# A general framework for tropical differential equations

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# Foreword

The vast majority of the material present here is a joint work: with Jeffrey H. Giansiracusa for what concerns the first part, appearing in [GM21] and with Tomasz Brzeziński and Bernard Rybołowicz for the second part, appearing in [BMR20].

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*"Theorems are like sausages, it is better not to see them being made"*

*- (almost) Otto von Bismarck*

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# Chapter 1

# Introduction

This first part of the present thesis constitutes the main body of the work I conducted during my PhD. The idea that gave the start to everything is introduced and studied in the following is to build a bridge between the recently introduced theory of tropical differential equations and the more established theory of $p$-adic differential equations. In particular the research of tropical methods to compute the radius of convergence of the solutions to a $p$-adic differential equation.

Ordinary differential equations with real or complex valued functions as coefficients are fundamental in a multitude of applications of mathematics to real world situations. Their theory traces back to the very beginning of modern mathematics, with the first example of differential equations appearing in the work of Newton and Leibniz. In a few words, differential equations are at the center of our understanding of continuous processes.

On the other hand, given some prime number $p$, considering a $p$-adic field in the place of real or complex numbers leads to the theory of $p$-adic differential equations, whose theory sheds light on problems of discrete, arithmetic and number theoretic nature. The only comprehensive text on the topic is [Ked10].

The first appearance of $p$-adic differential equations can be dated back to the work of Dwork, who in [Dwo60] used them as a tool to prove the rationality of the zeta function of a variety in characteristic $p$: solving some particular $p$-adic differential equations gives formulas for counting the number of points of varieties over finite fields. In its first years, the theory was then pushed forward mainly thanks to the pioneering contributions of Dwork and Robba [DR77, DR79, Rob94, Dwo12, ...] and developed links in many mathematical directions: studying zeta functions making use of $p$-adic analysis allows for numerical methods to be used, and these has found applications in the cryptography based on elliptic and hyperelliptic curves (see [CFA$^+$05] for an introduction to the subject); the theory of $p$-adic differential equations is intimately related to the development and the consolidation of $p$-adic rigid cohomology by Berthelot [Ber74, Ber86], motivated by previous works by Dwork and Monsky and Washnitzer [MW68, Mon68, Mon71], whose in turn has proven to be a valuable tool in computations for cryptography, as well. For general surveys and comprehensive texts on the subject see [Ill94, Ked09, LS07]. Furthermore $p$-adic differential equations play an important role in the theory of $p$-adic Galois representations, which have been studied by Fontaine [Fon94, Fon04, Fon07, FO08] by introducing a series of auxiliary period rings in order to classify them. The study of $p$-adic Galois representations is central for the development of a $p$-adic Hodge theory. More recently, the work of Berger [Ber02, Ber08a] highlighted a deep role of $p$-adic differential equations in this theory, through the use of $(\phi, \Gamma)$-modules. Surveys on the subject are [Ber08b, BC09].

An interesting feature of $p$-adic differential equations is that, in contrast to the complex case, the convergence radius of their solutions is not controlled by some "visible" object, such as, in the complex case, the poles of the coefficients of the equation: in this context in fact even equations as easy as that of the exponential give solutions with finite radius of convergence at any point. The topology of the space itself is an obstacle to the convergence.

The language of Berkovich geometry, introduced in [Ber90], has proven to be the right one to describe phenomena related to this convergence radii. The radius of convergence of the solutions of a $p$-adic differential equation on a Berkovich curve as a function of the expansion point has been proven to be a continuous piecewise linear function (see [CD94, Bal10, BDV07, Ked10]) with finitely many changes of slope [Chr11, PP15b, PP15a, PP13, Pul15], whose behaviour is actually controlled by a finite skeleton on which the curve retracts on. In general, although an explicit iterative formula to compute the radius of converge exists (see [Chr11]), it is difficult to calculate it.

Tropical methods are becoming more and more influential in many areas of mathematics and of sciences in general, and they often can be seen as a process shifting a problem from a geometric or algebraic framework to a discrete, combinatorial or polyhedral one.

Tropical geometry historically developed in at least two independent ways: as geometry over the tropical semifield and as the study of logarithmic limit sets of classical algebraic varieties. The first approach derives from problems of optimization in computer science [Sim78, Sim87, Eil74], solved using min-plus (or max-plus) algebra, thus tropical geometry is geometry over the tropical semiring $\mathbb{R} \cup \infty$, where usual sum and product are replaced respectively by minimum and sum (or by maximum and sum, isomorphically). Tropical polynomials give piecewise linear functions and the tropical varieties associated to them, i.e. the locus of points where the minimum is attained at least twice, are assembled by convex polyhedra. The second approach, undertaken in [Ber71, GB84, GKZ08], and reintroduced more recently by Kapranov, consists in considering the map $\log_t\colon (\mathbb{C}^*)^n \to \mathbb{R}^n$ sending a vector to the vector of the logarithms of the absolute value of its entries. The image of an algebraic variety via this map is called *amoeba* and taking the limit $\lim_{t\to\infty} X_t$ of a family $X_t$ of algebraic varieties gives a tropical variety, associated to the tropical polynomial obtained by the polynomial $F_t \in \mathbb{C}(\!(t)\!)$ defining the family $X_t$ by taking the leading order of the coefficients.

This point of view generalises to any valued field, i.e. field equipped with a map similar to the $t$-adic valuation on $\mathbb{C}(\!(t)\!)$ ("taking the leading order"). Maps with these properties are called non-archimedean valuations and are the necessary tool that allows us to move from the classical world of algebraic geometry to the polyhedral world of tropical geometry, through what is called tropicalization. This is interesting as many features of the geometric objects we start with are preserved by this process.

More recently, several algebraic foundations for tropical geometry have been developed, in order to see tropical geometry more classically as algebraic geometry on a certain category of objects, such as hyperfields [Vir11, BB19, Lor19, Jun21], Lorscheid's blueprints [Lor12, Lor15], and idempotent semirings [GG16, GG14, GG18, MR18, MR14, MR20, JM18b, JM18a, BE17, Yag16, CGM20]. We give a more in depth treatment of tropical geometry in Chapter 2 where we also recall plenty of references to the subject and its applications.

It is well known that a vast part of the theory of ODE's is algebraic, it is enough to cite the example of Picard-Vessiot theory. Algebraic ODEs are systems of differential equations formed from polynomial expressions in an indeterminate function $f$ and its derivatives. The algebraic theory was first established by Ritt [Rit50] and Kolchin [Kol73]. Many important classes of models from the natural sciences, such as chemical reaction networks, are algebraic ODEs, and in pure mathematics algebraic ODEs appear in many parts of geometry, including periods and monodromy. Understanding their solutions and singularities has many important consequences in pure and applied mathematics.

Here, we pursue the further development of the tropical mathematics tool set for studying differential equations. In [Gri17] Grigoriev first introduced a theory of tropical differential equations and defined a framework for tropicalizing algebraic ODEs over a ring of formal power series $R[\![t]\!]$. In this framework, one tropicalizes a differential equation by recording the leading power of $t$ in each coefficient, and one tropicalizes a power series solution simply recording the powers of $t$ that are present.

Solutions to a differential equation tropicalize to solutions to its tropicalization, and Grigoriev asked if *all* solutions to the tropicalization of an equation arise as tropicalizations of classical solutions; i.e., is the map from classical solutions to tropical solutions surjective? This is the differential equation analogue of the Fundamental Theorem of Tropical Geometry [MS15, Theorem 3.2.3], and this question was answered positively by Aroca et al. in [AGT16] (assuming $R$ is an uncountable algebraically closed field of characteristic 0). These ideas have also been extended to the case of algebraic *partial* differential equations in [FGLH$^+$20].

Paralleling the role of Gröbner theory in defining tropical varieties in the non-differential setting, [FT20] and [HG21] define initial forms and develop a Gröbner-theoretic approach to Grigoriev's tropical differential equations. A similar approach is also presented in [CGL20], which also gives an illuminating account of tropical ordinary and partial differential equations (in part based on a preliminary report of the algebraic perspective presented here). We recall most of the results on the subject of the aforementioned papers in Chapter 3.

A limitation present in all of the aforementioned works is that the tropicalization construction studied there records only the powers of $t$ present in a power series solution; it does not record any information about the valuations of the coefficients. Thus any information about convergence of power series solutions is lost when using Grigoriev's tropicalization, and to get an understanding of the radii of convergence of formal power series solutions of $p$-adic differential equations, retaining the valuations of the coefficients is of cardinal importance.

Even if we are far from this objective, the long term goal of the study undertaken here is a better comprehension and easier computability of the radii of convergence for $p$-adic differential equation using tropical methods.

# Results

The main purpose of this first part of my work is to build a refinement of Grigoriev's framework that records and incorporates the valuations of the coefficients in a power series solution so that convergence information is encoded in tropical solutions. This requires developing a theory of differentials on idempotent semirings in which the usual Leibniz rule is weakened to a tropical Leibniz rule, and this development includes constructing free tropical differential algebras (a tropical analogue of Ritt algebras) with differential variables coming from a differential $\mathbb{F}_1$-module, that we define in the following.

We give a brief explanation of our framework here. A *tropical pair* $\mathbf{S} = (S_1 \to S_0)$ is a tropical differential semiring $S_1$ and a homomorphism to a semiring $S_0$. The coefficients of tropical differential equations live in $S_0$. Solutions live in $S_1$ (where they can be differentiated), but the condition that tests if something is a solution takes place in $S_0$. We think of $S_0$ as recording the leading behaviour of elements of $S_1$. The primary example of a tropical pair has $S_1 = \mathbb{T}[\![t]\!]$ (the semiring of formal power series with tropical real number coefficients), $S_0 = \mathbb{R}^2_{lex} \cup \{\infty\}$ is a rank 2 version of the tropical semiring, and the map $S_1 \to S_0$ sends $at^n + \cdots$ to $(n, a)$.

We now state our main results informally, in the case of free differential $\mathbb{F}_1$-modules with $n$ generators.

**Theorem A.** *We construct a category of* $\mathbf{S}$*-algebras, and to a set $E$ of tropical differential equations over* $\mathbf{S}$ *we associate an object of this category such that morphisms to an* $\mathbf{S}$*-algebra* $\mathbf{T}$ *are in natural bijection with solutions to $E$ with values in* $\mathbf{T}$*.*

A system of algebraic differential equations over a field $K$ is represented in coordinate-free form by a differential $K$-algebra $A$. To tropicalize $A$, we need two pieces of additional data:

(1) A non-archimedean valuation on $K$ taking values in an idempotent semiring $S_0$, and a *differential enhancement* of the valuation, which is a lifting to a map $A \to S_1$ that commutes with the differential. (These notions are defined in Section 2.4 and Section 6.5.)

(2) A system of generators $x_i \in A$ so that $A$ is presented as a quotient of a Ritt algebra $K\{x_1, \ldots, x_n\} \twoheadrightarrow A$.

Any differential algebra $A$ admits a universal presentation $K\{x_a \mid a \in A\} \twoheadrightarrow A$. Tropicalizing this presentation, we find:

**Theorem B.** *The tropicalization of $A$ with respect to its universal presentation is the colimit of its tropicalizations with respect to finite presentations.*

Finally, we provide evidence for the appropriateness of our definitions and framework by proving a differential analogue of Payne's inverse limit theorem [Pay09]. Recall that, given an algebra $A$ over a non-archimedean field $K$, the underlying set of the Berkovich analytification of Spec $A$ is the set of all multiplicative seminorms on $A$ that are compatible with the valuation on $K$. Now suppose that $K$ is a differential ring, the valuation $v$ on $K$ has a differential enhancement $\widetilde{v}$ taking values in a pair $\mathbf{S}$, and $A$ is a differential algebra over $K$. In this setting, given an $\mathbf{S}$-algebra $\mathbf{T} = (T_1 \to T_0)$, we can consider the set of all pairs $(w, \widetilde{w})$ where $w\colon A \to T_0$ is a multiplicative valuation on $A$ compatible with $v$ and $\widetilde{w}\colon A \to T_1$ is a differential enhancement of $w$ compatible with $\widetilde{v}$. We call this the $\mathbf{T}$-valued *differential Berkovich space of $A$*, denoted $Berk_{\mathbf{T}}(A)$.

**Theorem C.** *There is a universal valuation with differential enhancement on $A$, and it takes values in the tropicalization of the universal presentation of $A$. Hence the tropicalization of the universal presentation corepresents the functor $\mathbf{T} \mapsto Berk_{\mathbf{T}}(A)$.*

Combining this with Theorem B, we immediately obtain our differential analogue of Payne's inverse limit theorem.

**Corollary D.** *Let $k$ be a differential ring equipped with a non-archimedean valuation and differential enhancement taking values in $\mathbf{S}$, let $A$ be a differential algebra over a $k$, and let $\mathbf{T}$ be an $\mathbf{S}$-algebra. The set $Berk_{\mathbf{T}}(A)$ is isomorphic to the inverse limit of the $\mathbf{T}$-valued solution sets of the tropicalizations of all finite presentations of $A$.*

Most of the material presented in the first part of this thesis appears in [GM21], in the terms we introduced it in the previous lines, i.e. restricting our treatment to free differential $\mathbb{F}_1$-modules with $n$ generators. In addition to take this more general approach here, we discuss the notion of differential $\mathbb{F}_1$-algebra and interactions between some free base change functors on these categories of differential $\mathbb{F}_1$-objects. In the last chapter, we present a generalisation of all the results introduced in the following to the case of partial differential equations, generalising the framework of [FGLH$^+$20].

# Chapter 2

# Tropical geometry and tropical scheme theory

As mentioned in the introduction, the basic object to develop tropical geometry on is the idempotent semiring (with min-plus convention) of tropical numbers: $\mathbb{T} := (\mathbb{R} \cup \{\infty\}, \oplus, \otimes)$. Addition is replaced by minimum and multiplication by usual addition. Applications of tropical geometry range from enumerative geometry [Shu06, Mik05b, GM08, AB13, ...] to neural networks [ZNL18, CM18, MRZ21, ...], from mirror simmetry [Gro11, Abo09, Gro10, ...] to mathematical biology and algebraic statistics [MLYK18, PS05, PS04, Man11, ...], from optimization [JS19, AGG12, TY19, DKLV17, But10, ...] to representation theory [FZ02, GL10, ...] and many other areas. One of the first results that brought tropical geometry to the attention of the mathematical community is the tropical computation of Gromov-Witten invariants of $\mathbb{CP}^2$ by Mikhalkin in [Mik05a]. A comprehensive introduction to tropical geometry is given in [MS15].

In the first section of this chapter we will briefly go through the basic notions needed to develop tropical geometry. We start with the definition of a (non-archimedean) valuation on a field $K$ and we give some examples, then we define the tropicalization of points in $K^n$, of Laurent polynomials over $K$ and the tropical hypersurface of a Laurent polynomial in $\mathbb{T}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. After discussing quickly the theory of initial forms and Gröbner basis for homogeneous ideals in $K[x_0, \dots, x_n]$, we conclude by recalling the fundamental theorem of tropical geometry and two structure theorems describing the polyhedral nature of tropical hypersurfaces and tropicalization of varieties.

In Section 2.2 we describe how the tropicalization process links with one of the main constructions in non-archimedean analytic geometry, namely that of the Berkovich analytification of a variety. Firstly we will recall the process of tropicalization of a toric variety as introduced by Payne and Kajiwara, then we will recall Payne's inverse limit theorem which states that the Berkovich analytification is the inverse limit of all the tropicalization with respect to toric embedding.

After introducing semirings in general and the so-called bend congruence over a polynomial semiring in Section 2.3, in Section 2.4 we get into tropical scheme theory as introduced by Giansiracusa and Giansiracusa which is scheme theory over semirings. In this context we recall a more general definition of valuation that will be the one we will be using all along this work, the categories of $\mathbb{F}_1$-modules and -algebras and some property of the functor of scheme theoretic tropicalization. We will conclude by going through a scheme theoretic generalisation of Payne's inverse limit theorem, also realising the Berkovich analytification as the tropical points of a universal tropicalization.

The analogues to most of the ideas of this last two sections will be developed for differential equations in the following, and will constitute the core of the first part of this work.

## 2.1 Tropical geometry: basic definitions, fundamental and structure theorems

In this section we will briefly go through the fundamental notions and main theorems in tropical geometry. We start by introducing non-archimedean valuations, the fundamental datum needed to perform a tropicalization:

**Definition 2.1.1.** Given a field $K$, a map $v : K \to \mathbb{T}$ is a valuation on $K$ if it satisfies the following properties:

- $v(a) = \infty \iff a = 0$;
- $v(ab) = v(a) \otimes v(b)$ for all $a, b \in K$ ;
- $v(a + b) \geq v(a) \oplus v(b)$ for all $a, b \in K$;

Let us denote by $\Gamma_v$ the additive subgroup $v(K^\times)$ of $\mathbb{R}$, as $O_K := \{a \in K \mid v(a) \geq 0\}$ the ring of integers of $K$, as $m_K := \{a \in K \mid v(a) > 0\}$ its maximal ideal and as $k$ the residue field $O_K/m_K$.

Furthermore, given a valuation $v$ on a field, by defining $|\ \ |\colon K \to \mathbb{R}_{\geq 0}$ as $|x| = e^{-v(x)}$ for all $x \in K$, $|\ \ |$ is a norm on $K$, thus making it into a topological space.

*Example* 2.1.2. (1) Let $K := F(\!(t)\!)$ be the field of Laurent series over a field $F$, then the map $v\colon F(\!(t)\!) \to \mathbb{T}$ sending a power series $A = \sum_n a_n t^n$ to $\min\{n \mid a_n \neq 0\}$ and 0 to $\infty$ is a valuation called the $t$-adic valuation. In this case $\Gamma_v = \mathbb{Z}$, $O_K = F[\![t]\!]$, $m_K = (t)$ and $k = F$;

(2) Fix a prime number $p$, then given any integer $n = \prod_q q^{e_q}$, written in its prime decomposition where all but finitely of the exponents are 0, let $v_p(n) = e_p$. Given any $a/b \in \mathbb{Q}$ let $v_p(a/b) = v_p(a) - v_p(b)$ and $v_p(0) = \infty$, then $v_p\colon \mathbb{Q} \to \mathbb{T}$ is a valuation called $p$-adic valuation. The completion of $\mathbb{Q}$ with respect to the topology induced by $v_p$ is denoted by $\mathbb{Q}_p$ and called the field of $p$-adic numbers, its ring of integers will be denoted as $\mathbb{Z}_p$, the maximal ideal of which is $(p)$ and the residue field $k$ is the field with $p$ elements $\mathbb{F}_p$.

Let now $K$ be an algebraically closed field and $v$ a surjective valuation on $K$. Given an ideal $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ we can define three objects:

- there is a way to map subsets of $K^n$ into subsets of $\mathbb{T}^n$ by sending an element $x = (x_1, \dots, x_n)$ in $K^n$ to the vector of the valuations of the entries of $x$, i.e. $(v(x_1), \dots, v(x_n)) \in \mathbb{T}^n$. Let us denote by $\mathrm{trop}_v : K^n \to \mathbb{T}^n$ this coordinatewise valuation. Considering the associated subvariety $X := V(I)$ of the torus $T_K^n$ over $K$, we will say that $\mathrm{trop}_v(X) \subset \mathbb{R}^n$ is the tropicalization of $X$.

- given any $F \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ we can consider $\mathrm{trop}_v(F) \in \mathbb{T}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ obtained by applying $v$ on the coefficients of $F$. Any $f = \sum_{\mathbf{m} \in \mathbb{N}^n} a_{\mathbf{m}} x^{\mathbf{m}} \in \mathbb{T}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ determines a piecewise linear function $\mathbb{R}^n \to \mathbb{R}$ that, on a vector $\mathbf{w} \in \mathbb{R}^n$, acts as:

$$f(\mathbf{w}) = \bigoplus_{\substack{\mathbf{m} \in \mathbb{N}^n \\ a_{\mathbf{m}} \neq \infty}} a_{\mathbf{m}} \odot \mathbf{w}^{\odot \mathbf{m}} = \min_{\substack{\mathbf{m} \in \mathbb{N}^n \\ a_{\mathbf{m}} \neq \infty}} \{a_{\mathbf{m}} + \mathbf{w}\mathbf{m}\}$$

  Define the *tropical variety* associated to $f$ as:

$$V(f) := \{\mathbf{w} \in \mathbb{R}^n : \text{the minimum in } f(\mathbf{w}) \text{ is attained at least twice}\} \subset \mathbb{R}^n$$

  It is the locus in which the function associated to $f$ is not linear, for this reason is also named the *bend locus* of $f$.

- assuming there is a splitting $\Gamma_v \ni c \mapsto t^c \in K$ of the valuation map for some element $t \in K$, a theory of Gröbner basis and initial forms of homogeneous ideals of $K[x_0, \dots, x_n]$ can be developed in the valued setting as well. When $K$ is a local field, $t$ can be chosen to be a uniformizer for $K$. This splitting exists for example whenever $K$ is algebraically closed. Defining the initial term of an homogeneous polynomial $f = \sum_{\mathbf{m} \in \mathbb{N}^{n+1}} a_{\mathbf{m}} x^{\mathbf{m}} \in K[x_0, \dots, x_n]$ with respect to $\mathbf{w} \in \mathbb{R}^{n+1}$ as:

$$\mathrm{in}_{\mathbf{w}}(f) := \sum_{\substack{\mathbf{m} \in \mathbb{N}^{n+1} \text{s.t. minimum of} \\ \mathrm{trop}_v(f) \text{ is achieved at } x^{\mathbf{m}}}} \overline{a_{\mathbf{m}} t^{-v(a_{\mathbf{m}})}} x^{\mathbf{m}} \in k[x_0, \dots, x_n]$$

  many well-known facts about Gröbner basis extend to this context as well.

  Given a polynomial $f = \sum_{\mathbf{m} \in \mathbb{N}^n} a_{\mathbf{m}} x^{\mathbf{m}} \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and a vector $\mathbf{w} \in \mathbb{R}^n$ it is still possible to define the initial term of $f$ with respect to $\mathbf{w}$ in an analogous way as above. Given an ideal $I \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, in this case its initial term could be a unit, and this precisely happens when $\mathrm{in}_{\mathbf{w}}(I)$ is generated by monomials.

A basic result in tropical geometry is the so-called fundamental theorem, also know as Kapranov's theorem in the case of hypersurfaces, which links the three notions we just introduced:

**Theorem 2.1.3** ([MS15] Theorem 3.2.5). *Let $K$ a non-trivially valued algebraically closed field and $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ an ideal, let $X = V(I) \subset (K^\times)^n$. Then the following three sets coincide:*

- *the intersection $\bigcap_{F \in I} V(\mathrm{trop}_v(F)) \subset \mathbb{R}^n$ of the tropical hypersurfaces cut out by the tropicalization of the polynomials in $I$;*
- *the topological closure $\overline{\mathrm{trop}_v(X)} \subset \mathbb{R}^n$ of the tropicalization of $X$;*
- *the set $\{\mathbf{w} \in \mathbb{R}^n \mid in_{\mathbf{w}}(I) \neq (1)\}$ of weight vectors such that the corresponding initial ideal is not monomial.*

For what concerns the polyhedral structure of tropical hypersurfaces, it is closely related to the subdivision of the Newton polytope of the polynomial induced by the valuation of its coefficient:

**Theorem 2.1.4** ([MS15](Proposition 3.1.6)). *Let $f = \sum_{\boldsymbol{m} \in \mathbb{N}^n} a_{\boldsymbol{m}} x_{\boldsymbol{m}} \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then the tropical hypersurface $V(\mathrm{trop}_v(f))$ is the support of a pure $\Gamma_v$-rational polyhedral complex of dimension n-1 in $\mathbb{R}^n$. It is the (n-1)-skeleton of the polyhedral complex dual to the regular subdivision $\Delta$ of the Newton polytope of $f$ induced by the weights $v(a_{\boldsymbol{m}})$.*

Furthermore:

**Theorem 2.1.5** ([MS15](Theorem 3.3.6)). *Let $X$ be an irreducible d-dimensional subvariety of the torus $(K^\times)^n$. Then $\mathrm{trop}_v(X)$ is the support of a balanced weighted $\Gamma_v$-rational polyhedral complex pure of dimension d, connected through codimension one.*

In particular, given an ideal $I \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, let its minimal associated primes be $\mathrm{Ass}^{\min}(I) := \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ and for $i \in \{1, \ldots, s\}$, define the multiplicity $\mathrm{mult}(\mathfrak{p}_i, I)$ of $\mathfrak{p}_i$ to be the integer $\mathrm{length}((K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]/I)_{\mathfrak{p}_i})$, i.e. the length as a $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]_{\mathfrak{p}_i}$-module of the quotient $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]_{\mathfrak{p}_i}/I_{\mathfrak{p}_i}$

With this definition, given a top-dimensional cell $\sigma$ of $\mathrm{trop}_v(V(I))$, the ideal $\mathrm{in}_{\mathbf{w}}(I)$ is constant for $\mathbf{w} \in \sigma$ and it does not contain a monomial. Thus we associate to $\sigma$ the following weight:

$$\mathrm{mult}(\sigma) = \sum_{\mathfrak{p} \in \mathrm{Ass}^{\min}(\mathrm{in}_{\mathbf{w}}(I))} \mathrm{mult}(\mathfrak{p}_i, \mathrm{in}_{\mathbf{w}}(I))$$

When $I$ is principal and generated by a polynomial $f$, let $\Delta$ be the subdivision of the Newton polytope induced by the valuation of its coefficients as in 2.1.4. Given a top-dimensional cell $\sigma$ of $\mathrm{trop}_v(V(I))$, $\mathrm{mult}(\sigma)$ is equal to the lattice length of the edge of $\Delta$ dual to $\sigma$.

## 2.2 Payne's inverse limit theorem

This section is devoted to introduce all the ingredients needed to state Payne's inverse limit theorem: the two main ones will be the notion of Berkovich analityfication and that of tropicalization of a toric variety. We start with the former.

In [Ber90] Vladimir Berkovich introduced another procedure to obtain a tree-like object from an algebraic variety $X$ over a valued field, known as Berkovich analytification and denoted by $X^{an}$. This is a particular instance of a Berkovich space, analytic spaces over non-archimedean field which refine the notion, given by Tate in [Tat71], of rigid analytic space. This last class of objects, in turn, gives a non-archimedean analogue to complex analytic spaces. The approach given by Berkovich is central in non-archimedean geometry.

An easy way to introduce the Berkovich analytification relies on the definition of multiplicative seminorm:

**Definition 2.2.1.** A multiplicative seminorm $|\ \ |$ on a ring $R$ is a multiplicative map $R \to \mathbb{R}_{\geq 0}$ such that $|0| = 0$ and $|r_1 + r_2| \leq |r_1| + |r_2|$ for all $r_1, r_2 \in R$.

Given $v\colon K \to \mathbb{T}$ an algebraically closed valued field, complete with respect to $v$, we say that a seminorm $|\ \ |$ on a $K$-algebra $A$ is compatible with $v$ if $|a| = \exp(-v(a))$ for all $a \in K$.

**Definition 2.2.2** (*Berkovich analytification*)**.** Given an affine algebraic variety $X$ over $K$, let $K[X]$ be the its $K$-algebra of coordinates. The analytification $X^{an}$ of $X$ is the topological space

$$\{\text{seminorms } |\ \ |\colon K[X] \to \mathbb{R}_{\geq 0} \text{ compatible with } v\}$$

equipped with the weakest topology such that, for every $f \in K[X]$, the function $|\ \ | \to |f|$ is continuous.

In general, given two affine open subsets $U_1, U_2 \subset X$ the gluing is given by identifying $|\ \ |_1 \in U_1^{an}$ and $|\ \ |_2 \in U_2^{an}$ if there exists a seminorm $|\ \ | \in V^{an}$ for some $V \subset U_1 \cap U_2$, such that both $|\ \ |_1$ and $|\ \ |_2$ are a restriction of $|\ \ |$.

All the definitions given in Section 2.1 and the fundamental theorem can be extended to subvarieties of toric varieties, whose tropicalization was introduced by Kajiwara and Payne in [Kaj08] and [Pay09], respectively. Originally tropicalization was considered with respect to embeddings into algebraic tori, but in the two aformentioned works the authors introduced a straightforward generalisation to embedding into toric varieties by tropicalizing stratum by stratum and then assembling the result. This construction is also illustrated in Chapter 6 of [MS15].

For an exhaustive reference on toric varieties refer to [CLS11] or [Ful93].

Given a lattice $N \simeq \mathbb{Z}^n$ and $\Sigma$ a rational polyhedral fan in $N_{\mathbb{R}} := N \otimes \mathbb{R}$, we will denote by $X_\Sigma$ the associated toric variety and by $M$ the lattice $\mathrm{Hom}(N, \mathbb{Z})$ dual to $N$.

**Definition 2.2.3** (*Tropicalization of toric varieties*)**.** Given a toric variety $X_\Sigma$, for each cone $\sigma \in \Sigma$ let $N(\sigma) := N_\mathbb{R}/\mathrm{span}(\sigma)$, it is a $(n - \dim(\sigma))$-dimensional vector space. As a set the tropicalization $\mathrm{trop}(X_\Sigma)$ of the toric variety $X_\Sigma$ is:

$$\mathrm{trop}(X_\Sigma) := \coprod_{\sigma\in\Sigma} N(\sigma)$$

For what concerns its topology, let $U_\sigma$ be an affine toric variety, then its coordinate ring is $K[S_\sigma]$ where $S_\sigma := \sigma^\vee \cap M$. The preimage of $\mathbb{R}$ under a monoid homomorphism $\phi$ is $\tau^\perp \cap S_\sigma$ for some face $\tau \preceq \sigma$: this follows by observing that, given an element $u \in S_\sigma$ such that $\phi(u) = \infty$, then $\phi(u+v) = \infty$ for all $v \in S_\sigma$. On the other hand, if $\phi(u+v) = \infty$, either $\phi(u) = \infty$ or $\phi(v) = \infty$. Thus the disjoint union $\mathrm{trop}(U_\sigma) = \coprod_{\tau\preceq\sigma} N(\tau)$ is identified with $\mathrm{Hom}(S_\sigma, \mathbb{T})$, sending $v \in N(\tau)$ to the homomorphism $\phi_v : S_\sigma \to \mathbb{T}$ given by:

$$\phi_v(u) := \begin{cases} \langle u, v\rangle & \text{if } u \in \tau^\perp \\ \infty & \text{otherwise} \end{cases}$$

and we equip $\mathrm{trop}(U_\sigma)$ with the subspace topology inherited by $\mathbb{T}^{\sigma^\vee\cap M}$.

If $\tau \preceq \sigma$ is a face of $\sigma \in \Sigma$ then $S_\sigma$ is a submonoid of $S_\tau$ and the map $\mathrm{trop}(U_\tau) \to \mathrm{trop}(U_\sigma)$ given by restriction is injective, which follows from the fact that $S_\tau = S_\sigma + \tau^\perp \cap M$ and any $u \in \tau^\perp \cap M$ can be written as $u = u_1 + u_2$ for $u_1, u_2 \in \sigma^\vee \cap \tau^\perp$.

Notice that $S_\sigma$ is finitely generated, and any choice of generators determines an embedding of $\mathrm{Hom}(S_\sigma, \mathbb{T}) \hookrightarrow \mathbb{T}^{S_\sigma} = \mathbb{T}^m$: the subspace topology inherited this way is the same as the one described, in a more intrinsic way, above. Equivalently, a choice of generators determines an embedding of $U_\sigma \hookrightarrow \mathbb{A}^m$ and $\mathrm{trop}(U_\sigma)$ is the tropicalization of this embedding.

Furthermore, there is a natural continuous map $U_\sigma^{an} \to \mathrm{trop}(U_\sigma) = \mathrm{Hom}(S_\sigma, \mathbb{T})$ given by sending a seminorm $|\ \ |$ to the monoid homomorphism $u \mapsto -\log|u|$. These maps glue together to give a continuous morphism $X_\Sigma^{an} \to \mathrm{trop}(X_\Sigma)$

Finally, tropicalization of toric varieties is functorial: let $\overline{S}_\sigma := S_\sigma \cup \{\infty\}$ with $u + \infty = \infty$ for any $u \in S_\sigma$, then there is an identification between $\mathrm{Hom}(\overline{S}_\sigma, \mathbb{T})$ and $\mathrm{trop}(U_\sigma)$. If $f : U_\tau \to U_\sigma$ is an equivariant morphism between affine toric varieties, by pulling back regular functions we get a monoid map $f^* : \overline{S}_\sigma \to \overline{S}_\tau$ which is $\infty$ if the pullback of a regular function vanishes on $U_\tau$. This induces a continuous map $\mathrm{trop}(f) : \mathrm{trop}(U_\tau) \to \mathrm{trop}(U_\sigma)$ given by precomposition by $f^*$. In the non-affine case these maps glue together to give a continuous map between toric varieties.

**Definition 2.2.4** (*Kajiwara-Payne tropicalization*)**.** Let $Y$ be a variety over $K$ and let $\varphi\colon Y \hookrightarrow X_\Sigma$ a closed embedding of $Y$ into a toric variety. Define the tropicalization $\mathrm{trop}_\varphi(Y)$ to be the closure of the image of $Y(K)$ in $\mathrm{trop}(X_\Sigma)$.

Let $\varphi_1\colon Y \hookrightarrow X_\Sigma$ and $\varphi_2\colon Y \hookrightarrow X_{\Sigma'}$ be two toric embeddings of $Y$. A morphism $f$ of toric embeddings from $\varphi_1$ to $\varphi_2$ is an equivariant map $X_\Sigma \to X_{\Sigma'}$ such that

$f \circ \varphi_2 = \varphi_1$. Tropicalization is a functor from the category of toric embeddings to topological spaces. There are natural continuous maps $\pi_\varphi \colon Y^{\mathrm{an}} \to \operatorname{trop}_\varphi(Y)$ for every toric embedding $\varphi$ of $Y$, compatible with the tropicalization of morphisms between toric embeddings.

**Theorem 2.2.5** ([Pay09])**.** *Let $Y$ be a variety over $K$, then $\varprojlim \pi_\varphi$ is an homeomorphism between the Berkovich analytification $Y^{\mathrm{an}}$ and the limit $\varprojlim \operatorname{trop}_\varphi(Y)$ over the category of the toric embeddings of $Y$.*

This theorem intimately relates the two processes of analytification and tropicalization: as analytification is intrinsic while tropicalization depends on an embedding into a toric variety, we are somehow allowed to think at the analytification as an intrinsic tropicalization. This point of view has been made clear in [GG14], whose results are going to be explained more in detail towards the end of Section 2.4.

## 2.3 Semirings

For the present work, we find that idempotent semirings provide the most convenient language for the development of our theory. We recollect here basic definitions and facts about semirings.

**Definition 2.3.1.** A semiring $(S, \oplus, \otimes)$ is an algebraic structure satisfying all the axioms to be a ring but the requirement that there exist an additive inverse for every element in $S$. A semiring is said to be idempotent if $a \oplus a = a$ for every $a \in S$. An idempotent semiring carries a canonical partial order defined by $a \leq b$ if $a \oplus b = b$. The additive unit $0_S$ is the unique minimal element. In a semiring, we will often write the product $a \otimes b$ simply as $ab$.

*Example* 2.3.2. (1) Let $(\mathbb{T}_n, \oplus, \otimes)$ denote $(\mathbb{R})^n \cup \{\infty\}$. We define $a \oplus b = \min(a, b)$, where the minimum is taken with respect to the lexicographic ordering, and we define $a \otimes b$ to be component-wise multiplication. Notice that for $n = 1$ the canonical partial order induced by $\oplus$ is the opposite of the usual one on $\mathbb{R}$.

(2) The boolean semiring $\mathbb{B}$ is the sub-semiring $\{0, \infty\} \subset \mathbb{T}$. Note that $\mathbb{B}$ is the initial object in the category of semirings.

One of the main features distinguishing rings and semirings is that in the second category ideals are not in bijection with equivalence relations giving semiring quotients, in fact, a quotient of a ring $R$ is defined by an equivalence relation on $R$ such that the ring structure descends to the set of equivalence classes. Such equivalence relations are of course in bijection with ideals via the correspondence

$$\begin{aligned} \text{ideal } I &\mapsto \text{equivalence relation } \{a \sim b \text{ if } a - b \in I\}, \\ \text{equivalence relation } K &\mapsto \text{ideal } \{a - b \mid a \sim_K b\}. \end{aligned}$$

This correspondence does not hold for semirings in general, and so we must work with the equivalence relations themselves when defining quotients.

We say that an equivalence relation $K \subset S \times S$ on a semiring $S$ is a *congruence* if it is also a subsemiring of $S \times S$. If $K$ is a congruence on $S$, then the semiring structure on $S$ descends to a well-defined semiring structure on $S/K$. Moreover, if $f\colon S \twoheadrightarrow S'$ is a surjective homomorphism of semirings then its kernel congruence $\ker f = \{(a,b) \mid f(a) = f(b)\}$ is indeed a congruence and $S/\ker f \cong S'$.

Given a set of binary relations $X \subset S \times S$, the congruence it generates can be described concretely. First take the subsemiring of $S \times S$ generated by $X$, and then take the transitive and symmetric closure of this. See [GG16, Lemma 2.4.5].

We now come to a class of congruences on idempotent semirings that are essential in tropical geometry. Given an expression $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ in an idempotent semiring $S$, the *bend congruence* of this expression, written $\mathcal{B}(a_1 \oplus \cdots \oplus a_n)$, is the congruence on $S$ generated by the relations

$$B_j\colon \bigoplus_{i=1}^{n} a_i \sim \bigoplus_{i=1, i\neq j}^{n} a_i$$

for each $j = 1 \dots n$. These generating relations are called *bend relations*. As special cases, when $n = 2$, $\mathcal{B}(a \oplus b)$ is generated by the single relation $a \sim b$. When $n = 3$, $\mathcal{B}(a \oplus b \oplus c)$ is generated by the relations

$$a \oplus b \oplus c \sim a \oplus b \sim a \oplus c \sim b \oplus c.$$

The motivation for the bend relations stems from the following fact. Recall that the *tropical hypersurface* of a tropical polynomial $f \in \mathbb{T}[x_1, \dots, x_n]$ can be described away from the boundary of $\mathbb{T}^n$ as the locus where the graph of $f$ is non-linear. Then

**Proposition 2.3.3** ([GG16], Proposition 5.1.6)**.** *Given a tropical polynomial $f \in \mathbb{T}[x_1, \dots, x_n]$, the tropical hypersurface of $f$ is precisely the set of homomorphisms $\mathbb{T}[x_1, \dots, x_n]/\mathcal{B}(f) \to \mathbb{T}$.*

## 2.4 Tropical scheme theory

In its first years, tropical geometry has been lacking of a solid algebraic foundation, for instance of a way to see tropical varieties as actual varieties in a more traditional sense. A broader environment that allows to look for schemes over a semiring and to interpret tropical varieties as the set of points of a scheme over a semiring is that of $\mathbb{F}_1$-geometry. In this context one can look for schemes not only over rings, as in the classical theory as developed by Grothendieck, but also over semiring and monoids. The realisation that we can tropicalize a subscheme of a tropical variety thanks to the fact that the coordinate rings of the affine patches of a toric variety have a distinguished class of monomials, and that they glue by localizing monomials leads to consider schemes with a model over $\mathbb{F}_1$ instead of just embeddings into toric varieties. In this context we can consider schemes over a valued field $K$ that admit an open cover by spectra of monoid rings and such that the gluing is induced by localizations happening at the level of the monoids.

The theory of $\mathbb{F}_1$-schemes have been developed for example in [Dur07], [TV09] and [Lor12]. In [GG16] the authors considered a category of $\mathbb{F}_1$-schemes that is a subcategory of all the ones introduced in the aformentioned works and developed a scheme-theoretic approach to tropicalization and tropical geometry in general. The content of this section is mostly a recapitulation of some results of [GG16] and [GG14].

In addition to this first generalisation of the context in which tropicalization can be considered, by working with the following definition of valuation, which generalises (semi)valuations, it is possible to replace the field $K$ by any ring $R$ and to replace the tropical semifield $\mathbb{T}$ by any idempotent semiring $S$.

**Definition 2.4.1.** A valuation on a ring $R$ is a map $v\colon R \to S$ to an idempotent semiring satisfying the following conditions:

(1) $v(0_R) = 0_S$;

(2) $v(1_R) = v(-1_R) = 1_S$;

(3) $v(ab) = v(a) \otimes v(b)$ for all $a, b \in R$;

(4) $v(a+b) \oplus v(a) \oplus v(b) = v(a) \oplus v(b)$ for all $a, b \in R$.

All along this work we will refer to a valuation as a function satisfying Definition 2.4.1 above.

*Remark* 2.4.2. Condition (4) generalizes the ultrametric triangle inequality, as it is equivalent to $v(a+b) \geq v(a) \oplus v(b)$ in the canonical partial order on $S$ given by $a \leq b$ if and only if $a \oplus b = a$. This definition thus becomes equivalent to the usual definition of a Krull valuation when the partial order is a total order, such as when $S = \mathbb{T}_n$. This condition can also be written more symmetrically as $v(a) \oplus v(b) \oplus v(c) = v(a) \oplus v(b)$ whenever $a + b + c = 0$ in $A$, since $v(c) = v(a+b)$.

*Remark* 2.4.3. For use later on, we record the following simple observation. A rank 1 valuation $v\colon R \to \mathbb{T}$ can be extended to a rank 2 valuation on the ring $R\{\{t\}\}$ of Puiseux series (or the subrings of formal Laurent series or polynomials) by the formula

$$a_0 t^{n_0} + \cdots \mapsto (t^{n_0}, v(a_0)) \in \mathbb{T}_2.$$

We move now to introduce the categories of objects over $\mathbb{F}_1$ that we are going to consider to build polynomial algebras with coefficients in a (semi)ring.

**Definition 2.4.4.** The category $\mathbb{F}_1$**-Mod** of $\mathbb{F}_1$-modules is the category of pointed sets and morphisms of pointed sets.

**Definition 2.4.5.** The category $\mathbb{F}_1$-**Alg** is the category of commutative monoids with zero, i.e. monoids $(M, \cdot, 0_M)$ such that $m \cdot 0_M = 0_M$ for all $m \in M$, and morphisms of monoids $f : M \to N$ such that $f(0_M) = 0_N$. The category $\mathbb{F}_1$-**Alg** is a subcategory of the category of monoids **Mon**.

A base change functor to a (semi)ring is defined:

**Definition 2.4.6.** Given a (semi)ring $S$ and an $\mathbb{F}_1$-algebra $M$, define the $S$-algebra $S[M]$ as the free $S$-algebra generated by $M \setminus 0_M$. This assignment gives a functor:

$$S[-] : \mathbb{F}_1\textbf{-Alg} \to S\textbf{-Alg}$$

Furthermore, let $M(-) : S\textbf{-Alg} \to \mathbb{F}_1\textbf{-Mod}$ be the forgetful functor sending a $S$-algebra to its underlying $\mathbb{F}_1$-algebra, then there is an adjunction:

$$S[-] : \mathbb{F}_1\textbf{-Alg} \rightleftarrows S\textbf{-Alg} : M(-)$$

Affine schemes can be defined over $Q$ (where $Q$ is $\mathbb{F}_1$ or a semiring) in a very similar fashion to the usual setting of rings: let $A$ be a $Q$-algebra, then an ideal $I$ of $A$ is a subset, if $Q = \mathbb{F}_1$ (respectively submonoid if $Q$ is a semiring), such that $a \cdot b \in I$ for all $a \in A$, $b \in I$. The ideal $I$ is said to be prime if its complement is closed under multiplication. Given a prime ideal $\mathfrak{p}$, the localization $A_{\mathfrak{p}}$ can be formed in the usual way.

The prime spectrum of a $Q$-algebra $A$ is given as the topological space whose underlying set is that of prime ideals of $A$ and whose topology has a basis of open sets given by the collection $D(f) := \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$ for all $f \in A$. Given an $A$-module (or algebra) $M$, this determines a sheaf of $Q$-modules (or algebras) sending an open set $D(f)$ to the localization $M_f = A_f \otimes M$. In particular if $M = A$ we get the structure sheaf $\mathcal{O}_A$ of Spec($A$). An affine scheme over $Q$ is a pair $(X, \mathcal{O})$ where $X$ is a topological space and $\mathcal{O}$ is a sheaf of $Q$-algebras such that it is isomorphic to a pair of the form (Spec($A$), $\mathcal{O}_A$) for some $Q$-algebra $A$. A general scheme over $Q$ is locally isomorphic to an affine scheme. Finally, homomorphisms of schemes over $Q$ are morphisms given by homomorphisms of $Q$-algebras on affine patches.

When $Q$ is a ring this construction coincides with the usual construction of schemes over a ring in terms of locally ringed spaces.

The base change functor $S[-] : \mathbb{F}_1\textbf{-Alg} \to S\textbf{-Alg}$ behaves well with respect to localizations thus it gives a base change functor for schemes as well:

$$(-)_S : \textbf{Sch}_{\mathbb{F}_1} \to \textbf{Sch}_S$$

One can define what a sheaf of semiring congruences is and consider subschemes of a scheme over a semiring. Among all the semiring congruences a very important role in this story is played by congruences generated by the bend relations as introduced in Section 2.3.

Let $\mathcal{B}(P)$ be the bend congruence on $S[M]$ generated by the bend relations of $P$. Given an ideal $J \subset S[M]$ the congruence $\mathcal{B}(J)$ is generated by the bend relations of all $P \in J$. Furthermore, we have that the following result holds:

**Proposition 2.4.7** ([GG16], Proposition 6.6.1.). *Given a valuation $v\colon R \to S$, an integral $\mathbb{F}_1$-algebra $M$ and an ideal $I \subset R[M]$, the $S$-submodule* $\mathrm{trop}_v(I) \subset S[M]$ *is an ideal.*

A closed immersion of schemes is a morphism $\phi\colon Y \to X$ such that $\phi(Y)$ is closed in $X$, the map $Y \to \phi(Y)$ is an homeomorphism and the induced map of sheaves $\phi^{\#}\colon \mathcal{O}_X \to \phi_*\mathcal{O}_Y$ is surjective. Given two closed immersions $\phi\colon Y \to X$ and $\phi'\colon Y' \to X$, they are equivalent if there is an isomorphism $Y \cong Y'$ that commutes with $\phi$ and $\phi'$. A closed subscheme is an equivalence class of closed immersions with respect to this equivalence relation.

Over rings the above definition of closed immersion is equivalent to say that $\phi$ is an affine morphism of schemes and $\phi^{\#}$ is surjective. Over semirings, instead, this equivalence breaks down, and this is a consequence of the fact that in general there is no bijection between ideals and congruences for semirings.

Here, for a scheme $X$ over a semiring, a *closed immersion* $\phi\colon Y \to X$ is, as in the second case, an affine morphism of semiring schemes such that $\phi^{\#}$ is surjective. A *closed subscheme* is an equivalence class of closed immersions with respect to the same equivalence relation as above.

As in the world of semirings congruences are the right object to take quotients, we define a *congruence sheaf* $\mathcal{J}$ on a semiring scheme $X$ to be a subsheaf of $\mathcal{O}_X \times \mathcal{O}_X$ such that $\mathcal{J}(U)$ is a semiring congruence on $\mathcal{O}_X(U)$ for each open $U \subset X$.

Given a valued ring $v\colon R \to S$, an $\mathbb{F}_1$-scheme $X$ locally defined as $\mathrm{Spec}\,(M)$ and a subscheme $\varphi\colon Y \hookrightarrow X_R$ cut out by an ideal sheaf $\mathcal{I}$, we have all the ingredients to give the definition of scheme-theoretic tropicalization of $Y$:

**Definition 2.4.8** (*Scheme-theoretic tropicalization*)**.** The scheme-theoretic tropicalization of $Y$ inside $X$ is defined as the subscheme $\mathcal{T}rop^v_\varphi(Y)$ of $X_S$ locally given as the spectrum of the semiring $S[M]/\mathcal{B}(\mathrm{trop}_v(\mathcal{I}(R[M])))$.

This notion of tropicalization is functorial (see [GG16] Proposition 6.4.1) on the category $\mathcal{C}$ whose objects are closed embeddings $Y \hookrightarrow X_R$ of schemes over $R$ where $X$ is a scheme over $\mathbb{F}_1$. Morphisms $(Y, X) \to (Y', X')$ in $\mathcal{C}$ are morphisms $f\colon X \to X'$ of schemes over $\mathbb{F}_1$ such that $f_R(Y) \subset Y'$. This is a scheme-theoretic generalisation of the observation that tropicalization on toric varieties is functorial with respect to equivariant morphisms. Indeed, since toric varieties are a particular instance of schemes admitting an $\mathbb{F}_1$-model, it also makes sense to ask if the scheme-theoretic tropicalization reduces to the one introduces by Payne and Kajiwara on toric varieties. This is actually the case, the Kajiwara-Payne tropicalization is recovered as the $\mathbb{T}$-points of the scheme-theoretic one:

**Theorem 2.4.9** ([GG16] Theorem 6.3.1)**.** *Let $X$ be a toric variety over $\mathbb{F}_1$, let $v\colon K \to \mathbb{T}$ be an algebraically closed valued field. Given a scheme $Y$ over $K$ and a closed embedding $\varphi\colon Y \to X_K$, the set $\mathcal{T}rop(\varphi)(\mathbb{T})$ of $\mathbb{T}$-points of the scheme-theoretic tropicalization of $Y$ inside $X$ coincides with the Kajiwara-Payne tropicalization* $\mathrm{trop}_\varphi(Y)$ *as a subset of $X(\mathbb{T})$.*

*Remark* 2.4.10. In the affine case, let $v : R \to S$ be a valued ring and $I \subset R[x_1, \dots, x_n]$ an ideal, then there is a bijection between the set of solutions to $\mathrm{trop}_v(I)$ in an $S$-algebra $T$ and the set $\mathrm{Hom}_{S\text{-}\mathbf{Alg}}(S[x_1, \dots, x_n]/\mathcal{B}(\mathrm{trop}_v(I)), T)$.

