Bounding the Escape Time of a Linear Dynamical System over a Compact Semialgebraic Set

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Abstract

We study the Escape Problem for discrete-time linear dynamical systems over compact semialgebraic sets. We establish a uniform upper bound on the number of iterations it takes for every orbit of a rational matrix to escape a compact semialgebraic set defined over rational data. Our bound is doubly exponential in the ambient dimension, singly exponential in the degrees of the polynomials used to define the semialgebraic set, and singly exponential in the bitsize of the coefficients of these polynomials and the bitsize of the matrix entries. We show that our bound is tight by providing a matching lower bound.

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1 Introduction

An *invariant set* of a dynamical system is a set K such that every trajectory that starts in K remains in K. Dually, an *escape set* K is one such that every trajectory that starts in K eventually leaves K (either temporarily or permanently). While it is usually straightforward to establish that a given set K is invariant, it can be challenging to decide whether it is an escape set. Indeed, while the former problem amounts to showing that K is closed under the transition function, the latter potentially involves considering entire orbits. In particular, even in case K has a finite escape time (the maximum number of steps for an orbit to escape the set), it can be highly non-trivial to establish an explicit upper bound on the escape time.

In this paper we focus on escape sets for (discrete-time) linear dynamical systems. Given a rational matrix $A \in \mathbb{Q}^{n\times n}$ we say that $K \subseteq \mathbb{R}^n$ is an escape set for A if for all points $x \in K$, there exists $t \in \mathbb{N}$ such that $A^t x \notin K$. The *compact escape problem (CEP)* asks to decide whether a given compact semialgebraic set K is an escape set for a given matrix A.

Decidability of CEP was shown in [18] and its computational complexity was characterised in [9] as being interreducible with the decision problem for a certain fragment of the theory of real closed fields.

The present paper focusses exclusively on positive instances (A, K) of CEP, that is, we assume that we are given a compact semialgebraic escape set for a linear dynamical system. In this situation it turns out, due to compactness of K, that there exists a finite time T such that for all $x \in K$ there exists $t \leq T$ with $A^t x \notin K$. The least such T is called the *escape time* of (A, K). Our main result (Theorem 1, shown below) gives an explicit upper bound on the escape time of (A, K) as a function of the length of the description of the matrix A and semialgebraic set K. In general, it is recognised that bounded liveness is a more useful property than mere liveness. Theorem 1 can be used to establish bounded liveness of several kinds of systems. For example, the result gives an upper bound on the termination time of a single-path linear loop with compact guard (cf. [22, 5]); it also gives a bound on the number of steps to remain in a particular control location of a hybrid system before a given (compact) state invariant becomes false, forcing a transition.

We next introduce some terminology to formalise our main contribution. We say that a semialgebraic set S has complexity at most (n, d, τ) if it can be expressed by a boolean combination of polynomial equations and inequalities $P(x_1, \ldots, x_n) \bowtie 0$ with $\bowtie \in \{\leq, =\}$, involving polynomials $P \in \mathbb{Z}[x_1, \ldots, x_n]$ in at most n variables of total degree at most d with integer coefficients bounded in bitsize by τ. Our main result is as follows:

▶ **Theorem 1.** *There exists an integer function* $\text{CompactEscape}(n, d, \tau) \in 2^{(d\tau)^{n^{O(1)}}}$ *with the following property. If $K \subseteq \mathbb{R}^n$ is a compact semialgebraic set of complexity at most (n, d, τ) that is an escape set for a matrix $A \in \mathbb{Q}^{n\times n}$ with entries of bitsize at most τ, then the escape time of K is bounded by* $\text{CompactEscape}(n, d, \tau)$.

As explained in the proof sketch below, Theorem 1 relies on the availability of certain quantitative bounds within semialgebraic geometry and number theory, particularly concerning quantifier elimination and Diophantine approximation. The latter results are crucial to handling the case in which the matrix A has complex eigenvalues of absolute value one.

Note that the upper bound on the escape time in Theorem 1 is singly exponential in the degrees and the bitsize of the coefficients of the polynomials used to define K and the bitsize of the coefficients of A. It is doubly exponential in the dimension. In Section 8 we provide two examples, one where A is an isometry and another in which all eigenvalues of A have absolute value strictly greater than one, that yield a corresponding lower bound of this form. It is moreover straightforward to give examples of non-compact escape sets for which the escape time is infinite.

Proof Overview. Let us now give a high-level overview of the proof of Theorem 1. As in the statement of the theorem, let $K \subseteq \mathbb{R}^n$ be a compact semialgebraic set of complexity at most (n, d, τ) and let $A \in \mathbb{Q}^{n\times n}$ be a matrix with entries of bitsize bounded by τ, and such that for all $x \in K$ there exists $t \in \mathbb{N}$ such that $A^t x \notin K$.

To facilitate the analysis of the dynamical behaviour of A we first transform our system into real Jordan normal form. A theorem of Cai [6] ensures that this step does not significantly increase the complexity of the system.

The dynamics of A naturally decomposes into a rotational part, corresponding to eigenvalues of modulus one, and an expansive or contractive part, corresponding to eigenvalues of absolute value different from 1 and to generalised eigenvalues of arbitrary moduli. Accordingly, the ambient space $\mathbb{R}^n$ decomposes into two subspaces V_{rec} and $V_{\text{non-rec}}$, such that A

exhibits rotational behaviour on V_{rec} and expansive or contractive behaviour on $V_{\text{non-rec}}$. We start by considering the special cases where either $V_{\text{rec}} = 0$ or $V_{\text{non-rec}} = 0$, so that only one of the two types of behaviours occurs.

First, assume that A has no complex eigenvalues of modulus 1. Since every trajectory under A escapes K we have in particular that $0 \notin K$. A theorem due to Jeronimo, Perrucci and Tsigaridas [15] shows that K is bounded away from zero by a function of the form $2^{-(d\tau)^{n^{O(1)}}}$ and a theorem due to Vorobjov [23] establishes an upper bound on the absolute value of every coordinate of every point in K of the form $2^{(d\tau)^{n^{O(1)}}}$. Furthermore, thanks to a result of Mignotte [17], we can bound the eigenvalues of A away from 1 by a function of the form $2^{\tau^{n^{O(1)}}}$. This yields a doubly exponential bound on how long it takes for A to leave the set K (either by converging to 0 or by converging to infinity in some eigenspace).

