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Which Arithmetical Data Types Admit Fracterm Flattening?

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Abstract

The formal theory of division in arithmetical algebras reconstructs fractions as syntactic objects called fracterms. Basic to calculation is the simplification of fracterms to fracterms with one division operator, a process called fracterm flattening. We consider the equational axioms of a calculus for calculating with fracterms to determine what is necessary and sufficient for the fracterm calculus to allow fracterm flattening. For computation, arithmetical algebras require operators to be total for which there are several semantical methods. It is shown under what constraints up to isomorphism, the unique total and minimal enlargement of a field $Q(\div)$ of rational numbers equipped with a partial division operator $\div$ has fracterm flattening.

Keywords: fracterm, fracterm flattening, common meadow, rational numbers, equational specification, initial algebra semantics.

1 Introduction

The basic algebra for computing in arithmetical structures starts with rings and fields, with their operations $x + y$, $-x$, $x \cdot y$. Although a field has

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inverses for non-zero elements, to study division we must add an operator, either an inverse $\frac{1}{y}$ or a division $x \div y$; one can derive each from the other, of course, e.g., $x \div y = x \cdot \frac{1}{y}$. For rings and fields with inverse or division we have coined the term *meadow* [10].

For computing, we need algebras that are total so we have to deal with division by zero. Totalising division is standard in computer arithmetics, where *error*, unsigned infinity ∞, or signed infinities $+\infty$, $-\infty$ are familiar. For general information on division by zero see the survey paper [3].

We will focus on the fields Q of rational numbers in which division is made total. Two scenarios for totalisation can be distinguished when starting with a structure $Q(\div)$ of rational numbers equipped with a partial division operator, i.e., a partial meadow of rational numbers; we can use

1. *Internal methods*: No new elements are added to the domain of $Q(\div)$ so that $x/0$ is defined as some element of the field Q for each element x of the field; and

2. *External methods*: One or more new elements are added to the domain $Q(\div)$, so that the algebra is properly extended and is an enlargement.

Additional new elements are so-called *peripheral numbers*. In this paper, we will focus on external methods and especially the method of adding a single peripheral number to $Q(\div)$ that we denote $\perp$, for which $x \div 0 = \perp$ in Q, and which is *absorbtive*, i.e., if $\perp$ is an argument to an operation then the result is $\perp$. This method is quite general and applied to rings with division produces what we have chosen to call *common meadows*. Common meadows have a growing algebraic theory [7, 9].

Now, in the case of the rationals, we observe that "fractions" are everywhere in elementary measurements, calculation and teaching (though they tend to disappear as a concept in advanced mathematics). In [4], the informal term fraction is replaced with the formally defined syntactic concept of a *fracterm*. Central to using fracterms are calculations like

$$\frac{\frac{5}{4}}{\frac{3}{2}} = \frac{5}{4} \cdot \frac{2}{3} = \frac{10}{12} = \frac{5}{6}$$

where a complex fracterm is transformed into a simple fracterm containing only one division. Here are the definitions:

Definition 1 *Let Σ be a signature with a division operator. A* fracterm *is a Σ-term with division as its leading function symbol. A* flat fracterm *is* *a fracterm which contains no occurrences of division in its numerator and in its denominator, and contains no occurrences of constants other than* 0 *and* 1*.*

As a property of an arithmetical algebra we define [4]:

Definition 2 *Let* Σ *be a signature with a division operator and* B *a* Σ*-algebra. The algebra* B *has* fracterm flattening *if for every fracterm* t *over* Σ *there is a flat fracterm* r *such that*

$$B \models t = r.$$

Given a fracterm, the process of finding an equivalent flat fracterm is called fracterm flattening*.*

Totalising operators can have a dramatic effect on algebraic properties as certain classic laws of arithmetic fail; this is very noticeable in the case of the rationals. However, ideally, the method of totalising can be axiomatised in a meaningful and memorable way. For our totalising method of making common meadows there are several axiomatisations using equations [13]; and there are equational specifications of the common meadow of rational numbers under initial algebra semantics.

Thus, fracterm flattening can be considered relative to an axiomatic theory E, and then refers to finding a flat fracterm r which is provably equal to a given fracterm t using a theory E. When we work with fracterms, we call an axiomatisation a *fracterm calculus*. In this paper, we will use a fracterm calculus based on the axioms of a common meadow.

The fracterm representation and flattening are *essential* in practical calculation with rational numbers. Now in [6], it was shown that Internal Scenario 1 *cannot* lead to a totalisation of $Q(\div)$ for which fracterm flattening holds.

However, in External Scenario 2, when peripheral values are admitted, flattening becomes possible. The enlargement of $Q(\div)$ with an absortive peripheral number $\perp$ used to totalise division – $x/0 = \perp$ – makes the common meadow $\mathsf{Enl}_{\perp}(Q(\div))$. From [7], we know that the equational axioms of a common fracterm calculus – the fracterm form of the axioms of common meadows – enable all fracterms to be flattened. Here we analyse the equational axioms for calculating with fracterms to determine what is necessary and/or sufficient for a calculus to allow fracterm flattening. Our method is to discover small sets of axioms F for which we can prove:

Working Principle. Let B be an algebra with flattening. If B satisfies the equations in F then $B \cong \mathsf{Enl}_{\perp}(Q(\div))$.

The equations we find for F are surprisingly simple. The most prominent is the distribution law.

Thus, in this way, we will find under what conditions is a common meadow $\mathsf{Enl}_{\perp}(Q(\div))$ of rational numbers the unique total and minimal enlargement of the partial meadow $Q(\div)$ that has fracterm flattening (up to isomorphism).

