_3\Delta+\lambda_4\bigg(\left\lfloor\frac{1}{q}\right\rfloor+1\bigg)C^{\frac{\Delta}{q}}\Delta\bigg]\sum_{0\le j\le m-1}C^{(j+1)\Delta}|y(j)|^2\Delta\\
&+\bigg[\lambda_2\bigg(\left\lfloor\frac{1}{q}\right\rfloor+1\bigg)C^{\frac{\Delta}{q}}+\lambda_4\bigg(\left\lfloor\frac{1}{q}\right\rfloor+1\bigg)C^{\frac{\Delta}{q}}\Delta\bigg]\sum_{1\le j\le m}C^{j\Delta}|y(j)|^2\Delta+\sum_{0\le j\le m-1}M(j)\\
&\le x_0+\bigg[-\lambda_1+\alpha_0+2\lambda_2\bigg(\left\lfloor\frac{1}{q}\right\rfloor+1\bigg)C^{\frac{\Delta}{q}}\\
&\qquad+\lambda_3\Delta+2\lambda_4\bigg(\left\lfloor\frac{1}{q}\right\rfloor+1\bigg)C^{\frac{\Delta}{q}}\Delta\bigg]\sum_{0\le j\le m-1}C^{(j+1)\Delta}|y(j)|^2\Delta\\
&+\bigg[\lambda_2\bigg(\left\lfloor\frac{1}{q}\right\rfloor+1\bigg)C^{\frac{\Delta}{q}}+\lambda_4\bigg(\left\lfloor\frac{1}{q}\right\rfloor+1\bigg)C^{\frac{\Delta}{q}}\Delta\bigg]C^{m\Delta}|y(m)|^2\Delta+\sum_{0\le j\le m-1}M(j). \qquad (3.11)
\end{aligned}
$$

Set

$$
H(C,\Delta)=-\lambda_1+\alpha_0+2\lambda_2\bigg(\left\lfloor\frac{1}{q}\right\rfloor+1\bigg)C^{\frac{\Delta}{q}}+\lambda_3\Delta+2\lambda_4\bigg(\left\lfloor\frac{1}{q}\right\rfloor+1\bigg)C^{\frac{\Delta}{q}}\Delta. \qquad (3.12)
$$

Then,

$$
H(\bar{C},\lambda_0)=-\lambda_1+\alpha_0+2\lambda_2\bigg(\left\lfloor\frac{1}{q}\right\rfloor+1\bigg)\bar{C}^{\frac{\lambda_0}{q}}+\lambda_3\lambda_0+2\lambda_4\bigg(\left\lfloor\frac{1}{q}\right\rfloor+1\bigg)\bar{C}^{\frac{\lambda_0}{q}}\lambda_0. \qquad (3.13)
$$

By condition i), we have

$$
H(\bar{C},\lambda_0)\le 0.
$$

Using condition ii) and choosing a constant $\alpha>0$ with $e^{\eta}=\bar{C}$, one has

$$
e^{\eta m\Delta}|y(m)|^2\le 2x_0+2\sum_{0\le j\le m-1}M(j).
$$

Since $\sum_{0\le j\le m-1}M(j)$ is a martingale, we get

$$
\limsup_{m\to\infty}e^{\eta m\Delta}\mathbb{E}|y(m)|^2<\infty.
$$

Furthermore, by the semi-martingale convergence theorem in [11], we have

$$
\limsup_{m\to\infty}e^{\eta m\Delta}|y(m)|^2<\infty. \qquad (3.14)
$$

The proof is therefore complete. □

Remark 3.1. *Under the conditions of Theorem 3.1, Theorem 3.3 in [16] has shown that the analytical solution of* (1.1) *has the property of exponential stability. This means that the EM numerical solutions of* (2.1) *preserves the property of exponential stability.*

Now, we give an example to explain Theorem 3.1.

