Global well-posedness of stochastic nematic liquid crystals with random initial and boundary conditions driven by multiplicative noise *

Lidan Wang † Jiang-Lun Wu ‡ Guoli Zhou §

Abstract

The flow of nematic liquid crystals can be described by a highly nonlinear stochastic hydrodynamical model, thus is often influenced by random fluctuations, such as uncertainty in specifying initial conditions and boundary conditions. In this article, we consider a 2-D stochastic nematic liquid crystals with the velocity field perturbed by affine-linear multiplicative white noise, with random initial data and random boundary conditions. Our main objective is to obtain the global well-posedness of the stochastic equations under the sufficient Malliavin regularity of the initial condition. The Malliavin calculus techniques play important roles when we obtain the global existence of the solutions to the stochastic nematic liquid crystal model with random initial and boundary conditions.

Keywords: Stochastic nematic liquid crystals flows; Anticipating initial condition; Malliavin derivative; A priori estimates; Skorohod integral

1 Introduction

The liquid crystal is an intermediate state of a matter, which possesses some typical properties of a liquid as well as some crystalline properties. One can observe the flow of nematic liquid crystals as slowly moving particles where the alignment of particles and the velocity of the fluid sway each other. The history of the hydrodynamic theory for liquid crystals traces back to 1960's, Ericksen [8] and Leslie [12] expanded the continuum theory to design the dynamics of the nematic liquid crystals. The so-called Ericksen-Leslie system is well designed for describing many special flows for the materials, especially for those with small molecules, and is widely applied in the engineering and mathematical communities for studying liquid crystals.

Later on, the most fundamental form of dynamical system describing the orientation as well as the macroscopic motion for the nematic liquid crystals was introduced by Lin-Liu [13]:

$$d\mathbf{v} + [(\mathbf{v}\cdot\nabla)\mathbf{v} - \mu\Delta\mathbf{v} + \nabla p]dt = -\lambda\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d})dt, \nabla\cdot\mathbf{v} = 0,$$
$$d\mathbf{d} + (\mathbf{v}\cdot\nabla)\mathbf{d}dt = \gamma(\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d})dt, |\mathbf{d}|^2 = 1.$$

In order to avoid the nonlinear gradient in the above system, as suggested by Lin-Liu [13], one can use the Ginzburg-Landau approximation to ease the constraint $|\mathbf{d}|^2 = 1$, and the corresponding approximation energy is

$$\int_D \left[\frac{1}{2}|\nabla\mathbf{d}|^2 + \frac{1}{4\beta^2}(|\mathbf{d}|^2-1)^2\right]dx.$$

*This work was partially supported by NNSF of China(Grant No. 11971077, 11801283), Chongqing Key Laboratory of Analytic Mathematics and Applications, Chongqing University, Chongqing, 401331,China.

†School of Statistics and Data Science, Nankai University and KLMDASR, Tianjin, 300071, P.R. China
lidanw.math@gmail.com

‡Department of Mathematics, Computational Foundry, Swansea University, Swansea SA1 8EN, UK
E-mail: j.l.wu@swansea.ac.uk.

§Corresponding author. School of Statistics and Mathematics, Chongqing University, Chongqing, 400044, P.R. China
E-mail:zhouguoli736@126.com.

Then one arrives at the following approximating system

$$
\begin{aligned}
&d\mathbf{v} + [(\mathbf{v}\cdot\nabla)\mathbf{v} - \mu\Delta\mathbf{v} + \nabla p]dt = -\lambda\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d})dt,\\
&\nabla\cdot\mathbf{v} = 0,\\
&d\mathbf{d} + (\mathbf{v}\cdot\nabla)\mathbf{d}dt = \gamma\left(\Delta\mathbf{d} - \frac{1}{\eta^2}(|\mathbf{d}|^2-1)\mathbf{d}\right)dt.
\end{aligned}
$$

The above system can be viewed as the simplest mathematical model whilst still keeps the most important mathematical structure as well as essential difficulties of the original Ericksen-Leslie system (see [13]). This deterministic system with Dirichlet boundary conditions has been well studied in a series of work theoretically (see [13, 14]) and numerically.

Along with the developments of the deterministic system, the random case has also drawn a lot interests in recent years. In the papers [1, 2], Brzeźniak-Hausenblas-Razafimandimby studied the nematic crystal flow model perturbed by multiplicative Gaussian noise and gave the global well-posedness for the weak and strong solutions in 2-D case. For the pure jump noise case in 2-D, Brzeźniak-Manna-Panda [3] obtained the global well-posedness for the martingale solution. A weak martingale solution result was also established for three dimensional stochastic nematic liquid crystals with pure jump noise in [3]. Meanwhile, the authors in [3] also proved the Wentzell-Freidlin type large deviations principle for the two dimensional stochastic nematic liquid crystals using weak convergence method.

As far as we know, the present work is the first attempt to study stochastic nematic liquid crystal equations with random initial and boundary conditions. Our motivation firstly derives from the limitation of predicting dynamical behavior in nonlinear systems due to uncertainty in initial data, which has been widely investigated (see [10]). The related study has drawn a lot attention in the geophysical community (see [18, 19, 20]). Our main result in this article implies that each stationary point of the present stochastic model generates a pathwise anticipating stationary solution of the Stratonovich stochastic equations. Another motivation of our work is that, near stationary solutions, multiplicative ergodic theory techniques ensure the existence of local random invariant manifolds which necessarily anticipate the driven noise. One can refer to [7, 16] and related works for more details. Hence, the study of a dynamic characterization of semiflows as well as invariant manifolds will appeal to the analysis of the stochastic nematic liquid crystal equations with anticipating initial data and corresponding random boundary conditions. This can be viewed as a necessary first step in the analysis of the regularity of invariant manifolds.

In this article, we consider in $D\times\mathbb{R}_+$, where $D\subset\mathbb{R}^2$ is a bounded domain with smooth boundary, the stochastic version of the nematic liquid crystals flows with random initial and boundary conditions. The model is formalized in details as follows:

$$
\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} - \mu\Delta\mathbf{v} + \nabla p + \lambda\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d}) = \sum_{k=1}^{\infty}\sigma_k\mathbf{v}\circ\dot{W}_k + \sigma_0\dot{\widetilde{W}}_0, \tag{1.1}
$$

$$
\nabla\cdot\mathbf{v}(t) = 0, \tag{1.2}
$$

$$
\mathbf{d}_t + (\mathbf{v}\cdot\nabla)\mathbf{d} - \gamma\left(\Delta\mathbf{d} - \frac{1}{\eta^2}(|\mathbf{d}|^2-1)\mathbf{d}\right) = 0. \tag{1.3}
$$

The unknowns are the fluid velocity field $\mathbf{v} = (v^1, v^2)\in\mathbb{R}^2$, the averaged macroscopic/continuum molecular orientation field $\mathbf{d} = (d^1, d^2, d^3)\in\mathbb{R}^3$, and the pressure function $p(x,t)$. μ,λ,γ are positive constants and stand for viscosity, the competition between kinetic and potential energies, and macroscopic elastic relaxation time for $\mathbf{d}$, respectively. The operation $[\nabla\mathbf{d}\odot\nabla\mathbf{d}]_{ij}$ yields a 2×2 matrix whose entry is given by

$$
[\nabla\mathbf{d}\odot\nabla\mathbf{d}]_{ij} = \sum_{k=1}^{3}\partial_{x_i}d^k\partial_{x_j}d^k,\ \ i,j=1,2.
$$

For the stochastic term, $\{W_k(t)_{t\in[0,T]}\}_{k\geq 1}$ is a sequence of independent, identically distributed one dimensional Brownian motions which are also independent of a space-time noise $\widetilde{W_0}(t,x)$. The space-time noise $\tilde{W}_0(t,x)$ is a Brownian in the time variable $t\in\mathbb{R}_+$ and smooth in the space variable $x\in D$. $\dot{W}_k, \dot{\widetilde{W}}_0$ are the formal time derivatives. The random forces are all defined on the same completely filtered Wiener space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$. We also assume that $\sigma_0\in\mathbb{R}$ and $\sum_{k=1}^{\infty}\sigma_k^2<\infty$.

We supplement the stochastic nematic liquid crystals equations with the following random initial and boundary conditions:

$$\mathbf{v}(x,0)=R_v(x),\ \mathbf{d}(x,0)=R_d(x),\quad \text{for } x\in D; \tag{IC}$$

$$\mathbf{v}(x,t)=0,\ \mathbf{d}(x,t)=R_d(x),\quad \text{for } (x,t)\in \partial D\times\mathbb{R}^+, \tag{BC}$$

where the initial conditions R_v, R_d are $\mathcal{F}\otimes\mathcal{B}(D)$-measurable random fields on D.

In this article, our main objective is to establish the global well-posedness of the stochastic model (1.1)-(1.3), with random initial and boundary conditions (IC)-(BC). In the following, we would like to list some essential difficulties and novelties of this article.

1. Compared with 2-D Navier-Stokes equations [16], the stochastic nematic liquid crystal model is more complicated since there are three nonlinear terms with different forms, and this causes essential difficulties in obtaining moment estimates. To overcome this difficulty, we take advantage of the special geometric structure of the nematic liquid crystal equation to obtain the adjoint estimate of $\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d})$ and $(\mathbf{v}\cdot\nabla)\mathbf{d}$. This is essential to establish the *a priori* estimates for the solutions.

2. In [16], the random initial data is required to be in the strong solution space, so as to show the weak solution is Malliavin differentiable. With an approximating argument, they have achieved the Malliavin derivatve of the weak solution without extra assumption. The aim of our article is to obtain the global well-posedness of the strong solution to the nematic liquid crystal equation with random initial and boundary conditions. During the procedure, we observe that the regularities of the strong solution are sufficient for obtaining nice bounds of nonlinear and coupling terms. Therefore, we prove the Malliavian differentiability of the strong solution without using approximating method. Moreover, we conclude that, if the existence of the strong solution is not available for a stochastic nonlinear equation, one can not obtain the Malliavin derivative of the weak solution. Throughout our work, we infer that, the structure of the stochastic equation should be regular enough, so as to address the global well-posedness of the stochastic model with random initial and boundary conditions. In other words, for a nonlinear partial differential equation, if the global well-posedness is only valid for the weak solution, then one cannot achieve the global well-posedness of the corresponding stochastic version with random initial data and random boundary condition. This is a significant difference between the partial differential equations and stochastic partial differential equations.

3. As shown in (1.1)-(1.3), our model deals with the case that only the velocity field is disturbed by the noise. This is because one needs to make use of the particular geometric structure of the nematic liquid crystals equations, *the basic balance law* (see [13] or Lemma 2.4 for reference), to obtain the energy estimates of velocity field as well as orientation field in certain regular spaces. We would also like to point out that, according to our work, the structure of the stochatic equations should be regular enough in order to obtain the global well-posedness of the stochastic model with random initial conditions. That is to say, for a nonlinear partial differential equations, if the global well-posedness is only valid in the weak sense, then one can not obtain the global well-posedness for the corresponding stochastic model with random initial conditions.

4. In this article, we consider the initial and boundary problems for the nematic liquid crystal equations with multiplicative noise, here both the initial and boundary conditions are random, which leads to the stochastic integral defined via Skorohod integral, instead of Itô integral. Thus, in order to show the global well-posedness result for the random initial and boundary problems (see Theorem 2.10 or Theorem 5.1), we must establish the regularities of the solutions with respect to the initial data as well as the sample path. Specificaly saying, we need to show the solutions $\mathbf{v}(t,R_v,\omega),\mathbf{d}(t,R_d,\omega)$ are differentiable with respect to the random fields R_v, R_d and sample path ω. We would like to mention that the regularity results established in Theorem 3.7 and Theorem 4.2 are new and profound which do not exist in previous work even for the deterministic case. In the proving process, the main difficulties lie in bounding the highly nonlinear terms and the coupling terms. So in order to conquer that, we make full use of the geometric structure obtaining more delicate estimates: Proposition 2.5, Proposition 2.6, which are key *a priori* estimates to establish the regularities of the present stochastic system with random initial and boundary conditions.

The rest of this article is organized as follows: in Section 2, we define some functional spaces and give the abstract model expression for the stochastic model. The main result is also given in this section. In Section 3, the *a priori* estimates and new regularity properties of the solutions are established, according to which, we discuss the Fréchet differentiability of the stochastic model with deterministic initial conditions.Furthermore, Malliavin differentiability of the stochastic model with deterministic initial conditions is discussed in Section 4 upon Galerkin approximations. Finally, in Section 5, we get back to the anticipating model and prove the global well-posedness of the stochastic nematic liquid crystals flows with random initial and boundary conditions.

As usual, the constant C may change from one line to another except that we give a special declaration, we denote by $C(a)$ a constant that depends on some parameter a.

2 Preliminaries and the main result

We first set a space

$$V = \{\mathbf{v} \in (C_0^\infty(D))^2 : \nabla \cdot \mathbf{v} = 0\}.$$

Now we define spaces $\mathbf{H}, \mathbf{V}$, and $\mathbf{H}^m$ as the closure of V in $(L^2(D))^2, (H^1(D))^2$ and $(H^m(D))^2$, respectively. Let $|\cdot|_2$ and $\langle \cdot, \cdot \rangle$ be the norm and inner product in the space $\mathbf{H}$, and let $\|\cdot\|_1$ and $\langle , \rangle_{\mathbf{V}}$ stand for the norm and the inner product in the space $\mathbf{V}$, where $\langle , \rangle_{\mathbf{V}}$ is defined by

$$\langle \mathbf{v}, \mathbf{u} \rangle_{\mathbf{V}} := \int_D \nabla \mathbf{v} \cdot \nabla \mathbf{u} dx, \quad \text{for } \ \mathbf{v}, \mathbf{u} \in \mathbf{V}.$$

Moreover, by Poincaré's inequality, there exists a constant c such that for any $\mathbf{v} \in \mathbf{V}$ we have $\|\mathbf{v}\|_1 \leq c|\nabla \mathbf{v}|_2$.

Let $\mathbb{H}^m = (H^m(D))^3, m = 0, 1, 2, ...$ When $m = 0$, set $\mathbb{H} = \mathbb{H}^0 = (L^2(D))^3$ for simplicity. Then similarly, let $|\cdot|_2$ and $\langle \cdot, \cdot \rangle$ be the norm and inner product in the space $\mathbb{H}$, and let $\|\cdot\|_1$ and $\langle , \rangle_{\mathbb{H}^1}$ stand for the norm and the inner product in the space $\mathbb{H}^1$, where $\langle , \rangle_{\mathbb{H}^1}$ is defined by

$$\langle \mathbf{d}, \mathbf{b} \rangle_{\mathbb{H}^1} := \int_D \mathbf{d} \cdot \mathbf{b} dx + \int_D \nabla \mathbf{d} \cdot \nabla \mathbf{b} dx, \quad \text{for } \ \mathbf{d}, \mathbf{b} \in \mathbb{H}^1.$$

Denote by $\mathbf{V}'$ the dual space of $\mathbf{V}$, and define the linear operator $A_1 : \mathbf{V} \mapsto \mathbf{V}'$ as follows:

$$\langle A_1 \mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle_{\mathbf{V}}, \text{ for } \mathbf{v}, \mathbf{u} \in \mathbf{V}.$$

Since the operator A_1 is positive self-adjoint with compact resolvent, by the classical spectral theorem, A_1 admits an increasing sequence of eigenvalues $\{\alpha_j\}$ diverging to infinity with the corresponding eigenvectors $\{e_j\}$. Assume

$$\sum_{i=1}^{\infty} \lambda_i \alpha_i^2 < \infty. \tag{2.1}$$

Let $D(A_1) := \{\mathbf{v} \in \mathbf{V}, A_1 \mathbf{v} \in \mathbf{H}\}$, since A_1^{-1} is a self-adjoint compact operator as well, due to the classic spectral theory, we can define the power A_1^s for any $s \in \mathbb{R}$. Moreover, $D(A_1)' = D(A_1^{-1})$ is the dual space of $D(A_1)$. And we have the compact embedding relationship

$$D(A_1) \subset \mathbf{V} \subset \mathbf{H} \cong \mathbf{H}' \subset \mathbf{V}' \subset D(A_1)', \text{ and } \langle \cdot, \cdot \rangle_{\mathbf{V}} = \langle A_1 \cdot, \cdot \rangle = \langle A_1^{\frac{1}{2}} \cdot, A_1^{\frac{1}{2}} \cdot \rangle.$$

We define another operator $A_2 : \mathbb{H}^2 \to \mathbb{H}$ by $-\Delta$ satisfying $D(A_2) := \{\mathbf{d} \in \mathbb{H}^2; \mathbf{d} = R_d(x) \in \mathbb{H}^2 \text{ on } \partial D\}$. Obviously, we have the compact embedding relationship

$$\mathbb{H}^2 \subset \mathbb{H}^1 \subset \mathbb{H} \cong \mathbb{H}' \subset (\mathbb{H}^1)' \subset (\mathbb{H}^2)'.$$

Define the trilinear form b_1 by

$$b_1(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^{2} \int_D u^i \partial_{x_i} v^j w^j dx, \text{ for } \mathbf{u} \in \mathbf{H} \text{ and } \mathbf{v}, \mathbf{w} \in \mathbf{V} \text{ and integral exists.}$$

If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$, then

$$|b_1(\mathbf{u}, \mathbf{v}, \mathbf{w})| \le c|\mathbf{u}|_2 \|\mathbf{v}\|_1 \|\mathbf{w}\|_1.$$

Now we define a bilinear form $B_1(\mathbf{u}, \mathbf{v}) := b_1(\mathbf{u}, \mathbf{v}, \cdot)$, then $B_1(\mathbf{u}, \mathbf{v}) \in \mathbf{V}'$ for $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ and enjoys the following bound:

$$\|B_1(\mathbf{u}, \mathbf{v})\|_{\mathbf{V}'} \le c|\mathbf{u}|_2 \|\mathbf{v}\|_1.$$

Lemma 2.1. *The mapping $B_1 : \mathbf{V} \times \mathbf{V} \to \mathbf{V}'$ is bilinear and continuous, and b_1, B_1 have the following properties:*

$$\begin{aligned} &b_1(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b_1(\mathbf{u}, \mathbf{w}, \mathbf{v}), \quad && \langle B_1(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle = -\langle B_1(\mathbf{u}, \mathbf{w}), \mathbf{v}\rangle, \quad && \textit{for } \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}.\\ &b_1(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, && \langle B_1(\mathbf{u}, \mathbf{v}), \mathbf{v}\rangle = 0, && \textit{for } \mathbf{u}, \mathbf{v} \in \mathbf{V}.\end{aligned}$$

Moreover, if $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$, we have

$$|b_1(\mathbf{u}, \mathbf{v}, \mathbf{w})| = \langle B_1(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle \le 2|\mathbf{u}|_2^{\frac{1}{2}} \|\mathbf{u}\|_1^{\frac{1}{2}} \|\mathbf{v}\|_1 |\mathbf{w}|_2^{\frac{1}{2}} \|\mathbf{w}\|_1^{\frac{1}{2}}. \tag{2.2}$$

Define another trilinear form b_2 by

$$b_2(\mathbf{v}, \mathbf{d}, \mathbf{b}) = \sum_{j=1}^{3} \sum_{i=1}^{2} \int_D v^i \partial_{x_i} d^j b^j \, dx \text{ for } \mathbf{v} \in \mathbf{H}, \mathbf{d} \text{ and } \mathbf{b} \in \mathbb{H}^1.$$

Define another bilinear map B_2 on $\mathbf{H} \times \mathbb{H}^1$ taking values in $\mathbb{H}^{-1}$ such that $\langle B_2(\mathbf{v}, \mathbf{d}), \mathbf{b}\rangle := b_2(\mathbf{v}, \mathbf{d}, \mathbf{b})$.

Lemma 2.2. *For $\mathbf{v} \in \mathbf{V}, \mathbf{b} \in \mathbb{H}^1, \mathbf{d} \in \mathbb{H}^2$, there exists a constant c such that*

$$|b_2(\mathbf{v}, \mathbf{d}, \mathbf{b})| = |\langle B_2(\mathbf{v}, \mathbf{d}), \mathbf{b}\rangle| \le c|\mathbf{v}|_2 \|\mathbf{d}\|_1 \|\mathbf{b}\|_1.$$

Moreover, we have

$$\|B_2(\mathbf{v}, \mathbf{d})\|_{(\mathbb{H}^1)'} \le c|v|_2 \|\mathbf{d}\|_1, \qquad \langle B_2(\mathbf{v}, \mathbf{d}), \mathbf{d}\rangle = 0.$$

Now define the trilinear form m by setting

$$m(\mathbf{d}, \mathbf{b}, \mathbf{v}) = \sum_{i,j=1}^{2} \sum_{k=1}^{3} \int_D \partial_{x_i} d^k \partial_{x_j} b^k \partial_{x_i} v^j \, dx.$$

There exists a bilinear operator M defined on $\mathbb{H}^2 \times \mathbb{H}^2$ taking values in $\mathbf{V}'$ such that $\langle M(\mathbf{d}, \mathbf{b}), \mathbf{v}\rangle := m(\mathbf{d}, \mathbf{b}, \mathbf{v})$. By interpolation inequality, we can easily obtain

Lemma 2.3. *For any $\mathbf{d}, \mathbf{b} \in \mathbb{H}^2, \mathbf{v} \in \mathbf{V}$, there exists a constant c such that*

$$|m(\mathbf{d}, \mathbf{b}, \mathbf{v})| \le c\|\mathbf{d}\|_1^{\frac{1}{2}} \|\mathbf{d}\|_2^{\frac{1}{2}} \|\mathbf{b}\|_1^{\frac{1}{2}} \|\mathbf{b}\|_2^{\frac{1}{2}} \|\mathbf{v}\|_1.$$

Thus, for any $\mathbf{d}, \mathbf{b} \in \mathbb{H}^2$,

$$\|M(\mathbf{d}, \mathbf{b})\|_{\mathbf{V}'} \le c\|\mathbf{d}\|_1^{\frac{1}{2}} \|\mathbf{d}\|_2^{\frac{1}{2}} \|\mathbf{b}\|_1^{\frac{1}{2}} |\Delta \mathbf{b}|_2^{\frac{1}{2}}.$$

Now we arrive at the useful basic balance law and we include the proof here for reader's convenience.

