Convergence in Wasserstein Distance for Empirical Measures of Dirichlet Diffusion Processes on Manifolds*

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Abstract

Let M be a d-dimensional connected compact Riemannian manifold with boundary ∂M, let $V \in C^2(M)$ such that $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure, and let X_t be the diffusion process generated by $L := \Delta + \nabla V$ with $\tau := \inf\{t \geq 0 : X_t \in \partial M\}$. Consider the empirical measure $\mu_t := \frac{1}{t}\int_0^t \delta_{X_s}\mathrm{d}s$ under the condition $t < \tau$ for the diffusion process. If $d \leq 3$, then for any initial distribution not fully supported on ∂M,

$$c\sum_{m=1}^{\infty}\frac{2}{(\lambda_m-\lambda_0)^2} \leq \liminf_{t\to\infty}\inf_{T\geq t}\Big\{t\mathbb{E}\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big]\Big\}$$
$$\leq \limsup_{t\to\infty}\sup_{T\geq t}\Big\{t\mathbb{E}\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big]\Big\} \leq \sum_{m=1}^{\infty}\frac{2}{(\lambda_m-\lambda_0)^2}$$

holds for some constant $c \in (0,1]$ with $c = 1$ when ∂M is convex, where $\mu_0 := \phi_0^2\mu$ for the first Dirichet eigenfunction ϕ_0 of L, $\{\lambda_m\}_{m\geq 0}$ are the Dirichlet eigenvalues of $-L$ listed in the increasing order counting multiplicities, and the upper bound is finite if and only if $d \leq 3$. When $d = 4$, $\sup_{T\geq t}\mathbb{E}\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big]$ decays in the order $t^{-1}\log t$, while for $d \geq 5$ it behaves like $t^{-\frac{2}{d-2}}$, as $t \to \infty$.

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1 Introduction

Let M be a d-dimensional connected compact Riemannian manifold with a smooth boundary ∂M. Let $V \in C^2(M)$ such that $\mu(\mathrm{d}x) = \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure on M, where $\mathrm{d}x$ is the Riemannian volume measure. Let X_t be the diffusion process generated by $L := \Delta + \nabla V$ with hitting time

$$\tau := \inf\{t \geq 0 : X_t \in \partial M\}.$$

Denote by $\mathscr{P}$ the set of all probability measures on M, and let $\mathbb{E}^\nu$ be the expectation taken for the diffusion process with initial distribution $\nu \in \mathscr{P}$. We consider the empirical measure

$$\mu_t := \frac{1}{t}\int_0^t \delta_{X_s}\mathrm{d}s, \quad t > 0$$

under the condition that $t < \tau$. Since $\tau = 0$ when $X_0 \in \partial M$, to ensure $\mathbb{P}^\nu(\tau > t) > 0$, where $\mathbb{P}^\nu$ is the probability taken for the diffusion process with initial distribution ν, we only consider

$$\nu \in \mathscr{P}_0 := \big\{\nu \in \mathscr{P} :\ \nu(M^\circ) > 0\big\}, \quad M^\circ := M \setminus \partial M.$$

Let $\{\lambda_m\}_{m\geq 0}$ be all Dirichlet eigenfunctions of $-L$ on M, which are listed in the increasing order counting multiplicities. Let $\{\phi_m\}_{m\geq 0}$ be the associated unit Dirichlet eigenfunctions, i.e. $L\phi_m = -\lambda_m\phi_m, \phi_m|_{\partial M} = 0$ and $\{\phi_m\}_{m\geq 0}$ is an orthonormal basis of $L^2(\mu)$. Moreover, we take $\phi_0|_{M^\circ} > 0$ as ϕ_0 is non-zero in M°. It is well known (see [5]) that $\lambda_0 > 0$ and

$$\|\phi_m\|_\infty \leq \alpha_0\sqrt{m}, \quad \alpha_0^{-1}m^{\frac{2}{d}} \leq \lambda_m - \lambda_0 \leq \alpha_0 m^{\frac{2}{d}}, \quad m \geq 1 \tag{1.1}$$

holds for some constant $\alpha_0 > 1$.

Let $\mu_0 = \phi_0^2\mu$. We investigate the convergence rate of $\mathbb{E}^\nu[\mathbb{W}_2(\mu_t, \mu_0)^2|t < \tau]$ as $t \to \infty$, where $\mathbb{W}_2$ is the L^2-Wasserstein distance induced by the Riemannian metric ρ. In general, for any $p \geq 1$,

$$\mathbb{W}_p(\mu_1, \mu_2) := \inf_{\pi\in\mathscr{C}(\mu_1,\mu_2)}\left(\int_{M\times M}\rho(x,y)^p\pi(\mathrm{d}x,\mathrm{d}y)\right)^{\frac{1}{p}}, \quad \mu_1, \mu_2 \in \mathscr{P},$$

where $\mathscr{C}(\mu_1, \mu_2)$ is the set of all probability measures on $M \times M$ with marginal distributions μ_1 and μ_2, and $\rho(x, y)$ is the Riemannian distance between x and y, i.e. the length of the shortest curve on M linking x and y.

Recently, the convergence rate under $\mathbb{W}_2$ has been characterized in [24] for the empirical measures of the L-diffusion processes without boundary (i.e. $\partial M = \emptyset$) or with a reflecting boundary. Moreover, the convergence of $\mathbb{W}_2(\mu_t^\nu, \mu_0)$ for the conditional empirical measure

$$\mu_t^\nu := \mathbb{E}^\nu(\mu_t|t < \tau), \quad t > 0$$

is investigated in [20]. Comparing with $\mathbb{E}^\nu[\mathbb{W}_2(\mu_t, \mu_0)^2|t < \tau]$, in μ_t^ν the conditional expectation inside the Wasserstein distance. According to [20], $\mathbb{W}_2(\mu_t^\nu, \mu_0)^2$ behaves as t^{-2}, whereas the following result says that $\mathbb{E}[\mathbb{W}_2(\mu_t, \mu_0)^2|t < \tau]$ decays at a slower rate, which coincides with the rate of $\mathbb{E}[\mathbb{W}_2(\hat{\mu}_t, \mu)^2]$ given by [24, Theorems 1.1, 1.2], where $\hat{\mu}_t$ is the empirical measure of the reflecting diffusion process generated by L. See also [21] on the study for diffusion processes on non-compact manifolds, [22] for semi-linear SPDEs, and [23] for subordinated diffusions.

Theorem 1.1. *Let $\{\lambda_m\}_{m\ge 0}$ be the Dirichlet eigenvalues of $-L$ listed in the increasing order counting multiplicities. Then for any $\nu \in \mathscr{P}_0$, the following assertions hold.*

(1) *In general,*

$$\limsup_{t\to\infty}\Big\{t\sup_{T\ge t}\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big]\Big\}\le\sum_{m=1}^\infty\frac{2}{(\lambda_m-\lambda_0)^2},\tag{1.2}$$

and there exists a constant $c>0$ such that

$$\liminf_{t\to\infty}\Big\{t\inf_{T\ge t}\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big]\Big\}\ge c\sum_{m=1}^\infty\frac{2}{(\lambda_m-\lambda_0)^2}.\tag{1.3}$$

If ∂M is convex and $d\le 3$, then

$$\lim_{t\to\infty}\sup_{T\ge t}\bigg|t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big]-\sum_{m=1}^\infty\frac{2}{(\lambda_m-\lambda_0)^2}\bigg|=0.$$

(2) *When $d=4$, there exists a constant $c>0$ such that*

$$\sup_{T\ge t}\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big]\le ct^{-1}\log t,\quad t\ge 2.\tag{1.4}$$

(3) *When $d\ge 5$, there exist a constant $c>1$ such that*

$$c^{-1}t^{-\frac{2}{d-2}}\le\mathbb{E}^\nu\big[\mathbb{W}_1(\mu_t,\mu_0)^2\big|T<\tau\big]\le\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big]\le ct^{-\frac{2}{d-2}},\quad T\ge t\ge 2.$$

Let X_t^0 be the diffusion process generated by $L_0:=L+2\nabla\log\phi_0$ in M°. It is easy to see that for any initial distribution supported on M° and any $t_0>0$, the law of $\{X_t^0:t\in[0,t_0]\}$ is the weak limit of the conditional distribution of $\{X_t:t\in[0,t_0]\}$ given $T<\tau$ as $T\to\infty$. Indeed, for $T>t_0$ and $s\in[0,t_0]$, let $\psi_s=P_s^D 1$ and let $\{X_t^T:t\in[0,T)\}$ be the diffusion process on M° generated by $\Delta+2\nabla\log\psi_{T-t}, t\in[0,T)$. It is easy to see that for any $f\in C_0^\infty(M^\circ)$ and $t\in(0,t_0]$, the process

$$M_s:=\frac{P_{t-s}^D(f\psi_{T-t})}{P_{t-s}^D\psi_{T-t}}(X_s^T),\quad s\in[0,t]$$

is a martingale, so that

$$\mathbb{E}^x[f(X_t)|T<\tau]=\frac{P_t^D(f\psi_{T-t})}{P_t^D\psi_{T-t}}(x)=\mathbb{E}^xM_0=\mathbb{E}^xM_t=\mathbb{E}^x[f(X_t^T)],\quad x\in M^\circ.$$

By the Markov property, this implies that the law of $\{X_t^T:t\in[0,t_0]\}$ coincides with the conditional law of $\{X_t:t\in[0,t_0]\}$ given $T<\tau$. Since

$$\lim_{T\to\infty}\nabla\log\psi_{T-t}=\nabla\log\phi_0$$

locally uniformly on $M^\circ\times[0,t_0]$, as $T\to\infty$ the law of $\{X_t^T:t\in[0,t_0]\}$ converges weakly to the law of $\{X_t^0:t\in[0,t_0]\}$. In conclusion, the conditional distribution of $\{X_t:t\in[0,t_0]\}$ given $T<\tau$ converges weakly to the law of $\{X_t^0:t\in[0,t_0]\}$ as $T\to\infty$. Therefore, the following is a direct consequence of Theorem 1.1.

Corollary 1.2. *Let* $\mu_t^0 = \frac{1}{t}\int_0^t \delta_{X_s^0} \mathrm{d}s$. *Let* $\nu \in \mathscr{P}_0$ *with* $\nu(M^\circ) = 1$.

(1) *In general,*

$$\limsup_{t\to\infty} \Big\{ t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t^0, \mu_0)^2\big]\Big\} \le \sum_{m=1}^\infty \frac{2}{(\lambda_m - \lambda_0)^2},$$

and there exists a constant $c > 0$ *such that*

$$\liminf_{t\to\infty} \Big\{ t \inf_{T\ge t} \big[\mathbb{W}_2(\mu_t^0, \mu_0)^2\big]\Big\} \ge c\sum_{m=1}^\infty \frac{2}{(\lambda_m - \lambda_0)^2}.$$

If ∂M *is convex, then*

$$\lim_{t\to\infty} \Big\{ t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t^0, \mu_0)^2\big]\Big\} = \sum_{m=1}^\infty \frac{2}{(\lambda_m - \lambda_0)^2}.$$

(2) *When* $d = 4$, *there exists a constant* $c > 0$ *such that*

$$\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t^0, \mu_0)^2\big] \le ct^{-1}\log t, \quad t \ge 2.$$

(3) *When* $d \ge 5$, *there exists a constant* $c > 1$ *such that*

$$c^{-1}t^{-\frac{2}{d-2}} \le \mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t^0, \mu_0)^2\big] \le ct^{-\frac{2}{d-2}}, \quad t \ge 2.$$

In the next section, we first recall some facts on the Dirichlet semigroup and the diffusion semigroup P_t^0 generated by $L_0 := L + 2\nabla \log \phi_0$, then establish the Bismut derivative formula for P_t^0 which will be used to estimate the lower bound of $\mathbb{E}^\nu[\mathbb{W}_2(\mu_t, \mu_0)^2 | t < \tau)$. With these preparations, we prove Propositions 3.1 and 4.1 in Sections 3 and 4 respectively, which imply Theorem 1.1.

2 Some preparations

As in [24], we first recall some well known facts on the Dirichlet semigroup, see for instances [5, 6, 12, 19]. Let ρ_∂ be the Riemannian distance function to the boundary ∂M. Then $\phi_0^{-1}\rho_\partial$ is bounded such that

$$\|\phi_0^{-1}\|_{L^p(\mu_0)} < \infty, \quad p \in [1, 3). \tag{2.1}$$

The Dirichlet heat kernel has the representation

$$p_t^D(x, y) = \sum_{m=0}^\infty \mathrm{e}^{-\lambda_m t}\phi_m(x)\phi_m(y), \quad t > 0, x, y \in M. \tag{2.2}$$

Let $\mathbb{E}^x$ denote the expectation for the L-diffusion process starting at point x. Then Dirichlet diffusion semigroup generated by L is given by

$$\begin{aligned} P_t^D f(x) &:= \mathbb{E}^x[f(X_t)1_{\{t<\tau\}}] = \int_M p_t^D(x,y)f(y)\mu(\mathrm{d}y) \\ &= \sum_{m=0}^{\infty} \mathrm{e}^{-\lambda_m t}\mu(\phi_m f)\phi_m(x), \quad t>0, f\in L^2(\mu). \end{aligned} \tag{2.3}$$

Consequently,

$$\lim_{t\to\infty}\left\{\mathrm{e}^{\lambda_0 t}\mathbb{P}^\nu(t<\tau)\right\} = \lim_{t\to\infty}\left\{\mathrm{e}^{\lambda_0 t}\nu(P_t^D 1)\right\} = \mu(\phi_0)\nu(\phi_0), \quad \nu\in\mathscr{P}_0. \tag{2.4}$$

Moreover, there exists a constant $c>0$ such that

$$\|P_t^D\|_{L^p(\mu)\to L^q(\mu)} := \sup_{\mu(|f|^p)\le 1}\|P_t^D f\|_{L^q(\mu)} \le c\mathrm{e}^{-\lambda_0 t}(1\wedge t)^{-\frac{d(q-p)}{2pq}}, \quad t>0, q\ge p\ge 1. \tag{2.5}$$

Indeed, the Sobolev inequality implies

$$\|P_t^D\|_{L^1(\mu)\to L^\infty(\mu)} \le c(t\wedge 1)^{-\frac{d}{2}}, \quad t>0$$

for some constant $c>0$, which together with

$$\|P_t^D\|_{L^2(\mu)} \le \mathrm{e}^{-\lambda_0 t}, \quad t\ge 0$$

and the interpolation theorem (see for instance [6]), implies (2.5).