Now assume that all eigenvalues of A have modulus 1. This case is handled through a combination of two bounds. For the first bound we start by noting that for every $x \in K$ the closure of the orbit $\overline{\mathcal{O}_A(x)}$ is a compact semialgebraic set that is not entirely contained within K. In fact we show that for all $x \in K$ there exists a point $y \in \overline{\mathcal{O}_A(x)}$ whose distance to K is at least $2^{-(d\tau)^{n^{O(1)}}}$. This bound is achieved by applying [15, Theorem 1] to a suitable polynomial on an auxiliary semialgebraic set, which is constructed using quantifier elimination. The singly exponential bounds obtained in [14, 20] are crucial for this step to work. The second step of the argument combines Baker's theorem on linear forms in logarithms with a quantitative version of Kronecker's theorem on simultaneous Diophantine approximation to obtain a bound of the form $N_P \in 2^{(\tau P)^{n^{O(1)}}}$ such that for all positive integers P every point $z \in \overline{\mathcal{O}_A(x)}$ is within 2^{-P} of a point of the form $A^t x$ with $0 \leq t \leq N_P$. Combining the two bounds described above, we obtain a doubly exponential bound on the escape time.

In the presence of both types of behaviour, the analysis of each case becomes more involved. We select a parameter $\varepsilon > 0$ and partition K into three sets: $K_{\text{rec}} = K \cap V_{\text{rec}}$, $K_{\geq\varepsilon}$, and $K_{<\varepsilon}$. The matrix A exhibits purely rotational behaviour on K_{rec}. Intuitively, on $K_{\geq\varepsilon}$ the expansive or contractive behaviour of A dominates the overall dynamics, while on $K_{<\varepsilon}$ the rotational behaviour dominates the overall dynamics. We establish in Lemma 14 a bound N_{rec} such that for each initial point $x \in V_{\text{rec}}$, one of its first N_{rec} iterates is bounded away from K. In Lemma 15 we establish a bound $N_{\geq\varepsilon}$ such that every $x \in K_{\geq\varepsilon}$ either escapes or enters $K_{<\varepsilon} \cup K_{\text{rec}}$ within at most $N_{\geq\varepsilon}$ iterations. Finally, in Section 7, we establish a bound on how often the system can switch from a state where rotational behaviour dominates to one where expansive or non-expansive behaviour does and vice versa. We use this to combine the two bounds to an overall bound on the escape time, proving Theorem 1.

Main Contributions. While decidability of CEP was already established in [18], the proof given there was non-effective, combining two unbounded searches. To obtain a uniform quantitative bound on the escape time, the argument given in [18] needs to be refined and extended in two significant ways:

Firstly, one needs to establish non-trivial quantitative refinements of the techniques used in the decidability proof: to bound the escape time for purely expanding or retracting systems, we need to combine the sharp effective bounds on compact semialgebraic sets from real algebraic geometry established in [23, 15] with Mignotte's root separation bound [17]. The case of purely rotational systems requires an original combination of a quantitative version of Kronecker's theorem on simultaneous Diophantine approximation [12] and a quantitative version of Baker's theorem on linear forms in logarithms [1]. All of these techniques were completely absent from the decidability proof.

Secondly, to establish mere decidability of the problem, it was possible to study the possible behaviours of the system – rotating, expanding, or retracting – in isolation. For example, if the set K contains a point which has a non-zero component in an eigenspace of A for an eigenvalue whose modulus is strictly greater than one, then the system must eventually escape. However, no uniform bound on the escape time may be derived in this situation, for the component is allowed to be arbitrarily close to zero. Therefore, as outlined above, it is necessary in our proof to subdivide K into pieces where rotational, retractive, and expansive behaviour can be present simultaneously. The interaction of the three behaviours significantly increases the difficulty of the analysis and requires completely new ideas.

2 Mathematical Tools

We use the following singly exponential quantifier elimination result given in [2]. For a historical overview on this type of result see [2, Chapter 14, Bibliographical Notes].

▶ **Theorem 2** ([2, Theorem 14.16]). *Let $S \subseteq \mathbb{R}^{k+n_1+\cdots+n_\ell}$ be a semialgebraic set of complexity at most $(k + n_1 + n_2 + \cdots + n_\ell, d, \tau)$. Let $Q_1, \ldots, Q_\ell \in \{\exists, \forall\}$ be a sequence of alternating quantifiers. Consider the set $S' \subseteq \mathbb{R}^k$ of all $(x_1, \ldots, x_k) \in \mathbb{R}^k$ satisfying the first-order formula*

$$(Q_1(x_{1,1}, \ldots, x_{1,n_1})) \ldots (Q_\ell(x_{\ell,1}, \ldots, x_{\ell,n_\ell})) . \\ ((x_1, \ldots, x_k, x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{\ell,1}, \ldots, x_{\ell,n_\ell}) \in S)$$

Then S' is a semialgebraic set of complexity at most $(k, d^{O(n_1 \cdot \cdots \cdot n_\ell)}, \tau d^{O(n_1 \cdot \cdots \cdot n_\ell \cdot k)})$.

The next theorem is due to Vorobjov [23]. See also [13, Lemma 9] and [3, Theorem 4].

▶ **Theorem 3.** *There exists an integer function* $\mathrm{Bound}(n, d, \tau) \in 2^{\tau d^{O(n)}}$ *with the following property:*

Let K be a compact semialgebraic set of complexity at most (n, d, τ). Then K is contained in a ball centred at the origin of radius at most $\mathrm{Bound}(n, d, \tau)$.

A closely related result, due to [15], yields a lower bound on the minimum of a polynomial over a compact semialgebraic set, provided the minimum is non-zero. The result in [15] mentions explicit constants, which is more than we need.