Both the previously discussed points of view on Berkovich analytification, i.e. as a moduli space of valuations and as an intrinsic tropicalization, can be made rigorous using tropical scheme-theory, as worked out in [GG14]. It is fundamental here that we are working in the category of scheme with a model over $\mathbb{F}_1$. The key to this is to consider the universal embedding $Y \hookrightarrow \hat{Y}$ of a scheme over $R$:

**Definition 2.4.11.** Let $A$ an integral $R$-algebra and let $\hat{A} := R[M(A)]$ then there is a canonical surjection of $R$-algebras $ev\colon \hat{A} \to A$ sending $x_a$ to $a$ for every $a \in A$. We call $ev$ the evaluation map.

By definition of the adjunction between $R[-]$ and $M(-)$, for any $\mathbb{F}_1$-algebra $M$ and $R$-algebra homomorphism $f\colon R[M] \to A$, the adjoint $g\colon M \to M(A)$ is the unique $\mathbb{F}_1$-algebra homomorphism such that the diagram:

$$\begin{tikzcd}
 & \hat{A} \arrow[d, two heads, "ev"] \\
R[M] \arrow[ur, "R[g]"] \arrow[r, "f"'] & A
\end{tikzcd}$$

commutes.

Let $Y := \mathrm{Spec}\,(A)$ and $\hat{Y} := \mathrm{Spec}\,(\hat{A})$. As $ev\colon \hat{A} \to A$ is surjective we can regard it as an embedding $Y \hookrightarrow \hat{Y}$ of $R$-schemes. This, thanks to the good behaviour of $R[-]$ and $M(-)$ with respect to localizations, generalizes to non-affine schemes as well and is locally given by the above evaluation maps. Furthermore the embedding $\varphi_{\mathrm{univ}}\colon Y \hookrightarrow \hat{Y}$ is initial in the subcategory $\mathcal{C}_Y$ of $\mathcal{C}$ of embeddings of $Y$ into $R$-schemes with an $\mathbb{F}_1$ model, i.e. given any morphism $f\colon X_1 \to X_2$ of $\mathbb{F}_1$-schemes and closed embeddings $\varphi_i\colon Y \hookrightarrow (X_i)_R$ such that $f_R \circ \varphi_1 = \varphi_2$, we have a commutative diagram:

$$\begin{tikzcd}
 & (X_1)_R \arrow[dr, "f_R"] & \\
 & \hat{Y} \arrow[u] \arrow[r] & (X_2)_R \\
Y \arrow[uur, hook, "\varphi_1"] \arrow[ur, hook] \arrow[urr, hook, "\varphi_2"'] & &
\end{tikzcd}$$

Thanks to this fact and to the functoriality of tropicalization we have that the scheme-theoretic tropicalization $\mathcal{T}rop_{\mathrm{univ}}(Y) := \mathcal{T}rop_{\varphi_{\mathrm{univ}}}(Y)$ of the universal embedding is canonically isomorphic to $\varprojlim_{\varphi \in \mathcal{C}_Y} \mathcal{T}rop_{\varphi}(Y)$.

**Definition 2.4.12.** Given a scheme $Y$ over $R$, a valuation on $Y$ with values in an idempotent semiring $S$ is a valuation $v_U\colon \Gamma(U, \mathcal{O}_Y|U) \to S$ for every open affine $U \subset Y$. They are considered modulo the equivalence relation identifying $v_{U_1}$ and $v_{U_2}$ if they are both a restriction of a valuation $v_V$ for some open affine $V \subset U_1 \cap U_2$.

If $v\colon R \to S$ is a valued ring and $S'$ a $S$-algebra, a valuation on $Y$ is said to be compatible with $v$ if the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, "v"] \arrow[d] & S \arrow[d] \\ \Gamma(U, \mathcal{O}_Y|U) \arrow[r] & S' \end{tikzcd}$$

With this definition the following theorem holds:

**Theorem 2.4.13** ([GG14] Theorem 3.3.6)**.** *Given a valued ring $v\colon R \to S$ and an $R$-scheme $Y$, the universal tropicalization $\mathcal{T}rop_{univ}(Y)$ represents the controvariant functor on affine $S$-schemes sending* $\operatorname{Spec}(S')$ *to the set of valuations on $Y$ with values in $S'$ compatible with $v$, i.e.*

$$\mathcal{T}rop_{univ}(Y)(S') = \{\textit{valuations } Y \to S' \textit{ compatible with } v\}$$

In particular, when $S$ is equal to $\mathbb{T}$, we obtain a bijection between the set-theoretic tropicalization of the universal embedding of $Y$ and the Berkovich analytification $Y^{\mathrm{an}}$ of $Y$. This bijection can be refined into an homeomorphism by equipping $\mathcal{T}rop_{\mathrm{univ}}(Y)(\mathbb{T})$ with the so-called strong Zariski topology, where closed sets are given by the $\mathbb{T}$-points of closed subschemes of $Y$. Furthermore for any $\varphi \in \mathcal{C}_Y$, the map $\mathcal{T}rop_{\mathrm{univ}}(Y) \to \mathcal{T}rop_{\varphi}(Y)$ reduces to the projection $Y^{\mathrm{an}} \to \mathrm{trop}_{\varphi}(Y)$ upon passing to $\mathbb{T}$-points.

When $Y = \operatorname{Spec}(A)$ is an affine $R$-scheme, let $S_A := S[M(A)]/\mathcal{B}\mathrm{trop}(\ker \mathit{ev})$ then $\mathcal{T}rop_{\mathrm{univ}}(Y) = \operatorname{Spec}(S_A)$. We have:

**Corollary 2.4.14.** *There is a valuation $w\colon A \to S_A$ universal among all valuations compatible with $v$ in the following sense: given any valuation $w'\colon A \to S'$ compatible with $v$ there is a unique morphism of $S$-algebras $f\colon S_A \to S'$ such that the following diagram commutes:*

$$\begin{tikzcd} & S_A \arrow[d, "f"] \\ A \arrow[ur, "w"] \arrow[r, "w'"'] & S' \end{tikzcd}$$

*and it sends $a \in A$ to $x_a \in S_A$.*

Lastly, to generalise Payne's result in the affine case, let $v\colon R \to S$ a valued ring and for a finitely generated integral $R$-algebra $A$ let $Y := \operatorname{Spec}(A)$ and the category $\mathcal{C}_{Y,\mathrm{aff}}$ the subcategory of $\mathcal{C}_Y$ whose elements are embeddings into affine spaces. Then:

**Theorem 2.4.15** ([GG14] Theorem 4.1.1)**.** *The universal tropicalization of $Y$ is isomorphic as an $S$-scheme to the limit of the tropicalizations of embeddings into affine spaces:*

$$\mathcal{T}rop_{univ}(Y) \cong \varprojlim_{\varphi \in \mathcal{C}_{Y,aff}} \mathcal{T}rop_{\varphi}(Y)$$

*Remark* 2.4.16. Let us denote elements of the category $\mathcal{C}^{\mathrm{op}}_{Y,\mathrm{aff}}$ as $\psi\colon\ R[M] \twoheadrightarrow A$, then the statement of Theorem 2.4.15 is equivalent to say that

$$S_A \cong \operatorname*{colim}_{\psi \in \mathcal{C}^{\mathrm{op}}_{Y,\mathrm{aff}}} S[M]/\mathcal{B}\mathrm{trop}_v(\ker \psi)$$

as $S$-algebras.

# Chapter 3

# Tropical differential algebra and its fundamental theorem

For a field $K$, given a differential polynomial over $K[\![t]\!]$ (in the sense of [Rit50]) and given a solution to such a polynomial, relative associated notions of tropicalization are introduced in [Gri17]. There, the author introduces tropical differential equations, related notions of evaluation and solution and an algorithmic method to test the solvability of tropical differential linear systems, that, in case of positive answer, constructs a solution. Further computational aspects are investigated.

At the end of the paper, an open question was left: does an analogue of the fundamental theorem of tropical geometry (Theorem 2.1.3) hold in the differential setting? In other words, is the set of tropicalizations of solutions to the differential equation we started with equal to the set of solutions to the tropicalized one?

A positive answer to this question, at least when $K$ is a trivially valued algebraically closed field of characteristic 0, has been given in [AGT16] and extended to partial differential equations in [FGLH$^+$20].

In [FT20] the authors extended this analogy with the classical case by introducing a notion of initial ideal for a differential ideal $I$ of the Ritt polynomial algebra over $K[\![t]\!]$ and proving that the set of weight vectors such that the initial ideal does not contain a monomial coincides with the two sets in the statement of the main theorem of [AGT16].

The aim of Section 3.1 is to recall the setting of [Gri17] and the results of [AGT16] and [FT20]. The notation in which this results will be stated here will differ from the original one, and a dictionary table is given at the end of the section. In Section 3.2 we highlight the motivating ideas a generalisation of which will be the philosophy leading us in the following chapters.

## 3.1 The fundamental theorem of tropical differential algebra

All along this work a differential equation will be an element of a differential polynomial ring over a differential ring $R$, in the sense of Ritt:

**Definition 3.1.1.** Given a (semi)ring $R$ let

$$R\{x_1, \ldots, x_n\}_{basic} := R[x_i^{(j)} \mid i = 1, \ldots, n;\ j \in \mathbb{N}]$$

be the polynomial $R$-algebra generated by the family of variables $\{x_i^{(j)}\}_{i=1,\ldots,n;\, j\in\mathbb{N}}$

Recall that a differential ring is a ring $R$ equipped with an additive map $d_R \colon R \to R$ that satisfies the Leibniz relations $d(ab) = (da)b + a(db)$. This relation for 2-fold products easily implies that $d$ also satisfies an analogous relation for $n$-fold products. We will call these relations the *strict Leibniz relations*, and a map $d$ satisfying them will be called a *strict differential*.

**Definition 3.1.2** (*Ritt differential polynomial algebra*)**.** Given a differential ring $(R, d_R)$, let the differential $R$-algebra of Ritt polynomials in $n$ variables to be $R\{x_1, \ldots, x_n\}_{basic}$ equipped with the differential $d$ given by $d(x_i^{(j)}) = x_i^{(j+1)}$ and extending $d_R$. Denote $(R\{x_1, \ldots, x_n\}_{basic}, d)$ as $R\{x_1, \ldots, x_n\}$.

Ritt polynomial algebras are characterized by the following universal property: a homomorphism of differential rings $\varphi\colon R\{x_1, \ldots, x_n\} \to R'$ is uniquely determined by the images of the generators $\varphi(x_i) \in R'$.

**Definition 3.1.3.** A solution to $P \in R\{x_1, \ldots, x_n\}$ is a $n$-tuple $r = (r_1, \ldots, r_n) \in R^n$, such that

$$P|_{x_i^{(j)} = d_R^j(r_i)} = 0$$

*Example* 3.1.4. Let $R = \mathbb{C}[\![t]\!]$ be the differential ring of power series with complex coefficients, let $r = \sin(t^2) = \sum_{i=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{4n+2}$ and consider the polynomial

$$P = t^2 \sin(t^2) X + \frac{1}{4}(X')^2 - t^2 \in R\{X\}$$

Then $r$ is a solution for $P$, indeed:

$$\begin{aligned} P(r) &= t^2(\sin(t^2))^2 + \frac{1}{4}(2t\cos(t^2))^2 - t^2 = \\ &= t^2(\sin(t^2))^2 + t^2(\cos(t^2))^2 - t^2 = 0 \end{aligned}$$

Throughout the whole work we will look at the semiring $\mathbb{T}$ as the isomorphic semiring $(\{t^r \mid r \in \mathbb{R} \cup \{\infty\}\}, \oplus, \odot)$ with sum given by $t^r \oplus t^s = t^{\min\{r,s\}}$ and product $t^r \odot t^s = t^{r+s}$. This will give us a natural way to multiply boolean power series and tropical numbers.

Let $K$ be an uncountable algebraically closed characteristic 0 field equipped with the trivial valuation $v_K \colon K \to \mathbb{B}$. For the rest of this section, let $R := (K[\![t]\!], d/dt)$ and let $v\colon K[\![t]\!] \to \mathbb{T}$ be the $t$-adic valuation on $R$, sending a power series to its leading exponent.

**Definition 3.1.5.** The tropicalization of $P \in R\{x_1, \ldots, x_n\}$ is the polynomial $\operatorname{trop}_v(P) \in \mathbb{T}\{x_1, \ldots, x_n\}_{basic}$ obtained by applying the $t$-adic valuation to the coefficients of $P$.

*Example* 3.1.6. Let $P = t^2 \sin(t^2)X + \frac{1}{4}(X')^2 - t^2 \in \mathbb{C}(\!(t)\!)\{X\}$ as in Example 3.1.4, then

$$\operatorname{trop}_v(P) = t^4X + t^0(X')^2 + t^2$$

Let $\mathbb{B}[\![t]\!]$ be the semiring of power series with boolean coefficients. The linear map $d\colon \mathbb{B}[\![t]\!] \to \mathbb{B}[\![t]\!]$ mapping $t^n$ to $t^{n-1}$, satisfies the Leibniz rule. Thus $\mathbb{B}[\![t]\!]$ is the first example of a (strict) differential semiring, whose definition will be given in general in Definition 4.1.1.

The salient feature of the theory of tropical differential equations is that, as we look for solutions "at first order", we do not look for solutions to a polynomial $P \in \mathbb{T}\{x_1, \ldots, x_n\}_{basic}$ in a $\mathbb{T}$-algebra but in $\mathbb{B}[\![t]\!]$. Let $\pi\colon \mathbb{B}[\![t]\!] \to \mathbb{T}$ be the map sending a boolean power series to its leading term, it is an homomorphism of semirings.

**Definition 3.1.7.** Let $B \in \mathbb{B}[\![t]\!]^n$, the Grigoriev evaluation in $B$ is the semiring homomorphism

$$\operatorname{Gev}_B\colon \mathbb{T}\{x_1, \ldots, x_n\}_{basic} \to \mathbb{T}$$

mapping $x_i^{(j)}$ to $\pi(d^j(B_i))$.

The assignment $\operatorname{Gev}_B$ is a semiring homomorphism as it is the evaluation in the element:

$$\overline{B} := ((\pi(d^j(B_1)))_{j\in\mathbb{N}}, \ldots, (\pi(d^j(B_n)))_{j\in\mathbb{N}}) \in (\mathbb{T}^{\mathbb{N}})^n$$

**Definition 3.1.8.** Given $P \in \mathbb{T}\{x_1, \ldots, x_n\}_{basic}$, the $n$-tuple $B \in \mathbb{B}[\![t]\!]^n$ is a solution for $P$ if the minimum in $\operatorname{Gev}_B(P)$ is attained in at least two distinct monomials, or if it is $\infty$.

The last ingredient needed to state a fundamental theorem is a procedure to tropicalize $n$-tuples in $K[\![t]\!]^n$.

**Definition 3.1.9.** Let $\widetilde{v}\colon R \to \mathbb{B}[\![t]\!]$ be the coefficientwise application of the trivial valuation $v_K$. Given an $n$-tuple of elements $r = (r_1, \ldots, r_n) \in R^n$ its tropicalization is given by taking the $n$-tuple

$$\operatorname{trop}_{\widetilde{v}}(r) := (\widetilde{v}(r_1), \ldots, \widetilde{v}(r_n)) \in \mathbb{B}[\![t]\!]^n$$

*Remark* 3.1.10. The map $\widetilde{v}$ commutes with the differentials, i.e. for every $A \in K[\![t]\!]$, $\widetilde{v}(dA) = d(\widetilde{v}(A))$. The semiring of Boolean power series $(\mathbb{B}[\![t]\!], +, \cdot)$ is isomorphic to the semiring $(\mathcal{P}(\mathbb{N}), \cup, +_M)$, with Minkowski sum $+_M$, via

$$\mathcal{P}(\mathbb{N}) \ni S \mapsto \sum_{i\in\mathbb{N}} b_i t^i \in \mathbb{B}[\![t]\!] \text{ where } b_i = \begin{cases} 0 & \text{if } i \in S \\ \infty & \text{if } i \notin S \end{cases}$$

The inverse map is given by sending $b \in \mathbb{B}[\![t]\!]$ to $Supp(B) = \{n \in \mathbb{N} \mid a_n \neq \infty\}$. Through this isomorphism, $\widetilde{v}$ is mapping a power series $A$ to its support in $\mathcal{P}(\mathbb{N})$. As $K$ has characteristic 0:

$$Supp(dA) = \{n - 1 \mid n \in Supp(A)\} \cap \mathbb{N}$$

and the right hand side corresponds to $d(\widetilde{v}(A))$ via the isomorphism illustrated above.

Furthermore, $\widetilde{v}$ is not multiplicative, for example:

$$\widetilde{v}((1-t)(1+t)) = 0 + 0t^2 \neq 0 + 0t + 0t^2 = \tilde{v}(1-t)\nu(1+t)$$

*Example* 3.1.11. Consider $P$ as in Example 3.1.4, thus $\mathrm{trop}_v(P) = t^4X + t^0(X')^2 + t^2$ and let $r = \sin(t^2)$, so $B := \mathrm{trop}_\nu(r) = \sum_{i=0}^{\infty} t^{4n+2}$. Then:

$$\mathrm{Gev}_B(\mathrm{trop}_v(P)) = t^4 \cdot t^2 + t^0 \cdot (t)^2 + t^2$$

hence $B$ is a tropical solution to $\mathrm{trop}_v(P)$.

Given $\Sigma \subset R\{x_1, \ldots, x_n\}$, let us denote as $\mathrm{Sol}(\Sigma)$ the set of solutions to the elements in $\Sigma$, then:

**Theorem 3.1.12** (*Fundamental theorem of tropical differential algebra*, [AGT16], Theorem 8.1)**.** *Given a differential ideal $I \subset R\{x_1, \ldots, x_n\}$ the following equality holds:*

$$\mathrm{trop}_{\widetilde{v}}(\mathrm{Sol}(I)) = \mathrm{Sol}(\mathrm{trop}_v(I)) \tag{3.1.1}$$

In order to make the analogy with the fundamental theorem of tropical geometry even more appropriate let us see how to add the third term in the equality of the theorem above.

Given $m \in \mathbb{N}$, a differential polynomial $P \in R\{x_1, \ldots, x_n\}$ is of order $\leq m$ if $P \in R[x_i^{(j)} \mid i = 1, \ldots, n;\ j \leq m]$, thus it can be written as:

$$P = \sum_{M \in \Lambda} r_M \prod_{\substack{1 \leq i \leq n \\ 0 \leq j \leq m}} (x_i^{(j)})^{M_{i,j}}$$

for $\Lambda \subset M_{n \times (r+1)}(\mathbb{N})$ a finite subset and $r_M \in R$ for all $M \in \Lambda$.

In [FT20], given a differential polynomial $P \in R\{x_1, \ldots, x_n\}$ with order $\leq r$ and an element $B \in \mathbb{B}[\![t]\!]^n$, which here will act as a weight vector, the authors introduce the polynomial $P_B \in R\{x_1, \ldots, x_n\}$, defined as:

$$P_B((x_i^{(j)})_{i,j}) = \begin{cases} t^{-\mathrm{trop}(P)(B)} P((\pi(d^j(B_i))x_i^{(j)})_{i,j}) & \text{if } \mathrm{trop}_v(P)(B) \neq \infty \\ 0 & \text{if } \mathrm{trop}_v(P)(B) = \infty \end{cases}$$

The initial term of $P$ with respect to $B$ is $\mathrm{In}_B(P) := \overline{P_B}((x_i^{(j)})_{i,j}) \in K\{x_1, \ldots, x_n\}_{basic}$. Note that, when $\mathrm{trop}_v(P)(B) \neq \infty$:

$$\mathrm{In}_B(P) = \sum_{M \in \Xi} \overline{r_M t^{-v(r_M)}} \prod_{\substack{1 \leq i \leq n \\ 0 \leq j \leq m}} (x_i^{(j)})^{M_{i,j}}$$

where $\Xi = \{M \in \Lambda \mid \text{the minimum in } \operatorname{trop}_v(P)(B) \text{ is achieved in the } M\text{-th term}\}$.

Given a differential ideal $I \in R\{x_1, \dots, x_n\}$ and a weight vector $S \in \mathbb{B}[\![t]\!]^n$, let

$$\operatorname{In}_S(I) := \langle \operatorname{In}_S(P) \mid P \in I \rangle \subset K\{x_1, \dots, x_n\}_{basic}$$

the initial ideal of $I$ with respect to $S$. With these definitions a complete analogy between Theorem 3.1.12 and the Fundamental Theorem of Tropical Geometry can be established:

**Theorem 3.1.13** ([FT20], Theorem 3.9)**.** *Given a differential ideal $I \subset R\{x_1, \dots, x_n\}$ the following equalities hold :*

$$\operatorname{trop}_{\widetilde{v}}(\operatorname{Sol}(I)) = \operatorname{Sol}(\operatorname{trop}_v(I)) = \{S \in \mathbb{B}[\![t]\!]^n \mid \operatorname{In}_S(I) \textit{ does not contain a monomial}\} \tag{3.1.2}$$

To conclude, the correspondence between the notation used here and the notation in [AGT16] is summarized by the following table:

| [AGT16] notation | Present notation |
|---|---|
| $(\mathcal{P}(\mathbb{N}), \cup, +_M)$ | $(\mathbb{B}[\![t]\!], +, \cdot, d)$ |
| $Val_S(j)$ | $\pi(d^j(S))$ |
| $\operatorname{trop}\colon K[\![t]\!] \to \mathcal{P}(\mathbb{N})$ | $\widetilde{v}\colon K[\![t]\!] \to \mathbb{B}[\![t]\!]$ |

## 3.2 A first hint at differential enhancements and pairs

Two of the main ideas of the present work draw inspiration from features already visible in the theory developed by Gregoriev and illustrated above.

Firstly, in order to state the Fundamental Theorem 3.1.12, we made use of three maps: the valuation $v\colon K[\![t]\!] \to \mathbb{T}$, the homomorphism of semirings $\pi\colon \mathbb{B}[\![t]\!] \to \mathbb{T}$ and the differential map $\widetilde{v}\colon K[\![t]\!] \to \mathbb{B}[\![t]\!]$. In particular the valuation $v$ factors through the differential semiring $\mathbb{B}[\![t]\!]$ in the following way:

$$\begin{tikzcd} & \mathbb{B}[\![t]\!] \arrow[d, "\pi"] \\ R := K[\![t]\!] \arrow[ur, "\widetilde{v}"] \arrow[r, "v"] & \mathbb{T} \end{tikzcd}$$

The datum $(\pi, \widetilde{v})$ is the first instance of what we will define in Definition 6.5.1 as a *differential enhancement* of a valuation, in this case of $v$, and the map $\pi$ is the first instance of a *reduced pair*. This datum is what is needed in order to tropicalize differential equations and their solutions and it plays the same role in this context that is played in classical tropical geometry by valuations with respect to the tropicalization of polynomial equations and solutions.

Secondly, we can endow $\mathbb{B}[\![t]\!]\{x_1,\dots,x_n\}_{basic}$ with a (strict) differential exactly as in Definiton 3.1.2, let us call $d$ this differential. Then, given any $B \in \mathbb{B}[\![t]\!]^n$, the following diagram commutes:

$$\begin{tikzcd}
(\mathbb{B}[\![t]\!]\{x_1,\dots,x_n\}_{basic}, d) \arrow[r, "\mathrm{ev}_B"] \arrow[d, "\pi"'] & \mathbb{B}[\![t]\!] \arrow[d, "\pi"] \\
\mathbb{T}\{x_1,\dots,x_n\}_{basic} \arrow[r, "\mathrm{Gev}_B"] & \mathbb{T}
\end{tikzcd}$$

i.e. evaluating a differential polynomial $P \in \mathbb{B}[\![t]\!]\{x_1,\dots,x_n\}$ in $B \in \mathbb{B}[\![t]\!]^n$ (analogously to Definition 3.1.3) and taking the $t$-adic valuation of the result gives the same result as taking the coefficientwise $t$-adic valuation of $P$ and applying the Grigoriev evaluation in $B$.

By Definition 3.1.7 of solution to a tropical differential polynomial and by definition of the bend congruence introduced in Section 2.3, it is straightforward that for any differential ideal $I \in R\{x_1,\dots,x_n\}$ and $B \in \mathbb{B}[\![t]\!]^n$

$$B \text{ is a solution for } \mathrm{trop}_v(I) \iff \mathrm{Gev}_B \text{ factors through the quotient by } \mathcal{B}\mathrm{trop}_v(I)$$

thus, $B \in \mathbb{B}[\![t]\!]^n$ is a solution for $\mathrm{trop}_v(I)$ if and only if from the diagram above we get the following commutative diagram:

$$\begin{tikzcd}
(\mathbb{B}[\![t]\!]\{x_1,\dots,x_n\}_{basic}, d) \arrow[r, "\mathrm{ev}_B"] \arrow[d, "\pi"'] & \mathbb{B}[\![t]\!] \arrow[d, "\pi"] \\
\mathbb{T}\{x_1,\dots,x_n\}_{basic}/\mathcal{B}\mathrm{trop}_v(I) \arrow[r, "\mathrm{Gev}_B"] & \mathbb{T}
\end{tikzcd}$$

This is the first incarnation of Theorem 6.3.5 and it hints at us that in order to have a bijection between tropical solution and morphisms in some category, the right objects to consider are of the form of the two vertical arrows in the diagram above, which, as already pointed out above, are examples of (reduced) pairs, the definition and the property of which are the topic of Chapter 6. Pairs will play in this theory the role played in algebraic geometry and tropical scheme theory by (semi)rings and algebras.

*Remark* 3.2.1. Notice that even if the Grigoriev evaluation can be looked at as the evaluation map from $\mathbb{T}\{x_1,\dots,x_n\}_{basic}$ to $\mathbb{T}$ in a particular element of $(\mathbb{T}^{\mathbb{N}})^n$ (as pointed out after Definition 3.1.7), one can not consider any element in $(\mathbb{T}^{\mathbb{N}})^n$. We only want $n$-tuples of sequences of elements in $\mathbb{T}$ that are compatible in some way: in particular we want them to be the sequence of leading terms of the successive derivations of an $n$-tuple of power series. This condition is equivalent to say that every entry has to be a sequence of a certain decreasing sawtooth-like form. For example, sequences of the form $(t, t^2, \dots) \in \mathbb{T}^{\mathbb{N}}$ are not allowed, as there exists no element $B \in \mathbb{B}[\![t]\!]$ such that $\pi(B) = t$ and $\pi(dB) = t^2$. Thus if we want look at solutions to tropical differential equations as morphisms between some objects, it is not enough to consider morphisms of semirings from $\mathbb{T}\{x_1, \ldots, x_n\}_{basic}$ to some $\mathbb{T}$-algebra, as this would not take in account the compatibility discussed above. In the end, this compatibility request encodes the fact that we want tropical solutions to be solutions "at first order". This is why pairs are really the right category to look for morphisms in. Most importantly, working with pairs also allows us to move from the trivially valued case of [AGT16] to the non-trivially valued case and to look for solutions in any algebra over the pair we are going to fix as initial datum in a differential enhancement.

# Chapter 4

# Differential semirings

We start in this chapter to introduce original material, which, up to Chapter 7 included, appears in [GM21], restricted to the case of free $\mathbb{D}\mathbb{F}_1$-modules.

In the present chapter we introduce differential semirings, i.e. idempotent semiring equipped with an additive map satisfying a relaxed tropical Leibniz rule, we give some examples that will be relevant in the following and investigate some of their properties.

## 4.1 Tropical differential semirings

In this work, we propose that differentials on idempotent semirings should be required to satisfy a somewhat weaker condition than the strict Leibniz relations. Given an idempotent semiring $S$, an additive map $d\colon S \to S$ is said to be a *tropical differential* if it satisfies the *tropical Leibniz relations*: for any pairs of elements $x, y \in S$ the bend relations of the expression

$$d(xy) + xd(y) + yd(x)$$

hold. Note that we can view the tropical Leibniz relations as the tropicalization of the strict Leibniz relations.

**Definition 4.1.1.** A *tropical differential semiring* is an idempotent semiring equipped with a tropical differential.

We will denote as **DSRings** the category of differential semirings and, fixing a differential semiring $S$, as **D$S$-Alg** the category of differential $S$-algebras.

Just as the strict Leibniz relations for 2-fold products imply the strict Leibniz relations for $n$-fold products, such as

$$d(xyz) = d(x)yz + xd(y)z + xyd(z),$$

it is possible to derive $n$-fold tropical Leibniz relations from the 2-fold tropical Leibniz relations:

**Lemma 4.1.2.** *In a tropical differential semiring $S$ the bend relations of any expression*

$$d(x_1 \cdots x_n) + \sum_i x_1 \cdots x_{i-1} d(x_i) x_{i+1} \cdots x_n$$

*hold.*

*Proof.* Let us prove it by induction on the number of terms. For $n = 3$ we need to prove that the bend relations of the sum

$$d(xyz) + d(x)yz + xd(y)z + xyd(z)$$

hold. In $S$ the bend relations of the following expressions hold:

$$d(xyz) + d(x)yz + xd(yz) \tag{4.1.1}$$

$$xd(yz) + xd(y)z + xyd(z) \tag{4.1.2}$$

Thus, from the bend relations of 4.1.1 we have

$$d(xyz) + d(x)yz + xd(y)z + xyd(z) = xd(yz) + d(x)yz + xd(y)z + xyd(z)$$

and from the ones of 4.1.2 we get:

$$d(xyz) + d(x)yz + xd(y)z + xyd(z) = d(x)yz + xd(y)z + xyd(z)$$

On the other hand, from the bend relations of 4.1.1 we get:

$$d(xyz) + d(x)yz + xd(y)z + xyd(z) = d(xyz) + xd(yz) + xd(y)z + xyd(z)$$

and from the ones of 4.1.2:

$$d(xyz) + d(x)yz + xd(y)z + xyd(z) = d(xyz) + xd(y)z + xyd(z)$$

Analogously we can drop the terms $xd(y)z$ and $xyd(z)$, thus the thesis holds for $n = 3$.

Assuming that the tropical Leibniz relations hold for a product of $n - 1$ elements, let us prove they hold for a product of $n$ elements. Given a product $\prod_{i=1}^n x_i$ and an index $k = 1, \ldots, n$, denote by $\Theta_k$ the bend relations of the 2-fold product $x_k \cdot \prod_{i \neq k} x_i$ and by $\Lambda_k$ the product of $x_k$ with the $(n-1)$-fold tropical Leibniz relations for $\prod_{i \neq k} x_i$.

Then, we need to prove that the bend relations of the following expression

$$d\left(\prod_{i=1}^n x_i\right) + \sum_{j=1}^n d(x_j) \prod_{i \neq j} x_i \tag{4.1.3}$$

hold.

In order to prove that the expression above is equal to the expression:

$$\sum_{j=1}^n d(x_j) \prod_{i \neq j} x_i$$

it is sufficient to notice that, thanks to relations $\Theta_1$, expression 4.1.3 is equivalent to

$$x_1 d\left(\prod_{i=2}^{n}\right) + \sum_{j=1}^{n} d(x_j)\prod_{i\neq j} x_i$$

and by relations $\Lambda_1$ we have that expression 4.1.3 is equivalent to

$$\sum_{j=1}^{n} d(x_j)\prod_{i\neq j} x_i,$$

which is what we wanted. Given any $k = 1, \ldots, n$ we need to prove that expression 4.1.3 is equivalent to

$$d\left(\prod_{i=1}^{n} x_i\right) + \sum_{\substack{j=1 \\ j\neq k}}^{n} d(x_j)\prod_{i\neq j} x_i.$$

By relations $\Theta_k$ we have that expression 4.1.3 is equivalent to

$$d\left(\prod_{i=1}^{n} x_i\right) + x_k d\left(\prod_{i\neq k} x_i\right) + \sum_{\substack{j=1 \\ j\neq k}}^{n} d(x_j)\prod_{i\neq j} x_i$$

and by relations $\Lambda_k$ we get the equality we wanted. □

*Remark* 4.1.3. The tropical Leibniz relations are distinct from the strict Leibniz relations in two important ways. (1) A strict differential $d$ automatically satisfies the tropical Leibniz relations, but there are many tropical differentials that are not strict. (2) The differential of a product $xy$ is constrained by the tropical Leibniz relations and the differentials of $x$ and $y$, but it is not uniquely determined by them.

*Example* 4.1.4. (1) Let $S$ be an idempotent semiring and let $d$ be either the constant map $0_S$ or the identity; these are each strict differentials on $S$.

(2) Consider the idempotent semiring $\mathbb{B}[\![t]\!]$ of formal power series with coefficients in $\mathbb{B}$. As mentioned in Section 3.1, the map defined by $t^n \mapsto t^{n-1}$ (for any $n \geq 1$ ) is a strict differential. If $p$ is a prime, the map defined by

$$t^n \mapsto \begin{cases} t^{n-1} & n \geq 1 \text{ and } p \nmid n \\ \infty & n = 0 \text{ or } p \mid n \end{cases}$$

is a tropical differential that is not strict.

(3) Consider the idempotent semiring of formal tropical power series $\mathbb{T}[\![t]\!]$. It can be endowed with a strict differential, $d_0$, defined by

$$d_0(t^n) = \begin{cases} t^{n-1} & n \geq 1 \\ \infty & n = 0. \end{cases}$$

Let us denote the differential semiring $(\mathbb{T}[\![t]\!], d_0)$ as $\mathbb{T}[\![t]\!]_0$.

(4) More generally, if $v\colon \mathbb{N} \to \mathbb{T}$ is a valuation, then there is a non-strict tropical differential $d_v$ defined by,

$$d_v(t^n) = \begin{cases} v(n)t^{n-1} & n \geq 1 \\ \infty & n = 0. \end{cases} \tag{4.1.4}$$

Indeed, this satisfies the tropical Leibniz relations since

$$d_v(t^n t^m) \oplus d_v(t^n)t^m \oplus t^n d_v(t^m) = (v(n+m) \oplus v(n) \oplus v(m))t^{n+m-1},$$

and the coefficient $v(n+m) \oplus v(n) \oplus v(m)$ on the right satisfies the bend relations (the argument is essentially the same for $k$-fold products with $k > 2$). Note that $v$ could be either a $p$-adic valuation, or a degenerate $p$-adic valuation where $v(n) = \infty$ if $p$ divides $n$, and 0 otherwise. When $v$ equals a $p$-adic valuation for some prime number $p$, we will denote $(\mathbb{T}[\![t]\!], d_v)$ as $\mathbb{T}[\![t]\!]_p$.

*Remark* 4.1.5. While $\mathbb{B}$ is an initial objects in the category of idempotent semirings, tropical differential semirings do not admit an initial object because the tropical Leibniz rule does not determine $d(1)$.

## 4.2 Differential congruences

Let $(S, d)$ be a semiring equipped with an additive map $d\colon S \to S$. A *differential congruence* on $S$ is a congruence $K \subset S \times S$ that is closed under $d$; i.e., if $(a, b) \in K$ then $(da, db) \in K$. When $K$ is a differential congruence, the map $d$ descends to an additive map $\overline{d}\colon S/K \to S/K$, and if $d$ is a tropical differential then $\overline{d}$ is as well.

**Proposition 4.2.1.** *If $\{I_\lambda \subset S \times S\}$ is a set of differential congruences, then the congruence generated by them is a differential congruence.*

*Proof.* Let $K$ denote the congruence generated by the $I_\lambda$. It is the transitive and symmetric closure of the subsemiring $K_0$ generated by the $I_\lambda$.

We will first show that $d(K_0) \subset K$. Suppose that $(a_1, a_2)$ is a relation in some $I_i$ and $(b_1, b_2)$ is a relation in some $I_j$. Since $d$ is additive, it certainly sends the sum $(a_1 + b_1, a_2 + b_2)$ to a relation in $K$. For the product, we proceed as follows. In $I_i$ we have $(a_1 b_1, a_2 b_1)$ and hence $(d(a_1 b_1), d(a_2 b_1))$ since $I_i$ is a differential congruence. Likewise, in $I_j$ we have the relations $(b_1 a_2, b_2 a_2)$ and hence $(d(b_1 a_2), d(b_2 a_2))$. Hence the relation $(d(a_1 b_1), d(a_2 b_2))$ is indeed contained in the transitive closure $K$. Since any element of $K_0$ is produced by a finite sequence of sums and products of elements in the $I_\lambda$, it follows that $d(K_0) \subset K$, as desired.

Now, any relation in $K$ can be decomposed as a finite transitive chain of relations in $K_0$. Thus is follows that $d(K) \subset K$. $\square$

# Chapter 5

# Differential $\mathbb{F}_1$-modules and differential polynomials over a differential semiring

As in the non-differential case, in order to define a functorial tropicalization for differential equation we need some kind of information over $\mathbb{F}_1$. It is for this reason that in the first part of this chapter we introduce the categories of differential $\mathbb{F}_1$-objects whose datum we are going to use to determine the class of differential presentations that can be tropicalized via the differential tropicalization functor we will introduce in Chapter 7.

We start by introducing the categories of differential $\mathbb{F}_1$-modules and -algebras, we notice that the former is equivalent to the category of $\mathbb{F}_1[d]$-sets (as introduced in [CLS12]), moving then to explaining the construction of the free differential $\mathbb{F}_1$-algebra generated by a differential $\mathbb{F}_1$-module and its definition in terms of trees, which will be the point of view we will use in most of the proof hereinafter. In the second subsection the construction of differential polynomial algebras with coefficients in a differential (semi)ring $R$ will be presented as a base change functor from the category of differential $\mathbb{F}_1$-modules to that of differentia $R$-algebras: this will extend the familiar construction of Ritt differential polynomials in the case of rings and will give rise to a new class of polynomials over differential semirings, whose monomials will be represented by rooted trees. As last thing, we prove the functoriality and freeness of the aformentioned base change functor.

## 5.1 Differential $\mathbb{F}_1$-modules and -algebras

The category of differentially enriched $\mathbb{F}_1$-objects that will be the main object of study in the present section is that of differential $\mathbb{F}_1$-modules:

**Definition 5.1.1.** A differential $\mathbb{F}_1$-module (also D$\mathbb{F}_1$-module in the following) is a tuple $(\mathcal{M}, \star_{\mathcal{M}}, d_{\mathcal{M}})$ where $(\mathcal{M}, \star_{\mathcal{M}})$ is an $\mathbb{F}_1$-module, and $d_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is a map of $\mathbb{F}_1$-modules. Let us denote the category of differential $\mathbb{F}_1$-modules as **D$\mathbb{F}_1$-Mod**.

*Example* 5.1.2. Examples of $\mathbb{DF}_1$-modules are:

- the singleton set $\{\star\}$, it is the zero object in the category $\mathbf{D}\mathbb{F}_1\textbf{-Mod}$;
- the set $\mathbb{F}_1 := \{0, 1\}$ with distinguished point 0, equipped with the constant map 0 or with the identity;
- the set $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ equipped with $d\colon\ n \mapsto n+1$ for $n \in \mathbb{N}$ and $d(\infty) = \infty$;
- every $\mathbb{F}_1$-module $M$ is trivially a $\mathbb{DF}_1$-module equipping it with the trivial differential, mapping every element to $\star_M$.

Looking at $\overline{\mathbb{N}}$ as an $\mathbb{F}_1$-algebra with respect the usual sum of natural numbers, extended to $\infty$ by $n + \infty = \infty$ for all $n \in \overline{\mathbb{N}}$, notice that the datum of a differential $\mathbb{F}_1$-module is the same as that of a $\overline{\mathbb{N}}$-set, as introduced in [CLS12], Section 2.2. The action is given by sending $1 \in \mathbb{N}$ to $d_{\mathcal{M}}$ (as the operation of $A = \overline{\mathbb{N}}$ is the sum, in this case $0_{\mathcal{A}} = \infty$ and $1_{\mathcal{A}} = 0$). With this definition, morphisms of $\overline{\mathbb{N}}$-sets are morphisms commuting with $d$, thus there is an equivalence of categories between $\mathbf{D}\mathbb{F}_1\textbf{-Mod}$ and $\overline{\mathbb{N}}\textbf{-Sets}$.

As $\overline{\mathbb{N}}$ is isomorphic to the $\mathbb{F}_1$-algebra $\mathbb{F}_1[d]$, several features of $\mathbb{DF}_1$-modules are illustrated in [CLS12], Example 2.2.8. Nonetheless we find it useful to recap them here, adding some other interesting considerations.

Given a directed graph $\mathcal{G} := (V, E)$, where $V$ and $E$ are allowed to be infinite sets, if $\mathcal{G}$ satisfies the following conditions:

(1) $|\{e \in E \mid \text{source}(e) = x\}| = 1$ for all $x \in V$ ;

(2) $\mathcal{G}$ has a loop edge, i.e. an edge $e$ such that $\text{source}(e) = \text{target}(e)$;

it can be endowed with the structure of a $\mathbb{DF}_1$-module. Indeed, denoting as $e_x$ the only edge having $x$ as source, let $d(x) = \text{target}(e_x)$ for all $x \in \mathbb{N}$. The distinguished point can be chosen among elements for which $e_x$ is a loop. Different choices of distinguished point in general give rise to non-isomorphic $\mathbb{DF}_1$-module structures. Conversely, any $\mathbb{DF}_1$-module can be visualized as a directed graph satisfying the conditions above.

*Example* 5.1.3.
- The $\mathbb{DF}_1$-module $(\overline{\mathbb{N}}, \infty, d)$ introduced in Example 5.1.2 can be visualised as the directed graph:

$$\infty \circlearrowleft \qquad 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \to \cdots$$

- Consider the directed graph $\mathcal{G}$:

$$\begin{tikzcd} \bullet \arrow[loop right] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[loop right] \end{tikzcd}$$

then the two possible $\mathbb{D}\mathbb{F}_1$-module structures on $\mathcal{G}$:

$$\begin{tikzcd} \bullet \arrow[loop right] & \bullet \arrow[r] & \bullet \arrow[r] & \star \arrow[loop right] \end{tikzcd} \qquad \begin{tikzcd} \star \arrow[loop right] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[loop right] \end{tikzcd}$$

are not isomorphic.

Given two $\mathbb{D}\mathbb{F}_1$-modules $\mathcal{M}$ and $\mathcal{N}$, their coproduct $\mathcal{M} \vee \mathcal{N}$ is given as a set by their disjoint union $\mathcal{M} \sqcup \mathcal{N}$ modulo the relation $\star_{\mathcal{M}} \sim \star_{\mathcal{N}}$. By endowing $\mathcal{M} \vee \mathcal{N}$ with the differential $d$ defined as:

$$d(x) := \begin{cases} d_{\mathcal{M}}(x) & \text{if } x \in \mathcal{M} \\ d_{\mathcal{N}}(x) & \text{if } x \in \mathcal{N} \end{cases}$$

for any $x \in \mathcal{M} \vee \mathcal{N}$, we obtain a $\mathbb{D}\mathbb{F}_1$-module.

Dually, consider the product $\mathcal{M} \times \mathcal{N}$ equipped with the differential $d$ acting componentwise, then $(\mathcal{M} \times \mathcal{N}, d)$ is a $\mathbb{D}\mathbb{F}_1$-module with distinguished point $(\star_{\mathcal{M}}, \star_{\mathcal{N}})$. It is easy to prove that the constructions above are respectively the coproduct and the product of two objects in $\mathbf{D}\mathbb{F}_1$**-Mod**.

**Definition 5.1.4.** Let $\Delta\colon \mathbb{F}_1\textbf{-Mod} \to \mathbf{D}\mathbb{F}_1\textbf{-Mod}$ be the functor associating to an $\mathbb{F}_1$-module $(M, \star_M)$ the $\mathbb{D}\mathbb{F}_1$-module $\Delta M$ defined as:

$$\Delta M := (\{\mathrm{d}^n m \mid m \in M, n \in \mathbb{N}\} / \langle \{\mathrm{d}^n \star_M \sim \star_M\}_{n \in \mathbb{N}} \rangle, \mathrm{d})$$

We will identify $\mathrm{d}^0 m$ with $m$ for all $m \in M$ and often denote the elements $\mathrm{d}^n m$ as $m^{(n)}$.

*Remark* 5.1.5. It is easy to see that the functor $\Delta$ is left adjoint to the forgetful functor $\mathbf{D}\mathbb{F}_1\textbf{-Mod} \to \mathbb{F}_1\textbf{-Mod}$, thus it is a free functor.

*Example* 5.1.6. Let $X = \{\star_X, x\}$, then $\Delta X = \{\star_X, x, \mathrm{d}x, \mathrm{d}^2 x, \dots\}$ is isomorphic to the $\mathbb{D}\mathbb{F}_1$-module $(\overline{\mathbb{N}}, \infty, d)$.

From the example above, a $\mathbb{D}\mathbb{F}_1$-module is *free of rank* $n \in \mathbb{N}$ if it is isomorphic to $\bigvee_n \overline{\mathbb{N}}$. Let us now introduce notions of subobjects and quotients in $\mathbf{D}\mathbb{F}_1$**-Mod**:

**Definition 5.1.7.** Given a $\mathbb{D}\mathbb{F}_1$-module $\mathcal{M}$:

- a $\mathbb{D}\mathbb{F}_1$-submodule of $\mathcal{M}$ is an $\mathbb{F}_1$-submodule $(\mathcal{N}, \star_{\mathcal{M}})$ of $\mathcal{M}$ that is closed with respect to $d_{\mathcal{M}}$;

- a congruence on $\mathcal{M}$ is an equivalence relation $\sim$ on $\mathcal{M}$ that is also a $\mathbb{DF}_1$-submodule of $\mathcal{M} \times \mathcal{M}$. A quotient of $\mathcal{M}$ is a $\mathbb{DF}_1$-module isomorphic to $\mathcal{M}/\sim$ for some congruence on $\mathcal{M}$;

- $\mathcal{M}$ is *finitely generated of rank $n$* if it is isomorphic to a quotient of a free module of rank $n$.

*Example* 5.1.8. (1) Given a $\mathbb{DF}_1$-module $\mathcal{M}$, for every $n \in \mathbb{N}$ the image of $d^n_{\mathcal{M}}$ is a $\mathbb{DF}_1$-submodule of $\mathcal{M}$.

(2) The coproduct $\mathcal{M} \vee \mathcal{N}$ of two $\mathbb{DF}_1$-modules $\mathcal{M}$ and $\mathcal{N}$ is isomorphic to the submodule $\mathcal{M} \times \{\star_{\mathcal{N}}\} \cup \{\star_{\mathcal{M}}\} \times \mathcal{N}$ of the product $\mathcal{M} \times \mathcal{N}$;

(3) If $\mathcal{M} \cong \bigvee_n \overline{\mathbb{N}}$ is free or rank $n$, then the set of its $\mathbb{DF}_1$-submodules is a $\mathbb{DF}_1$-module isomorphic to $\prod_n \overline{\mathbb{N}}$. Indeed, notice that every submodule $\mathcal{N}$ of $\mathcal{M}$ is free and to characterise it it is enough to specify a minimal set of generators. Such a minimal set will have at most one element in every distinct copy of $\overline{\mathbb{N}}$, thus the isomorphism is given by sending each submodule in the $n$-tuple $(m_1, \ldots, m_n)$ where $m_i$ is equal to the generator of $\mathcal{N}$ lying in the $i$-th copy of $\overline{\mathbb{N}}$ if there is any, or $m_i = \infty$ otherwise. The trivial submodule $\star_{\mathcal{M}}$ corresponds to the distinguished element of $\prod_n \overline{\mathbb{N}}$ and $\mathcal{M}$ to the element $(0, \ldots, 0)$;

(4) Any $\mathcal{M}$ whose directed graph is a disjoint union of loops has $\mathcal{M}$ and $\star_{\mathcal{M}}$ as only $\mathbb{DF}_1$-submodules.

Notice that every $\mathbb{DF}_1$-submodule $\mathcal{N}$ of a given one defines a quotient by the congruence generated by the relations $\{n \sim \star_{\mathcal{M}}\}_{n \in \mathcal{N}}$. However not every congruence can be realised this way: let $\mathcal{M}$ be free of rank 2 with generators $x$ and $y$ and $\mathcal{N}$ the quotient of $\mathcal{M}$ by the congruence generated by $\mathrm{d}x \sim \mathrm{d}y$. Then $\mathcal{N}$ can be represented graphically as:

$$\begin{tikzcd} & x \arrow[dr] & & & & \\ \star \arrow[loop right] & & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \cdots \\ & y \arrow[ur] & & & & \end{tikzcd}$$

and it is straightforward that $\mathcal{N}$ cannot be obtained by collapsing a $\mathbb{DF}_1$-submodule of $\mathcal{M}$.