Lemma 2.4 (Basic balance law)**.** *For $\mathbf{u} \in \mathbf{V}, \mathbf{d} \in \mathbb{H}^2$, we have*

$$\langle M(\mathbf{d}, \mathbf{d}), \mathbf{u}\rangle = \langle B_2(\mathbf{u}, \mathbf{d}), \Delta \mathbf{d}\rangle.$$

Proof. By integration by parts and the boundary conditions (BC), we have

$$\langle M(\mathbf{d}, \mathbf{d}), \mathbf{u}\rangle = \langle \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}), \mathbf{u}\rangle = \int_D \partial_{x_i}(\partial_{x_i} d^k \partial_{x_j} d^k) u^j \, dx$$

$$= -\int_D \partial_{x_i} d^k \partial_{x_j} d^k \partial_{x_i} u^j dx,$$

and

$$\begin{aligned}\langle B_2(\mathbf{u},\mathbf{d}), \Delta\mathbf{d}\rangle =& \langle \mathbf{u}\cdot\nabla\mathbf{d}, \Delta\mathbf{d}\rangle = \int_D u^i \partial_{x_i} d^k \partial_{x_j x_j} d^k dx \\ =& -\int_D \partial_{x_j} u^i \partial_{x_i} d^k \partial_{x_j} d^k dx - \int_D u^i \partial_{x_i x_j} d^k \partial_{x_j} d^k dx \\ =& -\int_D \partial_{x_j} u^i \partial_{x_i} d^k \partial_{x_j} d^k dx = \langle M(\mathbf{d},\mathbf{d}), \mathbf{u}\rangle.\end{aligned}$$

□

In the following, we will state two important results that are used several times in the rest of the article.

Proposition 2.5. *For* $\mathbf{d}, \mathbf{b} \in \mathbb{H}^2$ *and* $\mathbf{u} \in \mathbf{V}$, *we have*

$$\langle M(\mathbf{d},\mathbf{b}), \mathbf{u}\rangle + \langle M(\mathbf{b},\mathbf{d}), \mathbf{u}\rangle = \langle B_2(\mathbf{u},\mathbf{d}), \Delta\mathbf{b}\rangle + \langle B_2(\mathbf{u},\mathbf{b}), \Delta\mathbf{d}\rangle.$$

Proof. By the bilinear property of the operator M, and the basic balance law in Lemma 2.4,

$$\begin{aligned}&\langle M(\mathbf{d},\mathbf{b}), \mathbf{u}\rangle + \langle M(\mathbf{b},\mathbf{d}), \mathbf{u}\rangle \\ =& \langle M(\mathbf{d},\mathbf{d}), \mathbf{u}\rangle - \langle M(\mathbf{d}-\mathbf{b},\mathbf{d}-\mathbf{b}), \mathbf{u}\rangle + \langle M(\mathbf{b},\mathbf{b}), \mathbf{u}\rangle \\ =& \langle B_2(\mathbf{u},\mathbf{d}), \Delta\mathbf{d}\rangle - \langle B_2(\mathbf{u},\mathbf{d}-\mathbf{b}), \Delta(\mathbf{d}-\mathbf{b})\rangle + \langle B_2(\mathbf{u},\mathbf{b}), \Delta\mathbf{b}\rangle \\ =& \langle B_2(\mathbf{u},\mathbf{d}), \Delta\mathbf{b}\rangle + \langle B_2(\mathbf{u},\mathbf{b}), \Delta\mathbf{d}\rangle.\end{aligned}$$

□

Proposition 2.6. *For* $\mathbf{d}, \mathbf{b} \in \mathbb{H}^3$ *and* $\mathbf{u} \in \mathbf{V}$, *and continuous functions* $\alpha(s), \beta(s), s \in [0,t]$, *we get*

$$\begin{aligned}&\int_0^t \alpha(s)\langle M(\mathbf{d},\mathbf{b}), \mathbf{u}\rangle ds + \int_0^t \alpha(s)\langle M(\mathbf{b},\mathbf{d}), \mathbf{u}\rangle ds - \int_0^t \beta(s)\langle B_2(\mathbf{u},\mathbf{b}), \Delta\mathbf{d}\rangle ds \\ \leq& 2(|\alpha|_\infty + |\beta|_\infty)\int_0^t \|\mathbf{d}\|_1^{1/2}\|\mathbf{d}\|_2^{1/2}\|\mathbf{b}\|_1^{1/2}\|\mathbf{b}\|_2^{1/2}\|\mathbf{u}\|_1 ds + |\beta|_\infty \int_0^t |\mathbf{u}|_2\|\mathbf{d}\|_1\|\mathbf{b}\|_3 ds;\end{aligned}$$

or

$$\begin{aligned}&\int_0^t \alpha(s)\langle M(\mathbf{d},\mathbf{b}), \mathbf{u}\rangle ds + \int_0^t \alpha(s)\langle M(\mathbf{b},\mathbf{d}), \mathbf{u}\rangle ds - \int_0^t \beta(s)\langle B_2(\mathbf{u},\mathbf{b}), \Delta\mathbf{d}\rangle ds \\ \leq& |\alpha|_\infty \int_0^t |\mathbf{u}|_2\|\mathbf{d}\|_1\|\mathbf{b}\|_3 ds + (|\alpha|_\infty + |\beta|_\infty)\int_0^t |\mathbf{u}|_2\|\mathbf{b}\|_1\|\mathbf{d}\|_3 ds,\end{aligned}$$

where $|\alpha|_\infty := \sup_{0\leq s\leq t} |\alpha(s)|, |\beta|_\infty := \sup_{0\leq s\leq t} |\beta(s)|$.

Proof. With different time function coeffecients, we apply the identity in Proposition 2.5, together with Lemma 2.2, Lemma 2.3,

$$\begin{aligned}&\int_0^t \alpha(s)\langle M(\mathbf{d},\mathbf{b}), \mathbf{u}\rangle ds + \int_0^t \alpha(s)\langle M(\mathbf{b},\mathbf{d}), \mathbf{u}\rangle ds - \int_0^t \beta(s)\langle B_2(\mathbf{u},\mathbf{b}), \Delta\mathbf{d}\rangle ds \\ =& \int_0^t (\alpha(s)-\beta(s))\langle M(\mathbf{d},\mathbf{b}), \mathbf{u}\rangle ds + \int_0^t (\alpha(s)-\beta(s))\langle M(\mathbf{b},\mathbf{d}), \mathbf{u}\rangle ds\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \beta(s)\langle M(\mathbf{d},\mathbf{b}),\mathbf{u}\rangle ds+\int_0^t \beta(s)\langle M(\mathbf{b},\mathbf{d}),\mathbf{u}\rangle ds-\int_0^t \beta(s)\langle B_2(\mathbf{u},\mathbf{b}),\Delta\mathbf{d}\rangle ds\\
=&\int_0^t (\alpha(s)-\beta(s))\langle M(\mathbf{d},\mathbf{b}),\mathbf{u}\rangle ds+\int_0^t (\alpha(s)-\beta(s))\langle M(\mathbf{b},\mathbf{d}),\mathbf{u}\rangle ds+\int_0^t \beta(s)\langle B_2(\mathbf{u},\mathbf{d}),\Delta\mathbf{b}\rangle ds\\
\le&2(|\alpha|_\infty+|\beta|_\infty)\int_0^t \|\mathbf{d}\|_1^{1/2}\|\mathbf{d}\|_2^{1/2}\|\mathbf{b}\|_1^{1/2}\|\mathbf{b}\|_2^{1/2}\|\mathbf{u}\|_1 ds+|\beta|_\infty\int_0^t |\mathbf{u}|_2\|\mathbf{d}\|_1\|\mathbf{b}\|_3 ds.
\end{aligned}
$$

Or, direclty applying Proposition 2.5 and Lemma 2.2,

$$
\begin{aligned}
&\int_0^t \alpha(s)\langle M(\mathbf{d},\mathbf{b}),\mathbf{u}\rangle ds+\int_0^t \alpha(s)\langle M(\mathbf{b},\mathbf{d}),\mathbf{u}\rangle ds-\int_0^t \beta(s)\langle B_2(\mathbf{u},\mathbf{b}),\Delta\mathbf{d}\rangle ds\\
=&\int_0^t \alpha(s)\langle B_2(\mathbf{u},\mathbf{d}),\Delta\mathbf{b}\rangle ds+\int_0^t \alpha(s)\langle B_2(\mathbf{u},\mathbf{b}),\Delta\mathbf{d}\rangle ds-\int_0^t \beta(s)\langle B_2(\mathbf{u},\mathbf{b}),\Delta\mathbf{d}\rangle ds\\
=&\int_0^t \alpha(s)\langle B_2(\mathbf{u},\mathbf{d}),\Delta\mathbf{b}\rangle ds+\int_0^t (\alpha(s)-\beta(s))\langle B_2(\mathbf{u},\mathbf{b}),\Delta\mathbf{d}\rangle ds\\
\le&|\alpha|_\infty\int_0^t |\mathbf{u}|_2\|\mathbf{d}\|_1\|\mathbf{b}\|_3 ds+(|\alpha|_\infty+|\beta|_\infty)\int_0^t |\mathbf{u}|_2\|\mathbf{b}\|_1\|\mathbf{d}\|_3 ds.
\end{aligned}
$$

□

Remark 2.7. *Proposition 2.5 and Proposition 2.6 are very important to bound the nonlinear terms when we try to obtain the regularities of the solutions with respect to initial data and sample path, please see the results in Section 3 and Section 4. In fact, these kinds of regularities are profound results which do not exist in previous work even for the deterministic case. In the proving process of these results (see Proposition 3.3, Proposition 3.5, Proposition 3.6, Theorem 3.7, Proposition 4.1, and Theorem 4.2), the difficulties lie in bounding the highly nonlinear term which obliges us to take full advantage of the delicate geometric structure of the stochastic nematic liquid crystals equations. Hence, Proposition 2.5 and Proposition 2.6 are the key observations to study the regularities of this stochastic model with random initial data and random boundary condition.*

Finally, $f(\mathbf{d})$ and $F(\mathbf{d})$ are given by

$$
f(\mathbf{d})=\frac{1}{\eta^2}(|\mathbf{d}|^2-1)\mathbf{d}\quad\text{and}\quad F(\mathbf{d})=\frac{1}{4\eta^2}(|\mathbf{d}|^2-1)^2.
$$

We define a function $\widetilde{f}:[0,\infty)\to\mathbb{R}$ by

$$
\widetilde{f}(x)=\frac{1}{\eta^2}(x-1),\qquad x\in\mathbb{R}_+,
$$

then $f(\mathbf{d})=\widetilde{f}(|\mathbf{d}|^2)\mathbf{d}$ and denote by $F:\mathbb{R}^3\to\mathbb{R}$ the Fréchet differentiable map such that for any $\mathbf{d}\in\mathbb{R}^3$ and $\xi\in\mathbb{R}^3$,

$$
F'(\mathbf{d})[\xi]=f(\mathbf{d})\cdot\xi.
$$

Set $\widetilde{F}$ to be an antiderivative of $\widetilde{f}$ such that $\widetilde{F}(0)=0$. Then

$$
\widetilde{F}(x)=\frac{1}{2\eta^2}(x^2-2x),\qquad x\in\mathbb{R}_+.
$$

Definition 2.8. *We say a continuous $\mathbf{H}\times\mathbb{H}^1$ valued random field $(\mathbf{v}(.,t),\mathbf{B}(.,t))_{t\in[0,T]}$ defined on $(\Omega,\mathcal{F},\mathbb{P})$ is a weak solution to problem (1.1)-(1.3) with initial and boundary conditions (IC) and (BC) if for $(\mathbf{v}_0,\mathbf{d}_0)\in\mathbf{H}\times\mathbb{H}^1$ the following conditions hold:*

$$
\mathbf{v}\in C([0,T];\mathbf{H})\cap L^2([0,T];\mathbf{V}),
$$

$$\mathbf{d} \in C([0,T];\mathbb{H}^1) \cap L^2([0,T];\mathbb{H}^2),$$

and the integral relation

$$\begin{aligned}
\langle \mathbf{v}(t), v\rangle + &\int_0^t \langle A_1\mathbf{v}(s), v\rangle ds + \int_0^t \langle \mathbf{v}(s)\cdot\nabla\mathbf{v}(s), v\rangle ds \\
&+ \int_0^t \langle \nabla\cdot(\nabla\mathbf{d}(s)\odot\nabla\mathbf{d}(s)), v\rangle ds = \langle \mathbf{v}_0, v\rangle + \int_0^t \langle \sum_{k=1}^{\infty}\sigma_k\mathbf{v}\circ dW_k(s), v\rangle + \langle W_0(t), v\rangle, \\
\langle \mathbf{d}(t), d\rangle + &\int_0^t \langle A_2\mathbf{d}(s), d\rangle ds + \int_0^t \langle \mathbf{v}(s)\cdot\nabla\mathbf{d}(s), d\rangle ds \\
&= \langle \mathbf{d}_0, d\rangle - \int_0^t \langle f(\mathbf{d}(s)), d\rangle ds,
\end{aligned}$$

hold a.s. for all $t \in [0,T]$ *and* $(v,d) \in \mathbf{V}\times\mathbb{H}$.

Definition 2.9. *We say a continuous* $\mathbf{V}\times\mathbb{H}^2$ *valued random field* $(\mathbf{v}(.,t), \mathbf{B}(.,t))_{t\in[0,T]}$ *defined on* $(\Omega,\mathcal{F},\mathbb{P})$ *is a strong solution to problem* (1.1)-(1.3) *with initial and boundary conditions* (IC) *and* (BC) *if for* $(\mathbf{v}_0,\mathbf{d}_0) \in \mathbf{V}\times\mathbb{H}^2$ *the following conditions hold:*

$$\begin{aligned}
&\mathbf{v} \in C([0,T];\mathbf{V}) \cap L^2([0,T];\mathbf{H}^2), \\
&\mathbf{d} \in C([0,T];\mathbb{H}^2) \cap L^2([0,T];\mathbb{H}^3),
\end{aligned}$$

and the integral relation

$$\begin{aligned}
\mathbf{v}(t) + &\int_0^t A_1\mathbf{v}(s)ds + \int_0^t \mathbf{v}(s)\cdot\nabla\mathbf{v}(s)ds \\
&+ \int_0^t \nabla\cdot(\nabla\mathbf{d}(s)\odot\nabla\mathbf{d}(s))ds = \mathbf{v}_0 + \int_0^t \sum_{k=1}^{\infty}\sigma_k\mathbf{v}\circ dW_k(s) + W_0(t), \\
\mathbf{d}(t) + &\int_0^t A_2\mathbf{d}(s)ds + \int_0^t \mathbf{v}(s)\cdot\nabla\mathbf{d}(s)ds = \mathbf{d}_0 - \int_0^t f(\mathbf{d}(s))ds,
\end{aligned}$$

hold a.s. for all $t \in [0,T]$.

Now the equations (1.1)-(1.3) can be written as

$$d\mathbf{v}(t) + [A_1\mathbf{v}(t) + B_1(\mathbf{v}(t)) + M(\mathbf{d}(t))]dt = \sum_{k=1}^{\infty}\sigma_k\mathbf{v}(t)\circ dW_k(t) + \sigma_0 dW_0(t), \tag{2.3}$$

$$\nabla\cdot\mathbf{v}(t) = 0, \tag{2.4}$$

$$d\mathbf{d}(t) + [A_2\mathbf{d}(t) + B_2(\mathbf{v}(t),\mathbf{d}(t)) + f(\mathbf{d}(t))]dt = 0, \tag{2.5}$$

with the initial conditions $\mathbf{v}(0) = R_v, \mathbf{d}(0) = R_d$.

Throughout the paper, we denote by $\mathcal{D}$ the Malliavin differentiation of random variables on the Wiener space $(\Omega,\mathcal{F},\mathbb{P})$. And we denote by $\mathcal{D}^{1,2}(\mathbf{H})$ the Malliavin Sobolev space of all $\mathcal{F}$-measurable and Malliavin differentiable random variables $\Omega\to\mathbf{H}$ with Malliavin derivatives owing second order moments. Correspondingly, $\mathcal{D}^{1,2}_{\text{loc}}(\mathbf{H})$ represents the space of random variables $\xi:\Omega\to\mathbf{H}$ that are locally in $\mathcal{D}^{1,2}(\mathbf{H})$.

We end this section with our main theorem, which gives the existence and uniqueness of solutions to the stochastic model (1.1)-(1.3), or (2.3)-(2.5), with random boundary conditions (BC) and random initial conditions (IC).

Theorem 2.10. *Assume the initial random field* $R_v \in \mathcal{D}^{1,2}_{loc}(\mathbf{H})\cap\mathbf{V}$, $R_d \in \mathcal{D}^{1,2}_{loc}(\mathbb{H}^1)\cap\mathbb{H}^2$, *then the stochastic nematic liquid crystals flows have a unique strong solution* $(\mathbf{v}(t,R_v),\mathbf{d}(t,R_d))$ *for all* $t\in[0,T]$.

Moreover, $\mathbf{v}(t,R_v) \in \mathcal{D}^{1,2}_{loc}(\mathbf{H}), \mathbf{d}(t,R_d) \in \mathcal{D}^{1,2}_{loc}(\mathbb{H}^1)$ *for all* $t\in[0,T]$.

3 Regularity of Fréchet derivatives

3.1 Decomposition

Consider the stochastic model with a deterministic initial condition $(\mathbf{v}_0, \mathbf{d}_0) \in \mathbf{V} \times \mathbb{H}^2$,

$$d\mathbf{v}(t,\mathbf{v}_0) + [A_1\mathbf{v}(t,\mathbf{v}_0) + B_1(\mathbf{v}(t,\mathbf{v}_0)) + M(\mathbf{d}(t,\mathbf{d}_0))]dt = \sum_{k=1}^{\infty}\sigma_k\mathbf{v}(t)\circ dW_k(t) + \sigma_0 dW_0(t), \tag{3.1a}$$

$$d\mathbf{d}(t,\mathbf{d}_0) + [A_2\mathbf{d}(t,\mathbf{d}_0) + B_2(\mathbf{v}(t,\mathbf{v}_0),\mathbf{d}(t,\mathbf{d}_0)) + f(\mathbf{d}(t,\mathbf{d}_0))]dt = 0, \tag{3.1b}$$

$$\mathbf{v}(0,\mathbf{v}_0) = \mathbf{v}_0 \in \mathbf{V},\ \ \mathbf{d}(0,\mathbf{d}_0) = \mathbf{d}_0 \in \mathbb{H}^2. \tag{3.1c}$$

The global well-posedness for the strong solution of (3.1) has been studied in [2] and [9], and it is known that under the condition (2.1), for any $T > 0$, $\mathbf{v}(\cdot,\mathbf{v}_0) \in C([0,T];\mathbf{V}) \cap L^2([0,T];\mathbf{H}^2)$, $\mathbf{d}(\cdot,\mathbf{d}_0) \in C([0,T];\mathbb{H}^2) \cap L^2([0,T];\mathbb{H}^3)$.

Define

$$Q(t) := \exp\left\{\sum_{k=1}^{\infty}\sigma_k W_k(t)\right\},$$

then $Q(0) = 1$, by Novikov condition and Doob's maximal inequality, for any fixed $T > 0$, we have

$$\mathbb{E}\left[\sup_{0\le t\le T} Q^n(t)\right] < \infty, \text{ for any } n \in \mathbb{Z}.$$

For simplicity of notations, we use $|Q|_\infty$ represent $\sup_{0\le s\le t}|Q(s)|$.

Let $Z(t)$ be the unique solution to the stochastic equation:

$$\begin{aligned} &dZ(t) = -A_1 Z(t)dt + \sigma_0 Q(t)^{-1}dW_0(t);\\ &Z(0) = 0;\\ &Z(t,x) = 0,\ x \in \partial D, t \ge 0. \end{aligned} \tag{3.2}$$

Now define $\mathbf{u}(t,\mathbf{v}_0) := \mathbf{v}(t,\mathbf{v}_0)Q(t)^{-1} - Z(t)$, $t \ge 0$, then by Itô's formula, $\mathbf{u}, \mathbf{d}$ satisfy the following equations:

$$d\mathbf{u}(t) + [A_1\mathbf{u}(t) + Q(t)B_1\big(\mathbf{u}(t) + Z(t)\big) + Q(t)^{-1}M(\mathbf{d}(t))]dt = 0, \tag{3.3a}$$

$$d\mathbf{d}(t) + \Big[A_2\mathbf{d}(t) + Q(t)B_2\big(\mathbf{u}(t) + Z(t), \mathbf{d}(t)\big) + f(\mathbf{d}(t))\Big]dt = 0, \tag{3.3b}$$

$$\mathbf{u}(0) = \mathbf{v}_0,\ \ \mathbf{d}(0) = \mathbf{d}_0. \tag{3.3c}$$

Using the estimates in [1] or [9], we obtain the following estimates,

Proposition 3.1. *For $\mathbf{v}_0 \in \mathbf{V}, \mathbf{d}_0 \in \mathbb{H}^2$ and $\omega \in \Omega$. Denote by $(\mathbf{u}(t,\mathbf{v}_0,\omega), \mathbf{d}(t,\mathbf{d}_0,\omega))$ the unique solution to (3.3) on $[0,T]$. Then the following estimates hold:*

$$\begin{aligned} &\sup_{0\le t\le T}[|\mathbf{u}(t,\mathbf{v}_0,\omega)|_2^2 + \|\mathbf{d}(t,\mathbf{d}_0,\omega)\|_1^2] + \int_0^T \|\mathbf{u}(t,\mathbf{v}_0,\omega)\|_1^2 dt + \int_0^T \|\mathbf{d}(t,\mathbf{d}_0,\omega)\|_2^2 dt\\ \le& c(|\mathbf{v}_0|_2, \|\mathbf{d}_0\|_1, |Q|_\infty, \sup_{0\le t\le T}\|Z\|_2, T), \end{aligned}$$

and

$$\begin{aligned} &\sup_{0\le t\le T}[\|\mathbf{u}(t,\mathbf{v}_0,\omega)\|_1^2 + \|\mathbf{d}(t,\mathbf{d}_0,\omega)\|_2^2] + \int_0^T \|\mathbf{u}(t,\mathbf{v}_0,\omega)\|_2^2 dt + \int_0^T \|\mathbf{d}(t,\mathbf{d}_0,\omega)\|_3^2 dt\\ \le& c(\|\mathbf{v}_0\|_1, \|\mathbf{d}_0\|_2, |Q|_\infty, \sup_{0\le t\le T}\|Z\|_2, \int_0^T \|Z\|_3^2 dt, T). \end{aligned}$$

Remark 3.2. *Now set* $\eta := \sqrt{\sum_{k=1}^{\infty}\sigma_k^2}$ *and define*

$$W(t) := \frac{1}{\eta}\sum_{k=1}^{\infty}\sigma_k W_k(t), t \geq 0.$$

Then $W(t)$ *is a new one-dimensional standard Brownian motion with*

$$\sum_{k=1}^{\infty}\sigma_k \mathbf{v}(t, v_0)\circ dW_k(t) = \eta\mathbf{v}(t, v_0)\circ dW(t).$$

Without loss of generality, here and in the future, we assume the stochastic model (3.1) *is driven by Brownian motion* W *(with* $\eta = 1$*), and* $Q(t) = \exp\{W(t)\}$.

We now further discuss the continuity property of $\mathbf{u}(t, \mathbf{v}_0), \mathbf{d}(t, \mathbf{d}_0)$ with respect to the initial data $(\mathbf{v}_0, \mathbf{d}_0)$.