▶ **Theorem 4** ([15, Theorem 1]). *There exists an integer function* $\mathrm{LowerBound}(n, d, \tau) \in 2^{(\tau d)^{n^{O(1)}}}$ *such that the following holds true:*

Let $P \in \mathbb{Q}[x_1, \ldots, x_n]$ be a polynomial of degree at most d, whose coefficients have bitsize at most τ. Let K be a compact semialgebraic set of complexity at most (n, d, τ). If $\min_{x \in K} P(x) > 0$ *then* $\min_{x \in K} P(x) > 1/\mathrm{LowerBound}$.

With the help of Theorem 2, Theorem 4 can be generalised to yield a lower bound on the distance of two disjoint compact semialgebraic sets. A very similar result is proved in [21] under more general assumptions. Unfortunately, the complexity bound stated there is not sufficiently fine-grained for our purpose, since the author do not distinguish the dimension of a set from the other complexity parameters.

▶ **Lemma 5.** *There exists an integer function* $\mathrm{Sep}(n, d, \tau) \in 2^{(\tau d)^{n^{O(1)}}}$ *with the following property:*

Let K and L be compact semialgebraic sets of complexity at most (n, d, τ). Assume that every $x \in K$ has positive euclidean distance to L. Then $\inf_{x \in K} d(x, L) > 1/\mathrm{Sep}(n, d, \tau)$.

Proof. See [10, Appendix E]. ◀

We require a version of Kronecker’s theorem on simultaneous Diophantine approximation. See [19, Corollary 3.1] for a proof.

▶ **Theorem 6.** *Let $(\lambda_1, \dots, \lambda_m)$ be complex algebraic numbers of modulus 1. Consider the free Abelian group*

$$L = \{(n_1, \dots, n_m) \in \mathbb{Z}^m \mid \lambda_1^{n_1} \cdot \dots \cdot \lambda_m^{n_m} = 1\}.$$

Let $(\beta_1, \dots, \beta_s)$ be a basis of L. Let $\mathbb{T}^m = \{(z_1, \dots, z_m) \in \mathbb{C}^m \mid |z_j| = 1\}$ denote the complex unit m-torus. Then the closure of the set $\left\{(\lambda_1^k, \dots, \lambda_m^k) \in \mathbb{T}^m \mid k \in \mathbb{N}\right\}$ is the set $S = \left\{(z_1, \dots, z_m) \in \mathbb{T}^m \mid \forall j \leq s.(z_1, \dots, z_m)^{\beta_j} = 1\right\}$.

Moreover, for all $\varepsilon > 0$ and all $(z_1, \dots, z_m) \in S$ there exist infinitely many indexes k such that $|\lambda_j^k - z_j| < \varepsilon$ for $j = 1, \dots, m$.

Moreover, the integer multiplicative relations between given complex algebraic numbers in the unit circle can be elicited in polynomial space. For a proof see [7, 16]. We assume the standard encoding of algebraic numbers, see [8] for details.

▶ **Theorem 7.** *Let $(\lambda_1, \dots, \lambda_m)$ be complex algebraic numbers of modulus 1. Consider the free Abelian group*

$$L = \{(n_1, \dots, n_m) \in \mathbb{Z}^m \mid \lambda_1^{n_1} \cdot \dots \cdot \lambda_m^{n_m}\}.$$

Then one can compute in polynomial space a basis $(\beta_1, \dots, \beta_s) \in (\mathbb{Z}^m)^s$ for L. Moreover, the integer entries of the basis elements β_j are bounded polynomially in the size of the encodings of $\lambda_1, \dots, \lambda_m$ and singly exponentially in m.

We need to be able to bound away the modulus of eigenvalues that fall outside the unit circle away from 1. This is achieved by combining a classic result due to Mignotte [17] on the separation of algebraic numbers with a bound on the height of the resultant of two polynomials, proved in [4, Theorem 10].

▶ **Lemma 8.** *Let λ be a complex algebraic number whose minimal polynomial has degree at most d and coefficients bounded in bitsize by τ. Assume that $|\lambda| \neq 1$. Then we have $||\lambda| - 1| > 2^{-(\tau d)^{O(1)}}$.*

Proof. See [10, Appendix C]. ◀

3 Preliminaries

3.1 Converting the matrix to real Jordan normal form

To obtain a bound on the escape time it will be important to work with instances of the Escape Problem in real Jordan normal form. In the following, let $\mathbb{A}$ denote the field of algebraic numbers. We establish the following reduction to this case:

▶ **Lemma 9.** *Let (K, A) be an instance of the Compact Escape Problem. Assume that K is given by a formula involving s polynomial equations and equalities $P \bowtie 0$ where $P \in \mathbb{Z}[x_1, \dots, x_n]$ is a polynomial in n variables of degree at most d whose coefficients are bounded in bitsize by τ.*

Let $\gamma_1, \dots, \gamma_m \in \mathbb{R}$ denote the real and imaginary parts of the eigenvalues of A. Let δ be a bound on the degrees of $\gamma_1, \dots, \gamma_m$.

Then there exists an equivalent instance (J, K') *of the Compact Escape Problem where* $J \in \mathbb{A}^{(n+m)\times(n+m)}$ *is in real Jordan normal form and* K' *is given by a formula involving at most* $s + 3m$ *polynomial equations and equalities* $P \bowtie 0$ *where* $P \in \mathbb{Z}[x_1, \ldots, x_{n+m}]$ *is a polynomial in* $n+m$ *variables of degree at most* $\delta \cdot d$ *whose coefficients are bounded in bitsize by* $\tau + d(\log(2n) + \log(\delta+1) + \sigma)$*, where* σ *depends polynomially on* n *and the bitsize of the entries of* A*.*

Proof. See [10, Appendix B]. ◀

3.2 Decomposing K

Let $K \subseteq \mathbb{R}^n$ be a compact semialgebraic set. Let $A \in \mathbb{R}^{n\times n}$ be a matrix in real Jordan normal form,

$$A = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix}.$$

Here, each J_i is a real Jordan block of the form

$$J_i = \begin{pmatrix} \Lambda_i & I_i & & \\ & \ddots & \ddots & \\ & & \Lambda_i & I_i \\ & & & \Lambda_i \end{pmatrix},$$

where $\Lambda_{i,1}$ is either a real number or a 2×2 real matrix of the form $\begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix}$ and, accordingly, I_i is either the real number 1 or the 2×2 identity matrix. The elements Λ_i correspond to real or complex eigenvalues $\lambda_i \in \mathbb{C}$ of A. By slight abuse of language we call $|\lambda_i|$ the modulus of Λ_i. By further slight abuse of language we define the "eigenspace" of Λ_i as the one- or two-dimensional space spanned by the vectors that correspond to the first entry of the Jordan block J_i. The "generalised eigenspaces" for Λ_i are defined analogously.