Finally, continuing to explore External Scenario 2, we discuss in detail the alternative totalisation method for division called wheels [21, 16], which uses ∞ and $\perp$. We show that a wheel of rational numbers does *not* have fracterm flattening.

The structure of the paper is this: Section 2 contains some algebraic preliminaries that we need. In Section 3 we set up the basics of common meadows. In Section 4 we prove the theorems. In Section 5 we focus on the necessity of the distribution law. In Section 6 we discuss the case of wheels. In Section 7 we discuss related work and offer some concluding remarks.

2 Algebraic Preliminaries

2.1 Basics

We assume the reader is familiar with the theory of algebraic specifications for data types [17, 19].

Definition 3 *A* partial data type *is an algebra A of signature Σ that is minimal, i.e., each element of the domain A is the interpretation in A of a closed Σ term. A* partial abstract data type *is the isomorphism class* $\mathbf{A}$ *of a partial data type A.*

The condition of minimality means that every element of the data type can be constructed by applying the operations of the data type to its named constants. As we are focussed on arithmetic we need only consider single-sorted data types with signature Σ.

Let $T(\Sigma)$ be the set of all closed terms made from the operations and constants of Σ, and let $T(\Sigma, X)$ be the set of all terms made with variables from X.

For simplicity, we will work with data types rather than abstract data types. To specify a data type, we use equations with the standard initial algebra semantics for total data types. To specify a total data type B by an equational specification (Σ, E) we make an initial algebra using the concrete construction:

$$T(\Sigma, E) = T(\Sigma)/\equiv_E \;\cong\; B$$

where $t \equiv_E t' \iff E \vdash t = t'$. The initial algebra semantics of (Σ, E) is the isomorphism class $\mathbf{I}(\Sigma, E)$ of $T(\Sigma, E)$.

2.2 Transforming Data Types

A data type A can be changed by adding or removing constants and operations *without* changing its domain A; these internal changes we call *expansions, enrichments or reductions*. A data type A can be also be changed by adding, or removing, constants and operations that require changing its domain A; these external changes we call *extensions or restrictions*. Extensions followed by expansions we call enlargements. We will use the following transformations to increase and reduce domains.

At the heart of our theorising are ways to make a total algebra from a partial algebra. Recalling our internal and external scenarios, two methods stand out as quite general.

Internally, the easiest way is to use a single element from the partial algebra to totalise the partial operations; we do not have to adjust the values of the operators if they are already defined. Let A be a partial minimal algebra of signature Σ. Let $t \in T(\Sigma)$ be a closed term such that t has a value in A. Then we define $B = \mathsf{Tot}_t(A)$ to be the total algebra obtained by using the value of t in A to make the partial operations of A total by returning the value t. Typically, we can use a constant from Σ.

Alternately, externally, we can add a single new element to make the partial operations total; however, we must define all the operators on any such new element. An easy way to do this adjustment is as follows:

Let A be a partial minimal algebra of signature Σ. Let $\perp \notin A$ be an element new to A that is absorbtive, i.e., if $\perp$ is an argument to an operation of A then its value is $\perp$. Then we define

$$B = \mathsf{Enl}_{\perp}(A)$$

to be the total algebra obtained by using the new value of $\perp$ in A to make all partial operations of A total by returning the value $\perp$.

3 Arithmetic Data Types and Common Meadows

Let N, Z and Q be sets of natural numbers, integers and rational numbers, respectively.

3.1 Arithmetic Data Types

The basic algebra of arithmetic is the algebra of rings and fields. These axiomatise the operations of addition $x + y$, its additive inverse $-x$, and multiplication $x \cdot y$. Let Σ_{cr} be the signature of rings and fields.

Although each non-zero element $x \in K$ of a field has multiplicative inverse, K does not have an inverse operator x^{-1} or a division operator $\div$. On adding a unary inverse operator $^{-1}$, or binary division operator $\div$, the algebras $K(^{-1})$ or $K(\div)$ become partial.

To make division $\div$ total, consider the two methods of Section 2.2. Suppose we do this by using an element of K such as the constant 0 then we get an involutive meadow $Tot_0(K) = K_0(\div)$ of signature Σ_m.

Alternately, if we add an absorbtive element $\perp$ to $K(\div)$ then we get a common meadow $\mathsf{Enl}_\perp(K(\div))$ with the signature $\Sigma_{cm} = \Sigma_m \cup \{\perp\}$.

Let $Q(\div)$ be the field Q of rational numbers with division $\div$. The algebra $Q(\div)$ is partial meadow with signature Σ_m. Note that the field Q does not qualify as a data type for computation as it is not a minimal algebra. However, the meadow $\mathsf{Enl}_\perp(Q(\div))$ is minimal and so does qualify as a data type for computation.