It can be proved fairly straightforwardly that finitely generated $\mathbb{DF}_1$-modules can be characterized in the following way:

**Lemma 5.1.9.** *A $\mathbb{DF}_1$-module is finitely generated if and only if its graph has finitely many connected components and finitely many roots i.e. elements not in the image of $d$.*

**Definition 5.1.10.** A differential $\mathbb{F}_1$-algebra (also $\mathbb{DF}_1$-algebra) is a tuple $(\mathcal{A}, 0_{\mathcal{A}}, \cdot, 1_{\mathcal{A}}, d_{\mathcal{A}})$ where $(\mathcal{A}, 0_{\mathcal{A}}, d_{\mathcal{A}})$ is a differential $\mathbb{F}_1$-module and $(\mathcal{A}, 0_{\mathcal{A}}, \cdot, 1_{\mathcal{A}})$ is an $\mathbb{F}_1$-algebra. Let us denote the category of differential $\mathbb{F}_1$-algebras as $\mathbf{D}\mathbb{F}_1\textbf{-Alg}$.

Equivalently, a $\mathbb{DF}_1$-algebra is an $\mathbb{F}_1$-algebra $(\mathcal{A}, 0_{\mathcal{A}}, \cdot, 1_{\mathcal{A}})$ with an action of $\mathbb{F}_1[d]$ making it into an $\mathbb{F}_1[d]$-set.

*Remark* 5.1.11. It is worth noticing that for any $a, b \in \mathcal{A}$, there is no relation required between the terms $d_{\mathcal{A}}(ab)$, $d_{\mathcal{A}}(a)b$ and $ad_{\mathcal{A}}(b)$ appearing in the usual Leibniz rule.

*Example* 5.1.12.

- The $D\mathbb{F}_1$-module $\overline{\mathbb{N}}$ is a $D\mathbb{F}_1$-algebra with respect to the usual sum of natural numbers, extended to $\infty$ by $n + \infty = \infty$ for all $n \in \overline{\mathbb{N}}$.
- Every $D\mathbb{F}_1$-algebra $A$ is trivially a $D\mathbb{F}_1$-algebra by endowing it with the differential mapping every element to $\star_A$.

Given two $D\mathbb{F}_1$-modules $\mathcal{M}$ and $\mathcal{N}$, the set $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$ has the structure of a $D\mathbb{F}_1$-module as well, with distinguished point $\star\colon m \mapsto \star_{\mathcal{N}}$ and differential d defined as:

$$\mathrm{d}f(m) := f(d_{\mathcal{M}}(m)) = d_{\mathcal{N}}(f(m))$$

When $\mathcal{M} = \mathcal{N}$, the identity map $id_{\mathcal{M}}$ and the differential $d_{\mathcal{M}}$ of $\mathcal{M}$ belong to $\mathrm{Hom}(\mathcal{M}, \mathcal{M})$. The operation of composition makes this set into a (in general non-commutative) $D\mathbb{F}_1$-algebra with multiplicative identity $id_{\mathcal{M}}$.

**Definition 5.1.13.** Let $F\colon \mathbb{F}_1\textbf{-Mod} \to \mathbb{F}_1\textbf{-Alg}$ be the free functor sending $(X, \star_X)$ into the free $\mathbb{F}_1$-algebra $(F(X), \star_X, \cdot, 1)$ generated by it.

**Definition 5.1.14.** Given a differential $\mathbb{F}_1$-module $(\mathcal{M}, d_{\mathcal{M}})$, let us define a sequence

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \dots$$

of differential $\mathbb{F}_1$-modules as:

$$\mathcal{M}_0 := (\mathcal{M}, d_{\mathcal{M}}) \quad \mathcal{M}_n := (\Delta F(\mathcal{M}_{n-1})/\langle \mathrm{d}m \sim d_{\mathcal{M}}m\rangle_{m\in\mathcal{M}}, \mathrm{d}) \text{ for all } n \geq 1$$

By definition we get an injective map of differential $\mathbb{F}_1$-modules $\mathcal{M}_n \hookrightarrow \mathcal{M}_{n+1}$ for every $n \in \mathbb{N}$. Let $\mathcal{M}_\infty = \bigcup_{n=0}^{\infty} \mathcal{M}_n$, it is a $D\mathbb{F}_1$-algebra.

A perhaps more concrete realisation of the $D\mathbb{F}_1$-algebra $\mathcal{M}_\infty$ can be given. Let us first establish some terminology:

**Definition 5.1.15.** A *forest* is a finite set $V$, called vertices, together with a parent map $P\colon V \to V$ such that for $n$ large enough $P^n$ sends every vertex to a fixed point of $P$. The fixed points of $P$ are called roots. The *valence* of a vertex $v \in V$ is the cardinality of $P^{-1}(v) \setminus \{v\}$ A forest with a single root will be called a *tree* and vertices of valence 0 will be called *leaves*.

Consider the monoid of isomorphism classes of trees with leaves labelled by elements of $\mathcal{M}$, with monoid structure given by joining two trees at the root and where the multiplicative identity is the tree consisting of a single root. The differential d is given by adding a segment to the stem of a tree. Given an element $m \in \mathcal{M}$, we denote by $[m]$ the tree consisting of the single leaf labelled by $m$. Denote as $\mathcal{F}(\mathbb{F}_1, \mathcal{M})$ the quotient of this monoid by the relations generated by $\{\mathrm{d}[m] \sim [d_{\mathcal{M}}m]\}_{m\in\mathcal{M}}$. Then the following holds:

**Lemma 5.1.16.** *The map sending $m \in \mathcal{M}$ to $[m] \in \mathcal{F}(\mathbb{F}_1, \mathcal{M})$ induces an isomorphism of $D\mathbb{F}_1$-algebras between $\mathcal{M}_\infty$ and $\mathcal{F}(\mathbb{F}_1, \mathcal{M})$.*

For any $n \in \mathbb{N}$, the isomorphism above gives the following isomorphism of $\mathrm{D}\mathbb{F}_1$-modules:

$$\mathcal{M}_n \cong \left\{ \begin{array}{c} \textit{classes of trees with at most n branching} \\ \textit{points on any path from the root to a leaf} \end{array} \right\} \subset \mathcal{F}(\mathbb{F}_1, \mathcal{M})$$

*Remark* 5.1.17. Notice that, from the definition of $\Delta$ and $F$, every tree with a leaf labelled by the distinguished point $\star_{\mathcal{M}}$ of $\mathcal{M}$ is identified with the empty tree, i.e. the distinguished point of $\mathcal{M}_\infty$.

**Definition 5.1.18.** Given a tree $t \in \mathcal{M}_\infty$, let the complexity of $t$

$$c(t) := \min\{n \in \mathbb{N} \mid t \in \mathcal{M}_n\}$$

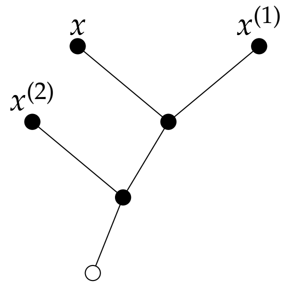
or, equivalently:

$$c(t) := \max\{\# \textit{ of branching points along paths from the root to a leaf in } t\}$$

By definition:

$$\mathcal{M}_n = \{t \in \mathcal{M}_\infty \mid c(t) \leq n\}$$

*Example* 5.1.19. Let $\mathcal{M} := \Delta X$ with $X$ as in Example 5.1.6; an example of an element of $\mathcal{M}_\infty$ is $\mathrm{d}(x^{(2)}\mathrm{d}(xx^{(1)}))$, which is equivalent to the following tree:




In particular, this is an element of $\mathcal{M}_2$.

**Proposition 5.1.20.** *The assignment* $\mathcal{M} \mapsto \mathcal{M}_\infty$ *is a free functor*

$$(-)_\infty \colon \mathbf{D}\mathbb{F}_1\textbf{-Mod} \to \mathbf{D}\mathbb{F}_1\textbf{-Alg}.$$

*Proof.* Firstly, let us prove it is a functor. Given a $\mathrm{D}\mathbb{F}_1$-module morphism $f \colon \mathcal{M} \to \mathcal{N}$ we define $f_\infty$ as the map sending a tree in $\mathcal{M}_\infty$ with $n$ leaves labelled $m_1, \ldots, m_n$ to the tree of the same shape in $\mathcal{N}_\infty$ with leaves labelled $f(m_1), \ldots, f(m_n)$. By construction and by definition of $\mathcal{M}_\infty$ and $\mathcal{N}_\infty$ the map $f_\infty$ commutes with the differentials. It is also straightforward to check that it is a morphism of $\mathbb{F}_1$-algebras. Identity and composition checks are trivial, thus $(-)_\infty$ is a functor.

Let us prove the freeness of $(-)_\infty$, by proving that the forgetful functor $U \colon \mathbf{D}\mathbb{F}_1\textbf{-Alg} \to \mathbf{D}\mathbb{F}_1\textbf{-Mod}$ is right adjoint to $(-)_\infty$.

Given a $\mathbb{DF}_1$-module $\mathcal{M}$, let $\eta_{\mathcal{M}}\colon \mathcal{M} \to U(\mathcal{M}_\infty)$ the identity morphism sending $m \in \mathcal{M}$ to $[m] \in \mathcal{M}_0 \subset \mathcal{M}_\infty$; this commutes with the differential. For any $\mathbb{DF}_1$-algebra $\mathcal{A}$ and morphism $g\colon \mathcal{M} \to U(\mathcal{A})$ in $\mathbf{D}\mathbb{F}_1\text{-}\mathbf{Mod}$, there exists a unique $f\colon \mathcal{M}_\infty \to \mathcal{A}$ making the following diagram commute:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\eta_{\mathcal{M}}"] \arrow[dr, "g"'] & U(\mathcal{M}_\infty) \arrow[d, "U(f)"] \\ & U(\mathcal{A}) \end{tikzcd}$$

given by sending a tree with leaves $m_1, \ldots, m_n$ to the tree of the same shape (meaning series of products and differentiations in $\mathcal{A}$) with elements $g(m_1), \ldots, g(m_n) \in \mathcal{A}$ grafted in the place of the respective $m_i$'s. This proves the required universal property. $\square$

## 5.2 Failure of the naive construction of Ritt polynomials over a differential semiring

We observe that the naive definition of a differential polynomial algebra $S\{x_1, \ldots, x_n\}$ over a differential semiring $(S, d_S)$ does not satisfy the expect universal property, i.e. morphisms of differential algebra from it to any D$S$-algebra are not determined by the image of the variables.

In particular we shall now see that it impossible to endow $S\{x_1, \ldots, x_n\}_{basic}$ with a differential (either strict or tropical) for which the analogous universal property holds. In fact, we show that there is no longer a unique choice of differential, and for any choice of differential the universal property fails. We will show in the next section that the idempotent semiring $S\{x_1, \ldots, x_n\}_{basic}$ can be enlarged to a tropical differential semiring $S\{x_1, \ldots, x_n\}$ enjoying the universal property that justifies calling it the tropical Ritt algebra.

*Remark* 5.2.1. Notice that if $S$ is an idempotent semiring satisfying the strict Leibniz rule, equipping the $S$-algebra $S\{x_1, \ldots, x_n\}_{basic}$ with a differential exactly as we have done for rings in Definition 3.1.2, gives a strict differential $S$-algebra as result (as noted in Section 3.2 for $S = \mathbb{B}[\![t]\!]$). This algebra satisfies an analogous universal property in the category of strict differential $S$-algebras as $R\{x_1, \ldots, x_n\}$ does in $\mathbf{D}R\text{-}\mathbf{Alg}$, but in general it does satisfy an analogous property in $\mathbf{D}S\text{-}\mathbf{Alg}$, see Proposition 5.2.3 below.

Let us now attempt to extend the differential of $S$ to $S\{x_1, \ldots, x_n\}_{basic}$. Obviously we would like to send $x_i^{(j)}$ to $x_i^{(j+1)}$, and we would like the map to be additive. The difficulty is in choosing how to extend it to arbitrary products. In contrast to the case of coefficients in a differential ring, the tropical Leibniz relations allow more freedom in extending a partially-defined differential to all products; there is not a uniquely determined extension of $d_S$ to a map $d$ on all of $S\{x_1, \ldots, x_n\}_{basic}$ satisfying the tropical Leibniz relations. In fact, $S\{x_1, \ldots, x_n\}_{basic}$ admits many distinct differentials.

*Example* 5.2.2. Suppose $w\colon \mathbb{N} \to S$ is a valuation. Then we can define a differential $d_w$ on $S\{x_1, \ldots, x_n\}_{basic}$ by the following rule. First, for a pure power $(x_i^{(j)})^k$ , we define

$$d_w(x_i^{(j)})^k = w(k)(x_i^{(j)})^{k-1} x_i^{(j+1)}$$

Then, we extend this to monomials $c(x_{i_1}^{(j_1)})^{k_1} \cdots (x_{i_m}^{(j_m)})^{k_m}$ as a strict derivation. E.g.,

$$d_w(cx_1^a(x_2^{(3)})^b) = d_S(c)x_1^a(x_2^{(3)})^b + cw(a)x_1^{a-1}x_1'(x_2^{(3)})^b + cw(b)(x_2^{(3)})^{b-1}x_2^{(4)}x_1^a.$$

It is straightforward to check that this map $d_w$ does indeed satisfy the tropical Leibniz relations.

One can generalize this example by choosing a distinct valuation $w_i$ for each variable $x_i$, defining the differential on pure powers by the rule

$$d(x_i^{(j)})^k = w_i(k)(x_i^{(j)})^{k-1} x_i^{(j+1)},$$

and then extending to arbitrary monomials using the strict Leibniz rule.

The above example shows that there is at least one differential on $S\{x_1, \ldots, x_n\}_{basic}$ for each $n$-tuple of valuations $\mathbb{N} \to S$.

**Proposition 5.2.3.** *There is no tropical differential on $S\{x_1, \ldots, x_n\}_{basic}$ that extends the tropical differential on $S$ and makes this the free object on $n$ generators.*

*Proof.* We use proof by contradiction. Suppose $d$ is such a differential, and let $w_1$ and $w_2$ be distinct valuations $\mathbb{N} \to S$. The identity map must be a morphism of differential idempotent semirings

$$(S\{x_1, \ldots, x_n\}_{basic}, d) \to (S\{x_1, \ldots, x_n\}_{basic}, d_{w_i}).$$

for $i = 1, 2$. This implies that $d = d_{w_1}$ and $d = d_{w_2}$ on pure powers. But this is a contradiction since $d_{w_1} \neq d_{w_2}$ as $w_1$ and $w_2$ are distinct valuations. $\square$

## 5.3 Ritt polynomial algebras over a differential (semi)ring

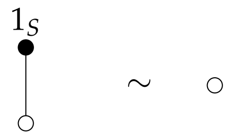
The aim of this section is to construct two free functors $S\{-\}\colon \mathbf{D}\mathbb{F}_1\textbf{-Mod} \to \mathbf{D}S\textbf{-Alg}$ and $(-)_S\colon \mathbf{D}\mathbb{F}_1\textbf{-Alg} \to \mathbf{D}S\textbf{-Alg}$ for any differential (semi)ring $(S, d_S)$, and to prove they give rise to the following commutative diagram:

$$\begin{tikzcd}
\mathbf{D}\mathbb{F}_1\textbf{-Mod} \arrow[rr, "(-)_\infty"] \arrow[dr, "S\{-\}"'] & & \mathbf{D}\mathbb{F}_1\textbf{-Alg} \arrow[dl, "(-)_S"] \\
& \mathbf{D}S\textbf{-Alg} &
\end{tikzcd}$$

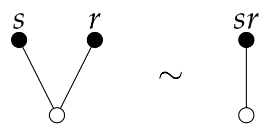
We start by the definition of the functor $S\{-\}\colon \mathbf{D}\mathbb{F}_1\textbf{-Mod} \to \mathbf{D}S\textbf{-Alg}$. This will give a generalisation of Ritt's construction, allowing coefficients in differential semirings and products of elements of an arbitrary differential $\mathbb{F}_1$-module as differential monomials. This construction is the general case of the one introduced in [GM21], Section 3.3, where only free $\mathrm{D}\mathbb{F}_1$-modules were considered.

Let $DM\colon \mathbf{D}S\textbf{-Alg} \to \mathbf{D}\mathbb{F}_1\textbf{-Mod}$ be the forgetful functor sending a differential $S$-algebra $(A, d_A)$ to the $\mathrm{D}\mathbb{F}_1$-module $(A, 0_A, d_A)$, forgetting about its $S$-algebra structure. Given a $\mathrm{D}\mathbb{F}_1$-module $(\mathcal{M}, d_{\mathcal{M}})$, consider the monoid of forests consisting of trees in $(DM(S) \vee \mathcal{M})_\infty$, where the monoid structure is given by disjoint union. We will denote as $\mathcal{F}(S, \mathcal{M})$ the quotient obtained from this monoid by the differential congruence generated by the following relations:

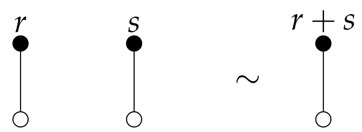
(1) The tree $[1_S]$ with a single leaf labelled by the unit of $S$ is equivalent to the identity of $(DM(S) \vee \mathcal{M})_\infty$, i.e. the tree consisting of the root only:




(2) Given any two elements $s, r \in S$, the tree $[s][r]$ is equivalent to the tree $[sr]$:




(3) Given any two elements $s, r \in S$, the disjoint union of the trees $[s]$ and $[r]$ is equivalent to the tree $[r + s]$:




From now on we need to distinguish between the case of differential rings and differential semirings, as the definition of the differential algebra generated by a $\mathrm{D}\mathbb{F}_1$-module $\mathcal{M}$ involves quotienting by Leibniz relations that will differ in the two cases. In the case of a differential ring $R$, it is clear that $\mathcal{F}(R, \mathcal{M})$ is an $R$-algebra as $R$ sits inside it as a subalgebra, and its product is inherited from the product of $(DM(R) \vee \mathcal{M})_\infty$ acting distributively on the disjoint union. Furthermore, $\mathcal{F}(R, \mathcal{M})$ comes with a linear map $d$, by linear extension of the differential of $(DM(R) \vee \mathcal{M})_\infty$. In order to make $(\mathcal{F}(R, \mathcal{M}), d)$ into a differential $R$-algebra, the differential $d$ is required to satisfy the Leibniz rule, thus:

**Definition 5.3.1.** Let $L$ be the differential ideal generated by the set

$$\{d(st) - sdt - tds \mid s, t \in \mathcal{F}(R, \mathcal{M})\}$$

i.e. the smallest ideal in $\mathcal{F}(R, \mathcal{M})$ containing this set and closed under applying $d$. We define $R\{\mathcal{M}\}$ as $\mathcal{F}(R, \mathcal{M})/L$. Then $R\{\mathcal{M}\}$ is a differential $R$-algebra, by definition of $L$.

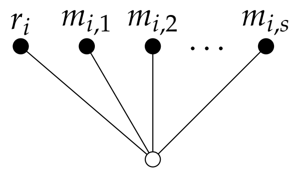
The following Proposition clarifies the $R$-algebra structure of $R\{\mathcal{M}\}$:

**Proposition 5.3.2.** *Given any differential ring $R$ and $\mathbb{DF}_1$-module $\mathcal{M}$, $R\{\mathcal{M}\}$ is isomorphic to $R[F(\mathcal{M})]$ as an $R$-algebra.*

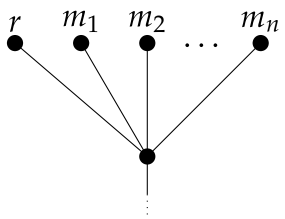
*Remark* 5.3.3. In particular, when $\mathcal{M}$ is free of rank $n$, the $R$-algebra $R[F(\mathcal{M})]$ underlying $R\{\mathcal{M}\}$ is isomorphic to $R\{x_1, \ldots, x_n\}_{basic}$, the $R$-algebra underlying the differential algebra of Ritt polynomials in $n$ variables $R\{x_1, \ldots, x_n\}$. In this case, this isomorphism extends to an isomorphism of differential $R$-algebras between $R\{\mathcal{M}\}$ and $R\{x_1, \ldots, x_n\}$. Thus the construction discussed here, taking a $\mathbb{DF}_1$-module as input, generalises Ritt's construction.

*Proof of Proposition 5.3.2.* Let $\iota$ be the homomorphism of $R$-algebras $R[F(\mathcal{M})] \to R\{\mathcal{M}\}$ given by:

$$\sum_i r_i \prod_j m_{i,j} \quad \longmapsto \quad \bigsqcup_i$$



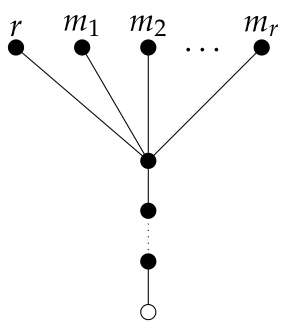

Clearly it is injective and we want to prove it is also surjective. Let the complexity of the class $T$ of a tree in $\mathcal{F}(R, \mathcal{M})$ to be the minimum of the complexity $c(t)$ of the elements of its class, where $c(t)$ is defined as in Definiton 5.1.18. Thanks to the relations imposed to get $\mathcal{F}(R, \mathcal{M})$, in any class $T$ of $\mathcal{F}(R, \mathcal{M})$ of positive complexity, there is a representative $t$ whose apical branchings have the form:




with $r \in R$ and $m_i \in \mathcal{M}$. For this representative we have $c(t) = c(T)$. Let $p(T) := |\{\text{paths in } t \text{ where } c(T) \text{ is achieved}\}|$.

We want to prove by induction on $c(T)$ and $p(T)$ that any class in $R\{\mathcal{M}\}$ is equivalent to an element in the image of $\iota$, thus any forest is as well.

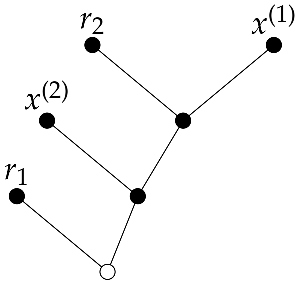
For $c(T) = 1$, the minimal representative $t$ is a product of trees of the form:



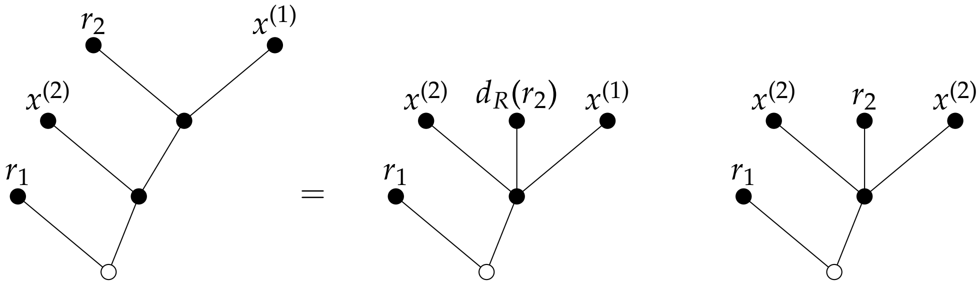
Thanks to linearity and Leibniz rule, every tree of this form is equivalent to an element in $R[F(\mathcal{M})]$. If $c(T) = 1$, either we have that $T$ is equivalent to an element of $R[F(\mathcal{M})]$ either $p(T)$ decreased by 1.

In general, if $c(T) \geq 2$, using the Leibniz relations, $T$ is equivalent to a sum of classes $\sum_{i=0}^{m} T_i$ such that either $c(T_i) \leq c(T) - 1$ or $p(T_i) \leq p(T) - 1$ for every $i$. Thus we get the thesis by inductive hypothesis. $\square$

*Example* 5.3.4. Let $\mathcal{M}$ be free of rank 1 with generator $x$ and consider the element $t := r_1\mathrm{d}(x^{(2)}\mathrm{d}(r_2x^{(1)}))$ in $R\{\mathcal{M}\} = R\{x\}$. The element $t$ is minimal in its class and can be represented as the following tree:



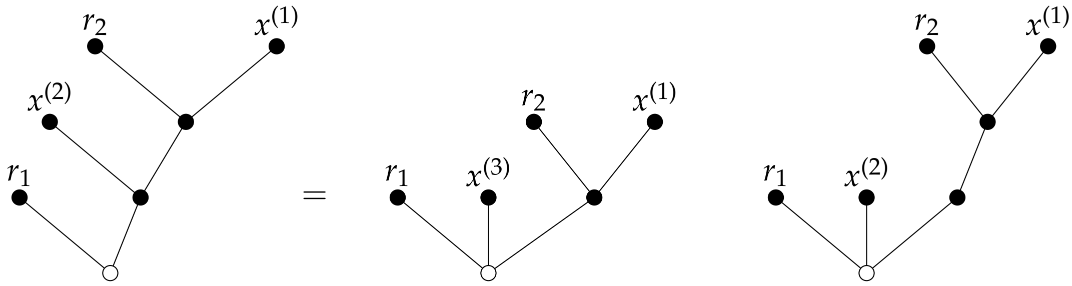

Its complexity is $c(t) = 2$ and, using the relations imposed to define $R\{\mathcal{M}\}$, as in the proof of Proposition 5.3.2, we have the following equality:




where the complexity dropped by 1. By the same relations again we obtain that $t$ is equivalent to an element in $R[F(\mathcal{M})]$, namely:

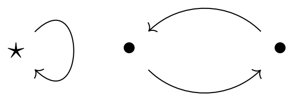
$$r_1x^{(1)}(d_R^2(r_2)x^{(2)} + d_R(r_2)x^{(3)}) + 2r_1d_R(r_2)x^{(2)}(x^{(2)} + x^{(3)})$$

It is worth noticing that there is another way to reduce $t$ to get an element in $R[F(\mathcal{M})]$: indeed since the ideal $\mathcal{L}$ is a differential ideal, the following equality holds in $R\{\mathcal{M}\}$ too:



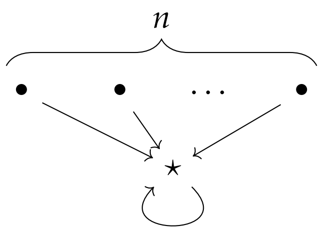
By applying again the Leibniz rule, it is easy to prove that this sum reduces to the element $r_1 x^{(1)}(d_R^2(r_2)x^{(2)} + d_R(r_2)x^{(3)}) + 2r_1 d_R(r_2)x^{(2)}(x^{(2)} + x^{(3)})$ of $R[F(\mathcal{M})]$ as in the previous case.

*Example* 5.3.5. (1) Let $\mathcal{M}$ be the $\mathbb{DF}_1$-module:



Notice that $\mathcal{M}$ is isomorphic to the quotient of the free $\mathbb{DF}_1$-module of rank 1 by the differential congruence $\langle x^{(2)} \sim x \rangle$. Thus the differential $R$-algebra $R\{\mathcal{M}\}$ is isomorphic to the quotient of the free differential algebra $R\{x\}$ by the differential ideal generated by the linear polynomial $x - x^{(2)}$.

(2) Let $\mathcal{M}$ be the $\mathbb{DF}_1$-module:




then, arguing in a similar way as the previous example, $R\{\mathcal{M}\}$ is isomorphic to the quotient of the Ritt algebra $R\{x_1, \ldots, x_n\}$ by the differential ideal generated by the elements $x_1', \ldots, x_n'$. This gives an isomorphism

$$(R\{\mathcal{M}\}, d) \cong (R[x_1, \ldots, x_n], d = 0)$$

Thus ordinary polynomial algebras can be recovered from this construction as well.

Let us now move to the construction of $S\{\mathcal{M}\}$ when $S$ is a differential idempotent semiring. As in the previous case we consider the commutative $S$-algebra $\mathcal{F}(S, \mathcal{M})$. Let

$$L_{\text{trop}} = \{d(st) + sdt + tds \mid s, t \in \mathcal{F}(S, \mathcal{M})\}$$

then:

**Definition 5.3.6.** Given a differential idempotent semiring $S$ and a $\mathbb{DF}_1$-module $\mathcal{M}$ define the differential $S$-algebra $S\{\mathcal{M}\}$ as the quotient:

$$S\{\mathcal{M}\} := \mathcal{F}(S, \mathcal{M}) / \mathcal{B}\langle L_{\text{trop}} \rangle$$

by the bend relations of the differential ideal generated by $L_{\text{trop}}$.

*Remark* 5.3.7. In contrast to what happens for rings, as proved in Proposition 5.3.2, in general there is just an injective homomorphism of $S$-algebras $S[F(\mathcal{M})] \hookrightarrow S\{\mathcal{M}\}$ with analogous definition as for differential rings. When $\mathcal{M}$ is a free $\mathbb{DF}_1$-module of rank $n$ the $S$-algebra $S[F(\mathcal{M})]$ is the algebra $S\{x_1, \ldots, x_n\}_{basic}$.

We finally prove the functoriality and freenes of the assignment $\mathcal{M} \mapsto S\{\mathcal{M}\}$ for a differential (semi)ring $S$:

**Proposition 5.3.8.** *Given a differential (semi)ring $S$, the assignment $\mathcal{M} \mapsto S\{\mathcal{M}\}$ is a free functor*

$$S\{-\}\colon \mathbf{D\mathbb{F}_1\text{-Mod}} \to \mathbf{D}S\mathbf{\text{-Alg}}.$$

*Proof.* Firstly, let us prove $S\{-\}$ is a functor. Given a morphism of $\mathrm{D}\mathbb{F}_1$-modules $f\colon \mathcal{M} \to \mathcal{N}$, the function $S\{f\}$ sends a forest with leaves labelled by $s_1, \ldots, s_n$ and $m_1, \ldots m_r$ to the same forest with leaves labelled by elements $s_1, \ldots, s_n$ and $f(m_1), \ldots f(m_r)$. It is straightforward to verify that this assignment is well defined and is an homomorphism of differential $S$-algebras $S\{\mathcal{M}\} \to S\{\mathcal{N}\}$. The identity morphism is sent to the identity morphism and composition is preserved, thus $S\{-\}$ is a functor.

Let us now prove this functor is left adjoint to the forgetful functor $DM\colon \mathbf{D}S\mathbf{\text{-Alg}} \to \mathbf{D\mathbb{F}_1\text{-Mod}}$. Given a $\mathrm{D}\mathbb{F}_1$-module $\mathcal{M}$, let $\eta_{\mathcal{M}}\colon \mathcal{M} \to DM(S\{\mathcal{M}\})$ the morphism sending $m \in \mathcal{M}$ to $[m] \in S\{\mathcal{M}\}$. It commutes with the differentials thanks to the definition of $(-)_\infty$. For any differential $S$-algebra $A$ and morphism $g\colon \mathcal{M} \to DM(A)$ in $\mathbf{D\mathbb{F}_1\text{-Mod}}$, there exists a unique $f\colon S\{\mathcal{M}\} \to A$ in $\mathbf{D}S\mathbf{\text{-Alg}}$ making the following diagram commute:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\eta_{\mathcal{M}}"] \arrow[dr, "g"'] & DM(S\{\mathcal{M}\}) \arrow[d, "DM(f)"] \\
& DM(A)
\end{tikzcd}$$

given by the map $\mathcal{F}(S, \mathcal{M}) \to A$ sending a forest with leaves labelled by $s_1, \ldots, s_n$ and $m_1, \ldots m_r$ to the same forest (meaning sum of successive products and differentiations in $A$) with leaves labelled by $s_1, \ldots, s_n$ and $g(m_1), \ldots g(m_r)$. Since $A$ is a differential $S$-algebra, the ideal $L$ (resp. the differential congruence $L_{\text{trop}}$) is contained in the kernel of $f$, making it into a well defined map $S\{\mathcal{M}\} \to A$. This proves the required universal property. $\square$

The freeness of $S\{-\}$ is equivalent to say that for every $\mathrm{D}\mathbb{F}_1$-module $\mathcal{M}$ and for every $\mathrm{D}S$-algebra $T$ there is a bijection

$$\Psi\colon \operatorname{Hom}_{\mathbf{D}S\mathbf{\text{-Alg}}}(S\{\mathcal{M}\}, T) \cong \operatorname{Hom}_{\mathbf{D\mathbb{F}_1\text{-Mod}}}(\mathcal{M}, DM(T)).$$

When $\mathcal{M}$ is free of rank $n$, any morphism of $\mathrm{D}\mathbb{F}_1$-modules $\mathcal{M} \to \mathcal{N}$ is determined by the image of the generators of $\mathcal{M}$ in $\mathcal{N}$, thus we get a bijection of sets

$$\operatorname{Hom}_{\mathbf{D\mathbb{F}_1\text{-Mod}}}(\mathcal{M}, \mathcal{N}) \cong \mathcal{N}^n.$$

In particular, when $\mathcal{N} = DM(T)$ this tells us that the tropical Ritt algebra $S\{\mathcal{M}\} = S\{x_1, \ldots, x_n\}$ enjoys a universal property in the category of differential idempotent semirings that is entirely analogous to the universal property of the classical Ritt algebra in the category of differential rings, i.e.:

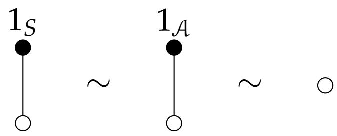
**Corollary 5.3.9.** *Given a differential $S$-algebra $T$, there is a bijection*

$$\mathrm{Hom}_{\mathbf{D}S\text{-}\mathbf{Alg}}(S\{x_1, \ldots, x_n\}, T) \cong (T)^n$$

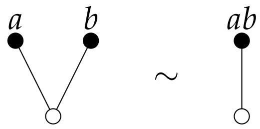
*implemented by sending a homomorphism $\varphi$ to the $n$-tuple $(\varphi(x_1), \ldots, \varphi(x_n))$.*

We move now to the definition of the functor $(-)_S\colon$ $\mathbf{D}\mathbb{F}_1\textbf{-Alg} \to \mathbf{D}S\textbf{-Alg}$. Given a $\mathrm{D}\mathbb{F}_1$-algebra $\mathcal{A}$ we proceed in a similar way as in the case of $S\{-\}$, by considering the monoid of forests of trees in $(DM(S) \vee U(\mathcal{A}))_\infty$ and by defining its quotient $\mathcal{F}(S, \mathcal{A})$ by the differential congruence generated by relations as in (2) and (3) above plus additional relations:

(1 bis) The trees $[1_S]$ and $[1_\mathcal{A}]$ are equivalent to the identity of $(DM(S) \vee U(\mathcal{A}))_\infty$, i.e. the tree consisting of the root only:



(4) Given any two elements $a, b \in \mathcal{A}$, the tree $[a][b]$ is equivalent to the tree $[ab]$:



As above we distinguish the case of a differential ring and a differential semiring. Analogously $\mathcal{F}(R, \mathcal{A})$ is an $R$-algebra with product inherited from the product of $(DM(R) \vee U(\mathcal{A}))_\infty$ acting distributively on the disjoint union and it is equipped with a linear map $d$, by linear extension of the differential of $(DM(R) \vee U(\mathcal{A}))_\infty$. In order to make $(\mathcal{F}(R, \mathcal{A}), d)$ into a differential $R$-algebra we proceed by imposing the Leibniz relations for $d$:

**Definition 5.3.10.** Let $L$ be the differential ideal generated by the set

$$\{d(st) - sdt - tds \mid s, t \in \mathcal{F}(R, \mathcal{A})\}$$

We define $\mathcal{A}_R$ as $\mathcal{F}(R, \mathcal{A})/L$. Then $\mathcal{A}_R$ is a differential $R$-algebra, by definition of $L$.

*Remark* 5.3.11. In this case, an analogous statement to Proposition 5.3.2 does not hold: there is no isomorphism of $R$-algebras between $R[\mathcal{A}]$ and $\mathcal{A}_R$. The kernel of the homomorphism of $R$-algebras $\psi\colon$ $R[\mathcal{A}] \to \mathcal{A}_R$ sending every element $a \in \mathcal{A}$ to its class in $\mathcal{A}_R$ is the ideal generated by the set:

$$\{d_\mathcal{A}(a_1a_2) - a_1d_\mathcal{A}(a_2) - a_2d_\mathcal{A}(a_1) \mid a_1, a_2 \in \mathcal{A}\}$$

thus $\psi$ is not injective.

When $S$ is a differential idempotent semiring, $\mathcal{F}(S, \mathcal{A})$ is a commutative $S$-algebra. As for $\mathrm{D}\mathbb{F}_1$-modules, let

$$L_{\text{trop}} = \{d(st) + sdt + tds \mid s, t \in \mathcal{F}(S, \mathcal{A})\}$$

and we define:

**Definition 5.3.12.** Given a differential idempotent semiring $S$ and a $\mathrm{D}\mathbb{F}_1$-module $\mathcal{A}$, the differential $S$-algebra $\mathcal{A}_S$ as the quotient:

$$\mathcal{A}_S := \mathcal{F}(S, \mathcal{A})/\mathcal{B}\langle L_{\text{trop}}\rangle$$

As in the case of $\mathrm{D}\mathbb{F}_1$-modules, let $DA\colon \mathbf{D}S\textbf{-Alg} \to \mathbf{D}\mathbb{F}_1\textbf{-Alg}$ be the forgetful functor assigning to a differential $S$-algebra $(A, d_A)$ the $\mathrm{D}\mathbb{F}_1$-algebra $(A, 0_A, \cdot_A, 1_A, d_A)$.

**Proposition 5.3.13.** *Given a differential (semi)ring $S$, the assignment $\mathcal{A} \to \mathcal{A}_S$ is a free functor*

$$(-)_S\colon \mathbf{D}\mathbb{F}_1\textbf{-Alg} \to \mathbf{D}S\textbf{-Alg}.$$

*Proof.* Given a morphism of differential $\mathbb{F}_1$-algebras $f\colon \mathcal{A} \to \mathcal{B}$, the morphism $(id_S \vee f)_\infty$, induces a well defined morphism of differential $S$-algebras $f_S\colon \mathcal{A}_S \to \mathcal{B}_S$, with analogous action to the one describe in the proof of Proposition 5.3.8. It is straightforward to prove that identity and composition are preserved.

Let us now prove this functor is left adjoint to the forgetful functor $DA\colon \mathbf{D}S\textbf{-Alg} \to \mathbf{D}\mathbb{F}_1\textbf{-Alg}$. Given a differential $\mathbb{F}_1$-algebra $\mathcal{A}$, let $\eta_\mathcal{A}\colon \mathcal{A} \to DA(\mathcal{A}_S)$ the morphism sending $m \in \mathcal{A}$ to $[m] \in \mathcal{A}_S$, it commutes with the differentials. For any differential $S$-algebra $B$ and morphism $g\colon \mathcal{A} \to DA(B)$ in $\mathbf{D}\mathbb{F}_1\textbf{-Alg}$, there exists a unique $f\colon \mathcal{A}_S \to B$ in $\mathbf{D}S\textbf{-Alg}$ making the following diagram commute:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\eta_\mathcal{A}"] \arrow[dr, "g"'] & DA(\mathcal{A}_S) \arrow[d, "DA(f)"] \\ & DA(B) \end{tikzcd}$$

The definition of $f$ is analogous to that in the proof of Proposition 5.3.8. This prove the freeness of the functor $(-)_S$. $\square$

**Proposition 5.3.14.** *The following diagram of free functors commutes:*

$$\begin{tikzcd} \mathbf{D}\mathbb{F}_1\textbf{-Mod} \arrow[rr, "(-)_\infty"] \arrow[dr, "S\{-\}"'] & & \mathbf{D}\mathbb{F}_1\textbf{-Alg} \arrow[dl, "(-)_S"] \\ & \mathbf{D}S\textbf{-Alg} & \end{tikzcd}$$

*Proof.* Given a $D\mathbb{F}_1$-module $\mathcal{M}$, forests of $(\mathcal{M}_\infty)_S$ have leaves labelled by elements of $S$ and $\mathcal{M}_\infty$, and elements of the latter are trees with leaves labelled by elements of $\mathcal{M}$. So elements of $(\mathcal{M}_\infty)_S$ are elements of $S\{\mathcal{M}\}$ and viceversa every element of $S\{\mathcal{M}\}$ can be seen as an element of $(\mathcal{M}_\infty)_S$. It is easy to check that these assignments are well defined with respect to the quotients imposed to define $\mathcal{F}(S, \mathcal{M}_\infty)$ and $\mathcal{F}(S, \mathcal{M})$ and with respect to the quotient by the Leibniz ideal (respectively tropical Leibniz congruence), and that they are morphisms of differential $S$-algebras inverse to each other. $\square$

*Remark* 5.3.15. If $R$ is a differential ring, from the commutativity of the diagram in Proposition 5.3.14 and from Proposition 5.3.2 it follows that when $\mathcal{A}$ is a free $D\mathbb{F}_1$-algebra, i.e. $\mathcal{A}$ is of the form $\mathcal{M}_\infty$ for some $D\mathbb{F}_1$-module $\mathcal{M}$, we get an isomorphism of $R$-algebras:

$$\mathcal{A}_R \cong R\{\mathcal{M}\} \cong R[F(\mathcal{M})]$$

and the map $\psi$ of Remark 5.3.11 gives an isomorphism between $R[\mathcal{M}_\infty]/\ker\psi$ and $R[F(\mathcal{M})]$.

# Chapter 6

# Tropical pairs

In the classical world, a differential equation over a differential ring $R$ is an element $f \in R\{x_1, \dots, x_n\}$, and a solution to $f$ in a differential $R$-algebra $A$ is an element $p \in A^n$ such that $f(p) = 0$. Equivalently, $p$ is a solution if the corresponding homomorphism $p\colon R\{x_1, \dots, x_n\} \to A$ factors through the quotient $R\{x_1, \dots, x_n\}/(f)$.

In the tropical world, we have introduced differential idempotent semirings, but these objects on their own are not sufficient to describe solutions to tropical differential equations. A tropical differential equation over a differential idempotent semiring $S$ is an element $f \in S\{x_1, \dots, x_n\}$ (where this is the tropical Ritt algebra defined in the previous section). Solutions to this differential equation will live in $S^n$, but asking that $f$ vanish or tropically vanish at $p \in S^n$ turns out to be too restrictive. Following the idea of Grigoriev's framework, $p$ should be considered a solution to $f$ if $f$ tropically vanishes at *$p$ to leading order* (rather than to all orders). This suggests that we must equip our differential idempotent semirings with something like a valuation that provides a way of measuring the leading order of elements. To this end, we will now define and study the category of tropical pairs, morphisms of idempotent semirings whose domain is a differential semiring.

We start in Section 6.1 by introducing them and giving a collection of examples that will be useful in the following of this work. We introduce a way to build a Ritt differential polynomial algebra over a pair with coefficients in a $\mathbb{D}\mathbb{F}_1$-module and we prove that, as in the case of differential semirings, this is a free functor. Then, in Section 6.2 we introduce the notion of reduction of a pair and prove its functoriality and that the subcategory of reduced pairs is reflective. In Section 6.3 we finally introduce tropical differential equations and prove Theorem 6.3.5, giving a bijection between solutions to a set of tropical differential equations in a pair $\mathbf{T}$ and morphisms of pairs between a quotient pair, given by bend relations, and the pair $\mathbf{T}$. We conclude with Example 6.3.6, that shows that this new setting gives stricly more refined information that the one studied by Grigoriev. In the following Section 6.4 we study the adjunction properties of the functors sending a pair to its top and bottom parts, for future use in Chapter 7. Section 6.5 is devoted to the introduction of the notion of differential enhancement, which will be fundamental to introduce a tropicalization functor in the differential setting. A differential enhancement of a valuation is an additional data that allows to preserve information about taking derivative while moving from the classical to the tropical world. In Example 6.5.5 we study the case of a $p$-adic differential enhancement that allows us to tropicalize $p$-adic differential equations, extending the theory of Grigoriev to encompass this case. Finally, in Section 6.6 we introduce and explore a differential notion of Berkovich analytification as the space of differential enhancements compatible with a given one.

## 6.1 The categories of tropical pairs and of S-algebras

A *tropical pair* $\mathbf{S}$ consists of a tropical differential semiring $S_1$, an idempotent semiring $S_0$, and a homomorphism of idempotent semirings $\pi\colon\ S_1 \to S_0$.

*Remark* 6.1.1. We think of $S_1$ as a space of functions, and we think of $S_0$ as a space of leading exponents of the series expansions of these functions. The map $\pi$, like the usual valuation on Puiseux series, sends a function to its leading exponent.

In category theoretic terms, if

$$F\colon\ \mathbf{DSemiRings} \to \mathbf{SemiRings}$$

is the forgetful functor from differential idempotent semirings to idempotent semirings, then the category of pairs is the simply the comma category ($F \downarrow$ **SemiRings**). Explicitly, a morphism of pairs $\varphi$ from $(S_1 \to S_0)$ to $(T_1 \to T_0)$ is a commutative diagram of idempotent semirings

$$\begin{array}{ccc} S_1 & \xrightarrow{\varphi_1} & T_1 \\ \downarrow & & \downarrow \\ S_0 & \xrightarrow{\varphi_0} & T_0 \end{array}$$

in which the upper horizontal arrow $\varphi_1$ is a morphism of differential idempotent semirings.

A pair $(S_1 \xrightarrow{\pi} S_0)$ is said to be *reduced* if $S_1$ admits no nontrivial quotient differential idempotent semiring over $S_0$; i.e., it is reduced if there is no nontrivial differential congruence contained in the congruence $\ker(\pi)$.

*Example* 6.1.2. (1) For any morphism of idempotent semirings $\mu\colon\ S \to T$ we have a pair

$$(S, d = 0) \to T,$$

and it is reduced if and only if $\mu$ is injective. If $\mu$ is not injective, then we can replace $S$ with $\operatorname{Im}(\mu)$ to obtain a reduced pair.

(2) Endow $\mathbb{B}[\![t]\!]$ with the differential $t^n \mapsto t^{n-1}$ and consider the homomorphism

$$\pi\colon\ \mathbb{B}[\![t]\!] \to \mathbb{T}$$

defined by $t^n \mapsto n$, as in section 3.1. This is a pair, and it is reduced by the following argument. Suppose $a \neq b \in \mathbb{B}[\![t]\!]$. If $a = \sum a_i t^i$ and $b = \sum b_i t^i$, then there exists a minimal $n$ such that $a_n \neq b_n$. It then follows that $\pi(d^n a) \neq \pi(d^n b)$, and so $(d^n(a), d^n(b)) \notin \ker(\pi)$. Hence any non-trivial differential congruence containing $(a, b)$ is not contained in $\ker(\pi)$.

(3) Consider $\mathbb{T}[\![t]\!]_0$ defined as in point (3) of Example 4.1.4 and

$$\pi' \colon \mathbb{T}[\![t]\!]_0 \to \mathbb{T}$$

sending a tropical power series to its leading exponent, as in the previous item. The pair $\pi'$ is not reduced as the support map $\varphi \colon \mathbb{T}[\![t]\!] \to \mathbb{B}[\![t]\!]$, which coefficientwise sends any $r \in \mathbb{R}$ to 0 and $\infty$ to $\infty$, is a nontrivial quotient differential semiring of $\mathbb{T}[\![t]\!]_0$ and $\pi' = \pi \circ \varphi$.

(4) Consider

$$\pi \colon \mathbb{T}[\![t]\!] \to \mathbb{T}_2,$$

where the source has any of the differentials from Example 4.1.4 and the morphism $\pi$ is given by

$$(a_{n_0} t^{n_0} \oplus a_{n_1} t^{n_1} \oplus \cdots) \mapsto (n_0, a_{n_0}).$$

This is a pair, and a modification of the argument of point (2) above shows that it is also reduced.

We let $\mathrm{Pairs}_{\mathrm{red}}$ denote the full subcategory of reduced pairs. We will show below in Section 6.2 that $\mathrm{Pairs}_{\mathrm{red}}$ is a reflective subcategory, and so any pair $\mathbf{S} = (S_1 \to S_0)$ has a functorial reduction $\mathbf{S}^{red} = (S_1^{red} \to S_0)$.

Finally we are ready to define the category that will describe tropical differential equations and their solutions.

**Definition 6.1.3.** Given a reduced pair $\mathbf{S}$, an $\mathbf{S}$-algebra is a reduced pair under $\mathbf{S}$, and we let **S-Alg** denote the category of $\mathbf{S}$-algebras.

An important example of an $\mathbf{S}$-algebras comes from the construction illustrated in Section 5.3. Given a pair $\mathbf{S} = (S_1 \to S_0)$ and a $\mathbb{DF}_1$-module $\mathcal{M}$, we first define an idempotent semiring $(S_0|S_1)\{\mathcal{M}\}$ by taking the pushout:

$$\begin{tikzcd} S_1 \arrow[r] \arrow[d] & S_1\{\mathcal{M}\} \arrow[d, dotted] \\ S_0 \arrow[r, dotted] & (S_0|S_1)\{\mathcal{M}\} \end{tikzcd}$$

in the category of semirings. This pushout can be described explicitly as the algebra of trees $S_1\{\mathcal{M}\}$ modulo the congruence generated by the relations that identify $a, b \in S_1 \subset S_1\{\mathcal{M}\}$ if they have the same image in $S_0$; i.e. leaves incident at the root labelled by elements of $S_1$ are replaced by leaves labelled by their image in $S_0$. Note that $(S_0|S_1)\{\mathcal{M}\}$ contains the polynomial $S_0$-algebra $S_0[F(\mathcal{M})]$.