Write $\mathbb{R}^n$ as the direct sum of two spaces $\mathbb{R}^n = V_{\text{rec}} \oplus V_{\text{non-rec}}$ where V_{rec} is the direct sum of the eigenspaces for eigenvalues of modulus 1, and $V_{\text{non-rec}}$ is the direct sum of the eigenspaces and generalised eigenspaces for eigenvalues of modulus $\neq 1$ and the generalised eigenspaces for eigenvalues of modulus 1. By convention, if A has no eigenvalues of modulus 1 we let $V_{\text{rec}} = 0$. Similarly, if A has only eigenvalues of modulus 1 and no generalised eigenvalues we let $V_{\text{non-rec}} = 0$. Thus, we decompose the state space $\mathbb{R}^n$ into a part V_{rec} on which A exhibits purely rotational behaviour, and a part $V_{\text{non-rec}}$ where A is additionally expansive or contractive.

We will work with several different norms throughout this paper. In addition to the familiar ℓ^2 and ℓ^∞ norms we introduce a third norm, depending on the matrix A, that combines features of the two. It facilitates block-wise arguments while ensuring that the restriction of A to V_{rec} is an isometry.

Write $\mathbb{R}^n$ as a direct sum $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_s \oplus W_1 \oplus \cdots \oplus W_t$, where $V_1, \ldots, V_s$ correspond to the Jordan blocks of A associated with real eigenvalues and $W_1, \ldots, W_t$ correspond to the Jordan blocks of A associated with non-real eigenvalues. Let $\pi_{W_j} : \mathbb{R}^n \to W_j$ and $\pi_{V_j} : \mathbb{R}^n \to V_j$ denote the orthogonal projections onto W_j and V_j respectively.

For a vector $x \in V_i$, let $\|x\|_J^{V_i} = \|x\|_\infty$. For a vector $x = (x_1, y_1, \ldots, x_k, y_k) \in W_i$, let

$$\|x\|_J^{W_i} = \max_{j=1,\ldots,k} \left(\sqrt{x_j^2 + y_j^2} \right).$$

For a vector $x \in \mathbb{R}^n$, let

$$\|x\|_J = \max\left\{\max_{j=1,\dots,s} \left\|\pi_{V_j}(x)\right\|_J^{V_j}, \max_{j=1,\dots,t} \left\|\pi_{W_j}(x)\right\|_J^{W_j}\right\}.$$

Call $\|x\|_J$ the Jordan norm of x. Observe that $\|x\|_J$ depends on the choice of the V_i's and W_i's. The Jordan norm compares to the ℓ^2- and ℓ^∞- norms as follows:

$$n^{-1/2}\|x\|_J \le n^{-1/2}\|x\|_2 \le \|x\|_\infty \le \|x\|_J \le \|x\|_2 \le n^{1/2}\|x\|_\infty \le n^{1/2}\|x\|_J.$$

Let $\varepsilon > 0$. Consider the ball $B_J(0,\varepsilon) \subseteq \mathbb{R}^n$ about 0 with respect to the distance induced by the $\|\cdot\|_J$-norm. We partition K into three sets:

$$K_{\text{rec}} = K \cap V_{\text{rec}}$$
$$K_{<\varepsilon} = K \cap (V_{\text{rec}} \oplus ((V_{\text{non-rec}} \cap B_J(0,\varepsilon)) \setminus \{0\}))$$
$$K_{\ge\varepsilon} = K \cap (V_{\text{rec}} \oplus (V_{\text{non-rec}} \setminus B_J(0,\varepsilon)))$$

4 A quantitative version of Kronecker's theorem for complex algebraic numbers

Our central tool for bounding the escape time in the recurrent case is a quantitative version of Kronecker's theorem for complex algebraic numbers.

Let $(\lambda_1,\dots,\lambda_m)$ be complex algebraic numbers of modulus 1. Our goal is to find for all $\varepsilon > 0$ a bound N such that for all $(\alpha_1,\dots,\alpha_m) \in \mathbb{T}^m$ contained in the closure of the sequence $(\lambda_1^t,\dots,\lambda_m^t)_{t\in\mathbb{N}}$ there exists $t \le N$ such that $|\lambda_j^t - \alpha_j| < \varepsilon$ for all $j = 1,\dots,m$.