Proposition 3.3. *For* $(\mathbf{v}_0, \mathbf{d}_0), (\mathbf{u}_0, \mathbf{b}_0) \in \mathbf{V}\times\mathbb{H}^2$*, and any* $h \in \mathbb{R}$*,*

$$\lim_{h\to 0}\sup_{\|\mathbf{u}_0\|_1+\|\mathbf{b}_0\|_2\leq 1}\Big\{\sup_{0\leq t\leq T}[\|\mathbf{u}(t, \mathbf{v}_0 + h\mathbf{u}_0) - \mathbf{u}(t, \mathbf{v}_0)\|_1^2 + \|\mathbf{d}(t, \mathbf{d}_0 + h\mathbf{b}_0) - \mathbf{d}(t, \mathbf{d}_0)\|_2^2] \\ + \int_0^T \|\mathbf{u}(t, \mathbf{v}_0 + h\mathbf{u}_0) - \mathbf{u}(t, \mathbf{v}_0)\|_2^2 dt + \int_0^T \|\mathbf{d}(t, \mathbf{d}_0 + h\mathbf{b}_0) - \mathbf{d}(t, \mathbf{d}_0)\|_3^2 dt\Big\} = 0. \quad (3.4)$$

Proof. We use the equations satisfied by $(\mathbf{u}(t, \mathbf{v}_0 + h\mathbf{u}_0, \omega), \mathbf{d}(t, \mathbf{d}_0 + h\mathbf{b}_0, \omega))$ and $(\mathbf{u}(t, \mathbf{v}_0, \omega), \mathbf{d}(t, \mathbf{d}_0, \omega))$, then multiply $\mathbf{u}(t, \mathbf{v}_0 + h\mathbf{u}_0, \omega) - \mathbf{u}(t, \mathbf{v}_0, \omega)$ with $A_1(\mathbf{u}(t, \mathbf{v}_0 + h\mathbf{u}_0, \omega) - \mathbf{u}(t, \mathbf{v}_0, \omega))$ and integrate over $\mathcal{D}$, for simplicity, we denote by $\bar{\mathbf{u}}(t, \omega) = \mathbf{u}(t, \mathbf{v}_0 + h\mathbf{u}_0) - \mathbf{u}(t, \mathbf{v}_0)$, $\bar{\mathbf{d}}(t, \omega) = \mathbf{d}(t, \mathbf{d}_0 + h\mathbf{b}_0) - \mathbf{d}(t, \mathbf{d}_0)$,

$$\begin{aligned}\|\bar{\mathbf{u}}(t, \omega)\|_1^2 =& h^2\|\mathbf{u}_0\|_1^2 - 2\int_0^t |A_1\bar{\mathbf{u}}(s, \omega)|_2^2 ds \\ & - 2\int_0^t Q(s)\langle B_1(\mathbf{u}(s, \mathbf{v}_0 + h\mathbf{u}_0) + Z(s), \bar{\mathbf{u}}(s, \omega)), A_1\bar{\mathbf{u}}(s, \omega)\rangle ds \\ & - 2\int_0^t Q(s)\langle B_1(\bar{\mathbf{u}}(s, \omega), \mathbf{u}(s, \mathbf{v}_0)) + Z(s), A_1\bar{\mathbf{u}}(s, \omega)\rangle ds \\ & - 2\int_0^t Q(s)^{-1}\langle M(\mathbf{d}(s, \mathbf{d}_0 + h\mathbf{b}_0), \bar{\mathbf{d}}(s, \omega)), A_1\bar{\mathbf{u}}(s, \omega)\rangle ds \\ & - 2\int_0^t Q(s)^{-1}\langle M(\bar{\mathbf{d}}(s, \omega), \mathbf{d}(s, \mathbf{d}_0)), A_1\bar{\mathbf{u}}(s, \omega)\rangle ds \\ =: & k_1 + \cdots + k_6. \end{aligned} \quad (3.5)$$

By Lemma 2.1 and Young's inequality,

$$\begin{aligned} k_3 =& 2\int_0^t Q(s)\langle B_1(\nabla(\mathbf{u}(s, \mathbf{v}_0 + h\mathbf{u}_0) + Z(s)), \bar{\mathbf{u}}(s, \omega)), \nabla\bar{\mathbf{u}}(s, \omega)\rangle ds \\ \leq & \varepsilon\int_0^t \|\bar{\mathbf{u}}(s, \omega)\|_2^2 ds \\ & + c|Q|_\infty^{4/3}\sup_{0\leq t\leq T}\|\mathbf{u}(t, \mathbf{v}_0 + h\mathbf{u}_0) + Z(t)\|_1^{2/3}\int_0^t \|\mathbf{u}(s, \mathbf{v}_0 + h\mathbf{u}_0) + Z(s)\|_2^{2/3}\|\bar{\mathbf{u}}(s, \omega)\|_1^2 ds. \end{aligned}$$

In view of Sobolev's embedding theorem and Young's inequality, we obtain that

$$k_4 \leq c\sup_{0\leq t\leq T}|Q(t)|\int_0^t |\bar{\mathbf{u}}|_\infty\|\mathbf{u}(s, \mathbf{v}_0) + Z(s)\|_1|A_1\bar{\mathbf{u}}(s, \omega)|_2 ds$$

$$\leq \varepsilon \int_0^t |A_1 \bar{\mathbf{u}}(s,\omega)|_2 ds + C \sup_{0\leq t\leq T} |Q(t)|^2 \int_0^t \|\bar{\mathbf{u}}\|_1^2 \|\mathbf{u}(s,\mathbf{v}_0) + Z(s)\|_1^2 ds.$$

By Proposition 2.5,

$$\begin{aligned} k_5 + k_6 =& 2\int_0^t Q(s)^{-1} \langle B_2(\Delta \bar{\mathbf{u}}(s,\omega), \bar{\mathbf{d}}(s,\omega)), \Delta \mathbf{d}(s, \mathbf{d}_0 + h\mathbf{b}_0)\rangle ds \\ &- 2\int_0^t Q(s)^{-1} \langle B_2(\Delta \bar{\mathbf{u}}(s,\omega), \mathbf{d}(s,\mathbf{d}_0)), \Delta \bar{\mathbf{d}}(s,\omega)\rangle ds \\ \leq& \varepsilon \int_0^t |\Delta \bar{\mathbf{u}}|_2^2 ds + Ch^2 \|\mathbf{b}_0\|_2^2 \sup_{s\in[0,T]} |Q^{-2}| (|\Delta \mathbf{d}(s,\mathbf{d}_0)|_2^2 + |\Delta \mathbf{d}(s,\mathbf{d}_0 + h\mathbf{b}_0)|_2^2). \end{aligned} \tag{3.6}$$

Now taking inner product between $\Delta \bar{\mathbf{d}}_t$ and $\Delta \bar{\mathbf{d}}$, we obtain that

$$\begin{aligned} \|\bar{\mathbf{d}}(t,\omega)\|_2^2 =& h^2 \|\mathbf{d}_0\|_2^2 - 2\int_0^t \langle \Delta A_2 \bar{\mathbf{d}}(s,\omega), \Delta \bar{\mathbf{d}}(s,\omega)\rangle ds \\ &- 2\int_0^t Q(s) \langle \Delta B_2(\mathbf{u}(s,\mathbf{v}_0 + h\mathbf{u}_0) + Z(s), \bar{\mathbf{d}}(s,\omega)), \Delta \bar{\mathbf{d}}(s,\omega)\rangle ds \\ &- 2\int_0^t Q(s) \langle \Delta B_2(\bar{\mathbf{u}}(s,\omega), \mathbf{d}(s,\mathbf{d}_0)), \Delta \bar{\mathbf{d}}(s,\omega)\rangle ds \\ &- 2\int_0^t \langle \Delta f(\mathbf{d}(s,\mathbf{d}_0 + h\mathbf{b}_0)), \Delta \bar{\mathbf{d}}(s,\omega)\rangle ds + 2\int_0^t \langle \Delta f(\mathbf{d}(s,\mathbf{d}_0)), \Delta \bar{\mathbf{d}}(s,\omega)\rangle ds \\ &=: l_1 + \cdots + l_6. \end{aligned} \tag{3.7}$$

First we have $l_2 = -2\int_0^t \|\bar{\mathbf{d}}(s,\omega)\|_3^2 ds$. By Lemma 2.2, and $\langle B_2(\mathbf{u},\mathbf{d}),\mathbf{d}\rangle = 0$, we have

$$\begin{aligned} l_3 =& -2\int_0^t Q(s) \langle B_2(\Delta(\mathbf{u}(s,\mathbf{v}_0 + h\mathbf{u}_0) + Z(s)), \bar{\mathbf{d}}(s,\omega)), \Delta \bar{\mathbf{d}}(s,\omega)\rangle ds \\ &- 2\int_0^t Q(s) \langle B_2(\nabla(\mathbf{u}(s,\mathbf{v}_0 + h\mathbf{u}_0) + Z(s)), \nabla \bar{\mathbf{d}}(s,\omega)), \Delta \bar{\mathbf{d}}(s,\omega)\rangle ds \\ \leq& \varepsilon \int_0^t \|\bar{\mathbf{d}}(s,\omega)\|_3^2 ds + c|Q|_\infty^2 \int_0^t \|\mathbf{u}(s,\mathbf{v}_0 + h\mathbf{u}_0) + Z(s)\|_2^2 \|\bar{\mathbf{d}}(s,\omega)\|_1^2 ds \\ &+ c|Q|_\infty^2 \sup_{0\leq t\leq T} \|\mathbf{u}(t,\mathbf{v}_0 + h\mathbf{u}_0) + Z(t)\|_1^2 \int_0^t \|\bar{\mathbf{d}}(s,\omega)\|_2^2 ds. \end{aligned}$$

With (3.6), and by Proposition 2.5,

$$\begin{aligned} k_6 + l_4 \leq& \varepsilon \int_0^t \|\bar{\mathbf{u}}(s,\omega)\|_2^2 ds + \varepsilon \int_0^t \|\bar{\mathbf{d}}(s,\omega)\|_3^2 ds \\ &+ c(|Q|_\infty + |Q^{-1}|_\infty)^4 \sup_{0\leq t\leq T} \|\mathbf{d}(t,\mathbf{d}_0 + h\mathbf{b}_0)\|_1^2 \|\mathbf{d}(t,\mathbf{d}_0 + h\mathbf{b}_0)\|_2^2 \int_0^t \|\bar{\mathbf{d}}(s,\omega)\|_2^2 ds \\ &+ c(|Q|_\infty + |Q^{-1}|_\infty)^2 \sup_{0\leq t\leq T} \|\mathbf{d}(t,\mathbf{d}_0 + h\mathbf{b}_0)\|_2 \int_0^t \|\mathbf{d}(s,\mathbf{d}_0 + h\mathbf{b}_0)\|_3 \|\bar{\mathbf{d}}(s,\omega)\|_1 \|\bar{\mathbf{d}}(s,\omega)\|_2 ds \\ &+ c(|Q^{-1}|_\infty + |Q|_\infty)^4 \sup_{0\leq t\leq T} \|\mathbf{d}(t,\mathbf{d}_0)\|_1^2 \|\mathbf{d}(t,\mathbf{d}_0)\|_2^2 \int_0^t \|\bar{\mathbf{d}}(s,\omega)\|_2^2 ds \\ &+ c(|Q^{-1}|_\infty + |Q|_\infty)^2 \sup_{0\leq t\leq T} \|\mathbf{d}(t,\mathbf{d}_0)\|_2 \int_0^t \|\mathbf{d}(s,\mathbf{d}_0)\|_3 \|\bar{\mathbf{d}}(s,\omega)\|_1 \|\bar{\mathbf{d}}(s,\omega)\|_2 ds \\ &+ c|Q|_\infty^2 \int_0^t \|\mathbf{d}(s,\mathbf{d}_0 + h\mathbf{b}_0)\|_3^2 \|\bar{\mathbf{d}}(s,\omega)\|_1^2 ds + c|Q|_\infty^2 \int_0^t \|\mathbf{d}(s,\mathbf{d}_0)\|_3^2 |\bar{\mathbf{u}}(s,\omega)|_2^2 ds \end{aligned}$$

$$+ c|Q|_\infty^2 \int_0^t \|\mathbf{d}(s,\mathbf{d}_0)\|_2^2 \|\bar{\mathbf{u}}(s,\omega)\|_1^2 ds.$$

Finally, we have

$$\begin{aligned} l_5 + l_6 =& -2\int_0^t \langle \nabla (f(\mathbf{d}(s,\mathbf{d}_0 + h\mathbf{b}_0)) - f(\mathbf{d}(s,\mathbf{d}_0))), \nabla\Delta\bar{\mathbf{d}}(s,\omega)\rangle ds \\ \leq& \varepsilon \int_0^t \|\bar{\mathbf{d}}(s,\omega)\|_3^2 ds + c\int_0^t \|\bar{\mathbf{d}}(s,\omega)\|_1^2 ds. \end{aligned}$$

Combining the estimates for terms in (3.5) and (3.7) and applying Gronwall's inequality yield that

$$\sup_{0\leq t\leq T}\{\|\bar{\mathbf{u}}(t)\|_1^2 + \|\bar{\mathbf{d}}(t)\|_2^2\} + \int_0^T \|\bar{\mathbf{u}}(s)\|_2^2 ds + \int_0^T \|\bar{\mathbf{d}}(s)\|_3^2 ds \leq h^2[\|\mathbf{u}_0\|_1^2 + \|\mathbf{b}_0\|_2^2] g_1(T),$$

where

$$\begin{aligned} g_1(T) := \exp c\Big\{& T + |Q|_\infty^{4/3} \sup_{0\leq t\leq T}\|\mathbf{u}(t,\mathbf{v}_0 + h\mathbf{u}_0) + Z(t)\|_1^{2/3} \int_0^T \|\mathbf{u}(t,\mathbf{v}_0 + h\mathbf{u}_0) + Z(t)\|_2^{2/3} dt \\ &+ |Q|_\infty^2 \sup_{0\leq t\leq T}\|\mathbf{u}(t,\mathbf{v}_0) + Z(t)\|_1^2 T + |Q|_\infty^4 \sup_{0\leq t\leq T}\|\mathbf{u}(t,\mathbf{v}_0) + Z(t)\|_1^2 \int_0^T \|\mathbf{u}(t,\mathbf{v}_0) + Z(t)\|_2^2 dt \\ &+ |Q|_\infty^2 \int_0^T \|\mathbf{u}(t,\mathbf{v}_0 + h\mathbf{u}_0) + Z(t)\|_2^2 dt + |Q|_\infty^2 \sup_{0\leq t\leq T}\|\mathbf{u}(t,\mathbf{v}_0 + h\mathbf{u}_0) + Z(t)\|_1^2 T \\ &+ (|Q|_\infty + |Q^{-1}|_\infty)^4 \sup_{0\leq t\leq T}\|\mathbf{d}(t,\mathbf{d}_0 + h\mathbf{b}_0)\|_1^2 \|\mathbf{d}(t,\mathbf{d}_0 + h\mathbf{b}_0)\|_2^2 T \\ &+ (|Q|_\infty + |Q^{-1}|_\infty)^2 \sup_{0\leq t\leq T}\|\mathbf{d}(t,\mathbf{d}_0 + h\mathbf{b}_0)\|_2 \int_0^T \|\mathbf{d}(t,\mathbf{d}_0 + h\mathbf{b}_0)\|_3 dt \\ &+ (|Q^{-1}|_\infty + |Q|_\infty)^4 \sup_{0\leq t\leq T}\|\mathbf{d}(t,\mathbf{d}_0)\|_1^2 \|\mathbf{d}(t,\mathbf{d}_0)\|_2^2 T \\ &+ (|Q^{-1}|_\infty + |Q|_\infty)^2 \sup_{0\leq t\leq T}\|\mathbf{d}(t,\mathbf{d}_0)\|_2 \int_0^T \|\mathbf{d}(t,\mathbf{d}_0)\|_3 dt \\ &+ |Q|_\infty^2 \int_0^T \|\mathbf{d}(t,\mathbf{d}_0 + h\mathbf{b}_0)\|_3^2 dt + |Q|_\infty^2 \int_0^T \|\mathbf{d}(t,\mathbf{d}_0)\|_3^2 dt + |Q|_\infty^2 \int_0^T \|\mathbf{d}(t,\mathbf{d}_0)\|_2^2 dt \Big\}. \end{aligned}$$

This indicates that for all $\mathbf{v}_0, \mathbf{u}_0 \in \mathbf{V}, \mathbf{d}_0, \mathbf{b}_0 \in \mathbb{H}^2$, and any $h \in \mathbb{R}$, (3.4) holds. □

3.2 Fréchet derivatives

In this subsection, we will show the regularity of Fréchet derivatives. We denote by $(\mathbf{u}(t,\mathbf{v}_0,\omega), \mathbf{d}(t,\mathbf{d}_0,\omega))$ or $(\mathbf{u}(t,\mathbf{v}_0), \mathbf{d}(t,\mathbf{d}_0))$ the solution to the random problem (3.1). Then for $(\mathbf{v}_0,\mathbf{d}_0) \in \mathbf{V}\times\mathbb{H}^2$, we aim to show that the solution map $(\mathbf{v}_0,\mathbf{d}_0) \mapsto (\mathbf{u}(t,\mathbf{v}_0,\omega), \mathbf{d}(t,\mathbf{d}_0,\omega)) \in \mathbf{V}\times\mathbb{H}^2$ has continuous Fréchet derivatives given by

$$D\mathbf{u}(t,\mathbf{v}_0,\omega) = \hat{\mathbf{u}}(t,\mathbf{v}_0,\omega)(\cdot), \qquad D\mathbf{d}(t,\mathbf{d}_0,\omega) = \hat{\mathbf{d}}(t,\mathbf{d}_0,\omega)(\cdot),$$

where $\hat{\mathbf{u}}(t,\mathbf{v}_0)(\mathbf{u}_0), \hat{\mathbf{d}}(t,\mathbf{d}_0)(\mathbf{b}_0)$, with $(\mathbf{u}_0,\mathbf{b}_0) \in \mathbf{V}\times\mathbb{H}^2$, satisfy the the following random equations:

$$\begin{aligned} \hat{\mathbf{u}}(t,\mathbf{v}_0)(\mathbf{u}_0) =& \mathbf{u}_0 - \int_0^t A_1\hat{\mathbf{u}}(s,\mathbf{v}_0)(\mathbf{u}_0) ds - \int_0^t Q(s)B_1(\hat{\mathbf{u}}(s,\mathbf{v}_0)(\mathbf{u}_0), \mathbf{u}(s,\mathbf{v}_0) + Z(s)) ds \\ &- \int_0^t Q(s)B_1(\mathbf{u}(s,\mathbf{v}_0) + Z(s), \hat{\mathbf{u}}(s,\mathbf{v}_0)(\mathbf{u}_0)) ds - \int_0^t Q(s)^{-1}M(\hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0), \mathbf{d}(s,\mathbf{d}_0)) ds \\ &- \int_0^t Q(s)^{-1}M(\mathbf{d}(s,\mathbf{d}_0), \hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0)) ds; \end{aligned} \tag{3.8}$$

$$\hat{\mathbf{d}}(t,\mathbf{d}_0)(\mathbf{b}_0) = \mathbf{b}_0 - \int_0^t A_2\hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0)ds - \int_0^t Q(s)B_2(\hat{\mathbf{u}}(s,\mathbf{v}_0)(\mathbf{u}_0),\mathbf{d}(s,\mathbf{d}_0))ds$$
$$- \int_0^t Q(s)B_2(\mathbf{u}(s,\mathbf{v}_0)+Z(s),\hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0))ds - \int_0^t \nabla_3 f(\mathbf{d}(s,\mathbf{d}_0))\cdot\hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0)ds, \quad (3.9)$$

where $\nabla_3 = (\partial_x,\partial_y,\partial_z)$. Obviously, the equations (3.8)-(3.9) are linear, the global well-posedness of the strong solutions is easy to show. We omit it here. One can see [1], [21] and other references.