On the other hand, let

$$L_0 = L + 2\nabla\log\phi_0.$$

Noting that $L_0 f = \phi_0^{-1}L(f\phi_0)+\lambda_0 f$, L_0 is a self-adjoint operator in $L^2(\mu_0)$ and the associated semigroup $P_t^0 := \mathrm{e}^{tL_0}$ satisfies

$$P_t^0 f = \mathrm{e}^{\lambda_0 t}\phi_0^{-1}P_t^D(f\phi_0), \quad f\in L^2(\mu_0), \quad t\ge 0. \tag{2.6}$$

So, $\{\phi_0^{-1}\phi_m\}_{m\ge 0}$ is an eigenbasis of L_0 in $L^2(\mu_0)$ with

$$L_0(\phi_m\phi_0^{-1}) = -(\lambda_m-\lambda_0)\phi_m\phi_0^{-1}, \quad P_t^0(\phi_m\phi_0^{-1}) = \mathrm{e}^{-(\lambda_m-\lambda_0)t}\phi_m\phi_0^{-1}, \quad m\ge 0, t\ge 0. \tag{2.7}$$

Consequently,

$$P_t^0 f = \sum_{m=0}^{\infty}\mu_0(f\phi_m\phi_0^{-1})\mathrm{e}^{-(\lambda_m-\lambda_0)t}\phi_m\phi_0^{-1}, \quad f\in L^2(\mu_0), \tag{2.8}$$

and the heat kernel of P_t^0 with respect to μ_0 is given by

$$p_t^0(x,y) = \sum_{m=0}^{\infty}(\phi_m\phi_0^{-1})(x)(\phi_m\phi_0^{-1})(y)\mathrm{e}^{-(\lambda_m-\lambda_0)t}, \quad x,y\in M, t>0. \tag{2.9}$$

By the intrinsic ultracontractivity, see for instance [13], there exists a constant $\alpha_1 \geq 1$ such that

$$\|P_t^0 - \mu_0\|_{L^1(\mu_0)\to L^\infty(\mu_0)} := \sup_{\mu_0(|f|)\leq 1} \|P_t^0 f - \mu_0(f)\|_\infty \leq \frac{\alpha_1 \mathrm{e}^{-(\lambda_1-\lambda_0)t}}{(1\wedge t)^{\frac{d+2}{2}}}, \quad t>0. \tag{2.10}$$

Combining this with the semigroup property and the contraction of P_t^0 in $L^p(\mu)$ for any $p \geq 1$, we find a constant $\alpha_2 \geq 1$ such that

$$\|P_t^0 - \mu_0\|_{L^p(\mu_0)} := \sup_{\mu_0(|f|^p)\leq 1} \|P_t^0 f - \mu_0(f)\|_{L^p(\mu_0)} \leq \alpha_2 \mathrm{e}^{-(\lambda_1-\lambda_0)t}, \quad t\geq 0, p\geq 1. \tag{2.11}$$

By the interpolation theorem, (2.10) and (2.11) yield that for some constant $\alpha_3 > 0$,

$$\|P_t^0 - \mu_0\|_{L^p(\mu_0)\to L^q(\mu_0)} \leq \alpha_3 \mathrm{e}^{-(\lambda_1-\lambda_0)t}\{1\wedge t\}^{-\frac{(d+2)(q-p)}{2pq}}, \quad t>0, \infty\geq q>p\geq 1. \tag{2.12}$$

By this and (2.7), there exists a constant $\alpha_4 > 0$ such that

$$\|\phi_m \phi_0^{-1}\|_\infty \leq \alpha_4 m^{\frac{d+2}{2d}}, \quad m\geq 1. \tag{2.13}$$

In the remainder of this section, we establish the Bismut derivative formula for P_t^0, which is not included by existing results due to the singularity of $\nabla \log \phi_0$ included in L_0: $|\nabla \log \phi_0| \sim \rho_\partial^{-1}$ and $\|\mathrm{Hess}_{\log \phi_0}\| \sim \rho_\partial^{-2}$ around the boundary, where Hess is the Hessian tensor. Let X_t^0 be the diffusion process generated by L_0, which solves the following Itô SDE on M°, see [8]:

$$\mathrm{d}^I X_t^0 = \nabla(V + 2\log\phi_0)(X_t^0)\mathrm{d}t + \sqrt{2}U_t \mathrm{d}B_t, \tag{2.14}$$

where d^I is Itô's differential, B_t is the d-dimensional Brownian motion, and $U_t \in O_{X_t^0}(M)$ is the horizontal lift of X_t^0 to the frame bundle $O(M)$. Let Ric and Hess be the Ricci curvature and the Hessian tensor on M respectively. Then the Bakry-Emery curvature of L_0 is given by

$$\mathrm{Ric}_{L_0} := \mathrm{Ric} - \mathrm{Hess}_{V+2\log\phi_0}.$$

Let $\mathrm{Ric}_{L^0}^{\#}(U_t) \in \mathbb{R}^d \otimes \mathbb{R}^d$ be defined by

$$\langle \mathrm{Ric}_{L^0}^{\#}(U_t)a, b\rangle_{\mathbb{R}^d} = \mathrm{Ric}_{L_0}(U_t a, U_t b), \quad a,b\in\mathbb{R}^d.$$

We consider the following ODE on $\mathbb{R}^d\otimes\mathbb{R}^d$:

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_t = -\mathrm{Ric}_{L^0}^{\#}(U_t)Q_t, \quad Q_0 = I, \tag{2.15}$$

where I is the identity matrix.

Lemma 2.1. *For any $\varepsilon > 0$, there exist constants $\delta_1, \delta_2 > 0$ such that*

$$\mathbb{E}^x\left[\mathrm{e}^{\delta_1\int_0^t \{\phi_0(X_s^0)\}^{-2}\mathrm{d}s}\right] \leq \delta_2 \phi_0^{-\varepsilon}(x)\mathrm{e}^{\delta_2 t}, \quad t\geq 0, x\in M^\circ. \tag{2.16}$$

Consequently,

(1) *For any $\varepsilon > 0$ and $p > 1$, there exists a constant $\kappa > 0$ such that*

$$|\nabla P_t^0 f(x)|^2 \leq \kappa \phi_0(x)^{-\varepsilon} e^{\kappa t} \{P_t^0 |\nabla f|^{2p}(x)\}^{\frac{1}{p}}, \quad f \in C_b^1(M).$$

(2) *For any $\varepsilon > 0$ and $p \geq 1$, there exists a constant $\kappa > 0$ such that for any stopping time τ',*

$$\mathbb{E}^x[\|Q_{t\wedge\tau'}\|^p] \leq \kappa \phi_0(x)^{-\varepsilon} e^{\kappa t}, \quad t \geq 0.$$

Proof. Since $L\phi_0 = -\lambda_0 \phi_0$, $\phi_0 > 0$ in M°, $\|\phi_0\|_\infty < \infty$ and $|\nabla \phi_0|$ is strictly positive in a neighborhood of ∂M, we find a constant $c_1, c_2 > 0$ such that

$$L_0 \log \phi_0^{-1} = -\phi_0^{-1} L\phi_0 + \phi_0^{-2} |\nabla \phi_0|^2 - 2\phi_0^{-2} |\nabla \phi_0|^2 \leq c_1 - c_2 \phi_0^{-2}.$$

So, by (2.14) and Itô's formula, we obtain

$$d \log \phi_0^{-1}(X_t^0) \leq \{c_1 - c_2 \phi_0^{-2}(X_t^0)\} dt + \sqrt{2} \langle \nabla \log \phi_0^{-1}(X_t^0), U_t dB_t \rangle.$$

This implies

$$\begin{aligned} \mathbb{E}^x \int_0^t [\phi_0^{-2}(X_s^0)] ds &\leq \frac{1}{c_2} \big(c_1 t + \log \phi_0^{-1}(x) + \mathbb{E} \log \phi_0(X_t^0)\big) \\ &\leq \frac{1}{c_2} \big(c_1 t + \log \phi_0^{-1}(x) + \log \|\phi_0\|_\infty\big) \leq ct + c \log(1 + \phi_0^{-1})(x), \quad t \geq 0 \end{aligned} \tag{2.17}$$

for some constant $c > 0$, and for any constant $\delta > 0$,

$$\begin{aligned} &\mathbb{E}^x \Big[e^{\delta c_2 \int_0^t \phi_0^{-2}(X_s^0) \} ds} \Big] \leq \mathbb{E}^x \Big[e^{\delta \log \phi_0^{-1}(x) + \delta \log \phi_0(X_t^0) + c_1 \delta t - \delta \sqrt{2} \int_0^t \langle \nabla \log \phi_0(X_s^0), U_s dB_s \rangle} \Big] \\ &\leq e^{c_1 \delta t} \phi_0^{-\delta}(x) \|\phi_0\|_\infty^\delta \big(\mathbb{E}^x [e^{4\delta^2 \int_0^t |\nabla \log \phi_0|^2 (X_s^0) ds}] \big)^{\frac{1}{2}}. \end{aligned}$$

Letting $c_3 = 4\|\nabla \phi_0\|_\infty^2$ and taking $\delta \in (0, c_2/c_3]$, we derive

$$\mathbb{E}^x \Big[e^{\delta c_2 \int_0^t \phi_0^{-2}(X_s^0) \} ds} \Big] \leq e^{2c_1 \delta t} \phi_0^{-2\delta}(x), \quad \delta \in (0, c_2/c_3].$$

This implies (2.16) for $\varepsilon \in (0, 2c_2/c_3]$. Since for $\varepsilon > 2c_2/c_3$ we have

$$\phi_0^{-\varepsilon} \geq \phi_0^{-2c_2/c_3} \|\phi_0\|_\infty^{\varepsilon - 2c_2/c_3},$$

(2.16) also holds for $\varepsilon > 2c_2/c_3$. Below we prove assertions (1) and (2) respectively.

Since $V \in C_b^2(M)$, $\phi_0 \in C_b^2(M)$ with $\phi_0 > 0$ in M°, and

$$-\text{Hess}_{\log \phi_0} = -\frac{\text{Hess}_{\phi_0}}{\phi_0} + \frac{\nabla \phi_0 \otimes \nabla \phi_0}{\phi_0^2} \geq -\frac{\text{Hess}_{\phi_0}}{\phi_0},$$

we find a constant $\alpha_1 > 0$ such that

$$\text{Ric}_{L_0}(U, U) \geq -\alpha_1 \phi_0^{-1}(x) |U|^2, \quad x \in M^\circ, U \in T_x M. \tag{2.18}$$

By (2.14), (2.18), and the formulas of Itô and Bochner, for fixed $t>0$ this implies

$$\begin{aligned}
&\mathrm{d}|\nabla P^0_{t-s}f|^2(X^0_s)\\
&=\big\{L_0|\nabla P^0_{t-s}f|^2(X^0_s)-2\langle \nabla P^0_{t-s}f,\nabla L_0P^0_{t-s}f\rangle\big\}\mathrm{d}s+\sqrt{2}\langle\nabla|\nabla P^0_{t-s}f|^2(X^0_s),U_s\mathrm{d}B_s\rangle\\
&\ge 2\mathrm{Ric}_{L^0}(\nabla P^0_{t-s}f,\nabla P^0_{t-s}f)(X^0_s)\mathrm{d}s+\sqrt{2}\langle\nabla|\nabla P^0_{t-s}f|^2(X^0_s),U_s\mathrm{d}B_s\rangle\\
&\ge -2\alpha_1\{\phi_0^{-1}|\nabla P^0_{t-s}f|^2\}(X^0_s)\mathrm{d}s+\sqrt{2}\langle\nabla|\nabla P^0_{t-s}f|^2(X_s),U_s\mathrm{d}B_s\rangle\mathrm{d}s.
\end{aligned}$$

Then by a Gronwall type inequality, we obtain

$$\begin{aligned}
|\nabla P^0_tf(x)|^2&=\mathbb{E}^x|\nabla P^0_tf|^2(X^0_0)\le \mathbb{E}^x\Big[|\nabla f|^2(X^0_t)\mathrm{e}^{2\int_0^t2\alpha_1\phi_0^{-1}(X^0_u)\mathrm{d}u}\Big]\\
&\le\Big\{\mathbb{E}^x\Big[\mathrm{e}^{\frac{2\alpha_1p}{p-1}\int_0^t\phi_0^{-1}(X^0_u)\mathrm{d}u}\Big]\Big\}^{\frac{p-1}{p}}\{P^0_t|\nabla f|^{2p}(x)\}^{\frac1p}.
\end{aligned}$$

Noting that

$$\frac{2\alpha_1p}{p-1}\int_0^t\phi_0^{-1}(X^0_s)\mathrm{d}s\le\delta_1\int_0^t\phi_0^{-2}(X^0_s)\mathrm{d}s+\frac{\alpha_1^2p^2t}{\delta_1(p-1)^2},\quad\delta_1>0,\tag{2.19}$$

by combining this with (2.16), we prove (1).

Next, by (2.15) and (2.18), we obtain

$$\|Q_{t\wedge\tau'}\|\le\mathrm{e}^{\alpha_1\int_0^t\phi^{-1}(X^0_s)\mathrm{d}s},\quad t\ge0.$$

This together with (2.16) and (2.19) implies (2).

□

Lemma 2.2. *For any $t>0$ and $\gamma\in C^1([0,t])$ with $\gamma(0)=0$ and $\gamma(t)=1$, we have*

$$\nabla P^0_tf(x)=\mathbb{E}^x\bigg[f(X^0_t)\int_0^t\gamma'(s)Q^*_s\mathrm{d}B_s\bigg],\quad x\in M^\circ,f\in\mathscr{B}_b(M^\circ).\tag{2.20}$$

Consequently, for any $\varepsilon>0$ and $p>1$, here exists a constant $c>0$ such that

$$|\nabla P^0_tf|\le\frac{c\phi_0^{-\varepsilon}}{\sqrt{1\wedge t}}(P^0_t|f|^p)^{\frac1p},\quad t>0,f\in\mathscr{B}_b(M^\circ).\tag{2.21}$$

Proof. Since (2.21) follows from (2.20) with $\gamma(s):=\frac{t-s}{t}$ and Lemma 2.1(2), it suffices to prove the Bismut formula (2.20). By an approximation argument, we only need to prove for $f\in C^1_b(M)$. The proof is standard by Elworthy-Li's martingale argument [7], see also [15]. By $\|\nabla f\|_\infty<\infty$ and Lemma 2.1(1) for $\varepsilon=\frac14$, we find a constant $c_1>0$ such that

$$|\nabla P^0_sf|(x)\le c_1\phi_0^{-1/4}(x),\quad s\in[0,t],x\in M^\circ.\tag{2.22}$$

Next, since $L\phi_0=-\lambda_0\phi_0$ implies $L_0\phi_0^{-1}=\lambda_0\phi_0^{-1}$, by Itô's formula we obtain

$$\mathbb{E}^x[\phi_0^{-1}(X^0_{t\wedge\tau_n})]\le\phi_0^{-1}(x)\mathrm{e}^{\lambda_0t},\quad t\ge0,n\ge1,\tag{2.23}$$

where $\tau_n := \inf\{t \geq 0 : \phi_0(X_s^0) \leq \frac{1}{n}\} \uparrow \infty$ as $n \uparrow \infty$ by noting that the process X_t^0 is non-explosive in M°.

Moreover, by Itô's formula, for any $a \in \mathbb{R}^d$, we have

$$\begin{aligned} &\mathrm{d}\langle \nabla P_{t-s}^0 f(X_s^0), U_s Q_s a\rangle = \sqrt{2}\,\mathrm{Hess}_{P_{t-s}f}(U_s \mathrm{d}B_s, U_s Q_s a)(X_s^0),\\ &\mathrm{d}P_{t-s}^0 f(X_s^0) = \sqrt{2}\,\langle \nabla P_{t-s}^0 f(X_s^0), U_s \mathrm{d}B_s\rangle, \quad s \in [0,t]. \end{aligned}$$

Due to the integration by part formula, this and $\gamma(0) = 0$ imply

$$\begin{aligned} &-\frac{1}{\sqrt{2}}\mathbb{E}^x\bigg[f(X_{t\wedge\tau_n}^0)\int_0^{t\wedge\tau_n}\gamma'(s)\langle Q_s a, \mathrm{d}B_s\rangle\bigg]\\ &= \mathbb{E}\bigg[\int_0^{t\wedge\tau_n}\langle \nabla P_{t-s}^0 f(X_s^0), U_s Q_s a\rangle \mathrm{d}(1-\gamma)(s)\bigg]\\ &= \mathbb{E}\big[(1-\gamma)(t\wedge\tau_n)\langle \nabla P_{t-t\wedge\tau_n}^0 f(X_{t\wedge\tau_n}^0), Q_{t\wedge\tau_n} a\rangle\big] - \langle \nabla P_t^0 f(x), U_0 a\rangle\\ &\quad - \mathbb{E}\bigg[\int_0^{t\wedge\tau_n}(1-\gamma)(s)\mathrm{d}\langle \nabla P_{t-s}^0 f(X_s^0), U_s Q_s a\rangle\bigg]\\ &= \mathbb{E}\big[(1-\gamma)(t\wedge\tau_n)\langle \nabla P_{t-t\wedge\tau_n}^0 f(X_{t\wedge\tau_n}^0), Q_{t\wedge\tau_n} a\rangle\big] - \langle \nabla P_t f(x), U_0 a\rangle, \quad n \geq 1. \end{aligned} \tag{2.24}$$

Since γ is bounded with $\gamma(t) = 1$ such that $(1-\gamma)(t\wedge\tau_n) \to 0$ as $n \to \infty$, and (2.22), (2.23) and Lemma 2.1(2) imply

$$\sup_{n\geq 1}\mathbb{E}^x\big[\langle \nabla P_{t-t\wedge\tau_n}^0 f(X_{t\wedge\tau_n}^0), Q_{t\wedge\tau_n} a\rangle^2\big] \leq c_1 \sup_{n\geq 1}\big(\mathbb{E}[\phi_0^{-1}(X_{t\wedge\tau_n}^0)]\big)^{\frac{1}{2}}\big(\mathbb{E}^x\|Q_{t\wedge\tau_n}\|^4\big)^{\frac{1}{2}} < \infty,$$

by the dominated convergence theorem, we may take $n \to \infty$ in (2.24) to derive (2.20). □

3 Upper bound estimates

In this section we prove the following result which includes upper bound estimates in Theorem 1.1.