Example 3.2. Consider PSFDEs as follows:

$$x(t) = x_0 + \int_0^t b(x_s, s)\mathrm{d}s + \int_0^t \sigma(x_s, s)\mathrm{d}B(s), t \geq 0 \tag{3.15}$$

where

$$b(\varphi, t) = -1.1\varphi(1) + 0.04 \int_{\frac{3}{4}}^1 e^{-0.7t}|\varphi(q)|\mathrm{d}\nu(q).$$

and

$$\sigma(\varphi, t) = 0.2 \int_{\frac{3}{4}}^1 e^{-0.7t}|\varphi(q)|\mathrm{d}\nu(q).$$

Then,

$$\begin{aligned}
&2\langle\varphi(1) - \phi(1), b(\varphi, t) - b(\phi, t)\rangle + |\sigma(\varphi, t) - \sigma(\phi, t)|^2 \\
&= 2\langle\varphi(1) - \phi(1), -1.1(\varphi(1) - \phi(1)) + 0.04 \int_{\frac{3}{4}}^1 e^{-0.7t}(\varphi(q) - \phi(q))\mathrm{d}\nu(q)\rangle \\
&+ \left|0.2 \int_{\frac{3}{4}}^1 e^{-0.7t}(\varphi(qt) - \phi(qt))\mathrm{d}\nu(q)\right|^2 \\
&\leq -2.2|\varphi(1) - \phi(1)|^2 + 0.08(\varphi(1) - \phi(1)) \int_{\frac{3}{4}}^1 e^{-0.7t}(\varphi(q) - \phi(q))\mathrm{d}\nu(q) \\
&+ \left|0.2 \int_{\frac{3}{4}}^1 e^{-0.7t}(\varphi(qt) - \phi(qt))\mathrm{d}\nu(q)\right|^2 \\
&\leq -2.16|\varphi(1) - \phi(1)|^2 + 0.08 \int_{\frac{3}{4}}^1 |e^{-0.7t}(\varphi(q) - \phi(q))|^2\mathrm{d}\nu(q),
\end{aligned}$$

and

$$\begin{aligned}
&|b(\varphi, t) - b(\phi, t)|^2 \vee |\sigma(\varphi, t) - \sigma(\phi, t)|^2 \\
&\leq 1.23|\varphi(1) - \phi(1)|^2 + 0.17 \int_{\frac{3}{4}}^1 e^{-0.7t}|(\varphi(\theta) - \phi(\theta))|^2\mathrm{d}\nu(\theta).
\end{aligned}$$

We can find that

$$\lambda_1 = 2.16, \lambda_2 = 0.08, \lambda_3 = 1.23, \lambda_4 = 0.17, \underline{q} = \frac{3}{4}.$$

Choosing $\bar{C} = 1.1, \lambda_0 = \frac{1}{300}, \alpha_0 = \frac{1}{10}$, it is obvious that (i) and (ii) of Theorem 3.1 are satisfied. Then, we conclude that the numerical solutions of (3.15) are almost surely exponential stable, and exponential stable in mean square.

3.2 Polynomial Stable Properties of Numerical Solutions

Next, we will study polynomial stable properties of numerical solution to (2.1). In this subsection, we assume that there exists a fixed constant $\overline{q}$ satisfying $\frac{1}{2} \vee \underline{q} < \overline{q} < 1$. We need the following assumptions.

(H3) For any $\varphi_1, \varphi_2 \in \mathscr{C}([\underline{q}, \overline{q}]; \mathbb{R}^n)$, there exists a probability measure ν on $[\underline{q}, \overline{q}]$ and positive constants $\bar{\lambda}_1, \bar{\lambda}_2$ such that

$$\begin{aligned}&2\langle \varphi_1(1) - \varphi_2(1), b(\varphi_1, t) - b(\varphi_2, t)\rangle + |\sigma(\varphi_1, t) - \sigma(\varphi_2, t)|^2 \\ &\leq -\bar{\lambda}_1 |\varphi_1(1) - \varphi_2(1)|^2 + \bar{\lambda}_2 \int_{\underline{q}}^{\overline{q}} |\varphi_1(q) - \varphi_2(q)|^2 \mathrm{d}\nu(q). \end{aligned} \tag{3.16}$$

(H4) For any $\varphi_1, \varphi_2 \in \mathscr{C}([\underline{q}, \overline{q}]; \mathbb{R}^n)$, there exists a probability measure ν on $[\underline{q}, \overline{q}]$ and two positive constants $\bar{\lambda}_3, \bar{\lambda}_4$ such that

$$\begin{aligned}&|b(\varphi_1, t) - b(\varphi_2, t)|^2 \vee |\sigma(\varphi_1, t) - \sigma(\varphi_2, t)|^2 \\ &\leq \bar{\lambda}_3 |\varphi_1(1) - \varphi_2(1)|^2 + \bar{\lambda}_4 \int_{\underline{q}}^{\overline{q}} |\varphi_1(q) - \varphi_2(q)|^2 \mathrm{d}\nu(q).\end{aligned}$$