The general problem we are tackling is this:

Problem 1 *For which totalisation methods applied to $Q(\div)$ is there an appropriate fracterm calculus that admits fracterm flattening?*

3.2 Commutative Rings

Following [14, 13], to prepare for the common meadows we first apply the general process of adding an absorbtive element $\perp$ to a commutative ring R. The ring is total so its operators will only change on the new element $\perp$. Thus, given any commutative ring R, we define the transformation $\mathsf{Enl}_\perp(R)$ to be the algebra that results from R by extending the domain by adding $\perp$ as a constant and then extending all operations such that $\perp$ is an absorptive element of the new algebra.

$$(x + y) + z = x + (y + z) \tag{1}$$
$$x + y = y + x \tag{2}$$
$$x + 0 = x \tag{3}$$
$$x + (-x) = 0 \cdot x \tag{4}$$
$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \tag{5}$$
$$x \cdot y = y \cdot x \tag{6}$$
$$1 \cdot x = x \tag{7}$$
$$x \cdot (y + z) = (x \cdot y) + (x \cdot z) \tag{8}$$
$$-(-x) = x \tag{9}$$
$$0 \cdot x = 0 \cdot (x \cdot x) \tag{10}$$
$$x + \bot = \bot \tag{11}$$

Table 1: $E_{wcr,\bot}$: weakened equations for commutative rings

Already the properties of the commutative ring are compromised: $x \cdot 0 = 0$ can no longer hold as $x \cdot \bot = \bot$. However, this semantic construction can be axiomatised by the equations in Table 1. We take Theorem 2.1 from [13]:

Theorem 1 *Given an equation $t = r$: $E_{wcr,\bot} \vdash t = r$ if, and only if, for every ring R, $\mathsf{Enl}_\bot(R) \models t = r$.*

3.3 Common Meadows and Flattening

We have just added $\bot$ to rings and given an equational axiomatisation of the structures. Now we will add $\div$ and totalise it with $\bot$ to make common meadows.

The common meadows have a number of equivalent equational axiomatisations as discussed in [13]. For our purposes, we choose to work with fracterms and add the 6 equations of Table 2 to the equations from Table 1. This gives an equational axiomatisation

$$E_{wcm} = E_{wcr,\bot} \cup E_{\mathit{ffl}}.$$

This we refer to as the *weak axiomatisation of the common meadows*, or the *weak common fracterm calculus*. We notice that following [9] fracterm

$$x = \frac{x}{1} \tag{12}$$

$$-\frac{x}{y} = \frac{-x}{y} \tag{13}$$

$$\frac{x}{y} \cdot \frac{u}{v} = \frac{x \cdot u}{y \cdot v} \tag{14}$$

$$\frac{x}{y} + \frac{u}{v} = \frac{(x \cdot v) + (y \cdot u)}{y \cdot v} \tag{15}$$

$$\frac{x}{(\frac{u}{v})} = x \cdot \frac{v \cdot v}{u \cdot v} \tag{16}$$

$$\perp = \frac{1}{0} \tag{17}$$

Table 2: E_{ffl}: Equations for fracterm flattening

calculus results from its weak version by adding the equation $\frac{1}{1+0\cdot x} = 1+0\cdot x$. Below we will not make use of the latter axiom.

In [13], it was noted that the equations of E_{wcm} imply that fracterms admit flattening.

E_{wcm} is logically equivalent to the earliest axiom system $\mathrm{Md}_{\mathbf{a}}$ for common meadows of [7]. In [9] it is shown that the equation $0 \cdot (x \cdot x) = x$ is derivable from the other equations of $E_{wcr,\perp} \cup E_{ffl}$. It has been included as an axiom in view of the pleasant modularisation which is thus obtained, especially Theorem 1 above.

3.4 Homomorphic Images

An arithmetical data type is *simple* if it has no proper non-trivial homomorphic images; thus, a homomorphism is either a constant function or an epimorphism. (The same property is referred to as being final in the abstract data type literature, but we prefer speaking of a simple structure as that is closer to algebraic conventions.) For $\mathsf{Enl}_{\perp}(Q(\div))$ we will use the following fact:

Proposition 1 $\mathsf{Enl}_{\perp}(Q(\div))$ *is simple.*

Proof: Suppose $\phi\colon \mathsf{Enl}_{\perp}(Q(\div)) \to B$ is a surjective homomorphism with a non-trivial image B. We will distinguish two cases.

Case (i): For any $a \in Z$, let $\underline{a}$ be the corresponding numeral. Assume that for two different fracterms $\frac{\underline{a}}{\underline{b}}$ and $\frac{\underline{c}}{\underline{d}}$ with $a, c \in Z$ and $b, d > 0$, $b, d \in N$, $\phi(\frac{\underline{a}}{\underline{b}}) = \phi(\frac{\underline{c}}{\underline{d}})$. Let $t \equiv \frac{\underline{a}}{\underline{b}} - \frac{\underline{c}}{\underline{d}}$ and $r \equiv \frac{\underline{c}}{\underline{d}} - \frac{\underline{c}}{\underline{d}}$ then $\mathsf{Enl}_\perp(Q(\div)) \models t \neq 0$ so that $\mathsf{Enl}_\perp(Q(\div)) \models \frac{t}{t} = 1$ and $B \models \frac{t}{t} = 1$. Moreover, $\mathsf{Enl}_\perp(Q(\div)) \models r = 0$ and $B \models t = \frac{\underline{a}}{\underline{b}} - \frac{\underline{c}}{\underline{d}} = \frac{\underline{c}}{\underline{d}} - \frac{\underline{c}}{\underline{d}} = r = 0$ so that $B \models 1 = \frac{0}{0} = \perp$. Now $\mathsf{Enl}_\perp(Q(\div)) \models x \cdot \perp = \perp$ so that $B \models x \cdot \perp = \perp$ and therefore $B \models x = x \cdot 1 = x \cdot \perp = \perp$ which implies B is trivial, a contradiction which completes the proof for case (i).