Proposition 3.1. *Let $\nu \in \mathscr{P}_0$.*

(1) (1.2) *holds.*

(2) *When $d = 4$, there exists a constant $c > 0$ such that* (1.4) *holds.*

(3) *When $d \geq 5$, there exists a constant $c > 0$ such that*

$$\sup_{T\geq t}\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t, \mu_0)^2\big|T < \tau\big] \leq c t^{-\frac{2}{d-2}}, \quad t \geq 2.$$

The main tool in the study of the upper bound estimate is the following inequality due to [1], see also [24, Lemma 2.3]: for any probability density $g \in L^2(\mu_0)$,

$$\mathbb{W}_2(g\mu_0, \mu_0)^2 \leq \int_M \frac{|\nabla L_0^{-1}(g-1)|^2}{\mathscr{M}(g,1)}\mathrm{d}\mu_0, \tag{3.1}$$

where $\mathscr{M}(a,b) := \frac{a-b}{\log a - \log b} 1_{\{a\wedge b>0\}}$. To apply this inequality, as in [24], we first modify μ_t by $\mu_{t,r} := \mu_t P_r^0$ for some $r > 0$, where for a probability measure ν on M°, νP_r^0 is the law of the L_0-diffusion process X_r^0 with initial distribution ν. Obviously, by (2.9) we have

$$
\begin{aligned}
\rho_{t,r} &:= \frac{\mathrm{d}\mu_{t,r}}{\mathrm{d}\mu_0} = \frac{1}{t}\int_0^t p_r^0(X_s, \cdot)\mathrm{d}s = 1 + \sum_{m=1}^\infty \mathrm{e}^{-(\lambda_m - \lambda_0)r}\psi_m(t)\phi_m\phi_0^{-1}, \\
\psi_m(t) &:= \frac{1}{t}\int_0^t \{\phi_m\phi_0^{-1}\}(X_s)\mathrm{d}s,
\end{aligned} \tag{3.2}
$$

which are well-defined on the event $\{t < \tau\}$.

Lemma 3.2. (1) *If $d \le 3$ and $\nu = h\mu$ with $h\phi_0^{-1} \in L^p(\mu_0)$ for some $p > \frac{d+2}{2}$, then there exists a constant $c > 0$ such that*

$$
\begin{aligned}
&\sup_{T\ge t}\Big| t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big| T<\tau\big] - 2\sum_{m=1}^\infty \frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2}\Big| \\
&\le ct^{-1}\big(r^{-\frac{(d-2)^+}{2}} + 1_{\{d=2\}}\log r^{-1}\big), \quad r\in(0,1], t\ge 1.
\end{aligned}
$$

(2) *If $d \ge 4$ and $\nu = h\mu$ with $\|h\phi_0^{-1}\|_\infty < \infty$, then for any $k > \frac{d-4}{6}$, there exists a constant $c > 0$ such that*

$$
\begin{aligned}
&\sup_{T\ge t} t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big| T<\tau\big] \\
&\le c\big\{r^{-\frac{d-4}{2}} + 1_{\{d=4\}}\log r^{-1} + t^{-1}r^{-k}\big\}, \quad t\ge 1, r\in(0,1).
\end{aligned}
$$

Proof. By the Markov property, (2.6) and (2.3), we have

$$
\begin{aligned}
\mathbb{E}^x[f(X_s)1_{\{T<\tau\}}] &= \mathbb{E}^x\big[1_{\{s<\tau\}}f(X_s)\mathbb{E}^{X_s}1_{\{T-s<\tau\}}\big] \\
&= P_s^D\{fP_{T-s}^D 1\}(x) = \mathrm{e}^{-\lambda_0 T}\big(\phi_0 P_s^0\{fP_{T-s}^0\phi_0^{-1}\}\big)(x), \quad s<T.
\end{aligned} \tag{3.3}
$$

By the same reason, and noting that $\mathbb{E}^\nu = \int_M \mathbb{E}^x\nu(\mathrm{d}x)$, we derive

$$
\begin{aligned}
&\mathbb{E}^\nu[f(X_{s_1})f(X_{s_2}))1_{\{T<\tau\}}] = \int_M \mathbb{E}^x\big[1_{\{s_1<\tau\}}f(X_{s_1})\mathbb{E}^{X_{s_1}}\{f(X_{s_2-s_1})1_{\{T-s_1<\tau\}}\}\big]\nu(\mathrm{d}x) \\
&= \mathrm{e}^{-\lambda_0 T}\nu\big(\phi_0 P_{s_1}^0[fP_{s_2-s_1}^0\{fP_{T-s_2}^0\phi_0^{-1}\}]\big), \quad s_1<s_2<T.
\end{aligned}
$$

In particular, the formula with $f = 1$ yields

$$
\mathbb{P}^\nu(T<\tau) = \mathrm{e}^{-\lambda_0 T}\nu(\phi_0 P_T^0\phi_0^{-1}). \tag{3.4}
$$

Since $\{\frac{\nabla(\phi_m\phi_0^{-1})}{\sqrt{\lambda_m-\lambda_0}}\}_{m\ge1}$ is othornormal in $L^2(\mu_0)$, we have

$$
\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2) = \sum_{m=1}^\infty \frac{\psi_m(t)^2}{(\lambda_m-\lambda_0)\mathrm{e}^{2(\lambda_m-\lambda_0)r}}.
$$

Combining these with (3.2), we obtain

$$t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big]=\sum_{m=1}^\infty \frac{t\mathbb{E}^\nu[\psi_m(t)^2|T<\tau]}{\mathrm{e}^{2(\lambda_m-\lambda_0)r}(\lambda_m-\lambda_0)}.$$

Noting that $\mathbb{E}^\nu(\xi|T<\tau):=\frac{\mathbb{E}^\nu[\xi 1_{\{T<\tau\}}]}{\mathbb{P}^\nu(T<\tau)}$ for an integrable random variable ξ, by (2.7) and and the symmetry of P_t^0 in $L^2(\mu_0)$, we deduce from this for $\nu=h\mu$ that

$$\begin{aligned}
&t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big]\\
&=\sum_{m=1}^\infty \frac{2\int_0^t\mathrm{d}s_1\int_{s_1}^t \mathbb{E}^\nu\big[1_{\{T<\tau\}}(\phi_m\phi_0^{-1})(X_{s_1})(\phi_m\phi_0^{-1})(X_{s_2})\big]\mathrm{d}s_2}{t\mathrm{e}^{2(\lambda_m-\lambda_0)r}(\lambda_m-\lambda_0)\nu(\phi_0P_T^0\phi_0^{-1})}\\
&=\sum_{m=1}^\infty \frac{2\int_0^t\mathrm{d}s_1\int_{s_1}^t \nu\big(\phi_0^{-1}P_{s_1}^0\{\phi_m\phi_0^{-1}P_{s_2-s_1}^0[\phi_m\phi_0^{-1}P_{T-s_2}^0\phi_0^{-1}]\}\big)\mathrm{d}s_2}{t\mathrm{e}^{2(\lambda_m-\lambda_0)r}(\lambda_m-\lambda_0)\nu(\phi_0P_T^0\phi_0^{-1})}\\
&=\sum_{m=1}^\infty \frac{2\int_0^t\mathrm{d}s_1\int_{s_1}^t \mu_0\big(\{P_{s_1}^0(h\phi_0^{-1})\}\phi_m\phi_0^{-1}P_{s_2-s_1}^0[\phi_m\phi_0^{-1}P_{T-s_2}^0\phi_0^{-1}]\big)\mathrm{d}s_2}{t\mathrm{e}^{2(\lambda_m-\lambda_0)r}(\lambda_m-\lambda_0)\mu_0(\phi_0^{-1}P_T^0(h\phi_0^{-1}))}.
\end{aligned}\tag{3.5}$$

Since $\|\phi_0^{-1}\|_{L^2(\mu_0)}=1$ and $\|h\phi_0^{-1}\|_{L^1(\mu_0)}=\mu(h\phi_0)\le\|\phi_0\|_\infty<\infty$, by (2.12) we find a constant $c_1>0$ such that

$$\begin{aligned}
&\big|\mu_0(\phi_0^{-1}P_T^0(h\phi_0^{-1}))-\mu(\phi_0)\nu(\phi_0)\big|\le\|\phi_0^{-1}(P_T^0-\mu_0)(h\phi_0^{-1})\|_{L^1(\mu_0)}\\
&\le\|P_T^0-\mu_0\|_{L^1(\mu_0)\to L^2(\mu_0)}\|h\phi_0^{-1}\|_{L^1(\mu_0)}\le c_1\mathrm{e}^{-(\lambda_1-\lambda_0)T},\quad T\ge1.
\end{aligned}\tag{3.6}$$

On the other hand, write

$$\begin{aligned}
&\mu_0\big(\{P_{s_1}^0(h\phi_0^{-1})\}\phi_m\phi_0^{-1}P_{s_2-s_1}^0[\phi_m\phi_0^{-1}P_{T-s_2}^0\phi_0^{-1}]\big)\\
&=\nu(\phi_0)\mu(\phi_0)\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}+J_1(s_1,s_2)+J_2(s_1,s_2)+J_3(s_1,s_2),
\end{aligned}\tag{3.7}$$

where, due to (2.7),

$$\begin{aligned}
&J_1(s_1,s_2):=\mu_0\big(\{P_{s_1}^0(h\phi_0^{-1})-\mu(h\phi_0)\}\phi_m\phi_0^{-1}P_{s_2-s_1}^0[\phi_m\phi_0^{-1}(P_{T-s_2}^0\phi_0^{-1}-\mu(\phi_0))]\big),\\
&J_2(s_1,s_2):=\mu(\phi_0)\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\mu_0\big(\{P_{s_1}^0(h\phi_0^{-1})-\mu(h\phi_0)\}\{\phi_m\phi_0^{-1}\}^2\big),\\
&J_3(s_1,s_2):=\mu(h\phi_0)\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\mu_0\big(\{\phi_m\phi_0^{-1}\}^2\{P_{T-s_2}^0\phi_0^{-1}-\mu(\phi_0)\}\big).
\end{aligned}$$

By (3.5), (3.6), (3.7) and

$$\int_0^t\mathrm{d}s_1\int_{s_1}^t\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\mathrm{d}s_2=\frac{t}{\lambda_m-\lambda_0}-\frac{1-\mathrm{e}^{-(\lambda_m-\lambda_0)t}}{(\lambda_m-\lambda_0)^2},$$

we find a constant $\kappa>0$ such that

$$\begin{aligned}
&\sup_{T\ge t}\Big|t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big]-2\sum_{m=1}^\infty\frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2}\Big|\\
&\le\frac{\kappa}{t}\sum_{m=1}^\infty\bigg(\frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2}+\frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{\lambda_m-\lambda_0}\int_0^t\mathrm{d}s_1\int_{s_1}^t|J_1+J_2+J_3|(s_2,s_2)\mathrm{d}s_2\bigg),\quad t\ge1.
\end{aligned}\tag{3.8}$$

Below we prove assertions (1) and (2) respectively.

(1) Let $d \le 3$. Since $\|h\phi_0^{-1}\|_{L^p(\mu_0)} < \infty$, $\|\phi_0^{-1}\|_{L^\theta(\mu_0)} < \infty$ for $\theta < 3$ due to (2.1), $\|\phi_m\phi_0^{-1}\|_{L^2(\mu_0)} = 1$, by (2.12), for any $\theta \in (\frac{5}{2}, 3)$, we find constants $c_1, c_2 > 0$ such that

$$\begin{aligned}|J_1(s_1,s_2)| &\le c_1\|P_{s_1}^0 - \mu_0\|_{L^p(\mu_0)\to L^\infty(\mu_0)}\| \cdot \|\phi_m\phi_0^{-1}(P_{T-s_2}^0\phi_0^{-1} - \mu(\phi_0))\|_{L^2(\mu_0)}\\ &\le c_1\|P_{s_1}^0 - \mu_0\|_{L^p(\mu_0)\to L^\infty(\mu_0)}\|P_{T-s_2}^0 - \mu_0\|_{L^\theta(\mu_0)\to L^\infty(\mu_0)}\\ &\le c_2 \mathrm{e}^{-(\lambda_1-\lambda_0)(s_1+T-s_2)}(1\wedge s_1)^{-\frac{d+2}{2p}}\{1\wedge(T-s_2)\}^{-\frac{d+2}{2\theta}},\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&|(J_2+J_3)(s_1,s_2)|\\ &\le c_1\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\big(\|P_{s_1}^0-\mu_0\|_{p\to\infty} + \|P_{T-s_2}^0 - \mu_0\|_{L^\theta(\mu_0)\to L^\infty(\mu_0)}\big)\\ &\le c_2\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\big(\{1\wedge s_1\}^{-\frac{d+2}{2p}}\mathrm{e}^{-(\lambda_1-\lambda_0)s_1} + \{1\wedge(T-s_2)\}^{-\frac{d+2}{2\theta}}\mathrm{e}^{-(\lambda_1-\lambda_0)(t-s_2)}\big).\end{aligned} \tag{3.10}$$

Since $\theta > \frac{5}{2}$ and $p > \frac{d+2}{2}$ imply $\frac{d+2}{2\theta} \vee \frac{d+2}{2p} < 1$ for $d \le 3$, by (3.9) and (3.10), we find a constant $c > 0$ such that

$$\int_0^t \mathrm{d}s_1 \int_{s_1}^t |J_1+J_2+J_3|(s_1,s_2)\mathrm{d}s_2 \le c, \quad T \ge t \ge 1, m \ge 1.$$

Combining this with (3.8) and (1.1), we find constants $c_3, c_4, c_5, c_6 > 0$ such that

$$\begin{aligned}&\sup_{T\ge t}\Big|t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big] - \sum_{m=0}^\infty \frac{\mathrm{e}^{-(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2}\Big|\\ &\le \frac{c_3}{t}\sum_{m=1}^\infty \frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{\lambda_m-\lambda_0} \le \frac{c_4}{t}\int_1^\infty s^{-\frac{2}{d}}\mathrm{e}^{-c_5 s^{\frac{2}{d}}r}\mathrm{d}s \le c_6 t^{-1}\big(r^{-\frac{(d-2)^+}{2}} + 1_{\{d=2\}}\log r^{-1}\big), \quad t \ge 1.\end{aligned}$$

(2) Let $d \ge 4$. Let $\nu = h\mu$ with $|h\phi_0^{-1}| \le C$ for a constant $C > 0$, we have

$$\mathbb{E}^\nu = \int_{\mathbb{R}^d}\mathbb{E}^x\nu(\mathrm{d}x) \le C\mu(\phi_0)^{-1}\mathbb{E}^{\nu_0}. \tag{3.11}$$

On the other hand, by (3.4), we see that

$$\frac{P^\nu(t<\tau)}{\mathbb{P}^{\nu_0}(t<\tau)} = \frac{\nu(\phi_0 P_t^0\phi_0^{-1})}{\nu_0(\phi_0 P_t^0\phi_0^{-1})} > 0$$

is continuous in $t \ge 0$, and it converges to $\frac{\nu(\phi_0)}{\nu_0(\phi_0)} \in (0,\infty)$ due to (2.4). So, there exists a constant $c > 0$ such that

$$P^\nu(t<\tau) \ge c\mathbb{P}^{\nu_0}(t<\tau), \quad t \ge 0.$$

Combining this with (3.11), we find a constant $K > 0$ such that

$$\mathbb{E}^\nu(\cdot|t<\tau) = \frac{\mathbb{E}^\nu(1_{t<\tau}\cdot)}{\mathbb{P}^\nu(t<\tau)} \le \frac{K\mathbb{E}^{\nu_0}(1_{t<\tau}\cdot)}{\mathbb{P}^{\nu_0}(t<\tau)} = K\mathbb{E}^{\nu_0}(\cdot|t<\tau), \quad t \ge 0. \tag{3.12}$$