The right vertical arrow in the above diagram gives an $\mathbf{S}$-algebra that we will denote by $\mathbf{S}\{\mathcal{M}\}$. These pairs will play the role of tropical Ritt algebras in the category of $\mathbf{S}$-algebras.

**Proposition 6.1.4.** *Given a reduced pair* $\mathbf{S} = (S_1 \to S_0)$*, the assignment* $\mathcal{M} \mapsto \mathbf{S}\{\mathcal{M}\}$ *is a free functor*

$$\mathbf{S}\{-\} \colon \mathbf{D}\mathbb{F}_1\text{-}\mathbf{Mod} \to \mathbf{S}\text{-}\mathbf{Alg}.$$

*Proof.* The functoriality of $\mathbf{S}\{-\}$ descends from Proposition 5.3.8 and from the properties of pushouts.

Let us now prove this functor is left adjoint to the forgetful functor $V\colon \mathbf{S}\text{-}\mathbf{Alg} \to \mathbf{D}\mathbb{F}_1\text{-}\mathbf{Mod}$ sending a pair $(Y_1 \to Y_0)$ to $DM(Y_1)$. Given a $\mathrm{D}\mathbb{F}_1$-module $\mathcal{M}$, let $\eta_{\mathcal{M}}\colon \mathcal{M} \to V(\mathbf{S}\{\mathcal{M}\}) = DM(S_1\{\mathcal{M}\})$ the morphism sending $m \in \mathcal{M}$ to $[m] \in S_1\{\mathcal{M}\}$ as in the proof of Proposition 5.3.8. For any $\mathbf{S}$-algebra $\mathbf{T} = (T_1 \to T_0)$ and morphism $g\colon \mathcal{M} \to V(\mathbf{T}) = DM(T_1)$ in $\mathbf{D}\mathbb{F}_1\text{-}\mathbf{Mod}$, from Proposition 5.3.8, there exists a unique $f_1\colon S_1\{\mathcal{M}\} \to T_1$ in $\mathbf{D}S_1\text{-}\mathbf{Alg}$ making the following diagram commute:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\eta_{\mathcal{M}}"] \arrow[dr, "g"'] & DM(S_1\{\mathcal{M}\}) \arrow[d, "DM(f_1)"] \\ & DM(T_1) \end{tikzcd} \tag{6.1.1}$$

From the universal property of the pushout, $f_1$ induces an arrow

$$f_0\colon (S_0|S_1)\{\mathcal{M}\} \to T_0$$

such that $f := (f_0, f_1)\colon \mathbf{S}\{\mathcal{M}\} \to \mathbf{T}$ is a morphism of $\mathbf{S}$-algebras. This morphism is unique since $f_1$ is unique and it makes the following diagram commute:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\eta_{\mathcal{M}}"] \arrow[dr, "g"'] & V(\mathbf{S}\{\mathcal{M}\}) \arrow[d, "V(f)"] \\ & V(\mathbf{T}) \end{tikzcd}$$

as this diagram is the same as the previous one. This proves that $\mathbf{S}\{-\}$ is a free functor. $\square$

Similarly to what we said above for $S\{-\}$, from the freeness of $\mathbf{S}\{-\}$, for every $\mathrm{D}\mathbb{F}_1$-module $\mathcal{M}$ and for every $\mathbf{S}$-algebra $\mathbf{T}$ there is a bijection

$$\Phi\colon \mathrm{Hom}_{\mathbf{S}\text{-}\mathbf{Alg}}(\mathbf{S}\{\mathcal{M}\}, \mathbf{T}) \cong \mathrm{Hom}_{\mathbf{D}\mathbb{F}_1\text{-}\mathbf{Mod}}(\mathcal{M}, V(\mathbf{T}))$$

which, when $\mathcal{M}$ is free of rank $n$, tells us that the $\mathbf{S}$-algebra $\mathbf{S}\{x_1, \dots, x_n\}$ enjoys a universal property in $\mathbf{S}\text{-}\mathbf{Alg}$ analogous again to the universal property of the classical Ritt algebra in the category of differential rings. Explicitly:

**Corollary 6.1.5.** *Let* $\mathbf{S} = (S_1 \to S_0)$ *be a reduced pair and* $\mathbf{T} = (T_1 \to T_0)$ *an* $\mathbf{S}$*-algebra. There is a bijection*

$$\mathrm{Hom}_{\mathbf{S}\text{-}\mathbf{Alg}}(\mathbf{S}\{x_1, \dots, x_n\}, \mathbf{T}) \cong (T_1)^n$$

*implemented by sending a morphism* $\varphi = (\varphi_1, \varphi_0)$ *to* $(\varphi_1(x_1), \dots \varphi_1(x_n))$*.*

*Remark* 6.1.6. Let $\mathbf{S} = (S_1 \to S_0)$ be a strict pair, i.e. a pair such that $S_1$ is a strict differential semiring. We have seen in Remark 5.2.1 that the strict differential $S$-algebra $(S_1\{x_1, \ldots, x_n\}_{basic}, d)$ satisfies a universal property, analogous to that of the Ritt algebra over a ring, in the category of strict differential $S_1$-algebras. Similarly, the pair resulting from the pushout of the diagram

$$\begin{tikzcd} S_1 \arrow[r] \arrow[d] & (S_1\{x_1, \ldots, x_n\}_{basic}, d) \\ S_0 & \end{tikzcd}$$

satisfies a universal property analogous to that of Corollary 6.1.5 in the category of strict $\mathbf{S}$-algebras, but in general not in $\mathbf{S}$-$\mathbf{Alg}$. In particular, when $\mathbf{S} = (\mathbb{B}[\![t]\!] \to \mathbb{T})$ as in the Grigoriev setting of Section 3.2, the pushout above gives us the strict pair

$$(\mathbb{B}[\![t]\!]\{x_1, \ldots, x_n\}_{basic}, d) \to \mathbb{T}\{x_1, \ldots, x_n\}_{basic}$$

appearing in Section 3.2.

## 6.2 The reduction functor

Given a pair $\mathbf{S} = (S_1 \xrightarrow{\pi} S_0)$, it follows from Proposition 4.2.1 that the set of differential congruences contained in $\ker \pi$ has a unique maximal element $R(\pi)$, and hence the pair $\mathbf{S}^{red} := (S_1/R(\pi) \to S_0)$ is reduced.

**Proposition 6.2.1.** *The morphism $\mathbf{S} \to \mathbf{S}^{red}$ is initial among morphisms from $\mathbf{S}$ to reduced pairs.*

*Proof.* Suppose $\mathbf{T} = (T_1 \xrightarrow{\psi} T_0)$ is a reduced pair and

$$\begin{tikzcd} S_1 \arrow[r, "\varphi_1"] \arrow[d, "\pi"'] & T_1 \arrow[d, "\psi"] \\ S_0 \arrow[r, "\varphi_0"] & T_0 \end{tikzcd}$$

is a morphism of pairs. There are inclusions

$$R(\pi) \subset \ker \pi \subset \ker \pi \circ \varphi_0.$$

The map $\varphi_1$ sends $\ker \pi \circ \varphi_0$ into $\ker \psi$, and the image of a differential congruence by a homomorphism of tropical differential semirings is again a differential congruence, so $\varphi_1$ must send $R(\pi)$ to a differential congruence contained in $\ker \psi$. Since $\mathbf{T}$ is reduced, the only such differential congruence on $T_1$ is the diagonal, and so $\varphi_1$ factors uniquely through the quotient map $S_1 \to S_1/R(\pi)$. $\square$

We now show that the above reduction construction exhibits $\mathbf{Pairs}_{\mathrm{red}}$ as a reflective subcategory of $\mathbf{Pairs}$.

**Proposition 6.2.2.** *Sending* $\mathbf{S}$ *to* $\mathbf{S}^{red}$ *defines a functor*

$$\mathscr{R}\colon\ \mathbf{Pairs} \to \mathbf{Pairs}_{\mathrm{red}}$$

*and the quotient map* $\mathbf{S} \to \mathbf{S}^{red}$ *is a natural transformation* $\mathrm{Id} \to \mathscr{R}$*. Moreover* $\mathscr{R}$ *is left adjoint to the inclusion* $\iota\colon\ \mathbf{Pairs}_{\mathrm{red}} \hookrightarrow \mathbf{Pairs}$*.*

*Proof.* Suppose $f\colon\ \mathbf{S} \to \mathbf{T}$ is a morphism of pairs and consider the composition $\mathbf{S} \to \mathbf{T} \to \mathbf{T}^{red}$. By Proposition 6.2.1, there is a unique factorization $\mathbf{S} \to \mathbf{S}^{red} \to \mathbf{T}^{red}$, and hence we obtain a morphism $\mathscr{R}(f)\colon\ \mathbf{S}^{red} \to \mathbf{T}^{red}$. It is straightforward the check that this respects compositions: $\mathscr{R}(f \circ g) = \mathscr{R}(f) \circ \mathscr{R}(g)$. Hence $\mathscr{R}$ is a functor.

It is a straightforward verification that the quotient map $\mathbf{S} \to \mathbf{S}^{red}$ defines a natural transformation from the identity on Pairs to $\iota \circ \mathscr{R}$. Clearly if $\mathbf{S}$ is reduced, then $\mathbf{S}^{red} = \mathbf{S}$, and there is trivially a natural transformation from $\mathscr{R} \circ \iota$ to the identity on $\mathbf{Pairs}_{\mathrm{red}}$. It is now elementary to check that these two natural transformations give the claimed adjunction. □

As a consequence of reduction being a left adjoint functor, it commutes with colimits.

## 6.3 Tropical differential equations and their solutions

In this section we firstly introduce quotients of pairs and tropical differential equations and a notion of solution for such an equation. We then prove that solutions to tropical differential equations in an $\mathbf{S}$-algebra $\mathbf{T}$ correspond to morphisms of $\mathbf{S}$-algebras between a quotient of $\mathbf{S}\{\mathcal{M}\}$ and $\mathbf{T}$.

Let us get started by introducing the definition of a quotient of a pair.

**Definition 6.3.1.** Let $\mathbf{S} = (S_1 \to S_0)$ be a pair. A quotient of $\mathbf{S}$ is a morphism of pairs

$$\begin{array}{ccc} S_1 & \xrightarrow{\varphi_1} & T_1 \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ S_0 & \xrightarrow{\varphi_0} & T_0 \end{array}$$

such that both $\varphi_1$ and $\varphi_0$ are surjective.

Notice that the kernel of $\varphi_1$ is a differential congruence $\ker \varphi_1$ on $S_1$, the kernel of $\varphi_0$ is a congruence $\ker \varphi_0$ on $S_0$, and $\pi$ sends $\ker \varphi_1$ into $\ker \varphi_0$. Conversely, a pair of congruences $(K_1 \subset S_1 \times S_1, K_0 \subset S_0 \times S_0)$ satisfying $\pi(K_1) \subset K_0$ defines a quotient of $\mathbf{S}$.

We now describe an important class of quotients. Suppose we are given a pair $\mathbf{S} = (S_1 \to S_0)$, a $\mathbb{DF}_1$-module $\mathcal{M}$ and a congruence $K$ on the polynomial semiring

$S_0[F(\mathcal{M})]$. By a slight abuse of notation, let $(S_0|S_1)\{\mathcal{M}\}/K$ denote the induced quotient, and then let

$$\mathbf{S}\{\mathcal{M}\}/\!\!/K$$

denote the reduction of the pair $S_1\{\mathcal{M}\} \to (S_0|S_1)\{\mathcal{M}\}/K$. Quotients of this form will be used when we define the tropicalization of a system of differential equations in Section 7.1.

**Proposition 6.3.2.** *Let $\mathbf{T} = (T_1 \to T_0)$ be a reduced $\mathbf{S}$-algebra, then $\mathrm{Hom}_{\mathbf{S\text{-}Alg}}(\mathbf{S}\{\mathcal{M}\}/\!\!/K, \mathbf{T})$ is in bijection with the set:*

$$\{f \in \mathrm{Hom}_{\mathbf{D\mathbb{F}_1\text{-}Mod}}(\mathcal{M}, T_1) \mid \Phi(f)_0 \textit{ factors through } (S_0|S_1)\{\mathcal{M}\}/K\}$$

*where $\Phi$ is the* Hom*-set bijection induced by the freeness of $\mathbf{S}\{-\}$, as in Section 6.1.*

*Proof.* As in Proposition 6.2.2 we proved that the reduction functor is left adjoint to the inclusion functor $\iota\colon \mathbf{Pairs}_{\mathrm{red}} \hookrightarrow \mathbf{Pairs}$, applying $\mathscr{R}$ gives a bijection between the set of morphisms of (unreduced) $\mathbf{S}$-algebras from $S_1\{\mathcal{M}\} \to (S_0|S_1)\{\mathcal{M}\}/K$ to $\mathbf{T}$ and $\mathrm{Hom}_{\mathbf{S\text{-}Alg}}(\mathbf{S}\{\mathcal{M}\}/\!\!/K, \mathbf{T})$. From Proposition 6.1.4, we have a bijection

$$\Phi\colon \mathrm{Hom}_{\mathbf{D\mathbb{F}_1\text{-}Mod}}(\mathcal{M}, T_1) \cong \mathrm{Hom}_{\mathbf{S\text{-}Alg}}(\mathbf{S}\{\mathcal{M}\}, \mathbf{T})$$

and from the observation at the beginning of the proof we get the thesis. $\square$

When $\mathcal{M}$ is free of rank $n$, Proposition 6.3.2 above translates to the following:

**Corollary 6.3.3.** *Let $\mathbf{T} = (T_1 \xrightarrow{\pi} T_0)$ be a reduced $\mathbf{S}$-algebra and $K$ a congruence on $S_0\{x_1, \ldots, x_n\}_{basic}$. Morphisms of $\mathbf{S}$-algebras*

$$\mathbf{S}\{x_1, \ldots, x_n\}/\!\!/K \to \mathbf{T}$$

*correspond bijectively with $n$-tuples $y_1, \ldots, y_n \in T_1$ such that the elements $\pi(d^j y_i) \in T_0$ define an $S_0$-algebra homomorphism $S_0\{x_1, \ldots, x_n\}_{basic}/K \to T_0$.*

Let $\mathbf{S}$ be a reduced pair and $\mathbf{T} = (T_1 \xrightarrow{\pi} T_0)$ an $\mathbf{S}$-algebra. In order to introduce tropical differential equations and their solutions we consider at first the case of a free $\mathbb{D}\mathbb{F}_1$-module of rank $n$. In this case a tropical differential equation is a polynomial $f \in S_0\{x_1, \ldots, x_n\}_{basic}$. Let us write $f = \sum_\alpha f_\alpha x^\alpha$, where $x^\alpha$ runs over the differential monomials in $f$. If $x^\alpha$ has any factors of the form $d^n x_i$ for $n > 0$ then it does not make sense to evaluate $x^\alpha$ at an element $c \in T_0^n$ because $T_0$ is not a differential semiring. However, we can evaluate $x^\alpha$ at an element $C \in T_1^n$ and then push down to $T_0$ via $\pi$. Thus we can evaluate $f \in S_0\{x_1, \ldots, x_n\}_{basic}$ at $C \in T_1^n$ by the expression

$$f(C) = \bigoplus_\alpha f_\alpha \pi(C^\alpha).$$

and the set of solutions in $\mathbf{T}$ of a differential polynomial $f = \sum_\alpha f_\alpha x^\alpha \in S_0\{x_1, \ldots, x_n\}_{basic}$ is the subset of $T_1^n$ consisting of all elements $C = (C_1, \ldots, C_n)$ such that the sum

$$\bigoplus_\alpha f_\alpha \pi(C^\alpha)$$

tropically vanishes.

When the pair $S_1 \to S_0$ is $\mathbb{B}[\![t]\!] \xrightarrow{\pi} \mathbb{T}$, the above definition recovers Grigoriev's framework. A subset $W \subset \mathbb{N}$ corresponds to the boolean formal power series $\sum_{i \in W} t^i$, and Grigoriev's map $\mathrm{Val}_W(j)$ is precisely $\pi(d^j W)$, as highlighted in the table at the end of Section 3.1.

Fix a reduced pair $\mathbf{S} = (S_1 \xrightarrow{\pi} S_0)$ and a $D\mathbb{F}_1$-module $\mathcal{M}$. In view of Corollary 6.1.5, Corollary 6.3.3 and Proposition 6.3.2, the appropriate general definitions of tropical differential equation and solution are the following: a tropical differential equation is a polynomial $f \in S_0[F(\mathcal{M})]$ and:

**Definition 6.3.4.** Let $\mathbf{T} = (T_1 \to T_0)$ be an $\mathbf{S}$-algebra. The set of solutions in $\mathbf{T}$ of a differential polynomial $f \in S_0[F(\mathcal{M})]$, denoted $\mathrm{Sol}_{\mathbf{T}}(f)$, is the set of morphisms $p \in \mathrm{Hom}_{\mathbf{D\mathbb{F}_1\text{-}Mod}}(\mathcal{M}, T_1)$ such that $\Phi(p)_0$ factors through $(S_0|S_1)\{\mathcal{M}\}/\mathcal{B}(f)$.

Given $E \subset S_0[F(\mathcal{M})]$, it follows directly from Proposition 6.3.2 and the definition of solutions to tropical differential equations that $\mathbf{S}$-algebra morphisms

$$\mathbf{S}\{\mathcal{M}\} /\!\!/ \mathcal{B}(E) \to \mathbf{S}$$

are in bijection with the solution set $\mathrm{Sol}_{\mathbf{S}}(E)$. More in general, we have:

**Theorem 6.3.5.** *The functor* **S-Alg** $\to$ **Sets** *sending an* $\mathbf{S}$*-algebra* $\mathbf{T}$ *to* $\mathrm{Sol}_{\mathbf{T}}(E)$ *is corepresented by* $\mathbf{S}\{\mathcal{M}\} /\!\!/ \mathcal{B}(E)$.

We see from the following example that the general framework we just introduced provides more information than Grigoriev's one.

*Example* 6.3.6. Consider the pair $\mathbf{S} = \mathbb{T}[\![t]\!] \xrightarrow{\pi} \mathbb{T}_2$, where $\mathbb{T}[\![t]\!]$ has the differential from Example 4.1.4 part (4) corresponding to the 2-adic valuation, $d(t^n) = v_2(n)t^{n-1}$. Over this pair we consider solutions to the differential equation

$$f = (4,0)x + (0,-3)x' + (1,-3)x'' \in \mathbb{T}_2\{x\}_{basic}.$$

Let us look for solutions of the form

$$x = 0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \cdots .$$

We have

$$\begin{aligned} x' &= a_1 + (a_2+1)t + a_3 t^2 + (a_4+2)t^3 + a_5 t^4 + \cdots \\ x'' &= (a_2+1) + (a_3+1)t + (a_4+2)t^2 + (a_5+2)t^3 + \cdots . \end{aligned}$$

If $a_1 \neq \infty$ then

$$\pi(x) = (0,0), \quad \pi(x') = (0,a_1), \quad \pi(x'') = (0,a_2+1),$$

and so evaluating $f$ at $x$ gives the expression

$$\begin{aligned} f(x) &= (4,0)(0,0) \oplus (0,-3)(0,a_1) \oplus (1,-3)(0,a_2+1) \\ &= (4,0) \oplus (0,a_1-3) \oplus (1,a_2-2). \end{aligned}$$

The maximum occurs only in the middle term, so there is no solution with $a_1 \neq \infty$.

Assuming next that $a_1 = \infty$ and $a_2 \neq \infty$, we have

$$\pi(x) = (0,0), \quad \pi(x') = (1, a_2 + 1), \quad \pi(x'') = (0, a_2 + 1),$$

and

$$\begin{aligned} f(x) &= (4,0)(0,0) \oplus (0,-3)(1, a_2+1) \oplus (1,-3)(0, a_2+1) \\ &= (4,0) \oplus (1, a_2 - 2) \oplus (1, a_2 - 2). \end{aligned}$$

The second and third terms are equal and maximal, so this is a solution for any finite value of $a_2$.

If $a_1 = a_2 = \infty$ and $a_3 \neq \infty$, then

$$\pi(x) = (0,0), \quad \pi(x') = (2, a_3), \quad \pi(x'') = (1, a_3 + 1),$$

and $f(x) = (4,0) \oplus (2, a_3 - 3) \oplus (2, a_3 - 2)$. The middle term is the sole maximal term, so this is not a solution.

If $a_1 = a_2 = a_3 = \infty$ and $a_4 \neq \infty$ then

$$\pi(x) = (0,0), \quad \pi(x') = (3, a_4 + 2), \quad \pi(x'') = (2, a_4 + 2),$$

and $f(x) = (4,0) \oplus (3, a_4 - 1) \oplus (3, a_4 - 1)$, so we have a solution since the second and third terms are jointly maximal.

The last case we will look at is $a_1 = a_2 = a_3 = a_4 = \infty$ and $a_5 \neq \infty$. Now

$$\pi(x) = (0,0), \quad \pi(x') = (4, a_5), \quad \pi(x'') = (3, a_5 + 2),$$

and $f(x) = (4,0) \oplus (4, a_5 - 3) \oplus (4, a_5 - 1)$.

If $a_5 = 3$, then the first two terms are jointly maximal and we have a solution but when $a_5 \neq 3$ either the first or second term is the sole maximum. In this case we see for the first time that the tropical framework here provides additional information about solutions beyond the information contained in Grigoriev's framework.

## 6.4 Colimits of pairs

In this section we show that colimits in the category of pairs can be computed by computing the colimits of the top and bottom individually. In order for this to be useful, it is helpful to note the following.

**Proposition 6.4.1.** *The categories of idempotent semirings and differential idempotent semirings are cocomplete.*

*Proof.* The category of idempotent semirings is cocomplete for the same reason as the category of rings; one can easily check that arbitrary coproducts and coequalizers exist. For differential idempotent semirings, one must only verify that tropical differentials $d_i$ on $S_i$ induce a tropical differential on the coproduct $\bigoplus_i S_i$, and likewise for coequalizers. Both of these verifications are elementary and straightforward. $\square$

**Proposition 6.4.2.** *The forgetful functors*

$$\begin{tikzcd} & \textbf{Pairs} \arrow[dl, "\pi_b"'] \arrow[dr, "\pi_t"] & \\ \textbf{SemiRings} & & \textbf{DSemiRings} \end{tikzcd}$$

*sending, respectively, a pair to its bottom and to its top, commute with colimits, and $\pi_t$ also commutes with limits.*

*Proof.* It suffices to show that $\pi_b$ admits a right adjoint and $\pi_t$ admits both a left and a right adjoint.

We start with $\pi_t$. Let

$$L_t \colon \textbf{DSemiRings} \to \textbf{Pairs}$$

be the functor sending a differential idempotent semiring $S$ to the pair $S \xrightarrow{id} S$, and let $R_t$ be the functor sending $S$ to the pair $S \to *$, where $*$ denotes the trivial idempotent semiring consisting of a single element. Given a pair $A \xrightarrow{p} B$, a morphism of differential idempotent semirings $f \colon S \to A$ uniquely determines, and is uniquely determined by, a morphism of pairs

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "id"'] & A \arrow[d, "p"] \\ S \arrow[r, "p \circ f"] & B \end{tikzcd}$$

that is evidently natural in the semiring $S$ and the pair $A \to B$. Thus $L_t$ is left adjoint to $\pi_t$. For $R_t$, observe that a morphism of differential semirings $f \colon A \to S$ is equivalent to a morphism of pairs:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & S \arrow[d] \\ B \arrow[r] & * \end{tikzcd}$$

For $\pi_b$, we will construct a right adjoint $R_b$. Consider the subcategory

$$\textbf{Pairs}_T \subset \textbf{Pairs}$$

of pairs $S \to T$, where a morphism is a morphism of pairs that is the identity on $T$. The colimit $\operatorname{colim}_{\textbf{Pairs}_T} \pi_t$ comes with a natural semiring homomorphism to $T$, and this defines a pair $R_b(T)$. It is straightforward to verify that $R_b(T)$ is functorial in $T$. A morphism of pairs

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p"'] & \mathrm{colim}_{\mathbf{Pairs}_T} \pi_t \arrow[d] \\
B \arrow[r] & T
\end{tikzcd}$$

clearly provides a semiring homomorphism $B \to T$. Conversely, given a semiring homomorphism $B \to T$, the composition $A \to B \to T$ is an object of $\mathbf{Pairs}_T$ and hence it has a canonical map to $R_b(T)$. $\square$

Finally, note that since the reduction functor is idempotent and has a left adjoint (Proposition 6.2.2), the colimit of a diagram of reduced pairs is reduced.

## 6.5 Differential enhancements of valuations

Given a valuation $v$ on $R$ as in Definition 2.4.1, $v(r)$ does not in general determine the valuation of derivatives of $r$. In order to define the tropicalization of solutions to differential equations, we must enhance the seminorm with some additional information in order to determine the valuation of sequences $r, dr, d^2r, \ldots$. For this reason, we now introduce differential enhancements of valuations.

**Definition 6.5.1.** Given a differential ring $R$ and a valuation $v\colon R \to S_0$, a *differential enhancement of* $v$ is a reduced pair $\mathbf{S} = (S_1 \to S_0)$ and a map of sets $\widetilde{v}\colon R \to S_1$ such that

(1) $\widetilde{v}(0) = 0 \in S_1$ and $\widetilde{v}(1) = 1 \in S_1$;

(2) it commutes with the differentials: $d_{S_1}\widetilde{v}(x) = \widetilde{v}(d_R x)$ for any $x \in R$;

(3) the following diagram commutes:

$$\begin{tikzcd}
 & S_1 \arrow[d] \\
R \arrow[ur, "\widetilde{v}"] \arrow[r, "v"] & S_0
\end{tikzcd}$$

We will use the term *differentially enhanced valuation* $\mathbf{v} = (v, \widetilde{v})\colon A \to \mathbf{S}$ to mean a seminorm $v$ together with a differential enhancement $\widetilde{v}$.

*Remark* 6.5.2. Notice that the map $\widetilde{v}\colon R \to S_1$ is a morphism of $\mathbb{D}\mathbb{F}_1$-modules.

If $(v, \widetilde{v})\colon A \to \mathbf{S}$ is a differentially enhanced valuation, $\mathbf{T}$ is a reduced pair and $(\varphi_0, \varphi_1)\colon \mathbf{S} \to \mathbf{T}$ is a morphism of pairs, then the composition

$$\begin{tikzcd}
 & S_1 \arrow[r, "\varphi_1"] \arrow[d] & T_1 \arrow[d] \\
R \arrow[ur, "\widetilde{v}"] \arrow[r, "v"] & S_0 \arrow[r, "\varphi_0"] & T_0
\end{tikzcd}$$

is also a differentially enhanced valuation.

*Example* 6.5.3. Given a valued ring $v : R \to T$, endowing both $R$ and $T$ with the constant differentials $0_R$ and $0_T$ respectively, the valuation $v$ admits a trivial differential enhancement:

$$\begin{tikzcd} & T \arrow[d, "id_T"] \\ R \arrow[ur, "v"] \arrow[r, "v"] & T \end{tikzcd}$$

*Example* 6.5.4. Let $K$ be a field and consider the differential ring of formal power series $K[\![t]\!]$ with differential $d/dt$. The $t$-adic valuation $K[\![t]\!] \to \mathbb{T}$ admits a differential enhancement

$$\begin{tikzcd} & \mathbb{B}[\![t]\!] \arrow[d] \\ K[\![t]\!] \arrow[ur, "\widetilde{v}"] \arrow[r, "v"] & \mathbb{T} \end{tikzcd}$$

which is the one considered in Section 3.2, in which the map $\mathbb{B}[\![t]\!] \to \mathbb{T}$ sends a boolean power series $t^n + \cdots$ to $n$. Note that while $v$ is multiplicative, its differential enhancement $\widetilde{v}$ is not, as notice in Section 3.1.

This is the differentially enhanced valuation used by Grigoriev [Gri17] in his framework and subsequent works [AGT16], [CGL20] and [FT20].

*Example* 6.5.5. Consider the $p$-adic valuation $v_p\colon \mathbb{Q} \to \mathbb{T}$ and extend this to a valuation $\mathbb{Q}[\![t]\!] \to \mathbb{T}_2$ as in Remark 2.4.3. This admits a differential enhancement

$$\begin{tikzcd} & \mathbb{T}[\![t]\!] \arrow[d] \\ \mathbb{Q}[\![t]\!] \arrow[ur, "\widetilde{u}"] \arrow[r, "u"] & \mathbb{T}_2 \end{tikzcd}$$

where the differential on $\mathbb{T}[\![t]\!]$ is by (4.1.4), and the vertical arrow sends $a_0 t^{n_0} \oplus \cdots$ to $(n_0, a_0)$. Let $\mathbf{u} = (u, \widetilde{u})$. There is a morphism of pairs

$$\begin{tikzcd} \mathbb{T}[\![t]\!] \arrow[r] \arrow[d] & \mathbb{B}[\![t]\!] \arrow[d] \\ \mathbb{T}_2 \arrow[r] & \mathbb{T} \end{tikzcd}$$

given on the top by the sending all finite coefficients to 0, and on the bottom by projection onto the first component. This morphism of pairs sends differentially enhanced valuation $\mathbf{u}$ to the $\mathbf{v}$ of Example 6.5.4. Thus $\mathbf{u}$ provides a refinement of the structure considered by Grigoriev.

In a differential ring $R$, an element $a \in R$ is said to be a *constant* if $d(a) = 0$. The constants form a subring of $R$.

**Proposition 6.5.6.** *Given a differentially enhanced valuation* $\mathbf{v} = (\widetilde{v}\colon R \to S_1, v\colon R \to S_0)$ *on* $R$, $\widetilde{v}$ *restricts to a valuation on the subring of constants in* $R$.

*Proof.* Suppose $a, b$ are constants and consider the semiring congruence $K$ on $S_1$ generated by the relations

$$\begin{aligned} \widetilde{v}(0) &\sim 0_{S_1} \\ \widetilde{v}(1) &\sim \widetilde{v}(-1) \\ \widetilde{v}(ab) &\sim \widetilde{v}(a)\widetilde{v}(b) \\ \widetilde{v}(a+b) \oplus \widetilde{v}(a) \oplus \widetilde{v}(b) &\sim \widetilde{v}(a) \oplus \widetilde{v}(b). \end{aligned}$$

Since $v$ is a valuation, the relations $v(a+b) \oplus v(a) \oplus v(b) = v(a) \oplus v(b)$ and $v(ab) = v(a)v(b)$ hold in $S_0$, and hence the semiring homomorphism $S_1 \to S_0$ factors through the quotient semiring $S_1/K$ because each of the generators of $K$ (as a semiring congruence) is a relation that holds in $S_0$. Since $\widetilde{v}$ commutes with the differentials, $\widetilde{v}(1)$, $\widetilde{v}(-1)$, $\widetilde{v}(a)$, $\widetilde{v}(b)$, $\widetilde{v}(a+b)$ and $\widetilde{v}(ab)$ are each constants in $S_1$. From this we see that $K$ is in fact a congruence of differential semirings. If $K$ were nontrivial then the factorization $S_1 \to S_1/K \to S_0$ would contradict the fact that $S_1 \to S_0$ is reduced. Thus the equalities

$$\begin{aligned} \widetilde{v}(0) &= 0_{S_1} \\ \widetilde{v}(1) &= v(-1) \\ \widetilde{v}(a)\widetilde{v}(b) &= \widetilde{v}(ab), \\ \widetilde{v}(a+b) \oplus \widetilde{v}(a) \oplus \widetilde{v}(b) &= \widetilde{v}(a) \oplus \widetilde{v}(b) \end{aligned}$$

must hold in $S_1$. □

## 6.6 The differential Berkovich space

We now propose a generalization to the differential setting of the notion of Berkovich analytification introduced in Section 2.2. Suppose $(R, d)$ is a differential ring equipped with a differentially enhanced valuation $\mathbf{v}$ to $\mathbf{S} = (S_1 \to S_0)$, and let $A$ be a differential $R$-algebra. Given an $\mathbf{S}$-algebra $\mathbf{T}$, a differentially enhanced valuation $\mathbf{w} = (\widetilde{w}, w)$ is said to be compatible with $\mathbf{v}$ if the diagram

$$\begin{tikzcd} & S_1 \arrow[r] \arrow[d] & T_1 \arrow[dd] \\ A \arrow[ur, "\widetilde{v}"] \arrow[urr, "\widetilde{w}"'] \arrow[dr, "v"'] \arrow[drr, "w"] & S_0 \arrow[r] & T_0 \end{tikzcd} \tag{6.6.1}$$

commutes.

**Definition 6.6.1.** Given an $\mathbf{S}$-algebra $\mathbf{T}$, the *differential Berkovich space over* $\mathbf{T}$ of $A$, denoted $Berk_{\mathbf{T}}(A)$, is the set of differentially enhanced valuations $\mathbf{w}\colon A \to \mathbf{T}$ that are compatible with $\mathbf{v}$.

With the notation introduced in Section 2.4, recall that

$$\mathcal{T}rop_{univ}(\mathrm{Spec}\, A)(T_0) = \{w : A \to T_0 \mid w \text{ is a valuation compatible with } v\}$$

Then, note that there is a natural map

$$\Lambda\colon\ Berk_{\mathbf{T}}(A) \to \mathcal{T}rop_{univ}(\mathrm{Spec}\, A)(T_0)$$

induced by sending a differentially enhanced valuation $\mathbf{w} = (\widetilde{w}, w)$ to its underlying ordinary valuation $w$.

**Lemma 6.6.2.** *Given a valued differential ring $v : R \to S_0$, fix a reduced pair $\mathbf{S} : S_1 \xrightarrow{\pi} S_0$ and let $\widetilde{v}$ and $\widetilde{v}'$ be two maps $R \to S_1$ such that both $(\mathbf{S}, \widetilde{v})$ and $(\mathbf{S}, \widetilde{v}')$ are differential enhancements of $v$. Then $\widetilde{v} = \widetilde{v}'$.*

*Proof.* Let us assume $\widetilde{v} \neq \widetilde{v}'$ and let $a \in R$ an element such that $\widetilde{v}(a) \neq \widetilde{v}'(a)$. As both $(\mathbf{S}, \widetilde{v})$ and $(\mathbf{S}, \widetilde{v}')$ are differential enhancements of $v$, we have $\pi(\widetilde{v}(a)) = v(a) = \pi(\widetilde{v}'(a))$, thus $(\widetilde{v}(a), \widetilde{v}'(a)) \in \ker \pi$. Furthermore, for the same reason and since $\widetilde{v}$ and $\widetilde{v}'$ commute with the differential, we have $(d^n(\widetilde{v}(a)), d^n(\widetilde{v}'(a))) \in \ker \pi$ for every $n \in \mathbb{N}$. As $\pi$ is a reduced pair, this implies $d^n(\widetilde{v}(a)) = d^n(\widetilde{v}'(a))$ for every $n$. So $\widetilde{v}(a) = \widetilde{v}'(a)$ which is a contradiction, thus $\widetilde{v} = \widetilde{v}'$. □

It follows directly from the above Lemma that the map $\Lambda$ is injective:

**Proposition 6.6.3.** *Let $A$ be a differential $R$-algebra and let $(v, \widetilde{v}) : R \to \mathbf{S}$ a differentially enhanced valuation. Then for every $\mathbf{S}$-algebra $\mathbf{T} = (T_1 \to T_0)$ the natural map*

$$\Lambda : (w, \widetilde{w}) \mapsto w$$

*from the differential Berkovich space $Berk_{\mathbf{T}}(A)$ to the $T_0$-points of the (classical) universal tropicalization $\mathcal{T}rop_{univ}(\mathrm{Spec}\, A)(T_0)$ of* Spec $A$ *is injective.*

*Remark* 6.6.4. If $\mathbf{S}$ is not reduced Lemma 6.6.2 does not hold. Indeed, consider the non-reduced pair $\mathbb{T}[\![t]\!]_0 \xrightarrow{\pi} \mathbb{T}$ as in Example 6.1.2 part (3), sending a tropical power series to its leading exponent. Then for a field $K$, let $R = (K[[s]])[[t]]$ equipped with the $d/dt$ differential and let $v_t$ (resp. $v_s$) the $t$-adic (resp. $s$-adic) valuation on $R$ and $v : K[[s]] \to \mathbb{B}$ the trivial valuation. It is possible to complete the following diagram:

$$\begin{array}{ccc} & & \mathbb{T}[\![t]\!]_0 \\ & & \big\downarrow \pi \\ R & \xrightarrow{\ v_t\ } & \mathbb{T} \end{array}$$

to a commutative diagram with a differential map $R \to \mathbb{T}[\![t]\!]_0$ in two different ways by the two maps given by coefficientwise application of $v$ and $v_s$.

Notice that for every $\mathbf{S}$-algebra $\mathbf{T}$ with $T_0 = \mathbb{T}$, the map $\Lambda$ gives an inclusion $Berk_{\mathbf{T}}(A) \hookrightarrow (\mathrm{Spec}\, A)^{an}$ of the differential Berkovich space over $\mathbf{T}$ into the Berkovich analytification of Spec $A$. In the following we give examples of the possible mutual behaviour of the differential Berkovich space $Berk_{\mathbf{T}}(A)$ and the classical one.

*Example* 6.6.5. In this first example we will see the extreme case in which the differential Berkovich space and the classical one are in bijection. Let $v\colon K \to \mathbb{T}$ be a valued field equipped with the trivial differential $d = 0$, and consider the trivial differential enhancement

$$\begin{tikzcd} & (\mathbb{T}, d = \infty) \arrow[d, "id_{\mathbb{T}}"] \\ K \arrow[ur, "v"] \arrow[r, "v"] & \mathbb{T} \end{tikzcd}$$

of Example 6.5.3. Given the differential algebra $(K[x], d = 0)$ over $K$, the Berkovich analytification of Spec $(K[x])$ is the Berkovich affine line over $K$(see for example [Ber90, BR10] for a detailed treatement of its geometry and topology). Given a valuation $w \in (\mathrm{Spec}\, K[x])^{an}$, this is by definition a valuation $w\colon K[x] \to \mathbb{T}$ extending the valuation $v$, thus the following trivial differential enhancement is extending the trivial differential enhancement of $v$ above:

$$\begin{tikzcd} & (\mathbb{T}, d = \infty) \arrow[d, "id_{\mathbb{T}}"] \\ K[x] \arrow[ur, "w"] \arrow[r, "w"] & \mathbb{T} \end{tikzcd}$$

Denoting as $\mathbf{T}$ the pair $\mathbb{T} \xrightarrow{\mathrm{id}_{\mathbb{T}}} \mathbb{T}$, in this case the map $\Lambda\colon\ Berk_{\mathbf{T}}(K[x]) \to (\mathrm{Spec}\, K[x])^{an}$ is a bijection. This is a desirable result, since we are considering the zero differential, thus every point of the Berkovich analytification should be a differential point.

*Example* 6.6.6. Let $v\colon\ \mathbb{C}(\!(t)\!) \to \mathbb{T}$ be the $t$-adic valuation and equip $K := \mathbb{C}(\!(t)\!)$ with the differential $\frac{d}{dt}$. Consider the differentially enhanced valuation

$$\begin{tikzcd} & \mathbb{B}(\!(t)\!) \arrow[d, "\pi"] \\ \mathbb{C}(\!(t)\!) \arrow[ur, "\widetilde{v}"] \arrow[r, "v"] & \mathbb{T} \end{tikzcd}$$

where $\mathbf{S} := (\mathbb{B}(\!(t)\!) \xrightarrow{\pi} \mathbb{T})$ is given analogously to the pair introduced in Example 6.1.2. Let the differential algebra $(K[x], d)$ over $K$, where $d(X) = 1$. Again, the Berkovich analytification of Spec $(K[x])$ is the Berkovich affine line over $K$. Given a point of type II or III on the Berkovich affine line, associated to a ball of center 0 and radius $e^r$ for some $r \in \mathbb{R}$, denote the associated valuation as $w_{0,r}$. It is given as the map

$$f = \sum_{i=0}^{n} a_i X^i \mapsto \bigoplus_i (v(a_i) + ir) \in \mathbb{T}.$$

The pair

$$\mathbf{T} := (\mathbb{B}(\!(t)\!)\{X\}/\langle X' \sim 0\rangle \xrightarrow{\rho_r} \mathbb{T})$$

sending $F$ to $\pi(F(r))$ is an $\mathbf{S}$-algebra and the following diagram is a differentially enhancement of $w_{0,r}$ compatible with $(v, \widetilde{v})$:

$$\begin{tikzcd}
 & \mathbb{B}(\!(t)\!)\{X\}/\langle X' \sim 0\rangle \arrow[d, "\rho_r"] \\
K[X] \arrow[ur, "\widetilde{w}"] \arrow[r, "w_{0,r}"'] & \mathbb{T}
\end{tikzcd}$$

where the map $\widetilde{w}$ is the coefficientwise application of $\widetilde{v}$. This proves that, for every $r \in \mathbb{R}$, the points of the form $w_{0,r}$ can be obtained as differential points.

# Chapter 7

# Differential tropicalization functor and colimit theorem

The main aim of this chapter is the definition of a tropicalization procedure for differential equations with coefficients in a differential ring $R$ equipped with a differentially enhanced valuation to a pair $\mathbf{S}$, and for their solutions. This is the topic of Sections 7.1 and 7.2, where we also prove, in Proposition 7.1.5, that the easy containment of the fundamental theorem of tropical differential algebra holds in this generalised context. In the last section we will introduce the universal differential presentation of a differential algebra $A$ over $R$, in analogy to [GG14] as recalled in Section 2.4. We will prove that its tropicalization is the colimit of the tropicalization functor over the category of presentations of $A$ and that there exists a differentially enhanced valuation from $A$ to its universal tropicalization that is universal, thus getting a differential version of Corollary 2.4.14. Furthermore, when the differential algebra $A$ is finitely generated the colimit can be taken over the category of finite presentations and this statement is a differential analogue of Payne's inverse limit theorem (Theorem 2.2.5, Theorem 2.4.15) This also implies that maps from the universal tropicalization to any other $\mathbf{S}$-algebra $\mathbf{T}$ are in bijection with the points of the differential Berkovich space over $\mathbf{T}$, an analogue of Theorem 2.4.13.

## 7.1 Differential tropicalization

In the familiar non-differential setting, as seen in Section 2.1 and Section 2.4, one starts with a ring $R$ with a valuation $v\colon R \to S$, and then defines three tropicalization maps:

(1) Tropicalization of points is the map $\mathrm{trop}_v\colon R^n \to S^n$ given by applying the valuation $v$ coordinate-wise.

(2) Tropicalization of equations is the map $\mathrm{trop}_v\colon R[x_1, \ldots, x_n] \to S[x_1, \ldots, x_n]$ given by applying the valuation $v$ coefficient-wise. This extends to a map sending ideals in $R[x_1, \ldots, x_n]$ to ideals in $S[x_1, \ldots, x_n]$.

(3) Tropicalization of varieties sends $V(I)$ to the subset of $S^n$ defined by the intersection of the tropical hypersurfaces of all $f \in \mathrm{trop}(I)$.

An ideal $I \subset R[x_1, \ldots, x_n]$ is of course a system of polynomial equations, and a solution to this system in an $R$-algebra $A$ is the same as a homomorphism $R[x_1, \ldots, x_n]/I \to A$. Since the tropical hypersurface of a tropical polynomial $f$ is exactly the solution set of the bend relations of $\mathrm{trop}(f)$, it follows that the tropicalization of a variety is the set of solutions to the bend relations of the tropicalization of its defining ideal. Moreover, solutions to these bend relations are precisely homomorphisms of $S$-algebras $S[x_1, \ldots, x_n]/\mathcal{B}\mathrm{trop}(I) \to T$, as in Remark 2.4.10. One can thus think of the semiring $S[x_1, \ldots, x_n]/\mathcal{B}\mathrm{trop}(I)$ as the coordinate algebra of the tropical variety, and hence tropicalization of varieties has an incarnation at the level of algebras given by

$$R[x_1, \ldots, x_n]/I \mapsto S[x_1, \ldots, x_n]/\mathcal{B}\mathrm{trop}(I)$$

We now turn to the differential setting. Let $R$ be a differential ring equipped with a differentially enhanced seminorm $\mathbf{v} = (\widetilde{v}, v) \colon R \to \mathbf{S} = (S_1 \xrightarrow{\pi} S_0)$.

(1) We tropicalize points $p \in R^n$ via the map $\mathrm{trop}_{\widetilde{v}} \colon R^n \to S_1^n$ defined by applying $\widetilde{v}$ component-wise.

(2) We tropicalize differential equations by applying $v$ coefficient-wise to define a map
$$\mathrm{trop}_v \colon R\{x_1, \ldots, x_n\} \to S_0\{x_1, \ldots, x_n\}_{basic}$$
so there is an induced a map sending ideals in $R\{x_1, \ldots, x_n\}$ to ideals in $S_0\{x_1, \ldots, x_n\}_{basic}$.

(3) We use the tropicalization of equations map to define a construction sending quotients $\alpha \colon R\{x_1, \ldots, x_n\} \twoheadrightarrow R\{x_1, \ldots, x_n\}/I$ to quotients $\mathbf{trop}(\alpha)$ of the pair $\mathbf{S}\{x_1, \ldots, x_n\}$. Define
$$\mathbf{trop}(\alpha) := \mathbf{S}\{x_1, \ldots, x_n\} /\!\!/ \mathcal{B}\mathrm{trop}_v(I),$$
where $\mathcal{B}\mathrm{trop}_v(I)$ is the congruence on $(S_0|S_1)\{x_1, \ldots, x_n\}$ generated by the bend relations of $\mathrm{trop}_v(I)$ and we use the quotient construction of Section 6.3.

We will extend what we just sketched to any $\mathbb{DF}_1$-module, not just considering free ones, and we will prove the functoriality of the assignment $\alpha \mapsto \mathbf{trop}(\alpha)$.

*Remark* 7.1.1. Since for any $\mathbb{DF}_1$-module $\mathcal{M}$ the algebra $(S_0|S_1)\{\mathcal{M}\}$ is not a polynomial algebra, we cannot form the bend relations of an arbitrary element in it. The above construction uses the fact that applying $v$ coefficient-wise lands in $S_0[F(\mathcal{M})] \subset (S_0|S_1)\{\mathcal{M}\}$, and this algebra is a polynomial algebra so we can form bend relations in it.

So far we introduced the definition of solution to a tropical differential equation $f \in S_0\{\mathcal{M}\}$ with monomials in a $D\mathbb{F}_1$-module $\mathcal{M}$ only in the world of pairs, in Definition 6.3.4. For differential rings, in Definition 3.1.3 we defined a notion of solution only when the differential equation $f \in R\{x_1, \dots, x_n\}$ has monomials in a free differential $D\mathbb{F}_1$-modules, let us do it now in general.

As in Section 5.3, given a differential (semi)ring $S$, for every $D\mathbb{F}_1$-module $\mathcal{M}$ and for every $DS$-algebra $T$, let us denote as $\Psi$ the Hom-set bijection

$$\Psi\colon \operatorname{Hom}_{\mathbf{D}S\textbf{-Alg}}(S\{\mathcal{M}\}, T) \cong \operatorname{Hom}_{\mathbf{D}\mathbb{F}_1\textbf{-Mod}}(\mathcal{M}, T)$$

induced by the freeness of $S\{-\}$ proved in Proposition 5.3.8. When $R$ is a differential ring, from Proposition 5.3.2 the differential $R$-algebra $R\{\mathcal{M}\}$ is isomorphic as an $R$-algebra to $R[F(\mathcal{M})]$ and we give the general definition of the solution set to a differential equation as:

**Definition 7.1.2.** Let $\mathcal{M}$ be a $D\mathbb{F}_1$-module and $A$ a differential algebra over $R$. The set of solutions in $A$ of a differential polynomial $f \in R\{\mathcal{M}\}$, denoted $\operatorname{Sol}_A(f)$, is the set of morphisms $p \in \operatorname{Hom}_{\mathbf{D}\mathbb{F}_1\textbf{-Mod}}(\mathcal{M}, A)$ such that its adjoint $\Psi(p)$ factors through $R\{\mathcal{M}\}/[f]$, where $[f]$ is the differential ideal generated by $f$.

*Remark* 7.1.3. Notice that when $\mathcal{M}$ is free of rank $n$ this new definition matches with Definition 3.1.3.