Case (ii): Assume that $\phi(\perp) = \phi(\frac{\underline{a}}{\underline{b}})$ for some $a \in Z, b \in N, b \neq 0$. Now we find that $B \models \perp = \perp - \perp = \frac{\underline{a}}{\underline{b}} - \frac{\underline{a}}{\underline{b}} = 0$ so that for $B \models x = x + 0 = x + \perp = \perp$ with as a consequence that B is trivial in case (ii) as well. □

4 Axioms Delivering Flattening

4.1 Extensions by Peripheral Elements

As we stated in the introduction, there are a number of semantic methods for making division total in $Q(\div)$. For example, as well as common meadows, there are wheels [21, 16] and transreals [2, 20], both of which add more than one peripheral elements. Our focus is on adding the absorbtive element $\perp$ for division by zero. We isolate a general form of extension of $Q(\div)$:

Definition 4 *We say that a Σ_{cm}-algebra B is an* extension of $Q(\div)$ by peripheral numbers *if B contains Q and*

(i) one of the new elements denotes the closed term $0 \div 0$; this element we denote $\perp$;

(ii) all of the new elements are interpretations of closed Σ_{cm}-terms;

(iii) all closed Σ_{cm}-terms have an interpretation in B.

As implied by the assumption that algebra B has signature Σ_{cm}, we do not make these new elements constants of B except $\perp$ and we do not add operations either.

Notice that (ii) and (iii) makes B a total minimal algebra and, in particular, implies that $\div$ is total on 0.

4.2 Flattening in Extensions

The result in Section 3.3 that $E_{wcm} = E_{wcr,\perp} \cup E_{ffl}$ enables fracterm flattening for all common meadows is theoretical fact whose importance grows as we seek the fate of flattening for other totalisation methods. Our results strengthen our theory of common meadows:

Proposition 2 *Let B be a Σ_{cm}-algebra that is a total minimal extension of $Q(\div)$ by peripheral numbers. Suppose that B admits flattening. If $B \models E_{wcr,\perp} \cup E_{ffl}$ then $B \cong \mathsf{Enl}_{\perp}(Q(\div))$.*

Proof: Let $\chi_0 = \{\underline{n}\div\underline{n} = 1 \mid n \in N, n > 0\}$. Because B is an enlargement of $Q(\div)$, $B \models \chi_0$. Now from combining results of [8, 7] and [13] we know that $E_{wcr,\perp} \cup \chi_0$ is an initial algebra specification of the abstract data type of $\mathsf{Enl}_{\perp}(Q(\div))$ so that B must be a homomorphic image of $\mathsf{Enl}_{\perp}(Q(\div))$. As B is an enlargement of $Q(\div)$ it is non-trivial, so that with Proposition 1 it follows that $B \cong \mathsf{Enl}_{\perp}(Q(\div))$. $\square$

We now dig into the axioms of $E_{wcr,\perp} \cup E_{ffl}$ that are at work:

Proposition 3 *Let B be a Σ_{cm}-algebra that is a total minimal extension of $Q(\div)$ by peripheral numbers. Suppose that B admits flattening. If B satisfies distribution equation 8 of Table 1, and the equations 14, 15 and 17 of Table 2, then $B \cong \mathsf{Enl}_{\perp}(Q(\div))$.*

Proof: Because of flattening each closed expression t equals in B an expression $r \div s$ with r and s closed and without division and without $\perp$. Because B is an enlargement of $Q(\div)$, for some $n \in Z$ and $m \in N$, $r = \underline{n}$ and $s = \underline{m}$ in B. From this observation it follows that all (and only) peripheral elements of B are interpretations of expressions $\underline{n} \div 0$ in B.

Lemma 1 *For $n \neq 0$, we have*

(i) $B \models \underline{n} \div 0 = 1 \div 0$

(ii) $B \models 1 \div 0 = 0 \div 0$

(iii) $B \models 1 \div (1 \div 0) = 1 \div 0$

(iv) $B \models -(1 \div 0) = 1 \div 0$

Proof: We notice first that for $n \neq 0$: $\underline{n} \div 0 = 1 \div 0$ as follows: $\underline{n} \div 0 = (\underline{n} \cdot 1) \div (\underline{n} \cdot 0) = (\underline{n} \div \underline{n}) \cdot (1 \div 0) = (1 \div 1) \cdot (1 \div 0) = (1 \cdot 1) \div (1 \cdot 0) = 1 \div 0$.

Next we will show (ii). Indeed: $1 \div 0 = (1 \cdot 1) \div (1 \cdot 0) = (1 \div 1) \cdot (1 \div 0) = 1 \cdot (1 \div 0) = ((1 \div \underline{2}) + (1 \div \underline{2})) \cdot (1 \div 0) = ((1 \div \underline{2}) \cdot (1 \div 0)) + ((1 \div \underline{2}) \cdot (1 \div 0)) = ((1 \cdot 1) \div (\underline{2} \cdot 0)) + ((1 \cdot 1) \div (\underline{2} \cdot 0)) = (1 \div 0) + (1 \div 0) = ((1 \cdot 0) + (0 \cdot 1)) \div (0 \cdot 0) = 0 \div 0$. It is now immediate to prove that $1 \div 0$ is absorptive for $\cdot$ and $+$ with the help of equations 14 and 15 which B both satisfies by assumption.