So, it suffices to prove for $\nu = \nu_0$. In this case, $h\phi_0^{-1} = \mu(\phi_0)^{-1}$ is a constant, so that $J_1 = J_2 = 0$. By (2.12), $\|\phi_m\phi_0^{-1}\|_{L^2(\mu_0)} = 1$ and $\|\phi_0\|_{L^\theta(\mu_0)} < \infty$ for $\theta \in (1,3)$, we find a constant $c_1 > 0$ such that

$$\begin{aligned}
&|J_2(s_1,s_2)| \le c_1 \mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1-r)} \big\| |(P^0_{r/2}-\mu_0)(\phi_m\phi_0^{-1})|^2 (P^0_{T-s_2}-\mu_0)\phi_0^{-1}\big\|_{L^1(\mu_0)} \\
&\le c_1 \mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1-r)} \|P^0_{r/2}-\mu_0\|^2_{L^{2p}(\mu_0)\to L^2(\mu_0)} \|P^0_{T-s_2}-\mu_0\|_{L^\theta(\mu_0)\to L^{\frac{p}{p-1}}(\mu_0)} \|\phi_0^{-1}\|_{L^\theta(\mu_0)} \\
&c_2 \mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1-r)-(\lambda_1-\lambda_0)(T-s_2)} r^{-\frac{(d+2)(p-1)}{2p}} [1\wedge(T-s_2)]^{-\frac{(d+2)[\theta-(\theta-1)p]}{2\theta p}}, \quad p \in [1, \theta/(\theta-1)].
\end{aligned}$$

Let $p_0 := \frac{3(d+2)}{2d+10}$. Since

$$\begin{aligned}
&\lim_{p\downarrow p_0} \frac{(d+2)(p-1)}{2p} = \frac{d-4}{6}, \\
&\lim_{\theta\uparrow 3} \frac{(d+2)[\theta-(\theta-1)p]}{2\theta p} = \frac{(d+2)(3-2p)}{6p} \in (0,1) \text{ if } p \in (p_0, 3/2),
\end{aligned}$$

for any $k > \frac{d-4}{6}$, there exists $\theta \in (1,3)$ and $p \in (p_0, \frac{\theta}{\theta-1})$ such that

$$\frac{(d+2)(p-1)}{2p} \le k, \quad \varepsilon := \frac{(d+2)[\theta-(\theta-1)p]}{2\theta p} < 1.$$

Thus,

$$|J_3(s_1,s_2)| \le c_2 \mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1-r)-(\lambda_1-\lambda_0)(T-s_2)} r^{-k} [1\wedge(T-s_2)]^{-\varepsilon}.$$

Since $J_1 = J_2 = 0$, this implies

$$\int_0^t \mathrm{d}s_1 \int_{s_1}^t |J_1+J_2+J_3|(s_1,s_2)\mathrm{d}s_2 \le \frac{c_3 \mathrm{e}^{(\lambda_m-\lambda_0)r} r^{-k}}{\lambda_m-\lambda_0}, \quad t \ge 1, r \in (0,1)$$

for some constant $c_3 > 0$. Substituting into (3.8) for $\nu = \nu_0$, we find a constant $c_4 > 0$ such that

$$\begin{aligned}
&\sup_{T\ge t} t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big| T<\tau\big] \\
&\le 2\sum_{m=1}^\infty \frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2} + \frac{c_4 r^{-k}}{t} \sum_{m=1}^\infty \frac{\mathrm{e}^{-(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2},
\end{aligned}$$

which implies the desired estimate since by (1.1),

$$\sum_{m=1}^\infty \frac{\mathrm{e}^{-(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2} \le c_5 \int_1^\infty s^{-\frac{1}{d}} \mathrm{e}^{-c_5 s^{\frac{2}{d}} r} \mathrm{d}r \le c_6 \big(r^{-\frac{d-4}{2}} + 1_{\{d=4\}} \log r^{-1}\big)$$

holds for some constants $c_5, c_6 > 0$ and all $r \in (0,1)$.

□

Lemma 3.3. *There exists a constant $c > 0$ such that for any $t > 0$ and nonnegative random variable $\xi \in \sigma(X_s : s \le t)$,*

$$\sup_{T\ge t} \mathbb{E}^\nu[\xi|T<\tau] \le c\mathbb{E}^\nu[\xi|t<\tau], \quad t \ge 1, \nu \in \mathscr{P}_0.$$

Proof. By the Markov property, (2.5) for $p = q = \infty$ and (2.4), we find constants $c_1, c_2 > 0$ such that

$$\mathbb{E}^\nu[\xi 1_{\{T<\tau\}}] = \mathbb{E}^\nu[\xi 1_{\{t<\tau\}} P^D_{T-t} 1(X_t)] \le c_1 \mathrm{e}^{-\lambda_0(T-t)} \mathbb{E}^\nu[\xi 1_{\{t<\tau\}}],$$
$$\mathbb{P}^\nu(T<\tau) \ge c_2 \mathbb{P}^\nu(t<\tau)\mathrm{e}^{-(T-t)\lambda_0}, \quad T \ge t \ge 1.$$

Then

$$\mathbb{E}^\nu[\xi|T<\tau] = \frac{\mathbb{E}^\nu[\xi 1_{\{T<\tau\}}]}{\mathbb{P}^\nu(T<\tau)} \le \frac{c_1 \mathbb{E}^\nu[\xi 1_{\{t<\tau\}}]}{c_2 \mathbb{P}^\nu(t<\tau)} = \frac{c_1}{c_2}\mathbb{E}^\nu[\xi|t<\tau].$$

□

Lemma 3.4. *Let $d \le 3$ and denote $\nu_0 = \frac{\phi_0}{\mu(\phi_0)}\mu$. For any $\varepsilon \in (\frac{d}{4} \vee \frac{d^2}{2d+4}, 1)$, there exists a constant $c > 0$ such that*

$$\sup_{T\ge t} \mathbb{E}^{\nu_0}\big[|\rho_{t,r}(y) - 1|^2 \big| T<\tau\big] \le c\phi_0^{-2}(y) t^{-1} r^{-\varepsilon}, \quad t \ge 1, r \in (0,1], y \in M^\circ.$$

Proof. By Lemma 3.3, it suffices to prove for $T = t$ replacing $T \ge t$. For fixed $y \in M^\circ$, let $f = p^0_r(\cdot, y) - 1$. We have

$$\rho_{t,r}(y) - 1 = \frac{1}{t}\int_0^t f(X_s)\mathrm{d}s.$$

Then

$$\mathbb{E}^{\nu_0}\big[|\rho_{t,r}(y) - 1|^2 1_{\{t<\tau\}}\big] = \frac{2}{t^2}\int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathbb{E}^{\nu_0}\big[1_{\{t<\tau\}} f(X_{s_1}) f(X_{s_2})\big]\mathrm{d}s_2. \tag{3.13}$$

By (3.3), $\mu_0(f) = 0$, and the symmetry of P^0_t in $L^2(\mu_0)$, we obtain

$$\begin{aligned} I &:= \mathrm{e}^{\lambda_0 t}\mathbb{E}^{\nu_0}\big[1_{\{t<\tau\}} f(X_{s_1}) f(X_{s_2})\big] = \mu(\phi_0)^{-1}\mu_0\big(P^0_{s_1}\{f P^0_{s_2-s_1}(f P^0_{t-s_2}\phi_0^{-1})\}\big) \\ &= \mu(\phi_0)^{-1}\mu_0\big(f P^0_{s_2-s_1}(f P^0_{t-s_2}\phi_0^{-1})\big) = \mu(\phi_0)^{-1}\mu_0\big(\{f P^0_{t-s_2}\phi_0^{-1}\} P^0_{s_2-s_1} f\big) \\ &= \mu(\phi_0)^{-1}\mu_0\big(\{f P^0_{t-s_2}\phi_0^{-1}\}\{P^0_{s_2-s_1} - \mu_0\} f\big). \end{aligned} \tag{3.14}$$

Taking $q \in (\frac{5}{2}, 3)$ so that $\varepsilon_1 := \frac{d+2}{2q} < 1$ for $d \le 3$ and $\|\phi_0^{-1}\|_{L^q(\mu_0)} < \infty$ due to (2.1), for any $p \in (1, 2]$ we deduce from this and (2.12) that

$$\begin{aligned} \mu(\phi_0) I &\le \|f\|_{L^p(\mu_0)} \|P^0_{t-s_2}\phi_0^{-1}\|_{L^\infty(\mu_0)} \|(P^0_{s_2-s_1} - \mu_0) f\|_{L^{\frac{p}{p-1}}(\mu_0)} \\ &\le \|f\|_{L^p(\mu_0)} \|P^0_{t-s_2}\|_{L^q(\mu_0)\to L^\infty(\mu_0)} \|\phi_0^{-1}\|_{L^q(\mu_0)} \|P^0_{s_2-s_1} - \mu_0\|_{L^2(\mu_0)\to L^{\frac{p}{p-1}}(\mu_0)} \|f\|_{L^2(\mu_0)} \\ &\le c_1 \|f\|_{L^p(\mu_0)} \|f\|_{L^2(\mu_0)} \{1 \wedge (t-s_2)\}^{-\varepsilon_1} \{1 \wedge (s_2 - s_1)\}^{-\frac{(d+2)(2-p)}{2p}} \mathrm{e}^{-(\lambda_1 - \lambda_0)(s_2 - s_1)} \end{aligned} \tag{3.15}$$

holds for some constants $c_1 > 0$. Since $f = p^0_r(\cdot, y) - 1$ and $\inf \phi_0^{-1} > 0$, by (2.5) and (2.6), we find constants $\beta_1, \beta_2 > 0$ such that

$$\begin{aligned} \|f\|_{L^p(\mu_0)} &\le 1 + \|p^0_r(\cdot, y)\|_{L^p(\mu_0)} \le 1 + \mathrm{e}^{r\lambda_0}\phi_0^{-1}(y)\|\phi_0^{-1} p^D_r(\cdot, y)\|_{L^p(\mu_0)} \\ &\le 1 + \beta_1 \phi_0^{-1}(y) \|\phi_0\|_\infty^{\frac{2-p}{p}} \|p^D_r(\cdot, y)\|_{L^p(\mu)} \le \beta_2 \phi_0^{-1}(y) r^{-\frac{d(p-1)}{2p}}, \quad r \in (0,1], p \in [1,2]. \end{aligned}$$

Combining this with (3.15) we find a constant $c_2 > 0$ such that

$$I \le c_2\phi_0^{-2}(y) r^{-\frac{d(p-1)}{2p}-\frac{d}{4}}\{1\wedge(t-s_2)\}^{-\varepsilon_1}\{1\wedge(s_2-s_1)\}^{-\frac{(d+2)(2-p)}{2p}}\mathrm{e}^{-(\lambda_1-\lambda_0)(s_2-s_1)},\quad p\in(1,2].$$

Letting $p_0 := 1\vee\frac{2(d+2)}{d+6}$ and taking $p > p_0$ such that

$$\varepsilon_2 := \frac{(d+2)(2-p)}{4p} \le \frac{5(2-p)}{4p} < 1,$$

we arrive at

$$I \le c_2\phi^{-2}(y) r^{-\frac{d(p-1)}{2p}-\frac{d}{4}}\{1\wedge(t-s_2)\}^{-\varepsilon_1}\{1\wedge(s_2-s_1)\}^{-\varepsilon_2}\mathrm{e}^{-(\lambda_1-\lambda_0)(s_2-s_1)}$$

for some constants $\varepsilon_1,\varepsilon_2\in(0,1)$. Combining this with (3.13), we obtain

$$\mathbb{E}^{\nu_0}\big[|\rho_{t,r}(y)-1|^2\big|t<\tau\big] \le c\phi_0^{-2}(y)t^{-1}r^{-\frac{d(p-1)}{2p}-\frac{d}{4}},\quad t\ge 1.$$

Noting that

$$\lim_{p\downarrow p_0}\Big\{\frac{d(p-1)}{2p}+\frac{d}{4}\Big\} = \frac{d}{4}\vee\frac{d^2}{2d+4} < 1 \text{ for } d\le 3,$$

for any $\varepsilon\in(\frac{d}{4}\vee\frac{d^2}{2d+4},1)$, there exists $p>p_0$ such that $\frac{d}{4}\vee\frac{d^2}{2d+4}\le\varepsilon$. Therefore, the proof is finished. □

Lemma 3.5. *Let $d\le 3$ and denote $\psi_m(t)=\frac{1}{t}\int_0^t(\phi_m\phi_0^{-1})(X_s)\mathrm{d}s$. Then there exists a constant $c>0$ such that for any $p\in[1,2]$,*

$$\sup_{T\ge t}\mathbb{E}^{\nu_0}\big[|\psi_m(t)|^{2p}\big|t<\tau\big] \le cm^{\frac{p(d+4)-d-8}{2d}}t^{-p},\quad t\ge 1, m\ge 1, r\in(0,1).$$

Proof. By Lemma 3.3, it suffices to prove for $T=t$ replacing $T\ge t$. By Hölder's inequality, we have

$$\begin{aligned}\mathbb{E}^{\nu_0}\big[|\psi_m(t)|^{2p}|T<\tau\big] &= \mathbb{E}^{\nu_0}\big[|\psi_m(t)|^{4-2p}|\psi_m(t)|^{4p-4}|T<\tau\big]\\ &\le \big\{\mathbb{E}^{\nu_0}\big[|\psi_m(t)|^2|T<\tau\big]\big\}^{2-p}\big\{\mathbb{E}^{\nu_0}\big[|\psi_m(t)|^4|T<\tau\big]\big\}^{p-1}.\end{aligned}$$

Combining this with (2.4), it suffices to find a constant $c>0$ such that

$$\mathbb{E}^{\nu_0}\big[|\psi_m(t)|^2 1_{\{t<\tau\}}\big] \le \frac{c\mathrm{e}^{-\lambda_0 t}}{tm^{\frac{2}{d}}},\quad t\ge 1, r\in(0,1), \tag{3.16}$$

$$\mathbb{E}^{\nu_0}\big[|\psi_m(t)|^4 1_{\{t<\tau\}}\big] \le c\sqrt{m}\,\mathrm{e}^{-\lambda_0 t}t^{-2},\quad t\ge 1, r\in(0,1). \tag{3.17}$$

(a) Proof of (3.16). Let $\hat\phi_m=\phi_m\phi_0^{-1}$. We have

$$\mathbb{E}^{\nu_0}\big[|\psi_m(t)|^2 1_{\{t<\tau\}}\big] = \frac{2}{t^2}\int_0^t \mathrm{d}s_1\int_{s_1}^t \mathbb{E}^{\nu_0}\big[1_{\{t<\tau\}}\hat\phi_m(X_{s_1})\hat\phi_m(X_{s_2})\big]\mathrm{d}s_2. \tag{3.18}$$

By (2.7), (3.3), $\mu_0(|\hat\phi_m|^2)=1$, and the symmetry of P_t^0 in $L^2(\mu_0)$, we find a constant $c_1>0$ such that

$$\begin{aligned}
&\mathrm{e}^{\lambda_0 t}\mathbb{E}^{\nu_0}\big[1_{\{T<\tau\}}\hat\phi_m(X_{s_1})\hat\phi_m(X_{s_2})\big]=\nu_0\big(\phi_0P_{s_1}^0\{\hat\phi_mP_{s_2-s_1}^0(\hat\phi_mP_{t-s_2}^0\phi_0^{-1})\}\big)\\
&=\frac{1}{\mu(\phi_0)}\mu_0\big(\hat\phi_mP_{s_2-s_1}^0(\hat\phi_mP_{t-s_2}^0\phi_0^{-1})\big)=\frac{\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}}{\mu(\phi_0)}\mu_0\big(|\hat\phi_m|^2P_{t-s_2}^0\phi_0^{-1}\big)\\
&\le c_1\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\|P_{t-s_2}\|_{L^p(\mu_0)\to L^\infty(\mu_0)}\|\phi_0^{-1}\|_{L^p(\mu_0)},\quad p>1.
\end{aligned}$$

Since $d\le 3$, we may take $p\in(1,3)$ such that $\varepsilon:=\frac{d+2}{2q}<1$ and $\|\phi_0^{-1}\|_{L^p(\mu_0)}<\infty$ due to (2.1), so that this and (2.12) imply

$$\mathrm{e}^{\lambda_0 t}\mathbb{E}^{\nu_0}\big[1_{\{t<\tau\}}\hat\phi_m(X_{s_1})\hat\phi_m(X_{s_2})\big]\le c_2\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\{1\wedge(t-s_2)\}^{-\varepsilon}$$

for some constant $c_3>0$. Therefore, (3.16) follows from (3.18) and (1.1).