We first consider the case where the λ_j's do not admit any integer multiplicative relations. In this case we can employ the following quantitative version of the continuous formulation of Kronecker's theorem, proved in [12]:

▶ **Theorem 10** ([12, Theorem 4.1]). *Let $\varphi_1,\dots,\varphi_N$ and $\zeta_1,\dots,\zeta_N$ be real numbers. Let $\varepsilon_1,\dots,\varepsilon_N$ be positive real numbers with $\varepsilon_j < 1/2$ for all j. Let $M_j = \left\lceil \frac{1}{\varepsilon_j}\log\frac{N}{\varepsilon_j}\right\rceil$. Let $\varphi = (\varphi_1,\dots,\varphi_N)$. Let $\delta = \min\left\{|\varphi\cdot m| \;\middle|\; m \in \mathbb{Z}^N, |m_j| < M_j, m \neq 0\right\}$. Assume that $\delta > 0$. Then in any interval I of length $T \ge 4/\delta$ there is a real number t such that $\|\varphi_j t - \zeta_j\| < \varepsilon_j$, where $\|\cdot\|$ denotes distance to the nearest integer.*

Intuitively, the number δ in Theorem 10 is a quantitative measure of the linear independence of the φ_j's, as it bounds away from zero all integer linear combinations of the φ_j's with suitably bounded coefficients. In our case we consider the numbers $\varphi_j = \log\lambda_j$. For our purpose we need to obtain a bound on t, and thus a bound on δ, in terms of the algebraic complexity of the numbers $\lambda_1,\dots,\lambda_m$. This is achieved by invoking a quantitative version of Baker's theorem on linear forms in logarithms due to Baker and Wüstholz [1]. Recall that any algebraic number μ is the root of a unique irreducible polynomial p_μ with pairwise coprime integer coefficients. The *height* of an algebraic number μ is the maximum of the absolute values of the coefficients of p_μ. The *degree* of μ is the degree of p_μ. Recall that a field E is called an *extension* of a field F if E contains F as a subfield. The *degree* of a field extension $E \supseteq F$ is the dimension of E as an F-vector space.

▶ **Theorem 11.** *Let $\mu_1, \dots, \mu_N$ be algebraic numbers, none of which is equal to* 0 *or* 1*. Let*

$$L(z_1, \dots, z_N) = b_1 z_1 + \cdots + b_N z_N$$

be a linear form with rational integer coefficients $b_1, \dots, b_N$. Let B be an upper bound on the absolute values of the b_j's. For $j = 1, \dots, N$, let $A_j \geq \exp(1)$ be a bound on the height of μ_j. Let d be the degree of the field extension $\mathbb{Q}(\mu_1, \dot{,} \mu_N)$ generated by $\mu_1, \dots, \mu_N$ over $\mathbb{Q}$. Fix a determination of the complex logarithm log*. Let $\Lambda = L(\log \mu_1, \dots, \log \mu_N)$. If $\Lambda \neq 0$ then*

$$\log |\Lambda| > -(16Nd)^{2(N+2)} \log A_1 \cdot \cdots \cdot \log A_N \log B.$$

Finally, in the case where the λ_j's admit integer multiplicative relations, we employ Theorem 7 to bound their complexity. We arrive at the following result:

▶ **Theorem 12.** *Let $(\lambda_1, \dots, \lambda_m)$ be complex algebraic numbers of modulus* 1*. Assume that the numbers $2\pi i, \log \lambda_1, \dots, \log \lambda_s$ are linearly independent over the rationals, where $0 \leq s \leq m$. Let d be the degree of the field extension $\mathbb{Q}(\lambda_1, \dots, \lambda_s)$. Let $A_1, \dots, A_s \geq \exp(1)$ be upper bounds on the heights of $\lambda_1, \dots, \lambda_s$. Let $\ell \in \mathbb{N}$, and $\varepsilon_{s+1}, \dots, \varepsilon_m \in \mathbb{Z}^s$ be such that*

$$\lambda_j^\ell = (\lambda_1, \dots, \lambda_s)^{\varepsilon_j}$$

for all $j = s+1, \dots, m$. By convention, if $s = 0$ the right-hand side of the above equation is to be taken equal to 1*.*

Let

$$L = \max \left\{ \ell, \sum_{k=1}^{s} |\varepsilon_{s+1,k}|, \dots, \sum_{k=1}^{s} |\varepsilon_{m,k}| \right\}.$$

Let $\alpha_1, \dots, \alpha_m \in \mathbb{T}^m$ be such that any rational linear relation between the numbers $2\pi i, \log \lambda_1, \dots, \log \lambda_m$ is also satisfied by the numbers $2\pi i, \log \alpha_1, \dots, \log \alpha_m$. Let $\varepsilon > 0$. Then there exists a positive integer

$$t \leq 8\pi\ell \left(\tfrac{2\pi L}{\varepsilon}\right)^s \left(2s \tfrac{2\pi L}{\varepsilon} \left\lceil \tfrac{4\pi L}{\varepsilon} \log \tfrac{4\pi s L}{\varepsilon} \right\rceil\right)^{(16(s+1)d)^{2(s+3)} \log A_1 \cdot \cdots \cdot \log A_s} + \ell$$

such that $\left|\lambda_j^t - \alpha_j\right| < \varepsilon$ for $j = 1, \dots, m$.

Proof. An outline of the proof is sketched above. See [10, Appendix D] for a full proof. ◀

For the purpose of bounding the escape time, the following coarse bound suffices:

▶ **Corollary 13.** *There exists an integer function $\mathrm{Kron}(n, \tau, P) \in 2^{(\tau P)^{n^{O(1)}}}$, such that the following holds true:*

Let $\lambda_1, \dots, \lambda_n$ be algebraic numbers of modulus 1*. Assume that the degree of each λ_j is bounded by n. Let τ be a bound on the bitsize of the coefficients of the minimal polynomials of the λ_j's. Let P be a positive integer. Let $\alpha_1, \dots, \alpha_n$ be complex numbers which are contained in the closure of the sequence $(\lambda_1^t, \dots, \lambda_n^t)_{t \in \mathbb{N}}$. Then there exists a $t \leq \mathrm{Kron}(n, \tau, P)$ such that $|\alpha_j - \lambda_j^t| < 2^{-P}$ for all $j \in \{1, \dots, n\}$.*

Proof. By Kronecker's theorem, any integer multiplicative relation between the λ_j's is also satisfied by the α_j's. Theorem 12 hence yields a bound on t such that $|\alpha_j - \lambda_j^t| < 2^{-P}$ holds for all $j \in \{1, \dots, n\}$.

This bound is given in terms of quantities s, d, ℓ, $\varepsilon_{s+1}, \dots, \varepsilon_m \in \mathbb{Z}^s$, $A_1, \dots, A_s$, and L. It remains to show that these quantities can be chosen to be suitably bounded in terms of n and τ.