What remains to be done is to show that in B, $1 \div (1 \div 0) = 1 \div 0$ (for which we may not make use of equation 16) and $-(1 \div 0) = 1 \div 0$ (for which equation 13 may not be used). Beginning with $1 \div (1 \div 0) = 1 \div 0$, we can show that in B, $1 \div (1 \div 0) = (1 \div 0) \cdot (1 \div (1 \div 0))$ from which it follows that (in B) $1 \div (1 \div 0)$ must be a peripheral element, i.e. equal to $1 \div 0$. Calculating in B we have: $1 \div (1 \div 0) = (1 \cdot 1) \div (1 \div 0) = (1 \cdot 1) \div ((1 \cdot 1) \div (1 \cdot 0)) = (1 \cdot 1) \div ((1 \div 1) \cdot (1 \div 0)) = (1 \div (1 \div 1)) \cdot (1 \div (1 \div 0))) = 1 \cdot (1 \div (1 \div 0))) = ((1 \div \underline{2}) + (1 \div \underline{2})) \cdot (1 \div (1 \div 0)) = ((1 \div \underline{2}) \cdot (1 \div (1 \div 0))) + ((1 \div \underline{2}) \cdot (1 \div (1 \div 0))) = (1 \div (\underline{2} \div 0)) + (1 \div (\underline{2} \div 0)) = (1 \div (1 \div 0)) + (1 \div (1 \div 0)) = (1 \cdot (1 \div 0)) + ((1 \div 0) \cdot 1) \div ((1 \div 0) \cdot (1 \div 0)) = (1 \div 0) \cdot (1 \div (1 \div 0))$. In order to show that $-(1 \div 0) = 1 \div 0$ we notice that the fracterm flattening assumption on B implies the existence terms $p(x)$ and $q(x)$ over the signature of rings with variable x such that $B \models -(1 \div x) = p \div q$. In particular for all non-zero integers n, $B \models p(\underline{n}) \cdot \underline{n} = -q(\underline{n})$. Viewed as polynomials over the integers $p(x) \cdot x$ and $-q(x)$ are the same for nonzero n so that they must take the same value on all n so that $q(0) = p(0) \cdot 0 = 0$ and therefore $B \models -(1 \div 0) = p(0) \div 0 = 1 \div 0$. □

With this lemma at hand we see that there is just one peripheral number in B which is its both own inverse and its own opposite. This number is the interpretation of $\perp = 0 \div 0$. Now with help of equations 15 and 14 it is immediate that $\div 0$ is absorptive. This concludes the proposition. □

Surprisingly, Proposition 3 can be strengthened further as follows:

Proposition 4 *Let B be a Σ_{cm}-algebra that is a total minimal extension of $Q(\div)$ by peripheral numbers. Suppose that B admits flattening. If B satisfies distribution equation 8 of Table 1, and the equations 14 and 17 of Table 2, then $B \cong \mathsf{Enl}_{\perp}(Q(\div))$.*

Proof: Just as in the proof of Proposition 3 it follows that all (and only) peripheral elements of B are interpretations of expressions $\underline{n} \div 0$ in B. It also follows in the same way that for $n \neq 0$: $\underline{n} \div 0 = 1 \div 0$. Although we have not yet obtained $1 \div 0 = 0 \div 0$, it is now known that $\underline{n} \div 0$ can have at most two different values: $0 \div 0$ or $1 \div 0$.

By the flattening assumption there are expressions $p \equiv p(x,y,u,v)$ and $q \equiv q(x,y,u,v)$ without $\div$ and without $\perp$, i.e., over the signature of rings, such that $B \models (x \div y) + (u \div v) = p \div q$. With $\hat{p}$ and $\hat{q}$ we indicate the multivariate polynomials denoted by p and q respectively. For these polynomials the following holds, for all $x,y,u,v \in Z$:

$$y \neq 0 \wedge v \neq 0 \wedge q \neq 0 \rightarrow ((x \cdot v) + (y \cdot u)) \cdot \hat{q} = (y \cdot v) \cdot \hat{p}.$$

We notice that $\hat{q}$ cannot be zero on all argument tuples.

To see this notice that otherwise for all substitutions of x,y,z,u with integer values $p \div 0$ with p taking an integer value. As has been noticed above $p \div 0$ can take at most two different values, and thus is would be the case that $(x \div y) + (u \div v)$ can take at most two different values on integer arguments, which is clearly not true.

Thus, for all $x,y,u,v \in Z$:

$$(x \cdot v \cdot \hat{q}) \cdot ((x \cdot v) + (y \cdot u)) \cdot \hat{q} = (x \cdot v \cdot \hat{q}) \cdot (y \cdot v) \cdot \hat{p}.$$

Unique factorisation of integer polynomials over Z yields that for all $x,y,u,v \in Z$:

$$((x \cdot v) + (y \cdot u)) \cdot \hat{q} = (y \cdot v) \cdot \hat{p}.$$

Using unique factorisation once more it follows that $y \cdot v$ is a factor of $\hat{q}$, and that $(x \cdot v) + (y \cdot u)$ is a factor of $\hat{p}$. From these factorisations it follows that $\hat{p}(1,0,1,0) = 0$ and $\hat{q}(1,0,1,0) = 0$. Now consider the case $x = 1, y = 0$, $u = 1, v = 0$. We know that $B \models (1 \div 0) + (1 \div 0) = p(1,0,1,0) \div q(1,0,1,0)$ and moreover $B \models p(1,0,1,0) = 0$ (because $\hat{p}(1,0,1,0) = 0$) and $B \models q(1,0,1,0) = 0$ (because $\hat{q}(1,0,1,0) = 0$) so we find $B \models (1 \div 0) + (1 \div 0) = 0 \div 0$. With the latter information available the proof that $1 \div 0 = 0 \div 0$ as given in the proof of Proposition 3 now works without the assumption that B satisfies equation 15 of Table 2.