Similarly to Theorem 6.3.5, we have

**Theorem 7.1.4.** *Given a differential ideal $I \subset R\{\mathcal{M}\}$, the functor* $\mathbf{D}R\textbf{-Alg} \to \textbf{Sets}$ *sending a differential $R$-algebra $A$ to* $\operatorname{Sol}_A(I)$ *is corepresented by $R\{\mathcal{M}\}/I$.*

As above, let $R$ be a differential ring equipped with a differentially enhanced valuation $\mathbf{v} = (\widetilde{v}, v)\colon R \to \mathbf{S} = (S_1 \xrightarrow{\pi} S_0)$, $A$ a differential $R$-algebra equipped with a differentially enhanced valuation $\mathbf{w} = (\widetilde{w}, w)\colon A \to \mathbf{T} = (T_1 \xrightarrow{\rho} T_0)$ compatible with $\mathbf{v}$ and $\mathcal{M}$ a $D\mathbb{F}_1$-module. Then:

(1) We tropicalize morphisms $p \in \operatorname{Hom}_{\mathbf{D}\mathbb{F}_1\textbf{-Mod}}\{\mathcal{M}, A\}$ via the map

$$\operatorname{trop}_{\widetilde{w}}\colon \operatorname{Hom}_{\mathbf{D}\mathbb{F}_1\textbf{-Mod}}\{\mathcal{M}, A\} \to \operatorname{Hom}_{\mathbf{D}\mathbb{F}_1\textbf{-Mod}}\{\mathcal{M}, T_1\}$$

defined by mapping $p$ to $\widetilde{w} \circ p$. It is well defined thanks to Remark 6.5.2. This map reduces to the previously introduced definition of $\operatorname{trop}_{\widetilde{v}}$ when $\mathcal{M}$ is free of rank $n$ and $A = R$.

(2) We tropicalize differential equations by applying $v$ coefficient-wise to define a map

$$\operatorname{trop}_v\colon R\{\mathcal{M}\} \to S_0[F(\mathcal{M})]$$

and again there is an induced a map sending ideals in $R\{\mathcal{M}\}$ to ideals in $S_0[F(\mathcal{M})]$.

(3) We use the tropicalization of equations map to define a construction sending quotients $\alpha\colon R\{\mathcal{M}\} \twoheadrightarrow R\{\mathcal{M}\}/I$ to quotients $\mathbf{trop}(\alpha)$ of the pair $\mathbf{S}\{\mathcal{M}\}$. Define

$$\mathbf{trop}(\alpha) := \mathbf{S}\{\mathcal{M}\} /\!\!/ \mathcal{B}\mathrm{trop}_v(I).$$

With the same notations as above:

**Proposition 7.1.5.** *Given a differential ideal $I \subset R\{\mathcal{M}\}$, the tropicalization map*

$$\mathrm{trop}_{\widetilde{w}}\colon \mathrm{Hom}_{\mathbf{D}\mathbb{F}_1\textbf{-Mod}}\{\mathcal{M}, A\} \to \mathrm{Hom}_{\mathbf{D}\mathbb{F}_1\textbf{-Mod}}\{\mathcal{M}, T_1\}$$

*sends* $\mathrm{Sol}_A(I)$ *into* $\mathrm{Sol}_{\mathbf{T}}(\mathrm{trop}_v(I))$.

*Proof.* It suffices to show that if $p \in \mathrm{Hom}_{\mathbf{D}\mathbb{F}_1\textbf{-Mod}}\{\mathcal{M}, A\}$ is a solution to $I$ in $A$, then $\widetilde{w} \circ p \in \mathrm{Hom}_{\mathbf{D}\mathbb{F}_1\textbf{-Mod}}\{\mathcal{M}, T_1\}$ is a solution to $\mathrm{trop}_v(I) \subset S_0\{F(\mathcal{M})\}$ in $\mathbf{T}$. I.e., that if

$$\Psi(p)\colon R\{\mathcal{M}\} \to A$$

sending $m$ to $p(m)$, factors through the quotient by $I$ then

$$\Phi(\widetilde{w} \circ p)_0\colon (S_0|S_1)\{\mathcal{M}\} \to T_0$$

acting on the $S_0$-subalgebra $S_0[F(\mathcal{M})]$ by sending $m$ to $\rho((\widetilde{w} \circ p)(m))$, factors through the quotient by $\mathcal{B}\mathrm{trop}_v(I)$.

Write $f = \sum_i f_i \prod_j m_{i,j}$ for an element of $R\{\mathcal{M}\}$, where the $f_i$'s are elements of $R$. As $p$ is a solution for $I$, given any $f \in I$ we have $\Psi(p)(f) = \sum_i f_i \prod_j p(m_{i,j}) = 0$, thus $w(\sum_i f_i \prod_j p(m_{i,j})) = 0_{T_0}$. Since $w\colon R \to T_0$ is a valuation this happens if and only if the sum

$$\sum_i w(f_i \prod_j p(m_{i,j}))$$

tropically vanishes in $T_0$. The sum above is equal to

$$\sum_i v(f_i) \prod_j \rho((\widetilde{w} \circ p)(m_{i,j}))$$

which is equal to $\mathrm{trop}_v(f)(\rho((\widetilde{w} \circ p)(m_{i,j})))$. This last expression is

$$\Phi(\widetilde{w} \circ p)_0(\mathrm{trop}_v(f))$$

by definition, as $\mathrm{trop}_v(f)$ belongs to $S_0[F(\mathcal{M})]$, and for all $f \in I$ it is tropically vanishing in $T_0$. This is equivalent to say that $\Phi(\widetilde{w} \circ p)_0$ descends to the quotient of $(S_0|S_1)\{\mathcal{M}\}$ by $\mathcal{B}\mathrm{trop}_v(I)$, by definition of this quotient. □

*Remark* 7.1.6. It is important to notice though that in order for a fundamental theorem to hold it might be necessary to add some hypothesis, as we require the valuation to be surjective and the coefficient field to be algebraically closed for the non-differential fundamental theorem to hold. In fact, as highlighted in [FGLH+20, Example 7.4], considering the differentially enhanced valuation $(v, \widetilde{v})\colon K[\![t]\!] \to \mathbf{S} = (\mathbb{B}[\![t]\!] \to \mathbb{T})$ of Section 3.2 and the differential $K[\![t]\!]$-algebra of Puiseux series $K\{\{t\}\}$ equipped with the differentially enhanced valuation $(w, \widetilde{w})\colon K\{\{t\}\} \to \mathbf{T} = (\mathbb{B}\{\{t\}\} \to \mathbb{T})$ defined analogously, for the differential ideal $I$ generated by $2tx' - x \in K[\![t]\!]\{x\}$, the containment

$$\mathrm{trop}_{\widetilde{w}}(\mathrm{Sol}_{K\{\{t\}\}}(I)) \subset \mathrm{Sol}_{\mathbf{T}}(\mathrm{trop}_v(I))$$

is strict.

Furthermore, the fundamental theorem continues to fail also when considering a field $K$ with a nontrivial valuation: for example if we choose $K = \mathbb{Q}$, equipped with the differentially enhanced valuation of Example 6.5.5, for $p = 2$, from the morphism of pairs

$$\begin{tikzcd} \mathbb{T}\{\{t\}\} \arrow[r] \arrow[d] & \mathbb{B}\{\{t\}\} \arrow[d] \\ \mathbb{T}_2 \arrow[r] & \mathbb{T} \end{tikzcd}$$

we have that every solution to the tropicalization of the differential ideal $I$ generated by $P = 2tx' - x \in \mathbb{Q}[\![t]\!]\{x\}$ in $\mathbf{T} = (\mathbb{T}\{\{t\}\} \to \mathbb{T}_2)$ has to be supported on a solution in Grigoriev's setting. In particular, as computed in the aformentioned example of [FGLH+20], the possible supports for tropical solutions are only of the form $t^\alpha$ for $\alpha \in \mathbb{Q} \cap (0,1)$. Thus let us assume that $A = ct^\alpha \in \mathbb{T}\{\{t\}\}$ is a solution, then

$$\begin{aligned} \mathrm{trop}_v(P)(A) &= (1,1)(\alpha - 1, c + v_2(\alpha)) \oplus (0,0)(\alpha, c) \\ &= (\alpha, c + v_2(\alpha) + 1) \oplus (\alpha, c) \end{aligned}$$

so we get an additional condition on the exponent $\alpha$: $A$ is a solution if and only if $v_2(\alpha) = -1$. As there are infinitely many $\alpha \in \mathbb{Q} \cap (0,1)$ with $v_2(\alpha) = -1$, the fundamental theorem still fails.

If instead we choose $p \neq 2$, we get

$$\begin{aligned} \mathrm{trop}_v(P)(A) &= (1,0)(\alpha - 1, c + v_p(\alpha)) \oplus (0,0)(\alpha, c) \\ &= (\alpha, c + v_p(\alpha)) \oplus (\alpha, c) \end{aligned}$$

thus $A$ is a solution if and only if $v_p(\alpha) = 0$, and there are still infinitely many exponents allowed. This is another example showing that the setting we introduced is strictly refining the setting introduced by Grigoriev, although the fundamental theorem is still not holding.

The same as the above holds if we consider the equation $P = ptx' - x \in \mathbb{Q}[\![t]\!]\{x\}$ for $p \neq 2$. Equipping $\mathbb{Q}[\![t]\!]$ with the differentially enhanced valuation $(v, \widetilde{v})$ of Grigoriev's setting, by a similar computation as in [FGLH+20], we have that the containment $\mathrm{trop}_{\widetilde{w}}(\mathrm{Sol}_{K\{\{t\}\}}(I)) \subset \mathrm{Sol}_{\mathbf{T}}(\mathrm{trop}_v(I))$ is strict and all the Puiseux series tropical solutions are of the form $t^\alpha$ with $\alpha \in \mathbb{Q} \cap (0,1)$. Passing to the refined setting with respect to $p$, with similar computations as above, we get that $ct^\alpha$ is a solution to $\mathrm{trop}_v(P)$ if and only if $v_p(\alpha) = -1$ and that $ct^\alpha$ is a solution to $\mathrm{trop}_v(P)$ if and only if $v_\ell(\alpha) = 0$ when $\ell \neq p$ instead.

## 7.2 Functoriality of tropicalization

Tropicalization of differential equations sends a presentation of a differential algebra to a tropical pair. Here we show that this defines a functor from a category of presentations to the category of tropical pairs. Let us start by introducing the category **Pres**$(A)$:

**Definition 7.2.1.** Given a differential $R$-algebra $A$ let **Pres**$(A)$ be the category of differential $\mathbb{F}_1$-presentations of $A$ whose objects are differential $\mathbb{F}_1$-modules $\mathcal{M}$ together with a morphism $\mathcal{M} \to DM(A)$ in **D**$\mathbb{F}_1$**-Mod** such that its adjoint

$$R\{\mathcal{M}\} \twoheadrightarrow A$$

is a surjective morphisms of differential $R$-algebras, and arrows are morphisms $f$ in **D**$\mathbb{F}_1$**-Mod** such that the induced diagram of differential $R$-algebras is commutative:

$$\begin{tikzcd} R\{\mathcal{M}\} \arrow[rr, "R\{f\}"] \arrow[dr, two heads] & & R\{\mathcal{N}\} \arrow[dl, two heads] \\ & A & \end{tikzcd}$$

Furthermore, let **Pres**$^{fin}(A)$ be the subcategory of finite differential presentations of $A$ inside the category **Pres**$(A)$, whose objects are free $\mathrm{D}\mathbb{F}_1$-modules of rank $n$ for some $n \in \mathbb{N}$.

**Proposition 7.2.2.** *Given a differential $R$-algebra $A$, the tropicalization construction*

$$(R\{\mathcal{M}\} \overset{\alpha}{\twoheadrightarrow} A) \mapsto \mathbf{trop}(\alpha) := \mathbf{S}\{\mathcal{M}\} /\!/ \mathcal{B}\mathrm{trop}_v(\ker \alpha)$$

*yields a functor* **Pres**$(A) \to$ **S-Alg**.

*Proof.* Any morphism in **Pres**$(A)$

$$\begin{tikzcd} R\{\mathcal{M}\} \arrow[rr] \arrow[dr, two heads, "\alpha"'] & & R\{\mathcal{N}\} \arrow[dl, two heads, "\beta"] \\ & A & \end{tikzcd}$$

by definition is the adjoint of a morphism of $\mathrm{D}\mathbb{F}_1$-module $f\colon \mathcal{M} \to \mathcal{N}$ and by Proposition 6.1.4, $f$ gives a morphism of differential $\mathbf{S}$-algebras

$$\mathbf{S}\{f\}\colon \mathbf{S}\{\mathcal{M}\} \to \mathbf{S}\{\mathcal{N}\}$$

and this restricts to a morphism

$$S_0[F(\mathcal{M})] \to S_0[F(\mathcal{N})]$$

of the basic subalgebras on the bottom. Moreover, the congruence $\mathcal{B}\mathrm{trop}_v(\ker\alpha)$ on $S_0[F(\mathcal{M})]$ is sent into the congruence $\mathcal{B}\mathrm{trop}_v(\ker\beta)$ on $S_0[F(\mathcal{N})]$ thanks to Proposition 6.4.1 of [GG16]. Hence $\mathbf{S}\{f\}$ descends to a morphism of quotient pairs:

$$\begin{tikzcd}
S_1\{\mathcal{M}\} \arrow[r] \arrow[d] & S_1\{\mathcal{N}\} \arrow[d] \\
(S_0|S_1)\{\mathcal{M}\}/\mathcal{B}\mathrm{trop}_v(\ker\alpha) \arrow[r] & (S_0|S_1)\{\mathcal{N}\}/\mathcal{B}\mathrm{trop}_v(\ker\beta)
\end{tikzcd}$$

and by Proposition 6.2.2 this induces a morphism of their reductions, which is precisely the desired morphism

$$\mathbf{trop}(\alpha) \to \mathbf{trop}(\beta). \qquad \square$$

## 7.3 The universal presentation and the universal tropicalization

Given a differential $R$-algebra $A$, consider the differential presentation

$$Univ\colon\ R\{DM(A)\} \twoheadrightarrow A$$

induced by the $\mathbb{DF}_1$-modules morphism $id_A\colon\ DM(A) \to DM(A)$. Denoting as $x_a$ the differential variables in $R\{DM(A)\}$, the presentation *Univ* sends $x_a$ to $a$. It takes a formal differential polynomial in the elements of $A$ and evaluates it to an element of $A$ using the differential algebra structure of $A$. A similar morphism was studied in the non-differential setting of [GG14], presented in Section 2.4. In light of the following fact, we call this the *universal presentation* of $A$:

**Proposition 7.3.1.** *The presentation Univ*$\colon\ R\{DM(A)\} \twoheadrightarrow A$ *is*

*(1) the final object in* $\mathrm{Pres}(A)$

*(2) the colimit of the inclusion functor* $\iota\colon\ \mathrm{Pres}^{fin}(A) \hookrightarrow \mathrm{Pres}(A)$.

*Proof.* Part (1): Let $\alpha\colon\ R\{\mathcal{M}\} \twoheadrightarrow A$ be a presentation. We will show that the set of morphisms $\mathrm{Hom}_{\mathbf{Pres}(A)}(\alpha, Univ)$ contains exactly one element. Any morphism of presentations $f$ from $\alpha$ to *Univ* must send each element $m \in \mathcal{M}$ to an element of $A$ that is mapped to $\alpha(m)$ by *Univ*. One option is $f(m) = x_{\alpha(m)}$, and this is evidently the unique choice that gives a morphism of presentations $\alpha \to Univ$.

Part (2): By (1), any finite presentation $\alpha$ admits a unique morphism $\alpha \to Univ$, and hence there is a canonical morphism $u\colon\ \operatorname{colim}\iota \to Univ$. Given a finite presentation $\alpha\colon\ R\{y_1, \dots, y_n\} \to A$ and an element $a \in A$, we extend to a new finite presentation $\alpha'\colon\ R\{y_1, \dots, y_n, y_a\} \to A$ by $y_a \mapsto a$. The morphism $\alpha' \to Univ$ sends $y_a$ to $x_a$, and thus any element $x_a$ in the universal presentation is in the image of some finite presentation, so $u$ is surjective.

We turn now to the injectivity of $u$. If

$$R\{y_1, \dots, y_n\} \xrightarrow{\alpha} A \xleftarrow{\beta} R\{z_1, \dots, z_m\}$$

are two finite presentations with $\alpha(y_1) = \beta(z_1)$, then they each map to the presentation $R\{w, y_2 \dots, y_n, z_2, \dots, z_m\}$ by $y_1, z_1 \mapsto w$ and identity of all of the other generators. Hence $u$ is injective as well. $\square$

We define the *universal tropicalization of $A$* to simply be the tropicalization of the universal presentation, $\mathbf{trop}(Univ)$.

**Theorem 7.3.2.** *For any differential $R$-algebra $A$, there is a canonical isomorphism of pairs*

$$\mathbf{trop}(Univ) \cong \operatorname*{colim}_{\alpha \in \mathbf{Pres}(A)} \mathbf{trop}(\alpha),$$

*and if $A$ is finitely generated, i.e. admits a finite presentation $R\{x_1, \dots, x_n\} \twoheadrightarrow A$, then the colimit can be taken over the subcategory $\mathrm{Pres}^{fin}(A)$ of finite presentations.*

*Proof.* The first statement is an immediate consequence of Proposition 7.3.1. For the second statement, consider the canonical morphism

$$j = (j_1, j_0) \colon \operatorname*{colim}_{\alpha \in \mathrm{Pres}^{fin}(A)} \mathbf{trop}(\alpha) \to \mathbf{trop}(Univ).$$

Given any element $a \in A$, any finite presentation $\alpha \colon R\{x_1, \dots, x_n\} \to A$ maps to a finite presentation $\alpha' \colon R\{x_1, \dots, x_n, x_a\} \to A$, with $\alpha'(x_a) = a$. Hence it follows from Proposition 6.4.2 that $j_1$ and $j_0$ are both surjective. For what concerns injectivity, observe that the congruence $\mathcal{B}\mathrm{trop}_v(\ker Univ)$ on $S_0[F(DM(A))]$ is the transitive closure of the symmetric semiring generated by the bend relations of elements in $\mathrm{trop}_v(\ker Univ)$, and so for any relation $(f \sim g) \in \mathcal{B}\mathrm{trop}_v(\ker Univ)$ there exists a finite subset $\Lambda \subset A$ containing all variables appearing in either $f$ or $g$ and such that, for the restriction $Univ|_\Lambda \colon R\{x_a \mid a \in \Lambda\} \to A$, we have

$$(f \sim g) \in \mathcal{B}\mathrm{trop}_v(\ker Univ|_\Lambda).$$

If $\Lambda$ does not generate $A$ as a differential algebra then we may add finitely many elements so that it does. We thus have a finite presentation $\beta$ such that $(f \sim g)$ is in the image of the canonical map

$$\mathcal{B}\mathrm{trop}_v(\ker \beta) \to \mathcal{B}\mathrm{trop}_v(\ker Univ).$$

Therefore the map

$$j_0 \colon \operatorname*{colim}_{\alpha \in \mathbf{Pres}^{fin}(A)} (S_0|S_1)\{x_a \mid a \in \Lambda\}/\mathcal{B}\mathrm{trop}_v(\ker \beta) \to (S_0|S_1)\{DM(A)\}/\mathcal{B}\mathrm{trop}_v(\ker Univ)$$

is an isomorphism. The claim now follows from Proposition 6.4.2. $\square$

We now come to the main result of this section, which says that a differential algebra $A$ admits a universal differentially enhanced valuation which is valued in the tropicalization of the universal presentation of $A$ (c.f. [GG14, Theorem A]).

**Theorem 7.3.3.** *Given a differential $R$-algebra $A$, there is a differentially enhanced valuation*

$$\mathbf{u} = (u, \widetilde{u}) \colon A \to \mathbf{trop}(Univ)$$

*defined by sending $a \mapsto x_a$, and this is initial among differentially enhanced valuations on $A$ compatible with the differentially enhanced valuation $\mathbf{v}$ on $R$.*

As an immediate corollary, we have:

**Corollary 7.3.4.** *Let $\mathbf{T}$ be an $\mathbf{S}$-algebra. There is a bijection*

$$\mathrm{Hom}_{\mathbf{S}\text{-}\mathbf{Alg}}(\mathbf{trop}(Univ), \mathbf{T}) \cong Berk_{\mathbf{T}}(A).$$

*that is natural in $\mathbf{T}$. I.e., $\mathbf{trop}(Univ)$ co-represents the functor sending $\mathbf{T}$ to the set of differentially enhanced valuations on $A$ taking values in $\mathbf{T}$ and compatible with $\mathbf{v}$.*

The proof of Theorem 7.3.3 above requires an explicit description of the congruence $\mathcal{B}\mathrm{trop}_v(\ker Univ)$, which we provide below.

**Proposition 7.3.5.** *The differential ideal* $\ker Univ$ *is generated as an ideal by the following elements:*

*(1)* $x_1 - 1$;

*(2)* $\lambda x_a - x_{\lambda a}$ *for all* $a \in A$, $\lambda \in R$;

*(3)* $x_a + x_b + x_c$ *for all* $a, b, c \in A$ *such that* $a + b + c = 0$ *in* $A$;

*(4)* $x_{ab} - x_a x_b$ *for all* $a, b \in A$.

*Proof.* It is clear that all of these relations are in the kernel of $Univ$. Denote by $J$ the ideal generated by the elements of the four classes listed above. Thanks to Proposition 5.3.2, giving an isomorphism between $R\{DM(A)\}$ and $R[F(DM(A))]$ as $R$-algebras, we can denote elements of $\ker Univ$ as $\sum_i \lambda_i \prod_j x_{a_{i,j}}$ with $\sum_i \lambda_i \prod_j a_{i,j} = 0$ in $A$.

Firstly, let us prove that, for every $n \in \mathbb{N}$ and every $n$ elements $a_1, \ldots, a_n \in A$ the element $x_{\prod_j^n a_j} - \prod_j^n x_{a_j}$ belongs to $J$. The case $n = 2$ holds by definition of $J$, for $n \geq 2$ we have $x_{(\prod_j^{n-1} a_j) a_n} - x_{\prod_j^{n-1} a_j} x_{a_n} \in J$ again by definition and by inductive hypothesis we get:

$$\begin{aligned} &x_{(\prod_j^{n-1} a_j) a_n} - x_{\prod_j^{n-1} a_j} x_{a_n} + x_{a_n}(x_{\prod_j^{n-1} a_j} - \prod_j^{n-1} x_{a_j}) = \\ &= x_{(\prod_j^{n-1} a_j) a_n} - \prod_j^{n} x_{a_j} = x_{\prod_j^n a_j} - \prod_j^{n} x_{a_j} \in J \end{aligned}$$

Now, given an element $\sum_i \lambda_i \prod_j x_{a_{i,j}} \in \ker Univ$, by adding the element $\sum_i x_{\lambda_i \prod_j a_{i,j}} - \sum_i \lambda_i \prod_j x_{a_{i,j}} \in J$ to it we get $\sum_i x_{\lambda_i \prod_j a_{i,j}}$. By subtracting from $\sum_i x_{\lambda_i \prod_j a_{i,j}}$ the element

$x_{\lambda_1 \prod_j a_{1,j}} + x_{\lambda_2 \prod_j a_{2,j}} + x_{-\lambda_1 \prod_j a_{1,j} - \lambda_2 \prod_j a_{2,j}}$ and then again adding $x_{-\lambda_1 \prod_j a_{1,j} - \lambda_2 \prod_j a_{2,j}} + x_{\lambda_1 \prod_j a_{1,j} + \lambda_2 \prod_j a_{2,j}}$ we get a sum with one fewer terms. Repeating this process inductively yields $x_{\sum_i \lambda_i \prod_j x_{a_{i,j}}}$ but by hypothesis $\sum_i \lambda_i \prod_j x_{a_{i,j}} = 0$ thus $x_{\sum_i \lambda_i \prod_j x_{a_{i,j}}} = x_0 = 0$. □

Tropicalizing the above family of elements, we have:

**Lemma 7.3.6.** *The congruence $\mathcal{B}\mathrm{trop}_v(\ker Univ)$ on $S_0[F(DM(A))]$ is generated by the bend relations of the polynomials:*

*(1)* $x_1 + 1_{S_0}$*;*

*(2)* $x_{\lambda a} + v(\lambda) x_a$ *for* $a \in A$ *and* $\lambda \in k$*;*

*(3)* $x_a x_b + x_{ab}$ *for* $a, b \in A$*;*

*(4)* $x_a + x_b + x_c$ *for* $a, b, c \in A$ *satisfying* $a + b + c = 0$*.*

*Proof.* It suffices to show that the bend relations of listed expressions imply the bend relations of any element $g \in \mathrm{trop}_v(\ker Univ)$. As in the proof of Proposition 7.3.5 above, using the bend relations of (1)–(3) allows us reduce $g$ to an expression of the form

$$\sum_{a \in \Lambda} x_a, \text{ with } \sum_{a \in \Lambda} a = 0.$$

Let us call a finite set $\Lambda \subset A$ a *null set* if $\sum_\Lambda a = 0$. It remains to show that the bend relations of sums over null sets of size 3 (i.e., relation (4) from the list) imply the bend relations for sums as above over null sets of arbitrary size.

We prove this by induction on the cardinality $n$ of the null set $\Lambda$. The base case $n = 3$ is simply relation (4). Assume the bend relations hold for all sums over null sets of size $\leq n$, and consider a null set $\Lambda = \{a_1, \dots, a_{n+1}\}$. Let

$$b = a_1 + a_2 = -(a_3 + \cdots + a_{n+1}),$$

so we have null sets $\Lambda_1 = \{a_1, a_2, -b\}$ and $\Lambda_2 = \{b, a_3, \dots, a_{n+1}\}$ of size 3 and $n$. Then

$$\begin{array}{lllll} & x_{a_1} + & x_{a_2} + \cdots + x_{a_{n+1}} & & \\ \sim & x_{a_1} + & x_{a_2} + \cdots + x_{a_{n+1}} & + x_{-b} & \text{(using the bend relations of } \Lambda_1 \text{ to pull out } x_{-b}) \\ \sim & & x_{a_2} + \cdots + x_{a_{n+1}} & + x_{-b} & \text{(using the bend relations of } \Lambda_1 \text{ to delete } x_{a_1}) \\ \sim & & x_{a_2} + \cdots + x_{a_{n+1}} & + x_b & \text{(since } x_b \sim x_{-b} \text{ by (2))} \\ \sim & & x_{a_2} + \cdots + x_{a_{n+1}} & & \text{(using the bend relations of } \Lambda_2 \text{ to delete } x_b). \end{array}$$

Since $a_1$ was chosen arbitrarily, this shows that the bend relations of sums over null sets of size $n + 1$ hold. □

Note that relations (1)–(4) correspond to the four conditions defining a valuation in Definition 2.4.1.

Let us write $U_1 \xrightarrow{\pi} U_0$ for the pair $\mathbf{trop}(Univ)$, which is the reduction of the pair

$$S_1[F(DM(A))] \to (S_0|S_1)[F(DM(A))]/\mathcal{B}\mathrm{trop}_v(\ker Univ).$$

*Proof of Theorem 7.3.3.* It is clear from the definition that $\pi \circ \widetilde{u} = u$. By definition of $S_1\{-\}$, the map $\widetilde{u}\colon A \to U_1$ commutes with the differential, and by relations (1), (3) and (4) of Lemma 7.3.6, the map $u\colon A \to U_0$ is a valuation. Thus $\mathbf{u}$ is a differentially enhanced valuation. Moreover, relation (2) implies that $u$ is compatible with the valuation $v\colon A \to S_0$.

It remains to show that there is a unique morphism from $\mathbf{u}$ to any other differentially enhanced seminorm $\mathbf{w} = (w, \widetilde{w})\colon A \to \mathbf{T} : (T_1 \xrightarrow{\tau} T_0)$ compatible with $\mathbf{v}\colon A \to \mathbf{S}$. From Proposition 6.3.2, to define a morphism of pairs $\mathbf{trop}(Univ) \to \mathbf{T}$ it is enough to define a morphism of $\mathbb{DF}_1$-modules $f\colon DM(A) \to T_1$ such that the bottom component $\Phi(f)_0$ of its adjoint with respect to the functor $\mathbf{S}\{-\}$ factors through the quotient by $\mathcal{B}\mathrm{trop}_v(\ker Univ)$.

Let us denote by $x_a$ the elements of $DM(A)$. Define a $\mathbb{DF}_1$-module morphism $f\colon DM(A) \to T_1$ by $x_a \mapsto \widetilde{w}(a)$, then $\Phi(f)_0$ acts on elements of $S_0[F(DM(A))]$ by mapping $x_a$ to $\tau(f(x_a)) = w(a)$. Since $w$ is a valuation compatible with $v$, the map $\Phi(f)_0$ factors through the quotient by $\mathcal{B}\mathrm{trop}_v(\ker Univ)$, thus giving a morphism of pairs $\mathbf{trop}(Univ) \to \mathbf{T}$. It is clearly unique as $f$ is unique.

$\square$

# Chapter 8

# A general framework for tropical PDEs

The aim of the present chapter is to extend all the previously introduced ideas to the partial setting, in order to be able to tropicalize partial differential equations with coefficients in a partial differential ring $(R, \{\partial_1, \ldots, \partial_r\})$ (see [Rit50] and [Kol73]) and define tropical PDEs and their solutions. As the previously introduced notions are generalising Grigoriev's setting of [Gri17] and [AGT16], the framework we introduce here will be generalising the setting of [FGLH$^+$20]. We will introduce partial differential semirings, partial differential $\mathbb{F}_1$-modules and partial tropical pairs in the following pages, a construction of partial Ritt algebras in $n$ variables over a partial differential semiring and a generalised notion of differential enhancement tropicalization functor adapted to this context. All the notations and construction introduced in this chapter reduce to the one of the previous chapters when the number of differentials $r$ is equal to 1.

## 8.1 Partial differential semirings

We start by introducing the notion of a partial differential semiring: for every $r \in \mathbb{N}$, a *tropical partial differential semiring* with $r$ commuting differentials is an idempotent semiring equipped with a family $\{\partial_1, \ldots, \partial_r\}$ of mutually commuting tropical differential. We will denote the category of tropical partial differential semirings with $r$ differentials as $\mathbf{DSemiRings}_r$.

*Example* 8.1.1. The following are examples of partial differential semirings, respectively generalising the cases of Example 4.1.4:

(1) Let $S$ be an idempotent semiring. Then for any $r \in \mathbb{N}$ we can make it into a partial differential semiring $(S, \{\partial_1, \ldots, \partial_r\})$ by setting every $\partial_i$ be either the constant map $0_S$ or the identity; any such choice gives strict differentials on $S$.

(2) For any $r \in \mathbb{N}$, consider the idempotent semiring $\mathbb{B}[\![t_1, \ldots, t_r]\!]$ of multivariate formal power series with coefficients in $\mathbb{B}$. Endowing it with the set of

differentials $\partial/\partial t_i$, for $i = 1, \ldots, r$ defined as:

$$\left(\frac{\partial}{\partial t_i}\right) t_j^n = \begin{cases} t_i^{n-1} & n \geq 1 \text{ and } j = i \\ \infty & \text{otherwise} \end{cases}$$

and extended by Leibniz rule, gives it the structure of a strict partial differential semiring.

Similarly, if $p$ is a prime, for $i = 1, \ldots, r$, let

$$\left(\frac{\partial}{\partial t_i}\right) t_j^n := \begin{cases} t_i^{n-1} & n \geq 1, j = i \text{ and } p \nmid n \\ \infty & \text{otherwise} \end{cases}$$

and extend them strictly on mixed products, then $(\mathbb{B}[\![t_1, \ldots, t_r]\!], \{\partial/\partial t_1, \ldots, \partial/\partial t_r\},)$ is a partial differential semiring that is not strict.

(3) Consider the idempotent semiring of multivariate formal tropical power series $\mathbb{T}[\![t_1, \ldots, t_r]\!]$. As in example (2), it can be endowed with a family of strict differentials defined on pure powers as

$$\left(\frac{\partial}{\partial t_i}\right) t_j^n = \begin{cases} t_i^{n-1} & n \geq 1 \text{ and } j = i \\ \infty & \text{otherwise} \end{cases}$$

and extended strictly on mixed products. Let us denote the differential semiring $\mathbb{T}[\![t_1, \ldots, t_r]\!]$ endowed with this family as $\mathbb{T}[\![t_1, \ldots, t_r]\!]_0$.

(4) More generally, given a family of valuations $\{v_i \colon \mathbb{N} \to \mathbb{T}\}_{i=1,\ldots,r}$, let $\partial_i$ to be defined as

$$\partial_i(t_j^n) = \begin{cases} v_i(n) t_i^{n-1} & n \geq 1 \text{ and } j = i \\ \infty & \text{otherwise} \end{cases} \tag{8.1.1}$$

and extended strictly on mixed products. As in the ordinary case, $v_i$ could be either the trivial valuation, a $p$-adic valuation, or a degenerate $p$-adic valuation where $v_i(n) = 0$ if $p$ divides $n$, and $\infty$ otherwise. When $v_i$ equals a valuation $v$ for all $i$, we will denote $(\mathbb{T}[\![t_1, \ldots, t_r]\!], \{\partial_1, \ldots, \partial_r\})$ as $\mathbb{T}[\![t_1, \ldots, t_r]\!]_v$. If $v$ is the $p$-adic valuation for some prime number $p$ we will write $\mathbb{T}[\![t_1, \ldots, t_r]\!]_p$.

*Remark* 8.1.2. As in Lemma 4.1.2, in a tropical partial differential semiring $(S, \{\partial_i\})$ for any $i$, the bend relations of any expression

$$\partial_i(x_1 \cdots x_n) + \sum_j x_1 \cdots x_{j-1} \partial_i(x_j) x_{j+1} \cdots x_n$$

hold.

Let $(S, \{\partial_1, \ldots, \partial_r\})$ be a semiring equipped with a family of $r$ additive maps $\partial_i \colon S \to S$. A *differential congruence* on $S$ is a congruence $K \subset S \times S$ that is closed under the action of the commutative monoid generated by $\{\partial_1, \ldots, \partial_r\}$. Analogously to the ordinary case, when $K$ is a differential congruence, the maps $\partial_i$ descends to additive maps $\overline{\partial_i} \colon S/K \to S/K$, and if the $\partial_i$'s are tropical differentials then the family $\{\overline{\partial_1}, \ldots, \overline{\partial_r}\}$ is a family of tropical differentials on $S/K$. Proposition 4.2.1 holds in this context as well, with similar proof.

## 8.2 Ritt polynomial algebras over a partial differential (semi)ring

In this section we define a similar constructions to those of Chapter 5, in the case of coefficients in a partial differential (semi)ring $(S, \{\partial_1, \dots, \partial_r\})$. Most of the results will have analogous proofs to those of the ordinary case.

In the following, let $\mathbb{F}_1[\partial_1, \dots, \partial_r]$ be the free $\mathbb{F}_1$-algebra generated by the set $\{\partial_1, \dots, \partial_r\}$. Notice that $\mathbb{F}_1[\partial_1, \dots, \partial_r]$ is isomorphic to $\mathbb{N}^r \cup \{\infty\}$ as $\mathbb{F}_1$-algebras. Sometimes, we will denote $\mathbb{F}_1[\partial_1, \dots, \partial_r]$ as $\Xi_r$ for shortness.

We start with the definition of a partial differential $\mathbb{F}_1$-module:

**Definition 8.2.1.** The category $\mathbf{D}\mathbb{F}_1\textbf{-Mod}_r$ of partial differential $\mathbb{F}_1$-modules with $r$ commuting differentials is the category of $\mathbb{F}_1[\partial_1, \dots, \partial_r]$-sets. When $m = 1$ we will omit to write $r$ in the previous notation as we recover the previously introduced category $\mathbf{D}\mathbb{F}_1\textbf{-Mod}$, thus the notation is coherent.

It is straightforward to see that a partial $\mathrm{D}\mathbb{F}_1$-module $\mathcal{M}$ with $r$ differential is the same as an $\mathbb{F}_1$-module equipped with a set of $r$ morphisms $\delta_1, \dots, \delta_r \colon \mathcal{M} \to \mathcal{M}$ of $\mathbb{F}_1$-modules commuting with each other. We will denote the free $\mathbb{F}_1$-algebra generated by the $\delta_i$ as $\Xi_{\mathcal{M}}$.

As for ordinary ones, partial differential $\mathbb{F}_1$-modules can be represented as graphs with some additional properties: every object in $\mathbf{D}\mathbb{F}_1\textbf{-Mod}_r$ can be represented as a directed graph $\mathcal{G} := (V, E)$, where $V$ and $E$ are allowed to be infinite sets and we allow multiple edges between any two vertices, satisfying the following conditions:

(1) $|\{e \in E \mid \mathrm{source}(e) = x\}| = r$ for all $x \in V$ ;

(2) For any $x \in V$, $r(r-1)$ out of the $r^2$ possible concatenation of 2 edges have pairwise the same target;

(3) $\mathcal{G}$ has at least one vertex $x$ such that every edge with source $x$ has also $x$ as target;

On the other hand a graph satisfying the conditions above can be endowed with the structure of a partial $\mathrm{D}\mathbb{F}_1$-module with $m$ commuting differentials. Indeed, denoting as $e_1(x), \dots, e_r(x)$ the edges having $x$ as source, let $\partial_i(x) = \mathrm{target}(e_i(x))$ for all $x \in \mathbb{N}$. The distinguished point can be chosen among elements for which the $e_i$'s are all loops. Since these choices are arbitrary, different choices give rise to a priori non-isomorphic partial $\mathrm{D}\mathbb{F}_1$-module structures.

*Example* 8.2.2. For any $r$, the partial $\mathrm{D}\mathbb{F}_1$-module $\mathbb{F}_1[\partial_1, \dots, \partial_r]$ is isomorphic to $\mathbb{N}^r \cup \{\infty\}$ equipped with differentials $\partial_i((a_1, \dots, a_r)) = (a_1, \dots a_i + 1, \dots, a_r)$. Then $\mathbb{F}_1[\partial_1, \partial_2]$ can be visualised graphically as:

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & & & & \iddots \\
 & & & (2,0) \arrow[ur] \arrow[dr] & \\
 & & (1,0) \arrow[ur] \arrow[dr] & & \cdots \\
\infty \arrow[loop above] \arrow[loop below] & (0,0) \arrow[ur] \arrow[dr] & & (1,1) \arrow[ur] \arrow[dr] & \\
 & & (0,1) \arrow[ur] \arrow[dr] & & \cdots \\
 & & & (0,2) \arrow[ur] \arrow[dr] & \\
 & & & & \ddots
\end{tikzcd}
$$

**Definition 8.2.3.** Given $r \in \mathbb{N}$, let $\Delta_r\colon\ \mathbb{F}_1\text{-}\mathbf{Mod} \to \mathbf{D}\mathbb{F}_1\text{-}\mathbf{Mod}_r$ be the free functor associating to an $\mathbb{F}_1$-module $(M, \star_M)$ the partial $\mathrm{D}\mathbb{F}_1$-module with $r$ commuting differentials $\Delta_r M$ defined as:

$$\Delta_r M := (\{\partial m \mid \partial \in \Xi_r,\, m \in M\}/\langle\{\partial\star_M \sim \star_M\}_{\partial\in\Xi_r}\rangle, \{\partial_1, \dots, \partial_r\})$$

We will identify $\star_{\Xi_r} m$ with $m$ for all $m \in M$.

*Example* 8.2.4. Let $X = \{\star_X, x\}$, $r \in \mathbb{N}$, then $\Delta_r X$ is isomorphic to the partial $\mathrm{D}\mathbb{F}_1$-module $(\mathbb{F}_1[\partial_1, \dots, \partial_r], \{\partial_1, \dots \partial_r\})$.

Products and coproducts in $\mathbf{D}\mathbb{F}_1\text{-}\mathbf{Mod}_r$ are defined in a similar way as in Section 5.1. A partial $\mathrm{D}\mathbb{F}_1$-module with $r$ differentials is *free of rank $n \in \mathbb{N}$* if it is isomorphic to $\bigvee_n \mathbb{F}_1[\partial_1, \dots, \partial_r]$. We define partial $\mathrm{D}\mathbb{F}_1$-submodules, congruences and quotients analogously to Definition 5.1.7.

**Definition 8.2.5.** A partial differential $\mathbb{F}_1$-algebra with $r$ commuting differentials is an $\mathbb{F}_1$-algebra $\mathcal{A}$ with an action of $\mathbb{F}_1[\partial_1, \dots, \partial_r]$ making it into an $\mathbb{F}_1[\partial_1, \dots, \partial_r]$-set. Let us denote the category of differential $\mathbb{F}_1$-algebras as $\mathbf{D}\mathbb{F}_1\text{-}\mathbf{Alg}_r$.

For every $r \in \mathbb{N}$, we now introduce the free functor

$$(-)_{\infty,r} : \mathbf{D}\mathbb{F}_1\text{-}\mathbf{Mod}_r \to \mathbf{D}\mathbb{F}_1\text{-}\mathbf{Alg}_r$$

mapping a partial $\mathrm{D}\mathbb{F}_1$-module with $r$ differentials to the free partial $\mathrm{D}\mathbb{F}_1$-algebras with $r$ differentials on it.

**Definition 8.2.6.** Given a partial $\mathrm{D}\mathbb{F}_1$-module $(\mathcal{M}, \{\delta_1, \dots, \delta_r\})$, let us define a sequence

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \dots$$

of partial differential $\mathbb{F}_1$-modules as:

$$\mathcal{M}_0 := (\mathcal{M}, \{\delta_1, \dots, \delta_r\})$$

$$\mathcal{M}_n := (\Delta_r F(\mathcal{M}_{n-1})/\langle \partial_i m \sim \delta_i m\rangle_{m\in\mathcal{M}}, \{\partial_1, \dots, \partial_r\}) \text{ for all } n \geq 1.$$

Then as in the case $r = 1$, we get an injective map of partial $\mathbb{F}_1$-modules $\mathcal{M}_n \hookrightarrow \mathcal{M}_{n+1}$ for every $n \in \mathbb{N}$ and we define $\mathcal{M}_{\infty,r} := \bigcup_{n=0}^{\infty} \mathcal{M}_n$. It is a partial $D\mathbb{F}_1$-algebra with $r$ differentials.

Again, we can give a more concrete realisation of $\mathcal{M}_{\infty,r}$, this time in terms of $r$-colored trees and forests, which we are now going to define: forgetting the labels of the differentials we can see elements of $\mathcal{M}_{\infty,r}$ as trees as we did in Section 5.1 when we defined $(-)_\infty$. Using the same terminology as in Definition 5.1.15, let us denote as $G(P)$ the graph $\{(v_1, v_2) \mid v_2 = P(v_1)\} \subset V \times V$ of the parent function $P$ of such trees. Then, given $r \in \mathbb{N}$, an *r-colour map* on the forest $(V, P)$ is a function

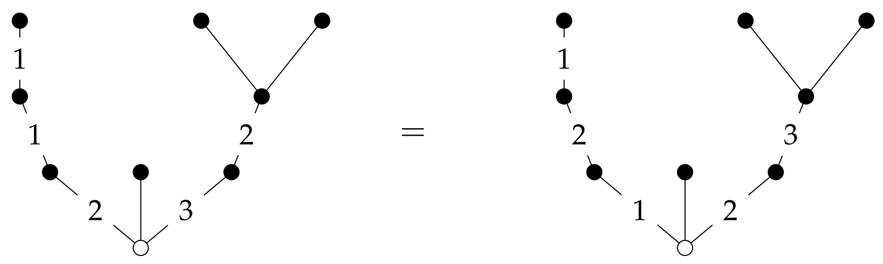
$$c\colon\ G(P) \to \{1, \dots, r\}$$

For $n \in \mathbb{N}$, a tuple of vertices $v_0, \dots, v_n, v_{n+1} \in V$ such that $P(v_i) = v_{i+1}$ is said to be a *defining sequence* for $(V, P)$ if $v_0$ is not univalent, the vertices $v_1, \dots, v_n$ are univalent and $v_{n+1}$ is not univalent or it is a root.

Two $r$-colour maps $c_1$ and $c_2$ on $(V, P)$ are equivalent if given any defining sequence $v_0, \dots, v_{n+1}$ the images of $\{(v_i, P(v_i)) \mid i = 0, \dots, n\}$ via $c_1$ and $c_2$ are equal as multisets. This is an equivalence relation $\sim$ on the set $C_r$ of $r$-colour maps on $(V, P)$.

**Definition 8.2.7.** An *r-colouring* on a forest is an element of the quotient $C_r/{\sim}$. An *r-coloured forest* is a forest $(V, P)$ equipped with a $r$-colouring. We will say that two $r$-coloured forests are equivalent if they have the same underlying forest and equivalent $r$-colourings.

*Example* 8.2.8. We will visualise $r$-coloured forests as forests where each segment whose vertices are in a defining sequence is labelled with an index $i \in \{1, \dots, r\}$. We focus on this set of segments as the value of an $r$-coloured function outside this set does not influence the $r$-colouring class it belongs to. As an example, the following two 3-coloured forests are equivalent:




Consider the monoid of isomorphism classes of $r$-coloured trees with leaves labelled by elements of $\mathcal{M}$, with monoid structure given as in the ordinary case. We can make it into a partial $D\mathbb{F}_1$-algebra by defining the set of differentials $\{\partial_1, \dots, \partial_r\}$ as $\partial_i$ being the map adding an $i$-labelled segment to the stem of an $r$-coloured tree. Thanks to the definition of $r$-colouring, we have that $\partial_i\partial_j = \partial_j\partial_i$ for any $i, j \in \{1, \dots, r\}$.

Given an element $m \in \mathcal{M}$, we denote by $[m]$ the tree consisting of the single leaf labelled by $m$. Denote as $\mathcal{F}_r(\mathbb{F}_1, \mathcal{M})$ the quotient of the partial $\mathrm{D}\mathbb{F}_1$-module described above by the relations generated by $\{\partial_i[m] \sim [\delta_i m]\}_{m \in \mathcal{M}, i \in \{1,\ldots,r\}}$. Analogously to Lemma 5.1.16, we have that $\mathcal{M}_{\infty,r}$ and $\mathcal{F}_r(\mathbb{F}_1, \mathcal{M})$ are isomorphic as partial $\mathrm{D}\mathbb{F}_1$-algebras with $r$ differentials. Furthermore, for any $n \in \mathbb{N}$, the isomorphism above gives the following isomorphism of partial $\mathrm{D}\mathbb{F}_1$-modules with $r$ differentials:

$$\mathcal{M}_n \cong \left\{ \begin{array}{c} \textit{classes of } r\textit{-coloured trees with at most } n \textit{ branching} \\ \textit{points on any path from the root to a leaf} \end{array} \right\} \subset \mathcal{F}_r(\mathbb{F}_1, \mathcal{M})$$

With analogue proof to that of Proposition 5.1.20, we have the following:

**Proposition 8.2.9.** *For every $r \in \mathbb{N}$, the assignment $\mathcal{M} \mapsto \mathcal{M}_{\infty,r}$ is a free functor*

$$(-)_{\infty,r} \colon \mathbf{D}\mathbb{F}_1\textbf{-Mod}_r \to \mathbf{D}\mathbb{F}_1\textbf{-Alg}_r.$$

As in Section 5.3, for any partial differential (semi)ring $(S, \{\partial_1, \ldots, \partial_r\})$ with $r$ commuting differentials, we will construct two free functors $S\{-\} \colon \mathbf{D}\mathbb{F}_1\textbf{-Mod}_r \to \mathbf{D}S\textbf{-Alg}$ and $(-)_S \colon \mathbf{D}\mathbb{F}_1\textbf{-Alg}_r \to \mathbf{D}S\textbf{-Alg}$ such that for every $r$ the following diagram commutes:

$$\begin{tikzcd} \mathbf{D}\mathbb{F}_1\textbf{-Mod}_r \arrow[rr, "(-)_{\infty,r}"] \arrow[dr, "S\{-\}"'] & & \mathbf{D}\mathbb{F}_1\textbf{-Alg}_r \arrow[dl, "(-)_S"] \\ & \mathbf{D}S\textbf{-Alg} & \end{tikzcd} \tag{8.2.1}$$

Let $DM \colon \mathbf{D}S\textbf{-Alg} \to \mathbf{D}\mathbb{F}_1\textbf{-Mod}_r$ be the forgetful functor. Given a partial $\mathrm{D}\mathbb{F}_1$-module $(\mathcal{M}, \{\partial_1, \ldots, \partial_r\})$, let $\mathcal{F}_r(S, \mathcal{M})$ to be defined in complete analogy to $\mathcal{F}(S, \mathcal{M})$, as in Section 5.3, substituting $(-)_\infty$ with $(-)_{\infty,r}$. As $\mathcal{F}_r(S, \mathcal{M})$ comes equipped with a set of linear maps $\partial_1, \ldots, \partial_r$ as in the $r = 1$ case, we make it into a differential algebra by taking the following quotients, in the case of rings:

**Definition 8.2.10.** If $R$ is a partial differential ring, let $L_r$ be the differential ideal generated by the set

$$\{\partial_i(st) - s\partial_i t - t\partial_i s \mid s, t \in \mathcal{F}_r(R, \mathcal{M}),\, i \in \{1, \ldots, r\}\}$$

i.e. the smallest ideal in $\mathcal{F}_r(R, \mathcal{M})$ containing this set and closed under applying $\partial_1, \ldots, \partial_r$. We define the partial differential $R$-algebra $R\{\mathcal{M}\}$ as $\mathcal{F}_r(R, \mathcal{M})/L_r$ equipped with differentials $\partial_1, \ldots, \partial_r$.