Proposition 26 in [10, Appendix D], which is mainly based on Theorem 7, shows that numbers ℓ and $\varepsilon_1, \ldots, \varepsilon_m$ can be computed in polynomial space. In particular, the absolute size of L and ℓ is of the form $2^{(n\tau)^{O(1)}}$. The numbers $\log A_i$ are bounded by τ by assumption. We have $s \leq m \leq n$ by definition. Finally, we have assumed that each λ_j has degree at most n. It follows that the degree d of the field extension $\mathbb{Q}(\lambda_1, \ldots, \lambda_s)$ is bounded by n^n. The result follows from Theorem 12. ◀

5 The recurrent eigenspace

The next lemma establishes as a special case an escape bound for all initial values $x \in K_{\text{rec}}$. In order to combine the recurrent and the non-recurrent case we need a stronger result, however. Thus, we establish not only a bound on the escape time for all initial values $x \in K_{\text{rec}}$, but a bound N such that every $x \in V_{\text{rec}}$ – not just in K_{rec} – has distance at least $1/N$ – not just positive distance – from K. Further, note that Lemma 14 is still applicable in the special cases where $K_{\text{rec}} = \emptyset$ or $V_{\text{rec}} = 0$.

▶ **Lemma 14.** *There exists an integer function* $\text{Rec}(n, d, \tau) \in 2^{(\tau d)^{n^{O(1)}}}$ *with the following property:*

Let $A \in \mathbb{A}^{n \times n}$ be a matrix in real Jordan normal form with algebraic entries. Assume that the minimal polynomial of A has rational coefficients whose bitsize is bounded by τ. Let $K \subseteq \mathbb{R}^n$ be a compact semialgebraic set of complexity at most (n, d, τ). If every point $x \in K_{\text{rec}}$ escapes K under iterations of A then for all $x \in V_{\text{rec}}$ there exists $t \leq \text{Rec}(n, d, \tau)$ such that

$$\text{dist}_{\ell^2}(A^t x, K) > \frac{\sqrt{n}}{\text{Rec}(n, d, \tau)}.$$

Proof. The full proof is given in [10, Appendix F]. We only sketch an outline here.

We first prove the result for initial points $x \in K_{\text{rec}}$. For these points, the closure of the orbit $\overline{\mathcal{O}_A(x)}$ of x under A is a compact semialgebraic set. We employ Corollary 13 to obtain for all $\varepsilon > 0$ a doubly exponential bound N such that for all $x \in K_{\text{rec}}$ and all $y \in \overline{\mathcal{O}_A(x)}$ there exists $t \leq N$ such that $\|A^t x - y\|_2 < \varepsilon$. We then use Theorem 4 to obtain a uniform at most doubly exponentially small lower bound on the quantity

$$\inf_{x \in K_{\text{rec}}} \sup_{y \in \overline{\mathcal{O}_A(x)}} \inf_{z \in K} \|y - z\|_2^2 .$$

In order to apply this theorem we construct an auxiliary semialgebraic set, whose complexity is controlled by Theorem 2. Combining these two steps, we obtain a function Rec_0 that satisfies the statement of the lemma for all initial points $x \in K_{\text{rec}}$.

Finally, we extend the result to all initial points $x \in V_{\text{rec}}$. The special case where $K_{\text{rec}} = \emptyset$ is treated using Theorem 4.

In the case where K_{rec} is non-empty we obtain from Lemma 5 that every $x \in V_{\text{rec}}$ which is doubly exponentially close to K with a sufficiently large constant in the third exponent is already doubly exponentially close to K_{rec}, with a slightly smaller constant in the third exponent. Now, any point that is sufficiently far away from K trivially satisfies the claim. By the preceding discussion, points $x \in V_{\text{rec}}$ that are sufficiently close to K are already sufficiently close to K_{rec}, so that there exists an escaping orbit $\overline{\mathcal{O}_A(x')}$ with $x' \in K_{\text{rec}}$ which is close to the orbit of x since A is an isometry on V_{rec}. This allows us to reduce the result to the already established result for initial values in K_{rec}. ◀

6 The non-recurrent eigenspace

The next lemma concerns the subset $K_{\geq\varepsilon}$ of K containing the points in K that are bounded away from V_{rec} by some $\varepsilon > 0$.

For any such point, there exist coordinates (or pairs of coordinates if the corresponding eigenvalues are not real) whose contribution to the Jordan norm is greater than ε. Moreover, the contribution to the Jordan norm of these coordinates does not stay constant under applications of A. If the contribution to the norm of at least one such coordinate is increasing under applications of A, the orbit will eventually leave K, since K is compact. Moreover, Theorem 3 yields an upper bound on the escape time.

Coordinates whose contribution to the norm is decreasing under applications of A will, after sufficiently many iterations, contribute less than ε. We establish a uniform upper bound on the number of iterations required to ensure this for all such coordinates. Combining this with the previous bound, we obtain a number N such that after at most N applications of A, every $x \in K_{\geq\varepsilon}$ has either escaped K, entered $K_{<\varepsilon} \cup K_{\mathrm{rec}}$, or it remains in $K_{\geq\varepsilon}$ because it has a component whose contribution to the norm was initially smaller than ε, but grew beyond ε under iteration of A. In the last case, the point will grow in norm beyond the bound established in Theorem 3 and thus escape K after a further N applications of A. This yields a uniform bound on the number of iterations that are required for any point $x \in K_{\geq\varepsilon}$ to either leave K entirely or move into $K_{<\varepsilon} \cup K_{\mathrm{rec}}$.

The overall structure of this proof closely follows the one given in [11], where the assumptions allow the authors to restrict the discussion to real eigenvalues.

▶ **Lemma 15.** *There exists an integer function* $\mathrm{NonRec}(n, d, \tau, P) \in 2^{(d\tau P)^{n^{O(1)}}}$ *with the following property:*

Let K be a compact semialgebraic set of complexity at most (n, d, τ). Let $A \in \mathbb{A}^{n\times n}$ be a matrix in real Jordan normal form. Assume that the characteristic polynomial of A has rational coefficients whose bitsize is bounded by τ. Let P be a positive integer.