Theorem 3.3. *Assume* (H3) *and* (H4)*. If the following conditions hold:*

1° $\bar{\lambda}_1 - 2\bar{\lambda}_2(\lfloor \frac{1}{\underline{q}} \rfloor + 1) > 0.$

2° *Assume that*

$$\bar{\lambda}_1 - \zeta = 2\bar{\lambda}_2 \left(\left\lfloor \frac{1}{\underline{q}} \right\rfloor + 1 \right) \underline{q}^{-\zeta}. \tag{3.17}$$

has a unique solution $\zeta^* > 0$.

3° Δ *is sufficiently small such that:*

$$\left[\bar{\lambda}_2 \left(\left\lfloor \frac{1}{\underline{q}} \right\rfloor + 1 \right) \underline{q}^{-\zeta^* - 1} + \bar{\lambda}_4 \left(\left\lfloor \frac{1}{\underline{q}} \right\rfloor + 1 \right) \underline{q}^{-\zeta^* - 1} \Delta \right] \Delta < \frac{1}{2}. \tag{3.18}$$

Then, $\forall\ \varepsilon \in (0, \frac{\zeta^*}{2})$*, there exists a sufficiently small* $\Delta^* \in (0, 1)$ *such that the numerical solution* $y(m)$ *satisfies*

$$\limsup_{m \to \infty} \frac{\ln |y(m)|}{\ln |(m+1)\Delta|} \leq -\frac{\zeta^*}{2} + \varepsilon, a.s.$$

and

$$\limsup_{m \to \infty} \frac{\ln \mathbb{E}[|y(m)|^2]}{\ln |(m+1)\Delta|} \leq -\zeta^* + 2\varepsilon, a.s.$$

Proof. According to (3.2), (H3) and (H4), we get

$$|y(m+1)|^2 - |y(m)|^2 \le \Big(-\bar{\lambda}_1 |y(m)|^2 + \bar{\lambda}_2 \int_{\underline{q}}^{\bar{q}} |y_m(mq\Delta)|^2 \mathrm{d}\nu(q) \Big)\Delta + \Big(\bar{\lambda}_3 |y(m)|^2 + \bar{\lambda}_4 \int_{\underline{q}}^{\bar{q}} |y_m(mq\Delta)|^2 \mathrm{d}\nu(q) \Big)\Delta^2 + M(m). \tag{3.19}$$

Multiplying $(1+(1+m)\Delta)^\gamma$ on both sides of the inequality (3.19) yields that

$$\begin{aligned}
&(1+(1+m)\Delta)^\gamma |y(m+1)|^2 - (1+m\Delta)^\gamma |y(m)|^2 \\
&\le (1+(1+m)\Delta)^\gamma \Big(1 - \frac{(1+m\Delta)^\gamma)}{(1+(1+m)\Delta)^\gamma}\Big)|y(m)|^2 \\
&+ \Big(-\bar{\lambda}_1 (1+(1+m)\Delta)^\gamma |y(m)|^2 + \bar{\lambda}_2 \int_{\underline{q}}^{\bar{q}} (1+(1+m)\Delta)^\gamma |y_m(mq\Delta)|^2 \mathrm{d}\nu(q) \Big)\Delta \\
&+ \Big(\bar{\lambda}_3 (1+(1+m)\Delta)^\gamma |y(m)|^2 + \bar{\lambda}_4 \int_{\underline{q}}^{\bar{q}} (1+(1+m)\Delta)^\gamma |y_m(mq\Delta)|^2 \mathrm{d}\nu(q) \Big)\Delta^2 + M(m),
\end{aligned} \tag{3.20}$$

where γ is a positive constant. Observing that $1-|x|^\gamma \le -\gamma \ln |x|$, one has

$$1 - \frac{(1+m\Delta)^\gamma}{(1+(1+m)\Delta)^\gamma} \le \gamma \ln \frac{1+(1+m)\Delta}{1+m\Delta} \le \frac{\gamma\Delta}{1+m\Delta} \le \gamma\Delta.$$