With proof analogous to 5.3.2 we have the following:

**Proposition 8.2.11.** *Given any partial differential ring $R$ and partial* $\mathrm{D}\mathbb{F}_1$*-module $\mathcal{M}$, both equipped with $r$ differentials, $R\{\mathcal{M}\}$ is isomorphic to $R[F(\mathcal{M})]$ as an $R$-algebra.*

*Remark* 8.2.12. When $\mathcal{M}$ is free of rank $n$, this isomorphism extends to an isomorphism of partial differential $R$-algebras

$$(R\{\mathcal{M}\}, \{\partial_1, \dots, \partial_r\}) \cong (R\{x_1, \dots, x_n\}, \{\partial_1, \dots, \partial_r\}).$$

Thus this construction generalises Ritt's one in the partial case as well.

*Example* 8.2.13. Let $(R, \{\partial_1, \partial_2\})$ be a partial differential ring and $(\mathcal{M}, \{\delta_1, \delta_2\})$ be the partial $\mathbb{DF}_1$-module associated to the following directed graph:

$$\begin{tikzcd}
 & (1,0) \arrow[dr] \arrow[drr, bend left] & & \\
(0,0) \arrow[ur] \arrow[dr] & & (1,1) \arrow[r, bend left] \arrow[r, bend right] & \infty \arrow[loop above] \arrow[loop below] \\
 & (0,1) \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

where the choice of the differentials is uninfluent as in this case the two possible choices for $\delta_1$ and $\delta_2$ return isomorphic $\mathbb{DF}_1$-modules.

Since $\mathcal{M}$ is isomorphic to the quotient of $\mathbb{F}_1[\partial_1, \partial_2]$ by the free submodule generated by $\partial_1^2$ and $\partial_2^2$, the differential $R$-algebra $R\{\mathcal{M}\}$ is isomorphic to the quotient of the free differential algebra $R\{x\}$ by the differential ideal generated by the monomials $\partial_1^2 x$ and $\partial_2^2 x$.

Let us now move to the construction of $S\{\mathcal{M}\}$ when $(S, \{\partial_1, \dots, \partial_r\})$ is a partial differential idempotent semiring. Analogously to the ordinary case, let

$$L_{\text{trop},r} = \{\partial_i(st) + s\partial_i t + t\partial_i s \mid s, t \in \mathcal{F}_r(S, \mathcal{M}),\ i \in \{1, \dots, r\}\}$$

and:

**Definition 8.2.14.** Given a partial $\mathbb{DF}_1$-module $(\mathcal{M}, \{\delta_1, \dots, \delta_r\})$ define the differential $S$-algebra $S\{\mathcal{M}\}$ as the following quotient:

$$S\{\mathcal{M}\} := \mathcal{F}_r(S, \mathcal{M}) / \mathcal{B}\langle L_{\text{trop},r} \rangle.$$

*Remark* 8.2.15. In this case too, when $S$ is a partial differential semiring, in general there is just an injective homomorphism of $S$-algebras $S[F(\mathcal{M})] \hookrightarrow S\{\mathcal{M}\}$.

With analogous proof to that of Proposition 5.3.8, we have:

**Proposition 8.2.16.** *For every $r \in \mathbb{N}$, given a partial differential (semi)ring $S$ with $r$ commuting differentials, the assignment $\mathcal{M} \mapsto S\{\mathcal{M}\}$ is a free functor*

$$S\{-\}\colon \mathbf{D}\mathbb{F}_1\text{-}\mathbf{Mod}_r \to \mathbf{D}S\text{-}\mathbf{Alg}.$$

From the freeness of $S\{-\}$ it follows that:

**Corollary 8.2.17.** *Given a differential $S$-algebra $T$, there is a bijection*

$$\mathrm{Hom}_{\mathbf{D}S\text{-}\mathbf{Alg}}(S\{x_1, \dots, x_n\}, T) \cong (T)^n$$

Given any partial differential (semi)ring, the definition of the functor

$$(-)_S\colon \mathbf{D}\mathbb{F}_1\text{-}\mathbf{Alg} \to \mathbf{D}S\text{-}\mathbf{Alg}$$

the proof of its freeness and of the commutativity of diagram 8.2.1 are completely analogous to the ordinary case, as in Section 5.3

## 8.3 Partial tropical pairs

Similar to the ordinary case, in order to talk about tropical partial differential equations we introduce the category of partial tropical pairs. This section is devoted to introduce in the partial setting all the main objects of Chapter 6.

For every $r \in \mathbb{N}$, a *partial tropical pair* $\mathbf{S}$ with $r$ differentials consists of a partial tropical differential semiring $S_1$ with $r$ differentials, an idempotent semiring $S_0$, and a homomorphism of idempotent semirings $\pi\colon S_1 \to S_0$. Let us denote as $\mathbf{Pairs}_r$ the category of partial tropical pairs with $r$ differentials.

We define reduced partial pairs as for ordinary ones and we denote the full subcategory of $\mathbf{Pairs}_r$ of reduced partial pairs as $\mathbf{Pairs}_{r,\mathrm{red}}$. Analogously we define $\mathbf{S}$-algebras for a partial pair $\mathbf{S}$ and denote their category as $\mathbf{S}$**-Alg**: this category will describe tropical partial differential equations and their solutions.

*Example* 8.3.1. (1) For any morphism of idempotent semirings $\mu\colon S \to T$ we have a pair

$$(S, \{\partial_1 = \cdots = \partial_r = 0\}) \to T,$$

and it is reduced if and only if $\mu$ is injective. If $\mu$ is not injective, then we can replace $S$ with $\mathrm{Im}(\mu)$ to obtain a reduced pair.

(2) For every $r \in \mathbb{N}$, let $(\mathrm{Conv}_r(\mathbb{B}), \oplus, \otimes)$ be the semiring whose underlying set is that of convex hulls of finite (possibly empty) sets of points in $\mathbb{R}^r$, where $\oplus$ is given by taking the convex hull of the union and $\otimes$ is Minkowski sum. With these operation $0_{\mathrm{Conv}_r(\mathbb{B})} = \emptyset$, which is the convex hull of itself, and $1_{\mathrm{Conv}_r(\mathbb{B})}$ is the convex hull of the singleton set $\{0_{\mathbb{R}^r}\}$, i.e. the positive orthant of $\mathbb{R}^r$.

Let $(\mathbb{B}[\![t_1, \ldots, t_r]\!], \{\partial/\partial t_i\})$ as in Example 8.1.1 and consider the partial pair

$$\pi_r\colon \mathbb{B}[\![t_1, \ldots, t_r]\!] \to \mathrm{Conv}_r(\mathbb{B}).$$

sending an element $a \in \mathbb{B}[\![t_1, \ldots, t_r]\!]$ to the convex hull of its support. Notice that $\pi_r(A)$ is the convex hull of a finite set of exponents, namely its vertices, by definition of multivariate power series. We will call leading terms the terms whose exponents are the vertices of $\pi_r(a)$. As the information of the convex hull of a subset or that of its vertices are equivalent we will often not distinguish between the two. It is straightforward to see that when $r = 1$, $\mathrm{Conv}_1(\mathbb{B}) \cong \mathbb{T}$ and $\pi_1$ is the pair $\pi$ of point (2) of Example 6.1.2.

This partial pair is reduced. Indeed if $a \neq b \in \mathbb{B}[\![t_1, \ldots, t_r]\!]$ then there exists an exponent $\mathbf{n} = (n_1, \ldots, n_r)$ such that the $\mathbf{n}$-th term of $a$ and of $b$ are different, and without loss of generality we can assume $a_{\mathbf{n}} = 0$ and $b_{\mathbf{n}} = \infty$. Then, letting

$$\partial^{\mathbf{n}} := \prod_{i=1}^{r} (\partial/\partial t_i)^{n_i},$$

we have that $\pi_r(\partial^{\mathbf{n}}(a))$ is the positive orthant while $\pi_r(\partial^{\mathbf{n}}(b))$ is strictly contained in it.

(3) Similarly to the ordinary case, for every $r \in \mathbb{N}$ the partial pair

$$\pi'_r \colon \mathbb{T}[\![t_1, \ldots, t_r]\!]_0 \to \mathrm{Conv}_r(\mathbb{B})$$

sending a tropical power series to its convex hull, as above, is not reduced. In fact it reduces to the partial pair of point (2).

(4) For every $r \in \mathbb{N}$, let $(\mathrm{Conv}_r(\mathbb{T}), \oplus, \otimes)$ be the semiring whose underlying set is the set of convex hulls of finite (possibly empty) subsets of $\mathbb{R}^r$ with vertices weighted in $\mathbb{T}$. The sum $\oplus$ is given by taking the convex hull of the union with weights given by the sum in $\mathbb{T}$ of the weights. Similarly, the product $\otimes$ is the Minkowski sum with weights given by the product in $\mathbb{T}$ of the weights.

Consider the partial pair

$$\pi_r \colon \mathbb{T}[\![t_1, \ldots, t_r]\!] \to \mathrm{Conv}_r(\mathbb{T}),$$

where the source has any of the differentials as in Example 8.1.1 for a single valuation $v$. The morphism $\pi_r$ is given by sending $a \in \mathbb{T}[\![t_1, \ldots, t_r]\!]$ to the convex hull of its support with vertices weighted by the valuation $v(a_{\mathbf{n}})$ of the coefficients of the corresponding leading terms. A modification of the argument used in point (2) above shows that $\pi_r$ is reduced, furthermore, when $r = 1$, we recover the pair of point (4) of Example 6.1.2.

(5) For $r \in \mathbb{N}$, given an ordering $t_{l_1} \preceq t_{l_2} \preceq \cdots \preceq t_{l_r}$ of the variables of $\mathbb{T}[\![t_1, \ldots, t_r]\!]$, and endowing $\mathbb{T}[\![t_1, \ldots, t_r]\!]$ with any of the differentials of Example 8.1.1 let

$$\rho_r \colon \mathbb{T}[\![t_1, \ldots, t_r]\!] \to \mathbb{T}_{r+1}$$

be the map sending a multivariate power series $\sum_{\mathbf{w} \in \mathbb{N}} a_{\mathbf{w}} t^{\mathbf{w}}$ to $(\overline{\mathbf{w}}, a_{\overline{\mathbf{w}}})$ where $\overline{\mathbf{w}} = \min_{\preceq}\{\mathbf{w} \in \mathbb{N} \mid a_{\mathbf{w}} \neq \infty\}$. As in (2) and (4), the pair $\rho_r$ is reduced for any $r \in \mathbb{N}$, and when $r = 1$ the pair $\rho_1$ recovers that of point (4) and $\rho_1|_{\mathbb{B}[\![t]\!]}$ recovers the pair $\pi$ of point (2) of Example 6.1.2.

For every $r \in \mathbb{N}$, given a partial pair $\mathbf{S} = (S_1 \to S_0)$ with $r$ differentials, we define now the functor $\mathbf{S}\{-\} \colon \mathbf{D}\mathbb{F}_1\textbf{-Mod}_r \to \mathbf{Pairs}_r$ generalising the functor introduced in Section 6.1 for $r = 1$. Let the idempotent semiring $(S_0|S_1)\{\mathcal{M}\}$ be the pushout:

$$\begin{tikzcd}
S_1 \arrow[r] \arrow[d] & S_1\{\mathcal{M}\} \arrow[d, dotted] \\
S_0 \arrow[r, dotted] & (S_0|S_1)\{\mathcal{M}\}
\end{tikzcd}$$

in the category of semirings. As for $r = 1$ this pushout can be described as the algebra of $r$-coloured trees $S_1\{\mathcal{M}\}$ modulo the congruence generated by the relations that identify $a, b \in S_1 \subset S_1\{\mathcal{M}\}$ if they have the same image in $S_0$. In this case too we have that $(S_0|S_1)\{\mathcal{M}\}$ contains the polynomial $S_0$-algebra $S_0[F(\mathcal{M})]$ and we denote as $\mathbf{S}\{\mathcal{M}\}$ the $\mathbf{S}$-algebra $S_1\{\mathcal{M}\} \to (S_0|S_1)\{\mathcal{M}\}$.

With analogous proof as that of 6.1.4, the following holds:

**Proposition 8.3.2.** *For every $r \in \mathbb{N}$, given a reduced partial pair $\mathbf{S} = (S_1 \to S_0)$, the assignment $\mathcal{M} \mapsto \mathbf{S}\{\mathcal{M}\}$ is a free functor*

$$\mathbf{S}\{-\}\colon\ \mathbf{D}\mathbb{F}_1\textbf{-Mod}_r \to \mathbf{S}\textbf{-Alg}.$$

Which for free partial $\mathrm{D}\mathbb{F}_1$-modules implies a generalisation of Corollary 6.1.5 to any $r \in \mathbb{N}$.

Proceeding as in Section 6.2, we have a reduction functor in the partial setting as well:

**Proposition 8.3.3.** *For every $r \in \mathbb{N}$, the assignment $\mathbf{S} \mapsto \mathbf{S}^{red}$ defines a functor*

$$\mathscr{R}\colon\ \mathbf{Pairs}_r \to \mathbf{Pairs}_{r,\mathrm{red}}$$

*and the quotient map $\mathbf{S} \to \mathbf{S}^{red}$ is a natural transformation* $\mathrm{Id} \to \mathscr{R}$*. Moreover $\mathscr{R}$ is left adjoint to the inclusion $\iota\colon\ \mathbf{Pairs}_{r,\mathrm{red}} \hookrightarrow \mathbf{Pairs}_r$.*

A quotient of a partial differential pair $\mathbf{S} : S_1 \xrightarrow{\pi} S_0$ is a morphism $\mathbf{S} \to \mathbf{T}$ in $\mathbf{Pairs}_r$ such that both components are surjective or equivalently a pair of congruences $(K_1 \subset S_1 \times S_1, K_0 \subset S_0 \times S_0)$ where $K_1$ is a differential congruence and $K_0$ is a semiring congruence, such that $\pi(K_1) \subset K_0$.

Given a partial $\mathrm{D}\mathbb{F}_1$-module $\mathcal{M}$ and a congruence $K$ on the polynomial semiring $S_0[F(\mathcal{M})]$ as in the ordinary case we let $(S_0|S_1)\{\mathcal{M}\}/K$ denote the induced quotient, and

$$\mathbf{S}\{\mathcal{M}\}/\!\!/K$$

denote the reduction of the pair $S_1\{\mathcal{M}\} \to (S_0|S_1)\{\mathcal{M}\}/K$. Analogous statements to Proposition 6.3.2 and Corollary 6.3.3 holds.

Fix a reduced partial pair $\mathbf{S} = (S_1 \xrightarrow{\pi} S_0)$ and a partial $\mathrm{D}\mathbb{F}_1$-module $\mathcal{M}$, then a *tropical partial differential equation* is a polynomial $f \in S_0[F(\mathcal{M})]$ and, as in Definition 6.3.4, given an $\mathbf{S}$-algebra $\mathbf{T} = (T_1 \to T_0)$, the *set of solutions* to $f$ in $\mathbf{T}$, denoted $\mathrm{Sol}_{\mathbf{T}}(f)$, is the set of morphisms $p \in \mathrm{Hom}_{\mathbf{D}\mathbb{F}_1\textbf{-Mod}_r}(\mathcal{M}, T_1)$ such that $\Phi(p)_0$ factors through $(S_0|S_1)\{\mathcal{M}\}/\mathcal{B}(f)$.

*Remark* 8.3.4. When the pair $S_1 \to S_0$ is $\mathbb{B}[\![t_1, \dots, t_r]\!] \xrightarrow{\pi_r} \mathrm{Conv}_r(\mathbb{B})$, the above definition recovers the framework considered in [FGLH+20].

Finally, a generalisation of 6.3.5 holds:

**Theorem 8.3.5.** *For every $r \in \mathbb{N}$, given a partial pair $\mathbf{S} : S_1 \to S_0$ with $r$ differentials and given any set $E \subset S_0[F(\mathcal{M})]$ of tropical PDEs, the functor $\mathbf{S}\textbf{-Alg} \to \mathbf{Sets}$ sending an $\mathbf{S}$-algebra $\mathbf{T}$ to $\mathrm{Sol}_{\mathbf{T}}(E)$ is corepresented by $\mathbf{S}\{\mathcal{M}\}/\!\!/\mathcal{B}(E)$.*

By a similar proof to that of Proposition 6.4.1 we have that the category of differential idempotent semirings with $r$ commuting differentials is cocomplete for any $r \in \mathbb{N}$, and proceeding as in Section 6.4 it is possible to prove that colimits in the category $\mathbf{Pairs}_r$ can be computed by computing the colimits of the top and bottom parts separately.

**Definition 8.3.6.** Given a differential ring $(R, \{\partial_{R,1}, \dots, \partial_{R,r}\})$ with $r$ commuting differentials and a valuation $v\colon R \to S_0$, a *differential enhancement of $v$* is a reduced partial pair $\mathbf{S} = (S_1 \to S_0)$ in $\mathbf{Pairs}_{r,\mathrm{red}}$ and a map of sets $\widetilde{v}\colon R \to S_1$ such that

(1) $\widetilde{v}(0) = 0 \in S_1$ and $\widetilde{v}(1) = 1 \in S_1$;

(2) it commutes with the differentials: $\partial_{S_1,i}\widetilde{v}(x) = \widetilde{v}(\partial_{R,i}x)$ for any $x \in R$, for any $i \in \{1, \dots, r\}$;

(3) the following diagram commutes:

$$\begin{tikzcd} & S_1 \arrow[d] \\ R \arrow[ur, "\widetilde{v}"] \arrow[r, "v"] & S_0 \end{tikzcd}$$

Proposition 6.5.6 and all the properties highlighted in Section 6.5 about differentially enhanced valuations are valid for every $r \in \mathbb{N}$.

*Example* 8.3.7. (1) Given a field $K$, the valuation $v\colon K[\![t_1, \dots, t_r]\!] \to \mathrm{Conv}_r(\mathbb{B})$ sending a power series to the convex hull of its support admits a differential enhancement to the pair of Example 8.3.1 part (2):

$$\begin{tikzcd} & \mathbb{B}[\![t_1, \dots, t_r]\!] \arrow[d] \\ K[\![t_1, \dots, t_r]\!] \arrow[ur, "\widetilde{v}"] \arrow[r, "v"] & \mathrm{Conv}_r(\mathbb{B}) \end{tikzcd}$$

where $\widetilde{v}$ is the map $K[\![t_1, \dots, t_r]\!] \to \mathbb{B}[\![t_1, \dots, t_r]\!]$ given by coefficientwise trivial valuation, i.e. taking the support. This is the differentially enhanced valuation used in [FGLH+20].

(2) Given a prime number $p$, consider the valuation $v\colon \mathbb{Q}_p[\![t_1, \dots, t_r]\!] \to \mathrm{Conv}_r(\mathbb{T})$, defined by sending a power series to the convex hull of its support, with weights given by the $p$-adic valuations of the coefficients of the vertices. This admits a differential enhancement to the pair of Example 8.3.1 part (4)

$$\begin{tikzcd} & \mathbb{T}[\![t_1, \dots, t_r]\!]_p \arrow[d] \\ \mathbb{Q}_p[\![t_1, \dots, t_r]\!] \arrow[ur, "\widetilde{u}"] \arrow[r, "u"] & \mathrm{Conv}_r(\mathbb{T}) \end{tikzcd}$$

where $\widetilde{u}$ sends a power series to its coefficientwise $p$-adic valuation. Let $\mathbf{u} := (u, \widetilde{u})$, there is a morphism of pairs

$$\begin{tikzcd} \mathbb{T}[\![t_1, \dots, t_r]\!]_p \arrow[r] \arrow[d] & \mathbb{B}[\![t_1, \dots, t_r]\!] \arrow[d] \\ \mathrm{Conv}_r(\mathbb{T}) \arrow[r] & \mathrm{Conv}_r(\mathbb{B}) \end{tikzcd}$$

given on the top by the sending all finite coefficients to 0, and on the bottom by sending all finite weights to 0. This morphism of pairs sends differentially enhanced valuation $\mathbf{u}$ to the $\mathbf{v} := (v, \widetilde{v})$ of the previous point. Thus $\mathbf{u}$ provides a refinement of the structure considered by Falkensteiner at al. in [FGLH$^+$20].

The same construction gives a differential enhancement also in the case of a valued field $w\colon K \to \mathbb{T}$ in place of $\mathbb{Q}_p$ and with $\mathbb{T}[\![t_1, \dots, t_r]\!]$ equipped with the differential associated to $w$.

(3) Let $w\colon K \to \mathbb{T}$ be a valued field, given an ordering $t_{l_1} \preceq t_{l_2} \preceq \cdots \preceq t_{l_r}$ of the variables of $K[\![t_1, \dots, t_r]\!]$, consider the valuation

$$v\colon K[\![t_1, \dots, t_r]\!] \to \mathbb{T}_{r+1}$$

sending a power series $\sum_{\mathbf{w}\in\mathbb{N}} b_{\mathbf{w}} t^{\mathbf{w}} \in K[\![t_1, \dots, t_r]\!]$ to $(\overline{\mathbf{w}}, v(b_{\overline{\mathbf{w}}}))$ where $\overline{\mathbf{w}} = \min_{\preceq}\{\mathbf{w} \in \mathbb{N} \mid a_{\mathbf{w}} \neq 0\}$. This admits a differential enhancement to the pair of Example 8.3.1 part (5)

$$\begin{tikzcd} & \mathbb{T}[\![t_1, \dots, t_r]\!]_w \arrow[d] \\ K[\![t_1, \dots, t_r]\!] \arrow[ur, "\widetilde{v}"] \arrow[r, "v"] & \mathbb{T}_{r+1} \end{tikzcd}$$

where again $\widetilde{v}$ is the coefficientwise application of $w$.

Let $(R, \{\partial_1, \dots, \partial_r\})$ a partial differential ring equipped with a differentially enhanced valuation $\mathbf{v}$ to a partial pair $\mathbf{S} = (S_1 \to S_0)$ and let $A$ be a partial differential $R$-algebra. We conclude this section by noticing that it is possible to define differential enhancements compatible with $\mathbf{v}$ on $A$ and the differential Berkovich space over an $\mathbf{S}$-algebra $\mathbf{T}$ of $A$ exactly as in Section 6.6. Furthermore we can still define the natural map

$$\Lambda\colon \mathit{Berk}_{\mathbf{T}}(A) \to \mathcal{T}\!\mathit{rop}_{univ}(\operatorname{Spec} A)(T_0)$$

sending a differentially enhanced valuation $\mathbf{w} = (\widetilde{w}, w)$ to its underlying valuation $w$ and with analogous proof to Lemma 6.6.2 we have that $\Lambda$ is injective for every $r \in \mathbb{N}$.

## 8.4 Tropicalization of partial differential equations

In this section we quickly go through a generalisation of the results of Chapter 7 to the partial case. We start by giving the notion of solution to a partial differential equation over a differential algebra $R\{\mathcal{M}\}$.

**Definition 8.4.1.** Let $\mathcal{M}$ be a partial $\mathbb{DF}_1$-module and $A$ a differential algebra over a partial differential ring $(R, \{\partial_1, \dots, \partial_r\})$. The set of solutions in $A$ of a differential polynomial $f \in R\{\mathcal{M}\}$, denoted $\mathrm{Sol}_A(f)$, is the set of morphisms $p \in \mathrm{Hom}_{\mathbf{D}\mathbb{F}_1\text{-}\mathbf{Mod}_r}(\mathcal{M}, A)$ such that its adjoint $\Psi(p)$ factors through $R\{\mathcal{M}\}/[f]$, where $[f]$ is the differential ideal generated by $f$.

As in Theorem 7.1.4, given a differential ideal $I \subset R\{\mathcal{M}\}$, the functor of solutions from $\mathbf{D}R$**-Alg** to the category of sets is corepresented by $R\{\mathcal{M}\}/I$.

Given a partial differential ring $R$ equipped with a differentially enhanced valuation $\mathbf{v} = (\widetilde{v}, v)\colon R \to \mathbf{S} = (S_1 \xrightarrow{\pi} S_0)$, $A$ a differential $R$-algebra equipped with a differentially enhanced valuation $\mathbf{w} = (\widetilde{w}, w)\colon A \to \mathbf{T} = (T_1 \xrightarrow{\rho} T_0)$ compatible with $\mathbf{v}$ and $\mathcal{M}$ a partial $\mathrm{D}\mathbb{F}_1$-module, we tropicalize morphisms $p \in \mathrm{Hom}_{\mathbf{D}\mathbb{F}_1\text{-}\mathbf{Mod}_r}\{\mathcal{M}, A\}$ by composition with $\widetilde{w}$, we tropicalize partial differential equations $f \in R\{\mathcal{M}\}$ by applying $v$ coefficient-wise, landing in $S_0[F(\mathcal{M})]$. Given a quotients $\alpha\colon R\{\mathcal{M}\} \twoheadrightarrow R\{\mathcal{M}\}/I$ we define $\mathbf{trop}(\alpha)$ to be the partial pair

$$\mathbf{trop}(\alpha) := \mathbf{S}\{\mathcal{M}\} /\!\!/ \mathcal{B}\mathrm{trop}_v(I).$$

The easy containment of the fundamental theorem of tropical differential algebra holds for every $r$ with the same proof of Proposition 7.1.5.

Let $\mathbf{Pres}(A)$ be the category of $\mathbb{F}_1$ partial differential presentations of $A$, with analogous definition as in the case $r = 1$, thus

$$(R\{\mathcal{M}\} \overset{\alpha}{\twoheadrightarrow} A) \mapsto \mathbf{trop}(\alpha) := \mathbf{S}\{\mathcal{M}\} /\!\!/ \mathcal{B}\mathrm{trop}_v(\ker \alpha)$$

is a functor $\mathbf{Pres}(A) \to \mathbf{S}$**-Alg** and defining the universal presentation of $A$ as

$$\mathit{Univ}\colon\ R\{DM(A)\} \twoheadrightarrow A$$

induced by morphism $id_A\colon\ DM(A) \to DM(A)$ in $\mathbf{D}\mathbb{F}_1\text{-}\mathbf{Mod}_r$ we have that analogues to Proposition 7.3.1 and Theorem 7.3.2 hold for every $r \in \mathbb{N}$.

Finally we notice that we have a partial differential version of [GG14, Theorem A]), in fact it is possible to prove Theorem 7.3.3 and consequent Corollary 7.3.4 for arbitrary $r \in \mathbb{N}$ following the same steps of Section 7.3.

# Chapter 9

# Future directions

The material introduced in the previous pages opens up many different paths of research and here we will list some possible future directions that we identified:

- The most natural application to think of is perhaps the proof of an extended version of the fundamental theorems of [AGT16] and [FGLH$^+$20] to this generalized context, this would allow to directly give informations about the convergence of solutions to differential equations by using combinatorial methods in the tropical world. This would also link to the computation of the convergence radii of solutions to $p$-adic differential equations, which is described in the next bullet.

- As sketched in the introductive chapter, investigate the possibility to recover information about the radius of convergence of the solution to a $p$-adic differential equation by computations taking place in the tropical world. Furthermore, explore a possible way to give a notion of radius of convergence for $p$-adic differential equations in higher dimensions: in the one dimensional case the radii of convergence are in relation with the slopes of the Newton polygon associated to the differential polynomial defining the equation: it is just a matter of rephrasing to show that these slopes are the tropical roots of the tropicalization of the polynomial. Generalising this line of thought, it would be interesting to find a definition for these radii in higher dimension such that they are encoded in the tropical variety of the polynomial associated with the equation;

- Moving from the purely algebraic approach of treated here to a geometric one: an extension of the tropicalization functor to non-affine differential schemes over $\mathbb{F}_1$ in analogy with the classical case treated in [GG16], combining differential scheme-theory [Kei75, CF78, Kov02, Bar10] and the work in [CLS12]. Once given this formulation, a more thorough study of the geometric and combinatoric aspects of this new kind of tropical schemes can be undertaken;

- A development of a theory of differential tropical ideals: which would at first require a development of a notion of "differential valued matroid", in analogy to the work carried out in [MR18]. Tropical ideals are ideals in the polynomial ring over the tropical semifield such that each bounded degree piece is the set of vectors of a valuated matroid; the variety associated to a tropical ideal is still a finite balanced polyhedral complex. Furthermore the tropicalization of an ideal is a tropical ideal, so this new, intrinsic notion proves to give the right class of ideals cutting out well-behaved tropical varieties, also satisfying several other nice properties (ascending chain condition, eventual polynomiality of Hilbert function, tropical Nullstellensatz). As a first step, it would be interesting to develop a notion of tropical differential module and investigate its links with linear tropical differential equations.

- A definition of differential tropical basis: in tropical geometry a tropical basis for an ideal is a finite generating set such the intersection of the tropical hypersurfaces of its polynomials is equal to the tropical variety associated to the ideal, thus giving a finite intersection instead that an infinite one. A tropical basis exists for every ideal. In [FT20] a natural definition of a differential analogue has been given and its non-existence has been proven in an example, thus another, less straightforward idea has to be found.

- Developing an algorithmic way to compute solutions to linear tropical differential equations in the setting we introduce in the present work, generalising the algorithm introduced in [Gri17], and implement it as a package for mathematical softwares.

- Investigating the possible interplay between the theory of tropical (partial) differential equations and box-ball systems (also known as soliton cellular automata). These are integrable cellular automata that were introduced in [TS90] as a discrete counterpart of the Korteweg-de Vries equation, which is a mathematical model of waves on shallow water surface. Box-ball systems arise both from classical and quantum integrable systems via discretization procedures, this endows them with several aspects related to Yang-Baxter integrable models. Tropical geometry has already proven to be a valuable tool in their study, thus we believe that the framework we introduced here could give new instights in the study of these systems. For a general overview on the subject see [IKT12].

# Part II

# Trusses and braces

# Chapter 10

# Introduction

In the 1920s H. Prüfer and R. Baer defined heaps as algebraic objects consisting of a set with a ternary operation which fulfils conditions that allows to associate a class of isomorphic groups, one for every element of the heap; conversely, every group gives rise to a heap by taking the operation $(a, b, c) \mapsto ab^{-1}c$ (see [Bae29] and [Prü24]). In 2007 W. Rump introduced braces as algebraic systems corresponding to set-theoretic solutions of the Yang-Baxter equation [Rum07]. A brace is a triple $(G, +, \cdot)$ where $(G, +)$ is an Abelian group, $(G, \cdot)$ is a group and the following distributive law holds, for all $a, b, c \in G$,

$$a \cdot (b + c) = a \cdot b - a + a \cdot c;$$

see [CJO14]. Through their connection with set-theoretical solutions of the Yang-Baxter equation, braces have become an intensive field of studies. In particular it has been shown that a brace allows one to construct a non-degenerate involutive set-theoretic solution of Yang-Baxter equation (see for example, [CJO14], [Rum07], [CGIS17] and [Smo18]). In 2017 L. Guarnieri and L. Vendramin introduced the notion of a skew brace. This is a generalisation of a brace in which $(G, +)$ is not required to be Abelian [GV17]. It has been shown to correspond closely to non-degenerate set-theoretic solutions of the Yang-Baxter equation; one can construct such a solution from any skew brace, while to any non-degenerate bijective solution one can associate a skew brace that satisfies a universal property (see [GV17], [ESS99], [Sol00], [LYZ00], [SV18] or [Bac18]). In recent years there has been a vast progress in the research on set-theoretical solutions of the Yang-Baxter equation, but, even though we know that every skew brace provides us with such a solution, it is not an easy task to construct skew braces (for a list of problems on skew braces and a literature review see [Ven18]). In 2018, T. Brzezinski in [Brz19] observed that it is possible to unify the distributive laws of rings and braces in a single more general algebraic structure, that of a truss. A skew left truss $T$ is a heap $(T, [-,-,-])$ with an additional binary operation $\cdot \colon T \times T \to T$ which is associative and which distributes over the ternary operation from the left, i.e., for all $a, b, c, d \in T$,

$$a \cdot [b, c, d] = [a \cdot b, a \cdot c, a \cdot d].$$

The theory of trusses have been developed further in [Brz20, BR21, BMR20, BRS20, ABR21, BR20].

Every skew brace can be associated with an appropriate skew left truss: here we will call such trusses brace-type trusses. This leads to the main questions that motivated the work presented in this second part, result of a collaboration with Tomasz Brzeziński and Bernard Rybołowicz, appearing in [BMR20]. What are exactly brace-type trusses? How to construct them starting from a not necessarily brace-type truss? When is such a construction possible? Here we present two approaches to answer questions of this kind. The first approach is to take quotients of trusses by some special congruence and the second one relies on a localisation procedure.

We give answers to the questions above in a more general context, considering pre- and near-trusses, skew rings and skew braces, the definition of which is in Chapter 11. Congruences in pre- and near-trusses are shown to arise from normal sub-heaps with an additional closure property of equivalence classes that involves both the ternary and binary operations. Such sub-heaps, introduced in [Brz20], are called paragons. A necessary and sufficient criterion on paragons under which the quotient of a unital near-truss corresponds to a skew brace is derived. Regular elements in a pre-truss are defined as elements with left and right cancellation properties; following the ring-theoretic terminology pre-trusses in which all non-absorbing elements are regular are termed domains. The latter are described as quotients by completely prime paragons, also defined hereby. Regular pre-trusses and near-trusses as domains that satisfy the Ore condition are introduced and the pre-trusses of fractions are constructed through localisation. In particular, it is shown that near-trusses of fractions without absorber correspond to skew braces.

# Chapter 11

# Background

The present chapter contains definitions and facts about near-rings, skew braces and heaps which that are necessary for the following. It concludes with Lemma 11.2.1 which describes fully all equivalence classes for a sub-heap relation $\sim_S$ as mutually isomorphic heaps with an explicitly given isomorphism in each case.

## 11.1 Near-rings, skew-rings and skew braces

A *near-ring* (see [Pil11]) is a set $N$ with two associative binary operations $+$ and $\cdot$, such that $(N, +)$ is a group and, for all $n, m, m' \in N$,

$$n(m + m') = nm + nm'.$$

Analogously to the case of rings a *near-field* is a near-ring such that $(N \setminus \{0\}, \cdot)$ is a group, where 0 is the neutral element for $+$.

A *homomorphism of near-rings* is a function $f\colon N \to N'$ that commutes with both near-ring operations, that is, for all $a, b \in N$,

$$f(ab) = f(a)f(b) \quad \text{and} \quad f(a + b) = f(a) + f(b).$$

A *skew-ring* [Rum19, Definition 3 and Corollary] is a triple $(B, +, \cdot)$, where $(B, +)$ is a group $(B, \cdot)$ is a monoid and the following distributive law holds

$$a(b + c) = ab - a + ac,$$

for all $a, b, c \in B$. A skew-ring $(B, +, \cdot)$ in which $(B, \cdot)$ is a group is called a *skew left brace* [GV17]. A *homomorphism of skew braces* is a function that commutes with both group operations. A close connection between skew left braces and near-rings is revealed in [SV18, Proposition 2.20], which states that any construction subgroup of a near-ring is a skew left brace. In what follows, we drop the adjective "left", and hence skew brace means skew left brace. An *ideal* in a skew brace $B$ is a subset $B' \subset B$ such that $(B', +)$ is a normal subgroup, $aB' = B'a$ and $ab - a \in B'$, for all $a \in B$ and $b \in B'$.

*Remark* 11.1.1. Obviously, "right" versions of all the notions discussed in this text can be defined and developed symmetrically, and in fact in [Rum19] Rump gives the definition of a skew-ring in the right-sided convention.

## 11.2 Heaps

A *heap* is a set $H$ together with a ternary operation,

$$[-,-,-]\colon\ H \times H \times H \to H,$$

that is associative and satisfies Mal'cev identities. Explicitly this means that, for all $a_1, a_2, a_3, a_4, a_5 \in H$,

$$[a_1, a_2, [a_3, a_4, a_5]] = [[a_1, a_2, a_3], a_4, a_5] \quad \text{and} \quad [a_1, a_1, a_2] = a_2 = [a_2, a_1, a_1].$$

These conditions imply that, for all $a_1, a_2, a_3, a_4, a_5 \in H$,

$$[[a_1, a_2, a_3], a_4, a_5] = [a_1, [a_4, a_3, a_2], a_5]. \tag{11.2.1}$$

We say that $H$ is an *Abelian heap* if $[a, b, c] = [c, b, a]$, for all $a, b, c \in H$.

A *homomorphism of heaps* is a map that commutes with the ternary operations, that is, $f\colon\ H \to H'$ is a heap morphism if, for all $a, b, c \in H$,

$$f([a, b, c]) = [f(a), f(b), f(c)].$$

Every non-empty heap can be associated with a group by fixing the middle entry of the ternary operation: explicitly, for all $a \in H$, $+_a := [-, a, -]$ is a group operation on $H$. This group is called a *retract* of $H$ at $a$ and is denoted by $\mathrm{G}(H; a)$. Retracts at two different elements are isomorphic. Starting with a group $G$ one can assign a heap to it by setting $[a, b, c] := ab^{-1}c$, for all $a, b, c \in G$. This *heap associated to a group $G$* will be denoted by $\mathrm{H}(G)$.

A subset $S$ of a heap $H$ is a *sub-heap* if it is closed under the heap operation of $H$. A non-empty sub-heap $S$ of a heap $H$ is said to be *normal* if there exists $e \in S$ such that, for all $a \in H$ and $s \in S$, there exists $t \in S$ such that $[a, e, s] = [t, e, a]$. This is equivalent to say that for all $a \in H$ and $e, s \in S$, $[[a, e, s], a, e] \in S$. Every non-empty sub-heap of an Abelian heap is normal. The retract of a normal sub-heap at an element $e$ is a normal subgroup of the retract of the heap at the same element $e$. Furthermore, for any heap homomorphism $f\colon\ H \to H'$ and any $b \in \mathrm{Im}\ f$, $f^{-1}(b)$ is a normal sub-heap of $H$; see e.g. [Brz20, Lemma 2.12].

If $S$ is a sub-heap of $H$, then the relation $\sim_S$ on $H$ given by

$$a \sim_S b \iff \exists s \in S\ [a, b, s] \in S \iff \forall s \in S\ [a, b, s] \in S$$

is an equivalence relation. The set of equivalence classes is denoted by $H/S$. The equivalence class of any $s \in S$ is equal to $S$. If $S$ is a normal sub-heap, then $\sim_S$ is a congruence and thus the canonical map $\pi : H \to H/S$ is a heap epimorphism; see [Brz20, Proposition 2.10].

The following Lemma summarises properties of the sub-heap equivalence relation and gives an explicit description of all equivalence classes and relations between them.

**Lemma 11.2.1.** *Let $S$ be a non-empty sub-heap of $(H, [-,-,-])$, and consider the sub-heap relation $\sim_S$.*

(1) *For all $a, b \in H$, define the* translation map*:*
$$\tau_a^b\colon\ H \to H, \qquad z \mapsto [z, a, b]. \tag{11.2.2}$$

(a) *The map $\tau_a^b$ is an isomorphism of heaps.*

(b) *The equivalence classes of $\sim_S$ are related by the formula:*
$$\bar{b} = \tau_a^b(\bar{a}) = \{[z, a, b] \mid z \sim_S a\}.$$

(c) *For all $e \in S$ and $a \in H$, set $S_e^a := \tau_e^a(S)$. Then $\bar{a} = S_e^a$.*

(2) *For all $a \in H$, the equivalence class $\bar{a}$ is a sub-heap of $H$. Furthermore, if $S$ is a normal sub-heap of $H$, then so are the $\bar{a}$'s.*

(3) *Equivalence classes of $\sim_S$ are mutually isomorphic as heaps.*

(4) *For all $a \in H$, the sub-heap equivalence relation $\sim_S$ coincides with the sub-heap equivalence relation $\sim_{\bar{a}}$. Consequently $H/S = H/\bar{a}$.*

*Proof.* (1) (a) First we need to check that $\tau_a^b$ preserves the ternary operation. Using the associativity and Mal'cev identities, we can compute, for all $z, z', z'' \in H$,
$$\begin{aligned}
\left[\tau_a^b(z), \tau_a^b(z'), \tau_a^b(z'')\right] &= \left[[z,a,b],[z',a,b],[z'',a,b]\right] \\
&= \left[z, a, \left[b, [z',a,b], [z'',a,b]\right]\right] \\
&= \left[z, a, \left[[b,b,a], z', [z'',a,b]\right]\right] \qquad \text{(by (11.2.1))} \\
&= \left[[z,z',z''], a, b\right] = \tau_a^b([z,z',z'']).
\end{aligned}$$

Therefore, the $\tau_a^b$ preserve ternary operations and thus each one of them is a homomorphism of heaps. The inverse of $\tau_a^b$ is $\tau_b^a$.

(1)(b) Assume that $z \sim_S a$, that is, that $[z,a,s] \in S$, for all $s \in S$. If $z' = \tau_a^b(z) = [z,a,b]$, then $[z',b,s] = [z,a,s]$, by the associativity and the Mal'cev property. Hence $z' \sim_S b$, that is, $\tau_a^b(\bar{a}) \subseteq \bar{b}$. On the other hand, if $z' \in \bar{b}$, then set $z = \tau_b^a(z') = [z',b,a]$. Since $\tau_b^a$ is the inverse of $\tau_a^b$, $z' = \tau_a^b(z)$. Furthermore, for all $s \in S$, $[z,a,s] = [z',b,s]$, and so $[z,a,s] \in S$, since $z' \sim_S b$. This proves the second inclusion $\bar{b} \subseteq \tau_a^b(\bar{a})$, and hence the required equality.

Assertion (1)(c) follows by 1(b) and the fact that $\bar{e} = S$.

Statement (2) follows by (1) and the observation that heap isomorphisms preserve the normality. Statement (3) is a straightforward consequence of (1) and (2).

(4) Using (1)(c) we can argue as follows: $b \sim_S c$ if, and only if, there exist $s, s' \in S$ such that $[b,c,s] = s'$. This is equivalent to the equality $[[b,c,s],e,a] = [s',e,a]$, for any $a \in H$ and $e \in S$, which, by associativity, is equivalent to $[b,c,[s,e,a]] = [s',e,a]$. The fact that $\bar{a} = S_e^a$ implies that $b \sim_{\bar{a}} c$. $\square$

# Chapter 12

# Quotient pre-trusses, near-trusses and skew braces

This chapter focuses on trusses. We start with the introduction of pre-trusses, near-trusses and skew trusses. A pre-truss is a heap with an additional semigroup operation. A near-truss is a pre-truss in which the semigroup operation distributes over the ternary operation from the left. The best known examples of these objects are near-trusses with left absorbers associated with near-rings (see Example 12.1.4) or unital near-trusses which can be associated to recently introduced skew-rings [Rum19] (see Example 12.1.5). The notion of near-truss was introduced in [Brz19, Definition 2.1] under the name of skew left truss; the present terminology is intended to be coherent with that of the near-ring theory. Another example of a near-truss which is of particular interest is that of a near-truss associated with a skew brace (see Example 12.1.8); these near-trusses are said to be brace-type. Finally, a skew truss is a near-truss for which the right distributive law holds. We start by defining trusses, then in Section 12.2 we focus on the characterisation of algebraic structures that correspond to congruences in a pre-truss. For that, we give the definition of a paragon as a normal sub-heap, the equivalence classes of which have a particular closure property, and in Theorem 12.2.6 we show that paragons fully describe all the congruences in a pre-truss. We conclude this theorem with Corollary 12.2.7 and Corollary 12.2.9 which tell us that congruence equivalence classes in near-rings and skew braces are in fact paragons in the associated near-trusses. In Section 12.3, we introduce the definition of an ideal to determine, in Proposition 12.3.5, whether a unital near-truss is associated with a skew brace or a near-ring.

Finally, in Section 12.4, combining the most natural concept of a maximal paragon with the analysis of the ideal structure of a truss we give a full description of those paragons whose quotient is a brace-type near-truss: in Theorem 12.4.5 we show that a quotient near-truss is brace-type if and only if all equivalence classes are not subsets of any ideal in a near-truss. We conclude this Chapter with two examples of paragons that fulfil the hypothesis of the theorem.

## 12.1 Trusses

**Definition 12.1.1.**

(1) A *pre-truss* is a heap $(T, [-,-,-])$ together with an associative binary operation (denoted by juxtaposition of elements or by $\cdot$).

(2) A pre-truss $T$ satisfying the left distributive law:

$$a[b,c,d] = [ab, ac, ad], \qquad \text{for all } a,b,c,d \in T,$$

is called a *near-truss*.

(3) A near-truss $T$ satisfying the right distributive law

$$[b,c,d]a = [ba, ca, da], \qquad \text{for all } a,b,c,d \in T,$$

is called a *skew truss*.

(4) A skew truss such that the underlying heap is Abelian is called a *truss*.

Every one of the above notions is said to be *unital* provided the binary operation has an identity (denoted by 1).

A homomorphism of (pre-, near-, skew) trusses is a homomorphism of heaps that is also a homomorphism of semigroups (or monoids in the unital case).

It is clear from this definition that the image of a homomorphism of (pre-, near-, skew) trusses is itself a (pre-, near-, skew) truss.

*Remark* 12.1.2. Except for a pre-truss all the notions listed in Definition 12.1.1 have been introduced in [Brz19] and [Brz20]. Note, however, that the terminology introduced there was motivated by braces, and thus what we call a near-truss here was named a skew left truss there. In this paper we are adopting a terminology more aligned with the ring (or near-ring) theory one. Of course, a right distributive version of a near-truss can be considered, but in line with the convention of Section 11.1 we only consider the left distributive version (with no qualifier).

A *left (resp. right) absorber* is an element $a$ of a pre-truss $T$ such that, for all $t \in T$, $ta = a$ (resp. $at = a$). We say that $a$ is an *absorber* if it is a left and right absorber. It is worth noting that if a pre-truss $T$ has both a left and a right absorber, then they necessarily coincide, in particular an absorber is unique. We denote by $T^{\mathrm{Abs}} := T \setminus \{a\}$, if $a$ is the unique absorber with tacit understanding that $T^{\mathrm{Abs}} = T$ when $T$ has no absorbers. Furthermore, since homomorphisms of pre-trusses preserve multiplication, if $f : T \to T'$ is a morphism and $e$ is a left (resp. right) absorber in $T$, then $f(e)$ is a left (resp. right) absorber in the pre-truss $f(T)$.

*Example* 12.1.3. If $T$ is a truss that has an absorber, then $T$ is a *ring-type truss*. This means that by taking the retract of $T$ at the absorber, say 0, we obtain a ring $(T, +_0, \cdot)$.

Conversely, if $R$ is a ring then one can associate to it the truss $(\mathrm{H}(R), \cdot)$ with absorber 0. This truss is denoted by $\mathrm{T}(R)$. If $R$ is unital, then $\mathrm{T}(R)$ is unital. Observe that if we start with a ring $R$, we assign to it the truss $\mathrm{T}(R)$ and then take the retract we necessarily obtain $R$ again, since the absorber is unique.

*Example* 12.1.4. Let $T$ be a near-truss such that there exists a left absorber $e$. Then a near-ring can be associated to $T$ by taking the retract of the heap $T$ at $e$ to obtain $(T, +_e, \cdot)$. We call such $T$ a *ring-type near-truss*

Conversely, if $N$ is a near-ring then one can associate to it the near-truss $(\mathrm{H}(N), \cdot)$ which we will denote by $\mathrm{T}(N)$. In contrast to rings, since left absorbers are not unique, if one associates a near-truss $\mathrm{T}(N)$ to $N$ and then take the retract at a left absorber, then not necessarily one obtains $N$.

*Example* 12.1.5. Let $T$ be a unital near-truss. Then a skew-ring can be associated to it by taking the retract of the heap $T$ at the identity 1 of the multiplication, that is $(T, +_1, \cdot)$ is a skew-ring.

Conversely, if $B$ is a skew-ring, then one can assign to it the unital near-truss $(\mathrm{H}(B), \cdot)$ which we will denote by $\mathrm{T}(B)$. Observe that if we start with a skew-ring $B$, we assign to it the near-truss $\mathrm{T}(B)$ and then take the retract at the identity, we obtain the same skew-ring as identity is unique.

Recall from [Rum19] that an element $u$ in a skew-ring $B$ is called a *unit* if, for all $a \in B$,

$$a \cdot u = a + u + a.$$

**Lemma 12.1.6.** *Units in a skew-ring $B$ are in one-to-one correspondence with left absorbers in the associated unital near-truss* $\mathrm{T}(B)$*.*

*Proof.* The correspondence is given by $u = [1, e, 1]$. That is, $u$ is a unit in $B$ (resp. absorber in $\mathrm{T}(B)$) provided $e$ is an absorber in $\mathrm{T}(B)$ (resp. unit in $B$). □

*Remark* 12.1.7. Combining Example 12.1.4 with Example 12.1.5 and Lemma 12.1.6 we are led to the correspondence between skew-rings with units and unital near-rings. If $u$ is a unit in a skew-ring $B$, then $(T(B), +_{[1,u,1]}, \cdot, 1)$ is a unital near-ring, and vice versa, if $(N, +, \cdot, 1)$ is a unital near-ring with zero $e$, then $(T(N), +_1, \cdot)$ is a skew-ring with unit $[1, e, 1] = -e$ (cf. [Rum19, Example 3]). This correspondence is seemingly different from the one described in [Rum19, Proposition 2] as it changes the additive structure keeping the multiplication fixed, while in [Rum19, Proposition 2] one considers a new multiplication with addition unchanged. However, if $e$ is a left absorber in a unital near-truss $(T, [-,-,-], \cdot, 1)$, then using the translation heap automorphism (11.2.2) one can induce a new associative product on $T$ by the formula:

$$a *_e b = \tau_e^1\left(\tau_1^e(a) \cdot \tau_1^e(b)\right), \qquad \text{for all } a, b \in T.$$

Then $(T, [-,-,-], *_e, [1,e,1])$ is a unital near-truss isomorphic to $(T, [-,-,-], \cdot, 1)$ in which 1 is a left absorber. Consequently, $(T, +_1, *_e)$ is a unital near-ring corresponding to the skew-ring $(T, +_1, \cdot)$. In particular, if $B$ is a skew-ring with unit $u$, then $(T(B), +_1, *_{[1,u,1]}, u) = (B, +, *_{-u}, u)$ is the unital near-ring described in [Rum19, Proposition 2].