Remark 3.4. *Since the stochastic equations (1.1)-(1.3) is a coupled system of linear momentum (1.1) and and angular momentum (1.3), then when we consider the differentiability of the solutions to (1.1)-(1.3) with respect to the initial data, we need to calculate the derivative of equation (1.1) and (1.3) with respect to the initial data $(\mathbf{v}_0,\mathbf{d}_0)$ at the same time. In other words, if we only consider the derivative of the equation of $\mathbf{u}$ with respect to $\mathbf{v}_0$ or the derivative of the equation of $\mathbf{d}$ with respect to $\mathbf{d}_0$ is not true. See, for example, (3.8)) and (3.9). Therefore, the coupled system (1.1)-(1.3) is more difficult than 2D stochastic Navier-Stokes equations and other stochastic hydrodynamic systems. We should carefully deal with this kind of stochastic coupled system.*

Proposition 3.5. *For $(\mathbf{v}_0,\mathbf{d}_0),(\mathbf{u}_0,\mathbf{b}_0)\in\mathbf{V}\times\mathbb{H}^2$, $(\hat{\mathbf{u}}(t,\mathbf{v}_0)(\mathbf{u}_0),\hat{\mathbf{d}}(t,\mathbf{d}_0)(\mathbf{b}_0)\in L(\mathbf{V}\times\mathbb{H}^2)$ for any $t\in[0,T]$, where $L(\mathbf{V}\times\mathbb{H}^2)$ represents the space of bounded linear operators from $\mathbf{V}\times\mathbb{H}^2$ to $\mathbf{V}\times\mathbb{H}^2$.*

Proof. Multiplying (3.8) with $A_1\hat{\mathbf{u}}(t,\mathbf{v}_0)(\mathbf{u}_0)$ and integrating over $\mathcal{D}$ yields that

$$\begin{aligned}\|\hat{\mathbf{u}}(t,\mathbf{v}_0)(\mathbf{u}_0)\|_1^2 =& \|\mathbf{u}_0\|_1^2 - 2\int_0^t \|\hat{\mathbf{u}}(s,\mathbf{v}_0)(\mathbf{u}_0)\|_2^2 ds\\
&- 2\int_0^t Q(s)\langle B_1(\hat{\mathbf{u}}(s,\mathbf{v}_0)(\mathbf{u}_0),\mathbf{u}(s,\mathbf{v}_0)+Z(s)),A_1\hat{\mathbf{u}}(s,\mathbf{v}_0)(\mathbf{u}_0)\rangle ds\\
&- 2\int_0^t Q(s)\langle B_1(\mathbf{u}(s,\mathbf{v}_0)+Z(s),\hat{\mathbf{u}}(s,\mathbf{v}_0)(\mathbf{u}_0)),A_1\hat{\mathbf{u}}(s,\mathbf{v}_0)(\mathbf{u}_0)\rangle ds\\
&- 2\int_0^t Q(s)^{-1}\langle M(\hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0),\mathbf{d}(s,\mathbf{d}_0)),A_1\hat{\mathbf{u}}(s,\mathbf{v}_0)(\mathbf{u}_0)\rangle ds\\
&- 2\int_0^t Q(s)^{-1}\langle M(\mathbf{d}(s,\mathbf{d}_0),\hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0)),A_1\hat{\mathbf{u}}(s,\mathbf{v}_0)(\mathbf{u}_0)\rangle ds. \quad (3.10)\end{aligned}$$

Taking inner product between $\Delta\hat{\mathbf{d}}$ and $\Delta\hat{\mathbf{d}}_t$, we get

$$\begin{aligned}\|\hat{\mathbf{d}}(t,\mathbf{d}_0)(\mathbf{b}_0)\|_2^2 =& \|\mathbf{b}_0\|_2^2 - 2\int_0^t \langle\Delta A_2\hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0),\Delta\hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0)\rangle ds\\
&- 2\int_0^t Q(s)\langle\Delta B_2(\hat{\mathbf{u}}(s,\mathbf{v}_0)(\mathbf{u}_0),\mathbf{d}(s,\mathbf{d}_0)),\Delta\hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0)\rangle ds\\
&- 2\int_0^t Q(s)\langle\Delta B_2(\mathbf{u}(s,\mathbf{v}_0)+Z(s),\hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0)),\Delta\hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0)\rangle ds\\
&- 2\int_0^t \langle\Delta\nabla f(\mathbf{d}(s,\mathbf{d}_0))\cdot\hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0),\Delta\hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0)\rangle ds. \quad (3.11)\end{aligned}$$

With similar discussion as it is in the proof of Proposition 3.3, we get the estimates for the terms in (3.10) and (3.11), then applying Gronwall's inequality yields that

$$\begin{aligned}&\sup_{0\le t\le T}[\|\hat{\mathbf{u}}(t,\mathbf{v}_0)(\mathbf{u}_0)\|_1^2 + \|\hat{\mathbf{d}}(t,\mathbf{d}_0)(\mathbf{b}_0)\|_2^2] + \int_0^T \|\hat{\mathbf{u}}(t,\mathbf{v}_0)(\mathbf{u}_0)\|_2^2 ds + \int_0^T \|\hat{\mathbf{d}}(t,\mathbf{d}_0)(\mathbf{b}_0)\|_3^2 ds\\
\le&(\|\mathbf{u}_0\|_1^2 + \|\mathbf{b}_0\|_2^2)g(T),\end{aligned}$$

where

$$g(T) := \exp c\Big\{T + |Q|_\infty^2 \sup_{0\le t\le T}\|\mathbf{u}(t,\mathbf{v}_0)+Z(t)\|_1^2 T + \sup_{0\le t\le T}\|\mathbf{u}(t,\mathbf{v}_0)+Z(t)\|_1^2 |Q|_\infty^4 \int_0^T \|\mathbf{u}(t,\mathbf{v}_0)+Z(t)\|_2^2 dt$$

$$+|Q|_\infty^{4/3}\sup_{0\le t\le T}\|\mathbf{u}(t,\mathbf{v}_0)+Z(t)\|_1^{2/3}\int_0^T\|\mathbf{u}(t,\mathbf{v}_0)+Z(t)\|_2^{2/3}dt$$

$$+|Q|_\infty^{4}\sup_{0\le t\le T}|\mathbf{u}(t,\mathbf{v}_0)+Z(t)|_2^{2}\|\mathbf{u}(t,\mathbf{v}_0)+Z(t)\|_1^{2}T+|Q|_\infty^2\int_0^T\|\mathbf{d}(t,\mathbf{d}_0)\|_3^2dt$$

$$+|Q|_\infty^2\int_0^T\|\mathbf{d}(t,\mathbf{d}_0)\|_3^2dt+|Q|_\infty^2\sup_{0\le t\le T}\|\mathbf{d}(t,\mathbf{d}_0)\|_2^2T+|Q|_\infty^2\int_0^T\|\mathbf{u}(t,\mathbf{v}_0)+Z(t)\|_2^2dt$$

$$+(|Q^{-1}|_\infty+|Q|_\infty)^4\sup_{0\le t\le T}\|\mathbf{d}(t,\mathbf{d}_0)\|_1^2\|\mathbf{d}(t,\mathbf{d}_0)\|_2^2T$$

$$+(|Q^{-1}|_\infty+|Q|_\infty)^2\sup_{0\le t\le T}\|\mathbf{d}(t,\mathbf{d}_0)\|_2\int_0^T\|\mathbf{d}(t,\mathbf{d}_0)\|_3dt$$

$$+|Q|_\infty^2\sup_{0\le t\le T}\|\mathbf{d}(t,\mathbf{d}_0)\|_1^2T+\sup_{0\le t\le T}\|\mathbf{d}(t,\mathbf{d}_0)\|_1^2T\Big\}.$$

Since $\hat{\mathbf{u}}(t,\mathbf{v}_0)(\mathbf{u}_0),\hat{\mathbf{d}}(t,\mathbf{d}_0)(\mathbf{b}_0)$ are linear with respect to $\mathbf{u}_0,\mathbf{b}_0$, respectively. The above discussion, together with Proposition 3.1, implies that $(\hat{\mathbf{u}}(t,\mathbf{v}_0)(\mathbf{u}_0),\hat{\mathbf{d}}(t,\mathbf{d}_0)(\mathbf{b}_0))\in L(\mathbf{V}\times\mathbb{H}^2)$ for any $t\in[0,T]$, and $\hat{\mathbf{u}}(t,\mathbf{v}_0)(\cdot)\in L(\mathbf{V},L^2([0,T];\mathbf{H}^2))$, $\hat{\mathbf{d}}(t,\mathbf{d}_0)(\cdot)\in L(\mathbb{H}^2,L^2([0,T];\mathbf{H}^3))$, that is,

$$\sup_{0\le t\le T}[\|\hat{\mathbf{u}}(t,\mathbf{v}_0)(\mathbf{u}_0)\|_{L(\mathbf{V})}^2+\|\hat{\mathbf{d}}(t,\mathbf{d}_0)(\mathbf{b}_0)\|_{L(\mathbb{H}^2)}^2]$$
$$+\int_0^T\|\hat{\mathbf{u}}(t,\mathbf{v}_0)(\mathbf{u}_0)\|_{L(\mathbf{V},L^2([0,T];\mathbf{H}^2))}^2dt++\int_0^T\|\hat{\mathbf{d}}(t,\mathbf{d}_0)(\mathbf{b}_0)\|_{L(\mathbb{H}^2,L^2([0,T];\mathbb{H}^3))}^2dt\le c(\mathbf{u}_0,\mathbf{d}_0,Q,Z,T).$$

□

In the following, we show that the aforementioned $\hat{\mathbf{u}}(t,\mathbf{v}_0)(\cdot),\hat{\mathbf{d}}(t,\mathbf{d}_0)(\cdot)$ are Fréchet derivatives.

Proposition 3.6. *For $(\mathbf{v}_0,\mathbf{d}_0)\in\mathbf{V}\times\mathbb{H}^2$, $(\mathbf{v}_0,\mathbf{d}_0)\mapsto(\mathbf{u}(t,\mathbf{v}_0,\omega),\mathbf{d}(t,\mathbf{d}_0,\omega))$ has Fréchet derivatives given by*

$$D\mathbf{u}(t,\mathbf{v}_0,\omega)=\hat{\mathbf{u}}(t,\mathbf{v}_0,\omega)(\cdot),\qquad D\mathbf{d}(t,\mathbf{d}_0,\omega)=\hat{\mathbf{d}}(t,\mathbf{d}_0,\omega)(\cdot).\tag{3.12}$$

Proof. To verify (3.12), it suffices to show

$$\lim_{h\to0}\sup_{\|\mathbf{u}_0\|_1+\|\mathbf{b}_0\|_2\le1}\Bigg\{\left\|\frac{\mathbf{u}(t,\mathbf{v}_0+h\mathbf{u}_0,\omega)-\mathbf{u}(t,\mathbf{v}_0,\omega)}{h}-\hat{\mathbf{u}}(t,\mathbf{v}_0)(\mathbf{u}_0)\right\|_1$$
$$+\left\|\frac{\mathbf{d}(t,\mathbf{d}_0+h\mathbf{b}_0,\omega)-\mathbf{d}(t,\mathbf{d}_0,\omega)}{h}-\hat{\mathbf{d}}(t,\mathbf{d}_0)(\mathbf{b}_0)\right\|_2\Bigg\}=0\tag{3.13}$$

holds. For $t\in[0,T],h\in\mathbb{R}\setminus\{0\}$, for simplicity of notations, set

$$U(t,\mathbf{v}_0,\mathbf{u}_0,h)=\frac{\mathbf{u}(t,\mathbf{v}_0+h\mathbf{u}_0,\omega)-\mathbf{u}(t,\mathbf{v}_0,\omega)}{h},\ X(t,\mathbf{v}_0,\mathbf{u}_0,h)=U(t,\mathbf{v}_0,\mathbf{u}_0,h)-\hat{\mathbf{u}}(t,\mathbf{v}_0)(\mathbf{u}_0);$$

$$D(t,\mathbf{d}_0,\mathbf{b}_0,h)=\frac{\mathbf{d}(t,\mathbf{d}_0+h\mathbf{b}_0,\omega)-\mathbf{d}(t,\mathbf{d}_0,\omega)}{h},\ Y(t,\mathbf{d}_0,\mathbf{b}_0,h)=D(t,\mathbf{d}_0,\mathbf{b}_0,h)-\hat{\mathbf{d}}(t,\mathbf{d}_0)(\mathbf{b}_0).$$

Then $X(t,\mathbf{v}_0,\mathbf{u}_0,h),Y(t,\mathbf{d}_0,\mathbf{b}_0,h)$ satisfy the following equations:

$$X(t,\mathbf{v}_0,\mathbf{u}_0,h)=-\int_0^tA_1X(s,\mathbf{v}_0,\mathbf{u}_0,h)ds-\int_0^tQ(s)B_1(\mathbf{u}(s,\mathbf{v}_0)+Z(s),X(s,\mathbf{v}_0,\mathbf{u}_0,h))ds$$
$$-\int_0^tQ(s)B_1(X(s,\mathbf{v}_0,\mathbf{u}_0,h),\mathbf{u}(s,\mathbf{v}_0+h\mathbf{u}_0)+Z(s))ds$$
$$-\int_0^tQ(s)B_1(\hat{\mathbf{u}}(s,\mathbf{v}_0)(\mathbf{u}_0),\mathbf{u}(s,\mathbf{v}_0+h\mathbf{u}_0)-\mathbf{u}(s,\mathbf{v}_0))ds$$

$$
\begin{aligned}
&- \int_0^t Q(s)^{-1} M(\mathbf{d}(s,\mathbf{d}_0), Y(s,\mathbf{d}_0,\mathbf{b}_0,h))ds - \int_0^t Q(s)^{-1} M(Y(s,\mathbf{d}_0,\mathbf{b}_0,h), \mathbf{d}(s,\mathbf{d}_0 + h\mathbf{b}_0))ds \\
&- \int_0^t Q(s)^{-1} M(\hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0), \mathbf{d}(s,\mathbf{d}_0 + h\mathbf{b}_0) - \mathbf{d}(s,\mathbf{d}_0))ds.
\end{aligned}
$$

$$
\begin{aligned}
Y(t,\mathbf{d}_0,\mathbf{b}_0,h) =& - \int_0^t A_2 Y(s,\mathbf{d}_0,\mathbf{b}_0,h)ds - \int_0^t Q(s) B_2(X(s,\mathbf{v}_0,\mathbf{u}_0,h), \mathbf{d}(s,\mathbf{d}_0))ds \\
&- \int_0^t Q(s) B_2(\mathbf{u}(s,\mathbf{v}_0 + h\mathbf{u}_0) + Z(s), Y(s,\mathbf{d}_0,\mathbf{b}_0,h))ds \\
&- \int_0^t Q(s) B_2(\mathbf{u}(s,\mathbf{v}_0 + h\mathbf{u}_0) - \mathbf{u}(s,\mathbf{u}_0), \hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0))ds \\
&- \int_0^t \frac{1}{h}[f(\mathbf{d}(s,\mathbf{d}_0 + h\mathbf{b}_0)) - f(\mathbf{d}(s,\mathbf{d}_0))]ds + \int_0^t \nabla f(\mathbf{d}(s,\mathbf{d}_0)) \cdot \hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0)ds.
\end{aligned}
$$

We multiply $X(t,\mathbf{v}_0,\mathbf{u}_0,h)$ with $A_1 X(t,\mathbf{v}_0,\mathbf{u}_0,h)$ and integrate over D,

$$
\begin{aligned}
\|X(t)\|_1^2 =& - 2\int_0^t \|X(s)\|_2^2 - 2\int_0^t Q(s)\langle B_1(\mathbf{u}(s,\mathbf{v}_0) + Z(s), X(s)), A_1 X(s)\rangle ds \\
&- 2\int_0^t Q(s)\langle B_1(X(s), \mathbf{u}(s,\mathbf{v}_0 + h\mathbf{u}_0) + Z(s)), A_1 X(s)\rangle ds \\
&- 2\int_0^t Q(s)\langle B_1(\hat{\mathbf{u}}(s,\mathbf{v}_0)(\mathbf{u}_0), \mathbf{u}(s,\mathbf{v}_0 + h\mathbf{u}_0) - \mathbf{u}(s,\mathbf{v}_0)), A_1 X(s)\rangle ds \\
&- 2\int_0^t Q(s)^{-1}\langle M(\mathbf{d}(s,\mathbf{d}_0), Y(s)), A_1 X(s)\rangle ds - 2\int_0^t Q(s)^{-1}\langle M(Y(s), \mathbf{d}(s,\mathbf{d}_0 + h\mathbf{b}_0)), A_1 X(s)\rangle ds \\
&- 2\int_0^t Q(s)^{-1}\langle M(\hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0), \mathbf{d}(s,\mathbf{d}_0 + h\mathbf{b}_0) - \mathbf{d}(s,\mathbf{d}_0)), A_1 X(s)\rangle ds. \qquad (3.14)
\end{aligned}
$$

Now we take inner product between $\Delta Y(t,\mathbf{d}_0,\mathbf{b}_0,h)$ and $\Delta Y_t(t,\mathbf{d}_0,\mathbf{b}_0,h)$ and integrate over D,

$$
\begin{aligned}
\|Y(t)\|_2^2 =& - 2\int_0^t \|Y(s)\|_3^2 ds - 2\int_0^t Q(s)\langle \Delta B_2(X(s), \mathbf{d}(s,\mathbf{d}_0)), \Delta Y(s)\rangle ds \\
&- 2\int_0^t Q(s)\langle \Delta B_2(\mathbf{u}(s,\mathbf{v}_0 + h\mathbf{u}_0) + Z(s), Y(s)), \Delta Y(s)\rangle ds \\
&- 2\int_0^t Q(s)\langle \Delta B_2(\mathbf{u}(s,\mathbf{v}_0 + h\mathbf{u}_0) - \mathbf{u}(s,\mathbf{u}_0), \hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0)), \Delta Y(s)\rangle ds \\
&- 2\int_0^t \langle \frac{1}{h}[\Delta f(\mathbf{d}(s,\mathbf{d}_0 + h\mathbf{b}_0)) - \Delta f(\mathbf{d}(s,\mathbf{d}_0))], \Delta Y(s)\rangle ds \\
&+ 2\int_0^t \langle \Delta(\nabla f(\mathbf{d}(s,\mathbf{d}_0)) \cdot \hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0)), \Delta Y(s)\rangle ds. \qquad (3.15)
\end{aligned}
$$

With similar discussion as it is in the proof of Proposition 3.3, we get the estimates for the terms in (3.14) and (3.15), then applying Gronwall's inequality yields that

$$
\sup_{0\le t\le T}\{\|X(t)\|_1^2 + \|Y(t)\|_2^2\} + \int_0^T \|X(s)\|_2^2 ds + \int_0^T \|Y(s)\|_3^2 ds \le c(|Q|_\infty + |Q^{-1}|_\infty)^2 g_2(T) g_3(T), \qquad (3.16)
$$

where

$$
\begin{aligned}
g_2(T) :=& \sup_{0\le t\le T}\|\hat{\mathbf{u}}(t,\mathbf{v}_0)\|_1 \|\bar{\mathbf{u}}(s,\omega)\|_1^2 \int_0^T \|\hat{\mathbf{u}}(t,\mathbf{v}_0)\|_2 dt + \sup_{0\le t\le T}|\hat{\mathbf{u}}(t,\mathbf{v}_0)|_2 \|\hat{\mathbf{u}}(t,\mathbf{v}_0)\|_1 \int_0^T \|\bar{\mathbf{u}}(t,\omega)\|_2^2 dt \\
&+ \sup_{0\le t\le T}\|\hat{\mathbf{d}}(t,\mathbf{d}_0)(\mathbf{b}_0)\|_1 \|\hat{\mathbf{d}}(t,\mathbf{d}_0)(\mathbf{b}_0)\|_2 \|\bar{\mathbf{d}}(t,\omega)\|_2 \int_0^T \|\bar{\mathbf{d}}(t,\omega)\|_3 dt
\end{aligned}
$$

$$
\begin{aligned}
&+ \sup_{0\le t\le T} \|\hat{\mathbf{d}}(t,\mathbf{d}_0)(\mathbf{b}_0)\|_2 \|\bar{\mathbf{d}}(t,\omega)\|_1 \|\bar{\mathbf{d}}(t,\omega)\|_2 \int_0^T \|\hat{\mathbf{d}}(t,\mathbf{d}_0)(\mathbf{b}_0)\|_3 dt \\
&+ \sup_{0\le t\le T} \|\hat{\mathbf{d}}(t,\mathbf{d}_0)(\mathbf{b}_0)\|_1^2 \int_0^T \|\bar{\mathbf{u}}(t,\omega)\|_2^2 dt + \sup_{0\le t\le T} |\bar{\mathbf{u}}(t,\omega)|_2^2 \int_0^T \|\hat{\mathbf{d}}(t,\mathbf{d}_0)(\mathbf{b}_0)\| dt,
\end{aligned}
$$

and

$$
\begin{aligned}
g_3(T) := \exp c \Big\{ & T + |Q|_\infty^{4/3} \sup_{0\le t\le T} \|\mathbf{u}(t,\mathbf{v}_0) + Z(t)\|_1^{2/3} \int_0^T \|\mathbf{u}(t,\mathbf{v}_0) + Z(t)\|_2^{2/3} dt \\
&+ |Q|_\infty^2 \sup_{0\le t\le T} \|\mathbf{u}(t,\mathbf{v}_0 + h\mathbf{u}_0) + Z(t)\|_1^2 T \\
&+ |Q|_\infty^4 \sup_{0\le t\le T} \|\mathbf{u}(t,\mathbf{v}_0 + h\mathbf{u}_0) + Z(t)\|_1^2 \int_0^T \|\mathbf{u}(t,\mathbf{v}_0 + h\mathbf{u}_0) + Z(t)\|_2^2 dt \\
&+ |Q|_\infty^2 \int_0^T \|\mathbf{d}(t,\mathbf{d}_0)\|_3^2 dt + |Q|_\infty^4 \sup_{0\le t\le T} \|\bar{\mathbf{d}}(t,\omega)\|_2^2 \int_0^T \|\bar{\mathbf{d}}(t,\omega)\|_3^2 dt \\
&+ |Q|_\infty^4 \sup_{0\le t\le T} \|\bar{\mathbf{d}}(t,\omega)\|_1^2 \|\bar{\mathbf{d}}(t,\omega)\|_2^2 T + |Q|_\infty^2 \sup_{0\le t\le T} \|\mathbf{d}(t,\mathbf{d}_0)\|_2^2 T \\
&+ (|Q|_\infty + |Q^{-1}|_\infty)^4 \sup_{0\le t\le T} \|\mathbf{d}(t,\mathbf{d}_0)\|_2^2 \int_0^T \|\mathbf{d}(t,\mathbf{d}_0)\|_3^2 dt \\
&+ (|Q|_\infty + |Q^{-1}|_\infty)^4 \sup_{0\le t\le T} \|\mathbf{d}(t,\mathbf{d}_0)\|_1^2 \|\mathbf{d}(t,\mathbf{d}_0)\|_2^2 T \\
&+ (|Q|_\infty + |Q^{-1}|_\infty)^4 \sup_{0\le t\le T} \|\mathbf{d}(t,\mathbf{d}_0 + h\mathbf{b}_0)\|_2^2 \int_0^T \|\mathbf{d}(t,\mathbf{d}_0 + h\mathbf{b}_0)\|_3^2 dt \\
&+ (|Q|_\infty + |Q^{-1}|_\infty)^4 \sup_{0\le t\le T} \|\mathbf{d}(t,\mathbf{d}_0 + h\mathbf{b}_0)\|_1^2 \|\mathbf{d}(t,\mathbf{d}_0 + h\mathbf{b}_0)\|_2^2 T \\
&+ |Q|_\infty^2 \int_0^T \|\mathbf{u}(t,\mathbf{v}_0 + h\mathbf{u}_0) + Z(t)\|_2^2 dt + \sup_h \sup_{0\le t\le T} \|\mathbf{d}(t,\mathbf{d}_0 + h\mathbf{b}_0)\|_1^2 T \Big\}.
\end{aligned}
$$

Note that by Proposition 3.3, $g_2(T) \to 0$ as $h \to 0$, by Proposition 3.1, $g_3(T) < \infty$, this verifies (3.13). ☐

Based on the above discussions, we summarize into the following result for Fréchet derivatives of $(\mathbf{v}(t,\mathbf{v}_0), \mathbf{d}(t,\mathbf{d}_0))$.

Theorem 3.7. *For $(\mathbf{v}_0, \mathbf{d}_0) \in \mathbf{V} \times \mathbb{H}^2$, $(\mathbf{v}(t,\mathbf{v}_0,\omega), \mathbf{d}(t,\mathbf{d}_0,\omega)) \in \mathbf{V} \times \mathbb{H}^2$, and the solution map $(\mathbf{v}_0, \mathbf{d}_0) \mapsto (\mathbf{v}(t,\mathbf{v}_0,\omega), \mathbf{d}(t,\mathbf{d}_0,\omega))$ is $C^{1,1}$ for all $\omega \in \Omega, t \ge 0$, and has bounded Fréchet derivatives on bounded sets in $\mathbf{V} \times \mathbb{H}^2$.*

Moreover, the Fréchet derivative $t \mapsto (D\mathbf{v}(t,\mathbf{v}_0,\omega), D\mathbf{d}(t,\mathbf{d}_0,\omega)) \in L(\mathbf{V} \times \mathbb{H}^2)$ is continuous in t, and the Fréchet derivative is compact for any $t > 0, \omega \in \Omega$, where $L(\mathbf{V} \times \mathbb{H}^2)$ represents the space of bounded linear operators from $\mathbf{V} \times \mathbb{H}^2$ to $\mathbf{V} \times \mathbb{H}^2$.

Proof. It remains to show $\mathbf{V} \times \mathbb{H}^2 \ni (\mathbf{v}_0, \mathbf{d}_0) \mapsto (\mathbf{v}(t,\mathbf{v}_0), \mathbf{d}(t,\mathbf{d}_0) \in \mathbf{V} \times \mathbb{H}^2$ is Fréchet $C^{1,1}$, and to see that, it suffices to show $\mathbf{V} \times \mathbb{H}^2 \ni (\mathbf{v}_0, \mathbf{d}_0) \mapsto (\hat{\mathbf{u}}(t,\mathbf{v}_0), \hat{\mathbf{d}}(t,\mathbf{d}_0)) \in L(\mathbf{V} \times \mathbb{H}^2)$ is Lipschitz continuous on bounded sets.