Summing both sides of (3.20) from $j=0$ to $j=m-1$, we obtain

$$\begin{aligned}
&(1+m\Delta)^\gamma |y(m)|^2 \\
&\le x_0 + \sum_{0\le j\le m-1} (1+(j+1)\Delta)^\gamma \gamma\Delta |y(j)|^2 \\
&+ \Big(-\bar{\lambda}_1 \sum_{0\le j\le m-1} (1+(j+1)\Delta)^\gamma |y(j)|^2 + \bar{\lambda}_2 \sum_{0\le j\le m-1} \int_{\underline{q}}^{\bar{q}} (1+(j+1)\Delta)^\gamma |y_j(jq\Delta)|^2 \mathrm{d}\nu(q) \Big)\Delta \\
&+ \Big(\bar{\lambda}_3 \sum_{0\le j\le m-1} (1+(j+1)\Delta)^\gamma |y(j)|^2 + \bar{\lambda}_4 \int_{\underline{q}}^{\bar{q}} \sum_{0\le j\le m-1} (1+(j+1)\Delta)^\gamma |y_j(jq\Delta)|^2 \mathrm{d}\nu(q) \Big)\Delta^2 \\
&\le x_0 + \sum_{0\le j\le m-1} (1+(j+1)\Delta)^\gamma \gamma\Delta |y(j)|^2 \\
&+ \Big(-\bar{\lambda}_1 \sum_{0\le j\le m-1} (1+(j+1)\Delta)^\gamma |y(j)|^2 + \bar{\lambda}_2 \sum_{0\le j\le m-1} \int_{\underline{q}}^{\bar{q}} (1+(j+1)\Delta)^\gamma |y(\lfloor jq \rfloor)|^2 \mathrm{d}\nu(q) \\
&\qquad + \bar{\lambda}_2 \sum_{0\le j\le m-1} \int_{\underline{q}}^{\bar{q}} (1+(j+1)\Delta)^\gamma |y(\lfloor jq \rfloor + 1)|^2 \mathrm{d}\nu(q) \Big)\Delta \\
&+ \Big(\bar{\lambda}_3 \sum_{0\le j\le m-1} (1+(j+1)\Delta)^\gamma |y(j)|^2 + \bar{\lambda}_4 \sum_{0\le j\le m-1} \int_{\underline{q}}^{\bar{q}} (1+(j+1)\Delta)^\gamma |y(\lfloor jq \rfloor)|^2 \mathrm{d}\nu(q)
\end{aligned}$$

$$+ \bar{\lambda}_4 \sum_{0\le j\le m-1} \int_{\underline{q}}^{\overline{q}} (1+(j+1)\Delta)^\gamma |y(\lfloor jq\rfloor +1)|^2 \mathrm{d}\nu(q) \Big) \Delta^2$$

$$+ \sum_{0\le j\le m-1} M(j). \tag{3.21}$$

Firstly, we compute

$$\sum_{0\le j\le m-1} \int_{\underline{q}}^{\overline{q}} (1+(j+1)\Delta)^\gamma |y(\lfloor jq\rfloor)|^2 \mathrm{d}\nu(q) = \int_{\underline{q}}^{\overline{q}} \sum_{0\le j\le m-1} (1+(j+1)\Delta)^\gamma |y(\lfloor jq\rfloor)|^2 \mathrm{d}\nu(q), \tag{3.22}$$

and

$$\sum_{0\le j\le m-1} \int_{\underline{q}}^{\overline{q}} (1+(j+1)\Delta)^\gamma |y(\lfloor jq\rfloor +1)|^2 \mathrm{d}\nu(q)$$
$$= \int_{\underline{q}}^{\overline{q}} \sum_{0\le j\le m-1} (1+(j+1)\Delta)^\gamma |y(\lfloor jq\rfloor +1)|^2 \mathrm{d}\nu(q). \tag{3.23}$$