(b) Proof of (3.17). For any $s>0$ we have

$$\begin{aligned}
&s^4\mathbb{E}^{\nu_0}\big[|\psi_m(s)|^4 1_{\{s<\tau\}}\big]\\
&=24\int_0^s\mathrm{d}s_1\int_{s_1}^s\mathrm{d}s_2\int_{s_2}^s\mathrm{d}s_3\int_{s_3}^s\mathbb{E}^{\nu_0}\big[1_{\{s<\tau\}}\hat\phi_m(X_{s_1})\hat\phi_m(X_{s_2})\hat\phi_m(X_{s_3})\hat\phi_m(X_{s_4})\big]\mathrm{d}s_4\\
&=24\int_0^s\mathrm{d}s_1\int_{s_1}^s\mathrm{d}s_2\int_{s_2}^s\mathrm{d}s_3\int_{s_3}^s\mathbb{E}^{\nu_0}\big[1_{\{s_3<\tau\}}\hat\phi_m(X_{s_1})\hat\phi_m(X_{s_2})g_s(s_3,s_4)\big]\mathrm{d}s_4,
\end{aligned}\tag{3.19}$$

where

$$g_s(s_3,s_4):=\mathbb{E}^{\nu_0}\big[1_{\{s<\tau\}}\hat\phi_m(X_{s_3})\hat\phi_m(X_{s_4})\big|X_r:r\le s_3\big].$$

By (3.3) and the Markov property,

$$\begin{aligned}
g_s(s_3,s_4)&=\hat\phi_m(X_{s_3})\mathbb{E}^{X_{s_3}}\big[1_{\{s-s_3<\tau\}}\hat\phi_m(X_{s_4-s_3})\big]\\
&=\mathrm{e}^{-\lambda_0(s-s_3)}\big\{\hat\phi_m\phi_0P_{s_4-s_3}^0(\hat\phi_mP_{s-s_4}^0\phi_0^{-1})\big\}(X_{s_3}),\quad 0<s_3<s_4\le s.
\end{aligned}\tag{3.20}$$

By Fubini's theorem and Schwarz's inequality, (3.19) and (3.20) yield

$$\begin{aligned}
&I(s):=s^4\mathrm{e}^{\lambda_0 s}\mathbb{E}^{\nu_0}\big[|\psi_m(s)|^4 1_{\{s<\tau\}}\big]\\
&=12\mathrm{e}^{\lambda_0 s}\int_0^s\mathrm{d}r_1\int_{r_1}^s\mathbb{E}^{\nu_0}\bigg[1_{\{r_1<\tau\}}g_s(r_1,r_2)\bigg|\int_0^{r_1}\hat\phi_m(X_r)\mathrm{d}r\bigg|^2\bigg]\mathrm{d}r_2\\
&\le 12\sup_{r\in[0,s]}\sqrt{I(r)}\int_0^s\mathrm{d}r_1\int_{r_1}^s\Big\{\mathrm{e}^{2\lambda_0 s-\lambda_0 r_1}\mathbb{E}^{\nu_0}\big[1_{\{r_1<\tau\}}g_s(r_1,r_2)^2\big]\Big\}^{\frac12}\mathrm{d}r_2.
\end{aligned}$$

Consequently,

$$I(t)\le\sup_{s\in[0,t]}I(s)\le\bigg(12\sup_{s\in[0,t]}\int_0^s\mathrm{d}r_1\int_{r_1}^s\Big\{\mathrm{e}^{\lambda_0(2s-r_1)}\mathbb{E}^{\nu_0}\big[1_{\{r_1<\tau\}}g_s(r_1,r_2)^2\big]\Big\}^{\frac12}\mathrm{d}r_2\bigg)^2.\tag{3.21}$$

On the other hand, by the definition of ν_0, (3.3), (3.20) and that μ_0 is P_t^0-invariant, we obtain

$$
\begin{aligned}
&\mathbb{E}^{\nu_0}\big[1_{\{r_1<\tau\}}|g_s(r_1,r_2)|^2\big]\\
&\le \frac{\mathrm{e}^{-2\lambda_0(s-r_1)-\lambda_0 r_1}}{\mu(\phi_0)}\mu_0\big(P_{r_1}^0\{\phi_0^{-1}|\hat\phi_m\phi_0P_{r_2-r_1}^0(\hat\phi_mP_{s-r_2}^0\phi_0^{-1})|^2\}\big)\\
&= \frac{\mathrm{e}^{-\lambda_0(2s-r_1)}}{\mu(\phi_0)}\mu_0\big(\phi_0|\hat\phi_mP_{r_2-r_1}^0(\hat\phi_mP_{s-r_2}^0\phi_0^{-1})|^2\big)\\
&\le \frac{2\mathrm{e}^{-\lambda_0(2s-r_1)}}{\mu(\phi_0)}\mu_0\big(\phi_0\{|\hat\phi_m(P_{r_2-r_1}^0\hat\phi_m)\mu(\phi_0)|^2+|\hat\phi_mP_{r_2-r_1}^0(\hat\phi_m[P_{s-r_2}^0-\mu_0]\phi_0^{-1})|^2\}\big).
\end{aligned}
\tag{3.22}
$$

Then, by (3.20), (2.7), (3.3), $\mu_0(|\hat\phi_m|^2)=1$, and noting that μ_0 is P_t^0-invariant, we find a constant $c_1>0$ such that

$$
\begin{aligned}
\mathbb{E}^{\nu_0}\big[1_{\{r_1<\tau\}}|g_s(r_1,r_2)|^2\big] &\le 2\mathrm{e}^{-\lambda_0(2s-r_1)-(\lambda_m-\lambda_0)(r_2-r_1)}\|\phi_m\|_\infty\|\phi_0\|_\infty\mu_0\big(|\hat\phi_m||P_{(r_r-r_1)/2}\hat\phi_m|^2\big)\\
&\quad+\frac{2\mathrm{e}^{-\lambda_0(2s-r_1)}\|\phi_m\|_\infty}{\mu(\phi_0)}\mu_0\big(|\hat\phi_m|\cdot\big|P_{r_2-r_1}^0(\hat\phi_m(P_{s-r_2}^0-\mu_0)\phi_0^{-1})\big|^2\big)\\
&\le c_1\mathrm{e}^{-\lambda_0(2s-r_1)}\Big\{\mathrm{e}^{-2(\lambda_m-\lambda_0)(r_2-r_1)}\|\phi_m\|_\infty\|P_{(r_2-r_1)/2}-\mu_0\|_{L^2(\mu_0)\to L^4(\mu_0)}^2\\
&\quad+\|\phi_m\|_\infty\|P_{r_2-r_1}^0(\hat\phi_m[P_{s-r_2}^0-\mu_0]\phi_0^{-1})\|_{L^4(\mu_0)}^2\Big\}.
\end{aligned}
$$

By (1.1), (2.12), $\|\hat\phi_m\|_{L^2(\mu_0)}=1$, $\|\phi_0^{-1}\|_{L^q(\mu_0)}<\infty$ and $\varepsilon:=\frac{d+2}{8}\vee\frac{d+2}{2q}<1$ for $q\in(\frac52,3)$ due to (2.1) and $d\le 3$, we find constants $c_2>0$ such that

$$
\|\phi_m\|_\infty\|P_{(r_2-r_1)/2}-\mu_0\|_{L^2(\mu_0)\to L^4(\mu_0)}^2\le c_2\sqrt{m}\{1\wedge(r_2-r_1)\}^{-\frac d4},
$$

and

$$
\begin{aligned}
&\|\phi_m\|_\infty\|P_{r_2-r_1}^0(\hat\phi_m[P_{s-r_2}^0-\mu_0]\phi_0^{-1})\|_{L^4(\mu_0)}^2\\
&\le\|\phi_m\|_\infty\|P_{r_2-r_1}^0\|_{L^2(\mu_0)\to L^4(\mu_0)}^2\|\hat\phi_m\|_{L^2(\mu_0)}^2\|(P_{s-r_2}^0-\mu_0)\phi_0^{-1}\|_{L^\infty(\mu_0)}^2\\
&\le\|\phi_m\|_\infty\|P_{r_2-r_1}^0\|_{L^2(\mu_0)\to L^4(\mu_0)}^2\|P_{s-r_2}^0-\mu_0\|_{L^q(\mu_0)\to L^\infty(\mu_0)}^2\|\phi_0^{-1}\|_{L^q(\mu_0)}^2\\
&\le c_2\sqrt{m}\,\mathrm{e}^{-2(\lambda_1-\lambda_0)(s-r_2)}\{1\wedge(r_2-r_1)\}^{-2\varepsilon}\{1\wedge(s-r_2)\}^{-2\varepsilon}.
\end{aligned}
$$

Therefore, there exist constants $c_3>0$ and $\varepsilon\in(0,1)$ such that

$$
\begin{aligned}
\mathbb{E}^{\nu_0}\big[1_{\{r_1<\tau\}}|g_s(r_1,r_2)|^2\big]&\le c_3\mathrm{e}^{-\lambda_0(2s-r_1)-(\lambda_m-\lambda_0)(r_2-r_1)}\sqrt{m}\{1\wedge(r_2-r_1)\}^{-\frac d4}\\
&\quad+c_3\sqrt{m}\,\mathrm{e}^{-\lambda_0(2s-r_1)-2(\lambda_1-\lambda_0)(s-r_2)}\{1\wedge(r_2-r_1)\}^{-2\varepsilon}\{1\wedge(t-r_2)\}^{-2\varepsilon}.
\end{aligned}
$$

Combining this with (3.21) and the definition of $I(t)$, we prove (3.17) for some constant $c>0$, and hence finish the proof. □

Lemma 3.6. *Let $d\le 3$. Then for any $p\in(1,\frac{3d+16}{5d+8}\wedge\frac{d+2}{d+1})$, there exists a constant $c>0$ such that*

$$
\sup_{r>0,T\ge t}\mathbb{E}^{\nu_0}\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^{2p})|T<\tau\big]\le ct^{-p},\quad t\ge 1.
$$

Proof. By Lemma 3.3, it suffices to prove for $T = t$ replacing $T \geq t$. Let $p \in (1, \frac{3d+16}{5d+8} \wedge \frac{d+2}{d+1})$. Remark that $p > 1$ implies

$$\frac{p}{2p-1} \in (0,1), \tag{3.23}$$

while $p < \frac{3d+16}{5d+8} \wedge \frac{d+2}{d+1}$ implies

$$\frac{(d+2)(2p-2)}{4} + \frac{d(p-1)}{2} + \Big(\frac{p(d+4)+d}{4} - 2\Big)^+ < 1,$$

hence there exists $\varepsilon \in (0,1)$ such that

$$\frac{(d+2)(2p-2+\varepsilon)}{4} + \frac{d(p-1)}{2} + \Big(\frac{p(d+4)+d}{4} - 2\Big)^+ < 1. \tag{3.24}$$

By (2.21), $L_0^{-1} = -\int_0^\infty P_s^0 \mathrm{d}s$, and applying Hölder's inequality, we find a constant $c_1, c_2 > 0$ such that

$$\begin{aligned}
&\int_M \big|\nabla L_0^{-1}(\rho_{t,r}-1)\big|^{2p} \mathrm{d}\mu_0 \leq \int_M \bigg(\int_0^\infty \big|\nabla P_s^0(\rho_{t,r}-1)\big| \mathrm{d}s\bigg)^{2p} \mathrm{d}\mu_0 \\
&\leq c_1 \int_M \bigg(\int_0^\infty \frac{1}{\sqrt{1\wedge s}} \big\{P_{\frac{s}{4}}^0 \big|P_{\frac{3s}{4}}^0(\rho_{t,r}-1)\big|^p\big\}^{\frac{1}{p}} \mathrm{d}s\bigg)^{2p} \phi_0^{-\varepsilon} \mathrm{d}\mu_0 \\
&\leq c_1 \bigg(\int_0^\infty (1\wedge s)^{-\frac{p}{2p-1}} \mathrm{e}^{-\frac{2p\theta s}{2p-1}} \mathrm{d}s\bigg)^{\frac{2p-1}{2p}} \\
&\quad \times \int_0^\infty \mathrm{e}^{\theta s} \mu_0\big(\phi_0^{-\varepsilon}\big\{P_{\frac{s}{4}}^0 |P_{\frac{3s}{4}}^0(\rho_{t,r}-1)|^p\big\}^2\big) \mathrm{d}s, \quad \theta > 0.
\end{aligned} \tag{3.25}$$

By (3.23), we obtain

$$\int_0^\infty (1\wedge s)^{-\frac{p}{2p-1}} \mathrm{e}^{-\frac{2p\theta s}{2p-1}} \mathrm{d}s < \infty, \quad \theta > 0. \tag{3.26}$$

Moreover, since $\|\phi_0^{-\varepsilon}\|_{L^{2\varepsilon^{-1}}(\mu_0)} = 1, \mu_0(\rho_{t,r}-1) = 0$, and P_t^0 is contractive in $L^p(\mu_0)$ for $p \geq 1$, by (2.12) and Hölder's inequality, we find a constant $c_2 > 0$ such that

$$\begin{aligned}
&\mu_0\big(\phi_0^{-\varepsilon}\big\{P_{\frac{s}{4}}^0 |P_{\frac{3s}{4}}^0(\rho_{t,r}-1)|^p\big\}^2\big) \leq \big\|P_{\frac{s}{4}}^0 |P_{\frac{3s}{4}}^0(\rho_{t,r}-1)|^p\big\|_{L^{\frac{4}{2-\varepsilon}}(\mu_0)}^2 \|\phi_0^{-\varepsilon}\|_{L^{2\varepsilon^{-1}}(\mu_0)} \\
&\leq \|P_{\frac{s}{4}}^0\|_{L^{\frac{4}{2-\varepsilon}}(\mu_0)}^2 \big\|(P_{\frac{s}{2}}^0 - \mu_0)(P_{\frac{s}{4}}^0 \rho_{t,r} - 1)\big\|_{L^{\frac{4p}{2-\varepsilon}}(\mu_0)}^{2p} \\
&\leq \|P_{\frac{s}{2}}^0 - \mu_0\|_{L^2(\mu_0)\to L^{\frac{4p}{2-\varepsilon}}(\mu_0)}^{2p} \|P_{\frac{s}{4}}^0 \rho_{t,r} - 1\|_{L^2(\mu_0)}^{2p} \\
&\leq c_2 (1\wedge s)^{-\frac{(d+2)(2p-2+\varepsilon)}{4}} \mathrm{e}^{-(\lambda_1-\lambda_0)ps} \|P_{\frac{s}{4}}^0 \rho_{t,r} - 1\|_{L^2(\mu_0)}^{2p}, \quad s > 0.
\end{aligned}$$