Now let $\mathbf{v}_0, \mathbf{v}_0', \mathbf{u}_0 \in \mathbf{V}, \mathbf{d}_0, \mathbf{d}_0', \mathbf{b}_0 \in \mathbb{H}^2$ with $\|\mathbf{v}_0\|_1 \le M, \|\mathbf{v}_0'\|_1 \le M, \|\mathbf{d}_0\|_2 \le M, \|\mathbf{d}_0'\|_2 \le M$ and $\|\mathbf{u}_0\|_1 + \|\mathbf{b}_0\|_2 \le 1$. For simplicity of notations, we denote by $\hat{\mathbf{u}}_\triangle(t) := \hat{\mathbf{u}}(t,\mathbf{v}_0)(\mathbf{u}_0) - \hat{\mathbf{u}}(t,\mathbf{v}_0')(\mathbf{u}_0)$ and $\hat{\mathbf{d}}_\triangle(t) := \hat{\mathbf{d}}(t,\mathbf{d}_0)(\mathbf{b}_0) - \hat{\mathbf{d}}(t,\mathbf{d}_0')(\mathbf{b}_0)$. By (3.8), we first take inner product between $\hat{\mathbf{u}}_\triangle(t)$ and $A_1\hat{\mathbf{u}}_\triangle(t)$, then integrate over D,

$$
\begin{aligned}
\|\hat{\mathbf{u}}_\triangle(t)\|_1^2 = &-2\int_0^t \|\hat{\mathbf{u}}_\triangle(s)\|_2^2 ds \\
&- 2\int_0^t Q(s)\langle B_1(\hat{\mathbf{u}}_\triangle(s), \mathbf{u}(s,\mathbf{v}_0) + Z(s)), A_1\hat{\mathbf{u}}_\triangle(s)\rangle ds
\end{aligned}
$$

$$
\begin{aligned}
&-2\int_0^t Q(s)\langle B_1(\mathbf{u}(s,\mathbf{v}_0)+Z(s),\hat{\mathbf{u}}_\triangle(s)),A_1\hat{\mathbf{u}}_\triangle(s)\rangle ds\\
&-2\int_0^t Q(s)\langle B_1(\hat{\mathbf{u}}(s,\mathbf{v}_0')(\mathbf{u}_0),\mathbf{u}(s,\mathbf{v}_0)-\mathbf{u}(s,\mathbf{v}_0')),A_1\hat{\mathbf{u}}_\triangle(s)\rangle ds\\
&-2\int_0^t Q(s)\langle B_1(\mathbf{u}(s,\mathbf{v}_0)-\mathbf{u}(s,\mathbf{v}_0'),\hat{\mathbf{u}}(s,\mathbf{v}_0')(\mathbf{u}_0)),A_1\hat{\mathbf{u}}_\triangle(s)\rangle ds\\
&-2\int_0^t Q(s)^{-1}\langle M(\hat{\mathbf{d}}_\triangle(s),\mathbf{d}(s,\mathbf{d}_0)),A_1\hat{\mathbf{u}}_\triangle(s)\rangle ds\\
&-2\int_0^t Q(s)^{-1}\langle M(\hat{\mathbf{d}}(s,\mathbf{d}_0')(\mathbf{b}_0),\mathbf{d}(s,\mathbf{d}_0)-\mathbf{d}(s,\mathbf{d}_0')),A_1\hat{\mathbf{u}}_\triangle(s)\rangle ds\\
&-2\int_0^t Q(s)^{-1}\langle M(\mathbf{d}(s,\mathbf{d}_0),\hat{\mathbf{d}}_\triangle(s)),A_1\hat{\mathbf{u}}_\triangle(s)\rangle ds\\
&-2\int_0^t Q(s)^{-1}\langle M(\mathbf{d}(s,\mathbf{d}_0)-\mathbf{d}(s,\mathbf{d}_0'),\hat{\mathbf{d}}(s,\mathbf{d}_0')(\mathbf{b}_0)),A_1\hat{\mathbf{u}}_\triangle(s)\rangle ds. \qquad (3.17)
\end{aligned}
$$

With (3.9), taking time derivative of $\Delta\hat{\mathbf{d}}_\triangle(t)$, then taking inner product with $\Delta\hat{\mathbf{d}}_\triangle(t)$ yields that

$$
\begin{aligned}
\|\hat{\mathbf{d}}_\triangle(t)\|_2^2 =&-2\int_0^t \|\hat{\mathbf{d}}_\triangle(s)\|_3^2 ds\\
&-2\int_0^t Q(s)\langle \Delta B_2(\hat{\mathbf{u}}_\triangle(s),\mathbf{d}(s,\mathbf{d}_0)),\Delta\hat{\mathbf{d}}_\triangle(s)\rangle ds\\
&-2\int_0^t Q(s)\langle \Delta B_2(\mathbf{u}(s,\mathbf{v}_0)+Z(s),\hat{\mathbf{d}}_\triangle(s)),\Delta\hat{\mathbf{d}}_\triangle(s)\rangle ds\\
&-2\int_0^t Q(s)\langle \Delta B_2(\hat{\mathbf{u}}(s,\mathbf{v}_0')(\mathbf{u}_0),\mathbf{d}(s,\mathbf{d}_0)-\mathbf{d}(s,\mathbf{d}_0'),\Delta\hat{\mathbf{d}}_\triangle(s)\rangle ds\\
&-2\int_0^t Q(s)\langle \Delta B_2(\mathbf{u}(s,\mathbf{v}_0)-\mathbf{u}(s,\mathbf{v}_0'),\hat{\mathbf{d}}(s,\mathbf{d}_0')),\Delta\hat{\mathbf{d}}_\triangle(s)\rangle ds\\
&-2\int_0^t \langle \Delta(\nabla f(\mathbf{d}(s,\mathbf{d}_0))\cdot\hat{\mathbf{d}}(s,\mathbf{d}_0)(\mathbf{b}_0)-\nabla f(\mathbf{d}(s,\mathbf{d}_0'))\cdot\hat{\mathbf{d}}(s,\mathbf{d}_0')(\mathbf{b}_0)),\Delta\hat{\mathbf{d}}_\triangle(s)\rangle ds. \qquad (3.18)
\end{aligned}
$$

With similar discussion as it is in the proof of Proposition 3.3, we get the estimates for the terms in (3.17) and (3.18), then applying Gronwall's inequality yields that

$$
\sup_{0\le t\le T}[\|\hat{\mathbf{u}}_\triangle(t)\|_1^2+\|\hat{\mathbf{d}}_\triangle(t)\|_2^2]++\int_0^T\|\hat{\mathbf{u}}_\triangle(t)\|_2^2ds+\int_0^T\|\hat{\mathbf{d}}_\triangle(t)\|_3^2ds\le c|Q|_\infty^2[g_4(T)g_5(T)+g_6(T)g_7(T)], \quad (3.19)
$$

where

$$
\begin{aligned}
g_4(T):=&\sup_{0\le t\le T}\|\mathbf{u}(t,\mathbf{v}_0)-\mathbf{u}(t,\mathbf{v}_0')\|_1^2+\int_0^T\|\mathbf{u}(t,\mathbf{v}_0)-\mathbf{u}(t,\mathbf{v}_0')\|_2^2dt\\
&+\sup_{0\le t\le T}\|\mathbf{u}(t,\mathbf{v}_0)-\mathbf{u}(t,\mathbf{v}_0')\|_1^{1/2}\int_0^T\|\mathbf{u}(t,\mathbf{v}_0)-\mathbf{u}(t,\mathbf{v}_0')\|_2^{1/2}dt\\
&+\sup_{0\le t\le T}|\mathbf{u}(t,\mathbf{v}_0)-\mathbf{u}(t,\mathbf{v}_0')|_2^{1/2}\|\mathbf{u}(t,\mathbf{v}_0)-\mathbf{u}(t,\mathbf{v}_0')\|_1^{1/2}+\sup_{0\le t\le T}\|\hat{\mathbf{d}}(t,\mathbf{d}_0')\|_1^2\int_0^T\|\mathbf{u}(t,\mathbf{v}_0)-\mathbf{u}(t,\mathbf{v}_0')\|_2^2dt\\
&+\sup_{0\le t\le T}\|\hat{\mathbf{d}}(t,\mathbf{d}_0')\|_2^2\|\mathbf{u}(t,\mathbf{v}_0)-\mathbf{u}(t,\mathbf{v}_0')\|_1^2T+\sup_{0\le t\le T}|\mathbf{u}(t,\mathbf{v}_0)-\mathbf{u}(t,\mathbf{v}_0')|_2^2\int_0^T\|\hat{\mathbf{d}}(t,\mathbf{d}_0')\|_3^2dt,\\
g_6(T):=&|Q^{-1}|_\infty^2\sup_{0\le t\le T}\|\mathbf{d}(t,\mathbf{d}_0)-\mathbf{d}(t,\mathbf{d}_0')\|_1\|\mathbf{d}(t,\mathbf{d}_0)-\mathbf{d}(t,\mathbf{d}_0')\|_2\|\hat{\mathbf{d}}(t,\mathbf{d}_0')(\mathbf{b}_0)\|_2\int_0^T\|\hat{\mathbf{d}}(t,\mathbf{d}_0')(\mathbf{b}_0)\|_3dt\\
&+|Q^{-1}|_\infty^2\sup_{0\le t\le T}\|\hat{\mathbf{d}}(t,\mathbf{d}_0')(\mathbf{b}_0)\|_1\|\hat{\mathbf{d}}(t,\mathbf{d}_0')(\mathbf{b}_0)\|_2\|\mathbf{d}(t,\mathbf{d}_0)-\mathbf{d}(t,\mathbf{d}_0')\|_2\int_0^T\|\mathbf{d}(t,\mathbf{d}_0)-\mathbf{d}(t,\mathbf{d}_0')\|_3dt
\end{aligned}
$$

$$+\sup_{0\le t\le T}\|\hat{\mathbf{d}}(t,\mathbf{d}_0')(\mathbf{b}_0)\|_1^2\int_0^T\|\mathbf{d}(t,\mathbf{d}_0)-\mathbf{d}(t,\mathbf{d}_0')\|_3^2dt$$
$$+\sup_{0\le t\le T}\|\mathbf{d}(t,\mathbf{d}_0)-\mathbf{d}(t,\mathbf{d}_0')\|_1^2\int_0^T\|\hat{\mathbf{d}}(t,\mathbf{d}_0')(\mathbf{b}_0)\|_3^2dt$$
$$+\sup_{0\le t\le T}\|\mathbf{d}(t,\mathbf{d}_0)-\mathbf{d}(t,\mathbf{d}_0')\|_1^2\int_0^T\|\hat{\mathbf{u}}(t,\mathbf{v}_0')\|_2^2dt$$
$$+\sup_{0\le t\le T}\|\hat{\mathbf{u}}(t,\mathbf{v}_0')\|_1^2\int_0^T\|\mathbf{d}(t,\mathbf{d}_0)-\mathbf{d}(t,\mathbf{d}_0')\|_2^2dt+\sup_{0\le t\le T}|\hat{\mathbf{u}}(t,\mathbf{v}_0')|_2^2\int_0^T\|\mathbf{d}(t,\mathbf{d}_0)-\mathbf{d}(t,\mathbf{d}_0')\|_3^2dt,$$

and

$$\begin{aligned}
g_5(T):=\exp c\Big\{&T+|Q|_\infty^2\sup_{0\le t\le T}\|\mathbf{u}(t,\mathbf{v}_0)+Z(t)\|_1^2T\\
&+|Q|_\infty^4\sup_{0\le t\le T}\|\mathbf{u}(t,\mathbf{v}_0)+Z(t)\|_1^2\int_0^T\|\mathbf{u}(t,\mathbf{v}_0)+Z(t)\|_2^2dt\\
&+|Q|_\infty^{4/3}\sup_{0\le t\le T}\|\mathbf{u}(t,\mathbf{v}_0)+Z(t)\|_1^{2/3}\int_0^T\|\mathbf{u}(t,\mathbf{v}_0)+Z(t)\|_2^{2/3}dt\\
&+|Q|_\infty^4\sup_{0\le t\le T}\|\hat{\mathbf{u}}(t,\mathbf{v}_0')(\mathbf{u}_0)\|_1^2\int_0^T\|\hat{\mathbf{u}}(t,\mathbf{v}_0')(\mathbf{u}_0)\|_2^2dt\\
&+|Q|_\infty^4\sup_{0\le t\le T}|\hat{\mathbf{u}}(t,\mathbf{v}_0')(\mathbf{u}_0)|_2^2\|\hat{\mathbf{u}}(t,\mathbf{v}_0')(\mathbf{u}_0)\|_1^2T+|Q|_\infty^4\sup_{0\le t\le T}\|\hat{\mathbf{u}}(t,\mathbf{v}_0')(\mathbf{u}_0)\|_1^4T\\
&+|Q|_\infty^4\int_0^T\|\hat{\mathbf{u}}(t,\mathbf{v}_0')(\mathbf{u}_0)\|_2^4dt+|Q|_\infty^2\int_0^T\|\mathbf{d}(t,\mathbf{d}_0)\|_3^2dt+|Q|_\infty^2\sup_{0\le t\le T}\|\mathbf{d}(t,\mathbf{d}_0)\|_2^2T\Big\};\\
g_7(T):=\exp c\Big\{&|Q|_\infty^2\int_0^T\|\mathbf{d}(t,\mathbf{d}_0)\|_3^2dt+(|Q|_\infty+|Q^{-1}|_\infty)^4\sup_{0\le t\le T}\|\mathbf{d}(t,\mathbf{d}_0)\|_1^2\|\mathbf{d}(t,\mathbf{d}_0)\|_2^2T\\
&+(|Q|_\infty+|Q^{-1}|_\infty)^2\sup_{0\le t\le T}\|\mathbf{d}(t,\mathbf{d}_0)\|_2\int_0^T\|\mathbf{d}(t,\mathbf{d}_0)\|_3dt\\
&+|Q|_\infty^2\int_0^T\|\mathbf{u}(t,\mathbf{v}_0)+Z(t)\|_2^2dt+|Q|_\infty^2\sup_{0\le t\le T}\|\mathbf{u}(t,\mathbf{v}_0)+Z(t)\|_1^2T\\
&+\sup_{0\le t\le T}\|\mathbf{d}(t,\mathbf{d}_0)\|_1^2T+\sup_{0\le t\le T}\|\hat{\mathbf{d}}(t,\mathbf{d}_0')\|_2\|\mathbf{d}(t,\mathbf{d}_0)+\mathbf{d}(t,\mathbf{d}_0')\|_1^2T\Big\}.
\end{aligned}$$

It was shown in [9], or using the same discussion in the proof of Proposition 3.3 yields that

$$\begin{aligned}
&\sup_{0\le t\le T}[\|\mathbf{u}(t,\mathbf{v}_0)-\mathbf{u}(t,\mathbf{v}_0')\|_1^2+\|\mathbf{d}(t,\mathbf{d}_0)-\mathbf{d}(t,\mathbf{d}_0')\|_2^2]\\
&+\int_0^T\|\mathbf{u}(t,\mathbf{v}_0)-\mathbf{u}(t,\mathbf{v}_0')\|_2^2dt+\int_0^T\|\mathbf{d}(t,\mathbf{d}_0)-\mathbf{d}(t,\mathbf{d}_0')\|_3^2dt\\
\le& c(T)[\|\mathbf{v}_0-\mathbf{v}_0'\|_1^2+\|\mathbf{d}_0-\mathbf{d}_0'\|_2^2],
\end{aligned}$$

where $c(T)$ is bounded given the initial value norms are bounded by M. Thus, we conclude that

$$\begin{aligned}
&\sup_{0\le t\le T}\|\hat{\mathbf{u}}(t,\mathbf{v}_0)-\hat{\mathbf{u}}(t,\mathbf{v}_0')\|_{L(\mathbf{V})}^2+\int_0^T\|\hat{\mathbf{u}}(t,\mathbf{v}_0)-\hat{\mathbf{u}}(t,\mathbf{v}_0')\|_{L(\mathbf{V};L^2([0,T];\mathbf{H}^2))}^2dt\\
&+\sup_{0\le t\le T}\|\hat{\mathbf{d}}(t,\mathbf{d}_0)-\hat{\mathbf{d}}(t,\mathbf{d}_0')\|_{L(\mathbb{H}^2)}^2+\int_0^T\|\hat{\mathbf{d}}(t,\mathbf{d}_0)-\hat{\mathbf{d}}(t,\mathbf{d}_0')\|_{L(\mathbb{H}^2;L^2([0,T];\mathbb{H}^3))}^2dt\\
\le& c[\|\mathbf{v}_0-\mathbf{v}_0'\|_1^2+\|\mathbf{d}_0-\mathbf{d}_0'\|_2^2],
\end{aligned}\tag{3.20}$$

where c is a positive constant that is independent of initial data provided that $\|\mathbf{v}_0\|_1 \le M, \|\mathbf{v}_0'\|_1 \le M, \|\mathbf{d}_0\|_2 \le M, \|\mathbf{d}_0'\|_2 \le M$. Hence, we prove that the map $\mathbf{V} \times \mathbb{H}^2 \ni (\mathbf{v}_0, \mathbf{d}_0) \mapsto (\hat{\mathbf{u}}(t, \mathbf{v}_0), \hat{\mathbf{d}}(t, \mathbf{d}_0) \in L(\mathbf{V} \times \mathbb{H}^2)$ is Lipschitz continuous on bounded sets.

To see the compactness of the Fréchet derivative, we can follow the method in [9] and use the Aubin-Lions Lemma as well as the regularity of solutions. One can also adopt the method in Theorem 3.1 of [15] to show the compactness of $(D\mathbf{v}(t, \mathbf{v}_0, \omega), D\mathbf{d}(t, \mathbf{d}_0, \omega)) : \mathbf{V} \times \mathbb{H}^2 \to \mathbf{V} \times \mathbb{H}^2$ for $t > 0$. ☐

4 Galerkin approximation and Malliavin regularities

In this section, we consider Galerkin approximation and write $\{e_k\}_{k\ge 1}$ as an orthonormal basis for $\mathbf{V}$, serving as eigenvectors of $-A_1$ subject to the boundary condition (BC), with corresponding eigenvalues $\{r_k\}_{k\ge1}$, that is, $A_1 e_k = -r_k e_k$. Let $\mathbf{V}_n$ be n-dimensional subspace spanned by $\{e_1, \dots, e_n\}$, and define

$$\mathbf{v}_{0,n} = \sum_{k=1}^{n} \langle \mathbf{v}_0, e_k \rangle e_k.$$

Similarly, let $\{\rho_k\}_{k\ge1}$ be an orthonormal basis for $\mathbb{H}^2$, which serves as eigenvectors of $-A_2$ subject to the boundary condition. Let $\mathbb{H}^2_n$ be n-dimensional subspace spanned by $\{\rho_1, \dots, \rho_n\}$ and define

$$\mathbf{d}_{0,n} = \sum_{k=1}^{n} \langle \mathbf{d}_0, \rho_k \rangle \rho_k.$$

Now we let $(\mathbf{u}_n(t, \mathbf{v}_{0,n}), \mathbf{d}_n(t, \mathbf{d}_{0,n}) \in \mathbf{V}_n \times \mathbb{H}^2_n$ be the unique solution to the following equations:

$$d\mathbf{u}_n(t, \mathbf{v}_{0,n}) = -A_1 \mathbf{u}_n(t, \mathbf{v}_{0,n})dt - Q(t)B_1(\mathbf{u}_n(t, \mathbf{v}_{0,n}) + Z(t))dt - Q(t)^{-1}M(\mathbf{d}_n(t, \mathbf{d}_{0,n}))dt, \tag{4.1a}$$
$$\nabla \cdot (\mathbf{u}_n(t, \mathbf{v}_{0,n}) + Z(t)) = 0, \tag{4.1b}$$
$$d\mathbf{d}_n(t, \mathbf{d}_{0,n}) = -A_2 \mathbf{d}_n(t, \mathbf{d}_{0,n})dt - Q(t)B_2(\mathbf{u}_n(t, \mathbf{v}_{0,n}) + Z(t), \mathbf{d}_n(t, \mathbf{d}_{0,n}))dt - f(\mathbf{d}_n(t, \mathbf{d}_{0,n})))dt, \tag{4.1c}$$
$$\mathbf{u}_n(t, \mathbf{v}_{0,n})|_{\partial\mathcal{D}} = 0,\ \mathbf{d}_n(t, \mathbf{d}_{0,n})|_{\partial\mathcal{D}} = \mathbf{d}_{0,n}, \tag{4.1d}$$
$$\mathbf{u}_n(0, \mathbf{v}_{0,n}) = \mathbf{v}_{0,n},\ \mathbf{d}_n(0, \mathbf{d}_{0,n}) = \mathbf{d}_{0,n}. \tag{4.1e}$$

It was shown in [9] that $(\mathbf{u}_n, \mathbf{d}_n) \to (\mathbf{u}, \mathbf{d})$ in $\mathbf{H}^1 \times \mathbb{H}^2$. We first discuss the Malliavin regularities of $\mathbf{u}_n(t, \mathbf{v}_{0,n}), \mathbf{d}_n(t, \mathbf{d}_{0,n})$.