Let $\Delta_0 = \frac{1-\overline{q}}{\overline{q}}$. Since $\overline{q} > \frac{1}{2}$, $\Delta_0 < 1$. Moreover, and for any $\Delta \in (0, \Delta_0), q \le \overline{q}$, we can derive $q + q\Delta \le 1$. Using the similar method in (3.9), (3.10), we have the follow two inequalities:

$$\sum_{0\le j\le m-1} (1+(j+1)\Delta)^\gamma |y(\lfloor qj\rfloor)|^2 \le \left(\left\lfloor \frac{1}{q}\right\rfloor +1\right) \sum_{0\le j\le \lfloor q(m-1)\rfloor} \left(1+\left(\frac{j+1}{q}+1\right)\Delta\right)^\gamma |y(j)|^2$$
$$\le \left(\left\lfloor \frac{1}{q}\right\rfloor +1\right) q^{-\gamma} \sum_{0\le j\le \lfloor q(m-1)\rfloor} (q(1+\Delta)+(j+1)\Delta)^\gamma |y(j)|^2$$
$$\le \left(\left\lfloor \frac{1}{q}\right\rfloor +1\right) q^{-\gamma} \sum_{0\le j\le m-1} (1+(j+1)\Delta)^\gamma |y(j)|^2$$
$$\le \left(\left\lfloor \frac{1}{\underline{q}}\right\rfloor +1\right) \underline{q}^{-\gamma} \sum_{0\le j\le m-1} (1+(j+1)\Delta)^\gamma |y(j)|^2, \tag{3.24}$$

and

$$\sum_{0\le j\le m-1} (1+(j+1)\Delta)^\gamma |y(\lfloor qj\rfloor +1)|^2$$
$$\le \left(\left\lfloor \frac{1}{q}\right\rfloor +1\right) \sum_{0\le j\le \lfloor q(m-1)\rfloor} \left(1+\left(\frac{j+1}{q}+1\right)\Delta\right)^\gamma |y(j+1)|^2$$
$$\le \left(\left\lfloor \frac{1}{\underline{q}}\right\rfloor +1\right) \underline{q}^{-\gamma} \sum_{1\le j\le m} (1+j\Delta)^\gamma |y(j)|^2. \tag{3.25}$$

Combining with (3.21), we arrive at

$$(1+m\Delta)^\gamma |y(m)|^2$$

$$
\begin{aligned}
&\le x_0 + \sum_{0\le j\le m-1} (1+(j+1)\Delta)^{\gamma}\gamma\Delta|y(j)|^2 \\
&+ \Bigg[-\bar{\lambda}_1 \sum_{0\le j\le m-1} (1+(j+1)\Delta)^{\gamma}|y(j)|^2 + \bar{\lambda}_2\bigg(\bigg\lfloor\frac{1}{\underline{q}}\bigg\rfloor+1\bigg)\underline{q}^{-\gamma}\sum_{0\le j\le m-1}(1+(j+1)\Delta)^{\gamma}|y(j)|^2 \\
&\qquad\qquad + \bar{\lambda}_2\bigg(\bigg\lfloor\frac{1}{\underline{q}}\bigg\rfloor+1\bigg)\underline{q}^{-\gamma}\sum_{1\le j\le m}(1+j\Delta)^{\gamma}|y(j)|^2\Bigg]\Delta \\
&+ \Bigg[\bar{\lambda}_3 \sum_{0\le j\le m-1} (1+(j+1)\Delta)^{\gamma}|y(j)|^2 + \bar{\lambda}_4\bigg(\bigg\lfloor\frac{1}{\underline{q}}\bigg\rfloor+1\bigg)\underline{q}^{-\gamma}\sum_{0\le j\le m-1}(1+(j+1)\Delta)^{\gamma}|y(j)|^2 \\
&\qquad\qquad + \bar{\lambda}_4\bigg(\bigg\lfloor\frac{1}{\underline{q}}\bigg\rfloor+1\bigg)\underline{q}^{-\gamma}\sum_{1\le j\le m}(1+j\Delta)^{\gamma}|y(j)|^2\Bigg]\Delta^2 \\
&+ \sum_{j=0}^{m-1} M(j) \\
&\le x_0 + \Bigg[-\bar{\lambda}_1 + \gamma + \bar{\lambda}_2\bigg(\bigg\lfloor\frac{1}{\underline{q}}\bigg\rfloor+1\bigg)\underline{q}^{-\gamma} + \bar{\lambda}_3\Delta \\
&\qquad\qquad + \bar{\lambda}_4\bigg(\bigg\lfloor\frac{1}{\underline{q}}\bigg\rfloor+1\bigg)\underline{q}^{-\gamma}\Delta\Bigg]\sum_{0\le j\le m-1}(1+(j+1)\Delta)^{\gamma}|y(j)|^2\Delta \\
&+ \Bigg[\bar{\lambda}_2\bigg(\bigg\lfloor\frac{1}{\underline{q}}\bigg\rfloor+1\bigg)\underline{q}^{-\gamma} + \bar{\lambda}_4\bigg(\bigg\lfloor\frac{1}{\underline{q}}\bigg\rfloor+1\bigg)\underline{\theta}^{-\gamma}\Delta\Bigg]\sum_{1\le j\le m}(1+j\Delta)^{\gamma}|y(j)|^2\Delta \\
&+ \sum_{0\le j\le m-1} M(j) \\
&\le x_0 + \Bigg[-\bar{\lambda}_1 + \gamma + 2\bar{\lambda}_2\bigg(\bigg\lfloor\frac{1}{\underline{q}}\bigg\rfloor+1\bigg)\underline{q}^{-\gamma} + \bar{\lambda}_3\Delta \\
&\qquad\qquad + 2\bar{\lambda}_4\bigg(\bigg\lfloor\frac{1}{\underline{q}}\bigg\rfloor+1\bigg)\underline{q}^{-\gamma}\Delta\Bigg]\sum_{0\le j\le m-1}(1+(j+1)\Delta)^{\gamma}|y(j)|^2\Delta \\
&+ \Bigg[\bar{\lambda}_2\bigg(\bigg\lfloor\frac{1}{\underline{q}}\bigg\rfloor+1\bigg)\underline{q}^{-\gamma} + \bar{\lambda}_4\bigg(\bigg\lfloor\frac{1}{\underline{q}}\bigg\rfloor+1\bigg)\underline{q}^{-\gamma}\Delta\Bigg](1+m\Delta)^{\gamma}|y(m)|^2\Delta \\
&+ \sum_{j=0}^{m-1} M(j).
\end{aligned}
\tag{3.26}
$$