Combining this with (3.26), we find a function $c: (0,\infty) \to (0,\infty)$ such that

$$\begin{aligned}
&\mathbb{E}^{\nu_0}\big[1_{\{t<\tau\}} \mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^{2p})\big] \\
&\leq c(\theta) \int_0^\infty \mathrm{e}^{\theta s} (1\wedge s)^{-\frac{(d+2)(2p-2+\varepsilon)}{4}} \mathrm{e}^{-(\lambda_1-\lambda_0)ps} \mathbb{E}^{\nu_0}\big[1_{\{t<\tau\}} \|P_{\frac{s}{4}}^0 \rho_{t,r} - 1\|_{L^2(\mu_0)}^{2p}\big] \mathrm{d}s, \quad \theta > 0.
\end{aligned} \tag{3.27}$$

By (2.7), (3.2) and Hölder's inequality, we obtain

$$\begin{aligned}\|P^0_{\frac{s}{4}}\rho_{t,r}-1\|^{2p}_{L^2(\mu_0)} &= \Big(\sum_{m=1}^\infty \mathrm{e}^{-(\lambda_m-\lambda_0)(2r+s/2)}|\psi_m(t)|^2\Big)^p\\ &\le \Big(\sum_{m=1}^\infty \mathrm{e}^{-(\lambda_m-\lambda_0)(2r+s/2)}\Big)^{p-1}\sum_{m=1}^\infty \mathrm{e}^{-(\lambda_m-\lambda_0)(2r+s/2)}|\psi_m(t)|^{2p}.\end{aligned}$$

Noting that (1.1) implies

$$\sum_{m=1}^\infty \mathrm{e}^{-(\lambda_m-\lambda_0)(2r+s/2)} \le \alpha_1\int_1^\infty \mathrm{e}^{-\alpha_2(2r+s/2)t^{\frac{2}{d}}}\mathrm{d}t \le \alpha_3(1\wedge s)^{-\frac{d}{2}}$$

for some constants $\alpha_1,\alpha_2,\alpha_3>0$, we derive

$$\mathbb{E}^{\nu_0}\big[\|P^0_{\frac{s}{4}}\rho_{t,r}-1\|^{2p}_{L^2(\mu_0)}\big|t<\tau\big] \le c_3(1\wedge s)^{-\frac{d(p-1)}{2}}\sum_{m=1}^\infty \mathrm{e}^{-(\lambda_m-\lambda_0)(2r+s/2)}\mathbb{E}^{\nu_0}\big[|\psi_m(t)|^{2p}\big|t<\tau\big]$$

for some constant $c_3>0$. Combining this with Lemma 3.5, (1.1), we find constants $c_4,c_5,c_6,c_7>0$ such that

$$\begin{aligned}&\mathbb{E}^{\nu_0}\big[\|P^0_{\frac{s}{4}}\rho_{t,r}-1\|^{2p}_{L^2(\mu_0)}\big|t<\tau\big] \le c_4t^{-p}(1\wedge s)^{-\frac{d(p-1)}{2}}\int_1^\infty \mathrm{e}^{-c_5su^{\frac{2}{d}}}u^{\frac{p(d+4)-d-8}{2d}}\mathrm{d}u\\ &\le c_6t^{-p}(1\wedge s)^{-\frac{d(p-1)}{2}}s^{2-\frac{p(d+4)+d}{4}}\int_s^\infty t^{\frac{p(d+4)+d}{4}-3}\mathrm{e}^{-t}\mathrm{d}t\\ &\le c_7t^{-p}(1\wedge s)^{-\frac{d(p-1)}{2}-(\frac{p(d+4)+d}{4}-2)^+}\log(2+s^{-1}),\end{aligned}$$

where the term $\log(2+s^{-1})$ comes when $\frac{p(d+4)+d}{4}-3=-1$. This together with (3.24) and (3.27) for $\theta\in(0,\lambda_1-\lambda_0)$ implies the desired estimate. □

Lemma 3.7. *Let $d\le 3$. If $r_t=t^{-\alpha}$ for some $\alpha\in(1,\frac{4}{d}\wedge\frac{2d+4}{d^2})$, then $\rho_{t,r_t,r_t}:=(1-r_t)\rho_{t,r_t}+r_t$ satisfies*

$$\lim_{t\to\infty}\sup_{T\ge t}\mathbb{E}^{\nu_0}\big[\mu_0(|\mathscr{M}(\rho_{t,r_t,r_t},1)^{-1}-1|^q)\big|T<\tau\big]=0,\quad q\ge 1.$$

Proof. By Lemma 3.3, it suffices to prove for $T=t$ replacing $T\ge t$. By the same reason leading to (3.16) in [24], for any $\eta\in(0,1), y\in M$, we have

$$\mathbb{E}^{\nu_0}\big[|\mathscr{M}(\rho_{t,r_t,r_t}(y),1)^{-1}-1|^q\big|t<\tau\big] \le \Big|\frac{1}{\sqrt{1-\eta}}-\frac{2}{2+\eta}\Big|^q+\mathbb{P}^{\nu_0}\big(|\rho_{t,r_t}(y)-1|>\eta|t<\tau\big).$$

Combining this with Lemma 3.4 we find constants $c>0$ and $\varepsilon\in(0,\alpha^{-1})$ such that

$$\mathbb{E}^{\nu_0}\big[|\mathscr{M}(\rho_{t,r_t,r_t}(y),1)^{-1}-1|^q\big|t<\tau\big] \le \Big|\frac{1}{\sqrt{1-\eta}}-\frac{2}{2+\eta}\Big|^q+c\eta^{-1}\phi_0(y)^{-2}t^{-1+\alpha\varepsilon}.$$

Since $\mu_0(\phi_0^{-2})=1$, we obtain

$$\mathbb{E}^{\nu_0}\big[\mu_0(|\mathscr{M}(\rho_{t,r_t,r_t},1)^{-1}-1|^q)\big|t<\tau\big] \le \Big|\frac{1}{\sqrt{1-\eta}}-\frac{2}{2+\eta}\Big|^q+c\eta^{-1}t^{-1+\alpha\varepsilon},\quad \eta\in(0,1),t\ge 1.$$

Noting that $\alpha\varepsilon<1$, by letting first $t\to\infty$ then $\eta\to 0$, we finish the proof. □

Lemma 3.8. *Let $\mu_{t,r,r} = (1+\rho_{t,r,r})\mu_0$, where $\rho_{t,r,r} := (1-r)\rho_{t,r} + r$ for $r \in (0,1]$. Assume that $\nu = h\mu$ with $h\phi_0^{-1} \in L^p(\mu_0)$ for some $p > 1$. Then there exists a constant $c > 0$ such that*

$$\sup_{T\ge t} \mathbb{E}^\nu\big[\mathbb{W}_2(\mu_{t,r,r},\mu_t)^2 \big| T<\tau\big] \le cr, \quad t>0, r\in(0,1].$$

Proof. By Lemma 3.3, it suffices to prove for $T = t$ replacing $T \ge t$. Firstly, letting D be the diameter of M, we have

$$\begin{aligned} \mathbb{W}_2(\mu,\nu)^2 &:= \inf_{\pi\in\mathscr{C}(\mu,\nu)} \int_{M\times M} \rho(x,y)^2 \pi(\mathrm{d}x,\mathrm{d}y) \\ &\le D^2 \inf_{\pi\in\mathscr{C}(\mu,\nu)} \pi(\{(x,y): x\neq y\}) = \frac{1}{2} D^2 \|\mu-\nu\|_{var}^2, \end{aligned} \tag{3.28}$$

where $\|\mu-\nu\|_{var} := \sup_{\|f\|_\infty\le 1} |\mu(f)-\nu(f)|$ is the total variation norm. Then

$$\mathbb{W}_2(\mu_{t,r,r},\mu_{t,r})^2 \le D^2\|\mu_{t,r,r}-\mu_{t,r}\|_{var} = D^2\mu_0(|\rho_{t,r,r}-\rho_{t,r}|) \le 2D^2 r, \quad r\in(0,1]. \tag{3.29}$$

Next, by the definition of $\mu_{t,r}$, we have

$$\pi(\mathrm{d}x,\mathrm{d}y) := \mu_t(\mathrm{d}x) P_r^0(x,\mathrm{d}y) \in \mathscr{C}(\mu_t,\mu_{t,r}),$$

where $P_r^0(x,\cdot)$ is the distribution of X_r^0 starting at x. So,

$$\mathbb{W}_2(\mu_t,\mu_{t,r})^2 \le \int_M \mathbb{E}^x[\rho(x,X_r^0)^2]\mu_t(\mathrm{d}x). \tag{3.30}$$

Moreover, by Itô's formula and $L_0 = L + 2\nabla\log\phi_0$, we find a constant $c_1 > 0$ such that

$$\mathrm{d}\rho(x,X_r^0)^2 = L_0\rho(x,\cdot)^2(X_r^0)\mathrm{d}r + \mathrm{d}M_r \le \big\{c_1 + c_1\phi_0^{-1}(X_r^0)\big\}\mathrm{d}r + \mathrm{d}M_r$$

holds for some martingale M_r. Combining this with (2.17), and noting that $\log(1+\phi_0^{-1}) \ge \log(1+\|\phi_0\|_\infty^{-1}) > 0$, we find a constant $c_2 > 0$ such that

$$\begin{aligned} \mathbb{W}_2(\mu_t,\mu_{t,r})^2 &\le c_1 r + c_1\int_M \bigg(\mathbb{E}^x\int_0^r \phi_0^{-1}(X_s^0)\mathrm{d}s\bigg)\mu_t(\mathrm{d}x) \\ &\le c_2 r\mu_t(\log(1+\phi_0^{-1})) = \frac{c_2 r}{t}\int_0^t \log\{1+\phi_0^{-1}(X_s)\}\mathrm{d}s, \quad r\in(0,1]. \end{aligned}$$

Combining this with (3.29), (3.3), $\|P_t^0\|_{L^p(\mu_0)} = 1$ for $t \ge 0$ and $p \ge 1$, and noting that

$$\inf_{t\ge 0} \mu_0(h\phi_0^{-1}P_t^0\phi_0^{-1}) > 0,$$

we find constants $c_3, c_4 > 0$ such that

$$\begin{aligned} \mathbb{E}^\nu[\mathbb{W}_2(\mu_{t,r,r},\mu_t)^2|t<\tau] &= \frac{\mathbb{E}^\nu[1_{\{t<\tau\}}\mathbb{W}_2(\mu_{t,r,r},\mu_t)^2]}{\mathbb{P}^\nu(t<\tau)} \\ &\le \frac{c_3 r}{t\mu_0(h\phi_0^{-1}P_t^0\phi_0^{-1})}\int_0^t \mu_0(h\phi_0^{-1}P_s^0\log\{1+\phi_0^{-1}\})\mathrm{d}s \\ &\le c_3 r\|h\phi_0^{-1}\|_{L^p(\mu_0)}\|\log(1+\phi_0^{-1})\|_{L^{\frac{p}{p-1}}(\mu_0)} \le c_4 r, \quad r\in(0,1]. \end{aligned} \tag{3.31}$$

Combining this with (3.29) we finish the proof. $\square$

We are now ready to prove the main result in this section.

Proof of Proposition 3.1(1). Since the upper bound is infinite for $d \geq 4$, it suffices to consider $d \leq 3$.

(a) We first assume that $\nu = h\mu$ with $h \leq C\phi_0$ for some constant $C > 0$. Let $\mu_{t,r_t,r_t} = \{(1-r_t)\rho_{t,r_t} + r_t\}\mu_0$ with $r_t = t^{-\alpha}$ for some $\alpha \in (1, \frac{4}{d} \wedge \frac{2d+4}{d^2})$. By Lemma 3.8 and the triangle inequality of $\mathbb{W}_2$, there exists a constant $c_1 > 0$ such that for any $t \geq 1$,

$$\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t, \mu_0)^2 \big| t < \tau\big] \leq (1+\varepsilon)\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_{t,r_t,r_t}, \mu_0)^2 \big| t < \tau\big] + c_1(1+\varepsilon^{-1})t^{-\alpha}, \quad \varepsilon > 0. \tag{3.32}$$

On the other hand, (3.1) implies

$$\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_{t,r_t,r_t}, \mu_0)^2 \big| t < \tau\big] \leq E^\nu\bigg[\int_M \frac{|\nabla L_0^{-1}(\rho_{t,r_t} - 1)|^2}{\mathscr{M}(\rho_{t,r_t,r_t}, 1)} \mathrm{d}\mu_0 \bigg| t < \tau\bigg] \leq I_1 + I_2, \tag{3.33}$$

where

$$\begin{aligned} I_1 &:= \mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r_t} - 1)|^2)\big| t < \tau\big], \\ I_2 &:= \mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r_t} - 1)|^2 |\mathscr{M}(\rho_{t,r_t,r_t}, 1)^{-1} - 1|)\big| t < \tau\big]. \end{aligned}$$

By Lemma 3.2(1), we have

$$\limsup_{t\to\infty} tI_1 \leq \sum_{m=1}^\infty \frac{2}{(\lambda_m - \lambda_0)^2},$$

while by Lemma 3.6 with $p \in (1, \frac{3d+16}{5d+8} \wedge \frac{d+2}{d+1})$, Lemma 3.7 for $q = \frac{p}{p-1}$, and (3.12), we derive

$$\begin{aligned} &\limsup_{t\to\infty} tI_2 \\ &\leq \limsup_{t\to\infty} t\big(\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r_t} - 1)|^{2p})\big| t < \tau\big]\big)^{\frac{1}{p}} \big(\mathbb{E}^\nu\big[\mu_0(|\mathscr{M}(\rho_{t,r_t,r_t}, 1)^{-1} - 1|^q)\big| t < \tau\big]\big)^{\frac{1}{q}} \\ &= 0. \end{aligned}$$

Combining these with (3.33) and (3.32) where $\alpha > 1$, we prove (1.2).