Proposition 4.1. *For $\mathbf{v}_0 \in \mathbf{V}, \mathbf{d}_0 \in \mathbb{H}^2$, the solution $(\mathbf{u}_n(t, \mathbf{v}_{0,n}), \mathbf{d}_n(t, \mathbf{d}_{0,n}))$ to (4.1) are Malliavin differentiable, and $\mathbf{u}_n(t, \mathbf{v}_{0,n}) \in \mathcal{D}^{1,2}_{loc}(\mathbf{V}), \mathbf{d}_n(t, \mathbf{d}_{0,n}) \in \mathcal{D}^{1,2}_{loc}(\mathbb{H}^2)$. In details,*

$$\begin{aligned}&\sup_{0\le t\le T} [|\mathcal{D}_v \mathbf{u}_n(t, \mathbf{v}_{0,n})|_2^2 + \|\mathcal{D}_v \mathbf{d}_n(t, \mathbf{d}_{0,n})\|_1^2] + \int_0^T \|\mathcal{D}_v \mathbf{u}_n(t, \mathbf{v}_{0,n})\|_1^2 dt + \int_0^T \|\mathcal{D}_v \mathbf{d}_n(t, \mathbf{d}_{0,n})\|_2^2 dt \\ \le& c(|\mathbf{v}_0|_2, \|\mathbf{d}_0\|_1, |Q|_\infty, \sup_{0\le t\le T} \|Z\|_2, T),\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}&\sup_{0\le t\le T} [\|\mathcal{D}_v \mathbf{u}_n(t, \mathbf{v}_{0,n})\|_1^2 + \|\mathcal{D}_v \mathbf{d}_n(t, \mathbf{d}_{0,n})\|_2^2] + \int_0^T \|\mathcal{D}_v \mathbf{u}_n(t, \mathbf{v}_{0,n})\|_2^2 dt + \int_0^T \|\mathcal{D}_v \mathbf{d}_n(t, \mathbf{d}_{0,n})\|_3^2 dt \\ \le& c(\|\mathbf{v}_0\|_1, \|\mathbf{d}_0\|_2, |Q|_\infty, \sup_{0\le t\le T} \|Z\|_2, \int_0^T \|Z(t)\|_3^2 dt, T).\end{aligned} \tag{4.3}$$

Proof. To show $\mathbf{u}_n(t, \mathbf{v}_{0,n}) \in \mathcal{D}^{1,2}_{\text{loc}}(\mathbf{V}), \mathbf{d}_n(t, \mathbf{d}_{0,n}) \in \mathcal{D}^{1,2}_{\text{loc}}(\mathbb{H}^2)$, if suffices to prove for any N, $\mathbf{u}_n^N(t, \mathbf{v}_{0,n}) \in \mathcal{D}^{1,2}(\mathbf{V}), \mathbf{d}_n^N(t, \mathbf{d}_{0,n}) \in \mathcal{D}^{1,2}(\mathbb{H}^2)$ on $\Omega_N = \{\sup_{0\le t\le T} (|W(t)| \vee \|Z(t)\|_2) \le N\}$, where $\mathbf{u}_n^N(t, \mathbf{v}_{0,n}), \mathbf{d}_n^N(t, \mathbf{d}_{0,n})$ are unique solutions to the equation (4.1) with $Q(t), Z(t)$ replaced by $Q_N(t) := \exp\left\{W(t)1_{\{|W|\leq N\}}\right\}, Z_N(t) := Z(t)1_{\{\|Z\|_2\leq N\}}$, respectively. For simplicity, we still use Q, Z to represent Q_N, Z_N.

Since $\mathbf{u}_n(t,\mathbf{v}_{0,n}), \mathbf{d}_n(t,\mathbf{d}_{0,n})$ are solutions to the finite-dimensional random ordinary differential equations, it is well known that $\mathbf{u}_n(t,\mathbf{v}_{0,n}), \mathbf{d}_n(t,\mathbf{d}_{0,n})$ are Malliavin differentiable and the corresponding Malliavin derivatives $\mathcal{D}_v\mathbf{u}_n(t,\mathbf{v}_{0,n}), \mathcal{D}_v\mathbf{d}_n(t,\mathbf{d}_{0,n})$ satisfy the following random ODEs:

$$
\begin{aligned}
\mathcal{D}_v\mathbf{u}_n(t,\mathbf{v}_{0,n}) =& -\int_0^t A_1\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})ds\\
&-\int_0^t Q(s)B_1\left(\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})+\mathcal{D}_vZ(s),\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s)\right)ds\\
&-\int_0^t Q(s)B_1\left(\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})+\mathcal{D}_vZ(s)\right)ds\\
&-\int_0^t \mathcal{D}_vQ(s)B_1(\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s))ds\\
&-\int_0^t Q(s)^{-1}M(\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n}),\mathbf{d}_n(s,\mathbf{d}_{0,n}))ds-\int_0^t Q(s)^{-1}M(\mathbf{d}_n(s,\mathbf{d}_{0,n}),\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n}))ds\\
&-\int_0^t \mathcal{D}_vQ(s)^{-1}M(\mathbf{d}_n(s,\mathbf{d}_{0,n}))ds. \qquad (4.4)\\
\mathcal{D}_v\mathbf{d}_n(t,\mathbf{d}_{0,n}) =& -\int_0^t A_2\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})ds\\
&-\int_0^t Q(s)B_2(\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})+\mathcal{D}_vZ(s),\mathbf{d}_n(s,\mathbf{d}_{0,n}))ds\\
&-\int_0^t Q(s)B_2\left(\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s),\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\right)ds\\
&-\int_0^t \mathcal{D}_vQ(s)B_2(\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s),\mathbf{d}_n(s,\mathbf{d}_{0,n}))ds\\
&-\int_0^t \nabla f(\mathbf{d}_n(s,\mathbf{d}_{0,n}))\cdot\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})ds. \qquad (4.5)
\end{aligned}
$$

Taking inner product of $\mathcal{D}_v\mathbf{u}_n, \mathcal{D}_v\mathbf{d}_n$ with $\mathcal{D}_v\mathbf{u}_n, A_2\mathcal{D}_v\mathbf{d}_n$, respectively, then integrating over D, we have

$$
\begin{aligned}
|\mathcal{D}_v\mathbf{u}_n(t,\mathbf{v}_{0,n})|_2^2 =& -2\int_0^t \|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\|_1^2ds\\
&-2\int_0^t Q(s)\langle B_1\left(\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})+\mathcal{D}_vZ(s),\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s)\right),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\rangle ds\\
&-2\int_0^t Q(s)\langle B_1\left(\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})+\mathcal{D}_vZ(s)\right),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\rangle ds\\
&-2\int_0^t \mathcal{D}_vQ(s)\langle B_1(\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s)),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\rangle ds\\
&-2\int_0^t Q(s)^{-1}\langle M(\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n}),\mathbf{d}_n(s,\mathbf{d}_{0,n})),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\rangle ds\\
&-2\int_0^t Q(s)^{-1}\langle M(\mathbf{d}_n(s,\mathbf{d}_{0,n}),\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\rangle ds\\
&-2\int_0^t \mathcal{D}_vQ(s)^{-1}\langle M(\mathbf{d}_n(s,\mathbf{d}_{0,n})),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\rangle ds\\
=:& P_1+\cdots+P_7. \qquad (4.6)\\
\|\mathcal{D}_v\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_1^2 =& -2\int_0^t \|\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\|_2^2ds
\end{aligned}
$$

$$
\begin{aligned}
&-2\int_0^t Q(s)\langle B_2(\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})+\mathcal{D}_vZ(s),\mathbf{d}_n(s,\mathbf{d}_{0,n})),A_2\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\rangle ds\\
&-2\int_0^t Q(s)\langle B_2\left(\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s),\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\right),A_2\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\rangle ds\\
&-2\int_0^t \mathcal{D}_vQ(s)\langle B_2(\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s),\mathbf{d}_n(s,\mathbf{d}_{0,n})),A_2\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\rangle ds\\
&-2\int_0^t \langle\nabla f(\mathbf{d}_n(s,\mathbf{d}_{0,n}))\cdot\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n}),A_2\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\rangle ds\\
=:&Q_1+\cdots+Q_5. \qquad (4.7)
\end{aligned}
$$

By Lemma 2.1, we have

$$
\begin{aligned}
P_2\le&\varepsilon\int_0^t\|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\|_1^2ds+c|Q|_\infty^2\int_0^t\|\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s)\|_1^2|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})|_2^2ds+c\int_0^t|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})|_2^2ds\\
&+c|Q|_\infty^2\int_0^t|\mathcal{D}_vZ(s)|_2\|\mathcal{D}_vZ(s)\|_1\|\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s)\|_1^2ds.
\end{aligned}
$$

Since $\langle B_1(\mathbf{u},\mathbf{v}),\mathbf{v}\rangle=0$, we have

$$
\begin{aligned}
P_3=&-2\int_0^t Q(s)\langle B_1\left(\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s),\mathcal{D}_vZ(s)\right),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\rangle ds\\
\le&\varepsilon\int_0^t\|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\|_1^2ds+c\int_0^t|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})|_2^2ds\\
&+c|Q|_\infty^2\int_0^t|\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s)|_2\|\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s)\|_1\|\mathcal{D}_vZ(s)\|_1^2ds.
\end{aligned}
$$

Similarly, one can get

$$
P_4\le\varepsilon\int_0^t\|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\|_1^2ds+c|\mathcal{D}_vQ|_\infty^2\int_0^t|\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s)|_2^2\|\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s)\|_1^2ds
$$

By Proposition 2.5, we obtain that

$$
\begin{aligned}
P_5+P_6=&-2\int_0^t Q(s)^{-1}\langle B_2(\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n}),\mathbf{d}_n(s,\mathbf{d}_{0,n})),\Delta\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\rangle ds\\
&-2\int_0^t Q(s)^{-1}\langle B_2(\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n}),\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})),\Delta\mathbf{d}_n(s,\mathbf{d}_{0,n})\rangle ds. \qquad (4.8)
\end{aligned}
$$

By (4.8), Proposition 2.6, we get

$$
\begin{aligned}
P_5+P_6+Q_2\le&\varepsilon\int_0^t\|\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\|_2^2ds+\varepsilon\int_0^t\|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\|_1^2ds\\
&+c(|Q^{-1}|_\infty+|Q|_\infty)^4\int_0^t\|\mathbf{d}_n(s,\mathbf{d}_{0,n})\|_1^2\|\mathbf{d}_n(s,\mathbf{d}_{0,n})\|_2^2\|\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\|_1^2ds\\
&+c|Q|_\infty^2\int_0^t\|\mathbf{d}_n(s,\mathbf{d}_{0,n})\|_2^2|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})|_2^2dsds+c|Q|_\infty^2\int_0^t\|\mathbf{d}_n(s,\mathbf{d}_{0,n})\|_2^2\|\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\|_1^2ds\\
&+c|Q|_\infty^2\int_0^t\|\mathcal{D}_vZ(s)\|_1^2\|\mathbf{d}(s,\mathbf{d}_0)\|_1^2ds+c|Q|_\infty^2\int_0^t|\mathcal{D}_vZ(s)|_2^2\|\mathbf{d}(s,\mathbf{d}_0)\|_2^2ds.
\end{aligned}
$$

Similarly, we have

$$
P_7\le\varepsilon\int_0^t\|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\|_1^2ds+c|\mathcal{D}_vQ^{-1}|_\infty^2\int_0^t\|\mathbf{d}_n(s,\mathbf{d}_{0,n})\|_1^2\|\mathbf{d}_n(s,\mathbf{d}_{0,n})\|_2^2ds
$$

By the chain rule and Lemma 2.2,

$$
\begin{aligned}
Q_3 =& 2\int_0^t Q(s)\langle B_2(\nabla(\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s)),\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})),\nabla\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\rangle ds\\
\le& \varepsilon\int_0^t \|\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\|_2^2 ds + c|Q|_\infty^2\int_0^t \|\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s)\|_1^2\|\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\|_1^2 ds.
\end{aligned}
$$

And

$$
\begin{aligned}
Q_4 \le& \varepsilon\int_0^t \|\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\|_2^2 ds + c|\mathcal{D}_v Q|_\infty^2\int_0^t \|\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s)\|_1^2\|\mathbf{d}_n(s,\mathbf{d}_{0,n})\|_1^2 ds\\
&+ c|\mathcal{D}_v Q|_\infty^2\int_0^t |\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s)|_2^2\|\mathbf{d}_n(s,\mathbf{d}_{0,n})\|_2^2 ds.
\end{aligned}
$$

Finally, since $|\nabla f(\mathbf{d})|_\infty \le c\|\mathbf{d}\|_1^2$, we have

$$
Q_5 \le \varepsilon\int_0^t \|\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\|_2^2 ds + c\int_0^t \|\mathbf{d}_n(s,\mathbf{d}_{0,n})\|_1^4|D_v\mathbf{d}_n(s,\mathbf{d}_{0,n})|_2^2 ds.
$$

With the above estimates for terms in (4.6) and (4.7), applying Gronwall's inequality yields that

$$
\begin{aligned}
&\sup_{0\le t\le T}[|\mathcal{D}_v\mathbf{u}_n(t,\mathbf{v}_{0,n})|_2^2+\|\mathcal{D}_v\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_1^2]+\int_0^T\|\mathcal{D}_v\mathbf{u}_n(t,\mathbf{v}_{0,n})\|_1^2 dt+\int_0^T\|\mathcal{D}_v\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_2^2 dt\\
\le& c h_1(T)h_2(T),
\end{aligned}
$$

where

$$
\begin{aligned}
h_1(T) :=& |Q|_\infty^2\int_0^T |\mathcal{D}_v Z(t)|_2\|\mathcal{D}_v Z(t)\|_1\|\mathbf{u}_n(t,\mathbf{v}_{0,n})+Z(t)\|_1^2 dt\\
&+|Q|_\infty^2\int_0^T |\mathbf{u}_n(t,\mathbf{v}_{0,n})+Z(t)|_2\|\mathbf{u}_n(t,\mathbf{v}_{0,n})+Z(t)\|_1\|\mathcal{D}_v Z(t)\|_1^2 dt\\
&+|\mathcal{D}_v Q|_\infty^2\int_0^T |\mathbf{u}_n(t,\mathbf{v}_{0,n})+Z(t)|_2^2\|\mathbf{u}_n(t,\mathbf{v}_{0,n})+Z(t)\|_1^2 dt+|Q|_\infty^2\int_0^T\|\mathcal{D}_v Z(t)\|_1^2\|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_1^2 dt\\
&+|Q|_\infty^2\int_0^T |\mathcal{D}_v Z(t)|_2^2\|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_2^2 dt+|\mathcal{D}_v Q^{-1}|_\infty^2\int_0^T\|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_1^2\|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_2^2 dt\\
&+|\mathcal{D}_v Q|_\infty^2\int_0^T\|\mathbf{u}_n(t,\mathbf{v}_{0,n})+Z(t)\|_1^2\|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_1^2 dt\\
&+|\mathcal{D}_v Q|_\infty^2\int_0^T\|\mathbf{u}_n(t,\mathbf{v}_{0,n})+Z(t)\|_1^2\|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_1^2 dt+|\mathcal{D}_v Q|_\infty^2\int_0^T |\mathbf{u}_n(t,\mathbf{v}_{0,n})+Z(t)|_2^2\|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_2^2 dt,
\end{aligned}
$$

and

$$
\begin{aligned}
h_2(T) := \exp c\Big\{& T+|Q|_\infty^2\int_0^T\|\mathbf{u}_n(t,\mathbf{v}_{0,n})+Z(t)\|_1^2 dt+(|Q^{-1}|_\infty+|Q|_\infty)^4\int_0^T\|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_1^2\|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_2^2 dt\\
&+|Q|_\infty^2\int_0^T\|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_2^2 dt+\int_0^T\|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_1^4 dt\Big\}.
\end{aligned}
$$

In view of Proposition 3.1, $h_1(T), h_2(T)$ are constants depending on $|\mathbf{v}_0|_2, \|\mathbf{d}_0\|_1, |Q|_\infty, \sup_{0\le t\le T}\|Z\|_2, T$.

Moreover, we have

$$
\|\mathcal{D}_v\mathbf{u}_n(t,\mathbf{v}_{0,n})\|_1^2 = -2\int_0^t\|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\|_2^2 ds
$$

$$
\begin{aligned}
&-2\int_0^t Q(s)\langle B_1\left(\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})+\mathcal{D}_vZ(s),\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s)\right),A_1\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\rangle ds\\
&-2\int_0^t Q(s)\langle B_1\left(\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})+\mathcal{D}_vZ(s)\right),A_1\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\rangle ds\\
&-2\int_0^t \mathcal{D}_vQ(s)\langle B_1(\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s)),A_1\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\rangle ds\\
&-2\int_0^t Q(s)^{-1}\langle M(\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n}),\mathbf{d}_n(s,\mathbf{d}_{0,n})),A_1\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\rangle ds\\
&-2\int_0^t Q(s)^{-1}\langle M(\mathbf{d}_n(s,\mathbf{d}_{0,n}),\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})),A_1\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\rangle ds\\
&-2\int_0^t \mathcal{D}_vQ(s)^{-1}\langle M(\mathbf{d}_n(s,\mathbf{d}_{0,n})),A_1\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})\rangle ds
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
\|\mathcal{D}_v\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_2^2=&-2\int_0^t \|\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\|_3^2ds\\
&-2\int_0^t Q(s)\langle\Delta B_2(\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})+\mathcal{D}_vZ(s),\mathbf{d}_n(s,\mathbf{d}_{0,n})),\Delta\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\rangle ds\\
&-2\int_0^t Q(s)\langle\Delta B_2\left(\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s),\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\right),\Delta\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\rangle ds\\
&-2\int_0^t \mathcal{D}_vQ(s)\langle\Delta B_2(\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s),\mathbf{d}_n(s,\mathbf{d}_{0,n})),\Delta\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\rangle ds\\
&-2\int_0^t \langle\Delta\nabla f(\mathbf{d}_n(s,\mathbf{d}_{0,n}))\cdot\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n}),\Delta\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})\rangle ds.
\end{aligned}
\tag{4.10}
$$

With similar discussion as before, we get the estimates for the terms in (4.9) and (4.10), then applying Gronwall's inequality yields that

$$
\begin{aligned}
&\sup_{0\le t\le T}[\|\mathcal{D}_v\mathbf{u}_n(t,\mathbf{v}_{0,n})\|_1^2+\|\mathcal{D}_v\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_2^2]+\int_0^T\|\mathcal{D}_v\mathbf{u}_n(t,\mathbf{v}_{0,n})\|_2^2dt+\int_0^T\|\mathcal{D}_v\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_3^2dt\\
\le& ch_3(T)h_4(T),
\end{aligned}
$$

where

$$
\begin{aligned}
h_3(T):=&|Q|_\infty^2\int_0^T\|\mathcal{D}_vZ(t)\|_1\|\mathcal{D}_vZ(t)\|_2\|\mathbf{u}_n(t,\mathbf{v}_{0,n})+Z(t)\|_1^2dt\\
&+|Q|_\infty^2\int_0^T|\mathcal{D}_vZ(t)|_2\|\mathcal{D}_vZ(t)\|_1\|\mathbf{u}_n(t,\mathbf{v}_{0,n})+Z(t)\|_2^2dt\\
&+|Q|_\infty^2\int_0^T\|\mathbf{u}_n(t,\mathbf{v}_{0,n})+Z(t)\|_1\|\mathbf{u}_n(t,\mathbf{v}_{0,n})+Z(t)\|_2\|\mathcal{D}_vZ(t)\|_1^2dt\\
&+|Q|_\infty^2\int_0^T|\mathbf{u}_n(t,\mathbf{v}_{0,n})+Z(t)|_2\|\mathbf{u}_n(t,\mathbf{v}_{0,n})+Z(t)\|_1\|\mathcal{D}_vZ(t)\|_2^2dt\\
&+|\mathcal{D}_vQ|_\infty^2\int_0^T\|\mathbf{u}_n(t,\mathbf{v}_{0,n})+Z(t)\|_1^3\|\mathbf{u}_n(t,\mathbf{v}_{0,n})+Z(t)\|_2dt\\
&+|\mathcal{D}_vQ|_\infty^2\int_0^T|\mathbf{u}_n(t,\mathbf{v}_{0,n})+Z(t)|_2\|\mathbf{u}_n(t,\mathbf{v}_{0,n})+Z(t)\|_2^3dt\\
&+|\mathcal{D}_vQ^{-1}|_\infty^2\int_0^T\|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_1\|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_2^2\|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_3ds\\
&+|Q|_\infty^2\int_0^T\|\mathcal{D}_vZ(t)\|_2^2\|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_1^2dt+|Q|_\infty^2\int_0^T|\mathcal{D}_vZ(t)|_2^2\|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_3^2dt
\end{aligned}
$$

$$+ |\mathcal{D}_v Q|_\infty^2 \int_0^T \|\mathbf{u}_n(t,\mathbf{v}_{0,n}) + Z(t)\|_2^2 \|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_1^2 dt + |\mathcal{D}_v Q|_\infty^2 \int_0^T |\mathbf{u}_n(t,\mathbf{v}_{0,n}) + Z(t)|_2^2 \|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_3^2 dt$$
$$+ |\mathcal{D}_v Q|_\infty^2 \int_0^T \|\mathbf{u}_n(t,\mathbf{v}_{0,n}) + Z(t)\|_1^2 \|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_2^2 dt,$$

and

$$\begin{aligned} h_4(T) := \exp c\Big\{ & T + |Q|_\infty^2 \int_0^T \|\mathbf{u}_n(t,\mathbf{v}_{0,n}) + Z(t)\|_1^2 dt + |Q|_\infty^{4/3} \int_0^T \|\mathbf{u}_n(t,\mathbf{v}_{0,n}) + Z(t)\|_2^{4/3} dt \\ & + |Q|_\infty^{4/3} \int_0^T \|\mathbf{u}_n(t,\mathbf{v}_{0,n}) + Z(t)\|_1^{2/3} \|\mathbf{u}_n(t,\mathbf{v}_{0,n}) + Z(t)\|_2^{2/3} dt \\ & + |Q|_\infty^2 \int_0^T \|\mathbf{u}_n(t,\mathbf{v}_{0,n}) + Z(t)\|_2^2 dt + |Q|_\infty^2 \int_0^T \|\mathbf{u}_n(t,\mathbf{v}_{0,n}) + Z(t)\|_1^2 dt \\ & + (|Q|_\infty + |Q^{-1}|_\infty)^4 \int_0^T \|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_1^2 \|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_2^2 dt \\ & + (|Q|_\infty + |Q^{-1}|_\infty)^2 \int_0^T \|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_2 \|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_3 dt + |Q|_\infty^2 \int_0^T \|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_3^2 dt \\ & + |Q|_\infty^2 \int_0^T \|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_2^2 dt + |Q|_\infty^2 \int_0^T \|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_3^2 dt + \int_0^T \|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_1^2 dt \Big\}. \end{aligned}$$

In view of Proposition 3.1, $h_3(T), h_4(T)$ are constants depending on $\|\mathbf{v}_0\|_1, \|\mathbf{d}_0\|_2, |Q|_\infty, \sup_{0\le t\le T} \|Z\|_2, \int_0^T \|Z(t)\|_3^2 dt, T$.
□

Now we are ready to discuss the Malliavin differentiability for the solution to the stochastic nematic liquid crystal equations.

Theorem 4.2. *For $\mathbf{v}_0 \in \mathbf{V}, \mathbf{d}_0 \in \mathbb{H}^2$ and $t \ge 0$, the solution maps $\omega \mapsto \mathbf{u}(t,\mathbf{v}_0,\omega), \omega \mapsto \mathbf{d}(t,\mathbf{d}_0,\omega)$ are Malliavin differentiable, and for all $t \in [0,T]$, almost surely their Malliavin derivatives $\mathcal{D}_v\mathbf{u}(t,\mathbf{v}_0), \mathcal{D}_v\mathbf{d}(t,\mathbf{d}_0)$ solve the random equations (4.4) and (4.5) with $\mathbf{u}(t,\mathbf{v}_0), \mathbf{d}(t,\mathbf{d}_0)$ in place of $\mathbf{u}_n(t,\mathbf{v}_{0,n}), \mathbf{d}_n(t,\mathbf{d}_{0,n})$.*

Proof. We can do the same localization as that in the proof of Proposition 4.1, and will show $\mathbf{u}(t,\mathbf{v}_0) \in \mathcal{D}^{1,2}_{\text{loc}}(\mathbf{H}), \mathbf{d}(t,\mathbf{d}_0) \in \mathcal{D}^{1,2}_{\text{loc}}(\mathbb{H}^1)$.