Set

$$
H(\gamma,\Delta) = -\bar{\lambda}_1 + \gamma + 2\bar{\lambda}_2\bigg(\bigg\lfloor\frac{1}{\underline{q}}\bigg\rfloor+1\bigg)\underline{q}^{-\gamma} + \bar{\lambda}_3\Delta + 2\bar{\lambda}_4\bigg(\bigg\lfloor\frac{1}{\underline{q}}\bigg\rfloor+1\bigg)\underline{q}^{-\gamma}\Delta. \tag{3.27}
$$

Immediately, one can see that

$$
\frac{\mathrm{d}H(\gamma,\Delta)}{\mathrm{d}\gamma} = 1 + 2\bar{\lambda}_2\bigg(\bigg\lfloor\frac{1}{\underline{q}}\bigg\rfloor+1\bigg)\underline{q}^{-\gamma}\log\frac{1}{\underline{q}} + 2\bar{\lambda}_4\bigg(\bigg\lfloor\frac{1}{\underline{q}}\bigg\rfloor+1\bigg)\underline{q}^{-\gamma}\log\frac{1}{\underline{q}}\Delta > 0,
$$

and

$$H(0,\Delta) = -\bar{\lambda}_1 + 2\bar{\lambda}_2\Big(\Big\lfloor \frac{1}{\underline{q}} \Big\rfloor + 1\Big) + \bar{\lambda}_3\Delta + 2\bar{\lambda}_4\Big(\Big\lfloor \frac{1}{\underline{q}} \Big\rfloor + 1\Big)\Delta < 0,$$

for $\Delta \in (0, \Delta_1 \wedge \Delta_0), \Delta_1 = \frac{\bar{\lambda}_1 - \bar{\lambda}_2(\lfloor \frac{1}{\underline{q}} \rfloor + 1)}{\bar{\lambda}_3 + \bar{\lambda}_4(\lfloor \frac{1}{\underline{q}} \rfloor + 1)}$. Then, for any $\Delta \in (0, \Delta_1 \wedge \Delta_0)$, there exists a constant γ^*_Δ such that $H(\gamma^*_\Delta, \Delta) = 0$. This together with condition 3° implies

$$(1 + k\Delta)^{\gamma^*_\Delta} |y(m)|^2 \leq 2|x_0|^2 + 2 \sum_{0 \leq j \leq m-1} M(j). \tag{3.28}$$

Since, $\sum_{0\leq j\leq m-1} M(j)$ is a martingale, we obtain

$$\limsup_{m\to\infty} (1 + m\Delta)^{\gamma^*_\Delta} \mathbb{E}|y(m)|^2 < \infty. \tag{3.29}$$