*Example* 12.1.8. Let $T$ be a near-truss such that $(T, \cdot)$ is a group with neutral element 1. Then $(T, +_1, \cdot)$ is a skew brace. We call such $T$ a *brace-type near-truss*.

Conversely, if $B$ is a skew brace, then one can assign to it the near-truss $(\mathrm{H}(B), \cdot)$ which we will denote by $\mathrm{T}(B)$. As was the case with the skew-rings, if we start with a skew brace $B$, assign to it the near-truss $\mathrm{T}(B)$ and then take the retract at identity, we obtain the same skew brace.

## 12.2 Paragons

Our goal is to describe the properties that a pre-truss $T$ and a congruence $\sim$ on it must have for the quotient near-truss $T/\sim$ to be a brace-type near-truss, i.e. a near-truss associated with a skew brace. The main theorem of this section is Theorem 12.4.5 which states when a near-truss $T/\sim$ can be associated with a skew brace. First we identify those normal sub-heaps of a pre-truss $T$ that faithfully correspond to congruences.

**Definition 12.2.1.** Let $T$ be a pre-truss.

(1) A sub-heap $S$ of $T$ is said to be *left-closed* (resp. *right-closed*) if, for all $s, s' \in S$ and $t \in T$,
$$[ts', ts, s] \in S \qquad (\text{resp. } [s't, st, s] \in S). \tag{12.2.1}$$

(2) A sub-heap $S$ that is left- and right-closed is said to be *closed*.

(3) A normal sub-heap $P$ of $T$ such that every equivalence class of the sub-heap relation $\sim_P$ is a closed (normal) sub-heap of $T$ is called a *paragon*.

Observe that Lemma 11.2.1 implies that if $P$ is a paragon in a pre-truss $T$, then all the equivalence classes of $\sim_P$ are mutually isomorphic paragons as well.

*Remark* 12.2.2. In the case of a non-empty sub-heap $S$ the quantifier 'for all $s \in S$' in the definition of the left or right closure property (12.2.1) can be equivalently replaced by the existential quantifier. Indeed, assume that there exists $q \in S$ such that, for all $s' \in S$ and $t \in T$, $[ts', tq, q] \in S$. Then, for all $s \in S$,
$$[ts', ts, s] = [[[ts', tq, q], q, tq], ts, s] = [[ts', tq, q], [ts, tq, q], s] \in S,$$
by the associativity, Mal'cev's identities and (11.2.1), and since $S$ is a sub-heap. Similarly for the right closure property.

**Lemma 12.2.3.** *A normal sub-heap $P$ of a pre-truss $T$ is a paragon if and only if, for all $a, b \in T$ and $p, e \in P$,*
$$[a[p,e,b], ab, e] \in P \quad \textit{and} \quad [[p,e,b]a, ba, e] \in P.$$

*Proof.* By Lemma 11.2.1, the equivalence class of $b \in T$ is $\bar{b} = P_e^b = \{[p, e, b] \mid p \in P\}$, for all $e \in P$. Hence $\bar{b}$ is left-closed if and only if, for all $p \in P$ and $a \in T$, there exists $q \in P$ such that

$$[a[p, e, b], ab, b] = [q, e, b],$$

that is, if and only if

$$[a[p, e, b], ab, e] = q \in P,$$

as required. By the same argument we obtain that $\bar{b}$ is right-closed. □

**Corollary 12.2.4.** *A normal sub-heap $P$ of a near-truss $T$ is a paragon if and only if $P$ is left-closed and all equivalence classes of the induced sub-heap relation are right-closed. In particular $P$ is a paragon in a skew truss if and only if it is a closed normal sub-heap.*

*Proof.* Since in a near-truss the left distributivity law holds, the left-closure property in Lemma 12.2.3 reduces to $[ap, ae, e] \in P$, that is, the left-closedness of $P$. In a skew truss the right-closure property is treated symmetrically. □

Corollary 12.2.4 shows that, in the case of skew trusses (and hence trusses) the notion of a paragon introduced in Definition 12.2.1 reduces to the notion introduced in [Brz20, Definition 3.15].

**Lemma 12.2.5.** *Let $f : T \to T'$ be a morphism of pre-trusses.*

*(1) For all $z \in \mathrm{Im}\ f$, $f^{-1}(z)$ is a paragon in $T$. In particular, if $P'$ is a paragon in* $\mathrm{Im}\ f$*, then $f^{-1}(P')$ is a paragon in $T$.*

*(2) If $P$ is a paragon in $T$ then $f(P)$ is a paragon in* $\mathrm{Im}\ f$.

*Proof.* (1) By [Brz20, Lemma 2.12], $f^{-1}(z)$ is a normal sub-heap which is non-empty (since $z \in \mathrm{Im}\ f$). For all $a, b \in T$ and $p, e \in f^{-1}(z)$,

$$\begin{aligned} f([a[p, e, b], ab, e]) &= [f(a)[f(p), f(e), f(b)], f(a)f(b), f(e)] \\ &= [f(a)[z, z, f(b)], f(a)f(b), z] = z, \end{aligned}$$

since $f$ preserves multiplication and ternary operations, and by Mal'cev identities. Thus $[a[p, e, b], ab, e] \in f^{-1}(z)$. By the same arguments, $[[p, e, b]a, ba, e] \in f^{-1}(z)$. In view of Lemma 12.2.3 this means that $f^{-1}(z)$ is a paragon.

Assume that $P'$ is a paragon. That the pre-image of a normal sub-heap is a normal sub-heap follows by the standard group-theoretic arguments. Since $f$ preserves multiplication and heap operation, for all $a, b \in T$ and $p, q \in f^{-1}(P')$,

$$\begin{aligned} f\left([a[p, q, b], ab, q]\right) &= \left[f(a)[f(p), f(q), f(b)], f(a)f(b), f(q)\right] \quad \text{and} \\ f\left([[p, q, b]a, ba, q]\right) &= \left[[f(p), f(q), f(b)]f(a), f(b)f(a), f(q)\right]. \end{aligned}$$

Since $P'$ is a paragon, and $f(p), f(q) \in P'$, both expressions are elements of $P'$. Therefore, $[a[p, q, b], ab, q], [[p, q, b]a, ba, q] \in f^{-1}(P')$, and hence $f^{-1}(P')$ is a paragon.

Statement (2) is proven by similar arguments. □

**Theorem 12.2.6.** *Let $P$ be a normal sub-heap of a pre-truss $T$. Then the canonical heap map $\pi : T \to T/P$ is a homomorphism of pre-trusses if and only if $P$ is a paragon.*

*Proof.* Assume that $\pi$ is a pre-truss homomorphism. Since $P = \pi^{-1}(P)$, $P$ is a paragon by Lemma 12.2.5.

For the proof of the opposite implication assume that $P$ is a paragon. Then $\sim_P$ is a congruence on the heap $T$, so we only need to show that this relation is a congruence on the pre-truss $T$ as well. Let $a, b \in T$ be such that $a \sim_P b$, so that $a, b \in \pi(b)$. Since $P$ is a paragon, for all $t \in T$, $[ta, tb, b] \in \pi(b)$. Hence, $[\pi(tb), \pi(ta), \pi(b)] = \pi(b)$, that is, $\pi(tb) = \pi(ta)$ or, equivalently, $ta \sim_P tb$. In the same way one can prove that $a \sim_P b$ implies $at \sim_P bt$ for all $t \in T$. Assume that $a \sim_P b$ and $c \sim_P d$. Then $ac \sim_P bc$, $bc \sim_P bd$ and $ac \sim_P bd$, since $\sim_P$ is an equivalence relation. Therefore, $\sim_P$ is a congruence and the canonical map $\pi : T \to T/P$ is a homomorphism of pre-trusses. This completes the proof. □

**Corollary 12.2.7.** *Let $N$ be a near-ring. Then $P \subseteq N$ is an equivalence class for a congruence on $N$ if and only if $P$ is a paragon in* $\mathrm{T}(N)$

*Proof.* Let us assume that $P$ is an equivalence class for a congruence on $N$, let $\bar{N}$ be the quotient near-truss with canonical homomorphism $\pi : N \to \bar{N}$. Since $\pi$ is also a homomorphism of associated near-rings, that is, $\pi \colon \mathrm{T}(N) \to \mathrm{T}(\bar{N})$, and $P = \pi^{-1}(P)$, $P$ is a paragon in $\mathrm{T}(N)$ by Lemma 12.2.5.

In the opposite direction, assume that $P$ is a paragon in $\mathrm{T}(N)$. Then there exists a near-truss homomorphism $\pi \colon \mathrm{T}(N) \to \mathrm{T}(N)/P$. Observe that the triple $(\mathrm{T}(N)/P, +_{\pi(e)}, \cdot)$, where $e$ is the neutral element of $N$, is a near-ring, since the image of a left absorber through a near-truss homomorphism is a left absorber. Therefore $\pi$ is also a homomorphism of the retracted near-rings and $P$ is an equivalence class of a congruence given by $\pi$ as $P = \pi^{-1}(P)$. □

**Lemma 12.2.8.** *Let* $\mathrm{T}(B)$ *be a near-truss associated to a skew brace $B$ (with identity 1). Then $P$ is a paragon in* $\mathrm{T}(B)$ *if and only if, for all $p \in P$, $P^1_p$ is an ideal in $B$.*

*Proof.* Assume that $P$ is a paragon in $\mathrm{T}(B)$. Then $1 \in P^1_p$, $(P^1_p, +_1)$ is a normal subgroup of $(B, +)$ as $P^1_p$ is a normal sub-heap and $+_1 = +$. Since $P^1_p$ is closed, for all $a \in B$ and $b \in P^1_p$,

$$ab - a = [ab, a1, 1] \in P^1_p \quad \& \quad ba - a = [ba, 1a, 1] \in P^1_p.$$

Therefore, $ba - ab = c \in P^1_p$, and, using the brace distributive law,

$$a^{-1}ba = a^{-1}(c + ab) = a^{-1}c - a^{-1} + b \in P^1_p,$$

since $P^1_p$ is left-closed. This implies that $a^{-1}P^1_p a = P^1_p$, that is, $aP^1_p = P^1_p a$, and completes the proof that $P^1_p$ is an ideal in $B$.

Conversely, if $P^1_p$ is an ideal in $B$, then $B/P^1_p$ is a skew-brace by [GV17, Lemma 2.3], and the canonical skew-brace epimorphism $\pi \colon B \to B/P^1_p$ induces a near-truss morphism $\pi\colon \mathrm{T}(B) \to \mathrm{T}(B/P_p^1)$. Since $P_p^1 = \pi^{-1}(P_p^1)$, $P_p^1$ and consequently also $P = \left(P_p^1\right)_1^p$ are paragons by Lemma 12.2.5. $\square$

**Corollary 12.2.9.** *Let $B$ be a skew brace, then $P \subseteq B$ is an equivalence class for some congruence on $B$ if and only if $P$ is a paragon in* $\mathrm{T}(B)$.

*Proof.* The proof of the left to right implication is the same as in Corollary 12.2.7. The other implication follows by Lemma 12.2.8. $\square$

## 12.3 Ideals in (pre-)trusses

To connect quotients of near-trusses with skew braces we need to determine which paragons do not produce absorbers in the quotients. To this end we introduce the notion of an ideal.

**Definition 12.3.1.** A normal sub-heap $I$ of a pre-truss $T$ is called a *left* (resp. *right*) *ideal* if, for all $t \in T$ and $i \in I$, $ti \in I$ (resp. $it \in I$). If $I$ is both left and right ideal, then it is called an *ideal*. A proper left (resp. right) ideal is said to be *maximal* if it is not strictly contained in any left (resp. right) proper ideal.

Note that an ideal is a closed sub-heap, but this does not yet make it into a paragon, since the equivalence classes of the corresponding sub-heap relations need not be closed. Also note that if $f : T \to T'$ is a homomorphism of pre-trusses, then the pre-image of an ideal in $\mathrm{Im}\ f$ is an ideal in $T$ and the image of an ideal in $T$ is an ideal in $\mathrm{Im}\ f$.

**Lemma 12.3.2.** *If a left-closed normal sub-heap of a pre-truss contains a left ideal, then it is a left ideal.*

*Proof.* Let $P$ be a left-closed normal sub-heap of $T$, and let $I$ be a left ideal such that $I \subseteq P$. Then, for all $p \in P$, $t \in T$ and $i \in I$, $tp = [[tp, ti, i], i, ti] \in P$, since $[tp, ti, i] \in P$ and $ti, i \in I \subseteq P$. $\square$

**Lemma 12.3.3.** *Let $T$ be a pre-truss and $P$ be a paragon. Then $T/P$ has a left absorber if and only if there exist $a \in P$ and $t \in T$ such that $P_a^t$ is a left ideal.*

*Proof.* The assertion follows from the fact that for every $a \in P$ and $t \in T$, $P_a^t = \pi(t)$, where $\pi$ is the canonical surjection onto the quotient $T/P$. $\square$

**Corollary 12.3.4.** *If $I$ is a paragon that is a right ideal in a pre-truss $T$, then for all $e \in T \setminus I$ and all $a \in I$, $I_a^e$ is not a left ideal.*

*Proof.* We know from Lemma 11.2.1 that $T/I = T/I_a^e$. Assume that $I$ is a right ideal and suppose that $I_a^e$ is a left ideal. Then, by Lemma 12.3.3, $I$ is a right absorber in $T/I$ and $I_a^e$ is a left absorber in $T/I_a^e$. Hence $I = I_a^e$. But $e \notin I$ and $e \in I_a^e$, which yields a required contradiction and completes the proof. $\square$

**Proposition 12.3.5.** *Let $T$ be a unital near-truss.*

*(1) $T$ is a near-truss associated with a skew brace if and only if $T$ has exactly one left ideal.*

*(2) $T$ is a near-truss associated with a near-field if and only if $T$ has a left absorber and exactly two left ideals.*

*Proof.* (1) Assume that $T$ has exactly one left ideal. For all $x \in T$ the left ideal $Tx := \{tx \mid t \in T\}$ has to be the whole of $T$ (in particular if $T$ has at least two elements, then it has no left absorbers). Therefore, there exists $y \in T$ such that $yx = 1$ and $y$ is a left inverse to $x$. As $x$ is an arbitrary element there exists $x'$ such that $x'y = 1$. Thus $(x'y)x = x$ and by associativity $x' = x$. The conclusion is that $y$ is the two-sided inverse of $x$ and the monoid $(T, \cdot)$ is a group. Therefore, the near-truss $T$ is a brace-type near-truss (see [Brz20, Corollary 3.10]).

Conversely, suppose that $T = \mathrm{T}(B)$ for a skew brace $B$ and that there exists a left ideal $I \subsetneq \mathrm{T}(B)$. Observe that if $x \in I$, then $x^{-1}x = 1 \in I$, therefore $I = T$. This contradicts the assumption that $I \neq T$. Thus $T$ has exactly one left ideal.

(2) Let us assume that $T$ has a left absorber and exactly two left ideals. Then there exists a near-ring $R$ such that $T = \mathrm{T}(R)$, to be precise $R$ is the retract $(T, +_e, \cdot)$, where $e$ is the left absorber. Seeking contradiction, suppose that $R$ is not a near-field. Then there exists a left ideal $\{e\} \neq I \subsetneq R$; but $I$ is also a left ideal of $\mathrm{T}(R)$, which contradicts with the assumption that $T$ has only two left ideals. Therefore, $R$ is a near-field.

Assume that $T = \mathrm{T}(F)$, where $F$ is a near-field, then 0 (the neutral element for the addition in $F$) is a left absorber in $T$. Suppose by contradiction that $\mathrm{T}(F)$ has a left ideal $\{0\} \neq I \subsetneq \mathrm{T}(F)$. Consider, for any $a \in I$ the ideal $I_a^0 := \{[b, a, 0] \mid b \in I\}$. The ideal $I_a^0$ is neither equal to $\{0\}$ nor to $T$, since the map $[-, a, 0]$ is a bijection. Furthermore, $I_a^0$ is an ideal in $F$, and hence $F$ is not a near-field. This contradicts with the assumption that $F$ is a near-field. □

**Lemma 12.3.6.** *Let $T$ be a near-truss. If $I$ is a paragon in $T$ that is a maximal left ideal, then $T/I$ has no ideals different from a singleton set and $T/I$.*

*Proof.* Suppose that $\mathfrak{J} \neq T/I$ is a left ideal in $T/I$ that is not a singleton set. Since $I$ is a left absorber in $T/I$, for any element $J \in \mathfrak{J}$, $\mathfrak{J}_J^I$ is a left ideal in $T/I$ by the left distributive law. Hence, $\pi^{-1}(\mathfrak{J}_J^I)$ is a left ideal in $T$, where $\pi\colon\ T \to T/I$ is the canonical surjection. Moreover, $I \subset \pi^{-1}(\mathfrak{J}_J^I)$, since $I \in \mathfrak{J}_J^I$. Therefore, since $I$ is a maximal left ideal, either $I = \pi^{-1}(\mathfrak{J}_J^I)$, and hence $\mathfrak{J}_J^I = \{I\}$, which implies that $\mathfrak{J} = \{J\}$, or $\pi^{-1}(\mathfrak{J}_J^I) = T$, which implies in turn that $\mathfrak{J} = T/I$. Thus both cases lead to a contradiction. □

## 12.4 Brace-type trusses as quotients

Although quotienting by a paragon which is a maximal left ideal yields a near-truss without proper left ideals, this near-truss always has an absorber. Therefore it is never a brace-type near-truss. The most straightforward idea to generalise maximality to paragons leads us to the following definition:

**Definition 12.4.1.** Let $T$ be a pre-truss. A left-closed (resp. right-closed) normal sub-heap $P \subsetneq T$ is said to be *maximal* if it is not contained in any left-closed (resp. right-closed) sub-heap other than $T$. A paragon $P$ is said to be *left maximal* (resp. *right-maximal*, *maximal*) if it is a maximal left-closed (resp. right-closed, left- and right-closed) sub-heap.

**Lemma 12.4.2.** *Let $T$ be a near-truss or a skew truss and $P$ be a left-closed normal sub-heap. Then $P$ is maximal if and only if, for all $a \in P$ and $t \in T$, $P_a^t$ is a maximal left-closed normal sub-heap.*

*Proof.* Note that by the normality of $P$ and the left distributive law, all the $P_a^t$ are left-closed normal sub-heaps. Seeking contradiction assume that $P$ is maximal and there exists $a \in P$ and $t \in T$ such that $P_a^t$ is not maximal. Then there exists a left-closed normal sub-heap $Q$ such that $P_a^t \subsetneq Q \subsetneq T$. Since $\tau_a^t$ is an isomorphism with the inverse $\tau_t^a$, this implies that $P \subsetneq Q_t^a \subsetneq T$. Hence $P$ is not maximal, contrary to the assumption.

The opposite implication is also easily deduced from the fact that $P = (P_a^t)_t^a$. □

*Remark* 12.4.3. In the case of rings the notion of maximal ideals and maximal paragons coincide as every paragon $P$ in the ring can be associated with an ideal $P_a^0$ for any $a \in P$ and an absorber 0.

**Lemma 12.4.4.** *Let $T$ be a near-truss or a skew truss and $P \subseteq T$ a left maximal paragon, then $T/P$ has no proper (i.e. different from singletons and the whole of $T/P$) left ideals.*

*Proof.* By the definition of maximality of $P$, $T/P$ has no proper left paragons. Therefore it has no proper left ideals as a left ideal is a left paragon. □

Observe that by dividing a near-truss without left absorbers by a paragon which is left-maximal one obtains a near-truss associated with a skew brace. If the quotient is a skew brace, then it is a simple skew brace, that is, it has no ideals in the sense of sub-braces different from the skew brace itself and singleton subsets of it. Maximal paragons do not characterise all the quotients which are brace-type near-trusses, since there exist skew braces that are non-simple.

**Theorem 12.4.5.** *Let $T$ be a unital near-truss and $P$ be a paragon, and let $\pi_P \colon\ T \to T/P$ be the canonical epimorphism. Then $T/P$ is a brace-type near-truss if and only if, for all left ideals $I \subsetneq T$ and $\bar{a} \in T/P$, $\pi_P^{-1}(\bar{a}) \not\subseteq I$.*

*Proof.* Let us assume that $T/P$ is a brace-type near-truss. Observe that should $\pi_P^{-1}(\bar{a}) \subseteq I$ for a left ideal $I$, then $\pi_P(I)$ would be a left ideal in $T/P$. Thus, $\pi_P(I) = T/P$, since $T/P$ is a brace-type near-truss. On the other hand, if $c \in T \setminus I$ then $\pi_P(c) \notin \pi_P(I)$. Indeed, should $\pi_P(c) \in \pi_P(I)$, then there would exist $i \in I$ and $p \in P$ such that $[c, i, p] \in P$. Thus, for all $a \in \pi_P^{-1}(\bar{a})$, $[c, i, a] = [[c, i, p], p, a] \in \pi_P^{-1}(\bar{a}) \subset I$ and $c \in I$. Therefore, $I = T$.

Now, assume that, for all left ideals $I \subsetneq T$ and $\bar{a} \in T/P$, $\pi_P^{-1}(\bar{a}) \nsubseteq I$ and $T/P$ is not a brace-type near-truss. Then there exists a left ideal $\mathfrak{J} \subsetneq T/P$. The pre-image $\pi_P^{-1}(\mathfrak{J}) \subsetneq T$ is a left ideal in $T$ and, obviously, for any $\|j \in \mathfrak{J}$, $\pi_P^{-1}(\|j) \subseteq \pi_P^{-1}(\mathfrak{J})$. This contradicts the assumption that, for all $\bar{a} \in T/P$, $\pi_P^{-1}(\bar{a}) \nsubseteq I$, so $T/P$ is a brace-type near-truss. The proof is completed. □

*Example* 12.4.6. Let $B$ be a skew brace and $R$ a ring. One can consider the product near-truss $\mathrm{T}(B) \times \mathrm{T}(R)$ with operation given by $(b, r)(b', r') = (bb', rr')$, for all $(b, r), (b', r') \in \mathrm{T}(B) \times \mathrm{T}(R)$. It is easy to check that, for any ideal $I$ in $R$, $\mathrm{T}(B) \times I$ is an ideal in $\mathrm{T}(B) \times \mathrm{T}(R)$ and that for any paragon $P$ in $\mathrm{T}(B)$, $P \times I$ is a paragon in $\mathrm{T}(B) \times \mathrm{T}(R)$. Every paragon of the form $P \times \mathrm{T}(R)$ fulfills conditions in Theorem 12.4.5 and one easily finds that $(\mathrm{T}(B) \times \mathrm{T}(R))/(P \times \mathrm{T}(R)) \cong \mathrm{T}(B)/P$

*Example* 12.4.7. Let $T = 2\mathbb{Z} + 1$ be the sub-truss of $\mathrm{T}(\mathbb{Z})$. The set $P = \{2^n m + 1 \mid m \in T\} \subset T$ is a paragon and the quotient $T/P$ is a brace-type truss isomorphic to $U(\mathbb{Z}/2^{n+1}\mathbb{Z})$, the sub-truss of all units in the quotient ring $\mathbb{Z}/2^{n+1}\mathbb{Z}$. To prove that this isomorphism holds it is first of all helpful to notice that $|T/P| = 2^n = |U(\mathbb{Z}/2^{n+1}\mathbb{Z})|$. Indeed, there are as many classes in the quotient as the odd numbers between $2^n m + 1$ and $2^n(m + 2) + 1$ (it is important to notice that, if $m$ is odd, then $m + 1$ is even), so exactly $2^n$. Then the isomorphism is given by sending $\overline{2m + 1} \in T/P$ to $2m + 1 \bmod 2^{n+1}$: this is evidently injective, so also surjective since the two sets have the same size, and it is easily proven to be a homomorphism.

# Chapter 13

# The (pre-)truss of fractions

The aim of the first Section 13.1 of this Chapter is to introduce the definition of completely prime paragons. This, in analogy to the case of rings, should lead to a quotient pre-truss that is a domain, i.e. a pre-truss in which cancellation properties hold. After describing this class of paragons, our next step is to consider the Ore localisation for pre-trusses, which is the subject of Section 13.2. By inverting all elements of a domain we obtain a pre-truss without proper left ideals and with no absorbers, so if the distributive law holds this will be a near-truss associated with a skew brace. Let us start with the definition of a domain. When working with rings, there is always an absorber which in many cases allows for simplification of some conditions. Not all pre-trusses have an absorber (in fact, having brace applications in mind, we are particularly interested in those that do not have absorbers), so many of the well-known definitions need to be in some sense generalised or stated without involving any absorber.

## 13.1 Domains and completely prime paragons

We begin with the definition of regular elements :

**Definition 13.1.1.** Let $T$ be a pre-truss. An element $a \in T^{\mathrm{Abs}}$ is said to be *left regular* (resp. *right regular*) if, for all $b \neq c$,

$$ab \neq ac \qquad \text{(resp. } ba \neq ca\text{)}. \tag{13.1.1}$$

If $a$ is both left and right regular element then it is said to be *regular*.

Observe that conditions (13.1.1) can be written in a way that makes them reminiscent of the closedness conditions (12.2.1) used in the definition of a paragon. The statement that $ac \neq ab$ is equivalent to saying that $[ac, ab, b] \neq b$. Similarly, $ba \neq ca$ is equivalent to say that $[ca, ba, b] \neq b$. This indicates that these conditions are closely related to the definition of paragon.

**Lemma 13.1.2.** *Let $T$ be a near-truss. Then $a \in T$ is a left regular element if and only if there exists an element $c$ such that, for all $b \in T \setminus \{c\}$,*

$$ab \neq ac. \tag{13.1.2}$$

*Proof.* If $a$ is left regular then, for all $c \in T$ and all $b \in T \setminus \{c\}$, the inequality (13.1.2) holds, which implies the existence of $c$.

Assume that there exists $c \in T$ such that, for all $b \neq c$, $ab \neq ac$. Thus $[ab, ac, ac'] = a[b, c, c'] \neq ac'$, for all $c' \in T$. Note that, for all $c, c' \in T$, the map

$$[-, c, c'] \colon T \setminus \{c\} \to T \setminus \{c'\}, \qquad b \mapsto [b, c, c'],$$

is a bijection. Therefore, for all $t \in T \setminus \{c'\}$, $at \neq ac'$. By the arbitrariness of $c'$, $a$ is a left regular element. This completes the proof. □

**Lemma 13.1.3.** *Let $R$ be a ring. Then $a \in R$ is a regular element if and only if $a$ is a regular element in* $\mathrm{T}(R)$.

*Proof.* The equivalence will be proven for left regularity only, the right regularity case in symmetric. Let us assume that $a \in R$ is a regular element. Then there is no $b \in R \setminus \{0\}$ such that $ab = 0$. Thus, by Lemma 13.1.2, if $c = 0$ in (13.1.2), then $a$ is a regular element in $\mathrm{T}(R)$, since $a$ is regular in $R$.

Suppose that $a$ is regular in $\mathrm{T}(R)$. Then $ab \neq ac$ implies $a(b - c) \neq 0$. Therefore, by substituting $b = t + c$, $at \neq 0$ for all $t \in R \setminus \{0\}$, which completes the proof. □

Now we are ready to introduce the definition of a domain in clear analogy with the usual notion for rings.

**Definition 13.1.4.** A pre-truss $T$ is called a *domain* if all elements of $T^{\mathrm{Abs}}$ are regular.

In view of Lemma 13.1.3, a ring $R$ is a domain if and only if $\mathrm{T}(R)$ is a domain.

**Lemma 13.1.5.** *A near-truss $T$ is a domain if and only if it satisfies the cancellation property, that is for all $a \in T^{\mathrm{Abs}}$ and $b, b' \in T$, each one of the equalities $ab = ab'$ or $ba = b'a$ implies that $b = b'$.*

*Proof.* This follows immediately for the definitions of a regular element and a domain. □

**Definition 13.1.6.** Let $T$ be a pre-truss. A non-empty paragon $P \subseteq T$ is said to be *completely prime* if, for all $p \in P$, $a, b, c \in T$,

$$[ab, ac, p] \in P \implies P^a_p \text{ is an ideal or } [b, c, p] \in P$$

and

$$[ba, ca, p] \in P \implies P^a_p \text{ is an ideal or } [b, c, p] \in P.$$

**Lemma 13.1.7.** *Let $T$ be a pre-truss and $P$ be a non-empty paragon. Then $P$ is completely prime if and only if, for all $p \in P$ and $t \in T$, $P^t_p$ is completely prime.*

*Proof.* Let us assume that $P$ is a completely prime paragon and let $p \in P$ and $t \in T$. We know that $P_p^t$ is a paragon (see comment that follows Definition 12.2.1). Then, for all $a, b, c \in T$ and $q \in P$, $[ab, ac, [q, p, t]] \in P_p^t$ implies $[[ab, ac, [q, p, t]], t, p] = [ab, ac, q] \in P$, since $(P_p^t)_t^p = P$. Thus, $P_q^a$ is an ideal or $[b, c, q] \in P$. In view of $(P_p^t)_{[q,p,t]}^a = P_q^a$, the first option is equivalent to $(P_p^t)_{[q,p,t]}^a$ being an ideal and the second to $[b, c, [q, p, t]] \in P_p^t$. Hence $P_p^t$ fulfils the left condition to be a completely prime paragon. Analogously one can prove that $P_p^t$ satisfies the right condition. Therefore, $P_p^t$ is a completely prime paragon. $\square$

Unsurprisingly, the distributive laws yield simplification of the definition of a completely prime paragon.

**Lemma 13.1.8.** *Let $T$ be a skew truss and $P$ be a paragon. Then $P$ is completely prime if and only if there exists $p \in P$ such that, for all $a, d \in T$,*

$$[ad, ap, p] \in P \implies P_p^a \text{ is an ideal or } d \in P$$

*and*

$$[da, pa, p] \in P \implies P_p^a \text{ is an ideal or } d \in P.$$

*Proof.* It is sufficient to observe that, for every $b \in T$, $[b, c, p]$ can be substituted by some $d \in T$ since $[-, c, p]\colon T \to T$ is a bijection with the inverse given by $[-, p, c]\colon T \to T$. Hence, if $b = [d, p, c]$, $d = [[d, p, c], c, p]$, and so

$$[ab, ac, p] = [a[d, p, c], ac, p] = [ad, ap, p] \text{ and } [ba, ca, p] = [[d, p, c]a, ca, p] = [da, pa, p],$$

by the distributive laws and the axioms of a heap. This completes the proof. $\square$

**Lemma 13.1.9.** *If $P \subsetneq T$ is a completely prime paragon in a pre-truss $T$, then, for all $p \in P$ and for all left (right) absorbers $a, a' \in T$, $P_p^a = P_p^{a'}$.*

*Proof.* Let $a$ be a left absorber. For all $b, c \in T$ and $p \in P$, $[ba, ca, p] = [a, a, p] = p \in P$, so $P_p^a$ is an ideal or $[b, c, p] \in P$. The second option is equivalent to $b \sim_P c$, for all $b, c \in T$. Observe, though, that since $P \neq T$, there exist $b, c \in T$ such that $b \not\sim_P c$. Therefore, $P_p^a$ is an ideal and $\|a \in T/P$ is an absorber. From the fact that if a truss has an absorber then it has only one left absorber one concludes that $P_p^a = P_p^{a'}$, for all left absorbers $a, a'$. $\square$

**Theorem 13.1.10.** *Let $T$ be a pre-truss. Then $P$ is a completely prime paragon if and only if $T/P$ is a domain.*

*Proof.* We write $\|a$ for the class of $a$ in $T/P$. The pre-truss $T/P$ is a domain if and only if, for all $\|a, \|b, \|c \in T/P$, $\|ab = \|ac$ implies that $\|b = \|c$ or $\|a$ is an absorber. The equality $\|ab = \|ac$ amounts to the existence of $p \in P$ such that $[ab, ac, p] \in P$. Observe that $\|b = \|c$ if and only if $[b, c, p] \in P$, and $\|a$ is an absorber if and only if $P_p^a$ is an ideal. The proof proceeds analogously for the right cancellation property. $\square$

*Remark* 13.1.11. Every paragon in a near-truss $\mathrm{T}(B)$ associated with a skew brace $B$ is completely prime.

**Corollary 13.1.12.** *Let $R$ be a ring. An ideal $I$ is completely prime in $R$ if and only if $I$ is a completely prime paragon in* $\mathrm{T}(R)$.

*Proof.* Let us assume that $I$ is a completely prime ideal in $R$. Then, for all $a, b \in R$ and absorber $0 \in I$,

$$[ab, a0, 0] = ab \in I \implies a \in I \text{ or } b \in I.$$

Thus, if $a \in I$, then $I_0^a = I$ is an ideal, and hence $I$ is a completely prime ideal in $\mathrm{T}(R)$.

Conversely, assume that $I$ is a completely prime paragon in $\mathrm{T}(R)$. For all $a, b \in \mathrm{T}(R)$,

$$ab = [ab, a0, 0] \in I \implies I_0^a \text{ is an ideal or } b \in I.$$

Observe that $I_0^a$ is an ideal if and only if $a \in I$. Therefore, $I$ is a completely prime ideal in $R$. This completes the proof. □

**Lemma 13.1.13.** *Let $f : T \to T'$ be a morphism of pre-trusses. If $P$ is a completely prime paragon in* $\mathrm{Im}\ f$*, then $f^{-1}(P)$ is a completely prime paragon in $T$.*

*Proof.* By Lemma 12.2.5, $f^{-1}(P)$ is a paragon. For all $a, b, c \in T$ and $p \in f^{-1}(P)$, if $[ab, ac, p] \in f^{-1}(P)$, then

$$f([ab, ac, p]) = [f(a)f(b), f(a)f(c), f(p)] \in P.$$

This implies that $P_{f(p)}^{f(a)}$ is an ideal or $f([b, c, p]) = [f(b), f(c), f(p)] \in P$. Therefore, $[b, c, p] \in f^{-1}(P)$ or $P_{f(p)}^{f(a)}$ is an ideal. Let us assume that

$$z \in f^{-1}\left(P_{f(p)}^{f(a)}\right) = \{x \in T \mid \exists q \in P \quad \text{s.t.} \quad f(z) = [q, f(p), f(a)]\}.$$

Then $f(z) = [q, f(p), f(a)]$, for some $q \in P$ and $f([z, a, p]) = [f(z), f(a), f(p)] = q \in P$. Hence $z = [[z, a, p], p, a] \in f^{-1}(P)_p^a$ and $f^{-1}(P_{f(p)}^{f(a)}) \subseteq f^{-1}(P)_p^a$. Therefore, $f^{-1}(P)_p^a \subseteq f^{-1}(P_{f(p)}^{f(a)})$ and by Lemma 12.3.2, $f^{-1}(P)_p^a$ is an ideal. This completes the proof. □

We conclude this section with an example of a completely prime paragon and the corresponding quotient domain.

*Example* 13.1.14. Let $O(x)$ be the set of all polynomials in $\mathbb{Z}[x]$ in which the sum of coefficients is odd. One can easily check that $O(x)$ is a sub-monoid of the multiplicative monoid $\mathbb{Z}[x]$ and a sub-heap of $\mathbb{Z}[x]$ with the standard operation $[p, q, r] = p - q + r$. All this means that $O(x)$ is a (commutative) truss.

Take any $t_0, t_1 \in O(x)$ and define

$$P(t_0, t_1) := \{p \in O(x) \mid (t_1 - t_0) \,\text{divides}\, (p - t_0)\}.$$

Then $P(t_0, t_1)$ is a paragon in $O(x)$ and it is a completely prime paragon provided that $t_1 - t_0$ is irreducible in $\mathbb{Z}[x]$.

*Proof.* Clearly, if $p - t_0$, $q - t_0$ and $r - t_0$ are divisible by $t_1 - t_0$, then so is $[p,q,r] - t_0 = p - q + r - t_0$. Hence $P(t_0, t_1)$ is a sub-heap of $O(x)$. Note that $t_0 \in P(t_0, t_1)$, and hence, for all $p \in P(t_0, t_1)$ and $q \in O(x)$,

$$[qp, qt_0, t_0] - t_0 = qp - qt_0 = q(p - t_0).$$

Therefore, $[qp, qt_0, t_0] = [pq, t_0q, t_0] \in P(t_0, t_1)$, which means that $P(t_0, t_1)$ is a paragon.

Now assume that $c = t_1 - t_0$ is irreducible in $\mathbb{Z}[x]$, and take $a, b \in O(x)$ for which there exists $p \in P(t_0, t_1)$ such that $[ab, ap, p] \in P(t_0, t_1)$, that is $c \mid a(b - t_0)$. Since $c$ is irreducible, then either $c \mid (b - t_0)$, in which case $b \in P(t_0, t_1)$, or $c \mid a$, that is, there exists $q \in \mathbb{Z}[x]$ such that $a = cq$. In this case,

$$P(t_0, t_1)_p^a = \{r - p + cq \mid r \in P(t_0, t_1)\}.$$

Thus $P(t_0, t_1)_p^a$ contains all elements of $O(x)$ divisible by $c$ (since $c \mid (r - p)$, for all $r, p \in P(t_0, t_1)$), and hence it is an ideal in $O(x)$. Combined with the commutativity of $O(x)$, Lemma 13.1.8 yields that $P(t_0, t_1)$ is a completely prime paragon. □

Note that in general in the situation described in Example 13.1.14,

$$\bar{a} = \bar{b} \in O(x)/P(t_0, t_1) \quad \text{if and only if} \quad (t_1 - t_0) \mid (a - b).$$

So, for example, take $t_0 = x$ and $t_1 = x^2 + x + 1$. Then $c = t_1 - t_0 = x^2 + 1$ is an irreducible polynomial in $\mathbb{Z}[x]$ and $O(x)/P(x, x^2 + x + 1)$ is a domain that can be identified with the sub-truss $O(i)$ of the truss (ring) of Gaussian integers $\mathbb{Z}[i]$, defined as

$$O(i) = \{m + ni \mid m + n \text{ is odd}\}.$$

## 13.2 Skew braces of fractions

As of now we have introduced the notions of a domain and a completely prime paragon, so that as long as we start with a pre-truss that has a completely prime paragon we can quotient out by it and obtain a domain. The next, and most important step, is to introduce localisation for pre-trusses. As the main goal of this section is to produce braces from near-trusses we will consider near-trusses without left absorbers and we will focus on localisation in the entire near-truss (to construct a "brace of fractions") following Ore's classic construction [Ore31]. First observe that since not every ring can be localised the same is true for trusses, thus, following [Ore31], we start by defining a regular pre-truss.

**Definition 13.2.1.** A pre-truss $T$ is said to be *left regular* if $T$ is a domain and it satisfies the left Ore condition, that is, for all $x, y \in T^{\mathrm{Abs}}$, there exist $r, s \in T^{\mathrm{Abs}}$ such that $rx = sy$.

In other words, a pre-truss is left (resp. right) regular if and only if $T^{\mathrm{Abs}}$ is a left Ore set. Next, we define the fraction relation on $T^{\mathrm{Abs}} \times T$, by $(b,a) \sim (b',a')$ if and only if there exists $\beta, \beta' \in T^{\mathrm{Abs}}$ such that $\beta b = \beta' b'$ and $\beta a = \beta' a'$. This is an equivalence relation by the same arguments as in [Ore31, Section 2]. The equivalence class of $(b,a)$ is denoted by $\frac{a}{b}$ and called a *fraction*, and the quotient set $T^{\mathrm{Abs}} \times T/\sim$ is denoted by $Q(T)$.

**Theorem 13.2.2.** *(Ore localisation for regular pre-trusses) Let T be a left regular pre-truss. Then $Q(T)$ is a pre-truss with the following operations*

*(1) For all $\frac{a}{b}, \frac{a'}{b'}, \frac{a''}{b''} \in Q(T)$, the ternary operation is defined by*

$$\left[\frac{a}{b}, \frac{a'}{b'}, \frac{a''}{b''}\right] := \frac{[\beta_1 a, \beta_2 a', \beta_3 a'']}{\beta_1 b} = \frac{[\beta_1 a, \beta_2 a', \beta_3 a'']}{\beta_2 b'} = \frac{[\beta_1 a, \beta_2 a', \beta_3 a'']}{\beta_3 b''}, \tag{13.2.1}$$

*where $\beta_1, \beta_2, \beta_3$ are any elements of $T^{\mathrm{Abs}}$ such that $\beta_1 b = \beta_2 b' = \beta_3 b''$.*

*(2) For all $\frac{a}{b}, \frac{a'}{b'} \in Q(T)$,*

$$\frac{a}{b} \cdot \frac{a'}{b'} := \frac{\gamma a'}{\gamma' b'}, \tag{13.2.2}$$

*where $\gamma, \gamma' \in T^{\mathrm{Abs}}$ are such that $\gamma b' = \gamma' a$.*

*Furthermore, $(Q(T)^{\mathrm{Abs}}, \cdot)$ is a group. We will call $Q(T)$ the* pre-truss of (left) fractions *of $T$.*

*Proof.* We follow closely the proof of [Ore31, Theorem 1]. The multiplication of fractions (13.2.2) is defined in such a way that $\frac{a}{b}$ can be interpreted as $b^{-1}a$. Since it relies entirely on the properties of the semigroup $(T, \cdot)$, the arguments of the proof of [Ore31, Theorem 1] (with no modification, apart from the conventions) yield that $(Q(T), \cdot)$ is a semigroup.

It remains to be proven that $Q(T)$ is a heap. In fact, by the Ore condition we may assume that all fractions in the definition of the ternary operation (13.2.1) on $Q(T)$ have common denominator, so that

$$\left[\frac{a}{b}, \frac{a'}{b}, \frac{a''}{b}\right] = \frac{[\beta a, \beta a', \beta a'']}{\beta b}, \tag{13.2.3}$$

since in this case we can choose $\beta := \beta_1 = \beta_2 = \beta_3$. Thus suffices it to prove that (13.2.3) is well-defined, as then the heap axioms for $T$ will imply the corresponding axioms for the derived operation (13.2.3). We proceed in two steps. At first, we show that the formula (13.2.3) does not depend on the choice of $\beta$; in the second stage we will prove that it is also independent of the choice of the representatives $a, b$ for the class $\frac{a}{b}$.

Choose another element $s \in T^{\mathrm{Abs}}$ such that

$$\left[\frac{a}{b}, \frac{a'}{b}, \frac{a''}{b}\right] = \frac{[sa, sa', sa'']}{sb}.$$

There exist $g, g' \in T^{\text{Abs}}$ such that $g\beta b = g' sb$, which implies

$$g\beta = g' s,$$

since $T$ is a domain. Therefore,

$$g[\beta a, \beta a', \beta a''] = g'[sa, sa', sa''], \quad g\beta b = g' sb.$$

Consequently,

$$\frac{[\beta a, \beta a', \beta a'']}{\beta b} = \frac{[sa, sa', sa'']}{sb},$$

which shows the independence of the formula (13.2.3) of the the choice of $\beta$.

To prove that the ternary operation (13.2.1) does not depend on the choice of the representatives in each equivalence class, let $(b, a), (b', a'), (b'', a''), (d, c), (d', c'), (d'', c'') \in T^{\text{Abs}} \times T$ be such that

$$\frac{a}{b} = \frac{c}{d}, \ \frac{a'}{b'} = \frac{c'}{d'}, \ \frac{a''}{b''} = \frac{c''}{d''},$$

and consider

$$\left[\frac{a}{b}, \frac{a'}{b'}, \frac{a''}{b''}\right] = \frac{[\beta_1 a, \beta_2 a', \beta_3 a'']}{\beta_1 b}, \qquad \left[\frac{a}{b}, \frac{a'}{b'}, \frac{c''}{d''}\right] = \frac{[s_1 a, s_2 a', s_3 c'']}{s_1 b}, \tag{13.2.4}$$

for suitable $\beta_1, \beta_2, \beta_3, s_1, s_2, s_3 \in T^{\text{Abs}}$. Then there exist $g, g' \in T$, such that

$$g\beta_1 b = g\beta_2 b' = g\beta_3 b'' = g' s_1 b = g' s_2 b' = g' s_3 d'',$$

and, since $T$ is a domain,

$$g\beta_1 = g' s_1, \ g\beta_2 = g' s_2.$$

Thus both fractions in the equation (13.2.4) are equal if and only if $g\beta_3 a'' = g' s_3 c''$. Observe, however, that since $g' s_3 d'' = g\beta_3 b''$, $g\beta_3 a'' = g' s_3 c''$ as $\frac{a''}{b''} = \frac{c''}{d''}$. Therefore,

$$\left[\frac{a}{b}, \frac{a'}{b'}, \frac{a''}{b''}\right] = \left[\frac{a}{b}, \frac{a'}{b'}, \frac{c''}{d''}\right].$$

The remaining equalities

$$\left[\frac{a}{b}, \frac{a'}{b'}, \frac{c''}{d''}\right] = \left[\frac{a}{b}, \frac{c'}{d'}, \frac{c''}{d''}\right] \ \text{ and } \ \left[\frac{a}{b}, \frac{c'}{d'}, \frac{c''}{d''}\right] = \left[\frac{c}{d}, \frac{c'}{d'}, \frac{c''}{d''}\right],$$

are proven in a similar way. This completes the proof that the definition of the ternary operation (13.2.1) does not depend on the choice of representatives.

Finally, observe that if $a$ is an absorber, then the class $\frac{a}{b}$ is an absorber and it is obviously unique. One can easily check that the class $\frac{b}{b}$ for $b \in T^{\text{Abs}}$ is a neutral element of $(Q(T)^{\text{Abs}}, \cdot)$ and that if $a \in T^{\text{Abs}}$ then $\frac{a}{b}$ is a two-sided inverse to $\frac{b}{a}$. Thus $(Q(T)^{\text{Abs}}, \cdot)$ is a group. This completes the proof of the theorem. $\square$

From the fact that one can find a common denominator to any system of fractions one can observe that additional properties of $T$ are carried over to $Q(T)$.

**Proposition 13.2.3.** *Let $T$ be a regular pre-truss.*

*(1) If $T$ is Abelian, then so is $Q(T)$.*

*(2) If $T$ is a near-truss, then $Q(T)$ is a near-truss.*

*(3) If $T$ is a skew truss, then $Q(T)$ is a skew truss.*

*Proof.* It is sufficient to consider heap operations of fractions with a common denominator, that is, those given by the formula (13.2.3). Statement (1) follows immediately from (13.2.3).

If $T$ is a near-truss, then

$$\left[\frac{a}{b}, \frac{a'}{b}, \frac{a''}{b}\right] = \frac{[\beta a, \beta a', \beta a'']}{\beta b} = \frac{\beta[a, a', a'']}{\beta b} = \frac{[a, a', a'']}{b}.$$

Take any $\frac{c}{d}, \frac{a}{b}, \frac{a'}{b}, \frac{a''}{b} \in Q(T)$ and $\gamma, \gamma' \in T^{\mathrm{Abs}}$ such that $\gamma b = \gamma' c$, and compute

$$\begin{aligned}\frac{c}{d} \cdot \left[\frac{a}{b}, \frac{a'}{b}, \frac{a''}{b}\right] &= \frac{c}{d} \cdot \frac{[a, a', a'']}{b} = \frac{\gamma[a, a', a'']}{\gamma' d} = \frac{[\gamma a, \gamma a', \gamma a'']}{\gamma' d} \\ &= \left[\frac{\gamma a}{\gamma' d}, \frac{\gamma a'}{\gamma' d}, \frac{\gamma a''}{\gamma' d}\right] = \left[\frac{c}{d} \cdot \frac{a}{b}, \frac{c}{d} \cdot \frac{a'}{b}, \frac{c}{d} \cdot \frac{a''}{b}\right].\end{aligned}$$

Hence the left distributive law holds, and this proves statement (2).