Again, it is immediate to prove that $1 \div 0$ is absorptive for $\cdot$ (i.e., $B \models (1 \div 0) \cdot (\underline{n} \div \underline{m}) = 1 \div 0$ with the help of equation 14). However, because equation 15 is unavailable, it requires an additional argument to see that $B \models (1 \div 0) + (\underline{n} \div \underline{m}) = 1 \div 0$. We will show that $B \models (0 \div 0) + (\underline{n} \div \underline{m}) = 0 \div 0$ which suffices.

Taking p and q as above we find that $\hat{p}(0,0,n,m) = 0$ because it has $(0 \cdot m) + (n \cdot 0)$ as a factor, and that $\hat{q}(0,0,n,m) = 0$ because it has $0 \cdot m$ as a factor. From these facts it follows that $B \models p(0,0,n,m) \div q(0,0,n,m) = 0 \div 0$.

What remains to be shown is that in B,

$$1 \div (1 \div 0) = 1 \div 0 \text{ and } -(1 \div 0) = 1 \div 0.$$

The argument for $-(1 \div 0) = 1 \div 0$ is the same as in the proof of Proposition 3.

Regarding $1 \div (1 \div 0) = 1 \div 0$, we will show that for no $n, m \in Z$ with $m \neq 0$ it is the case that $B \models 1 \div (1 \div 0) = \underline{n} \div \underline{m}$. Given that fact it follows that $B \models 1 \div (1 \div 0) = 1 \div 0$ because $1 \div 0$ is the unique peripheral element of B. To see the impossibility of $B \models 1 \div (1 \div 0) = \underline{n} \div \underline{m}$ for $n, m \in Z$ with $m \neq 0$ we consider once more the identity $B \models (x \div y) + (u \div v) = p \div q$. From the above arguments we recall that $(x \cdot v) + (y \cdot u)$ is a factor of $\hat{p}$ from which it follows that y occurs in P and that $y \cdot v$ is a factor of $\hat{q}$ from which it follows that y occurs in q. Now consider the case $x = 1, y = 1 \div 0, u = 0, v = 1$, then

$$B \models 1 \div (1 \div 0) + (0 \div 1) = p(1, 1 \div 0, 0, 1) \div q(1, 1 \div 0, 0, 1).$$

Because y occurs in p as well as in q we find, making use of the fact that $1 \div 0$ ($= 0 \div 0$) is absorptive for addition and multiplication (which has been established already above) that $B \models p(1, 1 \div 0, 0, 1) = 1 \div 0$ and $B \models q(1, 1 \div 0, 0, 1) = 1 \div 0$. Therefore $B \models p(1, 1 \div 0, 0, 1) \div q(1, 1 \div 0, 0, 1) = (1 \div 0) \div (1 \div 0)$. Moreover: $1 \div 0 = (1 \cdot 1) \div (1 \cdot 0) = (1 \div 1) \cdot (1 \div 0) = 1 \cdot (1 \div 0)$ and $1 \div 0 = (1 \cdot 1) \div (0 \cdot 1) = (1 \div 0) \cdot (1 \div 1) = (1 \div 0) \cdot 1$ so that $(1 \div 0) \div (1 \div 0) = ((1 \div 0) \div 1) \cdot (1 \div (1 \div 0))$. With $B \models 1 \div (1 \div 0) = \underline{n} \div \underline{m}$ for $m \neq 0$ we find:

$$B \models \underline{n} \div \underline{m} = ((1 \div 0) \div 1) \cdot \underline{n} \div \underline{m}.$$

In order to see that this cannot be the case consider expressions $r \equiv r(x)$ and $s \equiv s(x)$ such that $B \models (1 \div x) \div 1 = r \div s$. We find that for all $x \in Z$

$$x \neq 0 \wedge \hat{s} \neq 0 \rightarrow \hat{r} \cdot x = \hat{s}$$

which implies that for all $x \in Z$

$$x \cdot \hat{s} \cdot \hat{r} \cdot x = x \cdot \hat{s} \cdot \hat{s}$$

It cannot be the case that for all $x \in Z$, $\hat{s}(x) = 0$ because then $(1 \div x) \div 1$ takes only one value $(1 \div 0)$ on integers x. Thus q is nonzero and by unique factorisation for all $x \in Z$

$$\hat{r} \cdot x = \hat{s}$$

from which it follows that $\hat{s}(0) = 0$ so that $B \models (1 \div 0) \div 1 = r(0) \div 0 = 1 \div 0$ and then

$$B \models \underline{n} \div \underline{m} = (1 \div 0) \cdot \underline{n} \div \underline{m} = 1 \div 0$$

which is a contradiction as B is an enlargement of $Q(\div)$ which contains a value for $\underline{n} \div \underline{m}$ while $1 \div 0$ is a peripheral number in B i.e. not an element contained in $Q(\div)$ already. $\square$

Problem 2 *Can Proposition 4 be improved by weakening or even removing the assumption that B satisfies equation 14 of Table 2?*

The other equation 17 in Proposition 4 is the defining equation, $0^{-1} = \bot$, for the peripheral $\bot$.

5 On the Distribution Law

The condition that

$$B \models x \cdot (y + z) = (x \cdot y) + (x \cdot z)$$

is an important equation of commutative rings that is preserved as equation 8 of $E_{wcr,\bot}$. It plays a role in all the Propositions 2, 3, and 4. We will show that it cannot be removed.

We will construct a minimal algebra B of signature Σ_{cm} with these properties:

(i) B not isomorphic to $\mathsf{Enl}_\bot(Q(\div))$;

(ii) B has flattening;

(iii) distribution of $\cdot$ over $+$ fails.

To do this we take inspiration from wheels, which employ two peripheral numbers ∞ and $\bot$ – see Section 6 below.