Then for all $x \in K_{\geq 2^{-P}}$ there exists $t \leq \mathrm{NonRec}(n, d, \tau, P)$ such that $A^t x \notin K_{\geq 2^{-P}}$.

Proof. See [10, Appendix G] for details. ◀

7 Proof of Theorem 1

In the previous two sections, we successively showed how to establish a bound on the escape time for an instance (A, K) when the orbit remains in the recurrent eigenspace and how the orbit behaves when it starts away from the recurrent eigenspace. In this section, we show how to combine both results in order to establish an escape bound for any starting point in K. This will thus prove Theorem 1.

Let (A_0, K_0) be an instance of the compact escape problem, where $K_0 \subseteq \mathbb{R}^n$ is a compact semialgebraic set of complexity at most (n_0, d_0, τ_0) and $A_0 \in \mathbb{Q}^{n\times n}$ is a square matrix with rational entries whose bitsize is bounded by τ_0. Assume that every point $x \in K_0$ escapes K_0 under iterations of A_0.

Apply Lemma 9 to convert the instance (A_0, K_0) into an equivalent instance (A, K) such that $A \in \mathbb{A}^{n\times n}$ is in real Jordan normal form. Then the set K has complexity at most (n, d, τ), were $n = 2n_0$, $d = n_0 d_0$, and $\tau = (n_0\tau_0 d_0)^{C_\tau}$ for some absolute constant C_τ. By construction, the characteristic polynomial of A has rational coefficients of bitsize at most τ.

Let Rec be the function from Lemma 14. Let $\varepsilon = \frac{1}{\mathrm{Rec}(n,d,\tau)}$ and $N_{\mathrm{rec}} = \mathrm{Rec}(n, d, \tau)$. Let $x \in K$. If $x \in K_{\mathrm{rec}}$ then x escapes within N_{rec} steps. Suppose that $x \in K_{<\varepsilon}$.

Then there are two possibilities:

1. We have $A^t x \notin K_{\geq \varepsilon}$ for all $t \leq N_{\text{rec}}$.
2. We have $A^t x \in K_{\geq \varepsilon}$ for at least one $t \leq N_{\text{rec}}$.

In the first case, the orbit of x remains close to V_{rec} for long enough that we can rely on Lemma 14. Indeed, let x_0 denote the orthogonal projection of x onto V_{rec}. Let $t \leq N_{\text{rec}}$ be such that $\text{dist}_{\ell^2}(A^t x_0, K) > \sqrt{n}\varepsilon$. Since $A^t x \notin K_{\geq \varepsilon}$, we have $\left\|A^t x - A^t x_0\right\|_J < \varepsilon$, so that $\left\|A^t x - A^t x_0\right\|_2 < \sqrt{n}\varepsilon$. Let $y \in K$. Then

$$\left\|A^t x - y\right\|_2 \geq \left\|A^t x_0 - y\right\|_2 - \left\|A^t x - A^t x_0\right\|_2 > \sqrt{n}\varepsilon - \sqrt{n}\varepsilon = 0.$$

Thus, x escapes K under iterations of A.

In the second case, let t_1 be such that $A^{t_1} x \in K_{\geq \varepsilon}$. Let NonRec be the function from Lemma 15. Let $N_{\geq \varepsilon} = \text{NonRec}(n, d, \tau, \lceil \log(1/\varepsilon) \rceil)$. By Lemma 15 there exists $t_2 \leq N_{\geq \varepsilon}$ such that $A^{t_2} A^{t_1} x$ is contained either in $K_{<\varepsilon} \cup K_{\text{rec}}$ or in the complement of K. In the latter case we are done. In the former case we apply the initial case distinction: either for all $t \leq N_{\text{rec}}$ we have $A^t A^{t_2} A^{t_1} x \notin K_{\geq \varepsilon}$ or we have $A^{t_3} A^{t_2} A^{t_1} x \in K_{\geq \varepsilon}$ for at least one $t_3 \leq N_{\text{rec}}$. Once again, in the first case, the point has escaped. By repeating this reasoning, we construct a (finite or infinite) sequence $t_1, t_2, \ldots$ such that $t_i \leq N_{\text{rec}}$ if i is odd and $t_i \leq N_{\geq \varepsilon}$ if i is even and

$$A^{t_s} \cdot \cdots \cdot A^{t_1} x \in \begin{cases} K_{<\varepsilon} \cup K_{\text{rec}} & \text{if } s \text{ is even,} \\ K_{\geq \varepsilon} & \text{if } s \text{ is odd.} \end{cases}$$

We claim that the sequence $t_1, t_2, \ldots$ is finite and contains at most n^3 elements.

Consider a real Jordan block of A of size $m \leq n$ associated to the eigenvalue Λ. Denote by x_J the orthogonal projection of x onto the dimensions associated with this block.

Assume first that Λ is a real eigenvalue (as opposed to a 2×2 block representing a complex eigenvalue). If $\Lambda = 0$, then clearly $\left\|J^k x_J\right\|_J$ is monotonically decreasing. Thus, assume in the sequel that $\Lambda \neq 0$.

Let $j \in \{1, \ldots, m\}$. The $m - j + 1$'th component of the vector $J^k x_J$, viewed as a function of t, is an exponential polynomial $E_j(t) = \Lambda^t P(t)$, where $P \in \mathbb{R}[z]$ is a real polynomial of degree $j - 1$. Consider the real function

$$(E_j(\cdot))^2 \colon \mathbb{R} \to \mathbb{R},\ (E_j(t))^2 = |\Lambda|^{2t} \, |P(t)|^2.$$

This function is differentiable in t with derivative

$$\tfrac{d}{dt}(E_j(t))^2 = \Lambda^{2t} \left(\log(\Lambda^2)(P(t)^2) + 2P(t)P'(t) \right).$$

This derivative vanishes if and only if the factor $\left(\log(\Lambda^2)(P(t)^2) + 2P(t)P'(t)\right)$ vanishes. This factor is a polynomial of degree $2j - 2$, so that it has at most $2j - 2$ real zeroes. It follows that there exist numbers $t_{j,1}, \ldots, t_{j,m_j}$ with $m_j \leq 2j - 2$ such that the function $(E_j(t))^2 - \varepsilon^2$ does not change its sign in any of the open intervals

$$(0, t_{j,1}), (t_{j,1}, t_{j,2}), \ldots, (t_{j,m_j - 1}, t_{j,m_j}), (t_{j,m_j}, +\infty).$$

Thus, the norm $\|J^t x_J\|_J$ changes from smaller than ε to bigger than ε at most

$$\sum_{j=1}^{m} (2j - 2) = 2 \sum_{j=1}^{m} j - 2m = (m + 1)m - 2m = m^2 - m$$

times.