To prove (3) we take $\frac{c}{d}, \frac{a}{b}, \frac{a'}{b}, \frac{a''}{b} \in Q(T)$ and $\gamma, \gamma' \in T^{\mathrm{Abs}}$ such that $\gamma d = \gamma'[a, a', a'']$. Then

$$\left[\frac{a}{b}, \frac{a'}{b}, \frac{a''}{b}\right] \cdot \frac{c}{d} = \frac{[a, a', a'']}{b} \cdot \frac{c}{d} = \frac{\gamma c}{\gamma' b}.$$

On the other hand, using the definitions (13.2.1) and (13.2.2) and the right distributivity in $T$, we obtain

$$\left[\frac{a}{b} \cdot \frac{c}{d}, \frac{a'}{b} \cdot \frac{c}{d}, \frac{a''}{b} \cdot \frac{c}{d}\right] = \left[\frac{\gamma_1 c}{\gamma_1' b}, \frac{\gamma_2 c}{\gamma_2' b}, \frac{\gamma_3 c}{\gamma_3' b}\right] = \frac{[s_1\gamma_1, s_2\gamma_2, s_3\gamma_3]c}{s_1\gamma_1' b},$$

where $s_1, s_2, s_3, \gamma_1, \gamma_2, \gamma_3, \gamma_1', \gamma_2', \gamma_3' \in T^{\mathrm{Abs}}$ are such that

$$\gamma_1' a = \gamma_1 d, \ \gamma_2' a' = \gamma_2 d, \ \gamma_3' a'' = \gamma_3 d, \ s_1\gamma_1' = s_2\gamma_2' = s_3\gamma_3'. \tag{13.2.5}$$

Let $h, h' \in T^{\mathrm{Abs}}$ be such that

$$h\gamma' = h' s_1 \gamma_1'. \tag{13.2.6}$$

Then, using the distributive laws in $T$, (13.2.5) and (13.2.6), we find

$$\begin{aligned}h\gamma d &= h\gamma'[a, a', a''] = [h\gamma' a, h\gamma' a', h\gamma' a''] = [h' s_1\gamma_1' a, h' s_1\gamma_1' a', h' s_1\gamma_1' a''] \\ &= h'[s_1\gamma_1' a, s_2\gamma_2' a', s_3\gamma_3' a''] = h'[s_1\gamma_1 d, s_2\gamma_2 d, s_3\gamma_3 d] = h'[s_1\gamma_1, s_2\gamma_2, s_3\gamma_3]d.\end{aligned}$$

The right cancellation property yields

$$h\gamma = h'[s_1\gamma_1, s_2\gamma_2, s_3\gamma_3],$$

which in view of (13.2.6) implies that

$$\frac{\gamma c}{\gamma' b} = \frac{[s_1\gamma_1, s_2\gamma_2, s_3\gamma_3]c}{s_1\gamma_1' b}.$$

Therefore, also the right distributive law holds in the near-truss $Q(T)$. □

The construction of the truss of quotients is universal in the following sense.

**Proposition 13.2.4.** *Let $T$ be a regular pre-truss. Then*

*(1) For any $b \in T^{\mathrm{Abs}}$,*

$$\iota_b\colon\ T \to Q(T), \qquad a \mapsto \frac{ba}{b},$$

*is a monomorphism of semigroups, and it is a monomorphism of trusses provided $T$ is a near- or skew truss.*

*(2) If $T$ is a unital pre-truss then $\iota_1$ is a monomorphism of unital trusses. Furthermore, for any brace-type near-truss $B$ and any unital truss homomorphism $f : T \to B$, there exists a unique unital truss homomorphism $\hat{f}\colon\ Q(T) \to B$ rendering commutative the following diagram:*

$$\begin{tikzcd} T \arrow[rr, "\iota_1"] \arrow[dr, "f"'] & & Q(T) \arrow[dl, "\hat{f}"] \\ & B. & \end{tikzcd}$$

*Proof.* (1) Since $T$ is regular, $\iota_b$ is an injective map. For all $a, a' \in T$,

$$\iota_b\left(aa'\right) = \frac{baa'}{b} \quad \& \quad \iota_b\left(a\right)\cdot\iota_b\left(a'\right) = \frac{ba}{b}\cdot\frac{ba'}{b} = \frac{\gamma ba'}{\gamma' b},$$

where $\gamma, \gamma'$ are such that $\gamma b = \gamma' ba$. Take any $\beta, \beta' \in T$ such that $\beta b = \beta'\gamma' b$. Then

$$\beta baa' = \beta'\gamma' baa' = \beta'\gamma ba',$$

which means that $\iota_b\left(aa'\right) = \iota_b\left(a\right)\cdot\iota_b\left(a'\right)$ as required.

In the case of a near- or skew truss, that $\iota_b$ is a homomorphism of trusses follows by (13.2.3) and the left distributive law.

(2) The monomorphism of semigroups $\iota_1$ preserves the heap operation since 1 is the multiplicative identity in $T$.

Assume that $f : T \to B$ is a unital homomorphism of trusses and, for all fractions $\frac{a}{b} \in Q(T)$, define

$$\hat{f}\colon\ Q(T) \to B, \qquad \frac{a}{b} \mapsto f(b)^{-1}f(a).$$

This is well defined since two fractions $\frac{a}{b}$ and $\frac{a'}{b'}$ are identical if and only if there are $\beta, \beta'$ such that $\beta a = \beta' a'$ and $\beta b = \beta' b'$, in which case

$$\begin{aligned}\hat{f}\left(\frac{a}{b}\right) &= f(b)^{-1}f(a) = f(b)^{-1}f(\beta)^{-1}f(\beta)f(a)\\ &= f(\beta b)^{-1}f(\beta a) = f(\beta' b')^{-1}f(\beta' a') = f(b')^{-1}f(a') = \hat{f}\left(\frac{a'}{b'}\right),\end{aligned}$$

by the multiplicativity of $f$. By the same token, for all $\frac{a}{b}, \frac{a'}{b'} \in Q(T)$,

$$\hat{f}\left(\frac{a}{b}\cdot\frac{a'}{b'}\right) = \hat{f}\left(\frac{\gamma a'}{\gamma' b}\right) = f(\gamma' b)^{-1}f(\gamma a') = f(b)^{-1}f(\gamma')^{-1}f(\gamma)f(a'),$$

where $\gamma, \gamma' \in T$ are such that $\gamma b' = \gamma' a$. Applying $f$ to both sides of this equality and using the multiplicative property to $f$ we obtain

$$f(\gamma')^{-1}f(\gamma) = f(a)f(b')^{-1},$$

and hence

$$\hat{f}\left(\frac{a}{b}\cdot\frac{a'}{b'}\right) = f(b)^{-1}f(a)f(b')^{-1}f(a') = \hat{f}\left(\frac{a}{b}\right)\hat{f}\left(\frac{a'}{b'}\right),$$

that is $\hat{f}$ is a homomorphism of multiplicative groups. To check that $\hat{f}$ is a heap morphism it is enough to consider fractions with a common denominator and then

$$\begin{aligned}\hat{f}\left(\left[\frac{a}{b}, \frac{a'}{b}, \frac{a''}{b}\right]\right) &= f(b)^{-1}\left[f(a), f(a')f(a'')\right]\\ &= \left[f(b)^{-1}f(a), f(b)^{-1}f(a')f(b)^{-1}f(a'')\right]\\ &= \left[\hat{f}\left(\frac{a}{b}\right), \hat{f}\left(\frac{a'}{b}\right), \hat{f}\left(\frac{a''}{b}\right)\right],\end{aligned}$$

by the fact that $f$ is a heap homomorphism and the left distributive law in $B$. That $\hat{f} \circ \iota_1 = f$ follows by the unitality of $f$.

Suppose that there exists a unital truss homomorphism $\hat{g}\colon\ Q(T) \to B$ such that $\hat{g} \circ \iota_1 = f$. Note that

$$\frac{a}{b} = \frac{1}{b}\cdot\frac{a}{1}. \tag{13.2.7}$$

In particular,

$$1 = \hat{g}\left(\frac{1}{1}\right) = \hat{g}\left(\frac{1}{b}\cdot\frac{b}{1}\right) = \hat{g}\left(\frac{1}{b}\right)f(b),$$

where the last equality follows by the splitting assumption $\hat{g} \circ \iota_1 = f$. Hence $\hat{g}\left(\frac{1}{b}\right) = f(b)^{-1}$ and the equality $\hat{g} = \hat{f}$ follows by the multiplicativity of $\hat{g}$ and equations (13.2.7). □

The following corollary provides one with the method of constructing skew braces, which might be considered as one of the main results of this paper.

**Corollary 13.2.5.** *If $T$ is a regular near-truss without an absorber, then $Q(T)$ is a brace-type near-truss, that is, for all $b \in T$, the retract of $Q(T)$ at $\frac{b}{b}$ with the product* (13.2.2) *is a skew brace.*

*Proof.* Observe that if $T$ has no absorbers then $Q(T)$ has no absorbers either. Indeed, suppose that there exists $\frac{a}{b} \in Q(T)$ such that, for all $\frac{c}{d} \in Q(T)$, $\frac{c}{d} \cdot \frac{a}{b} = \frac{a}{b}$. Since $T$ has no absorbers, it has at least two elements, and hence, in particular we may consider $c \neq d$. Then there exist $\gamma, \gamma' \in T$, such that $\frac{\gamma a}{\gamma' d} = \frac{\gamma a}{\gamma b}$ and $\gamma' c = \gamma b$. Thus $\frac{\gamma a}{\gamma' d} = \frac{\gamma a}{\gamma' c}$, so there exist $\beta, \beta' \in T$ such that $\beta\gamma' d = \beta'\gamma' c$ and $\beta\gamma a = \beta'\gamma a$. By regularity, $\beta = \beta'$ and $c = d$, which is the required contradiction. Therefore, $\frac{a}{b}$ is not an absorber for all $a, b \in T$. Now, since $Q(T)$ is a group with multiplication and identity $\frac{b}{b}$, the retract of $Q(T)$ in $\frac{b}{b}$ is a skew brace by [Brz20, Remark 3.13]. □

Note in passing that if $T$ satisfies the same assumptions as in Corollary 13.2.5, but there exists an absorber in $T$, then $Q(T)$ is associated with a near-field.

*Example* 13.2.6. Let us consider the sub-truss $2\mathbb{Z}+1$ of $\mathrm{T}(\mathbb{Z})$. It is a domain satisfying the Ore condition, thus it is a regular truss and we can localise it in itself. Since $2\mathbb{Z}+1$ is commutative, the construction is much simpler than the one presented in the proof of Theorem 13.2.2. One can easily check that $Q(2\mathbb{Z}+1) = \frac{2\mathbb{Z}+1}{2\mathbb{Z}+1} := \left\{\frac{2p+1}{2q+1} \mid p, q \in \mathbb{Z}\right\}$. The two-sided brace associated with this truss is the retract in 1, i.e. the triple $(Q(2\mathbb{Z}+1), [-,1,-], \cdot)$.

Similarly, the truss $O(x)$ of integer polynomials with coefficients summing up to odd numbers considered in Example 13.1.14 is regular with no absorbers, and hence it can be localised to a brace-type truss of the following rational functions

$$Q(O(x)) = \frac{O(x)}{O(x)} := \left\{\frac{p(x)}{q(x)} \mid p(x), q(x) \in O(x)\right\}.$$

As a yet another example we can consider the truss $O(i)$ constructed as a special case of Example 13.1.14. Again this is a commutative domain satisfying the Ore condition and with no absorbers, and hence

$$\begin{aligned} Q(O(i)) &= \left\{\frac{m+ni}{p+qi} \mid m+n \text{ and } p+q \text{ are odd integers}\right\} = \\ &= \left\{\frac{m}{2p+1} + \frac{n}{2q+1} i \mid p, q \in \mathbb{Z},\ m+n \text{ is an odd integer}\right\}. \end{aligned}$$

The example of odd fractions described above is a special case of a more general construction.

*Example* 13.2.7. Let $T_n(\mathbb{Z})$ denote the set of all $n \times n$-matrices over $\mathbb{Z}$ with odd entries on the diagonal and even off diagonal entries. That is,

$$T_n(\mathbb{Z}) = \left\{(a_{ij})_{i,j=1}^n \mid a_{ii} \in 2\mathbb{Z}+1 \text{ and } a_{ij} \in 2\mathbb{Z},\ i \neq j\right\}.$$

(1) $T_n(\mathbb{Z})$ endowed with the matrix multiplication and the standard heap operation $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} - \mathbf{b} + \mathbf{c}$ is a unital regular truss with no absorbers.

(2) The brace-type truss of fractions $Q(T_n(\mathbb{Z}))$ can be identified with the set $T_n(\mathbb{Q})$ of $n \times n$-matrices over the rational numbers with diagonal entries made by the odd fractions (that is, fractions of both the numerator and denominator odd, $Q(2\mathbb{Z}+1)$) and with fractions with even numerator and odd denominator as off-diagonal entries. That is,

$$Q(T_n(\mathbb{Z})) \cong T_n(\mathbb{Q}) := \left\{ (q_{ij})_{i,j=1}^n \mid q_{ii} \in \frac{2\mathbb{Z}+1}{2\mathbb{Z}+1} \text{ and } q_{ij} \in \frac{2\mathbb{Z}}{2\mathbb{Z}+1},\ i \neq j \right\}.$$

It is clear that the set $T_n(\mathbb{Z})$ is closed under the described heap operation. That it is closed also under the matrix multiplication follows from an observation that in the product formula for the off-diagonal entries the sum involves the products of numbers of which at least one is even, while for the diagonal entry there is a single odd summand made out of the product of matching diagonal entries. Obviously $T_n(\mathbb{Z})$ has no absorber, as the zero matrix is not an element of $T_n(\mathbb{Z})$. Since the identity matrix has the prescribed form, $T_n(\mathbb{Z})$ is unital. The other statements of Example 13.2.7 can be justified by the following (elementary) lemma.

**Lemma 13.2.8.** *For all* $\mathbf{a} \in T_n(\mathbb{Z})$*,*

*(1) The determinant* $\det(\mathbf{a})$ *is an odd number.*

*(2) The matrix of cofactors* $\bar{\mathbf{a}}$ *of* $\mathbf{a}$ *and hence also its transpose* $\bar{\mathbf{a}}^t$ *are elements of* $T_n(\mathbb{Z})$*.*

*Proof.* Let $\mathbf{a}_{i,j}$ denote the matrix obtained from $\mathbf{a}$ by removing of the $i$-th row and the $j$-th column. Note that $\mathbf{a}_{i,i} \in T_{n-1}(\mathbb{Z})$ and that $\mathbf{a}_{i,j}$, $i \neq j$ has one row of even numbers.

The first statement is proven by induction on the size $n$ of matrices. For $n = 1$ the statement is obviously true. Assuming that the statement is true for $k$ we calculate the determinant of $\mathbf{a} \in T_{k+1}(\mathbb{Z})$ by expanding by the first row. Since $\mathbf{a}_{1,1}$ is an element of $T_k(\mathbb{Z})$, $\det(\mathbf{a}_{1,1})$ is odd by inductive assumption. In the expansion of $\det(\mathbf{a})$ this is multiplied by the first entry $a_{11}$ of $\mathbf{a}$ and thus it gives an odd number. All the remaining summands involve products of other entries of the first row, which are even. Hence the sum of all terms in the expansion is odd as required.

The diagonal entries of $\bar{\mathbf{a}}$ are given by $\det(\mathbf{a}_{i,i})$ which are odd by statement (i). Off-diagonal entries $(-1)^{i+j}\det(\mathbf{a}_{i,j})$ are even since one row of each of $\mathbf{a}_{i,j}$, $i \neq j$ consists entirely of even numbers. The transposition statement is obvious. □

With this lemma at hand we can now prove that $T_n(\mathbb{Z})$ is a domain satisfying the Ore condition. Since we can embed $T_n(\mathbb{Z})$ into a ring of matrices, the statement $\mathbf{ab} = \mathbf{ac}$, for some $\mathbf{a}, \mathbf{b}, \mathbf{c} \in T_n(\mathbb{Z})$ is equivalent to the statement that $\mathbf{a}(\mathbf{b} - \mathbf{c}) = 0$, hence

$$0 = \mathbf{a}(\mathbf{b} - \mathbf{c}) = \bar{\mathbf{a}}^t \mathbf{a}(\mathbf{b} - \mathbf{c}) = \det(\mathbf{a})(\mathbf{b} - \mathbf{c}),$$

which implies that $\mathbf{b} = \mathbf{c}$, since $\det(\mathbf{a}) \neq 0$ by Lemma 13.2.8(i). The regularity of the other side of each $\mathbf{a} \in T_n(\mathbb{Z})$ can be proven in a symmetric way.

To prove the Ore condition we take any $\mathbf{a}, \mathbf{b} \in T_n(\mathbb{Z})$ and set

$$\mathbf{r} = \mathbf{a}\bar{\mathbf{b}}^t \quad \& \quad \mathbf{s} = \det(\mathbf{b})\mathbf{1}.$$

Both these matrices are elements of $T_n(\mathbb{Z})$ by Lemma 13.2.8, and they satisfy the Ore condition $\mathbf{sa} = \mathbf{rb}$. Hence, $T_n(\mathbb{Z})$ is a left regular (in fact also right regular by similar arguments) truss.

For any element $\mathbf{q} \in T_n(\mathbb{Q})$ we write $q$ for the product of all denominators in entries of $\mathbf{q}$. This is an odd number and thus obviously $q\mathbf{q} \in T_n(\mathbb{Z})$. In particular, in view of Lemma 13.2.8, $\det(q\mathbf{q})$ is an odd number and its matrix of cofactors is an element of $T_n(\mathbb{Z})$. This in turn implies that the inverse of $\mathbf{q}$ is an element of $T_n(\mathbb{Z})$ divided by an odd number, hence an element of $T_n(\mathbb{Q})$. Consequently, $T_n(\mathbb{Q})$ is group with respect to multiplication of matrices. In order to identify $T_n(\mathbb{Q})$ with the truss of fractions $Q(T_n(\mathbb{Z}))$ we will explore the universal property described in Proposition 13.2.4(2). Thus consider a brace-type skew truss $B$ and a homomorphism of unital trusses $f\colon T_n(\mathbb{Z}) \to B$ and set

$$\hat{f}\colon T_n(\mathbb{Q}) \to B, \qquad \mathbf{q} \mapsto f(q\mathbf{1})^{-1} f(q\mathbf{q}).$$

Note that this definition does not depend on the way the fractions in $\mathbf{q}$ are represented, as the multiplication of the numerator and a denominator of an entry by a common (odd) factor results in multiplying both $q$ and $\mathbf{q}$ by the same factor which will cancel each other out in the formula for $\hat{f}$, by the multiplicative property of $f$. Since $q\mathbf{1}$ is a central element in $T_n(\mathbb{Z})$, $f(q\mathbf{1})^{-1}$ is central in the image of $f$ and, combined with the multiplicative property of $f$ this implies that $\hat{f}$ is a homomorphism of (multiplicative) groups. That $\hat{f}$ is a homomorphism of heaps follows by the distributivity. Obviously, $\hat{f} \circ \iota_{\mathbf{1}} = f$ and is a unique such morphisms. By the uniqueness of universal objects, $T_n(\mathbb{Q})$ is isomorphic to the truss of fractions $Q(T_n(\mathbb{Z}))$.

# Appendices

# Appendix A: Computing tropical solutions to the tropicalization of some linear $p$-adic ODEs

We compute here the tropical solutions for the tropicalization of three ordinary differential equations with coefficients in $\mathbb{Q}_p[\![t]\!]$ of a certain interest in the theory of $p$-adic differential equations and we notice that in both cases the fundamental theorem of differential tropical algebra holds. Let us fix the differentially enhanced valuation

$$(v, \widetilde{v}) : \mathbb{Q}_p[\![t]\!] \to \mathbf{S} = (\mathbb{T}[\![t]\!]_p \to \mathbb{T}_2)$$

of Example 6.5.5.

The first equation we are going to consider has as only solution the $p$-adic exponential function: as this is the easiest differential equation one can think of, if we want to test our methods in a concrete case and have hopes that a fundamental theorem holds,we need to start from this equation. Let $p$ a prime number and consider the differential ideal $I \subset \mathbb{Q}_p[\![t]\!]\{x\}$ generated by the differential polynomial $f = x' - x$. Solutions to this ideal are of the form:

$$c \exp(t) = c \sum_{n=0}^{\infty} \frac{1}{n!} t^n \tag{13.2.8}$$

for $c \in \mathbb{Q}_p$. The $p$-adic series $\sum_n \frac{1}{n!} t^n$ is called the $p$-adic exponential and it only converges for $|t|_p < 1/p^{(p-1)}$ (i.e. for $v_p(t) > p - 1$, if we normalise by $|p|_p = 1/p$).

We want to prove that all the tropical solutions to the tropicalization of the ideal $I$ are the tropicalization of some solution of the form 13.2.8, i.e. that in this particular case the fundamental theorem of differential tropical algebra holds.

Let $A = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{T}[\![t]\!]$ be a tropical solution. As adding the same real number to all the coefficients of a solution given another solution, i.e. the set of solutions is a tropical linear space, we can assume $a_0 = 0$. We need to prove inductively that $a_{n+1} = -v_p((n+1)!)$ for every $n \in \mathbb{N}$. As $A$ is a solution for $\mathrm{trop}_v(I)$ it must be in particular a tropical solution for the tropicalization of every generator of $I$, which means to the family of polynomials $f^{(n)} = x^{(n)} + x^{(n+1)} \in \mathbb{T}_2\{x\}_{basic}$ for every $n \in \mathbb{N}$. For $n = 0$ we have:

$$f(A) = (0, 0) \oplus (0, a_1)$$

thus $A$ is a solution to $f$ if and only if $a_1 = 0 = -v_p(1!)$. So the case $n = 0$ holds. The general case follows from noticing that for every $n \in \mathbb{N}$, $d^n t^n = v_p(n!)$, as the map $\mathbb{Q}_p[\![t]\!] \to \mathbb{T}[\![t]\!]_p$ sending a $p$-adic power series to its coefficientwise valuation commutes with the differentials. Thus assuming $a_n = 0 - v_p(n!)$ holds, we have:

$$f^{(n)}(A) = (0, a_n + v_p(n!)) \oplus (0, a_{n+1} + v_p((n+1)!) = (0,0) \oplus (0, a_{n+1} + v_p((n+1)!)$$

which implies the thesis as $A$ is a tropical solution. Since we know that the tropicalization of a solution is a tropical solutions, and we proved that every tropical solution is of that form, the two sets are the same.

Analogously one can prove that given an element $\pi$ in an algebraic extension $L$ of $\mathbb{Q}_p$ such that $\pi^{p-1} = -p$, we have that the fundament theorem holds for the equation $x' - \pi x \in L\{x\}$, whose solutions are multiples of $\exp(\pi t) \in L[\![t]\!]$. As the $p$-adic norm of $\pi$ is $|\pi|_p = p^{p-1}$, the radius of convergence of $\exp(\pi t)$ is 1 at any point, thus the radius of converge function is the constant function 1.

Let us consider now another equation, namely $f = x' - p\pi t^{p-1} x \in L\{x\}$. $p$-adic solutions to this equations are $p$-adic multiples of $\exp(\pi t^p) \in L[[t]]$. The interest of this $p$-adic differential equation lies in the fact that it is somehow the easiest one for which we can observe the piecewise linear behaviour of the radius of convergence as a function in the distance from the point 0, in particular, $\exp(\pi t^p)$ has radius of convergence 1 but there exists an $r > 1$ such that when $|x|_p > r$ the radius of converge of the solutions decreases. Another interesting fact is that the two differential modules related to this equation and the previous one are isomorphic via a Frobenius morphism, on the disc centered in 0 and of radius $r$, but are not outside this disk, as the radius of converge is invariant under isomorphisms.

Let us prove that the fundamental theorem holds for this equation too, considering the set of solutions at 0. The $n$-th derivative of $F$ can be expressed as:

$$d^n f = \begin{cases} x^{(n+1)} - \sum_{i=0}^{n} \binom{n}{i} \pi \prod_{k=0}^{n-i} (p-k) t^{p-1-n+i} x^{(i)} & \text{if } n < p \\ x^{(n+1)} - \sum_{i=0}^{p-1} \binom{n}{p-1-i} \pi \prod_{k=0}^{p-1-i} (p-k) t^{p-1-i} x^{(i+n-p+1)} & \text{if } n \geq p \end{cases}$$

which implies:

$$\operatorname{trop}(d^n f) = \begin{cases} x^{(n+1)} + \sum_{i=0}^{n} (p-1-n+i, v_p(\binom{n}{i})) + \frac{p}{p-1}) x^{(i)} & \text{if } n < p \\ x^{(n+1)} + \sum_{i=0}^{p-1} (i, v_p(\binom{n}{p-1-i})) + \frac{p}{p-1}) x^{(i+n-p+1)} & \text{if } n \geq p \end{cases}$$

In order to prove that the fundamental theorem holds, we prove that given a solution to the tropical system $\{\operatorname{trop}(d^n F)\}_{n \in N}$:

$$A = 0 + a_1 t + a_2 t^2 + \cdots \in \mathbb{T}[\![t]\!]_p$$

we have

$$a_k = \begin{cases} \infty & \text{if } p \nmid k \\ \frac{m}{p-1} - v_p(m!) & \text{if } k = mp \end{cases}$$

i.e. that $A$ is the tropicalization of $\exp(\pi t^p)$.

We will prove this by induction on $k$. For $k=0$ this is trivially satisfied as $p \mid 0$ and $0 = \frac{0}{p-1} - v_p(0!)$. For $0 < k \leq p$, consider $\mathrm{trop}(f) = x' + (p-1, \frac{p}{p-1})x$ and

$$\mathrm{trop}(f)(A) = (0, a_1) \oplus \left(p-1, \frac{p}{p-1}\right).$$

Being $A$ a solution, $a_k$ has to be equal to $\infty$ for every $0 < k < p$, thus we obtain:

$$\mathrm{trop}(f)(A) = (p-1, a_p + 1) \oplus \left(p-1, \frac{p}{p-1}\right)$$

which gives $a_p = \frac{1}{p-1} = \frac{1}{p-1} - v_p(1!)$.

In general let $Mp < k \leq (M+1)p$ for some $M \in \mathbb{N}$ and assume the inductive hypothesis holds for $Mp$. Notice that $v_p(\binom{Mp}{p-1-i}) = 1$ for all $M \in \mathbb{N}$ and for all $i \in \{0, \dots, p-2\}$, thus:

$$\mathrm{trop}(d^{Mp} f) = x^{(Mp+1)} + \left(p-1, \frac{p}{p-1}\right) x^{(Mp)} + \sum_{j=1}^{p-1} \left(p-1-j, \frac{p}{p-1} + 1\right) x^{(Mp-j)}$$

where $j := i - p + 1$. We obtain that $\mathrm{trop}(d^{Mp} f)(A)$ is equal to:

$$\left(0, a_{Mp+1} + \sum_{m=1}^{M} v_p(mp)\right) \oplus \left(p-1, \frac{p}{p-1}\right) \otimes \left(0, \frac{M}{p-1} - v_p(M!) + \sum_{m=1}^{M} v_p(mp)\right) \oplus$$
$$\bigoplus_{j=1}^{p-1} \left(p-1-j, \frac{p}{p-1} + 1\right) \otimes \left(j, \frac{M}{p-1} - v_p(M!) + \sum_{m=1}^{M} v_p(mp)\right)$$

As the first coordinate of every addendum of the sum above, except for the first one, is equal to $p-1$ and $A$ is a solution, we have as before that $a_k$ has to be equal to $\infty$ for $Mp < k < (M+1)p$. For $k = (M+1)p$, since

$$\sum_{m=1}^{M} v_p(mp) = \sum_{m=1}^{M} (1 + v_p(m)) = M + v_p(M!)$$

we have:

$$\mathrm{trop}(d^{Mp} f)(A) = \big(p-1, a_{(M+1)p} + M + 1 + v_p((M+1)!)\big) \oplus \left(p-1, \frac{M+p}{p-1} + M\right) \oplus$$
$$\oplus \left(p-1, \frac{M+p}{p-1} + M + 1\right)$$

which gives:

$$a_{(M+1)p} + 1 + v_p((M+1)!) = \frac{M+p}{p-1}$$

which is equivalent to $a_{(M+1)p} = \frac{M+1}{p-1} - v_p((M+1)!)$, as we wanted.

In general, in order to make tropical methods useful for the computation of the radius of convergence fucntion, it would be interesting to introduce an appropriate notion of tropical differential linear spaces, that would be the tropical shadow of differential modules and would encode the datum of a tropical differential equation.

# Résumé de thèse en français

## Première partie

La première partie de la présente thèse constitue le corps principal des travaux que j'ai menés au cours de ma thèse. L'idée qui a donné le départ à tout est introduite et étudiée dans ce qui suit est de construire un pont entre la théorie récemment introduite des équations différentielles tropicales et la théorie plus établie des équations différentielles $p$-adiques. En particulier la recherche de méthodes tropicales pour calculer le rayon de convergence des solutions à une équation différentielle $p$-adique.

Les équations différentielles ordinaires (EDO) avec des fonctions valuées réelles ou complexes comme coefficients sont fondamentales dans une multitude d'applications des mathématiques à des situations du monde réel. Leur théorie remonte au tout début des mathématiques modernes, avec le premier exemple d'équations différentielles apparaissant dans les travaux de Newton et Leibniz. En quelques mots, les équations différentielles sont au centre de notre compréhension des processus continus.

D'autre part, étant donné un certain nombre premier $p$, considérer un corps $p$-adique à la place des nombres réels ou complexes conduit à la théorie des équations différentielles $p$-adiques, dont la théorie éclaire les problèmes de l'arithmétique et la théorie des nombres. Le seul texte complet sur le sujet est [Ked10].

La première apparition des équations différentielles $p$-adiques remonte aux travaux de Dwork, qui dans [Dwo60] les a utilisées comme outil pour prouver la rationalité de la fonction zêta d'une variété en caractéristique $p$ : la résolution de certaines équations différentielles $p$-adiques particulières donne des formules pour compter le nombre de points de variétés sur des corps finis. Dans ses premières années, la théorie a ensuite été poussée principalement grâce aux contributions pionnières de Dwork et Robba [DR77, DR79, Rob94, Dwo12, ...] et a développé des liens dans de nombreuses directions mathématiques : étudier les fonctions zêta avec l'analyse $p$-adique permet d'utiliser des méthodes numériques, et celles-ci ont trouvé des applications dans la cryptographie basée sur les courbes elliptiques et hyperelliptiques (voir [CFA$^+$05] pour une introduction au sujet) ; la théorie des équations différentielles $p$-adiques est intimement liée au développement et à la consolidation de la cohomologie rigide $p$-adique par Berthelot [Ber74, Ber86], motivée par les travaux antérieurs de Dwork et Monsky et Washnitzer [MW68, Mon68, Mon71], qui s'est également avéré être un outil précieux dans les calculs pour la cryptographie. Pour des présentations générales et des textes complets sur le sujet, voir [Ill94, Ked09, LS07]. De plus, les équations différentielles $p$-adiques jouent un rôle important dans la théorie des représentations galoisiennes $p$-adiques, qui ont été étudiées par Fontaine [Fon94, Fon04, Fon07, FO08] en introduisant une série d'anneaux de périodes auxiliaires afin de les classer. L'étude des représentations galoisiennes $p$-adiques est centrale pour le développement d'une théorie de Hodge $p$-adique. Plus récemment, les travaux de Berger [Ber02, Ber08a] ont mis en évidence un rôle profond des équations différentielles $p$-adiques dans cette théorie, grâce à l'utilisation de $(\phi, \Gamma)$-modules. Ce sujet est presenter dans [Ber08b, BC09].

Une caractéristique intéressante des équations différentielles $p$-adiques est que, contrairement au cas complexe, le rayon de convergence de leurs solutions n'est pas contrôlé par un objet "visible", comme, dans le cas complexe, les pôles des coefficients de l'équation : dans ce contexte en effet même des équations aussi simples que celle de l'exponentielle donnent des solutions à rayon de convergence fini en tout point. La topologie de l'espace elle-même est un obstacle à la convergence.

Le langage de la géométrie de Berkovich, introduit dans [Ber90], s'est avéré être le bon pour décrire les phénomènes liés à ces rayons de convergence. Le rayon de convergence des solutions d'une équation différentielle $p$-adique sur une courbe de Berkovich en fonction du point d'expansion s'est avéré être une fonction linéaire continue par morceaux (voir [CD94, Bal10, BDV07, Ked10] ) avec un nombre fini de changements de pente [Chr11, PP15b, PP15a, PP13, Pul15], dont le comportement est en fait contrôlé par un squelette fini sur lequel la courbe se rétracte. En général, bien qu'une formule itérative explicite pour calculer le rayon de convergence existe (voir [Chr11]), il est difficile de la calculer.

Les méthodes tropicales deviennent de plus en plus influentes dans de nombreux domaines des mathématiques et des sciences en général, et elles peuvent souvent être considérées comme un processus faisant passer un problème d'un cadre géométrique ou algébrique à un cadre discret, combinatoire ou polyédrique.

La géométrie tropicale s'est historiquement développée d'au moins deux manières indépendantes: en tant que géométrie sur le semi-corp tropical et en tant qu'étude des ensembles logarithmiques limites des variétés algébriques classiques. La première approche dérive de problèmes d'optimisation en informatique [Sim78, Sim87, Eil74], résolus en utilisant l'algèbre min-plus (ou max-plus), ainsi la géométrie tropicale est la géométrie sur le semi-anneau tropical $\mathbb{R} \cup \infty$, où somme et produit usuels sont respectivement remplacés par minimum et somme (ou par maximum et somme, de manière isomorphe). Les polynômes tropicaux donnent des fonctions linéaires par morceaux et les variétés tropicales qui leur sont associées, c'est-à-dire le lieu des points où le minimum est atteint au moins deux fois, sont assemblées par des polyèdres convexes. La deuxième approche, entreprise dans [Ber71, GB84, GKZ08], et réintroduite plus récemment par Kapranov, consiste à considérer l'application $\log_t \colon (\mathbb{C}^*)^n \to \mathbb{R}^n$ envoyant un vector au vecteur des logarithmes de la valeur absolue de ses entrées. L'image d'une variété algébrique via cette fonction s'appelle *amibe* et prendre la limite $\lim_{t\to\infty} X_t$ d'une famille $X_t$ de variétés algébriques donne une variété tropicale, associée au polynôme tropical obtenu par le polynôme $F_t \in \mathbb{C}(\!(t)\!)$ définissant la famille $X_t$ en prenant l'ordre des coefficients.

Ce point de vue se généralise à tout champ valué, c'est-à-dire muni d'une fonction similaire à la valuation $t$-adique sur $\mathbb{C}(\!(t)\!)$ ("prendre l'ordre de tête"). Les fonction avec ces propriétés sont appelées valuations non archimédiennes et sont l'outil nécessaire qui nous permet de passer du monde classique de la géométrie algébrique au monde polyédrique de la géométrie tropicale, à travers ce qu'on appelle la tropicalisation. Ceci est intéressant car de nombreuses caractéristiques des objets géométriques avec lesquels nous commençons sont préservées par ce processus.

Plus récemment, plusieurs fondements algébriques de la géométrie tropicale ont été développés, afin de voir plus classiquement la géométrie tropicale comme géométrie algébrique sur une certaine catégorie d'objets, comme les hypercorps [Vir11, BB19, Lor19, Jun21], les *blueprints* de Lorscheid [Lor12, Lor15], et les semi-anneaux idempotents [GG16, GG14, GG18, MR18, MR14, MR20, JM18b, JM18a, BE17, Yag16, CGM20]. Nous donnons un traitement plus approfondi de la géométrie tropicale dans le chapitre 2 où nous rappelons également de nombreuses références sur le sujet et ses applications.

Il est bien connu qu'une grande partie de la théorie des EDO est algébrique, il suffit de citer l'exemple de la théorie de Picard-Vessiot. Les EDO algébriques sont des systèmes d'équations différentielles formés à partir d'expressions polynomiales dans une fonction indéterminée $f$ et ses dérivées. La théorie algébrique a d'abord été établie par Ritt [Rit50] et Kolchin [Kol73]. De nombreuses classes importantes de modèles des sciences naturelles, telles que les réseaux de réactions chimiques, sont des EDO algébriques, et en mathématiques pures, les EDO algébriques apparaissent dans de nombreuses parties de la géométrie, y compris les périodes et la monodromie. Comprendre leurs solutions et leurs singularités a de nombreuses conséquences importantes en mathématiques pures et appliquées.

Ici, nous poursuivons le développement de l'ensemble d'outils de mathématiques tropicales pour l'étude des équations différentielles. Dans [Gri17], Grigoriev a introduit pour la première fois une théorie des équations différentielles tropicales et a défini un cadre pour tropicaliser les EDO algébriques sur un anneau de séries formelles de puissances $R[[t]]$. Dans ce cadre, on tropicalise une équation différentielle en enregistrant la puissance dominante de $t$ dans chaque coefficient, et on tropicalise une solution en série de puissances en enregistrant simplement les puissances de $t$ qui sont présentes.

Les solutions à une équation différentielle se tropicalisent en solutions à sa tropicalisation, et Grigoriev demandé si *toutes* les solutions à la tropicalisation d'une équation se présentent comme des tropicalisations de solutions classiques; c'est-à-dire, la fonctions des solutions classiques aux solutions tropicales est-elle surjective ? C'est l'analogue pour les équations différentielles du théorème fondamental de la géométrie tropicale [MS15, Theorem 3.2.3], et cette question a été répondue positivement par Aroca et al. in [AGT16] (en supposant que $R$ est un corps indénombrable algébriquement clos de caractéristique 0). Ces idées ont également été étendues au cas des équations différentielles aux dérivées partielles algébriques dans [FGLH+20].

Parallèlement au rôle de la théorie de Gröbner dans la définition des variétés tropicales dans le cas non-différentiel , [FT20] et [HG21] définissent les formes initiales et développent une approche théorique de Gröbner des équations différentielles tropicales de Grigoriev. Une approche similaire est également présentée dans [CGL20], qui donne un résumé éclairant autour des équations différentielles ordinaires et partielles tropicales (basées en partie sur un rapport préliminaire du perspective algébrique présentée ici). Nous rappelons la plupart des résultats au sujet des articles précités au chapitre 3.

Une limitation présente dans tous les travaux ci-dessus est que la construction de tropicalisation qui y est étudiée n'enregistre que les puissances de $t$ présentes dans une solution en série de puissances ; il n'enregistre aucune information sur les valuations des coefficients. Ainsi, toute l'information sur la convergence des solutions de séries de puissances est perdue lors de l'utilisation de la tropicalisation de Grigoriev, et pour comprendre les rayons de convergence des solutions formelles de séries de puissances d'équations différentielles $p$-adiques, préserver les évaluations des coefficients est d'importance cardinale .

Même si nous sommes loin de cet objectif, le but à long terme de l'étude entreprise ici est une meilleure compréhension et une calculabilité plus aisée des rayons de convergence pour l'équation différentielle $p$-adique en utilisant des méthodes tropicales.

## Résultats

L'objectif principal de la première partie de mon travail est de construire un raffinement du cadre de Grigoriev qui enregistre et intègre les valuations des coefficients dans une solution de série de puissance afin que l'information de convergence soit codée dans les solutions tropicales. Cela nécessite de développer une théorie des différentielles sur les semi-anneaux idempotents dans laquelle la règle de Leibniz habituelle est affaiblie en une règle de Leibniz tropicale, et ce développement comprend la construction d'algèbres différentielles tropicales libres (un analogue tropical des algèbres de Ritt) avec des variables différentielles provenant d'un $\mathbb{F}_1$-module différentiel , que nous définissons dans ce qui suit.

Nous donnons ici une brève explication de notre context. Une *paire tropicale* $\mathbf{S} = (S_1 \to S_0)$ est un semi-anneau différentiel tropical $S_1$ et un homomorphisme à un semi-anneau $S_0$. Les coefficients des équations différentielles tropicales vivent dans $S_0$. Les solutions vivent dans $S_1$ (où elles peuvent être différenciées), mais la condition qui teste si quelque chose est une solution a lieu dans $S_0$. Nous pensons que $S_0$ enregistre le comportement au prèmier ordre des éléments de $S_1$. L'exemple principal d'une paire tropicale a $S_1 = \mathbb{T}[[t]]$ (le semi-anneau d'une série de puissances formelles avec des coefficients de nombres réels tropicaux), $S_0 = \mathbb{R}^2_{lex} \cup \{\infty\}$ est une version de rang 2 du semi-anneau tropical, et la fonction $S_1 \to S_0$ envoie $at^n + \cdots$ à $(n, a)$.

Nous énonçons maintenant les principaux résultats de manière informelle, dans le cas des $\mathbb{F}_1$-modules différentiels libres avec $n$ générateurs.

**Theorem E.** *On construit une catégorie de* $\mathbf{S}$*-algèbres, et un ensemble* $E$ *d'équations différentielles tropicales sur* $\mathbf{S}$ *on associe un objet de cette catégorie tel que les morphismes à une* $\mathbf{S}$*-algèbre* $\mathbf{T}$ *sont en bijection naturelle avec solutions de* $E$ *à valeurs dans* $\mathbf{T}$*.*

Un système d'équations différentielles algébriques sur un corps $K$ est représenté par une $K$-algèbre différentielle $A$. Pour tropicaliser $A$, nous avons besoin de deux données supplémentaires :

(1) Une valuation non archimédienne sur $K$ prenant des valeurs dans un semi-anneau idempotent $S_0$, et un *rehaussement différentiel* de la valuation, qui est une application $A \to S_1$ qui commute avec le différentiels. (Ces notions sont définies dans la Section 2.4 et la Section 6.5.)

(2) Un système de générateurs $x_i \in A$ tel que $A$ soit présenté comme un quotient d'une algèbre de Ritt $K\{x_1, \dots, x_n\} \twoheadrightarrow A$.

Toute algèbre différentielle $A$ admet une présentation universelle $K\{x_a \mid a \in A\} \twoheadrightarrow A$. Tropicalisant cette présentation, on retrouve :

**Theorem F.** *La tropicalisation de* $A$ *par rapport à sa présentation universelle est la colimite de ses tropicalisations par rapport aux présentations finies.*

Enfin, nous fournissons des preuves de la pertinence de nos définitions et de notre context en prouvant un analogue différentiel du théorème du limite inverse de Payne [Pay09]. Rappelons que, étant donnée une algèbre $A$ sur un corps non archimédien $K$, l'ensemble sous-jacent de l'analytification de Berkovich de Spec $A$ est l'ensemble de toutes les seminormes multiplicatives sur $A$ qui sont compatibles avec la valuation sur $K$. Supposons maintenant que $K$ soit un anneau différentiel, la valuation $v$ sur $K$ a un rehaussement différentielle $\widetilde{v}$ prenant des valeurs dans une paire tropicale $\mathbf{S}$, et $A$ est un algèbre différentielle sur $K$. Dans ce cadre, étant donné une $\mathbf{S}$-algèbre $\mathbf{T} = (T_1 \to T_0)$, nous pouvons considérer l'ensemble de toutes les paires $(w, \widetilde{w})$ où $w \colon A \to T_0$ est une valuation multiplicative sur $A$ compatible avec $v$ et $\widetilde{w} \colon A \to T_1$ est une rehaussement différentielle de $w$ compatible avec $\widetilde{v}$. Nous appelons cela l'espace *différentiel de Berkovich de* $A$ *à valeur en* $\mathbf{T}$, noté $Berk_{\mathbf{T}}(A)$.

**Theorem G.** *Il existe une valuation universelle avec un rehaussement différentiel sur* $A$*, et elle prend des valeurs dans la tropicalisation de la présentation universelle de* $A$*. Donc la tropicalisation de la présentation universelle coreprésente le foncteur* $\mathbf{T} \mapsto Berk_{\mathbf{T}}(A)$*.*

En combinant cela avec le théorème F, nous obtenons immédiatement notre analogue différentiel du théorème limite inverse de Payne.

**Corollary H.** *Soit* $k$ *un anneau différentiel muni d'une valuation non archimédienne et d'un enhaussement différentiel prenant des valeurs dans* $\mathbf{S}$*, soit* $A$ *une algèbre différentielle sur un* $k$*, et soit* $\mathbf{T}$ *soit une* $\mathbf{S}$*-algèbre. L'ensemble* $Berk_{\mathbf{T}}(A)$ *est isomorphe à la limite inverse des ensembles de solutions à valeur* $\mathbf{T}$ *des tropicalisations de toutes les présentations finies de* $A$ *.*

La majeure partie du matériel présenté dans la première partie de cette thèse apparaît en [GM21], dans les termes que nous l'avons présentés dans les lignes précédentes, c'est-à-dire en restreignant notre traitement aux $\mathbb{F}_1$-modules différentiels libres avec $n$ générateurs. En plus d'adopter cette approche plus générale ici, nous discutons de la notion de $\mathbb{F}_1$-algèbre différentielle et des interactions entre certains foncteurs de changement de base libre sur ces catégories d'objets différentiels $\mathbb{F}_1$. Dans le dernier chapitre, nous présentons une généralisation de tous les résultats introduits dans la thèse au cas des équations aux dérivées partielles, généralisant le cadre de [FGLH+20].

# Deuxième partie

Dans les années 1920, H. Prüfer et R. Baer définissent les *heaps* comme des objets algébriques constitués d'un ensemble avec une opération ternaire qui remplit des conditions permettant d'associer une classe de groupes isomorphes, un pour chaque élément du *heap* ; inversement , chaque groupe donne lieu à un *heap* en prenant l'opération $(a, b, c) \mapsto ab^{-1}c$ (voir [Bae29] et [Prü24]). En 2007, W. Rump a introduit les *braces* comme systèmes algébriques correspondant à des solutions de l'équation de Yang-Baxter [Rum07]. Un *brace* est un triplet $(G, +, \cdot)$ où $(G, +)$ est un groupe abélien, $(G, \cdot)$ est un groupe et la loi de distribution suivante s'applique, pour tout $a, b, c \in G$,

$$a \cdot (b + c) = a \cdot b - a + a \cdot c\,;$$

voir [CJO14]. Grâce à leur lien avec les solutions théoriques de l'équation de Yang-Baxter, les *braces* sont devenues un domaine d'études intensif. En particulier, il a été montré qu'un *brace* permet de construire une solution involutive non dégénérée de l'équation de Yang-Baxter (voir par exemple, [CJO14], [Rum07], [CGIS17] et [Smo18]). En 2017, L. Guarnieri et L. Vendramin ont introduit la notion de *skew brace*. C'est une généralisation d'une accolade dans laquelle $(G, +)$ n'est pas obligatoirement abélien [GV17]. Il a été démontré qu'il correspond aux solutions théoriques des ensembles non dégénérées de l'équation de Yang-Baxter; on peut construire une telle solution à partir de n'importe quel *skew brace*, tandis qu'à toute solution bijective non dégénérée on peut associer un *skew brace* qui satisfait une propriété universelle (voir [GV17], [ESS99], [Sol00], [LYZ00], [SV18] ou [Bac18]). Ces dernières années, il y a eu de grands progrès dans la recherche sur les solutions théoriques de l'équation de Yang-Baxter, mais, même si nous savons que chaque *skew brace* nous fournit une telle solution, ce n'est pas une tâche facile de construire des *skew braces* (pour une liste des problèmes sur les *skew braces* et une revue de la littérature, voir [Ven18]). En 2018, T. Brzezinski dans [Brz19] a observé qu'il est possible d'unifier les lois distributives des anneaux et des accolades dans une seule structure algébrique plus générale, celle d'un *truss*. Un *skew truss* à gauche $T$ est un *heap* $(T, [-, -, -])$ avec une opération binaire supplémentaire $\cdot\colon\ T \times T \to T$ qui est associative et qui distribue sur le ternaire opération depuis la gauche, c'est-à-dire pour tout $a, b, c, d \in T$,

$$a \cdot [b, c, d] = [a \cdot b, a \cdot c, a \cdot d].$$

La théorie des *trusses* a été développée dans [Brz20, BR21, BMR20, BRS20, ABR21, BR20].

Chaque *skew brace* peut être associé à un *skew truss* gauche approprié : nous appellerons ce *trusses brace-type*. Ceci conduit aux principales questions qui ont motivé le travail présenté dans cette deuxième partie, fruit d'une collaboration avec Tomasz Brzeziński et Bernard Rybołowicz, paru dans [BMR20]. Que sont exactement les *trusses brace-type* ? Comment les construire à partir d'un *truss* pas forcément *brace-type* ? Quand une telle construction est-elle possible ? Nous présentons ici deux approches pour répondre à des questions de ce type. La première approche consiste à prendre des quotients de *truss* par une congruence particulière et la seconde repose sur une procédure de localisation.

Nous donnons des réponses aux questions ci-dessus dans un contexte plus général, en considérant les *pre-trusses* et les *near-trusses*, les anneau non commutatives et les *skew braces*, dont la définition se trouve au chapitre 11. On montre que les congruences dans les *pre-trusses* et les *near-trusses* proviennent de sous-*heaps* normaux avec une propriété de fermeture supplémentaire des classes d'équivalence qui implique à la fois les opérations ternaires et binaires. De tels sous-*heaps*, introduits dans [Brz20], sont appelés *paragons*. Un critère nécessaire et suffisant sur les *paragons* sous lequel le quotient d'une *near-truss* unitaire correspond à un *skew brace* est dérivé. Les éléments réguliers d'une *pre-truss* sont définis comme des éléments avec des propriétés d'annulation gauche et droite ; suivant la terminologie de la théorie des anneaux, les *pre-trusses* dans lesquelles tous les éléments non absorbants sont réguliers sont appelées domaines. Ces derniers sont décrits comme des quotients par des *paragons* complètement premiers, également définis ici. Des *pre-* et *near-trusses* réguliers sont introduits comme domaines qui satisfont à la condition Ore et le *pre-truss* des fractions est construites par localisation. En particulier, on montre que les *pre-truss* des fractions sans absorbeur correspondent à un *skew brace*.

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