(b) In general, for any $t \geq 2$ and $\varepsilon \in (0,1)$, we consider

$$\mu_t^\varepsilon := \frac{1}{t-\varepsilon}\int_\varepsilon^t \delta_{X_s} \mathrm{d}s.$$

By (3.28), we find a constant $c_1 > 0$ such that

$$\begin{aligned} &\mathbb{W}_2(\mu_t^\varepsilon, \mu_t)^2 \leq D^2 \|\mu_t - \mu_t^\varepsilon\|_{var} \\ &\leq D^2 \int_\varepsilon^t \Big(\frac{1}{t-\varepsilon} - \frac{1}{t}\Big)\mathrm{d}s + \frac{D^2}{t}\int_0^\varepsilon \mathrm{d}s \leq c_1 \varepsilon t^{-1}, \quad t \geq 2, \varepsilon \in (0,1). \end{aligned} \tag{3.34}$$

On the other hand, by the Markov property we obtain

$$\mathbb{E}^\nu\big[1_{\{t<\tau\}}\mathbb{W}_2(\mu_t^\varepsilon, \mu_0)^2\big] = \mathbb{E}^\nu\big[1_{\{\varepsilon<\tau\}}\mathbb{E}^{X_\varepsilon}(1_{\{t-\varepsilon<\tau\}}\mathbb{W}_2(\mu_{t-\varepsilon}, \mu_0)^2)\big]$$

$$= \mathbb{P}^\nu(\varepsilon < \tau)\mathbb{E}^{\nu_\varepsilon}\big[1_{\{t-\varepsilon<\tau\}}\mathbb{W}_2(\mu_{t-\varepsilon},\mu_0)^2\big]$$
$$= \mathbb{P}^{\nu_\varepsilon}(t-\varepsilon < \tau)\mathbb{P}^\nu(\varepsilon < \tau)\mathbb{E}^{\nu_\varepsilon}\big[\mathbb{W}_2(\mu_{t-\varepsilon},\mu_0)^2\big|t-\varepsilon < \tau\big],$$

where $\nu_\varepsilon = h_\varepsilon \mu$ with

$$h_\varepsilon(y) := \frac{1}{\mathbb{P}^\nu(\varepsilon < \tau)} \int_M p_\varepsilon^D(x,y)\nu(\mathrm{d}x) \le c(\varepsilon,\nu)\phi_0(y)$$

for some constant $c(\varepsilon,\nu) > 0$. Moreover, by (2.2), (2.4) and $\nu_\varepsilon = h_\varepsilon\mu$, we have

$$\lim_{t\to\infty} \frac{\mathbb{P}^{\nu_\varepsilon}(t-\varepsilon<\tau)\mathbb{P}^\nu(\varepsilon<\tau)}{\mathbb{P}^\nu(t<\tau)} = 1.$$

So, (a) implies

$$\limsup_{t\to\infty}\Big\{t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t^\varepsilon,\mu_0)^2\big|t<\tau\big]\Big\}$$
$$= \limsup_{t\to\infty} \frac{\mathbb{P}^{\nu_\varepsilon}(t-\varepsilon<\tau)\mathbb{P}^\nu(\varepsilon<\tau)}{\mathbb{P}^\nu(t<\tau)} \Big\{t\mathbb{E}^{\nu_\varepsilon}\big[\mathbb{W}_2(\mu_{t-\varepsilon},\mu_0)^2\big|t-\varepsilon<\tau\big]\Big\}$$
$$\le \sum_{m=1}^\infty \frac{2}{(\lambda_m-\lambda_0)^2}.$$

Combining this with (3.34), we arrive at

$$\limsup_{t\to\infty}\Big\{t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|t<\tau\big]\Big\}$$
$$\le (1+\varepsilon^{\frac12})\limsup_{t\to\infty}\Big\{t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t^\varepsilon,\mu_0)^2\big|t<\tau\big]\Big\} + c_1\varepsilon(1+\varepsilon^{-\frac12})$$
$$\le (1+\varepsilon^{\frac12})\sum_{m=1}^\infty \frac{2}{(\lambda_m-\lambda_0)^2} + c_1\varepsilon(1+\varepsilon^{-\frac12}), \quad \varepsilon\in(0,1).$$

By letting $\varepsilon\to 0$, we derive (1.2). □

Proof of Proposition 3.1(2)-(3). Let $d\ge 4$. By (3.34), it suffices to prove the desired estimates for μ_t^1 replacing μ_t. Therefore, we may and do assume $\nu = h\mu$ with $\|h\phi_0^{-1}\|_\infty < \infty$. By Lemma 3.2(2) and the following inequality due to [11, Theorem 2] for $p=2$:

$$\mathbb{W}_2(f\mu_0,\mu_0)^2 \le 4\mu_0(|\nabla L_0^{-1}(f-1)|^2), \quad f\mu_0\in\mathscr{P}_0,$$

for any $k > \frac{d-4}{6}$, we find a constant $c>0$ such that

$$t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_{t,r,r},\mu_0)^2\big|T<\tau\big] \le c\big\{r^{-\frac{d-4}{2}} + 1_{\{d=4\}}\log r^{-1} + t^{-1}r^{-k}\big\}, \quad T\ge t\ge 1, r\in(0,1).$$

Combining this with Lemma 3.8, we find a function $c: (\frac{d-4}{6},\infty)\to(0,\infty)$ such that

$$\begin{aligned}&\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big] \le c(k)\big\{t^{-1}r^{-\frac{d-4}{2}} + t^{-1}1_{\{d=4\}}\log r^{-1} + t^{-2}r^{-k} + r\big\},\\ &\text{for } T\ge t\ge 1, r\in(0,1), k>\frac{d-4}{6}.\end{aligned} \tag{3.35}$$

(a) Let $d=4$. We take for instance $k=\frac12$ and $r=t^{-1}$ for $t>1$, such that (3.35) implies (1.4) for some constant $c>0$.

(b) When $d\ge 5$, we take for instance $k=\frac{d-4}{4}$ and $r=t^{-\frac{2}{d-2}}$ for $t>1$. Then (3.35) implies the inequality in (3). □

4 Lower bound estimate

This section devotes to the proof of the following result, which together with Proposition 3.1 implies Theorem 1.1.

Proposition 4.1. *Let $\nu \in \mathscr{P}_0$. There exists a constant $c > 0$ such that* (1.3) *holds, and when ∂M is convex it holds for $c = 1$. Moreover, when $d \geq 5$, there exists a constant $c' > 0$ such that*

$$\inf_{T \geq t} \big\{ t\mathbb{E}[\mathbb{W}_2(\mu_t, \mu_0)|T < \tau]\big\} \geq c' t^{-\frac{2}{d-2}}, \quad t \geq 1. \tag{4.1}$$

To estimate the Wasserstein distance from below, we use the idea of [1] to construct a pair of functions in Kantorovich's dual formula, which leads to the following lemma.

Lemma 4.2. *There exists a constant $c > 0$ such that*

$$\mathbb{W}_2(\mu_{t,r}, \mu_0)^2 \geq \mu_0(|\nabla L_0^{-1}(\rho_{t,r} - 1)|^2) - c\|\rho_{t,r} - 1\|_\infty^{\frac{7}{3}}(1 + \|\rho_{t,r} - 1\|_\infty^{\frac{1}{3}}), \quad t, r > 0.$$

Proof. Let $f = L_0^{-1}(\rho_{t,r} - 1)$, and take

$$\varphi_\theta^\varepsilon = -\varepsilon \log P^0_{\frac{\varepsilon\theta}{2}} \mathrm{e}^{-\varepsilon^{-1} f}, \quad \theta \in [0, 1], \varepsilon > 0.$$

We have $\varphi_0^\varepsilon = f$ and by [24, Lemma 2.9],

$$\begin{aligned} &\varphi_1^\varepsilon(y) - f(x) \leq \frac{1}{2}\big\{\rho(x,y)^2 + \varepsilon\|(L_0 f)^+\|_\infty + c_1 \varepsilon^{\frac{1}{2}}\|\nabla f\|_\infty^2\big\}, \\ &\mu_0(f - \varphi_1^\varepsilon) \leq \frac{1}{2}\mu_0(|\nabla f|^2) + c_1\varepsilon^{-1}\|\nabla f\|_\infty^4. \end{aligned}$$

Since $L_0 f = \rho_{t,r} - 1$, this and the integration by parts formula imply

$$\begin{aligned} &\frac{1}{2}\mathbb{W}_2(\mu_{t,r}, \mu_0)^2 + \varepsilon\|\rho_{t,r} - 1\|_\infty + c_1\varepsilon^{\frac{1}{2}}\|\nabla f\|_\infty^2 \geq \mu_0(\varphi_1^\varepsilon) - \mu_{t,r}(f) \\ &= \mu_0(\varphi_1^\varepsilon - f) - \mu_0(f L_0 f) \geq \frac{1}{2}\mu_0(|\nabla f|^2) - c_1\varepsilon^{-1}\|\nabla f\|_\infty^4, \quad \varepsilon > 0. \end{aligned} \tag{4.2}$$

Next, by Lemma 2.2 for $p = \infty$ and (2.11), we find constants $c_2, c_3, c_4 > 0$ such that

$$\begin{aligned} &\|\nabla f\|_\infty = \|\nabla L_0^{-1}(\rho_{t,r} - 1)\|_\infty \leq \int_0^\infty \|\nabla P_s^0(\rho_{t,r} - 1)\|_\infty \mathrm{d}s \\ &\leq c_2 \int_0^\infty (1 + s^{-\frac{1}{2}})\|P^0_{s/2}(\rho_{t,r} - 1)\|_\infty \mathrm{d}s \\ &\leq c_3\|\rho_{t.r} - 1\|_\infty \int_0^\infty (1 + s^{-\frac{1}{2}})\mathrm{e}^{-(\lambda_1 - \lambda_0)s/2}\mathrm{d}s \leq c_4\|\rho_{t.r} - 1\|_\infty. \end{aligned}$$

Combining this with (4.2) we find a constant $c_5 > 0$ such that for any $\varepsilon > 0$,

$$\mathbb{W}_2(\mu_{t,r}, \mu_0)^2 \geq \mu_0(|\nabla L_0^{-1}(\rho_{t,r} - 1)|^2) - c_5\big\{\varepsilon\|\rho_{t,r} - 1\|_\infty + \varepsilon^{\frac{1}{2}}\|\rho_{t,r} - 1\|_\infty^2 + \varepsilon^{-1}\|\rho_{t,r} - 1\|_\infty^4\big\}.$$

By taking $\varepsilon = \|\rho_{t,r} - 1\|_\infty^{\frac{4}{3}}$ we finish the proof. $\square$

By Lemma 4.2, to derive a sharp lower bound of $\mathbb{W}_2(\mu_{t,r},\mu_0)^2$, we need to estimate $\|\rho_{t,r}-1\|_\infty$ and $\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big]$, which are included in the following three lemmas.

Lemma 4.3. *For any $r>0$ and $\nu=h\mu$ with $\|h\phi_0^{-1}\|_\infty<\infty$, there exists a constant $c(r)>0$ such that*

$$\sup_{T\geq t}\mathbb{E}^\nu\big[\|\rho_{t,r}-1\|_\infty^4\big|T<\tau\big]\leq c(r)t^{-2},\quad t\geq 1.$$

Proof. By Lemma 3.3 and (3.12), it suffices to prove for $\nu=\nu_0$ and $T=t$ replacing $T\geq t$, i.e. for a constant $c(r)>0$ we have

$$\mathbb{E}^{\nu_0}\big[\|\rho_{t,r}-1\|_\infty^4\big|t<\tau\big]\leq c(r)t^{-2},\quad t\geq 1. \tag{4.3}$$

By (3.22), (2.7), (2.11), and $\|\phi_0^{-1}\|_{L^2(\mu_0)}=1$, we find a constant $c_1>0$ such that

$$\begin{aligned}&\mathbb{E}^{\nu_0}[1_{\{r_1<\tau\}}|g_s(r_1,r_2)|^2]\\&\leq c_1\mathrm{e}^{-\lambda_0(2s-\lambda_1)}\|\hat\phi_m\|_\infty^4\big\{\mathrm{e}^{-(\lambda_m-\lambda_0)(r_2-r_1)}+\mathrm{e}^{-(\lambda_1-\lambda_0)(s-r_2)}\big\},\quad s>r_2>r_1>0.\end{aligned}$$

By (3.21) and $\mathbb{P}^{\nu_0}(t<\tau)\geq c_0\mathrm{e}^{-\lambda_0 t}$ for some constant $c_0>0$ and all $t\geq1$, this implies

$$\mathbb{E}^{\nu_0}[|\psi_m(t)|^4|t<\tau]:=\frac{\mathbb{E}^{\nu_0}[|\psi_m(t)|^4 1_{\{t<\tau\}}]}{P^{\nu_0}(t<\tau)}\leq c_2\|\hat\phi_m\|_\infty^4 t^{-2},\quad m\geq1,t>1$$

for some constant $c_2>0$. Combining with (3.2) gives

$$\begin{aligned}&\mathbb{E}^{\nu_0}\big[\|\rho_{t,r}-1\|_\infty^4\big|t<\tau\big]\\&\leq\bigg(\sum_{m=1}^\infty\mathrm{e}^{-(\lambda_m-\lambda_0)r}\|\hat\phi_m\|_\infty^{\frac43}\bigg)^3\sum_{m=1}^\infty\mathrm{e}^{-(\lambda_m-\lambda_0)r}e^{\lambda_0 t}\mathbb{E}^{\nu_0}[1_{\{r_1<\tau\}}|\psi_m(t)|^4]\\&\leq\bigg(\sum_{m=1}^\infty\mathrm{e}^{-(\lambda_m-\lambda_0)r}\|\hat\phi_m\|_\infty^{\frac43}\bigg)^3c_2t^{-2}\sum_{m=1}^\infty\mathrm{e}^{-(\lambda_m-\lambda_0)r}\|\hat\phi_m\|_\infty^4.\end{aligned}$$

By (1.1) and (2.13), this implies (4.3) for some constant $c(r)>0$. □

Lemma 4.4. *Let $\nu=h\mu$ with $\|h\phi_0^{-1}\|_\infty<\infty$. Then for any $r>0$ there exists a constant $c(r)>0$ such that*

$$\sup_{T\geq t}\bigg|t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big]-2\sum_{m=1}^\infty\frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2}\bigg|\leq\frac{c(r)}{t},t\geq1.$$

Proof. Let $\{J_i:i=1,2,3\}$ be in (3.7). By (2.11), (2.13), and $\|\hat\phi_m\|_{L^2(\mu_0)}=1$, we find a constant $c_1>0$ such that for any $T\geq t\geq s_2\geq s_1>0$,

$$\begin{aligned}|J_1(s_1,s_2)|&\leq\|h\phi_0^{-1}\|_\infty\|P_{s_1}^0-\mu_0\|_{L^\infty(\mu_0)}\|\phi_m\phi_0^{-1}\|_\infty^2\|P_{T-s_2}^0-\mu_0\|_{L^1(\mu_0)}\|\phi_0^{-1}\|_{L^1(\mu_0)}\\&\leq c_1\|\phi_m\phi_0^{-1}\|_\infty^2\mathrm{e}^{-(\lambda_1-\lambda_0)(t+s_1-s_2)},\end{aligned}$$

$$
\begin{aligned}
|J_2(s_1,s_2)| &\le \|\phi_0\|_\infty \mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\|h\phi_0^{-1}\|_\infty \|P^0_{s_1}-\mu_0\|_{L^\infty(\mu_0)}\\
&\le c_1 \mathrm{e}^{-(\lambda_1-\lambda_0)s_2},\\
|J_3(s_1,s_2)| &\le \|\phi_0\|_\infty \mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\|\phi_m\phi_0^{-1}\|^2_\infty \|P^0_{T-s_2}-\mu_0\|_{L^1(\mu_0)}\|\phi_0^{-1}\|_{L^1(\mu_0)}\\
&\le c_1\|\phi_m\phi_0^{-1}\|^2_\infty \mathrm{e}^{-(\lambda_1-\lambda_0)(t-s_1)}.
\end{aligned}
$$

Substituting these into (3.8) and applying (1.1) and (2.13), we find a constant $c(r)>0$ such that the desired estimate holds. □

Lemma 4.5. *Let $\nu = h\mu$ with $\|h\phi_0^{-1}\|_\infty < \infty$. Then for any $r>0$ and $p\ge 2$, there exists a constant $c(r,p)>0$ such that*

$$
\|\nabla L_0^{-1}(\rho_{t,r}-1)|^{2p}\|_{L^{2p}(\mu_0)} \le c(r,p), \quad t>0.
$$

Proof. Since $\rho_{t,r} = \frac{1}{t}\int_0^t p_r^0(X_s,\cdot)\mathrm{d}s$, we have $\mu_0(\rho_{t,r})=1$ and $\|\rho_{t,r}\|_\infty \le \|p_r^0\|_\infty < \infty$. Then by (2.11) and $\|\phi_0^{-1}\|_{L^2(\mu_0)}=1$, we find a constant $c_1(r)>0$ such that

$$
\begin{aligned}
&\mu_0\big(\phi_0^{-1}\{P^0_{\frac{s}{4}}|P^0_{\frac{3s}{4}}(\rho_{t,r}-1)|^p\}^2\big) \le \|\phi_0^{-1}\|_{L^2(\mu_0)}\|(P^0_{\frac{3s}{4}}-\mu_0)\rho_{t,r}\|^{2p}_{L^{4p}(\mu_0)}\\
&\le \|P^0_{\frac{3s}{4}}-\mu_0\|^{2p}_{L^{4p}(\mu_0)}\|\rho_{t,r}\|^{2p}_\infty \le c_1(r)\mathrm{e}^{-3(\lambda_1-\lambda_0)s}.
\end{aligned}
$$

Combining this with (3.25) for $\varepsilon=1$ and $\theta\in(0,\frac{1}{\lambda_1-\lambda_0})$, we finish the proof.

□

Finally, since $\mu_{t,r}=\mu_t P_r^0$, to derive a lower bound of $\mathbb{W}_2(\mu_t,\mu_0)$ from that of $\mathbb{W}_2(\mu_{t,r},\mu_0)$, we present the following result.

Lemma 4.6. *There exist two constants $K_1,K_2>0$ such that for any probability measures μ_1,μ_2 on M°,*

$$
\mathbb{W}_2(\mu_1P_t^0,\mu_2P_t^0) \le K_1\mathrm{e}^{K_2t}\mathbb{W}_2(\mu_1,\mu_2), \quad t\ge 0. \tag{4.4}
$$

When ∂M is convex, this estimate holds for $K_1=1$.