From the semi-martingale convergence theorem in [11], we derive

$$\limsup_{m\to\infty} (1 + m\Delta)^{\gamma^*_\Delta} |y(m)|^2 < \infty. \tag{3.30}$$

Noting that

$$\lim_{\Delta\to 0} H(\gamma, \Delta) = -\bar{\lambda}_1 + \gamma + 2\bar{\lambda}_2\Big(\Big\lfloor \frac{1}{\underline{q}} \Big\rfloor + 1\Big)\underline{q}^{-\gamma},$$

and (3.17), one has $\lim_{\Delta\to 0} \gamma^*_\Delta = \zeta^*$. Thus, for any $\varepsilon \in (0, \frac{\zeta^*}{2})$, there exists Δ_2 such that $\gamma^*_\Delta > \zeta^* - 2\varepsilon$ for any $\Delta \in (0, \Delta_2)$. Then, for any $\Delta \in (0, \Delta_0 \wedge \Delta_1 \wedge \Delta_2)$, (3.28), (3.29) and (3.30) imply the result in the theorem. □

Remark 3.2. *Under the conditions in Theorem 4.3, from Theorem 3.6 in [16], we know that the analytical solution of* (1.1) *has the property of polynomial stability. This shows that the EM method inherits the polynomial stability of the true solution.*

Example 3.4. Consider the following PSFDEs:

$$x(t) = x_0 + \int_0^t b(x_s, s)\mathrm{d}s + \int_0^t \sigma(x_s, s)\mathrm{d}B(s), t \geq 0 \tag{3.31}$$

where

$$b(\varphi, t) = -0.4\varphi(1) + 0.04 \int_{\frac{3}{4}}^{\frac{4}{5}} |\varphi(\theta)|\mathrm{d}\nu(\theta).$$

and

$$\sigma(\varphi, t) = 0.3 \int_{\frac{3}{4}}^{\frac{4}{5}} |\varphi(\theta)|\mathrm{d}\nu(\theta).$$

Form the above definition, it follows that

$$
\begin{aligned}
&2\langle\varphi_1(1)-\varphi_2(1), b(\varphi_1,t)-b(\varphi_2,t)\rangle+|\sigma(\varphi_1,t)-\sigma(\varphi_2,t)|^2\\
&=2\langle\varphi_1(1)-\varphi_2(1), -0.4(\varphi_1(1)-\varphi_2(1))+0.04\int_{\frac{3}{4}}^{\frac{4}{5}}(\varphi_1(q)-\varphi_2(q))\mathrm{d}\nu(q)\rangle\\
&+\left|0.3\int_{\frac{3}{4}}^{\frac{4}{5}}(\varphi_1(qt)-\varphi_2(qt))\mathrm{d}\nu(q)\right|^2\\
&\le -0.8|\varphi_1(1)-\varphi_2(1)|^2+0.08(\varphi_1(1)-\varphi_2(1))\int_{\frac{3}{4}}^{\frac{4}{5}}(\varphi_1(q)-\varphi_2(q))\mathrm{d}\nu(q)\\
&+\left|0.3\int_{\frac{3}{4}}^{\frac{4}{5}}(\varphi_1(q)-\varphi_2(q))\mathrm{d}\nu(q)\right|^2\\
&\le -0.76|\varphi_1(1)-\varphi_2(1)|^2+0.13\int_{\frac{3}{4}}^{\frac{4}{5}}|(\varphi_1(q)-\varphi_2(q))|^2\mathrm{d}\nu(q),
\end{aligned}
$$

and

$$
\begin{aligned}
&|b(\varphi_1,t)-b(\varphi_2,t)|^2\vee|\sigma(\varphi_1,t)-\sigma(\varphi_2,t)|^2\\
&\le 0.19|\varphi_1(1)-\varphi_2(1)|^2+0.09\int_{\frac{3}{4}}^{\frac{4}{5}}|(\varphi_1(q)-\varphi_2(q))|^2\mathrm{d}\nu(q).
\end{aligned}
$$

Letting $\bar\lambda_1=0.76, \bar\lambda_2=0.13, \bar\lambda_3=0.19, \bar\lambda_1=0.09, \underline{q}=\frac{3}{4}, \overline{q}=\frac{4}{5}$, it is obvious that the conditions in Theorem 3.6 are satisfied. We derive that the numerical solutions of (3.31) are almost surely polynomial stable.

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