Firstly, It was shown in [9] that

$$\lim_{n\to\infty} \sup_{0\le t\le T} \left(\|\mathbf{u}_n(t,\mathbf{v}_{0,n}) - \mathbf{u}(t,\mathbf{v}_0)\|_1^2 + \|\mathbf{d}_n(t,\mathbf{d}_{0,n}) - \mathbf{d}(t,\mathbf{d}_0)\|_2^2 \right) = 0 \text{ a.s.}. \tag{4.11}$$

Now let ξ_v, η_v be the solution to the following random equations as well as the boundary conditions (BC)

$$\begin{aligned} \xi_v(t,\mathbf{v}_0) = & - \int_0^t A_1 \xi_v(s,\mathbf{v}_0) ds \\ & - \int_0^t Q(s) B_1 \left(\xi_v(s,\mathbf{v}_0) + \mathcal{D}_v Z(s), \mathbf{u}(s,\mathbf{v}_0) + Z(s) \right) ds \\ & - \int_0^t Q(s) B_1 \left(\mathbf{u}(s,\mathbf{v}_0) + Z(s), \xi_v(s,\mathbf{v}_0) + \mathcal{D}_v Z(s) \right) ds \\ & - \int_0^t \mathcal{D}_v Q(s) B_1(\mathbf{u}(s,\mathbf{v}_0) + Z(s)) ds \\ & - \int_0^t Q(s)^{-1} M(\eta_v(s,\mathbf{d}_0), \mathbf{d}(s,\mathbf{d}_0)) ds - \int_0^t Q(s)^{-1} M(\mathbf{d}(s,\mathbf{d}_0), \eta_v(s,\mathbf{d}_0)) ds \\ & - \int_0^t \mathcal{D}_v Q(s)^{-1} M(\mathbf{d}(s,\mathbf{d}_0)) ds. \end{aligned}$$

$$
\begin{aligned}
\eta_v(t,\mathbf{d}_0) = &-\int_0^t A_2\eta_v(s,\mathbf{d}_0)ds\\
&-\int_0^t Q(s)B_2(\xi_v(s,\mathbf{v}_0)+\mathcal{D}_vZ(s),\mathbf{d}(s,\mathbf{d}_0))ds\\
&-\int_0^t Q(s)B_2\left(\mathbf{u}(s,\mathbf{v}_0)+Z(s),\eta_v(s,\mathbf{d}_0)\right)ds\\
&-\int_0^t \mathcal{D}_vQ(s)B_2(\mathbf{u}(s,\mathbf{v}_0)+Z(s),\mathbf{d}(s,\mathbf{d}_0))ds-\int_0^t \nabla f(\mathbf{d}(s,\mathbf{d}_0))\cdot\eta_v(s,\mathbf{d}_0)ds,
\end{aligned}
$$

for any $t\in[0,T]$. The global well-posedness of the above equations have been studied in [5]. Since $\mathcal{D}$ is closed, it suffices to show that

$$
\lim_{n\to\infty}\mathbb{E}\sup_{0\le v\le t}\left[|\mathcal{D}_v\mathbf{u}_n(t,\mathbf{v}_{0,n})-\xi_v(t,\mathbf{v}_0)|_2^2+\|\mathcal{D}_v\mathbf{d}_n(t,\mathbf{d}_{0,n})-\eta_v(t,\mathbf{d}_0)\|_1^2\right]=0 \tag{4.12}
$$

For simplicity, we define the following norm notations:

$$
\begin{aligned}
&C_1^n:=\sup_{0\le t\le T}\left(|\mathbf{u}_n(t,\mathbf{v}_{0,n})|_2+|Z(t)|_2\right), && C_2^n:=\sup_{0\le t\le T}\left(\|\mathbf{u}_n(t,\mathbf{v}_{0,n})\|_1+\|Z(t)\|_1\right),\\
&C_1:=\sup_{0\le t\le T}\left(|\mathbf{u}(t,\mathbf{v}_0)|_2+|Z(t)|_2\right), && C_2:=\sup_{0\le t\le T}\left(\|\mathbf{u}(t,\mathbf{v}_0)\|_1+\|Z(t)\|_1\right),\\
&M_1^n:=\sup_{0\le t\le T}\left(|\mathcal{D}_v\mathbf{u}_n(t,\mathbf{v}_{0,n})|_2+|\mathcal{D}_vZ(t)|_2\right), && M_2^n:=\sup_{0\le t\le T}\left(\|\mathcal{D}_v\mathbf{u}_n(t,\mathbf{v}_{0,n})\|_1+\|\mathcal{D}_vZ(t)\|_1\right),\\
&M_1:=\sup_{0\le t\le T}|\xi_v(t,\mathbf{v}_0)|_2, && M_2:=\sup_{0\le t\le T}\|\xi_v(t,\mathbf{v}_0)\|_1;\\
&D_1^n:=\sup_{0\le t\le T}|\mathbf{d}_n(t,\mathbf{d}_{0,n})|_2,\ D_2^n:=\sup_{0\le t\le T}\|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_1, && D_3^n:=\sup_{0\le t\le T}\|\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_2,\\
&D_1:=\sup_{0\le t\le T}|\mathbf{d}(t,\mathbf{d}_0)|_2,\ D_2:=\sup_{0\le t\le T}\|\mathbf{d}(t,\mathbf{d}_0)\|_1, && D_3:=\sup_{0\le t\le T}\|\mathbf{d}(t,\mathbf{d}_0)\|_2,\\
&N_1^n:=\sup_{0\le t\le T}|\mathcal{D}_v\mathbf{d}_n(t,\mathbf{d}_{0,n})|_2,\ N_2^n:=\sup_{0\le t\le T}\|\mathcal{D}_v\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_1, && N_3^n:=\sup_{0\le t\le T}\|\mathcal{D}_v\mathbf{d}_n(t,\mathbf{d}_{0,n})\|_2,\\
&N_1:=\sup_{0\le t\le T}|\eta_v(t,\mathbf{d}_0)|_2,\ N_2:=\sup_{0\le t\le T}\|\eta_v(t,\mathbf{d}_0)\|_1, && N_3:=\sup_{0\le t\le T}\|\eta_v(t,\mathbf{d}_0)\|_2.
\end{aligned} \tag{4.13}
$$

We first estimate the following:

$$
\begin{aligned}
&|\mathcal{D}_v\mathbf{u}_n(t,\mathbf{v}_{0,n})-\xi_v(t,\mathbf{v}_0)|_2^2\\
=&-2\int_0^t\|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\|_1^2ds\\
&-2\int_0^t Q(s)\langle B_1\left(\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})+\mathcal{D}_vZ(s),\mathbf{u}_n(s,\mathbf{v}_{0,n})-\mathbf{u}(s,\mathbf{v}_0)\right),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\rangle ds\\
&-2\int_0^t Q(s)\langle B_1\left(\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0),\mathbf{u}(s,\mathbf{v}_0)+Z(s)\right),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\rangle ds\\
&-2\int_0^t Q(s)\langle B_1\left(\mathbf{u}_n(s,\mathbf{v}_{0,n})-\mathbf{u}(s,\mathbf{v}_0),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})+\mathcal{D}_vZ(s)\right),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\rangle ds\\
&-2\int_0^t Q(s)\langle B_1\left(\mathbf{u}(s,\mathbf{v}_0)+Z(s),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\right),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\rangle ds\\
&-2\int_0^t \mathcal{D}_vQ(s)\langle B_1(\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s),\mathbf{u}_n(s,\mathbf{v}_{0,n})-\mathbf{u}(s,\mathbf{v}_0)),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\rangle ds\\
&-2\int_0^t \mathcal{D}_vQ(s)\langle B_1(\mathbf{u}_n(s,\mathbf{v}_{0,n})-\mathbf{u}(s,\mathbf{v}_0),\mathbf{u}(s,\mathbf{v}_0)+Z(s)),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\rangle\\
&-2\int_0^t Q(s)^{-1}\langle M(\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n}),\mathbf{d}_n(s,\mathbf{d}_{0,n})-\mathbf{d}(s,\mathbf{d}_0)),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\rangle ds
\end{aligned}
$$

$$
\begin{aligned}
&-2\int_0^t Q(s)^{-1}\langle M(\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0),\mathbf{d}(s,\mathbf{d}_0)),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\rangle ds\\
&-2\int_0^t Q(s)^{-1}\langle M(\mathbf{d}_n(s,\mathbf{d}_{0,n})-\mathbf{d}(s,\mathbf{d}_0),\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\rangle ds\\
&-2\int_0^t Q(s)^{-1}\langle M(\mathbf{d}(s,\mathbf{d}_0),\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0)),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\rangle ds\\
&-2\int_0^t \mathcal{D}_vQ(s)^{-1}\langle M(\mathbf{d}_n(s,\mathbf{d}_{0,n}),\mathbf{d}_n(s,\mathbf{d}_{0,n})-\mathbf{d}(s,\mathbf{d}_0)),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\rangle ds\\
&-2\int_0^t \mathcal{D}_vQ(s)^{-1}\langle M(\mathbf{d}_n(s,\mathbf{d}_{0,n})-\mathbf{d}(s,\mathbf{d}_0),\mathbf{d}(s,\mathbf{d}_0)),\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\rangle ds\\
=&:I_1+\cdots+I_{13}. \qquad (4.14)
\end{aligned}
$$

By the estimate (2.2), we get that

$$
I_2 \le\varepsilon\int_0^t \|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\|_1^2ds+c|Q|_\infty^2M_1^nM_2^n(C_1^n+C_1)\int_0^t\|\mathbf{u}_n(s,\mathbf{v}_{0,n})-\mathbf{u}(s,\mathbf{v}_0)\|_1ds,
$$

and

$$
\begin{aligned}
I_3 \le&\varepsilon\int_0^t \|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\|_1^2ds+c|Q|_\infty^2[C_2]^2\int_0^t|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)|_2^2ds,\\
I_4 \le&\varepsilon\int_0^t \|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\|_1^2ds+c\int_0^t|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)|_2^2ds\\
&+c|Q|_\infty^2[M_2^n]^2(C_1^n+C_1)\int_0^t\|\mathbf{u}_n(s,\mathbf{v}_{0,n})-\mathbf{u}(s,\mathbf{v}_0)\|_1ds,
\end{aligned}
$$

According to Lemma 2.1, $\langle B_1(\mathbf{u},\mathbf{v}),\mathbf{v}\rangle=0$, we get $I_5=0$, and

$$
I_6 \le\varepsilon\int_0^t \|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\|_1^2ds+c|\mathcal{D}_vQ|_\infty^2C_1^nC_2^n(C_1^n+C_1)\int_0^t\|\mathbf{u}_n(s,\mathbf{v}_{0,n})-\mathbf{u}(s,\mathbf{v}_0)\|_1ds,
$$

and similarly we obtain that

$$
I_7 \le\varepsilon\int_0^t \|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\|_1^2ds+c|\mathcal{D}_vQ|_\infty^2C_1C_2(C_1^n+C_1)\int_0^t\|\mathbf{u}_n(s,\mathbf{v}_{0,n})-\mathbf{u}(s,\mathbf{v}_0)\|_1ds.
$$

By Lemma 2.3, we obtain that

$$
I_8+I_{10} \le\varepsilon\int_0^t \|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\|_1^2ds+c|Q^{-1}|_\infty^2N_2^nN_3^n\int_0^t\|\mathbf{d}_n(s,\mathbf{d}_{0,n})-\mathbf{d}(s,\mathbf{d}_0)\|_2^2ds.
$$

By Proposition 2.5, we get that

$$
\begin{aligned}
I_9+I_{11}=&-2\int_0^t Q(s)^{-1}\langle B_2(\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0),\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0)),\Delta\mathbf{d}(s,\mathbf{d}_0)\rangle ds\\
&-2\int_0^t Q(s)^{-1}\langle B_2(\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0),\mathbf{d}(s,\mathbf{d}_0),\Delta(\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0))\rangle ds. \qquad (4.15)
\end{aligned}
$$

By Lemma 2.3, we get

$$
I_{12} \le\varepsilon\int_0^t \|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\|_1^2ds+c|\mathcal{D}_vQ^{-1}|_\infty^2D_2^nD_3^n\int_0^t\|\mathbf{d}_n(s,\mathbf{d}_{0,n})-\mathbf{d}(s,\mathbf{d}_0)\|_2^2ds.
$$

Similarly, we obtain that

$$
I_{13} \le\varepsilon\int_0^t \|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\|_1^2ds+c|\mathcal{D}_vQ^{-1}|_\infty^2D_2D_3\int_0^t\|\mathbf{d}_n(s,\mathbf{d}_{0,n})-\mathbf{d}(s,\mathbf{d}_0)\|_2^2ds.
$$

Now taking inner product of $\mathcal{D}_v\mathbf{d}_n(t,\mathbf{d}_{0,n})-\eta_v(t,\mathbf{d}_0)$ with $\Delta(\mathcal{D}_v\mathbf{d}_n(t,\mathbf{d}_{0,n})-\eta_v(t,\mathbf{d}_0))$ yields that

$$\begin{aligned}
&\|\mathcal{D}_v\mathbf{d}_n(t,\mathbf{d}_{0,n})-\eta_v(t,\mathbf{d}_0)\|_1^2\\
=&-2\int_0^t\|\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0)\|_2^2ds\\
&-2\int_0^t Q(s)\langle\nabla B_2(\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})+\mathcal{D}_vZ(s),\mathbf{d}_n(s,\mathbf{d}_{0,n})-\mathbf{d}(s,\mathbf{d}_0),\nabla(\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0))\rangle ds\\
&+2\int_0^t Q(s)\langle B_2(\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0),\mathbf{d}(s,\mathbf{d}_0)),\Delta(\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0))\rangle\\
&-2\int_0^t Q(s)\langle\nabla B_2(\mathbf{u}(s,\mathbf{v}_0)+Z(s),\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0)),\nabla(\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0))\rangle ds\\
&-2\int_0^t Q(s)\langle\nabla B_2(\mathbf{u}_n(s,\mathbf{v}_{0,n})-\mathbf{u}(s,\mathbf{v}_0),\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})),\nabla(\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0))\rangle ds\\
&-2\int_0^t \mathcal{D}_vQ(s)\langle\nabla B_2(\mathbf{u}_n(s,\mathbf{v}_{0,n})-\mathbf{u}(s,\mathbf{v}_0),\mathbf{d}(s,\mathbf{d}_0)),\nabla(\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0))\rangle ds\\
&-2\int_0^t \mathcal{D}_vQ(s)\langle\nabla B_2(\mathbf{u}_n(s,\mathbf{v}_{0,n})+Z(s),\mathbf{d}_n(s,\mathbf{d}_{0,n})-\mathbf{d}(s,\mathbf{d}_0)),\nabla(\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0))\rangle ds\\
&+2\int_0^t\langle\nabla f(\mathbf{d}_n(s,\mathbf{d}_{0,n}))\cdot(\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0)),\Delta(\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0))\rangle ds\\
&+2\int_0^t\langle(\nabla f(\mathbf{d}_n(s,\mathbf{d}_{0,n}))-\nabla f(\mathbf{d}(s,\mathbf{d}_0)))\cdot\eta_v(s,\mathbf{d}_0),\Delta(\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0))\rangle ds\\
=:&K_1+\cdots K_9
\end{aligned}\tag{4.16}$$

By Lemma 2.2,

$$\begin{aligned}
K_2\le&\varepsilon\int_0^t\|\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_0)-\eta_v(s,\mathbf{d}_0)\|_2^2ds+c|Q|_\infty^2[M_2^n]^2\int_0^t\|\mathbf{d}_n(s,\mathbf{d}_{0,n})-\mathbf{d}(s,\mathbf{d}_0)\|_1^2ds\\
&+c|Q|_\infty^2[M_1^n]^2\int_0^t\|\mathbf{d}_n(s,\mathbf{d}_{0,n})-\mathbf{d}(s,\mathbf{d}_0)\|_2^2ds,
\end{aligned}$$

With (4.15), by Proposition 2.6, we have

$$\begin{aligned}
I_9+I_{11}+K_3\le&\varepsilon\int_0^t\|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)\|_1^2ds+\varepsilon\int_0^t\|\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0)\|_2^2ds\\
&+c|Q|_\infty^2D_3^2\int_0^t|\mathcal{D}_v\mathbf{u}_n(s,\mathbf{v}_{0,n})-\xi_v(s,\mathbf{v}_0)|_2^2ds+c|Q|_\infty^2D_3^2\int_0^t\|\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0)\|_1^2ds\\
&+c(|Q|_\infty+|Q^{-1}|_\infty)^4D_2^2D_3^2\int_0^t\|\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0)\|_1^2ds.
\end{aligned}$$

By Lemma 2.2, $\langle B_2(\mathbf{u},\mathbf{d}),\mathbf{d}\rangle=0$, and by Young's inequality,

$$K_4\le\varepsilon\int_0^t\|\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_0)-\eta_v(s,\mathbf{d}_0)\|_2^2ds+c|Q|_\infty^2[C_2]^2\int_0^t\|\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0)\|_1^2ds.$$

Then

$$\begin{aligned}
K_5\le&\varepsilon\int_0^t\|\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0)\|_2^2ds+c|Q|_\infty^2[N_2^n]^2\int_0^t\|\mathbf{u}_n(s,\mathbf{v}_{0,n})-\mathbf{u}(s,\mathbf{v}_0)\|_1^2ds\\
&+c|Q|_\infty^2[N_3^n]^2\int_0^t|\mathbf{u}_n(s,\mathbf{v}_{0,n})-\mathbf{u}(s,\mathbf{v}_0)|_2^2ds,\\
K_6\le&\varepsilon\int_0^t\|\mathcal{D}_v\mathbf{d}_n(s,\mathbf{d}_{0,n})-\eta_v(s,\mathbf{d}_0)\|_2^2ds+c|\mathcal{D}_vQ|_\infty^2[D_2]^2\int_0^t\|\mathbf{u}_n(s,\mathbf{v}_{0,n})-\mathbf{u}(s,\mathbf{v}_0)\|_1^2ds
\end{aligned}$$

$$
\begin{aligned}
&+ c|\mathcal{D}_v Q|_\infty^2 [D_3]^2 \int_0^t |\mathbf{u}_n(s,\mathbf{v}_{0,n}) - \mathbf{u}(s,\mathbf{v}_0)|_2^2 ds,\\
K_7 \le& \varepsilon \int_0^t \|\mathcal{D}_v \mathbf{d}_n(s,\mathbf{d}_0) - \eta_v(s,\mathbf{d}_0)\|_2^2 ds + c|\mathcal{D}_v Q|_\infty^2 [C_2^n]^2 \int_0^t \|\mathbf{d}_n(s,\mathbf{d}_{0,n}) - \mathbf{d}(s,\mathbf{d}_0)\|_1^2 ds\\
&+ c|\mathcal{D}_v Q|_\infty^2 [C_1^n]^2 \int_0^t \|\mathbf{d}_n(s,\mathbf{d}_{0,n}) - \mathbf{d}(s,\mathbf{d}_0)\|_2^2 ds,\\
K_8 \le& \varepsilon \int_0^t \|\mathcal{D}_v \mathbf{d}_n(s,\mathbf{d}_{0,n}) - \eta_v(s,\mathbf{d}_0)\|_2^2 ds + c[D_2^n]^2 \int_0^t |\mathcal{D}_v \mathbf{d}_n(s,\mathbf{d}_{0,n}) - \eta_v(s,\mathbf{d}_0)|_2^2 ds,
\end{aligned}
$$

where the last inequality follows from $|\nabla f(\mathbf{d})|_\infty \le c\|\mathbf{d}\|_1^2$.

$$
\begin{aligned}
K_9 \le& c \int_0^t |(\nabla f(\mathbf{d}_n(s,\mathbf{d}_{0,n})) - \nabla f(\mathbf{d}(s,\mathbf{d}_0))) \cdot \eta_v(s,\mathbf{d}_0)|_2 |\Delta(\mathcal{D}_v \mathbf{d}_n(s,\mathbf{d}_{0,n}) - \eta_v(s,\mathbf{d}_0))|_2 ds\\
\le& \varepsilon \int_0^t \|\mathcal{D}_v \mathbf{d}_n(s,\mathbf{d}_{0,n}) - \eta_v(s,\mathbf{d}_0)\|_2^2 ds + c[N_1]^2 \int_0^t \|\mathbf{d}_n(s,\mathbf{d}_{0,n}) - \mathbf{d}(s,\mathbf{d}_0)\|_1^2 ds,
\end{aligned}
$$

Combining the estimates for (4.14), (4.16), and applying Gronwall inequality we get that

$$
\begin{aligned}
&\sup_{v\le t\le T} \left\{|\mathcal{D}_v \mathbf{u}_n(t,\mathbf{v}_{0,n}) - \xi_v(t,\mathbf{v}_0)|_2^2 + \|\mathcal{D}_v \mathbf{d}_n(t,\mathbf{d}_{0,n}) - \eta_v(t,\mathbf{d}_0)\|_1^2\right\}\\
&+ \int_0^T \|\mathcal{D}_v \mathbf{u}_n(t,\mathbf{v}_{0,n}) - \xi_v(t,\mathbf{v}_0)\|_1^2 ds + \int_0^T \|\mathcal{D}_v \mathbf{d}_n(t,\mathbf{d}_{0,n}) - \eta_v(t,\mathbf{d}_0)\|_2^2 ds\\
\le& L_1(\omega) \int_0^t \|\mathbf{u}_n(s,\mathbf{v}_{0,n}) - \mathbf{u}(s,\mathbf{v}_0)\|_1 ds \times \exp\left\{cT(1 + |Q|_\infty^2 [C_2]^2 + D_3^2 |Q|_\infty^2)\right\}\\
&+ L_2(\omega) \int_0^t \|\mathbf{d}_n(s,\mathbf{d}_{0,n}) - \mathbf{d}(s,\mathbf{d}_0)\|_2^2 ds\\
&\quad \times \exp cT\left\{|Q|_\infty^2 D_3^2 + D_2^2 D_3^2 (|Q|_\infty + |Q^{-1}|_\infty)^4 + |Q|_\infty^2 [C_2]^2 + [D_2^n]^2\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
L_1(\omega) =& c(|Q|_\infty^2 (M_1^n M_2^n + [M_2^n]^2)(C_1^n + C_1) + |\mathcal{D}_v Q|_\infty^2 (C_1^n C_2^n + C_1 C_2)(C_1^n + C_1)\\
&+ |Q|_\infty^2 ([N_2^n]^2 + [N_3^n]^2) + |\mathcal{D}_v Q|_\infty^2 ([D_2]^2 + [D_3]^2)),\\
L_2(\omega) =& c(|Q^{-1}|_\infty^2 N_2^n N_3^n + |Q|_\infty^2 ([M_1^n]^2 + [M_2^n]^2)\\
&+ |\mathcal{D}_v Q|_\infty^2 (D_2^n D_3^n + D_2 D_3) + |\mathcal{D}_v Q|_\infty^2 ([C_1^n]^2 + [C_2^n]^2) + [N_1]^2).
\end{aligned}
$$

As we localize Q, Z at the beginning of the proof, they are bounded by N. Moreover, since the initial conditions are deterministic, by Theorem 3.7 and Proposition 4.1, all the norms defined in (4.13) are uniformly bounded with respect to ω, n. Hence by (4.11) and dominated convergence theorem,

$$
\lim_{n\to\infty} \mathbb{E} \sup_{v\le t\le T} \{|\mathcal{D}_v \mathbf{u}_n(t,\mathbf{v}_{0,n}) - \xi_v(t,\mathbf{v}_0)|_2^2 + \|\mathcal{D}_v \mathbf{d}_n(t,\mathbf{d}_{0,n}) - \eta_v(t,\mathbf{d}_0)\|_1^2\} = 0.
$$

Thus, (4.12) gets proved. □

5 The global well-posedness of stochastic nematic liquid crystals flows with random initial and boundary conditions

In this section, we replace the deterministic initial condition $(\mathbf{v}_0, \mathbf{d}_0)$ in (3.3) by the random field (R_v, R_d), and obtain the global well-posedness by adopting the method in [16, Theorem 4.1]. In details, we first operate parametrization and time-discretization to the anticipating model and obtain the expressions (5.3)-(5.4). Note that the pathwise continuity in time t of $(\mathbf{u}, \mathbf{d})$ will be preserved with random initial data in place of deterministic initial data, it remains to deal with the non-adapted Skorohod integral by using Malliavin-type integration by parts, see (5.5). On one hand, the convergence of K^n should be delicately handled such that both K^n and the Malliavin derivative of K^n converge, Proposition 3.1, Theorem 3.7 and Proposition 4.1 are applied to ensure this result; on the other hand, Fréchet derivative of $\mathbf{u}$ and Malliavin derivative of the random initial data are involved in estimating L^n, see (5.8). Pathwise continuity of Fréchet differentiation in time, as well as Malliavin calculus techniques are utilized to arrive at the required result for the Stratonovich integral, see (5.9).