First, we consider an enlargement of $Q(\div)$ by the single peripheral number $\widehat{\infty}$ to make a *partial* data type $B_0 = Q(\div, \widehat{\infty})$. The idea of $\widehat{\infty}$ is that

$$1 \div 0 = \widehat{\infty} \text{ but } 0 \div 0 \text{ is left undefined.}$$

Note that there is no $\bot$ in B_0 but shortly we will make the counter-example by setting $B = \mathsf{Enl}_\bot(B_0)$.

Addition and multiplication are commutative, though partial, in B_0.

In addition to $1 \div 0 = \widehat{\infty}$, we have in partial algebra B_0:

Partiality: $0 \cdot \widehat{\infty}$, $\widehat{\infty} \cdot 0$, $1 \div \widehat{\infty}$ and $\widehat{\infty} + \widehat{\infty}$ are undefined;

Inverse: $-\widehat{\infty} = \widehat{\infty}$;

For $n, m \in Z$ with $m \neq 0$,

Division: $(\underline{n} \div \underline{m}) + \widehat{\infty} = \widehat{\infty} + (\underline{n} \div \underline{m}) = \widehat{\infty}$.

Multiplication: $\widehat{\infty} \cdot \widehat{\infty} = \widehat{\infty}$, and $(\underline{n} \div \underline{m}) \cdot \widehat{\infty} = \widehat{\infty} \cdot (\underline{n} \div \underline{m}) = \widehat{\infty}$.

Now $B = \mathsf{Enl}_{\perp}(B_0)$, i.e, $B = Q_{\perp}(\div, \widehat{\infty}) = \mathsf{Enl}_{\perp}(Q(\div, \widehat{\infty})))$ which is a total minimal Σ_{cm}-algebra.

Proposition 5 *The algebra $Q_{\perp}(\div, \widehat{\infty})$ satisfies the axioms of $E_{wcr,\perp} \cup E_{ffl}$ with exception of distribution of $\cdot$ over $+$ (equation 8), but it is* not *isomorphic to $\mathsf{Enl}_{\perp}(Q(\div))$.*

Proof: By inspection of the various equations one may easily confirm the algebra satisfies the equations of $E_{wcr,\perp} \cup E_{ffl}$ with exception of distribution. Distribution fails because:

$$(1+1) \cdot \widehat{\infty} = 2 \cdot \widehat{\infty} = \widehat{\infty} \quad \text{but} \quad 1 \cdot \widehat{\infty} + 1 \cdot \widehat{\infty} = \widehat{\infty} + \widehat{\infty} = \perp$$

The algebras are not isomorphic because equations are preserved by homomorphisms and distribution is not preserved. □

It follows that $Q_{\perp}(\div, \widehat{\infty})$ is a minimal and total enlargement of $Q(\div)$, the fracterm calculus of which allows fracterm flattening, which implies the necessity of distributivity for Propositions 2, 3 and 4.

The structure $Q_{\perp}(\div, \widehat{\infty})$ we made is a modification of the wheel of rationals: we set $1 \div \widehat{\infty} = \perp$ instead of $1 \div \infty = 0$ as in wheels. The structure allows fracterm flattening but for wheels this is not the case.

6 The Wheel of Rational Numbers Does Not Allow Fracterm Flattening

Semantically, a wheel is an arithmetical structure containing an unsigned ∞ and a $\perp$ in which

$$\frac{1}{0} = \infty \text{ and } \frac{1}{\infty} = 0.$$

Wheels have an established algebraic theory [21, 16] and, in particular, the wheel of rational numbers has an equational specification under initial algebra semantics [12]. We will show that the wheel of rational numbers does not have flattening.

To see this we will use the following Proposition:

Proposition 6 *In a wheel (with $0 \neq 1$) each expression $p(x)$ which does not contain division, and no occurrences of variables different from x, and in which x actually occurs has the property that $p(\infty) = \infty$ or $p(\infty) = \perp$.*

Proof: With induction on the structure of P, for the constants 0 and 1 the required implication holds because the premise is false, there is no occurrence of x in a constant. For non-constant terms there are three cases.

If $p(\infty) = \infty$ then $-p(\infty) = -\infty = \infty$ according to the design of a wheel.

If $p(\infty) = \perp$ then $-p(\infty) = -\perp = \perp$ because $\perp$ is absorptive.

Thirdly, let $p \equiv r + s$, if $r\ (\infty) = \perp$ or $s(\infty) = \perp$ then $p(\infty) = \perp$, if $r(\infty) = s(\infty) = \infty$ then $p(\infty) = \infty + \infty = \perp$. Finally if $p \equiv r \cdot s$, if $r\ (\infty) = \perp$ or $s(\infty) = \perp$ then $p(\infty) = \perp$, if $r(\infty) = s(\infty) = \infty$ then $p(\infty) = \infty \cdot \infty = \infty$. □

Proposition 7 *There are* no *expressions p and q over the signature of commutative rings such that $1 + (1 \div x) = p \div q$ in a wheel of rationals.*

Proof: Assume that expressions p and q over the signature of rings exist such that

$$1 + (1 \div x) = p \div q$$

in a wheel of rationals. We will derive a contradiction from this assumption.

Clearly p and q need not contain any variable different from x. Further $p \div q$ is not a closed term, there must be at least one occurrence of x in either p or in q. If q contains no occurrence of x (so that $q \equiv q(0)$) the only way in which $1 + (1 \div 0) = p(0) \div q$ can hold is by having $q = 0$, which then fails on choosing $x = 1$. So we know that x has an occurrence in q because $1 + (1 \div 1) = p(1) \div 0$ cannot hold in a wheel for any value of $p(1)$.