Proof. When ∂M is convex, by [20, Lemma 2.16], there exists a constant K such that

$$
\mathrm{Ric}-\mathrm{Hess}_{V+2\log\phi_0} \ge -K,
$$

so that the desired estimate holds for $K_1=1$ and $K_2=K$, see [14].

In general, following the line of [18], we make the boundary from non-convex to convex by using a conformal change of metric. Let N be the inward normal unit vector field of ∂M. Then the second fundamental form of ∂M is a two-tensor on the tangent space of ∂M defined by

$$
\mathbb{I}(X,Y) := -\langle \nabla_X N, Y\rangle, \quad X,Y\in T\partial M.
$$

Since M is compact, we find a function $f\in C_b^\infty(M)$ such that $f\ge 1, N \parallel \nabla f$ (i.e. they are parralell each other) on ∂M, and $N\log f|_{\partial M}+\mathbb{I}(u,u)\ge 0$ holds on ∂M for any $u\in T\partial M$ with $|u|=1$. By [18, Lemma 2.1] or [19, Theorem 1.2.5], ∂M is convex under the metric

$$
\langle\cdot,\cdot\rangle' = f^{-2}\langle\cdot,\cdot\rangle.
$$

Let Δ', ∇' and Hess' be the Laplacian, gradient and Hessian induced by the new metric $\langle\cdot,\cdot\rangle'$. We have $\nabla' = f^2\nabla$ and (see (2.2) in [16])

$$L_0 = f^{-2}\Delta' + f^{-2}\nabla'\{V + 2\log\phi_0 + (d-2)f^{-1}\}.$$

Then the L_0-diffusion process X_t^0 with X_0^0 having distribution μ_1 can be constructed by solving the following Itô SDE on M° with metric $\langle\cdot,\cdot\rangle'$ (see [2])

$$\text{d}^I X_t^0 = \big\{f^{-2}\nabla'(V + 2\log\phi_0 + (d-2)f^{-1})\big\}(X_t^0)\text{d}t + \sqrt{2}f^{-1}(X_t^0)U_t\text{d}B_t, \tag{4.5}$$

where B_t is the d-dimensional Brownian motion, and U_t is the horizontal lift of X_t^0 to the frame bundle $O'(M)$ with respect to the metric $\langle\cdot,\cdot\rangle'$.

Let Y_0^0 be a random variable independent of B_t with distribution μ_2 such that

$$\mathbb{W}_2(\mu_1,\mu_2)^2 = \mathbb{E}[\rho(X_0^0, Y_0^0)^2]. \tag{4.6}$$

For any $x, y \in M^\circ$, let $P'_{x,y} : T_xM \to T_yM$ be the parallel transform along the minimal geodesic from x to y induced by the metric $\langle\cdot,\cdot\rangle'$, which is contained in M° by the convexity. Consider the coupling by parallel displacement

$$\text{d}^I Y_t^0 = \big\{f^{-2}\nabla'(V + 2\log\phi_0 + (d-2)f^{-1})\big\}(Y_t^0)\text{d}t + \sqrt{2}f^{-1}(Y_t^0)P'_{X_t^0,Y_t^0}U_t\text{d}B_t. \tag{4.7}$$

As explained in [2, Section 3], we may assume that $(M^\circ, \langle\cdot,\cdot\rangle')$ does not have cut-locus such that $P'_{x,y}$ is a smooth map, which ensures the existence and uniqueness of Y_t^0. Since the distributions of X_0^0 and Y_0^0 are μ_1, μ_2 respectively, the law of (X_t^0, Y_t^0) is in the class $\mathscr{C}(\mu_1 P_t^0, \mu_2 P_t^0)$, so that

$$\mathbb{W}_2(\mu_1 P_t^0, \mu_2 P_t^0)^2 \le \mathbb{E}[\rho(X_t^0, Y_t^0)^2], \quad t \ge 0. \tag{4.8}$$

Let $\rho'(x,y)$ be the Riemannian distance between x and y induced by $\langle\cdot,\cdot\rangle' := f^{-2}\langle\cdot,\cdot\rangle$. By $1 \le f \in C_b^\infty(M)$ we have

$$\|f\|_\infty^{-1}\rho \le \rho' \le \rho. \tag{4.9}$$

Since except the term $f^{-2}\nabla'\log\phi_0$, all coefficients in the SDEs are in $C_b^\infty(M)$, by Itô's formula, there exists a constant K such that

$$\text{d}\rho'(X_t^0, Y_t^0)^2 \le \big\{K\rho'(X_t^0, Y_t^0)^2 + I\big\}\text{d}t + \text{d}M_t, \tag{4.10}$$

where M_t is a martingale and

$$I := \langle (f^{-2}\nabla'\log\phi_0)(\gamma_1), \dot\gamma_1\rangle' - \langle (f^{-2}\nabla'\log\phi_0)(\gamma_0), \dot\gamma_0\rangle',$$

where $\gamma : [0,1] \to M$ is the minimal geodesic from X_t^0 to Y_t^0 induced by the metric $\langle\cdot,\cdot\rangle'$, which is contained in M° by the convexity, we obtain

$$I = \int_0^1 \frac{\text{d}}{\text{d}s}\langle (f^{-2}\nabla'\log\phi_0)(\gamma_s), \dot\gamma_s\rangle'\text{d}s$$

$$= \int_0^1 \Big\{ \frac{f^{-2}(\gamma_s)\mathrm{Hess}'_{\phi_0}(\dot\gamma_s,\dot\gamma_s) + \langle \nabla' f^{-2}(\gamma_s), \dot\gamma_s\rangle' \langle \nabla'\phi_0(\gamma_s),\dot\gamma_s\rangle'}{\phi_0(\gamma_s)} - \frac{\{\langle \nabla'\phi_0(\gamma_s),\dot\gamma_s\rangle'\}^2}{(f^2\phi_0^2)(\gamma_s)}\Big\}\mathrm{d}s$$
$$\le \int_0^1 \Big\{(\phi_0^{-1}f^{-2})(\gamma_s)\mathrm{Hess}'_{\phi_0}(\dot\gamma_s,\dot\gamma_s) + \frac{f^2}{4}\big[\langle \nabla' f^{-2}(\gamma_s),\dot\gamma_s\rangle'\big]^2\Big\}\mathrm{d}s \le C\rho'(X_t^0,Y_t^0)^2$$

for some constant $C>0$, where the last step is due to $\langle\dot\gamma_s,\dot\gamma_s\rangle' = \rho'(X_t^0,Y_t^0)^2$, $1\le f\in C_b^\infty(M)$, and that by the proof of [20, Lemma 2.1] the convexity of ∂M under $\langle\cdot,\cdot\rangle'$ implies $\mathrm{Hess}'_{\phi_0}\le c\phi_0$ for some constant $c>0$. This and (4.10) yield

$$\mathbb{E}[\rho'(X_t^0,Y_t^0)^2] \le \mathbb{E}[\rho'(X_0^0,Y_0^0)^2]\mathrm{e}^{(K+C)t},\quad t\ge 0.$$

Combining this with (4.6) and (4.9), we prove (4.4) for some constant $K_1, K_2>0$. □

We are now ready to prove the main result in this section.

Proof of Proposition 4.1. (a) According to (3.34), it suffices to prove for $\nu = h\mu$ with $\|h\phi_0^{-1}\|_\infty<\infty$. Let $r>0$ be fixed. By Lemma 4.2, we obtain

$$\begin{aligned} t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_{t,r},\mu_0)^2\big|T<\tau\big] &\ge t\mathbb{E}^\nu\big[1_{\{\|\rho_{t,r}-1\|_\infty\le\varepsilon\}}\mathbb{W}_2(\mu_{t,r},\mu_0)^2\big|T<\tau\big]\\ &\ge t\mathbb{E}^\nu\big[1_{\{\|\rho_{t,r}-1\|_\infty\le\varepsilon\}}\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big] - c\varepsilon^2\\ &\ge t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big]-c\varepsilon^2\\ &\quad - t\mathbb{E}^\nu\big[1_{\{\|\rho_{t,r}-1\|_\infty>\varepsilon\}}\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big],\quad \varepsilon>0, T\ge t.\end{aligned}\tag{4.11}$$

By Lemma 4.3 and Lemma 4.5 with $p=3$, we find some constants $c_1,c_2>0$ such that

$$\begin{aligned} &t\mathbb{E}^\nu\big[1_{\{\|\rho_{t,r}-1\|_\infty>\varepsilon\}}\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big] \le c_1 t\big\{\mathbb{P}^\nu\big(\|\rho_{t,r}-1\|_\infty>\varepsilon\big|T<\tau\big)\big\}^{\frac{2}{3}}\\ &\le c_1 t\varepsilon^{-\frac{8}{3}}\big\{\mathbb{E}^\nu\big(\|\rho_{t,r}-1\|_\infty^4\big|T<\tau\big)\big\}^{\frac{2}{3}} \le c_2\varepsilon^{-\frac{8}{3}}t^{-\frac{1}{3}},\quad T\ge t.\end{aligned}$$

Combining this with (4.11) and Lemma 4.4, we find a constant $c_3>0$ such that

$$\begin{aligned} t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_{t,r},\mu_0)^2\big|T<\tau\big] &\ge t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big]-\varepsilon_t\\ &\ge 2\sum_{m=1}^\infty \frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2} - \varepsilon_t - c_3t^{-1},\quad T\ge t\ge 1,\end{aligned}$$

where

$$\varepsilon_t := \inf_{\varepsilon>0}\{c\varepsilon^2 + c_2\varepsilon^{-\frac{8}{3}}t^{-\frac{1}{3}}\}\to 0 \text{ as } t\to\infty.$$

Therefore,

$$\liminf_{t\to\infty}\inf_{T\ge t}\Big\{tE^\nu\big[\mathbb{W}_2(\mu_{t,r},\mu_0)^2\big|T<\tau\big]\Big\} \ge 2\sum_{m=1}^\infty\frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2},\quad r>0.$$

Combining this with Lemma 4.6, we derive

$$\liminf_{t\to\infty}\inf_{T\ge t}\Big\{tE^\nu\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big]\Big\} \ge 2K_1^{-1}\mathrm{e}^{-K_1r}\sum_{m=1}^\infty\frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2},\quad r>0.$$

Letting $r \to 0$ we prove (1.3) for $c = K_1^{-1}$. By Lemma 4.6, we may take $c = 1$ when ∂M is convex.

(b) The second assertion can be proved as in [24, Subsection 4.2]. For any $t \geq 1$ and $N \in \mathbb{N}$, let $\bar{\mu}_N := \frac{1}{N}\sum_{i=1}^N \delta_{X_{t_i}}$, where $t_i := \frac{(i-1)t}{N}, 1 \leq i \leq N+1$. [10, Proposition 4.2] (see also [9, Corollary 12.14]) implies that for some constant $c_0 > 0$,

$$\mathbb{W}_1(\bar{\mu}_N, \mu_0)^2 \geq c_0 N^{-\frac{2}{d}}, \quad N \in \mathbb{N}, t \geq 1. \tag{4.12}$$

Write

$$\mu_t = \frac{1}{N}\sum_{i=1}^N \frac{N}{t}\int_{t_i}^{t_{i+1}} \delta_{X_s} \mathrm{d}s.$$

By the convexity of $\mathbb{W}_2^2$, which follows from the Kantorovich dual formula, we have

$$\mathbb{W}_2(\bar{\mu}_N, \mu_t)^2 \leq \frac{1}{N}\sum_{i=1}^N \frac{N}{t}\int_{t_i}^{t_{i+1}} \mathbb{W}_2(\delta_{X_{t_i}}, \delta_{X_s})^2 \mathrm{d}s = \frac{1}{t}\sum_{i=1}^N \int_{t_i}^{t_{i+1}} \rho(X_{t_i}, X_s)^2 \mathrm{d}s \tag{4.13}$$

On the other hand, by the Markov property,

$$\mathbb{E}^\nu[\rho(X_{t_i}, X_s)^2 1_{\{T<\tau\}}] = \mathbb{E}^\nu\big[1_{\{t_i<\tau\}} P^D_{s-t_i}\{\rho(X_{t_i}, \cdot)^2 P^D_{T-s}1\}(X_{t_i})\big]. \tag{4.14}$$

Since $P_t^D 1 \leq c_1 \mathrm{e}^{-\lambda_0 t}$ for some constant $c_1 > 0$ and all $t \geq 0$, (2.6) implies

$$\begin{aligned} &P^D_{s-t_i}\{\rho(x, \cdot)^2 P^D_{T-s}1\}(x) \\ &\leq c_1 \mathrm{e}^{-\lambda_0(T-s)} P^D_{s-t_i}\rho(x, \cdot)^2(x) \leq c_1 \mathrm{e}^{-\lambda_0(T-s)} \phi_0(x) P^0_{s-t_i}\{\rho(x, \cdot)^2 \phi_0^{-1}\}(x). \end{aligned} \tag{4.15}$$

It is easy to see that

$$L_0\{\rho(x, \cdot)^2 \phi_0^{-1}\} \leq c_2 \phi_0^{-2}$$

holds on M° for some constant $c_2 > 0$. So, by (2.17), we find a constant $c_3 > 0$ such that

$$P^0_{s-t_i}\{\rho(x, \cdot)^2 \phi_0^{-1}\}(x) \leq c_2 \mathbb{E}^x \int_0^{s-t_i} \phi_0^{-2}(X_r)\mathrm{d}r \leq c_3(s-t_i)\log(1+\phi_0^{-1}(x)).$$

Combining this with (4.14) and (4.15), and using $P_t^D 1 \leq c_1 \mathrm{e}^{-\lambda_0 t}$ observed above, we find a constant $c_5 > 0$ such that

$$\begin{aligned} &\mathbb{E}^\nu[\rho(X_{t_i}, X_s)^2 1_{\{T<\tau\}}] \leq c_4 \mathrm{e}^{-\lambda_0 T}\nu(\log(1+\phi_0^{-1}))(s-t_i) \\ &\leq c_4 \|h\phi_0^{-1}\|_\infty \mu(\phi_0 \log(1+\log\phi_0^{-1}))(s-t_i)\mathrm{e}^{-\lambda_0 T} \leq c_5(s-t_i)\mathrm{e}^{-\lambda_0 T}, \quad s \geq t_i. \end{aligned}$$

Since $\mathbb{P}^\nu(T < \tau) \geq c_0 \mathrm{e}^{-\lambda_0 T}$ for some constant $c_0 > 0$ and all $T \geq 1$, we find a constant $c > 0$ such that

$$\mathbb{E}^\nu[\rho(X_{t_i}, X_s)^2 | T < \tau] \leq c(s-t_i), \quad s \geq t_i.$$

Combining this with (4.12) and (4.13), we find a constant $c_6 > 0$ such that

$$\mathbb{E}^\nu[\mathbb{W}_1(\mu_t, \mu_0)^2 | T < \tau] \geq \frac{c_1}{2} N^{-\frac{2}{d}} - c_6 t N^{-1}, \quad T \geq t.$$

Taking $N = \sup\{i \in \mathbb{N} : i \le \alpha t^{\frac{d}{d-2}}\}$ for some $\alpha > 0$, we derive

$$t^{\frac{2}{d-2}} \inf_{T \ge t} \{\mathbb{E}^\nu[\mathbb{W}_1(\mu_0, \mu_t)^2 | T < \tau]\} \ge \frac{c_2}{2\alpha^{\frac{2}{d}}} - \frac{2c'}{\alpha}, \quad t \ge 1.$$

Therefore,

$$t^{\frac{2}{d-2}} \inf_{T \ge t} \mathbb{E}^\nu[\mathbb{W}_1(\mu_0, \mu_t)^2 | T < \tau] \ge \sup_{\alpha > 0} \left(\frac{c_2}{2\alpha^{\frac{2}{d}}} - \frac{2c'}{\alpha} \right) > 0, \quad t \ge 1.$$

□

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