In the following, we restate the main result of this article, the global well-posedness for the stochastic nematic liquid crystals flows with random initial and boundary conditions.

Theorem 5.1. *Let $R_v \in \mathcal{D}^{1,2}_{loc}(\mathbf{H}) \cap \mathbf{V}, R_d \in \mathcal{D}^{1,2}_{loc}(\mathbb{H}^1) \cap \mathbb{H}^2$, then there exists a unique strong solution $(\mathbf{v}(t, R_v), \mathbf{d}(t, R_d))$ of the following anticipating stratonovich model:*

$$\begin{aligned}\mathbf{v}(t, R_v) =& R_v - \int_0^t A_1\mathbf{v}(s, R_v)ds - \int_0^t B_1(\mathbf{v}(s, R_v))ds - \int_0^t M(\mathbf{d}(s, R_d))ds \\ &+ \int_0^t \mathbf{v}(s, R_v) \circ dW(s) + \sigma_0 W_0(t), \end{aligned} \tag{5.1}$$

$$\mathbf{d}(t, R_d) = R_d - \int_0^t A_2\mathbf{d}(s, R_d)ds - \int_0^t B_2(\mathbf{v}(s, R_v), \mathbf{d}(s, R_d))ds - \int_0^t f(\mathbf{d}(s, R_d))ds. \tag{5.2}$$

Proof. We will adopt the method in [16] to show the existence. For any fixed $t > 0$, denote by $\{0 = t_0 < t_1 < \cdots t_n = t\}$ an arbitrary partition such that $\tau_n := \max\limits_{1\le k\le n}(t_k - t_{k-1}) \to 0$ as $n \to \infty$. Then we have

$$\begin{aligned}&\mathbf{v}(t, R_v) - R_v - Q(t)Z(t) = Q(t)\mathbf{u}(t, R_v) - R_v \\ =&\sum_{k=1}^n (Q(t_k)\mathbf{u}(t_k, R_v) - Q(t_{k-1})\mathbf{u}(t_{k-1}, R_v)) \\ =&\sum_{k=1}^n Q(t_k)(\mathbf{u}(t_k, R_v) - \mathbf{u}(t_{k-1}, R_v)) + \sum_{k=1}^n (Q(t_k) - Q(t_{k-1}))\mathbf{u}(t_{k-1}, R_v) \\ =&-\sum_{k=1}^n \int_{t_{k-1}}^{t_k} Q(t_k)A_1\mathbf{u}(s, R_v)ds - \sum_{k=1}^n \int_{t_{k-1}}^{t_k} Q(t_k)Q(s)B_1(\mathbf{u}(s, R_v) + Z(s))ds \\ &-\sum_{k=1}^n \int_{t_{k-1}}^{t_k} Q(t_k)Q(s)^{-1}M(\mathbf{d}(s, R_d))ds + \sum_{k=1}^n \mathbf{u}(t_{k-1}, R_v)\int_{t_{k-1}}^{t_k} Q(s)dW(s) + \frac{1}{2}\sum_{k=1}^n \mathbf{u}(t_{k-1}, R_v)\int_{t_{k-1}}^{t_k} Q(s)ds \\ =:& I_1^n + \cdots + I_5^n;\end{aligned} \tag{5.3}$$

$$\begin{aligned}\mathbf{d}(t, R_d) - \mathbf{d}_0 =& \sum_{k=1}^n \mathbf{d}(t_k, R_d) - \mathbf{d}(t_{k-1}, R_d) \\ =& -\sum_{k=1}^n \int_{t_{k-1}}^{t_k} A_2\mathbf{d}(s, R_d)ds - \sum_{k=1}^n \int_{t_{k-1}}^{t_k} Q(s)B_2(\mathbf{u}(s, R_v) + Z(s), \mathbf{d}(s, R_d))ds - \sum_{k=1}^n \int_{t_{k-1}}^{t_k} f(\mathbf{d}(s, R_d))ds \\ =:& J_1^n + J_2^n + J_3^n.\end{aligned} \tag{5.4}$$

Since $\mathbf{u}(s, R_v), \mathbf{d}(s, R_d)$ and $Q(s)$ are contiuous with respect to time s, we have

$$\begin{aligned}&\lim_{n\to\infty} I_1^n = -\int_0^t A_1\mathbf{v}(s, R_v)ds + \int_0^t Q(s)A_1Z(s)ds, \\ &\lim_{n\to\infty} I_2^n = -\int_0^t Q(s)^2 B_1(\mathbf{u}(s, R_v) + Z(s))ds = -\int_0^t B_1(\mathbf{v}(s, R_v))ds, \\ &\lim_{n\to\infty} I_3^n = -\int_0^t M(\mathbf{d}(s, R_d))ds, \qquad \lim_{n\to\infty} I_5^n = \frac{1}{2}\int_0^t Q(s)\mathbf{u}(s, R_v)ds;\end{aligned}$$

$$\lim_{n\to\infty} J_1^n = -\int_0^t A_2\mathbf{d}(s,R_d)ds,$$
$$\lim_{n\to\infty} J_2^n = -\int_0^t Q(s)B_2(\mathbf{u}(s,R_v)+Z(s),\mathbf{d}(s,R_d))ds = -\int_0^t B_2(\mathbf{v}(s,R_v),\mathbf{d}(s,R_d))ds,$$
$$\lim_{n\to\infty} J_3^n = -\int_0^t f(\mathbf{d}(s,R_d))ds.$$

It remains to deal with I_4^n. By the property of the Skorohod integral (See Proposition 1.3.5 in [17]) we get

$$\begin{aligned}
I_4^n &= -\sum_{k=1}^n \int_{t_{k-1}}^{t_k} \mathbf{u}(t_{k-1},R_v)Q(s)dW(s) + \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \mathcal{D}_s(\mathbf{u}(t_{k-1},R_v))Q(s)ds \\
&= \int_0^t \sum_{k=1}^n \mathbf{u}(t_{k-1},R_v)Q(s)1_{(t_{k-1},t_k]}(s)dW(s) + \int_0^t \sum_{k=1}^n \mathcal{D}_s(\mathbf{u}(t_{k-1},R_v))Q(s)1_{(t_{k-1},t_k]}(s)ds \\
&=: \int_0^t K^n(s)dW(s) + \int_0^t L^n(s)ds.
\end{aligned} \tag{5.5}$$

Denote by $\mathbb{L}^{1,2}(\mathbf{H})$ the class of $\mathbf{H}$-valued process $\mathbf{v}(t)\in\mathcal{D}^{1,2}(\mathbf{H})$ for almost all t, and there always exists a measurable version of $\mathcal{D}_s\mathbf{v}(t)$ satsifying $\mathbb{E}\left(\int_0^T\int_0^T |\mathcal{D}_s\mathbf{v}(t)|_2^2 dsdt\right)<\infty$. We say $\mathbf{v}(t)\in\mathbb{L}^{1,2}_{\text{loc}}(\mathbf{H})$ if there exists a sequence $\{\Omega_n\}\subset\mathcal{F}$ such that Ω_n increases to Ω and $\mathbf{v}1_{\Omega_n}\in\mathbb{L}^{1,2}(\mathbf{H})$. Without loss of generality, we can assume $\|R_v\|_1\le M$, $\|Z(s)\|_2\le M$ and $Q=Q_N$, or we can always do truncation otherwise. As $\mathbf{u}(s,R_v)$ is continuous in s, we have for all $s>0$, $K^n(s)\to\mathbf{u}(s,R_v)Q(s)$, and by Proposition 3.1 and the localization, we have

$$\begin{aligned}
\sup_{0\le s\le t}|K^n(s)|_2 &\le \sup_{0\le s\le t}|Q(s)| \sup_{0\le s\le t}|\mathbf{u}(s,R_v)|_2 \\
&\le \sup_{0\le s\le t}|Q(s)|c(|R_v|_2,\|R_d\|_1,\|Z\|_2).
\end{aligned}$$

Applying dominated convergence theorem yields that

$$\lim_{n\to\infty}\mathbb{E}\left[\int_0^t |K^n(s)-\mathbf{u}(s,R_v)Q(s)|_2^2\right]=0. \tag{5.6}$$

Moreover, the Malliavin derivative of K^n is given by

$$\begin{aligned}
\mathcal{D}_vK^n(s) &= \sum_{k=1}^n [Q(s)\mathcal{D}_v\mathbf{u}(t_{k-1},R_v)+\mathcal{D}_vQ(s)\mathbf{u}(t_{k-1},R_v)]1_{(t_{k-1},t_k]}(s) \\
&= \sum_{k=1}^n \mathcal{D}_vQ(s)\mathbf{u}(t_{k-1},R_v)1_{(t_{k-1},t_k]}(s) \\
&\quad + \sum_{k=1}^n Q(s)[\mathcal{D}_v\mathbf{u}(t_{k-1},R_v)+D\mathbf{u}(t_{k-1},R_v)\mathcal{D}_vR_v]1_{(t_{k-1},t_k]}(s),
\end{aligned}$$

where $D\mathbf{u}(s,\gamma)$ represents the Fréchet derivative at $\gamma\in\mathbf{V}$, and $\mathcal{D}_v\mathbf{u}(t_{k-1},R_v):=\mathcal{D}_v\mathbf{u}(t_{k-1},\gamma)|_{\gamma=R_v}$. As $\mathbf{u}(s,R_v)$, $D\mathbf{u}(s,R_v)$ and $\mathcal{D}_v\mathbf{u}(s,R_v)$ are continuous in s, we have $\lim_{n\to\infty}\mathcal{D}_vK^n(s)=\mathcal{D}_v[\mathbf{u}(s,R_v)Q(s)]$ for any $s\ge 0$, and by Proposition 3.1, Theorem 3.7 and Proposition 4.1, we get

$$\begin{aligned}
|\mathcal{D}_vK^n(s)|_2 \le& c(|R_v|_2,\|Z\|_2)\sup_{0\le s\le t}|\mathcal{D}_vQ(s)| \\
&+ c(\|R_v\|_1,|Q|_\infty,\sup_{0\le t\le T}\|Z\|_2,\int_0^T\|Z\|_3^2ds,T)|Q|_\infty.
\end{aligned}$$

Hence, following the dominated convergence theorem yields that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^t \int_0^\infty |\mathcal{D}_v K^n(s) - \mathcal{D}_v[\mathbf{u}(s,R_v)Q(s)]|_2^2 dv ds\right] \tag{5.7}$$

From (5.6) and (5.7), we conclude that $K^n(\cdot) \to \mathbf{u}(\cdot, Y_0)Q(\cdot)$ in $\mathbb{L}^{1,2}_{\text{loc}}(\mathbf{H})$. Hence,

$$\lim_{n\to\infty} \int_0^t K^n(s)dW(s) = \int_0^t \mathbf{u}(s,R_v)Q(s)dW(s).$$

Now we estimate $L^n(s)$,

$$\begin{aligned} L^n(s) =& Q(s)\sum_{k=1}^n [\mathcal{D}_s\mathbf{u}(t_{k-1},R_v) + D\mathbf{u}(t_{k-1},R_v)\mathcal{D}_s R_v] 1_{(t_{k-1},t_k]}(s) \\ =& Q(s)\sum_{k=1}^n D\mathbf{u}(t_{k-1},R_v)\mathcal{D}_s R_v 1_{(t_{k-1},t_k]}(s). \end{aligned} \tag{5.8}$$

Since $D\mathbf{u}(s,R_v)$ is continuous in s, we get that

$$\lim_{n\to\infty} \int_0^t L^n(s)ds = \int_0^t Q(s)D\mathbf{u}(s,R_v)\mathcal{D}_s R_v ds.$$

Back to (5.3), sending $n \to \infty$ yields that

$$\begin{aligned} \mathbf{v}(t,R_v) - R_v - Q(t)Z(t) =& Q(t)\mathbf{u}(t,R_v) - R_v = \sum_{k=1}^n (Q(t_k)\mathbf{u}(t_k,R_v) - Q(t_{k-1})\mathbf{u}(t_{k-1},R_v)) \\ =& -\int_0^t A_1\mathbf{v}(s,R_v)ds - \int_0^t B_1(\mathbf{v}(s,R_v))ds - \int_0^t M(\mathbf{d}(s,R_d))ds \\ &+ \int_0^t Q(s)A_1 Z(s)ds + \int_0^t Q(s)\mathbf{u}(s,R_v)dW(s) \\ &+ \frac{1}{2}\int_0^t Q(s)\mathbf{u}(s,R_v)ds + \int_0^t Q(s)D\mathbf{u}(s,R_v)\mathcal{D}_s R_v ds, \end{aligned}$$

for all $t \geq 0$. By Itô's formula, we first have

$$\begin{aligned} Q(t)Z(t) =& \int_0^t Z(s) \circ dQ(s) + \int_0^t Q(s) \circ dZ(s) \\ =& -\int_0^t Q(s)A_1 Z(s)ds + \sigma_0 W_0(t) + \int_0^t Z(s)Q(s) \circ dW(s). \end{aligned}$$

To obtain the form (5.1), it remains to show for $t \geq 0$,

$$\begin{aligned} \int_0^t \mathbf{u}(s,R_v)Q(s) \circ dW(s) =& \int_0^t Q(s)\mathbf{u}(s,R_v)dW(s) + \frac{1}{2}\int_0^t Q(s)\mathbf{u}(s,R_v)ds \\ &+ \int_0^t Q(s)D\mathbf{u}(s,R_v)\mathcal{D}_s R_v ds \end{aligned} \tag{5.9}$$

In view of Theorem 3.1.1 in [17], the left hand side can be written as

$$\int_0^t \mathbf{u}(s,R_v)Q(s) \circ dW(s) = \int_0^t Q(s)\mathbf{u}(s,R_v)dW(s) + \frac{1}{2}\int_0^t (\nabla[\mathbf{u}(\cdot,R_v)Q(\cdot)])_s ds,$$

where

$$(\nabla[\mathbf{u}(\cdot,R_v)Q(\cdot)])(s) = \frac{1}{2}\left(\lim_{\varepsilon\to0+} \mathcal{D}_s[\mathbf{u}(s+\varepsilon,R_v)Q(s+\varepsilon)] + \lim_{\varepsilon\to0+} \mathcal{D}_s[\mathbf{u}(s-\varepsilon,R_v)Q(s-\varepsilon)]\right)$$

By chain rule, we know that

$$\mathcal{D}_s[\mathbf{u}(t,R_v)Q(t)] = \mathcal{D}_s\mathbf{u}(t,R_v)Q(t) + D\mathbf{u}(t,R_v)(\mathcal{D}_sR_v)Q(t) + \mathbf{u}(t,R_v)\mathcal{D}_sQ(t).$$

Now replacing t in the above identity by $s+\varepsilon$, $s-\varepsilon$, respectively, and using the face that $\mathcal{D}_s\mathbf{u}(s-\varepsilon,R_v) = 0, \mathcal{D}_sQ(s-\varepsilon) = 0$, one can get

$$\begin{aligned}\mathcal{D}_s[\mathbf{u}(s-\varepsilon,R_v)Q(s-\varepsilon)] =& D\mathbf{u}(s-\varepsilon,R_v)(\mathcal{D}_sR_v)Q(s-\varepsilon);\\ \mathcal{D}_s[\mathbf{u}(s+\varepsilon,R_v)Q(s+\varepsilon)] =& \mathcal{D}_s\mathbf{u}(s+\varepsilon,R_v)Q(s+\varepsilon) + D\mathbf{u}(s+\varepsilon,R_v)(\mathcal{D}_sR_v)Q(s+\varepsilon)\\ &+ \mathbf{u}(s+\varepsilon,R_v)\mathcal{D}_sQ(s+\varepsilon).\end{aligned}$$

Sending $\varepsilon \to 0+$, by the continuity of $\mathcal{D}_s\mathbf{u}(t,R_v)$ and $Q(t)$ in t, we get

$$(\nabla[\mathbf{u}(\cdot,R_v)Q(\cdot)])(s) = D\mathbf{u}(s,R_v)(\mathcal{D}_sR_v)Q(s) + \frac{1}{2}\mathbf{u}(s,R_v)Q(s).$$

This proves (5.9) and is the end of proving the existence result.

For the uniqueness result, with the arguments in Subsection 3.1, we note that the model (2.3)-(2.5) is equivalent to (3.3) when the initial random fields $R_v \in \mathcal{D}^{1,2}_{\text{loc}}(\mathbf{H}) \cap \mathbf{V}, R_d \in \mathcal{D}^{1,2}_{\text{loc}}(\mathbb{H}^1) \cap \mathbb{H}^2$. The proof of uniqueness is then very close to that in [11], [22], so we omit here. □

Acknowledgments. During the study of this work, Professor Z. Brzeźniak discussed with the last author of this article for many times, which encourage and motivate the authors to attack the random initial and boundary problems for nematic liquid crystals. The authors are deeply grateful for the valuable time and generous help from Professor Z. Brzeźniak. When this work was completed, the last author of this article was invited by Professor Tusheng Zhang to report the work in University of Science and Technology of China. The authors deeply appreciate Professor Tusheng Zhang's kind invitation and hospitality!

References

[1] Z. Brzeźniak, E. Hausenblas, and P. Razafimandimby, Some results on the penalised nematic liquid crystals driven by multiplicative noise, arxiv: 1310.8641v3 [math.PR] (2016).

[2] Z. Brzeźniak, E. Hausenblas, and P. Razafimandimby, Stochastic nonparabolic dissipative systems modelling the flow of liquid crystals: strong solution, RIMS Kokyuroku Proceeding of RIMS Symposium on Mathematical Analysis of Incompressible Flow, 1875, 41-72 (2014).

[3] Z. Brzeźniak, U. Manna, and A.-A. Panda , Martingale solutions of nematic liquid crystals driven by pure jump noise in the Marcus canonical form , *J. Differential Equations*, 266(10), 6204-6283 (2019).

[4] Z. Brzeźniak, U. Manna, and A.-A. Panda , Large deviations for stochastic nematic liquid crystals driven by multiplicative Gaussian noise, *Potential Analysis*, 53, 799–838 (2020).

[5] Z. Brzeźniak, E. Motyl, Existence of martingale solutions of the stochastic Navier-Stokes equations in unbounded 2D and 3D-domains, *J. Differential Equations.* **254**: 1627-1685 (2013).

[6] P. Constantin, C. Foias, Navier-Stokes Equations, Chicago Lectures in Mathematics, Univerisity of Chicago Press, 1988.

[7] J. Duan, K. Lu, B. Schmalfuss, Invariant manifolds for stochastic partial differential equations, Ann. Probab., 31, 2109-2135 (2003).

[8] J. L. Ericksen, Conservation laws for liquid crystals, *Trans. Soc. Rheology*, **5**, 23-34 (1961).

[9] B. Guo, Y. Han, G. Zhou, Random attractor for the 2D stochastic nematic liquid crystals flows accepted by Communications on Pure and Applied Analysis.

[10] J. Guckenheimer and P. Holmes, Nonlinear oscillations, dynamical systems and bifurcations of vector fields, Springer-Verlag, 1983.

[11] B. Guo, D. Huang: *3D stochastic primitive equations of the large-scale ocean: global well-posedness and attractors* . Comm. Math. Phys. 286, no. 2, 697-723 (2009).

[12] F. M. Leslie, Some constitutive equations for liquid crystals, *Arch. Rational Mech. Anal.* **28(04)**, 265-283 (1968).

[13] F.-H. Lin, C. Liu, Nonparabolic dissipative systems modelling the flow of liquid crystals, *Communications on Pure and Applied Mathematics*, Vol. **XLVIII**, 501-537 (1995).

[14] F.-H. Lin, C. Liu, Partial regularities of the nonlinear dissipative systems modeling the flow of liquid flows, *Discrete Contin. Dyn. Syst.*, 2, 1-23 (1995).

[15] S.-E.A. Mohammed, M.K.R. Scheutzow, The stable manifold theorem for nonlinear stochastic systems with memory, Part I: Existence of the semiflow, J. Funct. Anal. 205(2), 271-305 (2003), Part II: The local stable manifold theorem J. Funct. Anal. 206(2), 253-306 (2004).

[16] S. Mohammed, T. Zhang, Anticipating stochastic 2D Navier-Stokes equations, Journal of Functional Analysis, 264, 1380-1408 (2013).

[17] D. Nualart, The Malliavin Calculus and Related Topics, Springer-Verlag, 1995.

[18] D. Orrell, L. Smith, J. Barkmeijer and T. N. Palmer, Model error in weather forecasting, Nonlinear Processes Geophys., 8, 357-371 (2001).

[19] T. N. Palmer, A nonlinear dynamical perspective on model error: A proposal for non-local stochastic-dynamic parameterization in weather and climate prediction models, Q. J. R. Meteorol.Soc., 127, 279-304 (2001).

[20] T. N. Palmer and R. Hagedorn, Predictability of weather and climate, Cambridge University Press, 2006.

[21] G. Zhou, Random attractor for the 2D stochastic nematic liquid crystals flows with multiplicative noise, arxiv: 1803.11069 [math.PR](2018).

[22] G. Zhou, Random attractor of the 3D viscous primitive equations driven by fractional noises, *J.Differential Equations*, 266(11), 7569-7637 (2019).