The case where Λ represents a complex eigenvalue λ is similar. However, we now consider the evolution of the two coordinates corresponding to one Λ-block simultaneously.

For $j \in \{1, \dots, m\}$, write $E_j(t)$ for the $m-j+1$'th component of the vector $J^t x_J$, viewed as a function of t. We have for all $j \in \{1, \dots, m/2\}$ that the function

$$F_j(t) = (E_{2j}(t))^2 + (E_{2j-1}(t))^2$$

is an exponential polynomial $F_j(t) = |\lambda|^t P_j(t)$, where $P_j \in \mathbb{R}[z]$ is a real polynomial of degree $j-1$. Therefore, exactly as in case where Λ is a real eigenvalue, the derivative of F_j vanishes at most $2j-2$ times. From which we can deduce that the norm $\|J^t x_J\|_J$ crosses the ε-threshold at most $m^2 - m$ times.

Estimating generously, we have at most n Jordan blocks of size at most n, each of which crosses the ε-threshold at most $n^2 - n$ times. In total, we cross the threshold at most $n^3 - n^2$ times. The total escape bound is hence $n^3 \max\{N_{\mathrm{rec}}, N_{\geq\varepsilon}\}$. By the same argument, the same escape bound holds true when the initial point x lies in $K_{\geq\varepsilon}$.

Substituting the constants N_{rec}, $N_{\geq\varepsilon}$, n, d, and τ with their definitions, we obtain the upper bound

$$\begin{aligned}&\mathrm{CompactEscape}(n_0, d_0, \tau_0) = \\ &\quad (2n_0)^3 \max\Big\{ \mathrm{Rec}\left(2n_0, n_0 d_0, (n_0 d_0 \tau_0)^{C_\tau}\right), \\ &\qquad \mathrm{NonRec}\left(2n_0, n_0 d_0, (n_0 d_0 \tau_0)^{C_\tau}, \log\left\lceil \mathrm{Rec}\left(2n_0, n_0 d_0, (n_0 d_0 \tau_0)^{C_\tau}\right)\right\rceil\right) \Big\}.\end{aligned}$$

One easily verifies that $\mathrm{CompactEscape}(n, d, \tau) \in 2^{(d\tau)^{n^{O(1)}}}$ as claimed.

8 A matching lower bound on escape time

In Theorem 1 we established a uniform upper bound on the escape time for all positive instances of the Compact Escape Problem. Our bound is doubly exponential in the ambient dimension and singly exponential in the rest of the data. We will now show that this bound cannot be significantly improved by showing that a doubly exponential bound cannot be avoided even for purely rotational systems. A second example displaying a doubly exponential lower bound is presented in [10, Appendix H].

▶ **Example 16.** For $(n, d, \tau) \in \mathbb{N}^3$, let $K_{(n,d,\tau)} \subseteq \mathbb{R}^{n+2}$ be the set of all points $(x, y, u_1, \dots, u_n)$ satisfying the (in)equalities: $x^2 + y^2 = 1, u_1 = 2^{-\tau}$, $(x-1)^2 + y^2 \geq u_n$ and for $1 \leq i \leq n-1$, $u_{i+1} = (u_i)^d$.

Hence, $K_{(n,d,\tau)} = \left(S^1 \setminus B\left((1,0), 2^{-\tau d^{n-1}}\right)\right) \times \left\{\left(2^{-\tau}, 2^{-\tau d}, \dots, 2^{-\tau d^{n-1}}\right)\right\}$, where $S^1 \subseteq \mathbb{R}^2$ is the unit circle. Let $a = \frac{3}{5}$, $b = \frac{4}{5}$. Let

$$A_{(n,d,\tau)} = \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & I_n \end{pmatrix}$$

where I_n is the $n \times n$- identity matrix. It is easy to see that the complex number $\frac{3}{5} + i\frac{4}{5}$ has modulus 1 and is not a root of unity. It follows from Dirichlet's theorem on simultaneous Diophantine approximation that the orbit of A is equal to $S^1 \times \left\{\left(2^{-\tau}, 2^{-\tau d}, \dots, 2^{-\tau d^{n-1}}\right)\right\}$, so that every initial point escapes under A.

We claim that there exists a point $x \in K_{(n,d,\tau)}$ that requires $2^{\tau d^{n-1}}$ steps to escape. Indeed, let $x_0 \in K_{(n,d,\tau)}$ be an arbitrary initial point. Consider the orbit $x_t = A^t x_0$. Let $N < 2^{\tau d^{n-1}}$. By the pigeonhole principle, the finite set of points $x_0, \dots, x_N$ contains at least one consecutive pair of points x_i, x_j on the circle such that the points x_i and x_j are joined by an arc of the circle of length strictly greater than $2/N$. It follows that we can ensure that none of the points $x_1, \dots, x_N$ is outside of $K_{(n,d,\tau)}$ by applying a suitable planar rotation to all points. Since all planar rotations commute, there exists for each angle θ an initial point $x_\theta \in S^1 \times \left\{\left(2^{-\tau}, 2^{-\tau d}, \ldots, 2^{-\tau d^{n-1}}\right)\right\}$, such that the orbit of x_θ under A is equal to the orbit of x_0 under A rotated by θ. This proves the claim.

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