Suppose that x does not occur in p, then choosing $x = \infty$ we know that $1 = 1 + 0 = 1 + (1 \div \infty) = p \div q(\infty)$. Now using Proposition 6 there are two cases: if $q(\infty) = \perp$ then $p \div q(\infty) = \perp \neq 1$, and otherwise if $q(\infty) = \perp$ then one uses the fact that there is no possible value for p in a wheel so that $1 = p \div \infty$.

Now, in a wheel $\infty \div \infty = \bot$ and from this it follows, that x cannot occur both in p and in q. Indeed otherwise, by once more using Proposition 6, there are two cases for $p(\infty)$ and two cases for $q(\infty)$ and in each of the resulting four cases $1 = 1 + (1 \div \infty) = p(\infty) \div q(\infty) = \bot$. $\square$

7 Concluding Remarks

7.1 The Research Programme

The origin of fracterm flattening is the simplification of numerical fractions in school arithmetic – a tricky topic for young pupils and the subject of educational research (e.g., [18]). We take an interest in the formal details of these matters, under the assumption that investigating the details of arithmetic with division can be simplified by working with total functions. This, in turn, involves our peripheral numbers, i.e., values which do not qualify as conventional numbers.

It has long been known from computer arithmetics that adding peripheral numbers to make data types total causes familiar arithmetical laws to fail. We have embarked on a programme to explore how they can be controlled and axiomatised by equations. To date, we have explored and mapped some of the basic algebra and logic of computing with peripheral elements. Focussed by computing with the rational numbers, our agenda has included:

(i) axiomatisations by equations and conditional equations;

(ii) initial algebra specifications of data types of rational numbers;

(iii) logical issues of independence, completeness, and incompleteness;

(iv) the semantics of equality for partial terms;

(v) calculating with fractions as a theory of terms;

(vi) new methods of specifying data types with partial operators.

Starting in 2007, together with several colleagues, we have analysed in some detail four semantic options for the totalisation of division: involutive meadows [10], common meadows [9], wheels [12] and transreals [11].

This paper addresses the question when precisely fracterms can be flattened. It was shown in [6] that flattening fails if $1/0 = 0$ is adopted, which implies that flattening requires the presence of peripheral numbers. Following [5], flattening of fracterms fails in complicated arithmetical data types involving a plurality of peripheral numbers including signed infinities $\infty, -\infty$.

Whilst using no peripheral numbers or a single absorbtive peripheral number $\perp$ leads to rather stable general semantical theories, in the course of this paper we have seen that adding just two peripheral numbers $\perp$ and ∞ complicates the semantic options and changes their algebraic consequences. Crudely speaking, we have seen that on adding $\perp$ and ∞, if we choose $1 \div \infty = 0$, as in wheels (Proposition 7), then flattening fails for fracterms, whereas if we choose $1 \div \infty = \perp$ then flattening is possible (Proposition 5).

We notice that different views on the proper nature of algebraic simplification exist. For instance, in [22], it is argued that simplifying x/x can lead to 1 under the side condition that $x \neq 0$ thereby preventing an undesired extension of the domain. More generally, [22] claims that simplification of expressions should preferably not be treated in terms of equality alone. Undeniably our notion of fracterm flattening is highly specific for equational logic.

We have been able to find reasonably limiting conditions under which fracterm flattening can be done. We found that common meadows as proposed in [7] provide the most plausible option for fracterm flattening, though not the only option.

The work on fracterm calculus in this paper is focused on particular arithmetical data types and thereby it complements preceding work in [7, 9, 13] taking the axioms for fracterm calculus as an equational theory amenable to proof theory and model theory, with a focus on classes of models rather than on particular models.

7.2 Options for Future Work

As to future work, we notice that the results of this paper are not yet best possible. Problem 2 leaves open various options for improving upon the results obtained above. However, even without solving Problem 2, we see fracterm flattening as a desirable feature for any fracterm calculus. For that reason, we see the above results as providing an intrinsic motivation for work on the common meadow of rational numbers, which comes close to being the most plausible arithmetical datatype admitting fracterm flattening, and for work on the model theory and proof theory of the corresponding fracterm calculus.

Regarding fracterm calculus, three open problems stand out. An open issue is whether or not its equational axioms can be represented by means of a complete term rewriting system. A weak negative result on that matter was obtained in [8]. A second open question concerns the axiomatisation of conditional equations that hold in the common meadows of rationals. An initial result is reported in [13] where it is established that the equational axioms of fracterm calculus cannot be taken for an axiomatisation of the relevant conditional equations. We notice that in the case of involutive meadows (i.e., working with $1/0 = 0$, see [1] for historical information on that proposal and [3] for a brief survey) the situation differs, because in the case of involutive meadows the equational theory also axiomatises the relevant conditional equations. Finally, the decidability of the equational theory of the common meadow $\mathsf{Enl}_\perp(Q(\div))$ is unknown.

These three open problems on fracterm calculus seem to be quite difficult. Perhaps less demanding challenges are options for further work too, we mention:

(i) axiomatizing ordered, or rather signed, common meadows and revisiting flattening in the presence of a sign function;

(ii) reworking axioms of probability in the context of the common meadow $\mathsf{Enl}_\perp(Q(\div))$, i.e., extending fracterm calculus with probability mass functions;

(iii) characterising the finite models of E_{wcm}, a question which was settled for involutive meadows in [15], a model of E_{wcm} with 12 elements is described in detail in [8];

(iv) extending fracterm calculus with various forms of conditional operators.

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